MULTIPLICITIES OF COHOMOLOGICAL AUTOMORPHIC FORMS ON GL_2 AND MOD p REPRESENTATIONS OF $\mathrm{GL}_2(\mathbb{Q}_p)$

YONGQUAN HU

Abstract. We prove a new upper bound for the dimension of the space of cohomological automorphic forms of fixed level and growing parallel weight on GL_2 over a number field which is not totally real, improving the one obtained in [19]. The main tool of the proof is the mod p representation theory of $\mathrm{GL}_2(\mathbb{Q}_p)$ as started by Barthel-Livné and Breuil, and developed by Paškūnas.

1. Introduction

Let F be a finite extension of $\mathbb{Q}$ of degree r, and r_1 (resp. $2r_2$) be the number of real (resp. complex) embeddings. Let $F_\infty = F\otimes_{\mathbb{Q}}\mathbb{R}$, so that $\mathrm{GL}_2(F_\infty)=\mathrm{GL}_2(\mathbb{R})^{r_1}\times \mathrm{GL}_2(\mathbb{C})^{r_2}$. Let Z_∞ be the centre of $\mathrm{GL}_2(F_\infty)$, K_f be a compact open subgroup of $\mathrm{GL}_2(\mathbb{A}_f)$ and let

$$X=\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A})/K_fZ_\infty.$$

If $\mathbf{d}=(d_1,...,d_{r_1+r_2})$ is an (r_1+r_2)-tuple of positive even integers, we let $S_{\mathbf{d}}(K_f)$ denote the space of cusp forms on X which are of cohomological type with weight $\mathbf{d}$.

In this paper, we are interested in understanding the asymptotic behavior of the dimension of $S_{\mathbf{d}}(K_f)$ as $\mathbf{d}$ varies and K_f fixed. Define

$$\Delta(\mathbf{d})=\prod_{i\leq r_1} d_i\times\prod_{i>r_1} d_i^2.$$

When F is totally real, Shimizu [27] proved that

$$\dim_{\mathbb{C}} S_{\mathbf{d}}(K_f)\sim C\cdot\Delta(\mathbf{d})$$

for some constant C independent of $\mathbf{d}$. However, if F is not totally real, the actual growth rate of $\dim_{\mathbb{C}} S_{\mathbf{d}}(K_f)$ is still a mystery; see the discussion below when F is quadratic imaginary.

The main result of this paper is the following (see Theorem 6.1 for a slightly general statement).

Theorem 1.1. *If F is not totally real and $\mathbf{d}=(d,...,d)$ is a parallel weight, then for any fixed K_f, we have*

$$\dim_{\mathbb{C}} S_{\mathbf{d}}(K_f)\ll_\epsilon d^{r-1/2+\epsilon}.$$

To compare our result with the previous ones, let us restrict to the case when F is imaginary quadratic. In [13], Finis, Grunewald and Tirao has proven the bounds

$$d\ll \dim_{\mathbb{C}} S_{\mathbf{d}}(K_f)\ll \frac{d^2}{\ln d},\quad \mathbf{d}=(d,d)$$

Morningside Center of Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; University of the Chinese Academy of Sciences, Beijing 100049, China.
Email: yhu@amss.ac.cn
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using base change and the trace formula respectively (the lower bound is conditional on K_f). In [19], Marshall has improved the upper bound to be

$$\dim S_{\mathbf{d}}(K_f) \ll_\epsilon d^{5/3+\epsilon} \tag{1.1}$$

while our Theorem 1.1 gives

$$\dim S_{\mathbf{d}}(K_f) \ll_\epsilon d^{3/2+\epsilon},$$

hence a saving by a power $d^{1/6}$. It worths to point out that such a power saving is quite rare for tempered automorphic forms. Indeed, purely analytic methods, such as the trace formula, only allow to strengthen the trivial bound by a power of log, cf. [13]. We refer to the introduction of [19] for discussion on this point and a collection of known results.

Finally, let us mention that the experimental data of [13] (when F is quadratic imaginary) suggests that the actual growth rate of $\dim_{\mathbb{C}} S_{\mathbf{d}}(K_f)$ is probably d. We hope to return to this problem in future work.

Let us first explain Marshall's proof of the bound (1.1). It consists of two main steps, the first of which is to convert the problem to bounding the dimension of certain group cohomology of Emerton's completed cohomology spaces H^j (in mod p coefficients) and the second one is to establish this bound. For the first step, he used the Eichler-Shimura isomorphism, Shapiro's lemma and a fundamental spectral sequence due to Emerton. For the second, he actually proved a bound in a more general setting which applies typically to H^j. To make this precise, let us mention a key intermediate result in this step (stated in the simplest version). Let

$$K_1 = \begin{pmatrix} 1+p\mathbb{Z}_p & p\mathbb{Z}_p \\ p\mathbb{Z}_p & 1+p\mathbb{Z}_p \end{pmatrix}, \quad T_1(p^n) = \begin{pmatrix} 1+p\mathbb{Z}_p & p^n\mathbb{Z}_p \\ p^n\mathbb{Z}_p & 1+p\mathbb{Z}_p \end{pmatrix}$$

and $Z_1 \cong 1+p\mathbb{Z}_p$ be the center of K_1. Also let $\mathbb{F}$ be a sufficiently large finite extension of $\mathbb{F}_p$. By a careful and involved analysis of the structure of finitely generated torsion modules over the Iwasawa algebra $\Lambda := \mathbb{F}[[K_1/Z_1]]$, Marshall proved the following ([19, Prop. 5]): if Π is a smooth admissible $\mathbb{F}$-representation of K_1/Z_1 which is cotorsion[1], then for any $i \geq 0$,

$$\dim_{\mathbb{F}} H^i(T_1(p^n)/Z_1, \Pi) \ll p^{4n/3}. \tag{1.2}$$

Our proof of Theorem 1.1 follows closely the above strategy. Indeed, the first step is identical to Marshall's. Our main innovation is in the second step by improving the bound (1.2). The key observation is that Emerton's completed cohomology is not just an admissible representation of K_1, but also carries naturally a compatible action of $\mathrm{GL}_2(\mathbb{Q}_p)$, which largely narrows the possible shape of H^j. Indeed, this is already observed in [19] and used *once*[2] when deriving (1.1) from (1.2). However, the mod p representation theory of $\mathrm{GL}_2(\mathbb{Q}_p)$ developed by Barthel-Livné [2], Breuil [4] and Paškūnas [24, 25], allows us to make the most of this fact and prove the following result.

Theorem 1.2. *Let Π be a smooth admissible $\mathbb{F}$-representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ with a central character. Assume that Π is admissible and cotorsion. Then for any $i \geq 0$,*

$$\dim_{\mathbb{F}} H^i(T_1(p^n)/Z_1, \Pi) \ll np^n.$$

We obtain the bound by using numerous results of the mod p representation theory of $\mathrm{GL}_2(\mathbb{Q}_p)$. First, the classification theorems of [2] and [4] allow us to control the dimension of invariants for irreducible π, in which case we prove

$$\dim_{\mathbb{F}} H^i(T_1(p^n)/Z_1, \pi) \ll n. \tag{1.3}$$

In fact, to do this we also need more refined structure theorems due to Morra [21, 22]. Second, the theory of Paškūnas [24] allows us to pass to general admissible cotorsion representations. To explain this, let us assume moreover that all the Jordan-Hölder factors of Π are isomorphic

[1] that is, the Pontryagin dual $\Pi^\vee := \mathrm{Hom}_{\mathbb{F}}(\Pi, \mathbb{F})$ is torsion as an $\mathbb{F}[[K_1]]$-module

[2] we mean the trick of 'change of groups', see §5.3

to a given supersingular irreducible representation π. Paškūnas [24] studied the universal deformation of $\pi^\vee$ and showed that the universal deformation space is three dimensional. We show that the admissibility and cotorsion condition imposed on Π forces that $\Pi^\vee$ is a deformation of $\pi^\vee$ over a one-dimensional space. Knowing this, we deduce easily Theorem 1.2 from (1.3).

