Some Torsional Local Models for Heterotic Strings

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1 Introduction

In his study of heterotic superstrings with torsion, Strominger [Str86] derived a system of partial differential equations from considerations of supersymmetry and anomaly cancellation, which is later known as the *Strominger system*. In real dimension 6, the internal manifold X has to be a complex 3-fold with trivial canonical bundle. Moreover, X is equipped with an Hermitian metric ω and an Hermitian holomorphic vector bundle (E, h). Let Ω be a nowhere vanishing holomorphic 3-form on X, then the Strominger system consists of the following equations[1]:

$$(1) \qquad F \wedge \omega^2 = 0, \quad F^{0,2} = F^{2,0} = 0,$$

$$(2) \qquad \sqrt{-1}\partial\bar{\partial}\omega = \frac{\alpha'}{4}(\mathrm{Tr}(R \wedge R) - \mathrm{Tr}(F \wedge F)),$$

$$(3) \qquad \mathrm{d}^*\omega = i(\bar{\partial} - \partial)\log\|\Omega\|_\omega.$$

In the above equations, α' is a positive coupling constant, R and F are curvature 2-forms of $T^{1,0}X$ and E respectively, computed with respect to certain metric connections. As the Green-Schwarz anomaly cancellation mechanism works for arbitrary connection [Hul86], we have the freedom to choose the Chern connection to compute R, as suggested in [Str86, Section 4.3]. We will call Equations (1) and (2) the *Hermitian-Yang-Mills* equation and *anomaly cancellation* equation respectively.

It is very clear that the Strominger system generalizes the complex Monge-Ampère equation used in the torsion-free theory in which X is a Kähler Calabi-Yau. Assuming X is Kähler, if we further let E be the holomorphic tangent bundle $T^{1,0}X$, then $R = F$ and Equations (1)-(3) are solved automatically. These solutions will be called the *Kähler solutions*.

The first attempt to the Strominger system was made in [Str86, Section 4.2 & 4.3], where orbifold solutions and perturbative solutions (up to first order) of Kähler solutions were obtained.

A huge progress was achieved by Li-Yau [LY05] almost 20 years later. Their contribution has two aspects. Firstly, Li and Yau realized that Equation (3) is equivalent to

$$(4) \qquad \mathrm{d}(\|\Omega\|_\omega \cdot \omega^2) = 0,$$

whose geometric meaning is very clear. Recall from [Mic82] that an Hermitian metric ω is called *balanced* if it is co-closed. As Equation (4) says that ω is a balanced metric after a conformal change, we will call this equation the *conformally balanced equation*. Such an interpretation imposes a mild topological restriction on X, on which we refer to Michelsohn's intrinsic characterization of balanced manifolds [Mic82, Theorem 4.7]. Secondly, Li and Yau were able to obtain the first irreducible smooth solutions to the Strominger system by deforming Kähler solutions. Such techniques were further developed in [AGF12b, AGF12a] to deal with more general perturbations.

Another breakthrough was due to Fu, Yau and their collaborators. They observed that on the geometric models constructed by Goldstein-Prokushkin [GP04], a clever choice of ansatz reduces the

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[1]Strominger's original paper [Str86] uses a different normalization for Equation (2).

whole Strominger system to a Monge-Ampère type equation of a single dilaton function on the Calabi-Yau 2-fold base. Using this idea, they were able to construct genuine non-Kähler solutions on both compact backgrounds [BBF+06, FY07, FY08][2] and local models [FTY09]. Such a method can be further modified to yield more heterotic non-Kähler geometries [BTY09].

It should be mentioned that solutions to the Strominger system have also been found on blow-up of conifold [CI10], on various nilmanifolds [FIUV09, Gra11, FIUV14, UV14, UV15], on (quotients of) $SL(2, \mathbb{C})$ [FY15] and on Calabi-Gray models [Fei15a].

As a generalization of the complex Monge-Ampère equation solved by Yau [Yau78], the Strominger system provides a natural framework to find canonical metrics on non-Kähler Calabi-Yau manifolds. This is closely related to the moduli problem of Calabi-Yau manifolds via Reid's fantasy [Rei87]. Recall that by blowing down disjoint $(-1, -1)$-curves in a projective Calabi-Yau 3-fold, one gets a singular Calabi-Yau 3-fold with finitely many ordinary double points. These singularities can be further smoothed out to yield various topologically distinct 3-folds with trivial canonical bundle which are in general non-Kähler. Such a geometric manipulation is known as the *conifold transition*. Roughly speaking, Reid's fantasy says that all reasonably nice Calabi-Yau 3-folds can be connected via a sequence of conifold transitions.

The local model of conifold transition was studied by Candelas-de la Ossa [CdlO90] in great details, where explicit Ricci-flat Kähler metrics on both deformed and resolved conifold were constructed and their asymptotic behaviors were carefully analyzed. Because conifold transitions in general take place in the non-Kähler category, it is of great importance to understand the local model from a non-Kähler point of view. That is, to look for non-Kähler solutions to the Strominger system on both deformed and resolved conifold.

The first half was partially achieved in [FY15], where homogeneous solutions to the Strominger system on deformed conifold were found through an identification with $SL(2, \mathbb{C})$. The second half will be covered in this paper. Indeed, we are able to solve the Strominger system on the resolved conifold, and actually, on much more general local models. The ansatz we use comes from the twistor space construction, which is based on the ideas used in [FY08, Fei15a, Fei15b]. It should be noticed that our solution has a nontrivial $\mathrm{Tr}(F \wedge F)$ term in order to cancel the anomaly, which does not follow from simple topological consideration.

This paper is organized as follows. In Section 2 we give a brief account of preliminaries, including differential geometry of resolved conifolds, hyperkähler 4-manifolds and their twistor spaces. Section 3 is devoted to solving the Strominger system on a class of non-compact Calabi-Yau 3-folds which can be obtained by removing a fiber from the twistor fibration of hyperkähler 4-manifolds. Some special examples of these solutions are analyzed in Section 4. In particular, we obtain nontrivial solutions on $\mathbb{C}^3$ and on resolved conifold. Section 5 is of somewhat independent interest. We will make a discussion about Chern-Ricci flat balanced metrics on non-Kähler Calabi-Yau manifolds and give some examples. This can be viewed as a generalization of Calabi's work [Cal79]. In addition, we derive the Euler-Lagrange equation for extremal balanced metrics and show that Chern-Ricci flat balanced metrics are extremal (and in certain cases the only extremal ones).

Acknowledgement

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[2]The same ansatz on T^2-bundles over K3 surfaces was also discussed in [DRS99].

2 Preliminaries

2.1 Resolved Conifold

Let Q be the *conifold*, or in other words an affine quadric cone in $\mathbb{C}^4$. That is,

$$Q = \{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0\}.$$

It is quite obvious that Q has a unique singular point at the origin, which is known as an ordinary double point. The small resolution of ordinary double points was first discovered by Atiyah [Ati58] that we shall manifest.

By a linear transformation of coordinates

$$\begin{cases} w_1 = z_1 + iz_2 \\ w_2 = z_3 + iz_4 \\ w_3 = iz_4 - z_3 \\ w_4 = z_1 - iz_2 \end{cases},$$

we can identify Q as the zero locus of $w_1 w_4 - w_2 w_3$, or more suggestively,

$$\det \begin{pmatrix} w_1 & w_2 \\ w_3 & w_4 \end{pmatrix} = 0.$$

Now let $\mathbb{CP}^1$ be parameterized by $\lambda = [\lambda_1 : \lambda_2]$. Consider

$$\widetilde{Q} = \left\{ (w, \lambda) \in \mathbb{C}^4 \times \mathbb{CP}^1 : \begin{pmatrix} w_1 & w_2 \\ w_3 & w_4 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

It is not hard to see that $\widetilde{Q}$ is smooth and the first projection

$$p_1 : \widetilde{Q} \to Q$$

is an isomorphism away from $\{0\} \times \mathbb{CP}^1 \subset \widetilde{Q}$. Therefore we call $\widetilde{Q}$ the *resolved conifold* and $p_1 : \widetilde{Q} \to Q$ the small resolution of Q, as the exceptional set $\mathbb{CP}^1$ is of codimension 2. Moreover, the second projection $p_2 : \widetilde{Q} \to \mathbb{CP}^1$ allows us to identify $\widetilde{Q}$ with the total space of $\mathcal{O}(-1, -1) \to \mathbb{CP}^1$. As a consequence, the resolved conifold $\widetilde{Q}$ has trivial canonical bundle.

