AN APPLICATION OF A THEOREM OF EMERTON TO MOD p REPRESENTATIONS OF GL_2

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ABSTRACT. Let p be a prime and L be a finite extension of $\mathbb{Q}_p$. We study the ordinary parts of $\mathrm{GL}_2(L)$-representations arised in the mod p cohomology of Shimura curves attached to indefinite division algebras which splits at a finite place above p. The main tool of the proof is a theorem of Emerton [10].

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1. INTRODUCTION

Let p be a prime number and L a finite extension of $\mathbb{Q}_p$. Let $G = \mathrm{GL}_2(L)$. If $L = \mathbb{Q}_p$, the work of Barthel-Livné [1] and of Breuil [3] gave a complete classification of irreducible smooth representations of G over $\overline{\mathbb{F}}_p$ with a central character (this last restriction is now removed by Berger [2]). However, when $F \neq \mathbb{Q}_p$, the situation is much more complicated and a large part of the theory is still mysterious. The main difficulty lies in the study of supersingular representations, which are irreducible smooth representations of G which do not arise as subquotients of parabolic inductions. For example, when $[L : \mathbb{Q}_p] = 2$, Schraen has shown that supersingular representations are not of finite presentation [20] (similar result was first proved in [15] if L is a finite extension of $\mathbb{F}_p[[t]]$).

Though the general theory of smooth representations of G could be very weird, when L is unramified over $\mathbb{Q}_p$, Breuil and Paškūnas were able to construct some 'nicer' ones in [5]. Precisely, they constructed families of admissible smooth representations (by local methods) which are related to two dimensional continuous $\overline{\mathbb{F}}_p$-representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/L)$ via the Buzzard-Diamond-Jarvis conjecture [7]. Recent work of Emerton-Gee-Savitt [11] shows that this construction is indeed very important and tightly related to the mod p local Lancorollaryglands program.

To state our main result we need some notations. Let F be a totally real field and D a quaternion algebra with center F which splits at exactly one infinite place. One can associate to D a system of Shimura curves $(X_U)_U$ indexed by open compact subgroups of $(D\otimes_{\mathbb{Q}}\mathbb{A}_f)^\times$ (where $\mathbb{A}_f$ denotes the ring of finite adèles of $\mathbb{Q}$), which are projective and smooth over F. Put

$$S^D(\overline{\mathbb{F}}_p) := \varinjlim H^1_{\text{ét}}(X_{U,\overline{\mathbb{Q}}}, \overline{\mathbb{F}}_p)$$

where the inductive limit is taken over all open compact subgroups of $(D\otimes_{\mathbb{Q}}\mathbb{A}_f)^\times$. Let $\overline{\rho}: \mathrm{Gal}(\overline{\mathbb{Q}}/F)\to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be an irreducible, continuous, totally odd representation. Assume moreover $\overline{\rho}$ is modular, which means that

$$\pi^D(\overline{\rho}) := \mathrm{Hom}_{\mathrm{Gal}(\overline{\mathbb{Q}}/F)}(\overline{\rho}, S^D(\overline{\mathbb{F}}_p))$$

is non-zero. A conjecture of Buzzard, Diamond and Jarvis [7] says that the space $\pi^D(\overline{\rho})$ decomposes as a restricted tensor product

$$\pi^D(\overline{\rho}) \cong {\bigotimes_w}' \pi_w$$

where each factor π_w is an admissible smooth representation of $(D\otimes_F F_w)^\times$ and depends only on the restriction of $\overline{\rho}$ at w. Note that, when $w|p$, the local factor π_w is supposed to be the right representation in the mod p local Langlands (or Jacquet-Langlands) program, and many important properties about it have been proved, see e.g. [14], [4], [11].

In this article we prove some extra property about π_w, $w|p$, when the restriction of $\overline{\rho}$ at w is reducible indecomposable and generic (see §3), and when F is unramified at w. In this case, it is hoped that π_w has a filtration of length $f := [F_w : \mathbb{Q}_p]$ of the form (where $(\pi_i)_i$ denote the graded pieces of the filtration)

$$\pi_0 — \pi_1 — \cdots — \pi_f \tag{1.1}$$

such that π_i is a principal series if $i\in\{0,f\}$ and supersingular otherwise. Our main theorem is as follows.

Theorem 1.1. *Assume the decomposition $\pi^D(\overline{\rho})\cong\otimes'_w\pi_w$ holds. Let w be a place above p. Assume that D splits at w, and $\overline{\rho}_w$ is reducible indecomposable and generic. Assume that F is unramified at w and $[F_w:\mathbb{Q}_p]\geq 2$. Then π_w contains a unique sub-representation π which is of length 2 and which fits into an exact sequence*

$$0\to\pi_0\to\pi\to\pi_1\to 0$$

such that π_0 is a principal series and π_1 is a supersingular representation.

In other words, we prove that π_w contains the first two graded pieces in (1.1). The main observation in the proof of Theorem 1.1 is a theorem of Emerton [10], which allows us to control the ordinary part of π_w.

Back to the local situation, we get the following consequence.

Corollary 1.2. *Let L be a finite unramified extension of $\mathbb{Q}_p$ of degree $f\geq 2$. There exist a principal series π_0 and a supersingular representation π_1 such that*

$$\mathrm{Ext}^1_G(\pi_1,\pi_0)\neq 0.$$

If moreover $[L:\mathbb{Q}_p]=2$, and if π_2 denotes the f-th smooth dual of π_0 (see [17, Prop. 5.4]*), then*

$$\mathrm{Ext}^1_G(\pi_2,\pi_1)\neq 0.$$

The existence of extensions in Corollary 1.2 is not known before. Remark that such extensions do not exist when $L=\mathbb{Q}_p$ ([19]). We also prove, in §5, that when L is a local field of characteristic p there is no non-trivial extension of a principal series by a supersingular representation.

The paper is organized as follows. In §2 we prove some necessary results about smooth $\overline{\mathbb{F}}_p$-representations of G, especially the structure of (irreducible) principal series. In §3, we recall the construction of [5] and show, under certain assumption, that the representations they constructed behave well. In §4, we show that this assumption is satisfied in certain global situation. In the appendix (§5), we show the counterpart of Theorem 1.1 is false if L is of characteristic p.

2. Representation theoretic preparation

In this section, L denotes a finite extension of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}_L$, maximal ideal $\mathfrak{p}_L$, and residue field (identified with) $\mathbb{F}_q=\mathbb{F}_{p^f}$. Fix a uniformiser ϖ of L; we take $\varpi=p$ when L is unramified over $\mathbb{Q}_p$. For $\lambda\in\mathbb{F}_q$, $[\lambda]\in\mathcal{O}_L$ denotes its Teichmüller lifting.

Let $G=\mathrm{GL}_2(L)$, and define the following subgroups of G:

$$P=\begin{pmatrix}* & *\\ 0 & *\end{pmatrix},\quad \overline{P}=\begin{pmatrix}* & 0\\ * & *\end{pmatrix},\quad N=\begin{pmatrix}1 & *\\ 0 & 1\end{pmatrix},\quad T=\begin{pmatrix}* & 0\\ 0 & *\end{pmatrix},$$

and let Z be the center of G.

Let $K=\mathrm{GL}_2(\mathcal{O}_L)$ and K_1 be the kernel of the reduction morphism $K\twoheadrightarrow\mathrm{GL}_2(\mathbb{F}_q)$. Let $I\subset K$ denote the (upper) Iwahori subgroup and $I_1\subset I$ be the pro-p-Iwahori subgroup. Let $N_0=N\cap K$, $T_0=T\cap K$ and $T_1=T\cap K_1$.

We call a weight an irreducible representation of K over $\overline{\mathbb{F}}_p$, which is always an inflation of an irreducible representation of $\mathrm{GL}_2(\mathbb{F}_q)$. A weight is isomorphic to ([1, Prop. 1])

$$\mathrm{Sym}^{r_0}\overline{\mathbb{F}}_p^2\otimes_{\overline{\mathbb{F}}_p}(\mathrm{Sym}^{r_1}\overline{\mathbb{F}}_p^2)^{\mathrm{Fr}}\otimes\cdots\otimes_{\overline{\mathbb{F}}_p}(\mathrm{Sym}^{r_{f-1}}\overline{\mathbb{F}}_p^2)^{\mathrm{Fr}^{f-1}}\otimes_{\overline{\mathbb{F}}_p}\det{}^a$$

where $0\leq r_i\leq q-1$, $0\leq a\leq q-2$ and $\mathrm{Fr}:\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)\mapsto\left(\begin{smallmatrix}a^p & b^p\\ c^p & d^p\end{smallmatrix}\right)$ is the Frobenius on $\mathrm{GL}_2(\mathbb{F}_q)$. We denote this representation by $(r_0,...,r_{f-1})\otimes\det^a$.

2.1. **Principal series.** By the work of Barthel-Livné [1], smooth irreducible $\overline{\mathbb{F}}_p$-representations[1] of G with a central character fall into four classes:

(i) one-dimensional representations, i.e. characters.
(ii) principal series $\mathrm{Ind}_P^G\chi$, with $\chi\neq\chi^s$ where χ is a smooth character of T inflated to P and χ^s is the character of T defined as $\chi^s(\left(\begin{smallmatrix}a & 0\\ 0 & d\end{smallmatrix}\right))=\chi(\left(\begin{smallmatrix}d & 0\\ 0 & a\end{smallmatrix}\right))$.
(iii) special series, i.e. twists of the Steinberg representation Sp.
(iv) supersingular representations.

[1]Although we choose to work with $\overline{\mathbb{F}}_p$-representations in this section, all the results hold true if we replace $\overline{\mathbb{F}}_p$ by a sufficiently large finite extension of $\mathbb{F}_p$.

In this paper, only the second and fourth classes will be involved. When we talk about a principal series $\mathrm{Ind}_P^G\chi$, we implicitly mean $\chi\neq\chi^s$. Recall first the following result from [5, §8].

Proposition 2.1. *Let π be a principal series of G.*

(i) We have an isomorphism of $\overline{\mathbb{F}}_p$-vector spaces $\mathrm{Hom}(L^\times,\overline{\mathbb{F}}_p)\cong\mathrm{Ext}^1_{G,\zeta}(\pi,\pi)$.[2]

(ii) Assume $L\neq\mathbb{Q}_p$. Let π' be a smooth irreducible non-supersingular representation of G, then $\mathrm{Ext}^1_G(\pi',\pi)=0$ except when $\pi'\cong\pi$.