We point out that to prove Theorem 1.2 for $i \geq 1$ and to generalize it to a finite product of $\mathrm{GL}_2(\mathbb{Q}_p)$, we need to solve several complications caused by the additional requirement of carrying an action of $\mathrm{GL}_2(\mathbb{Q}_p)$. In doing so, we prove some results which might be of independent interest. We explain these in more detail below.

The first complication comes when we try to prove Theorem 1.2 for higher cohomology degrees. To apply the standard dimension-shifting argument, we need also consider admissible representations Π which are not necessarily cotorsion, that is, the Pontryagin dual $\Pi^\vee$ has a positive rank over Λ. Using the bound in the torsion case, one is reduced to consider torsion-free $\Pi^\vee$. The usual argument (as in [19, §3.2]) uses the existence of morphisms $\Lambda^s \to \Pi^\vee$ and $\Pi^\vee \to \Lambda^s$ with torsion cokernels, where s is the Λ-rank of $\Pi^\vee$. However, these are only morphisms of Λ-modules, so the bound for torsion modules does not apply to these cokernels. To solve this issue, we prove that under certain conditions a torsion-free Λ-module which carries a compatible action of $\mathrm{GL}_2(\mathbb{Q}_p)$ is actually free. The proof of this fact uses crucially a result of Kohlhaase [18].

To explain the second, we recall the following interesting result of Breuil-Paškūnas [6]: if Π is a smooth admissible $\mathbb{F}$-representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ with a central character, then there exists a $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant embedding

$$\Pi \hookrightarrow \Omega,$$

where $\Omega|_{\mathrm{GL}_2(\mathbb{Z}_p)}$ is an injective envelope of $\Pi|_{\mathrm{GL}_2(\mathbb{Z}_p)}$ in the category of smooth $\mathbb{F}$-representations of $\mathrm{GL}_2(\mathbb{Z}_p)$ with the (fixed) central character. Although this construction works for the group $\mathrm{GL}_2(F)$ for any local field F, it does not generalize (at least not obviously) to a finite product, say $\mathcal{G} = \mathrm{GL}_2(\mathbb{Q}_p) \times \cdots \times \mathrm{GL}_2(\mathbb{Q}_p)$. This causes an obstacle in generalizing Theorem 1.2 to $\mathcal{G}$. To overcome this we prove, using the theory of Serre weights, a weaker replacement of the construction of Breuil-Paškūnas. Roughly, it says that we may always embed Π into some Ω which, although *not* necessarily an injective envelope of $\Pi|_{\mathrm{GL}_2(\mathbb{Z}_p)}$, is an injective object. This statement generalizes to $\mathcal{G}$.

Notation. Throughout the paper, we fix a prime p and a finite extension $\mathbb{F}$ over $\mathbb{F}_p$ taken to be sufficiently large.

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2. Non-commutative Iwasawa algebras

Let G be a p-adic analytic group of dimension d and G_0 be an open compact subgroup of G. We assume G_0 is uniform and pro-p. Let

$$\Lambda = \mathbb{F}[[G_0]] = \varprojlim_{N \lhd G_0} \mathbb{F}[G_0/N]$$

be the *Iwasawa algebra* of G_0 over $\mathbb{F}$. A finitely generated Λ-module is said to have *codimension* c if $\mathrm{Ext}^i_\Lambda(M, \Lambda) = 0$ for all $i < c$ and is non-zero for $i = c$; the codimension of the zero module is defined to be ∞. We denote the codimension by $j_\Lambda(M)$. If M is non-zero, then $j_\Lambda(M) \leq d$. For our purpose, we set

$$\delta_\Lambda(M) = d - j_\Lambda(M)$$

and call it the *canonical dimension* of M. It is easy to see that if $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of finitely generated Λ-modules, then

$$\delta_\Lambda(M) = \max\{\delta_\Lambda(M'), \delta_\Lambda(M'')\}. \tag{2.1}$$

If M is a finitely generated Λ-module, we have the notion of *Gelfand-Kirillov* dimension of M, defined to be the growth rate of the function $\dim_{\mathbb{F}} M/J^n M$, where J denotes the maximal ideal of Λ. We have the following important fact ([1, §5.4]).

Theorem 2.1. *For all finitely generated Λ-modules M, the canonical dimension and the Gelfand-Kirillov dimension of M coincide.*

For $n \geq 0$, define inductively $G_{n+1} := \overline{G_n^p[G_n, G_0]}$ which are normal subgroups of G_0; the decreasing chain $G_0 \supseteq G_1 \supseteq \cdots$ is called the *lower p-series* of G_0, see [1, §2.4]. We have $|G_n : G_{n+1}| = p^d$. With this notation, the utility of the above theorem is the following result (see [9, Thm. 2.3]).

Corollary 2.2. *Let M be a finitely generated Λ-module of codimension c. Then*

$$\dim_{\mathbb{F}} H_0(G_n, M) = \lambda(M) \cdot p^{(d-c)n} + O(p^{(d-c-1)n}) \tag{2.2}$$

for some rational number $\lambda(M) > 0$.

Since the Artin-Rees property holds for the J-adic filtration of Λ (see [14, Lem. A.32]), by a standard argument we see that if $0 \to M_1 \to M \to M_2 \to 0$ is an exact sequence of finitely generated Λ-modules of codimension c, then $\lambda(M) = \lambda(M_1) + \lambda(M_2)$.

Proposition 2.3. *Let M be a finitely generated Λ-module and $\phi : M \to M$ be an endomorphism. Assume that $\bigcap_{n\geq 1} \phi^n(M) = 0$.*[3] *Then one the following holds:*

(i) *ϕ is nilpotent and $\delta_\Lambda(M) = \delta_\Lambda(M/\phi(M))$;*
(ii) *ϕ is not nilpotent and for $k_0 \gg 1$,*

$$\delta_\Lambda(M) = \max\left\{\delta_\Lambda(M/\phi(M)), \delta_\Lambda(\phi^{k_0}(M)/\phi^{k_0+1}(M)) + 1\right\}. \tag{2.3}$$

In any case, $\delta_\Lambda(M) \leq \delta_\Lambda(M/\phi(M)) + 1$.

Proof. We assume first that ϕ is nilpotent, say $\phi^{k_0} = 0$ for some $k_0 \geq 1$. Then M admits a finite filtration by $\phi^k(M)$ (for $k \leq k_0$). Since each of the graded pieces is a quotient of $M/\phi(M)$, the assertion follows from (2.1).