2.2 Hyperkähler 4-manifolds and Their Twistor Spaces

Let (N, g) be a Riemannian 4-manifold. If in addition M admits three integrable complex structures I, J and K with $IJK = -\mathrm{id}$ such that g is a Kähler metric with respect to any of $\{I, J, K\}$, then we call (N, g, I, J, K) a hyperkähler 4-manifold.

Now we assume that N is a hyperkähler 4-manifold and denote by ω_I, ω_J and ω_K the associated Kähler forms with respect to corresponding complex structures. One can easily check that $\omega_J + i\omega_K$ is a holomorphic nowhere vanishing 2-form with respect to the complex structure I, therefore (N, I) has trivial canonical bundle.

If N is compact, then by the Enriques-Kodaira classification of complex surfaces, N must be either a complex torus or a K3 surface. However, if we allow N to be noncompact, there are many more possibilities. An extremely important class of them is the so-called ALE (asymptotically locally Euclidean) spaces. These spaces were first discovered as gravitational (multi-)instantons by physicists [EH78, GH78] and finally classified completely by Kronheimer [Kro89a, Kro89b].

It is well-known that a hyperkähler 4-manifold N is anti-self-dual, therefore its twistor space Z is a complex 3-fold [AHS78]. Following [HKLR87], the twistor space Z can be described as follows. Let ζ parameterize $\mathbb{CP}^1$. We shall identify $\mathbb{CP}^1$ with $S^2 = \{(\alpha, \beta, \gamma) \in \mathbb{R}^3 : \alpha^2 + \beta^2 + \gamma^2 = 1\}$ via

$$(\alpha, \beta, \gamma) = \left(\frac{1 - |\zeta|^2}{1 + |\zeta|^2}, \frac{\zeta + \overline{\zeta}}{1 + |\zeta|^2}, \frac{i(\overline{\zeta} - \zeta)}{1 + |\zeta|^2} \right).$$

3

The twistor space Z of N is defined to be the manifold $Z = \mathbb{CP}^1 \times N$ with the almost complex structure $\mathfrak{I}$ given by

$$\mathfrak{I} = j \oplus (\alpha I_x + \beta J_x + \gamma K_x)$$

at point $(\zeta, x) \in \mathbb{CP}^1 \times N$, where j is the standard complex structure on $\mathbb{CP}^1$ with holomorphic coordinate ζ. It is a theorem of [HKLR87] that $\mathfrak{I}$ is integrable and the projection $p : Z \to \mathbb{CP}^1$ is a holomorphic fibration. We shall call $p : Z \to \mathbb{CP}^1$ the twistor fibration.

The twistor spaces of ALE spaces can be described in other ways. For instance, the twistor spaces of ALE spaces of type A can be constructed very concretely using algebraic geometry [Hit79]. For later use, we shall give a different characterization of the A_1 case. i.e. the twistor space of Eguchi-Hansen space, as Hitchin did in [Hit81, Section 8].

Let Q and $\widetilde{Q}$ be conifold and resolved conifold described above. Consider the map

$$\rho = z_4 \circ p_1 : \widetilde{Q} \xrightarrow{p_1} Q \xrightarrow{z_4} \mathbb{C}.$$

It is obvious that, when $z_4 \neq 0$, the fiber $\rho^{-1}(z_4)$ is a smooth affine quadric in $\mathbb{C}^3$. After a little work, we can see that $\rho^{-1}(0)$ is biholomorphically equivalent to the total space of $T^*\mathbb{CP}^1$. It follows that ρ is a fibration.

Now let $\rho' : \widetilde{Q}' \to \mathbb{C}$ be another copy of $\rho : \widetilde{Q} \to \mathbb{C}$. We may glue these two fibrations holomorphically by identifying $\rho^{-1}(\mathbb{C}^\times) \xrightarrow{\rho} \mathbb{C}^\times$ with $\rho'^{-1}(\mathbb{C}^\times) \xrightarrow{\rho'} \mathbb{C}^\times$ via

$$(z_1', z_2', z_3', z_4') = \left(\frac{z_1}{z_4^2}, \frac{z_2}{z_4^2}, \frac{z_3}{z_4^2}, \frac{1}{z_4} \right).$$

As a consequence, we get a holomorphic fibration over $\mathbb{CP}^1$, which is exactly the twistor fibration of Eguchi-Hansen space.

We conclude that, when performing hyperkähler rotations, there are exactly two complex structures on the Eguchi-Hansen space up to biholomorphism. There is a pair of two antipodal points on $\mathbb{CP}^1$, over which the fibers of twistor fibration are biholomorphic to the total space of $T^*\mathbb{CP}^1$. We shall call these fibers *special*. All the other fibers are biholomorphic to the smooth affine quadric in $\mathbb{C}^3$. A key observation from this construction is the following proposition.

Proposition 2.1. (Hitchin, [Hit81, Section 8]) If we remove a special fiber from the total space of the twistor fibration of Eguchi-Hansen space, then what is left is biholomorphic to the resolved conifold.

3 Local Torsional Models of Heterotic Strings

Let us first briefly recall the results in [Fei15a].

Given any hyperkähler 4-manifold N and any oriented minimal surface Σ_g of genus g in flat T^3, we can cook up a complex structure on $X = \Sigma_g \times N$ such that X is a non-Kähler Calabi-Yau 3-fold. Indeed, X can also be obtained as the total space of the pullback of the twistor fibration of N through the Gauss map of Σ_g. In the case of $N = T^4$, by making use of the fibration structure, we were able to solve the whole Strominger system on X. However, the solution metric is degenerate on the fibers over the branched points of the Gauss map.

This fact suggests that it is probably more natural to consider the Strominger system on the twistor space Z. However, the twistor space Z can never have trivial canonical bundle hence the Strominger system does not make sense. In order to remedy this problem, we need to remove some parts of Z to get a noncompact complex 3-fold $Y \subset Z$ with trivial canonical bundle. It turns out that we only need to remove one single fiber of the twistor fibration.

Now let N be an arbitrary hyperkähler 4-manifold and Z its twistor space with twistor fibration $p : Z \to \mathbb{CP}^1$. Without loss of generality, let F_∞ be the fiber of $p : Z \to \mathbb{CP}^1$ over $\zeta = \infty \in \mathbb{CP}^1$ and

we may remove F_∞ from Z to get a noncompact space $Y := Z \setminus F_\infty$. One can check directly that Y has trivial canonical bundle which can be trivialized by

$$\Omega := (-2\zeta\omega_I + (1 - \zeta^2)\omega_J + i(1 + \zeta^2)\omega_K) \wedge d\zeta.$$

The main theme of this section is to write down a solution to the Strominger system on Y. As we shall see, the recipe for the ansatz is almost identical to the one we used in [Fei15a]. However the curvature of N will come into play and our calculation will be much more complicated. Moreover, we have to take the Hermitian-Yang-Mills equation (1) into account as well.

3.1 The Conformally Balanced Equation (4)

Observe that Y is diffeomorphically a product $\mathbb{C} \times N$ with twisted complex structure. Let $h : N \to \mathbb{R}$ and $g : \mathbb{C} \to \mathbb{R}$ be arbitrary smooth functions. In addition, we use

$$\omega_{\mathbb{CP}^1} = \frac{2i}{(1 + |\zeta|^2)^2} d\zeta \wedge d\bar{\zeta}$$

to denote the round metric of radius 1 on $\mathbb{CP}^1$ and its restriction on $\mathbb{C} = \mathbb{CP}^1 \setminus \{\infty\}$.