Proof. (i) follows from [5, Cor. 8.2(ii)], noting that π is irreducible by our convention, hence the two cases excluded in *loc. cit.* can not happen. (ii) follows from [5, Thm.8.1] using [5, Thm. 7.16(i)]. □

We recall the construction of the isomorphism in Proposition 2.1(i). Let $\delta\in\mathrm{Hom}(L^\times,\overline{\mathbb{F}}_p)$. We lift δ to a homomorphism of P to $\overline{\mathbb{F}}_p$ via $P\twoheadrightarrow L^\times$ given by $\left(\begin{smallmatrix}a&b\\0&d\end{smallmatrix}\right)\mapsto ad^{-1}$. Write $\pi=\mathrm{Ind}_P^G\chi$ and let ϵ_δ be the extension

$$0\to\chi\to\epsilon_\delta\to\chi\to0 \tag{2.1}$$

corresponding to δ. Explicitly, ϵ_δ has a basis $\{v,w\}$ with the action of P represented by $\left(\begin{smallmatrix}\chi&\chi\delta\\0&\chi\end{smallmatrix}\right)$: $b\cdot v=\chi(b)v$ and $b\cdot w=\chi(b)w+\chi(b)\delta(b)v$ for $b\in P$. Then, inducing to G we obtain an exact sequence

$$0\to\mathrm{Ind}_P^G\chi\to\mathrm{Ind}_P^G\epsilon_\delta\to\mathrm{Ind}_P^G\chi\to0 \tag{2.2}$$

which is the element in $\mathrm{Ext}^1_G(\pi,\pi)$ corresponding to δ. Since by definition δ is trivial on Z, the representation $\mathrm{Ind}_P^G\epsilon_\delta$ has a central character.

Taking K_1-invariants, (2.2) induces a long exact sequence

$$0\to(\mathrm{Ind}_P^G\chi)^{K_1}\to(\mathrm{Ind}_P^G\epsilon_\delta)^{K_1}\to(\mathrm{Ind}_P^G\chi)^{K_1}\overset{\partial_\delta}{\to}H^1(K_1/Z_1,\mathrm{Ind}_P^G\chi).$$

Proposition 2.2. *Notations are as above. The map ∂_δ is zero if and only if δ is an unramified homomorphism, i.e. δ is trivial on $\mathcal{O}_L^\times$. Moreover, if ∂_δ is nonzero, then it is an injection.*

Proof. First, if δ is unramified, then it splits when restricted to $T\cap K$, hence the sequence (2.2) also splits when restricted to K. This shows that $\partial_\delta=0$ in this case.

Now assume δ is ramified. It suffices to show that the inclusion $(\mathrm{Ind}_P^G\chi)^{K_1}\hookrightarrow(\mathrm{Ind}_P^G\epsilon_\delta)^{K_1}$ is an equality. By definition, $\mathrm{Ind}_P^G\epsilon_\delta$ identifies with the vector space of smooth functions $f:G\to\overline{\mathbb{F}}_pv\oplus\overline{\mathbb{F}}_pw$ such that $f(pg)=p\cdot f(g)$, where $p\in P$, $g\in G$, and $\{v,w\}$ is the chosen basis of ϵ_δ as above; the action of G on $\mathrm{Ind}_P^G\epsilon_\delta$ is given by $(g'\cdot f)(g):=f(gg')$. Let f be such a function which is fixed by K_1. We need to show that $f(g)\in\overline{\mathbb{F}}_pv$ for any $g\in G$. Using the decomposition

$$G=PK=P\begin{pmatrix}0&1\\1&0\end{pmatrix}K_1\coprod\Big(\coprod_{\lambda\in\mathbb{F}_q}P\begin{pmatrix}1&0\\ [\lambda]&1\end{pmatrix}K_1\Big),$$

it suffices to check this for $g=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ and $g=\left(\begin{smallmatrix}1&0\\ [\lambda]&1\end{smallmatrix}\right)$, $\lambda\in\mathbb{F}_q$. If $h=\left(\begin{smallmatrix}1+\varpi a&0\\0&1\end{smallmatrix}\right)\in T_1$ and $g=\left(\begin{smallmatrix}1&0\\ [\lambda]&1\end{smallmatrix}\right)$, we have (as $h\cdot f=f$):

$$f(g)=(h\cdot f)(g)=f\Big(\begin{pmatrix}1+\varpi a&0\\ [\lambda](1+\varpi a)&1\end{pmatrix}\Big)=h\cdot\Big[f\Big(\begin{pmatrix}1&0\\ [\lambda](1+\varpi a)&1\end{pmatrix}\Big)\Big]=h\cdot[f(g)]$$

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[2] $\mathrm{Ext}^1_{G,\zeta}$ means that we consider extensions with a central character and ζ is the central character of π.

where the last equality holds because f is fixed by K_1. This shows that the vector $f(g)$ is fixed by T_1, hence lies in $\overline{\mathbb{F}}_p v$ because δ is non-trivial on T_1. Similar (and simpler) argument works for $g = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. This finishes the proof. $\square$

Since K_1 is normal in K, the space $H^1(K_1/Z_1, \pi)$ can be viewed naturally as a K-representation.

Corollary 2.3. *Let $d := \dim_{\overline{\mathbb{F}}_p} \mathrm{Hom}(1+\mathfrak{p}_L, \overline{\mathbb{F}}_p)$. There is a K-equivariant embedding $(\pi^{K_1})^{\oplus d} \hookrightarrow H^1(K_1/Z_1, \pi)$.*

Proof. The association $\delta \mapsto \partial_\delta$ defines an $\overline{\mathbb{F}}_p$-linear map

$$\mathrm{Hom}(F^\times, \overline{\mathbb{F}}_p) \to \mathrm{Hom}_K(\pi^{K_1}, H^1(K_1/Z_1, \pi)).$$

By Proposition 2.2, its kernel is the subspace of unramified homomorphisms, so we obtain an exact sequence

$$0 \to \mathrm{Hom}^{\mathrm{un}}(L^\times, \overline{\mathbb{F}}_p) \to \mathrm{Hom}(L^\times, \overline{\mathbb{F}}_p) \to \mathrm{Hom}_K(\pi^{K_1}, H^1(K_1/Z_1, \pi))$$

(where $\mathrm{Hom}^{\mathrm{un}}$ means unramified homomorphisms). The assertion follows from Proposition 2.2 and that $\mathrm{Hom}(\mathcal{O}_L^\times, \overline{\mathbb{F}}_p) \cong \mathrm{Hom}(1+\mathfrak{p}_L, \overline{\mathbb{F}}_p)$. $\square$

Let π be a principal series. For later use, we recall some basic results about its I_1-invariants. By [1, Thm. 34], $\mathrm{soc}_K \pi$ is not necessarily irreducible in general, but it contains a *unique* irreducible sub-representation which is of dimension ≥ 2, which we denote by σ. The decomposition $G = PI_1 \coprod P\Pi I_1$ (where $\Pi := \left(\begin{smallmatrix} 0 & 1 \\ \varpi & 0 \end{smallmatrix}\right)$) implies that π^{I_1} is always 2-dimensional, spanned over $\overline{\mathbb{F}}_p$ by f_1, f_2 characterized as follows:

$$f_1(\mathrm{Id}) = 1,\ f_1(\Pi) = 0; \quad f_2(\mathrm{Id}) = 0,\ f_2(\Pi) = 1. \tag{2.3}$$

Here Id denotes the identity matrix of G. It is easy to see that $\langle K.f_2 \rangle = \sigma$ and $\sum_{\lambda \in \mathbb{F}_q} \left(\begin{smallmatrix} \varpi & [\lambda] \\ 0 & 1 \end{smallmatrix}\right) f_2 = \chi(\left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi \end{smallmatrix}\right)) f_2$. Extending σ to be a KZ-representation by letting $\left(\begin{smallmatrix} \varpi & 0 \\ 0 & \varpi \end{smallmatrix}\right)$ act trivially and setting $\lambda := \chi(\left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi \end{smallmatrix}\right))$, we get $\pi \cong \text{c-Ind}_{KZ}^G \sigma/(T-\lambda) \otimes \chi' \circ \det$, for some (uniquely determined) character $\chi' : L^\times \to \overline{\mathbb{F}}_p^\times$. Here we have used the formula (5.1) (see §5 below) for the action of T.

2.2. **Computation of $H^1(K_1/Z_1, \pi)$.** Assume in this subsection that L is *unramified* over $\mathbb{Q}_p$ of degree f. Although it is not always needed, we assume for convenience $f \geq 2$.

Define the following elements in the completed Iwasawa algebra $\overline{\mathbb{F}}_p[[N_0]]$:

$$X_i := \sum_{\mu \in \mathbb{F}_q} \mu^{-p^i} \begin{pmatrix} 1 & [\mu] \\ 0 & 1 \end{pmatrix} \in \overline{\mathbb{F}}_p[[N_0]].$$

A similar proof as that of [20, Prop. 2.13] shows that $\overline{\mathbb{F}}_p[[N_0]] \cong \overline{\mathbb{F}}_p[[X_0, ..., X_{f-1}]]$. Let $\tau = (s_0, ..., s_{f-1}) \otimes \det^b$ be a weight. We can view τ as an $\overline{\mathbb{F}}_p[[N_0]]$-module. A direct generalization of [20, Prop. 2.14], using [1, Lem. 2], shows that τ is isomorphic to $\overline{\mathbb{F}}_p[[X_0, ..., X_{f-1}]]/(X_0^{s_0+1}, ..., X_{f-1}^{s_{f-1}+1})$ as a module over $\overline{\mathbb{F}}_p[[N_0]] \cong \overline{\mathbb{F}}_p[[X_0, ..., X_{f-1}]]$. Precisely, if $w \in \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \cdot \tau^{N_0}$ is non-zero (such a vector is unique up to a scalar), then τ is generated by w as an $\overline{\mathbb{F}}_p[[N_0]]$-module.

Lemma 2.4. *The $\overline{\mathbb{F}}_p$-vector space $\mathrm{Ext}^1_{N_0}(\tau, \overline{\mathbb{F}}_p)$ is of dimension f.*

Proof. We have the identification

$$\mathrm{Ext}^1_{N_0}(\tau,\overline{\mathbb{F}}_p)\cong \mathrm{Ext}^1_{\overline{\mathbb{F}}_p[[N_0]]}(M,\overline{\mathbb{F}}_p),$$

where M denotes $\overline{\mathbb{F}}_p[[X_0,...,X_{f-1}]]/(X_0^{s_0+1},...,X_{f-1}^{s_{f-1}+1})$. The assertion then follows from the theory of Koszul complex ([6, §1.6]). Explicitly, for each $0\le j\le f-1$, the extension

$$0\to\overline{\mathbb{F}}_pe_j\to\overline{\mathbb{F}}_p[[X_0,...,X_{f-1}]]/(X_0^{r_0+1},...,X_j^{r_j+2},...,X_{f-1}^{r_{f-1}+1})\to M\to 0,$$

is non split (where e_j is sent to $X_j^{r_j+1}$), and they form a basis of $\mathrm{Ext}^1_{\overline{\mathbb{F}}_p[[N_0]]}(M,\overline{\mathbb{F}}_p)$. ☐

We need take into account of the action of $\mathcal{H}$, where

$$\mathcal{H}:=\left\{\left(\begin{smallmatrix}[\lambda] & 0\\ 0 & [\mu]\end{smallmatrix}\right):\lambda,\mu\in\mathbb{F}_q^{\times}\right\}\subset K.$$

Note that the order of $\mathcal{H}$ is prime to p and $\mathcal{H}$ normalizes N_0.