Now assume that ϕ is not nilpotent, so by Lemma 2.4 below ϕ induces an injection $\phi^{k_0}(M) \to \phi^{k_0}(M)$ for some $k_0 \gg 1$ and the RHS of (2.3) does not depend on the choice of k_0. The above argument shows that

$$\delta_\Lambda(M/\phi^{k_0}(M)) = \delta_\Lambda(M/\phi(M)).$$

Hence, by (2.1) applied to the short exact sequence $0 \to \phi^{k_0}(M) \to M \to M/\phi^{k_0}(M) \to 0$, we need to show

$$\delta_\Lambda(\phi^{k_0}(M)) = \delta_\Lambda(\phi^{k_0}(M)/\phi^{k_0+1}(M)) + 1.$$

That is, by replacing M by $\phi^{k_0}(M)$, we may assume ϕ is injective and need to show $\delta_\Lambda(M) = \delta_\Lambda(M/\phi(M)) + 1$. Indeed, this follows from [14, Lem. A.15]. □

Lemma 2.4. *Let M be a finitely generated Λ-module. Let $\phi \in \mathrm{End}_\Lambda(M)$ be such that $\bigcap_{n\geq 1} \phi^n(M) = 0$. Then one of the following holds:*

(i) *ϕ is nilpotent;*

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[3] It would be more natural to impose the condition $\phi(M) \subset JM$. We consider the present one for the following reasons. On the one hand, in practice we do need consider ϕ such that $\bigcap_{n\geq 1} \phi^n(M) = 0$ but $\phi(M) \not\subseteq JM$. On the other hand, since M is finitely generated, the condition $\bigcap_{n\geq 1} \phi^n(M) = 0$ implies $\phi^n(M) \subset JM$ for $n \gg 1$, see the proof of Lemma 4.15.

(ii) *ϕ is not nilpotent and for $k_0 \gg 0$, ϕ induces an injection $\phi^{k_0}(M) \to \phi^{k_0}(M)$.*

Proof. For any $k \geq 1$, ϕ induces a surjective morphism

$$M/\phi(M) \twoheadrightarrow \phi^k(M)/\phi^{k+1}(M).$$

Since Λ is noetherian and M is finitely generated, any ascending chain of submodules of $M/\phi(M)$ is stable, so there exists $k_0 \gg 0$ such that

$$\phi^{k_0}(M)/\phi^{k_0+1}(M) = \phi^k(M)/\phi^{k+1}(M), \quad \forall k \gg k_0.$$

For this k_0, $\phi : \phi^{k_0}(M) \to \phi^{k_0}(M)$ is injective. If $\phi^{k_0}(M)/\phi^{k_0+1}(M) = 0$, then $\phi^{k_0}(M) = 0$ by Nakayama's lemma, that is, ϕ is nilpotent. □

Recall that the projective dimension, denoted by $\mathrm{pd}_\Lambda(M)$, is defined to be the length of a minimal projective resolution of M. It is proved in [28, Cor. 6.3] that $\mathrm{pd}_\Lambda(M)$ is equal to $\max\{i : \mathrm{Ext}^i_\Lambda(M, \Lambda) \neq 0\}$. We always have $\mathrm{pd}_\Lambda(M) \geq j_\Lambda(M)$ and say M is *Cohen-Macaulay* if $\mathrm{pd}_\Lambda(M) = j_\Lambda(M)$.

Lemma 2.5. *Let M be a finitely generated Λ-module. Let $\phi \in \mathrm{End}_\Lambda(M)$ be such that $\bigcap_{n\geq 1} \phi^n(M) = 0$. Assume ϕ is injective. Then M is Cohen-Macaulay if and only if $M/\phi(M)$ is Cohen-Macaulay.*

Proof. Since ϕ is injective, Proposition 2.3 implies that $j_\Lambda(M) = j_\Lambda(M/\phi(M)) - 1$. On the other hand, we also have $\mathrm{pd}_\Lambda(M) = \mathrm{pd}_\Lambda(M/\phi(M)) - 1$. □

2.1. **Torsion vs torsion free.** Assume now G_0 is a uniform and pro-p. Then Λ is a noetherian integral domain. Let $\mathcal{L}$ be the field of fractions of Λ. If M is a finitely generated Λ-module, then $M \otimes_\Lambda \mathcal{L}$ is a finite dimensional $\mathcal{L}$-vector space, and we define the rank of M to be the dimension of this vector space. We see that rank is additive in short exact sequences and that M has rank 0 if and only if M is torsion.

Let $\mathcal{O} = W(\mathbb{F})$ be the ring of Witt vectors with coefficients in $\mathbb{F}$. Similar to $\Lambda = \mathbb{F}[[G_0]]$, we may form the Iwasawa algebras

$$\widetilde{\Lambda} := \mathcal{O}[[G_0]] = \varprojlim_{N \lhd G_0} \mathcal{O}[G_0/N], \quad \widetilde{\Lambda}_{\mathbb{Q}_p} = \widetilde{\Lambda} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

They are both integral domains. Let $\mathcal{L}_{\mathbb{Q}_p}$ be the field of fractions of $\widetilde{\Lambda}_{\mathbb{Q}_p}$. If M is a finitely generated module over $\widetilde{\Lambda}_{\mathbb{Q}_p}$, we define its rank as above and the analogous facts hold.

Recall the following simple fact, see [10, Lem. 1.17].

Lemma 2.6. *Let M be a finite generated $\widetilde{\Lambda}$-module which is furthermore p-torsion free, then $M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is torsion as a $\widetilde{\Lambda}_{\mathbb{Q}_p}$-module if and only if $M \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ is torsion as an Λ-module.*

3. Mod p representations of $\mathrm{GL}_2(\mathbb{Q}_p)$

Notation. Let p be a prime[4] ≥ 5, $G = \mathrm{GL}_2(\mathbb{Q}_p)$, $K = \mathrm{GL}_2(\mathbb{Z}_p)$, Z be the center of G, T be the diagonal torus, and $B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ the upper Borel subgroup.

Let $\mathrm{Rep}_{\mathbb{F}}(G)$ denote the category of smooth $\mathbb{F}$-representations of G with a central character. Let $\mathrm{Rep}^{\mathrm{l,fin}}_{\mathbb{F}}(G)$ denote the subcategory of $\mathrm{Rep}_{\mathbb{F}}(G)$ consisting of locally finite objects. Here an object $\Pi \in \mathrm{Rep}_{\mathbb{F}}(G)$ is said to be *locally finite* if for all $v \in \Pi$ the $\mathbb{F}[G]$-submodule generated by v is of finite length.

[4]It is not always necessary, but for convenience we make this assumption throughout the paper.

If $\Pi, \Pi' \in \mathrm{Rep}_{\mathbb{F}}^{\mathrm{l,fin}}(G)$, we simply write $\mathrm{Ext}^i_G(\Pi, \Pi')$ for the extension groups computed in $\mathrm{Rep}_{\mathbb{F}}^{\mathrm{l,fin}}(G)$. In particular, the extension classes are required to carry a central character. In particular, $\mathrm{Ext}^i_G(\Pi, \Pi') = 0$ if Π, Π' have distinct central characters.

Let $\mathrm{Mod}_{\mathbb{F}}^{\mathrm{pro}}(G)$ be the category of compact $\mathbb{F}[[K]]$-modules with an action of $\mathbb{F}[G]$ such that the two actions coincide when restricted to $\mathbb{F}[K]$. It is anti-equivalent to $\mathrm{Rep}_{\mathbb{F}}(G)$ under Pontryagin dual $\Pi \mapsto \Pi^\vee := \mathrm{Hom}_{\mathbb{F}}(\Pi, \mathbb{F})$. Let $\mathfrak{C} = \mathfrak{C}(G)$ be the full subcategory of $\mathrm{Mod}_{\mathbb{F}}^{\mathrm{pro}}(G)$ anti-equivalent to $\mathrm{Rep}_{\mathbb{F}}^{\mathrm{l,fin}}(G)$.