Now consider the Hermitian metric

$$(5) \qquad \omega = \frac{e^{2h+g}}{(1 + |\zeta|^2)^2}(\alpha\omega_I + \beta\omega_J + \gamma\omega_K) + e^{2g}\omega_{\mathbb{CP}^1}$$

on $Y = \mathbb{C} \times N$. One can check that

$$(6) \qquad \|\Omega\|_\omega = c \cdot \frac{(1 + |\zeta|^2)^4}{e^{2h+2g}}$$

for some positive constant c and

$$(7) \qquad \omega^2 = 2\frac{e^{4h+2g}}{(1 + |\zeta|^2)^4}\text{vol}_N + 2\frac{e^{2h+3g}}{(1 + |\zeta|^2)^2}(\alpha\omega_I + \beta\omega_J + \gamma\omega_K) \wedge \omega_{\mathbb{CP}^1},$$

where vol_N is the volume form on N. It follows that ω solves the conformally balanced equation (4) for arbitrary g and h by direct computation.

3.2 Curvature Computation and the Anomaly Cancellation (2)

Now we proceed to solve the anomaly cancellation equation (2) using ansatz (5). The first step would be to compute the curvature term $\text{Tr}(R \wedge R)$, using the Chern connection, with respect to the metric (5). To do so, following the method used in [Fei15a], we had better first solve for a local holomorphic frame of $(1,0)$-forms on Y.

We fix I to be the background complex structure on the hyperkähler 4-manifold N. Locally, there exist holomorphic coordinates $\{z_1, z_2\}$ such that the holomorphic 2-form can be expressed as

$$\omega_J + i\omega_K = dz_1 \wedge dz_2.$$

Let κ be the Kähler potential, i.e., we have

$$\omega_I = i\partial\bar{\partial}\kappa = i(\kappa_{1\bar{1}}dz_1 \wedge d\bar{z}_1 + \kappa_{1\bar{2}}dz_1 \wedge d\bar{z}_2 + \kappa_{2\bar{1}}dz_2 \wedge d\bar{z}_1 + \kappa_{2\bar{2}}dz_2 \wedge d\bar{z}_2),$$

where lower indices represent partial derivatives with respect to holomorphic or anti-holomorphic coordinates. As N is hyperkähler, we must have

$$\kappa_{1\bar{1}}\kappa_{2\bar{2}} - \kappa_{1\bar{2}}\kappa_{2\bar{1}} = \frac{1}{4}.$$

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It is convenient to list the following equations

$$I\mathrm{d}z_1 = i\mathrm{d}z_1, \quad I\mathrm{d}z_2 = i\mathrm{d}z_2, \qquad\qquad I\mathrm{d}\bar{z}_1 = -i\mathrm{d}\bar{z}_1, \quad I\mathrm{d}\bar{z}_2 = -i\mathrm{d}\bar{z}_2,$$
$$J\mathrm{d}z_1 = -2(\kappa_{2\bar{1}}\mathrm{d}\bar{z}_1 + \kappa_{2\bar{2}}\mathrm{d}\bar{z}_2), \qquad\qquad J\mathrm{d}\bar{z}_1 = -2(\kappa_{1\bar{2}}\mathrm{d}z_1 + \kappa_{2\bar{2}}\mathrm{d}z_2),$$
$$J\mathrm{d}z_2 = 2(\kappa_{1\bar{1}}\mathrm{d}\bar{z}_1 + \kappa_{1\bar{2}}\mathrm{d}\bar{z}_2), \qquad\qquad J\mathrm{d}\bar{z}_2 = 2(\kappa_{1\bar{1}}\mathrm{d}z_1 + \kappa_{2\bar{1}}\mathrm{d}z_2),$$
$$K\mathrm{d}z_1 = -2i(\kappa_{2\bar{1}}\mathrm{d}\bar{z}_1 + \kappa_{2\bar{2}}\mathrm{d}\bar{z}_2), \qquad\qquad K\mathrm{d}\bar{z}_1 = 2i(\kappa_{1\bar{2}}\mathrm{d}z_1 + \kappa_{2\bar{2}}\mathrm{d}z_2),$$
$$K\mathrm{d}z_2 = 2i(\kappa_{1\bar{1}}\mathrm{d}\bar{z}_1 + \kappa_{1\bar{2}}\mathrm{d}\bar{z}_2), \qquad\qquad K\mathrm{d}\bar{z}_2 = -2i(\kappa_{1\bar{1}}\mathrm{d}z_1 + \kappa_{2\bar{1}}\mathrm{d}z_2),$$

which tell us how I, J and K act on 1-forms.

As $Y = \mathbb{C} \times N$, we can think of z_i as functions locally defined on Y, though they are no longer holomorphic. In the same manner, we shall record how the complex structure $\mathfrak{J}$ on Y acts on 1-forms

$$\mathfrak{J}\mathrm{d}\zeta = i\mathrm{d}\zeta, \qquad\qquad \mathfrak{J}\mathrm{d}\bar{\zeta} = -i\mathrm{d}\bar{\zeta},$$
$$\mathfrak{J}\mathrm{d}z_1 = i\alpha\mathrm{d}z_1 - 2(\beta + i\gamma)(\kappa_{2\bar{1}}\mathrm{d}\bar{z}_1 + \kappa_{2\bar{2}}\mathrm{d}\bar{z}_2), \qquad \mathfrak{J}\mathrm{d}\bar{z}_1 = -i\alpha\mathrm{d}\bar{z}_1 - 2(\beta - i\gamma)(\kappa_{1\bar{2}}\mathrm{d}z_1 + \kappa_{2\bar{2}}\mathrm{d}z_2),$$
$$\mathfrak{J}\mathrm{d}z_2 = i\alpha\mathrm{d}z_2 + 2(\beta + i\gamma)(\kappa_{1\bar{1}}\mathrm{d}\bar{z}_1 + \kappa_{1\bar{2}}\mathrm{d}\bar{z}_2), \qquad \mathfrak{J}\mathrm{d}\bar{z}_2 = -i\alpha\mathrm{d}\bar{z}_2 + 2(\beta - i\gamma)(\kappa_{1\bar{1}}\mathrm{d}z_1 + \kappa_{2\bar{1}}\mathrm{d}z_2).$$

Now consider a 1-form
$$\theta = L\mathrm{d}\zeta + A\mathrm{d}z_1 + B\mathrm{d}z_2 + C\mathrm{d}\bar{z}_1 + D\mathrm{d}\bar{z}_2.$$
If θ is of type (1,0), i.e., $\mathfrak{J}\theta = i\theta$, then we have

$$A = \frac{2i\kappa_{1\bar{2}}}{\zeta}C - \frac{2i\kappa_{1\bar{1}}}{\zeta}D,$$
$$B = \frac{2i\kappa_{2\bar{2}}}{\zeta}C - \frac{2i\kappa_{2\bar{1}}}{\zeta}D.$$

Let
$$\theta_1 = 2i\frac{\kappa_{1\bar{2}}}{\zeta}\mathrm{d}z_1 + 2i\frac{\kappa_{2\bar{2}}}{\zeta}\mathrm{d}z_2 + \mathrm{d}\bar{z}_1, \quad \theta_2 = 2i\frac{\kappa_{1\bar{1}}}{\zeta}\mathrm{d}z_1 + 2i\frac{\kappa_{2\bar{1}}}{\zeta}\mathrm{d}z_2 - \mathrm{d}\bar{z}_2.$$

It is easy to see that they are (1,0)-forms. Moreover, we have
$$\theta = L\mathrm{d}\zeta + C\theta_1 - D\theta_2.$$

Lemma 3.1. θ is a holomorphic $(1,0)$-form if and only if

$$(8) \qquad\qquad 2\zeta\bar{\partial}L = -C((1-\alpha)\theta_1 - 2\mathrm{d}\bar{z}_1) + D((1-\alpha)\theta_2 + 2\mathrm{d}\bar{z}_2)$$

holds, where C and D satisfy

$$(9) \qquad \begin{aligned} \bar{\partial}C &= i(\beta - i\gamma)[C(\kappa_{1\bar{1}\bar{2}}\bar{\theta}_1 - \kappa_{2\bar{1}\bar{2}}\bar{\theta}_2) - D(\kappa_{1\bar{1}\bar{1}}\bar{\theta}_1 - \kappa_{2\bar{1}\bar{1}}\bar{\theta}_2)] \\ \bar{\partial}D &= i(\beta - i\gamma)[C(\kappa_{1\bar{2}\bar{2}}\bar{\theta}_1 - \kappa_{2\bar{2}\bar{2}}\bar{\theta}_2) - D(\kappa_{1\bar{1}\bar{2}}\bar{\theta}_1 - \kappa_{2\bar{1}\bar{2}}\bar{\theta}_2)]. \end{aligned}$$