Proposition 2.5. *Let $\tau=(s_0,...,s_{f-1})\otimes\det^b$ be as above. Let ψ be a character of $\mathcal{H}$ such that $\mathrm{Ext}^1_{\mathcal{H}N_0}(\tau,\psi)\neq 0$. Then there exists $0\le j\le f-1$ such that $\psi=\psi_\tau^s\alpha^{(s_j+1)p^j}$, where ψ_τ is the character corresponding to the action of $\mathcal{H}$ on τ^{N_0}. Moreover, $\mathrm{Ext}^1_{\mathcal{H}N_0}(\tau,\psi)$ is of dimension 1.*

Proof. It is easy to see that if V is an $\overline{\mathbb{F}}_p[[\mathcal{H}N_0]]$-module, and $w\in V$ is an $\mathcal{H}$-eigenvector of character χ, then X_jw is also an $\mathcal{H}$-eigenvector, but of character $\chi\alpha^{p^j}$. So by the proof of Lemma 2.4, the characters ψ of $\mathcal{H}$ such that $\mathrm{Ext}^1_{\mathcal{H}N_0}(\tau,\psi)\neq 0$ are the characters $\{\psi_\tau^s\alpha^{(s_j+1)p^j},0\le j\le f-1\}$. ☐

From now on, we assume $\pi=\mathrm{Ind}_P^G\chi$ is a principal series satisfying:

(**H**) the K-socle of π is irreducible, and if $(r_0,...,r_{f-1})\otimes\det^a$ is the socle, then $0\le r_i\le p-2$.

Remark 1. *The first condition of (**H**) amounts to demand that if we write $\chi=\eta_1\otimes\eta_2$ for $\eta_i:L^\times\to\overline{\mathbb{F}}_p^{\times}$, then $\eta_1\eta_2^{-1}$ is a ramified character.*

Define a set of weights, depending on (the K-socle of) π, as follows: for $0\le j\le f-1$, let $\sigma_j(\pi):=(s_0,...,s_{f-1})\otimes\det^b$ where $s_i=r_i$ for $i\notin\{j-1,j\}$ and $s_{j-1}=p-2-r_{j-1}$, $s_j=r_j+1$, $b\equiv a+p^{j-1}(r_{j-1}+1)-p^j \pmod{q-1}$. They are well defined under the condition (**H**) and the assumption $f\ge 2$.

Proposition 2.6. *Let $\pi=\mathrm{Ind}_P^G\chi$ be a principal series satisfying (**H**) and denote $\sigma=\mathrm{soc}_K\pi=(r_0,...,r_{f-1})\otimes\det^a$. Let τ be a weight. Then $\mathrm{Ext}^1_{K/Z_1}(\tau,\pi|_K)\neq 0$ if and only if one of the following holds*

(i) *$\tau\cong\sigma_j(\pi)$ for some $0\le j\le f-1$; in this case $\dim\mathrm{Ext}^1_K(\sigma_j(\pi),\pi|_K)=1$.*
(ii) *$\tau\cong\sigma$; in this case $\dim\mathrm{Ext}^1_{K/Z_1}(\sigma,\pi|_K)=f$.*

Proof. Iwasawa decomposition implies that $(\mathrm{Ind}_P^G\chi)|_K\cong\mathrm{Ind}_{P\cap K}^K(\chi|_{T\cap K})$. To simplify the notation, we write $\psi=\chi|_{P\cap K}$ for its restriction to $P\cap K$. By Shapiro's lemma, we have an isomorphism $\mathrm{Ext}^1_K(\tau,\pi|_K)\cong\mathrm{Ext}^1_{P\cap K}(\tau,\psi)$. Consider a non-split extension of $P\cap K$-representations

$$0\to\psi\to V\to\tau\to 0. \tag{2.4}$$

First assume V remains non-split when restricted to $\mathcal{H}N_0$, so that $\mathrm{Ext}^1_{\mathcal{H}N_0}(\tau,\psi)\neq 0$. If we write $\tau=(s_0,...,s_{f-1})\otimes\det^b$, then Proposition 2.5 implies that $\psi=\psi_\tau^s\alpha^{(s_j+1)p^j}$ for some $0\leq j\leq f-1$. Using the relation $\psi^s=(r_0,...,r_{f-1})\otimes\det^a$, a simple calculation shows that $\psi_\tau=\psi_{\sigma_{j+1}(\pi)}$, hence $\tau\cong\sigma_{j+1}(\pi)$. That is we are in case (i) of the theorem. Again by Proposition 2.5, $\mathrm{Ext}^1_{P\cap K}(\tau,\psi)$ has dimension ≤ 1, so the same holds for $\mathrm{Ext}^1_K(\tau,\pi|_K)$. To conclude in this case, it suffices to construct a non-zero element in $\mathrm{Ext}^1_K(\sigma_{j+1}(\pi),\pi|_K)$. In fact, [5, Cor.5.6] says that $\mathrm{Ext}^1_K(\sigma_{j+1}(\pi),\sigma)$ is non-zero and has dimension 1. In view of the exact sequence

$$\mathrm{Hom}_K(\sigma_{j+1}(\pi),\pi/\sigma)\to\mathrm{Ext}^1_K(\sigma_{j+1}(\pi),\sigma)\to\mathrm{Ext}^1_K(\sigma_{j+1}(\pi),\pi|_K)$$

we are reduced to show $\mathrm{Hom}_K(\sigma_{j+1}(\pi),\pi/\sigma)=0$. If the later space were non-zero, we would get an inclusion $\Sigma\hookrightarrow\pi$, where Σ denotes the unique non-split extension of $\sigma_{j+1}(\pi)$ by σ. But K_1 acts trivially on Σ (see [5, Cor. 5.6]), we would get an inclusion $\Sigma\hookrightarrow\pi^{K_1}\cong\mathrm{Ind}_{P(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)}\psi$, which contradicts [5, Thm. 2.4].

Now assume V is split when restricted to $\mathcal{H}N_0$ and choose a $\mathcal{H}N_0$-splitting $s:\tau\hookrightarrow V$. This implies that V is fixed by $\left(\begin{smallmatrix}1&\mathfrak{p}_L\\0&1\end{smallmatrix}\right)$ since both ψ and τ are. If $n\in N_0$ and $h\in T_1$, a simple calculation shows that $hn=n'nh$, for some $n'\in\left(\begin{smallmatrix}1&\mathfrak{p}_L\\0&1\end{smallmatrix}\right)$, therefore the actions on V of T_1 and N_0 commute.

Claim: *if $x\in\tau$ lies in the radical of τ (as N_0-representation), then $s(x)\in V$ is fixed by T_1.*

Proof of Claim. Let $w\in\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\tau^{N_0}$ be a non-zero vector. We have seen that w generates τ as an N_0-representation. The condition that x lies in the radical of τ is equivalent to that there exists a finite set of elements $n_i\in N_0$ such that $x=\sum_i(n_i-1)w$. Let x be such an element and assume there exists $h\in T_1$ with $(h-1)s(x)\neq 0$. The remark above implies that

$$(h-1)s(x)=\sum_i(h-1)(n_i-1)s(w)=\sum_i(n_i-1)(h-1)s(w). \tag{2.5}$$

In particular, $(h-1)s(w)$ is non-zero. But, this vector lies in the underlying space of ψ on which N_0 acts trivially, so the equality (2.5) forces that $(h-1)s(x)=0$ as $n_i\in N_0$, contradiction. The claim follows.

By the claim, the extension (2.4) is the pullback of a (non-split) exact sequence

$$0\to\psi\to W\to\tau/\mathrm{rad}(\tau)\to 0$$

on which N_0 acts trivially but T_1 acts non trivially. This forces that $\psi=\psi_\tau^s$, so that $\tau\cong\sigma$ (since $\psi^s=\chi_\sigma$) and we are in case (ii) of the theorem. Moreover, because $T_1/Z_1\cong 1+\mathfrak{p}_L\cong\mathcal{O}_L\cong\mathbb{Z}_p^f$ (since L is unramified) , the space $\mathrm{Ext}^1_{T_1/Z_1}(\psi,\psi)$ has dimension f.

To conclude in this case, we need show $\dim\mathrm{Ext}^1_{K/Z_1}(\sigma,\pi|_K)$ has dimension $\geq f$, but it follows from Corollary 2.3. $\square$

Remark 2. *When $L=\mathbb{Q}_p$, the dimension part of Proposition 2.6(ii) is not always true, cf.* [5, Thm. 7.16(iii)].

The above proof has the following consequence.

Corollary 2.7. *Let π be as in Proposition 2.6 and $\tau=\sigma_j(\pi)$ for some $0\leq j\leq f-1$. Let $0\to\pi\to E\to\tau\to 0$ be the unique non-split K-extension. Then the induced sequence $0\to\pi^{I_1}\to E^{I_1}\to\tau^{I_1}\to 0$ is exact.*

Proof. With notations in the proof of Proposition 2.6, the extension E comes from the extension Σ of τ by σ. The result follows from the corresponding statement for Σ, see [5, Prop. 4.13]. $\square$

The next lemma will be used in the proof of Proposition 3.2.

Lemma 2.8. *Let π be a principal series satisfying (**H**) and σ be its K-socle. Assume that V is a smooth representation of G such that $\pi \hookrightarrow V$. Assume that $\mathrm{Hom}_K(\sigma, V|_K)$ is 1-dimensional and $\mathrm{Hom}_K(\sigma, V/\pi) \neq 0$. Then V contains a sub-G-representation V' which is a non-split extension of π by π.*

Proof. The assumption $\dim \mathrm{Hom}_K(\sigma, V) = 1$ means that σ appears in $\mathrm{soc}_K V$ with multiplicity one and is contained in π. Therefore V contains a sub-K-representation E which fits into a non-split extension

$$0 \to \pi|_K \to E \to \sigma \to 0.$$

We first describe the extension E more explicitly. Proposition 2.2 implies a surjective morphism $\mathrm{Ext}^1_{G,\zeta}(\pi,\pi) \twoheadrightarrow \mathrm{Ext}^1_{K,\zeta}(\sigma, \pi|_K)$, given by pullback via $\sigma \hookrightarrow \pi|_K$. Choose an extension

$$0 \to \pi \to \mathrm{Ind}_P^G \epsilon_\delta \to \pi \to 0$$

which lifts E. Choose a basis $\{v, w\}$ of ϵ_δ as in §2.1 and define two elements of $\mathrm{Ind}_P^G\epsilon_\delta$ as follows: let $f_v \in \pi \subset \mathrm{Ind}_P^G \epsilon_\delta$ be the vector f_2 defined in (2.3), and f_w be the element characterized by (write $\Pi = \left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)$):

$$\mathrm{Supp}(f_w) = P\Pi I_1 = P\Pi N_0, \quad f_w(b\Pi n) = b\cdot w, \ \forall b \in P, \forall n \in N_0.$$

Here $\cdot$ means the action of P on ϵ_δ. It is easy to see that f_w is well defined and fixed by N_0. If $h \in T_1$, then

$$hf_w = f_w + \delta(\Pi h \Pi^{-1}) f_v.$$

Moreover, the image of f_w in (the quotient) π lies in $\sigma = \mathrm{soc}_K \pi$, see §2.1; that is, f_w itself lies in E.