An object $M \in \mathfrak{C}$ is called *coadmissible* if $M^\vee$ is admissible in the usual sense. This is equivalent to requiring M to be finitely generated over $\mathbb{F}[[K]]$ (or equivalently, finitely generated over $\mathbb{F}[[H]]$ for any open compact subgroup $H \subset K$).

If H is a closed subgroup of K, we denote by $\mathrm{Rep}_{\mathbb{F}}(H)$ the category of smooth $\mathbb{F}$-representations of H such that $H \cap Z$ acts by a character. Let $\mathfrak{C}(H)$ be the dual category of $\mathrm{Rep}_{\mathbb{F}}(H)$.

For $n \geq 1$, let $K_n = \begin{pmatrix} 1+p^n\mathbb{Z}_p & p^n\mathbb{Z}_p \\ p^n\mathbb{Z}_p & 1+p^n\mathbb{Z}_p \end{pmatrix}$. Also let $Z_1 := K_1 \cap Z$. Since Z_1 is pro-p, any smooth character $\chi : Z \to \mathbb{F}^\times$ is trivial on Z_1, so any $\mathbb{F}$-representation of G (resp. K) with a central character can be viewed as a representation of G/Z_1 (resp. K/Z_1). Set

$$\Lambda := \mathbb{F}[[K_1/Z_1]].$$

Since K_1/Z_1 is uniform (as $p > 2$) and pro-p, the results in §2 apply to Λ. Note that $\dim(K_1/Z_1) = 3$. To simplify the notation, we write $j(\cdot) = j_\Lambda(\cdot)$, $\delta(\cdot) = \delta_\Lambda(\cdot)$ and $\mathrm{pd}(\cdot) = \mathrm{pd}_\Lambda(\cdot)$.

If H is a closed subgroup of G and σ is a smooth representation of H, we denote by $\mathrm{Ind}_H^G \sigma$ the usual smooth induction. When H is moreover open, we let $\text{c-Ind}_H^G\sigma$ denote the compact induction, meaning the subspace of $\mathrm{Ind}_H^G \sigma$ consisting of functions whose support is compact modulo H.

Let $\omega : \mathbb{Q}_p^\times \to \mathbb{F}^\times$ be the mod p cyclotomic character. If H is any group, we write $\mathbf{1}_H$ for the trivial representation of H (over $\mathbb{F}$).

3.1. **Irreducible representations.** The work of Barthel-Livné [2] shows that absolutely irreducible objects in $\mathrm{Rep}_{\mathbb{F}}(G)$ fall into four classes:

(1) one dimensional representations $\chi \circ \det$, where $\chi : \mathbb{Q}_p^\times \to \mathbb{F}^\times$ is a smooth character;
(2) (irreducible) principal series $\mathrm{Ind}_B^G \chi_1 \otimes \chi_2$ with $\chi_1 \neq \chi_2$;
(3) special series, i.e. twists of the Steinberg representation $\mathrm{Sp} := (\mathrm{Ind}_B^G \mathbf{1}_T)/\mathbf{1}_G$;
(4) supersingular representations, i.e. irreducible representations which are not isomorphic to sub-quotients of any parabolic induction.

For $0 \leq r \leq p-1$, let $\mathrm{Sym}^r\mathbb{F}^2$ denote the standard symmetric power representation of $\mathrm{GL}_2(\mathbb{F}_p)$. Up to twist by $\det^m$ with $0 \leq m \leq p-1$, any absolutely irreducible $\mathbb{F}$-representation of $\mathrm{GL}_2(\mathbb{F}_p)$ is isomorphic to $\mathrm{Sym}^r\mathbb{F}^2$. Inflating to K and letting $\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$ act trivially, we may view $\mathrm{Sym}^r\mathbb{F}^2$ as a representation of KZ. Let $I(\mathrm{Sym}^r\mathbb{F}^2) := \text{c-Ind}_{KZ}^G \mathrm{Sym}^r\mathbb{F}^2$ denote the compact induction to G. It is well-known that $\mathrm{End}_G(I(\mathrm{Sym}^r\mathbb{F}^2))$ is isomorphic to $\mathbb{F}[T]$ for a certain Hecke operator T ([2]). For $\lambda \in \mathbb{F}$ we define

$$\pi(r, \lambda) := I(\mathrm{Sym}^r\mathbb{F}^2)/(T - \lambda).$$

If $\chi : \mathbb{Q}_p^\times \to \mathbb{F}^\times$ is a smooth character, then let $\pi(r, \lambda, \chi) := \pi(r, \lambda) \otimes \chi \circ \det$. In [2], Barthel and Livné showed that any supersingular representation of G is a *quotient* of $\pi(r, 0, \chi)$ for suitable (r, χ). Later on, Breuil [4] proved that $\pi(r, 0, \chi)$ is itself irreducible, hence completes the classification of irreducible objects in $\mathrm{Rep}_{\mathbb{F}}(G)$. We will refer to (r, λ, χ) as above as a *parameter triple*.

Recall the link between non-supersingular representations and compact inductions: if $\lambda \neq 0$ and $(r,\lambda) \neq (0,\pm 1)$, then

$$\pi(r,\lambda) \cong \mathrm{Ind}_B^G \, \mu_{\lambda^{-1}} \otimes \mu_\lambda \omega^r,$$

where $\mu_x : \mathbb{Q}_p^\times \to \mathbb{F}^\times$ denotes the unramified character sending p to x. If $(r,\lambda) \in \{(0,\pm 1),(p-1,\pm 1)\}$, we have non-split exact sequences:

$$0 \to \mathrm{Sp} \otimes \mu_{\pm 1} \circ \det \to \pi(0,\pm 1) \to \mu_{\pm 1} \circ \det \to 0,$$

$$0 \to \mu_{\pm 1} \circ \det \to \pi(p-1,\pm 1) \to \mathrm{Sp} \otimes \mu_{\pm 1} \circ \det \to 0.$$

It is clear for non-supersingular representations and follows from [4] for supersingular representations that any absolutely irreducible $\pi \in \mathrm{Rep}_{\mathbb{F}}(G)$ is admissible. Therefore $\pi^\vee$ is coadmissible and it makes sense to talk about $\delta(\pi^\vee)$.

Theorem 3.1. *Let $\Pi \in \mathrm{Rep}_{\mathbb{F}}(G)$.*

(i) If Π is of finite length, then Π is admissible and $\delta(\Pi^\vee) \leq 1$.

(ii) Conversely, if Π is admissible and $\delta(\Pi^\vee) \leq 1$, then Π is of finite length.

Proof. (i) The first assertion is clear. For the second, we may assume Π is absolutely irreducible. Corollary 2.2 allows us to translate the problem to computing the growth of $\dim_{\mathbb{F}} \Pi^{K_n}$. If Π is non-supersingular, then it is easy, see [22, Prop. 5.3] for a proof. If Π is supersingular, this is first done in [23, Thm. 1.2] and later in [22, Cor. 4.15] (of course, both proofs are based on [4]).

(ii) If $\delta(\Pi^\vee) = 0$, then (up to enlarge $\mathbb{F}$) all the irreducible subquotients of Π are one-dimensional. Since $p \geq 5$ by assumption, if $\chi, \chi' : \mathbb{Q}_p^\times \to \mathbb{F}^\times$ are two smooth characters (distinct or not), we have $\mathrm{Ext}^1_G(\chi \circ \det, \chi' \circ \det) = 0$ by [23, Thm. 11.4]. Therefore Π is a direct sum of one-dimensional representations. But Π is admissible by assumption, so it is of finite length.