Proof. The proof follows from straightforward calculation. It is quite clear that θ is holomorphic if and only if
$$\bar{\partial}L \wedge \mathrm{d}\zeta + \bar{\partial}C \wedge \theta_1 - \bar{\partial}D \wedge \theta_2 + C\bar{\partial}\theta_1 - D\bar{\partial}\theta_2 = 0.$$

One can compute directly that

$$\partial\theta_1 = -\frac{1+\alpha}{2\zeta}\mathrm{d}\zeta \wedge \theta_1,$$

$$\bar{\partial}\theta_1 = -\frac{\mathrm{d}\zeta}{\zeta} \wedge \left(\frac{1-\alpha}{2}\theta_1 - \mathrm{d}\bar{z}_1\right) - \frac{2i}{\zeta}(\kappa_{1\bar{1}\bar{2}}\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 + \kappa_{1\bar{2}\bar{2}}\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_2 + \kappa_{2\bar{1}\bar{2}}\mathrm{d}z_2 \wedge \mathrm{d}\bar{z}_1 + \kappa_{2\bar{2}\bar{2}}\mathrm{d}z_2 \wedge \mathrm{d}\bar{z}_2),$$

$$\partial\theta_2 = -\frac{1+\alpha}{2\zeta}\mathrm{d}\zeta \wedge \theta_2,$$

$$\bar{\partial}\theta_2 = -\frac{\mathrm{d}\zeta}{\zeta} \wedge \left(\frac{1-\alpha}{2}\theta_2 + \mathrm{d}\bar{z}_2\right) - \frac{2i}{\zeta}(\kappa_{1\bar{1}\bar{1}}\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 + \kappa_{1\bar{1}\bar{2}}\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_2 + \kappa_{2\bar{1}\bar{1}}\mathrm{d}z_2 \wedge \mathrm{d}\bar{z}_1 + \kappa_{2\bar{1}\bar{2}}\mathrm{d}z_2 \wedge \mathrm{d}\bar{z}_2).$$

Therefore the holomorphicity of θ is equivalent to (8) and

$$
\text{(10)} \quad \frac{\zeta}{2i}(\bar{\partial}C \wedge \theta_1 - \bar{\partial}D \wedge \theta_2) = C(\kappa_{1\bar{1}2}dz_1 \wedge d\bar{z}_1 + \kappa_{1\bar{2}2}dz_1 \wedge d\bar{z}_2 + \kappa_{2\bar{1}2}dz_2 \wedge d\bar{z}_1 + \kappa_{2\bar{2}2}dz_2 \wedge d\bar{z}_2)
$$
$$
- D(\kappa_{1\bar{1}1}dz_1 \wedge d\bar{z}_1 + \kappa_{1\bar{1}2}dz_1 \wedge d\bar{z}_2 + \kappa_{2\bar{1}1}dz_2 \wedge d\bar{z}_1 + \kappa_{2\bar{1}2}dz_2 \wedge d\bar{z}_2).
$$

Observe that Equation (8) is (locally) solvable if and only if

$$
\bar{\partial}C \wedge ((1-\alpha)\theta_1 - 2d\bar{z}_1) = \bar{\partial}D \wedge ((1-\alpha)\theta_2 + 2d\bar{z}_2).
$$

Or in other words, there exists functions P, Q and R such that

$$
\bar{\partial}C = P((1-\alpha)\theta_1 - 2d\bar{z}_1) + Q((1-\alpha)\theta_2 + 2d\bar{z}_2),
$$
$$
\bar{\partial}D = -Q((1-\alpha)\theta_1 - 2d\bar{z}_1) + R((1-\alpha)\theta_2 + 2d\bar{z}_2).
$$

Plugging them in Equation (10) we get

$$
2P\kappa_{1\bar{2}} + 2Q\kappa_{1\bar{1}} = C\kappa_{1\bar{1}2} - D\kappa_{1\bar{1}1},
$$
$$
2R\kappa_{1\bar{1}} - 2Q\kappa_{1\bar{2}} = C\kappa_{1\bar{2}2} - D\kappa_{1\bar{1}2},
$$
$$
2P\kappa_{2\bar{2}} + 2Q\kappa_{2\bar{1}} = C\kappa_{2\bar{1}2} - D\kappa_{2\bar{1}1},
$$
$$
2R\kappa_{2\bar{1}} - 2Q\kappa_{2\bar{2}} = C\kappa_{2\bar{2}2} - D\kappa_{2\bar{1}2}.
$$

These equations can be solved explicitly

$$
P = 2C(\kappa_{1\bar{1}}\kappa_{2\bar{1}2} - \kappa_{2\bar{1}}\kappa_{1\bar{1}2}) + 2D(\kappa_{2\bar{1}}\kappa_{1\bar{1}1} - \kappa_{1\bar{1}}\kappa_{2\bar{1}1}),
$$
$$
Q = 2C(\kappa_{2\bar{2}}\kappa_{1\bar{1}2} - \kappa_{1\bar{2}}\kappa_{2\bar{1}2}) + 2D(\kappa_{1\bar{2}}\kappa_{2\bar{1}1} - \kappa_{2\bar{2}}\kappa_{1\bar{1}1}),
$$
$$
= 2C(\kappa_{2\bar{1}}\kappa_{1\bar{2}2} - \kappa_{1\bar{1}}\kappa_{2\bar{2}2}) + 2D(\kappa_{1\bar{1}}\kappa_{2\bar{1}2} - \kappa_{2\bar{1}}\kappa_{1\bar{1}2}),
$$
$$
R = 2C(\kappa_{2\bar{2}}\kappa_{1\bar{2}2} - \kappa_{1\bar{2}}\kappa_{2\bar{2}2}) + 2D(\kappa_{1\bar{2}}\kappa_{2\bar{1}2} - \kappa_{2\bar{2}}\kappa_{1\bar{1}2}).
$$

Substituting P, Q and R above back into (10), we get exactly (9). $\qquad\square$

Remark 3.2. The Equation (9) is consistent with itself in the sense that we do not get any further information from taking $\bar{\partial}$ of (9).

Now we are ready to compute the curvature term $\text{Tr}(R \wedge R)$ with respect to the metric (5). To do so, we shall let $s = 1 + |\zeta|^2$ and use $K = (\kappa^{i\bar{j}})$ to denote the inverse matrix of $(\kappa_{i\bar{j}})$. In addition, $\langle \cdot, \cdot \rangle$ is to be understood as the Hermitian metric associated with ω.

Let $L_1 d\zeta + C_1\theta_1 - D_1\theta_2$ and $L_2 d\zeta + C_2\theta_1 - D_2\theta_2$ be locally defined linearly independent holomorphic $(1,0)$-forms. It is not hard to compute that

$$
\langle d\zeta, d\zeta \rangle = \frac{s^2}{2e^{2g}},
$$
$$
\langle \theta_1, \theta_1 \rangle = \frac{s^2}{e^{2h+g}}\left(\kappa^{1\bar{1}} + \frac{4}{|\zeta|^2}\kappa_{2\bar{2}}\right) = \frac{s^3}{|\zeta|^2 e^{2h+g}}\kappa^{1\bar{1}},
$$
$$
\langle \theta_1, \theta_2 \rangle = -\frac{s^3}{|\zeta|^2 e^{2h+g}}\kappa^{1\bar{2}}
$$
$$
\langle \theta_2, \theta_2 \rangle = \frac{s^3}{|\zeta|^2 e^{2h+g}}\kappa^{2\bar{2}}.
$$

If we consider the local frame

$$
\{d\zeta, (\zeta L_1)d\zeta + C_1(\zeta\theta_1) + D_1(-\zeta\theta_2), (\zeta L_2)d\zeta + C_2(\zeta\theta_1) + D_2(-\zeta\theta_2)\},
$$