Now view f_w as a vector in V via the inclusion $E \subset V$. Consider the operator $S := \sum_{\lambda \in \mathbb{F}_q} \left(\begin{smallmatrix} p & [\lambda] \\ 0 & 1 \end{smallmatrix}\right) \in \overline{\mathbb{F}}_p[G]$. Because f_w is fixed by N_0, we get by Lemma 5.1(iii),

$$h(S(f_w)) = S(h(f_w)) = Sf_w + \delta(\Pi h\Pi^{-1}) Sf_v = Sf_w + \delta(\Pi h \Pi^{-1})\lambda f_v.$$

Hence $f'_w := S(f_w) - \lambda f_w$ is fixed by T_1. By Lemma 5.1(iv), for n large enough, $S^n f'_w$ is fixed by I_1. Moreover, by the proof of [18, Lem. 4.1], $S^{n+1}f'_w$ generates an irreducible K-representation which is isomorphic to σ. Since σ appears with multiplicity one in the K-socle of V by assumption, we deduce that $S^{n+1} f'_w \in \overline{\mathbb{F}}_p f_v$. In particular, $S^{n+2} f_w = \lambda S^{n+1} f_w$ in V/π, showing that $S^{n+1} f_w$ (which is non-zero) generates a principal series in V/π, which must be isomorphic to π. $\square$

2.3. **Ordinary part.** In this subsection L is a finite extension of $\mathbb{Q}_p$ of degree n.

Recall that Emerton has defined a functor, called ordinary parts and denoted by Ord_P, from the category of admissible smooth $\overline{\mathbb{F}}_p$-representations of G to the category of admissible smooth $\overline{\mathbb{F}}_p$-representations of T. Let $\mathbb{R}^i\mathrm{Ord}_P$ be its right derived functors for $i \geq 1$. It follows from [9, Prop. 3.6.1] and [12] that $\mathbb{R}^i\mathrm{Ord}_P$ vanishes for $i \geq n+1$.

Write $G_L = \mathrm{Gal}(\overline{\mathbb{Q}}_p/L)$. Let $\epsilon : G_L \to \mathbb{Z}_p^\times$ be p-adic the cyclotomic character ad ω be its reduction modulo p. View them as characters of $L^\times$ via the local Artin map normalized in such a way that uniformizers of L are sent to geometric Frobenii. Denote by α the character $\omega \otimes \omega^{-1} : T \to \mathbb{F}_p^\times$.

Proposition 2.9. *(i) If U is an admissible smooth representation of G and V is a smooth representation of G, then*

$$\mathrm{Hom}_G(\mathrm{Ind}_P^G U, V) \cong \mathrm{Hom}_T(U, \mathrm{Ord}_P(V)).$$

(ii) There is a canonical isomorphism $\mathbb{R}^n\mathrm{Ord}_P(V) \cong V_N \otimes \alpha^{-1}$, where π_N is the space of coinvariants (i.e. the usual Jacquet module of V with respect to P).

(iii) We have $\mathrm{Ord}_P(\mathrm{Ind}_P^G U) \cong U^s$ and $\mathbb{R}^n\mathrm{Ord}_P(\mathrm{Ind}_P^G U) \cong U \otimes \alpha^{-1}$.

(iv) If π is an absolutely irreducible supersingular representation of G over $\overline{\mathbb{F}}_p$, then $\mathrm{Ord}_P(\pi) = \mathbb{R}^n\mathrm{Ord}_P(\pi) = 0$.

Proof. (i) is [8, Theorem 4.4.6]. (ii) is [9, Prop.3.6.2]. (iii) follows from [8, Cor. 4.3.5] and [9, Prop. 3.6.2], using the natural isomorphism $\mathrm{Ind}_P^G U \cong \mathrm{Ind}_{\overline{P}}^G U^s$. For (iv), the first assertion follows from (i); the second follows from (ii). To see this, let $\pi(N) \subset \pi$ be the subspace spanned by vectors of the form $(n-1)v$, for all $n \in N$ and $v \in \pi$, so that $\pi_N = \pi/\pi(N)$. It is easily checked that $\pi(N)$ is stable under the group P and non-zero. The result follows from the main result of [18], which says that $\pi|_P$ is irreducible. □

2.4. **A definition.** We recall a definition due to Emerton [10, §3.6], which plays a crucial role below.

We normalize the local Artin map $L^\times \hookrightarrow G_L^{\mathrm{ab}}$ in such a way that uniformizers of L are sent to geometric Frobenii.

Denote by S the following subtorus of T

$$S := \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} | a \in L^\times \right\} \subset T$$

so that $S \cong L^\times$. The composite of this isomorphism with the local Artin map defines an injection $\iota : S \hookrightarrow G_L^{\mathrm{ab}}$, and hence an anti-diagonal embedding

$$S \hookrightarrow G_L^{\mathrm{ab}} \times S, \quad s \mapsto (\iota(s), s^{-1}). \tag{2.6}$$

Definition 1. *Let V be a representation of $G_L \times S$.*

(i) Let V^{ab} be the maximal sub-object of V on which G_L acts through its maximal abelian quotient G_L^{ab}. This is a $G_L \times S$-sub-representation of V.

(ii) Let $V^{\mathrm{ab},S}$ be the subspace of V^{ab} consisting of S-fixed vectors, where S acts through the anti-diagonal embedding (2.6) and the action of $G_L^{\mathrm{ab}} \times S$ on V^{ab}.

The space $V^{\mathrm{ab},S}$ is stable under the action of G_L and, of course, this action factors through G_L^{ab}.

Lemma 2.10. *Let V be a representation of $G_L \times S$. Assume that the action of G_L on $V/V^{\mathrm{ab},S}$ factors through G_L^{ab}. Then, for any subquotient W of V (as $G_L \times S$-representations), the action of G_L on $W/W^{\mathrm{ab},S}$ also factors through G_L^{ab}.*

Proof. By definition, if H_L denotes the kernel of the quotient map $G_L \twoheadrightarrow G_L^{\mathrm{ab}}$, then $V^{\mathrm{ab}} = V^{H_L}$ and $V^{\mathrm{ab},S} = (V^{H_L})^S$. If W is a sub-$G_L \times S$-representation of V, we deduce that $V^{\mathrm{ab},S} \cap W = W^{\mathrm{ab},S}$, hence a G_L-equivariant inclusion $W/W^{\mathrm{ab},S} \hookrightarrow V/V^{\mathrm{ab},S}$ and the result holds in this case. If W is a quotient of V, the result is obvious. The general case follows from this. □

3. Local results

We keep notations of Section 2. Assume L is *unramified* over $\mathbb{Q}_p$ of degree $f\geq 2$.

3.1. **Construction of Breuil-Paškūnas.** Let $\overline{\rho}:\mathrm{Gal}(\overline{\mathbb{Q}}_p/L)\rightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a continuous representation. Assume $\overline{\rho}$ is reducible and *generic* in the sense of [5, §11], i.e. $\overline{\rho}$ is of the form

$$\overline{\rho}\cong\begin{pmatrix}\mathrm{nr}(\mu)\omega_f^{(r_0+1)+\cdots+p^{f-1}(r_{f-1}-1)} & * \\ 0 & \mathrm{nr}(\mu^{-1})\end{pmatrix}\otimes\eta$$

where $\mu\in\overline{\mathbb{F}}_p^{\times}$, $r_i\in\{0,...,p-3\}$ with $(r_0,...,r_{f-1})\neq(0,...,0),(p-3,...,p-3)$, ω_f is Serre's fundamental character of level f, and $\eta:\mathrm{Gal}(\overline{\mathbb{Q}}_p/L)\rightarrow\overline{\mathbb{F}}_p^{\times}$ is a continuous character.

To $\overline{\rho}$ is associated a set of weights, called Serre weights and denoted by $\mathcal{D}(\overline{\rho})$, as follows (see [5, §11] or [7]). First, the genericity condition on $\overline{\rho}$ implies that $\overline{\rho}$ is in the category of Fontaine-Lafaille [13]. Writing down the associated Fontaine-Lafaille module, we define a subset $J_{\overline{\rho}}$ of $\mathcal{S}:=\{0,...,f-1\}$ which we refer to [16, §2] for its precise definition. We recall that $J_{\overline{\rho}}$ measures how far $\overline{\rho}$ is from splitting, in the sense that $J_{\overline{\rho}}=\mathcal{S}$ if and only if $\overline{\rho}$ is split. Second, we define $\mathcal{D}(x_0,...,x_{f-1})$ to be the set of f-tuples $\tau=(\tau_0(x_0),...,\tau_{f-1}(x_{f-1}))$ satisfying the following conditions:

(i) $\tau_i(x_i)\in\{x_i,x_i+1,p-2-x_i,p-3-x_i\}$
(ii) if $\tau_i(x_i)\in\{x_i,x_i+1\}$, then $\tau_{i+1}(x_{i+1})\in\{x_{i+1},p-2-x_{i+1}\}$
(iii) if $\tau_i(x_i)\in\{p-2-x_i,p-3-x_i\}$, then $\tau_{i+1}(x_{i+1})\in\{p-3-x_{i+1},x_{i+1}+1\}$
(iv) if $\tau_i(x_i)\in\{p-3-x_i,x_i+1\}$, then $i\in J_{\overline{\rho}}$

with the conventions $x_f:=x_0$ and $\tau_f(x_f):=\tau_0(x_0)$. Then $\mathcal{D}(\overline{\rho})$ can be explicitly described as

$$\mathcal{D}(\overline{\rho})=\big\{(\tau_0(r_0),...,\tau_{f-1}(r_{f-1}))\otimes\det{}^{e(\tau)(r_0,...,r_{f-1})},\ \tau\in\mathcal{D}(x_0,...,x_{f-1})\big\},$$

where $e(\tau)(x_0,...,x_{f-1})$ is defined as in [5, §4]. Remark that there are $2^{|J_{\overline{\rho}}|}$ elements in $\mathcal{D}(\overline{\rho})$, and it always contains the weight $\sigma_0:=(r_0,...,r_{f-1})\otimes\eta\circ\det$. For $\sigma\in\mathcal{D}(\overline{\rho})$ which corresponds to $\tau\in\mathcal{D}(x_0,...,x_{f-1})$, we set

$$\ell(\sigma):=\mathrm{Card}\{i\in\mathcal{S};\tau_i(x_i)\in\{p-2-x_i,p-3-x_i\}\},$$

and call it the *length* of σ.

Let $D_0(\overline{\rho})$ be the maximal representation of $\mathrm{GL}_2(\mathbb{F}_q)$ such that

(i) the $\mathrm{GL}_2(\mathbb{F}_q)$-socle of $D(\overline{\rho})$ is $\oplus_{\tau\in\mathcal{D}(\overline{\rho})}\tau$
(ii) each Serre weight $\tau\in\mathcal{D}(\overline{\rho})$ occurs exactly once in $D(\overline{\rho})$.