Assume $\delta(\Pi^\vee) = 1$. Consider the G-socle filtration with graded pieces $\mathrm{soc}_i\Pi$ $(i \geq 1)$ given by $\mathrm{soc}_1\Pi = \mathrm{soc}_G\Pi$, $\mathrm{soc}_2(\Pi) = \mathrm{soc}_G(\Pi/\mathrm{soc}_1\Pi)$, etc. Since Π is admissible, $\mathrm{soc}_i\Pi$ is non-zero and of finite length. Since there is no non-trivial extension between two characters, for any two successive pieces $\mathrm{soc}_i\Pi$, $\mathrm{soc}_{i+1}\Pi$, (at least) one of them contains an infinite dimensional irreducible representation of G. On the other hand, using Corollary 2.2 and the additivity of $\lambda(\cdot)$ with respect to short exact sequences, we deduce that the number of irreducible subquotients of $\Pi^\vee$ which have Gelfand-Kirillov dimension 1 is finite.[5] Putting these together, we see that the socle filtration of Π is finite, hence Π has finite length. □

Recall the following result of Kohlhaase.

Theorem 3.2. *Let $\pi \in \mathrm{Rep}_{\mathbb{F}}(G)$ be absolutely irreducible. Then $\pi^\vee$ is Cohen-Macaulay of codimension 2 (resp. codimension 3) if π is infinite dimensional (resp. one-dimensional).*

Proof. This is proved in [18, §5]. Precisely, see Prop. 5.4 when π is an (irreducible) principal series representation, Prop. 5.7 when π is special series, Thm. 5.13 when π is supersingular. The case when π is one-dimensional is trivial. □

Recall that a *block* in $\mathrm{Rep}_{\mathbb{F}}(G)$ is an equivalence class of irreducible objects in $\mathrm{Rep}_{\mathbb{F}}(G)$, where $\tau \sim \pi$ if and only if there exists a series of irreducible representations $\tau = \tau_0, \tau_1, \ldots, \tau_n = \pi$ such that $\mathrm{Ext}^1_G(\tau_i, \tau_{i+1}) \neq 0$ or $\mathrm{Ext}^1_G(\tau_{i+1}, \tau_i) \neq 0$ for each i.

[5]Strictly speaking, we also need to know that $\lambda(\cdot)$ is uniformly bounded below for any infinite dimensional irreducible representation. This can be seen by the result of Morra recalled in (i), or by the general theory of Hilbert polynomials.

Proposition 3.3. *The category* $\mathrm{Rep}_{\mathbb{F}}^{\mathrm{l,fin}}(G)$ *decomposes into a direct product of subcategories*

$$\mathrm{Rep}_{\mathbb{F}}^{\mathrm{l,fin}}(G) = \prod_{\mathfrak{B}} \mathrm{Rep}_{\mathbb{F}}^{\mathrm{l,fin}}(G)^{\mathfrak{B}}$$

where the product is taken over all the blocks $\mathfrak{B}$ *and the objects of* $\mathrm{Rep}_{\mathbb{F}}^{\mathrm{l,fin}}(G)^{\mathfrak{B}}$ *are representations with all the irreducible subquotients lying in* $\mathfrak{B}$*. Correspondingly, we have a decomposition of categories* $\mathfrak{C} = \prod_{\mathfrak{B}} \mathfrak{C}^{\mathfrak{B}}$*, where* $\mathfrak{C}^{\mathfrak{B}}$ *denotes the dual category of* $\mathrm{Rep}_{\mathbb{F}}(G)^{\mathfrak{B}}$*.*

Proof. See [24, Prop. 5.34]. □

The following theorem describes the blocks (when $p \geq 5$ as we are assuming).

Theorem 3.4. *Let* $\pi \in \mathrm{Rep}_{\mathbb{F}}(G)$ *be absolutely irreducible and let* $\mathfrak{B}$ *be the block in which* π *lies. Then one of the following holds:*

(I) *if* π *is supersingular, then* $\mathfrak{B} = \{\pi\}$*;*
(II) *if* $\pi \cong \mathrm{Ind}_B^G \chi_1 \otimes \chi_2\omega^{-1}$ *with* $\chi_1\chi_2^{-1} \neq \mathbf{1}, \omega^{\pm 1}$*, then*
$$\mathfrak{B} = \left\{ \mathrm{Ind}_B^G \chi_1 \otimes \chi_2\omega^{-1},\ \mathrm{Ind}_B^G \chi_2 \otimes \chi_1\omega^{-1} \right\};$$
(III) *if* $\pi = \mathrm{Ind}_B^G \chi \otimes \chi\omega^{-1}$*, then* $\mathfrak{B} = \{\pi\}$*;*
(IV) *otherwise,* $\mathfrak{B} = \{\chi \circ \det, \mathrm{Sp} \otimes \chi \circ \det, \mathrm{Ind}_B^G \alpha \otimes \chi \circ \det\}$*, where* $\alpha = \omega \otimes \omega^{-1}$*.*

Proof. See [24, Prop. 5.42]. □

Convention: By [24, Lem. 5.10], any smooth irreducible $\overline{\mathbb{F}}_p$-representation of G with a central character is defined over a finite extension of $\mathbb{F}_p$. Theorem 3.4 then implies that for a given block $\mathfrak{B}$, there is a common field $\mathbb{F}$ such that irreducible objects in $\mathfrak{B}$ are absolutely irreducible. Hereafter, given a finite set of blocks, we take $\mathbb{F}$ to be sufficiently large such that irreducible objects in these blocks are absolutely irreducible.

3.2. **Projective envelopes.** Fix $\pi \in \mathrm{Rep}_{\mathbb{F}}(G)$ irreducible and let $\mathfrak{B}$ be the block in which π lies. Let $\mathrm{Inj}_G\pi$ be an injective envelope of π in $\mathrm{Rep}_{\mathbb{F}}^{\mathrm{l,fin}}(G)$; the existence is guaranteed by [24, Cor. 2.3]. Let $P = P_{\pi^\vee} := (\mathrm{Inj}_G\pi)^\vee \in \mathfrak{C}$ and $E = E_{\pi^\vee} := \mathrm{End}_{\mathfrak{C}}(P)$. Then P is a projective envelope of $\pi^\vee$ in $\mathfrak{C}$ and is naturally a left E-module. Since P is indecomposable, Proposition 3.3 implies that (the dual of) every irreducible subquotient of P lies in $\mathfrak{B}$. Also, E is a local $\mathbb{F}$-algebra (with residue field $\mathbb{F}$). Paškūnas has computed E and showed in particular that E is commutative, except when $\mathfrak{B}$ is of type (III) listed in Theorem 3.4; in any case, we denote by $R = Z(E)$ the center of E. Hence $E = R$ except for blocks of type (III).