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then the Gram matrix H is given by

$$H = \frac{s^2}{2e^{2g}} \begin{pmatrix} 1 \\ L \end{pmatrix} \begin{pmatrix} 1 & \bar{L}^T \end{pmatrix} + \frac{s^3}{e^{2h+g}} \begin{pmatrix} 0 & 0 \\ 0 & EK\bar{E}^T \end{pmatrix},$$

where

$$L = \begin{pmatrix} \zeta L_1 \\ \zeta L_2 \end{pmatrix}$$

and

$$E = \begin{pmatrix} C_1 & D_1 \\ C_2 & D_2 \end{pmatrix}.$$

Let $U = EK\bar{E}^T$, notice that

$$\det H = \frac{s^8}{2e^{4g+4h}} \det U,$$

so we can compute that

$$H^{-1} = \frac{2e^{2g}}{s^2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{e^{2h+g}}{s^3} \begin{pmatrix} -\bar{L}^T \\ I_2 \end{pmatrix} U^{-1} \begin{pmatrix} -L & I_2 \end{pmatrix},$$

where I_2 is the 2×2 identity matrix. Write

$$A = \frac{s^2}{2e^{2g}} \quad \text{and} \quad B = \frac{s^3}{e^{2h+g}}.$$

It follows that

$$\partial\bar{H} = \partial A \begin{pmatrix} 1 \\ \bar{L} \end{pmatrix} \begin{pmatrix} 1 & L^T \end{pmatrix} + A \begin{pmatrix} 0 \\ \partial\bar{L} \end{pmatrix} \begin{pmatrix} 1 & L^T \end{pmatrix} + A \begin{pmatrix} 1 \\ \bar{L} \end{pmatrix} \begin{pmatrix} 0 & \partial L^T \end{pmatrix} + \partial B \begin{pmatrix} 0 & 0 \\ 0 & \bar{U} \end{pmatrix} + B \begin{pmatrix} 0 & 0 \\ 0 & \partial\bar{U} \end{pmatrix},$$

$$\bar{H}^{-1}\partial\bar{H} = \partial\log A \begin{pmatrix} 1 & L^T \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \partial L^T \\ 0 & 0 \end{pmatrix} + \frac{A}{B} \begin{pmatrix} -L^T \\ I_2 \end{pmatrix} \bar{U}^{-1}\partial\bar{L} \begin{pmatrix} 1 & L^T \end{pmatrix} + \partial\log B \begin{pmatrix} 0 & -L^T \\ 0 & I_2 \end{pmatrix}$$
$$+ \begin{pmatrix} -L^T \\ I_2 \end{pmatrix} \begin{pmatrix} 0 & \bar{U}^{-1}\partial\bar{U} \end{pmatrix}.$$

As a consequence,

$$R = \bar{\partial}(\bar{H}^{-1}\partial\bar{H})$$
$$= \bar{\partial}\partial\log A \begin{pmatrix} 1 & L^T \\ 0 & 0 \end{pmatrix} - \partial\log A \begin{pmatrix} 0 & \bar{\partial}L^T \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \bar{\partial}\partial L^T \\ 0 & 0 \end{pmatrix} + \bar{\partial}\left(\frac{A}{B}\right) \begin{pmatrix} -L^T \\ I_2 \end{pmatrix} \bar{U}^{-1}\partial\bar{L} \begin{pmatrix} 1 & L^T \end{pmatrix}$$
$$- \frac{A}{B} \begin{pmatrix} \bar{\partial}L^T \\ 0 \end{pmatrix} \bar{U}^{-1}\partial\bar{L} \begin{pmatrix} 1 & L^T \end{pmatrix} + \frac{A}{B} \begin{pmatrix} -L^T \\ I_2 \end{pmatrix} \bar{\partial}(\bar{U}^{-1}\partial\bar{L}) \begin{pmatrix} 1 & L^T \end{pmatrix} - \frac{A}{B} \begin{pmatrix} -L^T \\ I_2 \end{pmatrix} \bar{U}^{-1}\partial\bar{L} \begin{pmatrix} 0 & \bar{\partial}L^T \end{pmatrix}$$
$$+ \bar{\partial}\partial\log B \begin{pmatrix} 0 & -L^T \\ 0 & I_2 \end{pmatrix} + \partial\log B \begin{pmatrix} 0 & \bar{\partial}L^T \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} \bar{\partial}L^T \\ 0 \end{pmatrix} \begin{pmatrix} 0 & \bar{U}^{-1}\partial\bar{U} \end{pmatrix} + \begin{pmatrix} -L^T \\ I_2 \end{pmatrix} \begin{pmatrix} 0 & \bar{\partial}(\bar{U}^{-1}\partial\bar{U}) \end{pmatrix}.$$

A lengthy calculation shows that

(11) $$\mathrm{Tr}(R) = \bar{\partial}\partial\log A + 2\bar{\partial}\partial\log B + \mathrm{Tr}(\bar{\partial}(\bar{U}^{-1}\partial\bar{U}))$$

and

(12) $$\mathrm{Tr}(R \wedge R) = 2\partial\bar{\partial}\left(\frac{A}{B}W\right) + 2(\bar{\partial}\partial\log B)^2 + 2\bar{\partial}\partial\log B \wedge \mathrm{Tr}(\bar{\partial}(\bar{U}^{-1}\partial\bar{U})) + \mathrm{Tr}(\bar{\partial}(\bar{U}^{-1}\partial\bar{U})^2),$$

where $W = \bar{\partial}L^T\bar{U}^{-1}\partial\bar{L}$ and we have used the fact that $(\bar{\partial}\partial\log A)^2 = 0$.

Recall from (6) that if $\|\Omega\|_\omega$ is a constant, or equivalently, $e^{h+g} = s^2$, then $\mathrm{Tr}(R) = 0$. From this and (11) we can deduce that

$$\mathrm{Tr}(\overline{\partial}(\bar{U}^{-1}\partial\bar{U})) = 0. \tag{13}$$

From the (restricted) twistor fibration $p : Y \to \mathbb{C}$, we have a short exact sequence of holomorphic vector bundles

$$0 \to p^*T^*\mathbb{C} \to T^*Y \xrightarrow{q} E' = T^*Y/p^*T^*\mathbb{C} \to 0.$$

Moreover, the matrix U is the Gram matrix of the holomorphic frame of E' induced by

$$\{(\zeta L_1)\mathrm{d}\zeta + C_1(\zeta\theta_1) + D_1(-\zeta\theta_2), (\zeta L_2)\mathrm{d}\zeta + C_2(\zeta\theta_1) + D_2(-\zeta\theta_2)\}$$

with respect to the natural metric scaled by s. Let F' be the curvature form of E' with respect to this metric, then we have

$$F' = \overline{\partial}(\bar{U}^{-1}\partial\bar{U}). \tag{14}$$

By making use of (13) and (14), we can simplify (12) to

$$\mathrm{Tr}(R \wedge R) = 2\partial\overline{\partial}\left(\frac{A}{B}W\right) + 2(\overline{\partial}\partial\log B)^2 + \mathrm{Tr}(F' \wedge F'). \tag{15}$$

The quantity W can be computed explicitly using (8). Notice that

$$2\overline{\partial}L^T = \left(2\mathrm{d}\bar{z}_1 - (1-\alpha)\theta_1 \quad 2\mathrm{d}\bar{z}_2 + (1-\alpha)\theta_2\right)E^T,$$

it follows that

$$\begin{aligned}
W &= \frac{1}{4}\left(2\mathrm{d}\bar{z}_1 - (1-\alpha)\theta_1 \quad 2\mathrm{d}\bar{z}_2 + (1-\alpha)\theta_2\right)\bar{K}^{-1}\begin{pmatrix}2\mathrm{d}z_1 - (1-\alpha)\bar{\theta}_1 \\ 2\mathrm{d}z_2 + (1-\alpha)\bar{\theta}_2\end{pmatrix} \\
&= \frac{2i}{s}(\alpha\omega_I + \beta\omega_J + \gamma\omega_K).
\end{aligned}$$

Consequently, the anomaly cancellation equation (2) reduces to

$$\begin{aligned}
&i\partial\overline{\partial}\left(\frac{e^{2h+g}}{s^2}(\alpha\omega_I + \beta\omega_J + \gamma\omega_K)\right) \\
&= \frac{\alpha'}{4}\left[2i\partial\overline{\partial}\left(\frac{e^{2h-g}}{s^2}(\alpha\omega_I + \beta\omega_J + \gamma\omega_K)\right) + 2(\overline{\partial}\partial\log B)^2 + \mathrm{Tr}(F' \wedge F') - \mathrm{Tr}(F \wedge F)\right]
\end{aligned} \tag{16}$$

and we are free to choose functions g and h.