Let $D_1(\overline{\rho})=D_0(\overline{\rho})^{I_1}$ with the induced action of I and we choose an action of $\Pi=\left(\begin{smallmatrix}0&1\\p&0\end{smallmatrix}\right)$ such that Π^2 is the identity. The amalgam structure of G (more precisely, of $\mathrm{SL}_2(L)$) then allows Breuil and Paškūnas to construct a family of smooth admissible representations of G over $\overline{\mathbb{F}}_p$, with K-socle being $\oplus_{\sigma\in\mathcal{D}(\overline{\rho})}\sigma$. The construction is as follows (see [5]). We first embed K-equivariantly $D_0(\overline{\rho})$ inside an injective envelope $\Omega:=\mathrm{Inj}_K(\oplus_{\sigma\in\mathcal{D}(\overline{\rho})}\sigma)$. Then using the decomposition of $\Omega|_I$ we can give an action of $\left(\begin{smallmatrix}0&1\\p&0\end{smallmatrix}\right)$ on Ω which is compatible with the one on $D_1(\overline{\rho})$ via the embedding we have chosen. In such a way, a theorem of Ihara allows us to get a smooth action of G on Ω and we let V be the sub-representation generated by $D_0(\overline{\rho})$. In particular, V depends on the choice of the action of $\left(\begin{smallmatrix}0&1\\p&0\end{smallmatrix}\right)$ on Ω, and actually there are quite a lot of such choices. Finally, we twist V by $\eta \circ \det$ to recover the determinant of $\overline{\rho}$. Denote by $\mathcal{V}(\overline{\rho})$ the family of representations V obtained in this way.

It is expected that such a $V \in \mathcal{V}(\overline{\rho})$ has length f, or, at least, there exists one $V \in \mathcal{V}(\overline{\rho})$ which has length f. Precisely, we hope that V has a filtration of length f of the form (where $(\pi_i)_i$ denote the graded pieces of the filtration)

$$\pi_0 \text{ --- } \pi_1 \text{ --- } \cdots \text{ --- } \pi_f$$

such that π_i is a principal series if $i \in \{0, f\}$ and supersingular otherwise. This is the case when $\overline{\rho}$ is (reducible) split; in this case, V is a direct sum of π_i's, $0 \leq i \leq f$ (see [5, §19]). However, it is not even known whether such a V has finite length if $\overline{\rho}$ is non-split, except for the case $L = \mathbb{Q}_p$ (cf. [8, Conj. 2.3.7]).

In the following, we assume $\overline{\rho}$ is *non-split.*

Lemma 3.1. *For any $V \in \mathcal{V}(\overline{\rho})$, the G-socle of V is an irreducible principal series and isomorphic to* $\text{c-Ind}_{KZ}^G \sigma_0/(T-\lambda)$ *for some $\lambda \in \overline{\mathbb{F}}_p^\times$.*

Proof. Let π be an irreducible sub-representation of V. Let $\sigma \in \mathcal{D}(\overline{\rho})$ be a Serre weight which is contained in $\mathrm{soc}_K\pi$. The proof of [5, Thm. 15.4(ii)] shows that we can go from $D_{0,\sigma}(\overline{\rho})^{I_1}$ to $D_{0,\sigma_0}(\overline{\rho})^{I_1}$ using $\chi \mapsto \chi^s$ (with notations there). In particular, we see that σ_0 is also contained in π, hence $\langle G.\sigma_0\rangle \subset \pi$. But, by construction, $\langle G\cdot\sigma_0\rangle$ is an (irreducible) principal series, so that $\pi = \langle G\cdot\sigma_0\rangle$. The last assertion follows from [1, Thm. 30]. □

Remark 3. *In Lemma 3.1, any $\lambda \in \overline{\mathbb{F}}_p^\times$ could happen. In fact, the construction in* [5] *does not take into account of the whole information of $\overline{\rho}$. For the representations arising from the cohomology of Shimura curves, λ is uniquely determined by $\overline{\rho}$ (see* [4] *or §4).*

Proposition 3.2. *Let $V \in \mathcal{V}(\overline{\rho})$. Assume that $\mathrm{Ord}_P(V)$ is one-dimensional. Then V contains a sub-representation π which is of length 2 and fits into an exact sequence*

$$0 \to \pi_0 \to \pi \to \pi_1 \to 0$$

such that π_0 is a principal series and π_1 is a supersingular representation. Moreover, π_1 is uniquely determined (by V) in the following two cases:

(i) either $\mathcal{D}(\overline{\rho}) = \{\sigma_0\}$, in which case $\mathrm{soc}_K(\pi_1) = \oplus_{\sigma\in\mathcal{D}(\overline{\rho}^{\mathrm{ss}}),\ell(\sigma)=1}\sigma$;

(ii) or $V^{I_1} = D_1(\overline{\rho})$, in which case we only have an inclusion

$$\bigoplus_{\sigma\in\mathcal{D}(\overline{\rho}^{\mathrm{ss}}),\ell(\sigma)=1} \sigma \subseteq \mathrm{soc}_K(\pi_1), \tag{3.1}$$

which is an equality when $f = 2$.

Remark 4. *It is easy to check that the set $\{\sigma \in \overline{\rho}^{\mathrm{ss}} : \ell(\sigma) = 1\}$ is exactly the set $\{\sigma_j(\pi_0) : 0 \leq j \leq f-1\}$ (well defined thanks to the genericity condition on $\overline{\rho}$).*

Proof. Lemma 3.1 implies that the G-socle of V is a principal series; we denote it by π_0. By assumption, $\mathrm{Ord}_P(V)$ is one dimensional. We claim that V/π_0 does not admit principal series as a sub-representation. In fact, if $\pi' \hookrightarrow V/\pi_0$ is a principal series, then $\mathrm{Ext}^1_G(\pi', \pi_0) \neq 0$ since π_0 is the G-socle of V. But Proposition 2.1 implies that $\pi' \cong \pi_0$, and the corresponding extension is of the form (2.2) for certain $\delta \in \mathrm{Hom}(L^\times, \overline{\mathbb{F}}_p)$. The claim follows from the assumption on $\mathrm{Ord}_P(V)$ using Proposition 2.9(iii). By [22, Thm. 2], V/π_0 is still a smooth admissible representation, hence it contains at least one irreducible sub-representation; by the claim it must be supersingular. This shows the first assertion of the proposition.

We define π_1 and determine its K-socle under assumptions (i) or (ii). First assume $\mathcal{D}(\overline{\rho}) = \{\sigma_0\}$. By construction recalled above, V sits inside a certain Ω such that $\Omega|_K$ is an injective envelope of σ_0. Proposition 2.6 then implies that

$$\mathrm{soc}_K(\Omega/\pi_0) = \big(\bigoplus_{j=0}^{f-1}\sigma_0\big) \oplus \big(\bigoplus_{j=0}^{f-1}\sigma_j(\pi_0)\big).$$

By Lemma 2.8, we deduce that $\mathrm{soc}_K(V/\pi_0)$ does not contain σ_0 so that

$$\mathrm{soc}_K(V/\pi_0) \subseteq \big(\bigoplus_{j=0}^{f-1}\sigma_j(\pi_0)\big).$$

We claim that V/π_0 contains a unique irreducible sub-representation. In fact, let $\pi_1 \hookrightarrow V/\pi_0$ be an irreducible sub-representation and let σ be a weight contained in the K-socle of π_1. Then σ is of the form $\sigma_j(\pi_0)$ for some $0 \leq j \leq f-1$. We check that all other $\sigma_j(\pi_0)$'s are also contained in π_1 under the process $\chi \mapsto \chi^s$ (argument as in [5, Thm. 15.4]). To do this, we may assume $j = 1$ so that $\sigma = (p-2-r_0, r_1+1, r_2, ..., r_{f-1})$ (up to twist). If we let $I(\sigma_0, \sigma^{[s]})$ be the unique sub-representation of $D_0(\overline{\rho})$ with cosocle $\sigma^{[s]}$ (see [5, Cor. 3.12]), where $\sigma^{[s]} := (r_0+1, p-2-r_1, p-1-r_2, ..., p-1-r_{f-1})$ (up to a twist uniquely determined by σ) is as in [5, page 9]. Using [5, Cor. 4.11], $\sigma_0(\pi_0)$ occurs in $I(\sigma_0, \sigma^{[s]})$ as an irreducible constituent. This shows that $\sigma_0(\pi)$ also occurs in the K-socle of π. Repeating this argument gives the result and shows that $\mathrm{soc}_K \pi_1 = \oplus_{j=0}^{f-1}\sigma_j(\pi_0) = \oplus_{\sigma\in\mathcal{D}(\overline{\rho}),\ell(\sigma)=1}\sigma$ by Remark 4.

Now assume (ii) $V^{I_1} = D_1(\overline{\rho})$. By the construction, V sits inside certain $\Omega \in \mathrm{Mod}_G^{\mathrm{sm}}$ such that $\Omega|_K$ is an injective envelope of $\oplus_{\sigma\in\mathcal{D}(\overline{\rho})}\sigma$. Since $\mathrm{soc}_K(\pi_0) = \sigma_0$, we can K-equivariantly decompose $\Omega = \oplus_{\sigma\in\mathcal{D}(\overline{\rho})}\Omega_\sigma$ so that $\mathrm{soc}_K(\Omega_\sigma) = \sigma$ for each σ and that π_0 is contained in Ω_{σ_0}. Therefore $\Omega/\pi_0 = (\Omega_{\sigma_0}/\pi_0) \oplus (\oplus_{\sigma\neq\sigma_0}\Omega_\sigma)$ and Proposition 2.6 then implies that

$$\mathrm{soc}_K(V/\pi_0) \subseteq \mathrm{soc}_K(\Omega/\pi_0) = \big(\bigoplus_{j=0}^{f-1}\sigma_0\big) \oplus \big(\bigoplus_{j=0}^{f-1}\sigma_j(\pi_0)\big) \oplus \big(\bigoplus_{\sigma\in\mathcal{D}(\overline{\rho}),\sigma\neq\sigma_0}\sigma\big).$$

Here, although $\sigma_j(\pi_0)$ is possibly isomorphic to some $\sigma \in \mathcal{D}(\overline{\rho})$ in view of Remark 4 (automatically of length 1), we use $\sigma_j(\pi)$ to emphasize that it is a sub-representation of Ω_{σ_0}/π_0 and use $\sigma \in \mathcal{D}(\overline{\rho})$ to emphasize that it is contained in Ω_σ. Lemma 2.8 implies that V/π_0 does not admit σ_0 as a sub-K-representation, hence

$$\bigoplus_{\sigma\in\mathcal{D}(\overline{\rho}),\sigma\neq\sigma_0}\sigma \subseteq \mathrm{soc}_K(V/\pi_0) \subseteq \big(\bigoplus_{j=0}^{f-1}\sigma_j(\pi_0)\big) \oplus \big(\bigoplus_{\sigma\in\mathcal{D}(\overline{\rho}),\sigma\neq\sigma_0}\sigma\big).$$