Theorem 3.5. *(Paškūnas) Keep the above notation.*

(i) R *is naturally isomorphic to the Bernstein center of* $\mathfrak{C}^{\mathfrak{B}}$*. In particular,* R *acts on any object in* $\mathfrak{C}^{\mathfrak{B}}$ *and any morphism in* $\mathfrak{C}^{\mathfrak{B}}$ *is* R*-equivariant.*

(ii) We have the following facts:

(I) *If* $\mathfrak{B}$ *is of type (I), then* E *is commutative isomorphic to* $\mathbb{F}[[x,y,z]]$ *and* P *is a flat* E*-module.*
(II) *If* $\mathfrak{B}$ *is of type (II), then* E *is commutative isomorphic to* $\mathbb{F}[[x,y,z]]$ *and* P *is a flat* E*-module.*
(III) *If* $\mathfrak{B}$ *is of type (III), then* E *is non-commutative and its center* R *is isomorphic to* $\mathbb{F}[[x,y,z]]$*.* E *is a free* R*-module of rank 4 and carries an involution* $*$ *such that* $R = \{a \in E :\ a^* = a\}$*. Moreover,* P *is a flat* E*-module, hence also flat over* R*.*
(IV) *If* $\mathfrak{B}$ *is of type (IV), then* E *is commutative isomorphic to* $\mathbb{F}[[x,y,z,w]]/(xw - yz)$*.*

In particular, R *is a Cohen-Macaulay complete local noetherian* $\mathbb{F}$*-algebra of Krull dimension 3.*

Proof. (i) This is [24, Thm. 1.5].

(ii) These are proved in [24]. Precisely, see Prop. 6.3 for type (I), Cor. 8.7 for type (II), §9 for type (III) and Cor. 10.78, Lem. 10.93 for type (IV). The flatness of P over E (for blocks of type (I)-(III)) follows from Cor. 3.12. □

However, if $\mathfrak{B}$ is of type (IV), P is not flat over E. This causes quite a bit of complication in the proof of our main result. To solve this, we determine in §3.7 all the Tor-groups $\mathrm{Tor}_i^E(\mathbb{F}, P)$. We state a result which will be used there.

Lemma 3.6. *For $i \geq 1$, we have*

$$\mathrm{Hom}_{\mathfrak{C}}(P, \mathrm{Tor}_i^E(\mathbb{F}, P)) = 0.$$

Proof. Choose a resolution of $\mathbb{F}$ by finite free E-modules: $F_\bullet \to \mathbb{F} \to 0$. Then the homology of $F_\bullet \otimes_E P$ computes $\mathrm{Tor}_i^E(\mathbb{F}, P)$. It is clear that

$$\mathrm{Hom}_{\mathfrak{C}}(P, F_\bullet \otimes_E P) \cong F_\bullet.$$

Since $\mathrm{Hom}_{\mathfrak{C}}(P, -)$ is exact, this implies

$$\mathrm{Hom}_{\mathfrak{C}}(P, H_i(F_\bullet \otimes_E P)) \cong H_i(F_\bullet)$$

as required. □

Proposition 3.7. *(i) $\mathbb{F} \otimes_E P$ (resp. $\mathbb{F} \otimes_R P$) has finite length in $\mathfrak{C}$.*

(ii) If $\pi \notin \{\mathrm{Sp}, \pi_\alpha\} \otimes \chi \circ \det$ for any $\chi : \mathbb{Q}_p^\times \to \mathbb{F}^\times$,[6] then $\mathbb{F} \otimes_E P$ (resp. $\mathbb{F} \otimes_R P$) is Cohen-Macaulay.

Proof. (i) By definition, $\mathbb{F} \otimes_E P$ is characterized as the maximal quotient of P which contains $\pi^\vee$ with multiplicity one. This object is denoted by Q in [24, §3] and can be described explicitly. If $\mathfrak{B}$ is of type (I) or (III), Q is just $\pi^\vee$. If $\mathfrak{B}$ is of type (II), it has finite length by [25, Prop. 6.1]. If $\mathfrak{B}$ is of type (IV), it follows from Proposition 3.30 below in §3.7 where the explicit structure of $\mathbb{F} \otimes_E P$ is determined.

To see that $\mathbb{F} \otimes_R P$ has finite length, we may assume $\mathfrak{B}$ is of type (III). Then E is a free R-module of rank 4, so that $\mathbb{F} \otimes_R P \cong (\mathbb{F} \otimes_R E) \otimes_E P) \cong (\mathbb{F} \otimes_E P)^{\oplus 4}$.

(ii) the result follows from the explicit description of $\mathbb{F} \otimes_E P$, using Theorem 3.2 and Proposition 3.34 in the case $\pi = \mathbf{1}_G$. □

3.3. **Serre weights.** We keep the notation in the previous subsection. Let $\pi \in \mathrm{Rep}_{\mathbb{F}}(G)$ be irreducible. By a *Serre weight* of π we mean an isomorphism class of (absolutely) irreducible $\mathbb{F}$-representations of K, say σ, such that $\mathrm{Hom}_K(\sigma, \pi) \neq 0$. Denote by $\mathcal{D}(\pi)$ the set of Serre weights of π. The description of $\mathcal{D}(\pi)$ can be deduced from [2] and [4]; see [25, Rem. 6.2] for a summary.

Lemma 3.8. *If $\pi \neq \pi'$ are two objects in a block $\mathfrak{B}$, then $\mathcal{D}(\pi) \cap \mathcal{D}(\pi') = \emptyset$.*

Proof. The statement is trivial if $\mathfrak{B}$ is of type (I) or (III). For type (II) or type (IV), it is a direct check (using the assumption $p \geq 5$), see [25, Rem. 6.2]. □

Let (r, λ, χ) be a parameter triple. For any $n \geq 1$, set

$$\pi_n(r, \lambda, \chi) := I(\mathrm{Sym}^r \mathbb{F}^2)/(T - \lambda)^n \otimes \chi \circ \det,$$

so that $\pi_1(r, \lambda, \chi) = \pi(r, \lambda, \chi)$. Because $\mathbb{F}[T]$ acts freely on $I(\mathrm{Sym}^r \mathbb{F}^2)$ by [2, Thm. 19], for $m \leq n$ we have an exact sequence

$$0 \to \pi_m(r, \lambda, \chi) \xrightarrow{(T-\lambda)^{n-m}} \pi_n(r, \lambda, \chi) \to \pi_{n-m}(r, \lambda, \chi) \to 0.$$

[6] hereafter, we will often express this condition as $\pi \notin \{\mathrm{Sp}, \pi_\alpha\}$ up to twist

Put

$$\pi_\infty(r,\lambda,\chi) := \varinjlim_{n\geq 1} \pi_n(r,\lambda,\chi).$$

Then $\pi_\infty(r,\lambda,\chi)$ is a locally finite smooth $\mathbb{F}$-representation of G and we have an exact sequence

$$0 \to \pi(r,\lambda,\chi) \overset{T-\lambda}{\to} \pi_\infty(r,\lambda,\chi) \to \pi_\infty(r,\lambda,\chi) \to 0. \tag{3.1}$$

Proposition 3.9. *Assume $\lambda \neq 0$. The following statements hold.*

(i) We have $\mathrm{soc}_G\pi_\infty(r,\lambda,\chi) = \mathrm{soc}_G\,\pi(r,\lambda,\chi)$. In particular, there exists a G-equivariant embedding $\theta : \pi_\infty(r,\lambda,\chi) \hookrightarrow \mathrm{Inj}_G\pi(r,\lambda,\chi)$.

(ii) The morphism θ identifies $\pi_\infty(r,\lambda,\chi)$ with the largest G-stable subspace of $\mathrm{Inj}_G\pi(r,\lambda,\chi)$ which is generated by its I_1-invariants, where $I_1 = \left(\begin{smallmatrix} 1+p\mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & 1+p\mathbb{Z}_p\end{smallmatrix}\right)$ is the pro-p Iwahori subgroup. In particular, the image of θ does not depend on the choice of θ.