The simplest way to let (16) hold is to take $g = \frac{1}{2}\log\frac{\alpha'}{2}$, $F = F'$ and choose appropriate h such that

$$(\overline{\partial}\partial\log B)^2 = 0. \tag{17}$$

Recall that $B = s^3/e^{2h+g}$, hence (17) holds trivially if h is a constant say $h \equiv 0$. In this case, the metric (5) is conformal to the product of hyperkähler metric on N and the Euclidean metric on $\mathbb{C}$, hence conformally Ricci-flat. It should be pointed out that this metric is not complete, however this does not raise any problem for the use of gluing. Intuitively we can think of ω as a metric on the singular space $\mathbb{CP}^1 \times N/\{\infty\} \times N$.

It is also possible that (17) holds for nonconstant h. To find such h, one usually needs to know the explicit hyperkähler metric on N. We will give an example of this kind in Section 4.

3.3 The Hermitian-Yang-Mills Equation (1)

In [Fei15a], N is taken to be flat T^4 hence $F' = F = 0$ and (1) holds automatically. In general, neither N is flat nor F' trivial. In order to solve the whole Strominger, we shall prove

Theorem 3.3. F' solves the Hermitian-Yang-Mills equation, i.e.,

$$F' \wedge \omega^2 = 0$$

for arbitrary g and h.

By the product structure $Y = \mathbb{C} \times N$, we have the decomposition of space of 2-forms

$$\Omega^2(Y) = \Omega^2(\mathbb{C}) \oplus \Omega^1(\mathbb{C}) \otimes \Omega^1(N) \oplus \Omega^2(N).$$

Moreover the complex structure $\mathfrak{I}$ is compatible with this decomposition, so we have

$$\Omega^{1,1}(Y) = \Omega^{1,1}(\mathbb{C}) \oplus \Omega^{1,0}(\mathbb{C}) \otimes \Omega^{0,1}(N) \oplus \Omega^{0,1}(\mathbb{C}) \otimes \Omega^{1,0}(N) \oplus \Omega^{1,1}(N).$$

We first prove the following lemma

Lemma 3.4. Every entry of F' is contained in the space

$$\Omega^{1,0}(\mathbb{C}) \otimes \Omega^{0,1}(N) \oplus \Omega^{0,1}(\mathbb{C}) \otimes \Omega^{1,0}(N) \oplus \Omega^{1,1}(N)$$

.

Proof. Recall that $U = EK\bar{E}^T$, so we have

$$F' = \bar{\partial}(\bar{U}^{-1}\partial\bar{U}) = \bar{\partial}((E^T)^{-1}\bar{K}^{-1}\bar{E}^{-1}\partial\bar{E} \cdot \bar{K}E^T) + \bar{\partial}((E^T)^{-1}\bar{K}^{-1}\partial\bar{K} \cdot E^T) + \bar{\partial}((E^T)^{-1}\partial E^T)).$$

Observe that (9) can be rewritten as

$$\bar{\partial}E = i(\beta - i\gamma)E \cdot \begin{pmatrix} \kappa_{1\bar{1}\bar{2}}\bar{\theta}_1 - \kappa_{2\bar{1}\bar{2}}\bar{\theta}_2 & \kappa_{1\bar{2}\bar{2}}\bar{\theta}_1 - \kappa_{2\bar{2}\bar{2}}\bar{\theta}_2 \\ \kappa_{2\bar{1}\bar{1}}\bar{\theta}_2 - \kappa_{1\bar{1}\bar{1}}\bar{\theta}_1 & \kappa_{2\bar{1}\bar{2}}\bar{\theta}_2 - \kappa_{1\bar{1}\bar{2}}\bar{\theta}_1 \end{pmatrix},$$

which does not have any component from $\Omega^1(\mathbb{C})$. Similarly $\partial\bar{K}$ contains only components in $\Omega^{1,0}(N)$. The lemma follows directly from these two observations. $\qquad\square$

Now we proceed to prove Theorem 3.3

Proof. From (7), we see that ω^2 lives in the space

$$\Omega^2(\mathbb{C}) \otimes \Omega^2(N) \oplus \Omega^4(N).$$

Therefore the only component of F' that would contribute in $F' \wedge \omega^2$ is its $\Omega^{1,1}(N)$-part. The key point is that the $\Omega^{1,1}(N)$-part of F' can be computed fiberwise. Fix $\zeta \in \mathbb{C}$ and let N_ζ be the fiber of the twistor fibration over ζ. Notice that $(N_\zeta, \omega_\zeta := \alpha\omega_I + \beta\omega_J + \gamma\omega_K)$ is hyperkähler. Moreover, $E'|_{N_\zeta}$ is the cotangent bundle of N_ζ with hyperkähler metric. It is a well-known fact that hyperkähler 4-manifolds are anti-self-dual, thus

$$F'_{\Omega^{1,1}(N)} \wedge \omega_\zeta = 0.$$

The theorem follows directly. $\qquad\square$

In summary, for any hyperkähler 4-manifold N, we can construct a noncompact 3-fold Y with trivial canonical bundle, which can be obtained by removing an arbitrary fiber of the twistor fibration $p : Z \to \mathbb{CP}^1$ from Z. It is worth mentioning that the structure of Y depends on the choice of fiber. The main theorem we have proved is the following.

Theorem 3.5. The whole Strominger system (1, 2, 4) can be solved on Y for arbitrary N hyperkähler.

4 Some Examples

In this section, we will investigate the cases $N = \mathbb{R}^4$ and $N = T^*\mathbb{CP}^1$ in detail.

4.1 $N = \mathbb{R}^4$

Identify $\mathbb{R}^4$ with the space of quaternions $\mathbb{H}$, then left multiplication by i, j and k defines the standard hyperkähler structure on $\mathbb{R}^4$. We can construct the space Y as in Section 3 by removing the fiber with complex structure $-I$ at infinity. Actually, Y is biholomorphic to $\mathbb{C}^3$. An explicit isomorphism $f : \mathbb{C}^3 = \mathbb{C} \times \mathbb{C}^2 \to Y = \mathbb{C} \times \mathbb{R}^4$ can be written down as

$$(\zeta, u_1, u_2) = f(\zeta, w_1, w_2) = (\zeta, \frac{w_1 - i\zeta\overline{w}_2}{1 + |\zeta|^2}, \frac{w_2 + i\zeta\overline{w}_1}{1 + |\zeta|^2}).$$

Or conversely,

$$\begin{cases} w_1 &= u_1 + i\zeta\overline{u}_2 \\ w_2 &= u_2 - i\zeta\overline{u}_1 \end{cases},$$

where $u_1 = x^1 + ix^2$, $u_2 = x^3 + ix^4$ and $u^1 + u^2 j = x^1 + x^2 i + x^3 j + x^4 k$ parameterizes $\mathbb{H} = \mathbb{R}^4$. In terms of the holomorphic coordinate $\{\zeta, w_1, w_2\}$ on $\mathbb{C}^3$, the 2-form $\alpha\omega_I + \beta\omega_J + \gamma\omega_K$ can be expressed as

$$(18) \qquad \alpha\omega_I + \beta\omega_J + \gamma\omega_K = \frac{i}{2s}(\mathrm{d}w_1 \wedge \mathrm{d}\overline{w}_1 + \mathrm{d}w_2 \wedge \mathrm{d}\overline{w}_2 + (|u_1|^2 + |u_2|^2)\mathrm{d}\zeta \wedge \mathrm{d}\overline{\zeta}$$
$$+ iu_2\mathrm{d}w_1 \wedge \mathrm{d}\overline{\zeta} - iu_1\mathrm{d}w_2 \wedge \mathrm{d}\overline{\zeta} - i\overline{u}_2\mathrm{d}\zeta \wedge \mathrm{d}\overline{w}_1 + i\overline{u}_1\mathrm{d}\zeta \wedge \mathrm{d}\overline{w}_2).$$

Now let us consider Equation (17)
$$(\overline{\partial}\partial \log B)^2 = 0.$$

Notice that
$$\overline{\partial}\partial \log B = (3\overline{\partial}\partial \log s - \overline{\partial}\partial g) - 2\overline{\partial}\partial h,$$

hence (17) is equivalent to that

$$(19) \qquad (\overline{\partial}\partial h)^2 = \overline{\partial}\partial h \wedge (3\overline{\partial}\partial \log s - \overline{\partial}\partial g).$$