Let $\sigma \in \mathcal{D}(\overline{\rho})$ such that $\ell(\sigma) = 1$. We know $\sigma = \sigma_j(\pi_0)$ for some $0 \leq j \leq f-1$ by Remark 4. Let Σ be the unique non-split extension of $\sigma_j(\pi_0)$ by σ_0; then $\Sigma^{I_1} = \sigma_0^{I_1} \oplus \sigma_j(\pi_0)^{I_1}$ is 2-dimensional by [5, Prop. 4.13]. We have $\mathrm{Hom}_K(\Sigma, V) = 0$; if not, we would have $\Sigma \hookrightarrow V$ and therefore $\Sigma \oplus \sigma \hookrightarrow V$ which would contradict the assumption $V^{I_1} = D_1(\overline{\rho})$ which is multiplicity free ([5, Cor. 13.5]). In all we get

$$\mathrm{soc}_K(V/\pi_0) \subseteq \big(\bigoplus_{\sigma\in\mathcal{D}(\overline{\rho}^{\mathrm{ss}}),\ell(\sigma)=1}\sigma\big) \oplus \big(\bigoplus_{\sigma\in\mathcal{D}(\overline{\rho}),\ell(\sigma)\geq 2}\sigma\big).$$

As in case (i), we show that V/π_0 admits a unique irreducible sub-G-representation. For this, we show that any sub-representation π_1 of V/π_0 contains $\oplus_{\sigma\in\mathcal{D}(\overline{\rho}^{\mathrm{ss}}),\ell(\sigma)=1}\sigma$ in its K-socle. Indeed, if π_1 contains some $\sigma\in\mathcal{D}(\overline{\rho})$ with $\ell(\sigma)\geq 2$, then the same argument as in the proof of [5, Thm. 15.4], using the process $\chi\mapsto\chi^s$, shows that π_1 also contains another $\sigma'\in\mathcal{D}(\overline{\rho})$ with $\ell(\sigma')<\ell(\sigma)$. Remark that, although it is not necessary for us, we can guarantee that σ' belongs to $\mathcal{D}(\overline{\rho})$, not just to $\mathcal{D}(\overline{\rho}^{\mathrm{ss}})$. So we may assume $\ell(\sigma)=1$ at the beginning. Still using the process $\chi\mapsto\chi^s$, we claim that π_1 contains all other $\sigma'\in\mathcal{D}(\overline{\rho}^{\mathrm{ss}})$ with $\ell(\sigma')=1$. The argument is similar to that of case (i) above but slightly different, as follows.

Let $D_{0,0}(\overline{\rho}):=D_0(\overline{\rho})\cap\pi_0$ and let $D_{0,1}(\overline{\rho})$ be the maximal sub-K-representation of $D_0(\overline{\rho})/D_{0,0}(\overline{\rho})$ which does not contain any element of $\mathcal{D}(\overline{\rho}^{\mathrm{ss}})$ of length ≥ 2. By [16, Lem. 5.1], the K-socle of $D_{0,1}(\overline{\rho})$ is exactly $\oplus_{\sigma\in\mathcal{D}(\overline{\rho}^{\mathrm{ss}}),\ell(\sigma)=1}$. By [5, Thm. 13.1], $D_{0,1}(\overline{\rho})$ is a sub-K-representation of $D_{0,1}(\overline{\rho}^{\mathrm{ss}})$ (see [5, Thm. 15.4(ii)] for this notation), since it does not contain any element of $\mathcal{D}(\overline{\rho}^{\mathrm{ss}})$ of length 0 or ≥ 2. Moreover, $D_{0,1}(\overline{\rho})^{I_1}$ is stable under the action of $\left(\begin{smallmatrix}0&1\\p&0\end{smallmatrix}\right)$ and compatible with that of $D_{0,1}(\overline{\rho})^{I_1}$ because they are both multiplicity free. In all, we are reduced to check the claim in the case $\overline{\rho}=\overline{\rho}^{\mathrm{ss}}$, which is done in [5, Thm. 1.54(ii)].

To conclude, we just let π_1 be the sub-representation of V/π_0 generated by $\oplus_{\sigma\in\mathcal{D}(\overline{\rho}^{\mathrm{ss}}),\ell(\sigma)=1}$. The inclusion (3.1) follows from the claim and the equality when $f=2$ is obvious. □

Remark 5. *In* [11]*, it is shown that for some $V\in\mathcal{V}(\overline{\rho})$ coming from cohomology of Shimura curves, the condition (ii) of Proposition 3.2 is verified. Moreover, for such a V, it is hoped that (3.1) is always an equality.*

3.2. **The case $f=2$.** We have seen that non-split G-extensions of a supersingular representation by a principal series exist (under the conditions which will be checked in §4). In this section, we deduce from this the existence of non-split extensions of the converse type in the case $f=2$, namely extensions of the form

$$0\to\pi_1\to *\to\pi_2\to 0$$

with π_1 supersingular and π_2 a principal series.

We keep notations in the previous subsection.

Proposition 3.3. *Assume $f=2$. Let $\pi_0=\mathrm{Ind}_P^G\chi$ and π_1 be a supersingular representation. If $\mathrm{Ext}^1_G(\pi_1,\pi_0)\neq 0$, then $\mathrm{Ext}^1_G(\mathrm{Ind}_P^G\chi^s\alpha,\pi_1)\neq 0$.*

Proof. Let

$$0\to\pi_0\to V\to\pi_1\to 0 \tag{3.2}$$

be a non-split extension. Apply the functor Ord_P and using Lemma 3.4 below, we get a surjection

$$\mathbb{R}^1\mathrm{Ord}_P\pi_1\twoheadrightarrow\mathbb{R}^2\mathrm{Ord}_P\pi_0.$$

By Proposition 2.9(ii) (as $f=2$), we know $\mathbb{R}^2\mathrm{Ord}_P\pi_0\cong\chi\alpha^{-1}$, hence $\mathbb{R}^1\mathrm{Ord}_P\pi_1$ is non-zero and admits a quotient isomorphic to $\chi\alpha^{-1}$. Because there is no non-trivial extension between two non-isomorphic T-characters, we deduce that $\mathbb{R}^1\mathrm{Ord}_P\pi_1$ also contains a sub-character isomorphic to $\chi\alpha^{-1}$. The assertion follows from the long exact sequence [9, (3.7.5)] which implies that $\mathrm{Ext}^1_G(\mathrm{Ind}_{\overline{P}}^G\chi\alpha^{-1},\pi_1)\cong\mathrm{Hom}_T(\chi\alpha^{-1},\mathbb{R}^1\mathrm{Ord}_P(\pi_1))\neq 0$. □

Lemma 3.4. *We have $\mathbb{R}^2\mathrm{Ord}_PV=0$.*

Proof. By assumption $f = 2$, Proposition 2.9(iii) implies that $\mathbb{R}^2\mathrm{Ord}_P W = W_N \otimes \alpha^{-1}$ for any G-representation W. The sequence (3.2) induces an exact sequence $(\pi_0)_N \to V_N \to (\pi_1)_N \to 0$. Since $(\pi_1)_N = 0$ by Proposition 2.9(iv), it suffices to prove that the map $(\pi_0)_N \to V_N$ is zero. If not, then $V_N \cong (\pi_0)_N$ as $(\pi_0)_N$ is one dimensional, and the adjunction formula (see [8, §3.6]) implies that

$$\mathrm{Hom}_G(V, \mathrm{Ind}_P^G(\pi_0)_N) = \mathrm{Hom}_G(V, \pi_0) \neq 0,$$

As a consequence, the extension (3.2) splits, giving the desired contradiction. □

We have remarked that when $\overline{\rho}$ is reducible, we hope that any $V \in \mathcal{V}(\overline{\rho})$ is a successive extension of the irreducible representations $(\pi_i)_{0\leq i\leq 2}$, with π_0, π_2 being principal series. It is easily checked that if we write $\pi_0 = \mathrm{Ind}_P^G\chi$ for suitable χ, then $\pi_2 = \mathrm{Ind}_P^G\chi^s\alpha$, which is compatible with Proposition 3.3.

Remark 6. *Note that* π_2 *is exactly the* f*-th smooth dual of* π_0*, in the sense of* [17, Def. 3.12 & Prop. 5.4].

4. Global results

We prove the main result of this article in this section.

Let F be a totally real number field. For each finite place v of F, denote by F_v the completion of F at v. Write $G_F = \mathrm{Gal}(\overline{F}/F)$ and $G_{F_v} = \mathrm{Gal}(\overline{F_v}/F_v)$, and we identify G_{F_v} with a subgroup of G_F by fixing an embedding $\overline{F} \hookrightarrow \overline{F_v}$. We fix a finite extension E of $\mathbb{Q}_p$, which serves as the coefficient field and is allowed to be enlarged. Write $\mathcal{O}_E$ for the ring of integers of E, k_E its residue field, and ϖ_E a fixed uniformizer.

4.1. A theorem of Emerton.

Let D be a quaternion algebra over F which splits at exactly one infinite place denoted by τ. Fix an isomorphism $D_\tau \overset{\mathrm{def}}{=} D \otimes_{F,\tau} \mathbb{R} \cong \mathrm{M}_2(\mathbb{R})$. Let $D_f^\times = (D \otimes_{\mathbb{Q}} \mathbb{A}_f)^\times$.

For any open compact subgroup $U \subset D_f^\times$, let X_U be the projective smooth algebraic curve over F associated to U and consider the étale cohomology with coefficients in A

$$H^1_{\text{ét}}(X_{U,\overline{\mathbb{Q}}}, A),$$

where A denotes one of E, $\mathcal{O}_E$, or $\mathcal{O}_E/\varpi_E^s$ for some $s > 0$. For two open compact subgroups $V \subseteq U$ of $D_f^\times$, we have natural morphisms of algebraic curves $X_V \to X_U$ defined over F, which induces a $\mathrm{Gal}(\overline{F}/F)$-equivariant map

$$H^1_{\text{ét}}(X_{U,\bar{\mathbb{Q}}}, A) \to H^1_{\text{ét}}(X_{V,\bar{\mathbb{Q}}}, A).$$

Define

$$S^D(A) := \varinjlim H^1_{\text{ét}}(X_{U,\bar{\mathbb{Q}}}, A)$$

where the limit is taken over all the open compact subgroups $U \subset D_f^\times$. It carries a continuous action of $\mathrm{Gal}(\overline{F}/F)$ and a smooth admissible action of $D_f^\times$ commuting with each other.