(iii) For any irreducible $\sigma \in \mathrm{Rep}_{\mathbb{F}}(K)$, θ induces an isomorphism

$$\mathrm{Hom}_K(\sigma, \pi_\infty(r,\lambda,\chi)) \cong \mathrm{Hom}_K(\sigma, \mathrm{Inj}_G\pi(r,\lambda,\chi)).$$

Moreover, they are non-zero if and only if $\mathrm{Hom}_K(\sigma,\pi(r,\lambda,\chi)) \neq 0$.

(iv) If $\mathrm{Hom}_K(\sigma,\pi(r,\lambda,\chi)) \neq 0$, then $\mathrm{Hom}_K(P,\sigma^\vee)^\vee$ is a cyclic E-module isomorphic to $\mathbb{F}[[S]]$ (where S denotes $T-\lambda$), with the annihilator independent of σ.

Proof. If $(r,\lambda) \neq (p-1,\pm 1)$, this is proved in [16, §2]: Lem. 2.1 for (i), Prop. 2.3 for (ii), Cor. 2.5 for (iii), Prop. 2.9 for (iv).

If $(r,\lambda) = (p-1,\pm 1)$, the statements are still true and the proof can be adapted from the case of $(r,\lambda) = (0,\pm 1)$. □

Corollary 3.10. *Let $\pi \in \mathrm{Rep}_{\mathbb{F}}(G)$ be irreducible and $P = P_{\pi^\vee}$. If $\pi \notin \{\mathbf{1}_G, \mathrm{Sp}\}$ up to twist, then $\mathrm{Hom}_K(P,\sigma^\vee) \neq 0$ if and only if $\sigma \in \mathcal{D}(\pi)$. If $\pi \in \{\mathbf{1}_G, \mathrm{Sp}\}$, then $\mathrm{Hom}_K(P,\sigma^\vee) \neq 0$ if and only if $\sigma \in \{\mathrm{Sym}^0\mathbb{F}^2, \mathrm{Sym}^{p-1}\mathbb{F}^2\}$.*

Proof. The result is clear if $\mathfrak{B}$ is of type (I), and follows from Proposition 3.9 otherwise. □

Corollary 3.11. *Let $\pi \in \mathrm{Rep}_{\mathbb{F}}(G)$ be irreducible and $P = P_{\pi^\vee}$.*

(i) Let $\sigma \in \mathrm{Rep}_{\mathbb{F}}(K)$ be irreducible. Whenever non-zero, $\mathrm{Hom}_K(P,\sigma^\vee)^\vee$ is a cyclic E-module. If J_σ denotes the annihilator, there exists $x \notin J_\sigma$ such that

$$\mathrm{Hom}_K(P,\sigma^\vee)^\vee \cong E/J_\sigma \cong \mathbb{F}[[x]]. \tag{3.2}$$

(ii) Let $\widetilde{\sigma} = \oplus_\sigma \sigma$ where the sum is taken over all σ such that $\mathrm{Hom}_K(P,\sigma^\vee) \neq 0$. Then $\mathrm{Hom}_K(P,\widetilde{\sigma}^\vee)^\vee$ is a Cohen-Macaulay E-module of Krull dimension 1.

Proof. (i) If $\mathfrak{B}$ is not of type (I), the result is a reformulation of Proposition 3.9(iv). If $\mathfrak{B}$ is of type (I), it is proved in [25, Thm. 6.6, (38)].

(ii) Remark that although E is non-commutative when $\mathfrak{B}$ is of type (III), $E/J_{\widetilde{\sigma}}$ is commutative by Proposition 3.9, where $J_{\widetilde{\sigma}}$ denotes the annihilator of $\mathrm{Hom}_K(P,\widetilde{\sigma}^\vee)^\vee$. So it makes sense to talk about the Cohen-Macaulayness. That being said, if $\mathfrak{B}$ is not of type (I), the result follows from Proposition 3.9(iv). If $\mathfrak{B}$ is of type (I), it is a special case of [25, Lem. 2.33] via [25, Thm. 5.2]. □

We record a result in the context of commutative algebra which will be used in Section 5.

Lemma 3.12. *Let $\sigma \in \mathrm{Rep}_{\mathbb{F}}(K)$ be irreducible such that $\mathrm{Hom}_K(P,\sigma^\vee)$ is non-zero. View $\mathrm{Hom}_K(P,\sigma^\vee)^\vee$ as an R-module and let J'_σ be the annihilator. There exist $g,h \in J'_\sigma$ such that $J'_\sigma/(g,h)$ has finite length.*

Proof. First note that $R = E$ and $J'_\sigma = J_\sigma$ except when $\mathfrak{B}$ is of type (III).

If $\mathfrak{B}$ is of type (I) or (II), R is isomorphic to a power series ring over $\mathbb{F}$ in three variables, so we may even choose g, h such that $J_\sigma = (g, h)$. If $\mathfrak{B}$ is of type (III), $R = Z(E)$ is isomorphic to a power series ring in three variables and Proposition 3.28 proved in §3.6 below implies that the image of $R \to \mathbb{F}[[x]]$ is $\mathbb{F}[[x^2]]$ (with a suitable choice of x), the result is also clear. If $\mathfrak{B}$ is of type (IV), then R is isomorphic to $\mathbb{F}[[x, y, z, w]]/(xw - yz)$ and it is proved in [16, Lem. 3.9] that $J_\sigma = (y, z, w)$ with a suitable choice of variables. It suffices to take $g = y - z$ and $h = w$. □

The following general result is extracted from [15, Thm. 3.5].

Proposition 3.13. *Let $\widetilde{P} \in \mathfrak{C}$ and $f \in \mathrm{End}_{\mathfrak{C}}(\widetilde{P})$. Assume*

(a) *$\widetilde{P}$ is projective in $\mathfrak{C}(K)$;*
(b) *for any irreducible $\sigma \in \mathrm{Rep}_{\mathbb{F}}(K)$, the induced morphism*

$$f_* : \mathrm{Hom}_K(\widetilde{P}, \sigma^\vee)^\vee \to \mathrm{Hom}_K(\widetilde{P}, \sigma^\vee)^\vee$$

is injective.

Then f is injective and $\widetilde{P}/f\widetilde{P}$ is projective in $\mathfrak{C}(K)$.

Proof. Consider the exact sequence $\widetilde{P} \xrightarrow{f} \widetilde{P} \to \widetilde{P}/f\widetilde{P} \to 0$. Let $\sigma \in \mathrm{Rep}_{\mathbb{F}}(K)$ be irreducible. Applying $\mathrm{Hom}_K(-, \sigma^\vee)^\vee$ we obtain

$$\mathrm{Hom}_K(\widetilde{P}, \sigma^\vee)^\vee \xrightarrow{f_*} \mathrm{Hom}_K(\widetilde{P}, \sigma^\vee)^\vee \to \mathrm{Hom}_K(\widetilde{P}/f\widetilde{P}, \sigma^\vee)^\vee \to 0.$$

Denote by $\mathrm{Im}(f)$ the image of $f : \widetilde{P} \to \widetilde{P}$. Then f_* factors as

$$\mathrm{Hom}_K(\widetilde{P}, \sigma^\vee)^\vee \overset{\alpha}{\twoheadrightarrow} \mathrm{Hom}_K(\mathrm{Im}(f), \sigma^\vee)^\vee \xrightarrow{\beta} \mathrm{Hom}_K(\widetilde{P}, \sigma^\vee)^\vee, \tag{3.3}$$

with α surjective. Since f_* is injective by assumption, β is also injective and α is an isomorphism.