For simplicity, let us assume that $h : \mathbb{R}^4 \to \mathbb{R}$ is a radial function, i.e., $h = h(\rho)$ where

$$\rho = |u_1|^2 + |u_2|^2 = (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2.$$

Therefore
$$\mathrm{d}h = h'\mathrm{d}\rho$$

and

$$2i\overline{\partial}\partial h = \mathrm{d}\mathfrak{I}\mathrm{d}h$$
$$= h''\mathrm{d}\rho \wedge (\alpha I + \beta J + \gamma K)\mathrm{d}\rho + h'(\mathrm{d}\alpha \wedge I\mathrm{d}\rho + \mathrm{d}\beta \wedge J\mathrm{d}\rho + \mathrm{d}\gamma \wedge K\mathrm{d}\rho) - 4h'(\alpha\omega_I + \beta\omega_J + \gamma\omega_K).$$

One can verify that

$$-4(\overline{\partial}\partial h)^2 = 32h'(h' + h''\rho)\mathrm{vol}_{\mathbb{R}^4} - 2(h')^2(\alpha J\mathrm{d}\rho \wedge K\mathrm{d}\rho + \beta K\mathrm{d}\rho \wedge I\mathrm{d}\rho + \gamma I\mathrm{d}\rho \wedge J\mathrm{d}\rho) \wedge \omega_{\mathbb{CP}^1}$$
$$+ 8h'(h' + h''\rho)((\alpha\mathrm{d}\beta - \beta\mathrm{d}\alpha) \wedge *K\mathrm{d}\rho + (\beta\mathrm{d}\gamma - \gamma\mathrm{d}\beta) \wedge *I\mathrm{d}\rho + (\gamma\mathrm{d}\alpha - \alpha\mathrm{d}\gamma) \wedge *J\mathrm{d}\rho),$$

where $*$ is the Hodge star operator on $\mathbb{R}^4$. Also we have

$$4\overline{\partial}\partial h \wedge (3\overline{\partial}\partial \log s - \overline{\partial}\partial g) = (h''\mathrm{d}\rho \wedge (\alpha I + \beta J + \gamma K)\mathrm{d}\rho - 4h'(\alpha\omega_I + \beta\omega_J + \gamma\omega_K)) \wedge (2i\overline{\partial}\partial \log g + 3\omega_{\mathbb{CP}^1}).$$

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Notice that we have the identity

$$\mathrm{d}\rho \wedge I\mathrm{d}\rho + 4\rho \cdot \omega_I = J\mathrm{d}\rho \wedge K\mathrm{d}\rho$$

and the like. It follows that assuming g is a constant, then (19) holds if and only if $h' = 0$ or $h' = \dfrac{-3}{2\rho}$. In both cases we solve the Strominger system.

(a). The case $h' = 0$.

The metric on Y is essentially of the form

$$\omega = \frac{1}{s^2}\left(\alpha\omega_I + \beta\omega_J + \gamma\omega_K + \frac{i}{2}\mathrm{d}\zeta \wedge \mathrm{d}\bar\zeta\right),$$

which is conformal to the Euclidean metric on $Y = \mathbb{C} \times \mathbb{R}^4$. However this metric has rather complicated expression (see (18)) in standard complex coordinates on $\mathbb{C}^3$. Direct calculation shows that though its sectional curvature is not bounded from below, however it is bounded from above by a positive constant. Same conclusion applies to Ricci and scalar curvatures as well.

(b). The case $h' = \dfrac{-3}{2\rho}$.

In this case $e^h \sim \rho^{-3/2}$, so the metric ω is only defined on $\mathbb{C} \times (\mathbb{R}^4 \setminus \{0\}) \cong \mathbb{C} \times (\mathbb{C}^2 \setminus \{0\})$. On each copy of $\mathbb{R}^4 \setminus \{0\}$, the restricted metric is conformally flat, with non-positive sectional curvature. The curvature property of $\mathbb{C} \times (\mathbb{R}^4 \setminus \{0\})$ is like what we have in the $h' = 0$ case.

4.2 $N = T^*\mathbb{CP}^1$

The hyperkähler structure on $T^*\mathbb{CP}^1$ is known as the Eguchi-Hanson [EH78] geometry, which was also discovered by Calabi [Cal79] independently. This is the simplest example of ALE space which can be constructed by taking resolution of $\mathbb{C}^2/\mathbb{Z}_2$. Now let N be the Eguchi-Hanson space. As we have seen from Proposition 2.1 that if we choose the fiber removed to be special, then Y is can be identified with the resolved conifold.

As we know, setting $h \equiv$ const gives a solution to the Strominger system which is conformal to the product metric. Its curvature behaviors are very similar to the corresponding $N = \mathbb{R}^4$ case.

One may want to ask if there is a nonconstant h such that (17), or equivalently (19) holds. We may want to make a reduction as before by assuming that h depends only on R, where $R : T^*\mathbb{CP}^1 \to \mathbb{R}$ is the norm square function of cotangent fibers with respect to the Fubini-Study metric $\omega_{\mathbb{CP}^1}$. Unfortunately in such a case we do not get any extra solutions by this consideration.

5　On Chern-Ricci Flat Balanced Metrics

Let X be a Kähler manifold with trivial canonical bundle. A fundamental question is whether M admits a (complete) Ricci-flat Kähler metric or not. For X compact, this was answered affirmatively by Yau's famous solution to the Calabi conjecture [Yau77, Yau78]. However, we do not know enough about this problem when X is noncompact. For instance, let M be any Kähler manifold, then we know that the total space of the canonical bundle of M, denoted by K_M, is Kähler and has itself trivial canonical bundle. Hence we may ask when does K_M support a Kähler Ricci-flat metric. This problem was first attacked by Calabi [Cal79], where he showed that if M admits a Kähler-Einstein metric, then one can write down a Kähler Ricci-flat metric on K_M. A relative recent progress in this direction was made by Futaki [Fut07], where he showed that such a metric exists if M is toric Fano.

The Calabi ansatz can be rephrased as follows. Let (M, ω) be a Kähler manifold. By choosing a set of holomorphic coordinates $\{z^1, \ldots, z^n\}$ on M, we can trivialize K_M by $\mathrm{d}z^1 \wedge \cdots \wedge \mathrm{d}z^n$ locally. Let t

parameterizes the fiber of K_M under this trivialization, then $\{z^1, \ldots, z^n, t\}$ forms a set of holomorphic coordinates on K_M and

$$\Omega = \mathrm{d}z^1 \wedge \cdots \wedge \mathrm{d}z^n \wedge \mathrm{d}t$$

is a globally defined nowhere vanishing holomorphic volume form.

In terms of coordinates $\{z^1, \ldots, z^n\}$, the Kähler form ω can be written as

$$\omega = ih_{j\overline{k}}\mathrm{d}z^j \wedge \mathrm{d}\overline{z}^k.$$

It follows that $h = \det(h_{j\overline{k}})$ is a positive function. Notice that the Kähler metric on M naturally induces an Hermitian metric on K_M, which can be expressed as

$$\omega_0 = \omega + \frac{i}{h}(\mathrm{d}t - t\partial \log h) \wedge (\mathrm{d}\overline{t} - \overline{t}\overline{\partial} \log h).$$

Let $R : K_M \to \mathbb{R}$ be the norm square function of fibers of $K_M \to M$. Clearly $R = \dfrac{|t|^2}{h}$ and the metric ω_0 has the form

$$\omega_0 = \omega + i\frac{\partial R \wedge \overline{\partial} R}{R}.$$

In general, ω_0 is not a Kähler metric. It turns out that ω_0 is Kähler if and only if

$$\partial\overline{\partial} \log R = -\partial\overline{\partial} \log h = 0,$$

i.e., (M, ω) is Kähler Ricci-flat.