Let $\overline{\rho} : G_F \to \mathrm{GL}_2(k_E)$ be an irreducible, continuous, totally odd representation. Assume that $\overline{\rho}$ is modular, in the sense that $\mathrm{Hom}_{G_F}(\overline{\rho}, S^D(k_E)) \neq 0$. Let Σ be a finite set of finite places of F which contains all those places dividing p, or at which U_w is not maximal, or D or $\overline{\rho}$ is ramified. Define as usual the Hecke operators

$$T_v = [\mathrm{GL}_2(\mathcal{O}_{F_v})\begin{pmatrix}\varpi_v & 0\\ 0 & 1\end{pmatrix}\mathrm{GL}_2(\mathcal{O}_{F_v})],\quad S_v = [\mathrm{GL}_2(\mathcal{O}_{F_v})\begin{pmatrix}\varpi_v & 0\\ 0 & \varpi_v\end{pmatrix}\mathrm{GL}_2(\mathcal{O}_{F_v})]$$

and let $\mathbb{T}^{\Sigma}(U)$ denote the commutative A-algebra generated by T_v and S_v for $v \notin \Sigma$. We let $\mathfrak{m}_{\overline{\rho}}^{\Sigma} = \mathfrak{m}_{\overline{\rho}}^{\Sigma}(U)$ denote the maximal ideal of $\mathbb{T}^{\Sigma}$ corresponding to $\overline{\rho}$, i.e. satisfying

$$T_v \bmod \mathfrak{m}_{\overline{\rho}}^{\Sigma} \equiv \mathrm{trace}(\overline{\rho}(\mathrm{Frob}_v)) \ \text{ and } \ \mathbf{N}(v)S_v \bmod \mathfrak{m}_{\overline{\rho}}^{\Sigma} \equiv \det(\overline{\rho}(\mathrm{Frob}_v))$$

for all $v \notin \Sigma$. Let

$$S^D(U,k_E)[\mathfrak{m}_{\overline{\rho}}^{\Sigma}] = \{f \in S^D(U,k_E) | Tf = 0 \text{ for all } T \in \mathfrak{m}_{\overline{\rho}}^{\Sigma}\}.$$

By [7, Lemma 4.6], it is independent of Σ, so denote it by $S^D(U,k_E)[\mathfrak{m}_{\overline{\rho}}]$. We can consider the direct limit over U of the spaces $S^U(U,k_E)[\mathfrak{m}_{\overline{\rho}}]$ which yields $S^D(k_E)[\mathfrak{m}_{\overline{\rho}}]$. Recall the following conjecture due to Buzzard, Diamond and Jarvis [7, Conj. 4.9].

Conjecture 1. *The representation $S^D(k_E)[\mathfrak{m}_{\overline{\rho}}]$ of $G_F \times D_f^{\times}$ is isomorphic to a restricted tensor product*

$$S^D(k_E)[\mathfrak{m}_{\overline{\rho}}] \cong \overline{\rho} \otimes (\otimes'_w \pi_w), \tag{4.1}$$

where π_w is a smooth admissible representation of $D_w^{\times}$ such that

- *if w does not divide p, then π_w is the representation attached to $\overline{\rho}_w := \overline{\rho}|_{G_{F_w}}$ by the modulo ℓ local Langlands and Jacquet-Langlands correspondence, see [7, §4];*
- *if $w|p$, then $\pi_w \neq 0$; moreover if F and D are unramified at v, and σ is an irreducible k_E-representation of $\mathrm{GL}_2(\mathcal{O}_{F_w})$, then*

  $$\mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_{F_w})}(\sigma,\pi_w) \neq 0 \iff \sigma \in W(\overline{\rho}_w).$$

  Here $W(\overline{\rho}_w)$ is a certain set of Serre weights associated to $\overline{\rho}_w$ which, when $\overline{\rho}_w$ is generic, coincides with $\mathcal{D}(\overline{\rho}_w)$ (in §3) up to normalisation.

From now on, assume that D splits at some finite place v lying above p. For U^v an open subgroup of $\prod_{w\neq v} \mathcal{O}_{D_w}^{\times}$, we write

$$S^D(U^v,A) := \varinjlim S^D(U^vU_v,A)$$

where the inductive limit is taken over all compact open subgroups U_v of $D_v^{\times} \cong \mathrm{GL}_2(F_v)$. The following result is due to Emerton (see Definition 1 for the notation).

Theorem 4.1. *For any $n \geq 0$, the action of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$ on the cokernel of the embedding*

$$\mathrm{Ord}_P\big(S^D(U^v,k_E)_{\mathfrak{m}_{\overline{\rho}}}\big)^{\mathrm{ab},S_v} \hookrightarrow \mathrm{Ord}_P\big(S^D(U^v,k_E)_{\mathfrak{m}_{\overline{\rho}}}\big)$$

factors through $G_{F_v}^{\mathrm{ab}}$.

Proof. This is Theorem 5.6.11 of [10] in our setting. The proof of Emerton works equally in this case. In fact, Lemmas 5.6.7 and 5.6.8 *loc. cit.* hold true, with the only change being to replace the absolute value $|\ |_p$ by $|\ |_v$, the absolute value on F_v normalised as $|\varpi_v|_v := q_v^{-1}$. We also need

$$S^D(U^v,\mathcal{O}_E)_{\mathfrak{m}_{\overline{\rho}}}^{N_{0,v}}/\varpi_L S^D(U^v,\mathcal{O}_E)_{\mathfrak{m}_{\overline{\rho}}}^{N_{0,v}} \to S^D(U^v,k_E)_{\mathfrak{m}_{\overline{\rho}}}^{N_{0,v}}$$

to be surjective to apply Lemma 5.6.3 *loc.cit.*. This is a consequence, by taking inductive limit over r, of the isomorphisms (provided r is large enough so that $U^vI_{r,v}$ is neat)

$$S^D(U^v,\mathcal{O}_E)^{I_{r,v}}_{\mathfrak{m}_{\overline{\rho}}}/\varpi_L S^D(U^v,\mathcal{O}_E)^{I_{r,v}}_{\mathfrak{m}_{\overline{\rho}}}\to S^D(U^v,k_E)^{I_{r,v}}_{\mathfrak{m}_{\overline{\rho}}}.$$

Here, we denote by $I_{r,v}$ the open subgroup of $\mathrm{GL}_2(\mathcal{O}_{F_v})$ defined as $\{g\equiv\left(\begin{smallmatrix}1&*\\0&1\end{smallmatrix}\right)\mod \varpi_v^r\}$. □

4.2. **Application.** Assuming the existence of the decomposition as in Conjecture 1, we get the following information about the local factor π_v (for $v|p$) when $\overline{\rho}|_{G_{F_v}}$ is reducible with scalar endomorphisms.

Theorem 4.2. *Assume the decomposition (4.1) holds. If for some $v|p$, $\overline{\rho}_v$ is reducible indecomposable and isomorphic to $\left(\begin{smallmatrix}\psi_1&*\\0&\psi_2\end{smallmatrix}\right)$, with $\psi_1\neq\psi_2$, then $\mathrm{Ord}_P(\pi_{D,v}(\overline{\rho}))$ is an admissible semi-simple T_v-representation.*

Proof. The natural inclusion $S^D(U^v,k_E)[\mathfrak{m}_{\overline{\rho}}]\hookrightarrow S^D(U^v,k_E)_{\mathfrak{m}_{\overline{\rho}}}$ induces an inclusion $\mathrm{Ord}_P(S^D(U^v,k_E)[\mathfrak{m}_{\overline{\rho}}])\hookrightarrow\mathrm{Ord}_P(S^D(U^v,k_E)_{\mathfrak{m}_{\overline{\rho}}})$, hence Lemma 2.10 and Theorem 4.1 imply that the action of G_{F_v} on the cokernel of

$$\big(\mathrm{Ord}_P(S^D(U^v,k_E)[\mathfrak{m}_{\overline{\rho}}])\big)^{\mathrm{ab},S_v}\hookrightarrow\mathrm{Ord}_P(S^D(U^v,k_E)[\mathfrak{m}_{\overline{\rho}}])$$

factors through $G_{F_v}^{\mathrm{ab}}$.

Assume that the decomposition (4.1) holds. Then, as a representation of $G_{F_v}\times\mathrm{GL}_2(F_v)$, $S^D(U^v,k_E)[\mathfrak{m}_{\overline{\rho}}]$ is isomorphic to $(\overline{\rho}_v\otimes\pi_v)^{\oplus r}$ for some integer $r\geq 1$. Apply Lemma 2.10 again, we deduce that the action of G_{F_v} on the cokernel of

$$\mathrm{Ord}_P(\overline{\rho}_v\otimes\pi_v)^{\mathrm{ab},S_v}\hookrightarrow\mathrm{Ord}_P(\overline{\rho}_v\otimes\pi_v)$$

factors through $G_{F_v}^{\mathrm{ab}}$.

Now assume that $\mathrm{Ord}_P(\pi_v)$ is not semi-simple. Because there is no non-trivial extension between two non-isomorphic k_E-characters of T_v, there must exist some character $\chi:T_v\to k_E^\times$ and some non-trivial extension as (2.1)

$$0\to\chi\to\epsilon_\delta\to\chi\to 0$$

such that ϵ_δ appears as a *subquotient* of $\mathrm{Ord}_P(\pi_v)$. This implies that $\mathrm{Ord}_P(\overline{\rho}_v\otimes\pi_v)$, which is equal to $\overline{\rho}_v\otimes\mathrm{Ord}_P(\pi_v)$, has a $G_{F_v}\times T_v$-equivariant, hence $G_{F_v}\times S_v$-equivariant, subquotient of the form $\overline{\rho}_v\otimes\epsilon_\delta$. Applying Lemma 2.10 to it shows that the action of G_{F_v} on the cokernel of

$$(\overline{\rho}_v\otimes\epsilon_\delta)^{\mathrm{ab},S_v}\hookrightarrow\overline{\rho}_v\otimes\epsilon_\delta \tag{4.2}$$

factors through $G_{F_v}^{\mathrm{ab}}$. We claim that $(\overline{\rho}_v\otimes\epsilon_\delta)^{\mathrm{ab},S_v}$ is at most 1-dimensional over k_E, while $\overline{\rho}_v\otimes\epsilon_\delta$ is 4-dimensional, and the cokernel of (4.2) admits a quotient isomorphic to $\overline{\rho}_v\otimes\chi$. Because $\overline{\rho}_v$ is reducible and indecomposable by assumption, the action of G_{F_v} on $\overline{\rho}_v\otimes\chi$ does not factor through $G_{F_v}^{\mathrm{ab}}$, hence a contradiction which shows that $\mathrm{Ord}_P(\pi_v)$ is semi-simple.

To verify the claim, we choose a basis $\{v_1,v_2\}$ of $\overline{\rho}_v$ over k_E such that $g\cdot v_1=\psi_1(g)v_1$ for $g\in G_{F_v}$; also choose a basis $\{w_1,w_2\}$ of ϵ_δ such that $s\cdot w_1=\chi(s)w_1$ and $s\cdot w_2=\chi(s)(w_2+\delta(s)w_1)$ for $s\in T_v$. It is clear that $V^{\mathrm{ab}}\subset\psi_1\otimes\epsilon_\delta$, as $\overline{\rho}_v$ is indecomposable and $\psi_1\neq\psi_2$. Since the action of S_v on $\overline{\rho}_v\otimes\epsilon_\delta$ is via the anti-diagonal embedding $S_v\hookrightarrow G_{F_v}^{\mathrm{ab}}\times S_v$, $s\mapsto(\iota(s),s^{-1})$, we get

$$s\cdot(v_1\otimes w_1)=\psi_1(\iota(s))v_1\otimes\chi(s^{-1})w_1$$

$$s\cdot(v_1\otimes w_2)=\psi_1(\iota(s))v_1\otimes[\chi(s^{-1})(w_2+\delta(s^{-1})w_1)].$$

Because $\delta: F_v^\times\cong S_v\to k_E$ is non-trivial and the extension ϵ_δ admits a central character, there exists $s\in S_v$ such that $\delta(s^{-1})=-\delta(s)\neq 0$, hence

$$(\overline{\rho}_v\otimes\epsilon_\delta)^{\mathrm{ab},S_v}\subseteq k_E(v_1\otimes w_1).$$

The claim follows easily from this. □

Combined with results proved in §3, we get the following corollary.

Corollary 4.3. *Keep assumptions in Theorem 4.2. Assume moreover that $\overline{\rho}_v$ is generic and F_v is unramified over $\mathbb{Q}_p$ of degree ≥ 2. Then the $\mathrm{GL}_2(F_v)$-representation π_v contains a sub-representation π which is of length 2 and fits into a non-split exact sequence*

$$0\to\pi_0\to\pi\to\pi_1\to 0$$

with π_0 a principal series and π_1 supersingular. If moreover $[F_v:\mathbb{Q}_p]=2$, and if π_2 denotes the f_v-th smooth dual of π_0, then $\mathrm{Ext}^1_{\mathrm{GL}_2(F_v)}(\pi_2,\pi_1)\neq 0$.

Proof. Under our assumptions together with the genericity condition of $\overline{\rho}_v$, we can apply [4, Thm. 3.7.1] to get that $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_{F_v})}(\pi_v)$ is of multiplicity 1. Then Theorem 4.2 implies that $\mathrm{Ord}_{P_v}(\pi_v)$ is one-dimensional, so that the result follows from Propositions 3.2 and 3.3 and Remark 6. □

We can get rid of the assumption (4.1) in Corollary 4.3 as follows (note however that then π_v does not make sense).

Proof of Corollary 1.2. In [4], Breuil and Diamond explicitly constructed a certain k_E-representation π'_v of $\mathrm{GL}_2(F_v)$, and showed that π'_v must be the local factor π_v if (4.1) holds (see [4, Cor. 3.7.4]). We have a $G_{F_v}\times\mathrm{GL}_2(F_v)$-equivariant embedding $\overline{\rho}_v\otimes\pi'_v\hookrightarrow S^D(U^v,k_E)[\mathfrak{m}_{\overline{\rho}}]$, so that $\mathrm{Ord}_P(\pi'_v)$ is semi-simple by the same proof of Theorem 4.2. Moreover, under genericity assumption on $\overline{\rho}_v$, [4, Thm. 3.7.1] says that the K_v-socle of π'_v is of multiplicity one, hence $\mathrm{Ord}_P(\pi'_v)$ is of dimension 1. We deduce that π'_v contains a non-split extension of π_1 by π_0. The last assertion of Corollary 1.2 is deduced as in Corollary 4.3. □

5. Appendix

In this section, we prove, contrast to Proposition 3.3, that there is no non-trivial extension of principal series by supersingular representations, when L is a local field of characteristic p. Notations are the same as in Section 2.

We fix a uniformizer ϖ of L. For an irreducible smooth representation σ of K, we view it as a representation of KZ by letting $\left(\begin{smallmatrix}\varpi & 0\\ 0 & \varpi\end{smallmatrix}\right)$ act trivially. Consider the compact induction $\text{c-Ind}_{KZ}^G\sigma$ and recall that the $\overline{\mathbb{F}}_p$-algebra $\mathrm{End}_G(\text{c-Ind}_{KZ}^G\sigma)$ is isomorphic to $\overline{\mathbb{F}}_p[T]$ where T is a certain Hecke operator (normalized as in [1]). To describe the action of T, for $g\in G$ and $v\in\sigma$, denote by $[g,v]\in\text{c-Ind}_{KZ}^G\sigma$ the function supported on KZg^{-1} and such that $[g,v](g^{-1})=v$. Let $v_0\in\sigma^{I_1}$ be a non-zero vector and recall that Id denotes the identity matrix of $\mathrm{GL}_2(L)$, then the action of T on $\text{c-Ind}_{KZ}^G\sigma$ is characterized by the formula (and by the G-equivariance of T):

$$T([\mathrm{Id},v_0])=\sum_{\lambda\in\mathbb{F}_q}\left[\begin{pmatrix}\varpi & [\lambda]\\ 0 & 1\end{pmatrix},v_0\right]+\epsilon(\sigma)[\Pi,v_0] \tag{5.1}$$

where $\epsilon(\sigma) = 1$ if σ is of dimension 1 and $\epsilon(\sigma) = 0$ otherwise.

In [15, Def. 2.7] is defined an operator by the formula:

$$S = \sum_{\lambda \in \mathbb{F}_q} \begin{pmatrix} \varpi & [\lambda] \\ 0 & 1 \end{pmatrix} \in \overline{\mathbb{F}}_p[G]$$

which acts on any G-representation. By (5.1) we get immediately

$$T([\mathrm{Id}, v_0]) = S([\mathrm{Id}, v_0]) \tag{5.2}$$

when σ is of dimension ≥ 2. Set $S^1 = S$ and by induction $S^n = S \circ S^{n-1}$ for $n \geq 1$. We then have $S^{n+m} = S^n \circ S^m = S^m \circ S^n$. Recall some useful properties of S.

Lemma 5.1. *Let π be a smooth representation of G and let $v \in \pi$.*

(i) If v is an eigenvector of $\mathcal{H}$, then so is Sv for the same eigencharacter.

(ii) If v is fixed by N_0, then so is Sv.

(iii) If v is fixed by N_0, then so is $h \cdot v$ for $h \in T_1$, and we have the formula $h \cdot Sv = S(h \cdot v)$ for all $h \in T_1$.

(iv) If v is fixed by $\left(\begin{smallmatrix} 1+\mathfrak{p}_L & \mathcal{O}_L \\ \mathfrak{p}_L^{n+1} & 1+\mathfrak{p}_L \end{smallmatrix}\right)$ with $n \geq 1$, then Sv is fixed by $\left(\begin{smallmatrix} 1+\mathfrak{p}_L & \mathcal{O}_L \\ \mathfrak{p}_L^{n} & 1+\mathfrak{p}_L \end{smallmatrix}\right)$. In particular, if v is fixed by I_1, then so is Sv.

(v) Assume that L is of characteristic p and π is a supersingular representation of G. Then there exists an integer $n \gg 0$ such that $S^n v = 0$.

Proof. (i), (ii) and (iii) follow from an easy calculation, see [15, Lem. 2.8] for the details. (iv) is proved in [15, §4.2, (4.4)]. (v) is just [15, Thm. 5.1], which requires L to be of characteristic p. $\square$

Theorem 5.2. *Assume that L is of characteristic p. Let π_1 be a supersingular representation and π_2 be a principal series or a special series of G. Then $\mathrm{Ext}^1_G(\pi_2, \pi_1) = 0$.*

Proof. Consider an extension of G-representations

$$0 \to \pi_1 \to V \to \pi_2 \to 0.$$

Because π_2 is a principal series or a special series, it always contains a sub-K-representation of dimension ≥ 2, say σ, and π_2 is a quotient of $\text{c-Ind}_{KZ}^G \sigma/(T-\lambda) \otimes \chi \circ \det$ for some $\lambda \in \overline{\mathbb{F}}_p^\times$ and $\chi : L^\times \to \overline{\mathbb{F}}_p^\times$, see [1, Thm. 33]. Let $\bar{w} \in \sigma^{I_1} \subset \pi_2$ be a non-zero vector, which is unique up to a scalar and is automatically an eigenvector of $\mathcal{H}$. Choose a lifting $w \in V$ of $\bar{w}$ arbitrarily. Since the order of $\mathcal{H}$ is prime to p, we may choose w so that it is an eigenvector of $\mathcal{H}$. Denote by $M = \langle N_0.w \rangle \subset V$ the sub-N_0-representation generated by w and choose vectors $v_i \in \pi_1$, where i runs over a finite set, such that $\{w, v_i\}$ forms a basis of M. Then by [15, Lem. 5.2], for any $n > 0$, $\langle N_0.S^n w \rangle$ is spanned by the vectors $\{S^n w, S^n v_i\}$. But, since M is finite dimensional and π_1 is supersingular, Lemma 5.1(v) implies that $S^n v_i = 0$ for $n \gg 0$, hence $S^n w$ is fixed by N_0 for such n. Since $S\bar{w} \overset{(5.2)}{=} T\bar{w} = \lambda \bar{w}$ in π_2 and $\lambda \neq 0$, $\frac{1}{\lambda^n} S^n w$ is still a lifting of $\bar{w}$, so we may assume the chosen lifting w is fixed by N_0. Since $\bar{w}$ is fixed by I_1, we have (in particular) $(h-1)\bar{w} = 0$ for any $h \in T_1$, hence $(h-1)w \in \pi_1$. By Lemma 5.1(v), there exists $n_h \gg 0$ such that $S^{n_h}(h-1)w = 0$. The representation V being smooth, $\langle T_1.w \rangle$ is finite dimensional, so we may find n large enough so that $S^n(h-1)w = 0$ for all $h \in T_1$. But, Lemma 5.1(iii) implies

$$0 = S^n(h-1)w = (h-1)S^n w,$$

so that $S^n w$ is fixed by $I_1 \cap P$. Again, by Lemma 5.1(iv), up to enlarge n, $S^n w$ is fixed by I_1. Replacing w by $\frac{1}{\lambda^n} S^n w$, we obtain a lifting w of $\bar{w}$ which is fixed by I_1.

Next, consider the vector $(S-\lambda)w$ which belongs to π_1 as its image in π_2 is zero. By Lemma 5.1(v) again, there exists $n \gg 0$ such that $S^n(S-\lambda)w = 0$. Replacing w by $\frac{1}{\lambda^n} S^n w$, we get a lifting w of $\bar{w}$ satisfying $Sw = \lambda w$.

Summarizing, we obtain a lifting $w \in V^{I_1}$ of $\bar{w}$ which is an eigenvector of $\mathcal{H}$ and satisfies $Sw = \lambda w$. By the proof of [18, Lem. 4.1]

$$\tau := \langle K \cdot w \rangle \subset V$$

is irreducible and isomorphic to σ. Moreover, the fact $Sw = \lambda w$ implies that the G-morphism $\text{c-Ind}_{KZ}^G \tau \otimes \chi \circ \det \to V$ (here χ is the character appeared at the beginning of the proof) induced by Frobenius reciprocity must factor through (note that $Sw = Tw$ by (5.2))

$$\phi : \text{c-Ind}_{KZ}^G \tau/(T-\lambda) \otimes \chi \circ \det \to V.$$

This shows that V contains π_1 as a sub-representation and hence splits. $\square$

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