To get a better behaved metric, we can modify ω_0 by some conformal factors. Let u, v be smooth functions on M, and f, g be smooth functions of R. Then one can cook up a new Hermitian metric

$$\omega_{u,v,f,g} = e^{u+f}\omega + ie^{v+g}\frac{\partial R \wedge \overline{\partial} R}{R}.$$

It is not hard to check that Ω is of constant length if and only if

(20) $$v = -nu \quad \text{and} \quad g = -nf + c$$

for some constant c. Assuming this, if we further want the metric $\omega_{u,v,f,g}$ to be Kähler, then u must be a constant. Without loss of generality we may assume that $u = 0$, and we still need

$$e^f \partial f \wedge \omega - ie^{-nf+c}\partial R \wedge \partial\overline{\partial} \log R = 0.$$

In other words,

$$e^{(n+1)f-c}f'\omega = i\partial\overline{\partial} \log R = -i\partial\overline{\partial} \log h = Ric(\omega).$$

We see immediately that this equation has a solution if and only if ω is Kähler-Einstein, in which case we get the Calabi ansatz.

In this section, we shall consider the case that $\omega_{u,v,f,g}$ is balanced and Chern-Ricci flat. Clearly such a metric satisfies the conformally balanced equation (4), which geometrically means that the manifold has restricted holonomy contained in $SU(n)$ with respect to the Strominger-Bismut connection. However the balanced Chern-Ricci flat condition is far more stronger in the sense that it can be interpreted as that Ω is parallel under the Strominger-Bismut connection.

Now let us impose the balanced Chern-Ricci flat condition on $\omega_{u,v,f,g}$. For the "Chern-Ricci flat" part, i.e., $\|\Omega\|_{\omega_{u,v,f,g}} \equiv \text{const.}$, we still need (20) as before. Plug this in and we can compute that

$$\omega_{u,v,f,g}^n = e^{n(u+f)}\omega^n + ine^{c-u-f}\omega^{n-1} \wedge \frac{\partial R \wedge \overline{\partial} R}{R}.$$

Hence

$$\partial(\omega_{u,v,f,g}^n) = e^{n(u+f)}\partial f \wedge \omega^n - ine^{c-u-f}\omega^{n-1} \wedge \partial R \wedge \partial\overline{\partial}\log R - ine^{c-u-f}\omega^{n-1} \wedge \partial u \wedge \frac{\partial R \wedge \overline{\partial} R}{R}.$$

As there is no other terms to cancel the last term in the above equation, so we need $\partial u = 0$ to make $\omega_{u,v,f,g}$ balanced. By choosing $u = 0$, the balancing condition is reduced to

$$e^{nf}\partial f \wedge \omega^n = ine^{c-f}\omega^{n-1} \wedge \partial R \wedge \partial\overline{\partial}\log R,$$

or equivalently,

$$e^{(n+1)f-c}f'\omega^n = -in\omega^{n-1}\partial\overline{\partial}\log h = s \cdot \omega^n,$$

where s is the scalar curvature function of M up to a positive constant. From the calculation we conclude that this is possible if and only if (M, ω) has constant scalar curvature.

Constant scalar curvature Kähler metrics (cscK) has been studied extensively as a special case of extremal Kähler metrics [Cal82]. It is believed that the existence of cscK metrics is equivalent to certain stability condition in the sense of algebraic geometry.

Notice that in the derivation of Chern-Ricci flat balanced metric on K_M, what we actually need is that ω is balanced instead of being Kähler. In such a case, s is known as the Chern scalar curvature, which is in general different from the scalar curvature in Riemannian geometry. Thus we have proved the following generalization of Calabi's result.

Theorem 5.1. If M admits a balanced metric with constant Chern scalar curvature, then K_M admits a Chern-Ricci flat balanced metric.

Recall that on a complex n-fold X, a balanced metric ω defines the so-called balanced class

$$\left[\frac{\omega^{n-1}}{(n-1)!}\right] \in H_{BC}^{n-1,n-1}(X) = \frac{\text{d-closed } (n-1, n-1)\text{-forms}}{i\partial\overline{\partial}\text{-exact } (n-1, n-1)\text{-forms}}.$$

The balanced version of Gauduchon conjecture has been proposed for some time, see [Tos15, Conjecture 4.1] for instance. In particular, for balanced manifolds with trivial canonical bundle, this conjecture implies the existence of Chern-Ricci flat balanced metrics in any given balanced class. We refer to [STW15] for recent progresses on this problem.

Theorem 5.1 implies that Chern-Ricci flat balanced metrics should be viewed as a balanced analogue of extremal Kähler metrics. We shall justify this claim by the following consideration.

On a compact balanced manifold (X, ω), a very useful property is that the total Chern scalar curvature

$$\int_X s \cdot \frac{\omega^n}{n!} = \int_X \rho \wedge \frac{\omega^{n-1}}{(n-1)!}$$

depends only on the complex structure of X and the balanced class. Here $\rho = -i\partial\overline{\partial}\log h$ is the Chern-Ricci form. As an analogue of Kähler case, it is natural to consider the variational problem associated to the Calabi-type functional [Cal82]

$$\mathcal{S}(\omega) = \int_X s^2 \cdot \frac{\omega^n}{n!},$$

where $\frac{\omega^{n-1}}{(n-1)!}$ varies in the given balanced class. Since we can modify ω by $i\partial\overline{\partial}\alpha$, where α is any $(n-2, n-2)$-form, we expect the associated Euler-Lagrange equation is an equation of $(n-2, n-2)$-forms, or dually, a $(2, 2)$-form equation.

Indeed, it is not hard to derive the following

Theorem 5.2. A balanced metric ω is a critical point of $\mathcal{S}(\omega)$ if and only if it satisfies

$$(21) \qquad 2(n-1)i\partial\overline{\partial}s \wedge \rho = i\partial\overline{\partial}((2\Delta s + s^2)\omega),$$

where Δ_c is the complex Laplacian defined by

$$\Delta f = \Lambda(i\partial\overline{\partial}f) = h^{k\overline{j}}\frac{\partial^2 f}{\partial z^j \partial \overline{z}^k}.$$

We shall call such balanced metrics extremal.

From Equation (21), we have the following observations:

(a). If the background metric is non-Kähler, one can easily show that $i\partial\overline{\partial}\omega \neq 0$. Hence $s = \text{const.}$ is an extremal balanced metric if and only if $s = 0$. In particular, Chern-Ricci flat balanced metrics are extremal. An intuitive reason is that there is no analogue of Kähler-Einstein metrics with nonzero Einstein constant on non-Kähler balanced manifolds. Indeed, if there is a smooth function f such that

$$\rho = f \cdot \omega,$$

one can deduce that $\rho \equiv 0$.

(b). If there exists a (1,0)-form μ such that

$$(22) \qquad 2(n-1)s \cdot \rho = (2\Delta s + s^2)\omega + \overline{\partial}\mu + \partial\overline{\mu},$$

then (21) holds. On the other hand, if (21) holds, then condition (22) is automatically satisfied if $H_A^{1,1}(X) = 0$, where $H_A^*(X)$ is the Aeppli cohomology group of X. In this case, by taking trace of (22), we get

$$2(n-1)s^2 = 2n\Delta s + ns^2 + \Lambda(\overline{\partial}\mu + \partial\overline{\mu}).$$

Since the last term is of divergence form, by integration over X, we get

$$(n-2)\int_X s^2 \cdot \frac{\omega^n}{n!} = 0.$$

As we always assume that $n > 2$ (otherwise ω is Kähler), we conclude that $s \equiv 0$.

(c). Assuming $s \equiv 0$, if we further assume that $0 = c_1(X) \in H_{BC}^{1,1}(X)$, then $\rho = i\partial\overline{\partial}f$ for some globally defined real function f. Therefore

$$0 = s = \Lambda\rho = \Delta f.$$

Hence by maximal principle f is a constant and $\rho \equiv 0$.

A very important class of non-Kähler Calabi-Yau 3-folds is of the form $X_k = \#_k(S^3 \times S^3)$ for $k \geq 2$ [Fri91, LT96], which can be constructed from projective Calabi-Yau 3-folds by taking conifold transitions. Moreover, these manifolds satisfy the $\partial\overline{\partial}$-lemma and admit balanced metrics [FLY12]. Therefore conditions in (b) and (c) above are satisfied. Consequently we have

Corollary 5.3. Extremal balanced metrics on X_k are exactly those Chern-Ricci flat balanced metrics.

Hopefully this point of view will be useful in proving the balanced Gauduchon conjecture for X_k's.

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