Since $\widetilde{P}$ is projective in $\mathfrak{C}$, it remains projective in $\mathfrak{C}(K)$ by [12]. Applying $\mathrm{Hom}_K(-, \sigma^\vee)^\vee$ to $0 \to \mathrm{Im}(f) \to \widetilde{P} \to \widetilde{P}/f\widetilde{P} \to 0$, we get an exact sequence of E-modules:

$$0 \to \mathrm{Ext}^1_K(\widetilde{P}/f\widetilde{P}, \sigma^\vee)^\vee \to \mathrm{Hom}_K(\mathrm{Im}(f), \sigma^\vee)^\vee \xrightarrow{\beta} \mathrm{Hom}_K(\widetilde{P}, \sigma^\vee)^\vee \to \mathrm{Hom}_K(\widetilde{P}/f\widetilde{P}, \sigma^\vee)^\vee \to 0.$$

The injectivity of β implies $\mathrm{Ext}^1_K(\widetilde{P}/f\widetilde{P}, \sigma^\vee)^\vee = 0$. This being true for every irreducible $\sigma \in \mathrm{Rep}_{\mathbb{F}}(K)$, we deduce that $\widetilde{P}/f\widetilde{P}$ is projective in $\mathfrak{C}(K)$.

As a consequence, $\mathrm{Im}(f)$ is also projective in $\mathfrak{C}(K)$. To check $f : \widetilde{P} \to \widetilde{P}$ is injective, let N be the kernel. For any irreducible $\sigma \in \mathrm{Rep}_{\mathbb{F}}(K)$, the exact sequence $0 \to N \to \widetilde{P} \to \mathrm{Im}(f) \to 0$ induces

$$0 \to \mathrm{Hom}_K(N, \sigma^\vee)^\vee \to \mathrm{Hom}_K(\widetilde{P}, \sigma^\vee)^\vee \xrightarrow{\alpha} \mathrm{Hom}_K(\mathrm{Im}(f), \sigma^\vee)^\vee \to 0.$$

Since α is an isomorphism, we obtain $\mathrm{Hom}_K(N, \sigma^\vee)^\vee = 0$. This being true for any σ, we finally obtain $N = 0$, so $f : \widetilde{P} \to \widetilde{P}$ is injective. □

The following result complements [25, Thm. 5.2].[7]

Corollary 3.14. *If $\pi \in \{\mathbf{1}_G, \mathrm{Sp}\}$, there exists $f \in E$ such that $f : P \to P$ is injective and P/fP isomorphic to a projective envelope of $\mathrm{Sym}^0\mathbb{F}^2 \oplus \mathrm{Sym}^{p-1}\mathbb{F}^2$ in $\mathfrak{C}(K)$.*

[7] Although our result is stated for mod p coefficients, the p-adic case can be deduced from this by the proof of [25, Prop. 5.1].

Proof. Write $\widetilde{\sigma} = \mathrm{Sym}^0\mathbb{F}^2 \oplus \mathrm{Sym}^{p-1}\mathbb{F}^2$. By Proposition 3.9, $\mathrm{Hom}_K(P, \widetilde{\sigma}^\vee)^\vee$ is isomorphic to $\mathbb{F}[[S]]$ as E-modules. Let $f \in E$ be any lifting of S, then we obtain

$$\mathrm{Hom}_K(P/fP, \widetilde{\sigma}^\vee)^\vee \cong \mathrm{Hom}_K(P/fP, (\mathrm{Sym}^0\mathbb{F}^2)^\vee)^\vee \oplus \mathrm{Hom}_K(P/fP, (\mathrm{Sym}^{p-1}\mathbb{F}^2)^\vee)^\vee \cong \mathbb{F}^2.$$

In particular, P/fP is coadmissible and we conclude by Proposition 3.13. □

3.4. **Principal series and deformations.** Recall that T denotes the diagonal torus of G. If $\eta : T \to \mathbb{F}^\times$ is a smooth character, set $\pi_\eta = \mathrm{Ind}_B^G\,\eta$ (possibly reducible). Let $\mathrm{Inj}_T\eta$ be an injective envelope of η in $\mathrm{Rep}_{\mathbb{F}}(T)$ and set $\Pi_\eta = \mathrm{Ind}_B^G\,\mathrm{Inj}_T\eta$. Then Π_η is a locally finite smooth representation of G. It is easy to see that $\mathrm{soc}_G\Pi_\eta = \mathrm{soc}_G\pi_\eta$, which we denote by π. So there is a G-equivariant embedding $\Pi_\eta \hookrightarrow \mathrm{Inj}_G\pi$ and by [24, Prop. 7.1] the image does not depend on the choice of the embedding.

Let (r, λ, χ) be a parameter triple such that $\pi_\eta \cong \pi(r, \lambda, \chi)$. This is always possible by [2, Thm. 30] and we have $(r, \lambda) \neq (0, \pm 1)$.

Proposition 3.15. *We have $\pi_\infty(r, \lambda, \chi) \subset \Pi_\eta$, both identified with subspaces of $\mathrm{Inj}_G\pi$.*

Proof. This follows from [2, 3]. Recall that $\mathbb{F}[T]$ denotes the Hecke algebra associated to $I(\mathrm{Sym}^r\mathbb{F}^2)$. In [2, §6.1] is constructed an $\mathbb{F}[T, T^{-1}]$-linear morphism

$$P : I(\mathrm{Sym}^r\mathbb{F}^2) \otimes_{\mathbb{F}[T]} \mathbb{F}[T, T^{-1}] \to \mathrm{Ind}_B^G\, X_1 \otimes X_2$$

where $X_i : \mathbb{Q}_p^\times \to (\mathbb{F}[T, T^{-1}])^\times$ are tamely ramified characters given by

$$X_1 \text{ unramified}, \quad X_1(p) = T^{-1}, \quad X_1X_2 = \omega^r.$$

By [2, Thm. 25], P is an isomorphism except for $r = 0$, in which case P is injective and we have an exact sequence ([3, Thm. 20])

$$0 \to I(\mathrm{Sym}^0\mathbb{F}^2) \otimes_{\mathbb{F}[T]} (\mathbb{F}[T, T^{-1}]) \xrightarrow{P} \mathrm{Ind}_B^G\, X_1 \otimes X_2 \to \mathrm{Sp} \otimes \mathbb{F}[T, T^{-1}]/(T^{-2} - 1) \to 0. \tag{3.4}$$

Since $(r, \lambda) \neq (0, \pm 1)$, specializing (3.4) to $(T - \lambda)^n$ identifies $\pi_n(r, \lambda, \chi)$ with a subrepresentation of Π_η. Taking limit gives the result. □

Corollary 3.16. *Π_η is not admissible.*

Proof. Since $\pi_\infty(r, \lambda, \chi)$ is not admissible by Proposition 3.9(iv), the result is a consequence of Proposition 3.15. □

Let $M_{\eta^\vee} = (\Pi_\eta)^\vee \in \mathfrak{C}$ and $E_{\eta^\vee} = \mathrm{End}_{\mathfrak{C}}(M_{\eta^\vee})$.

Lemma 3.17. *$E_{\eta^\vee}$ is isomorphic to $\mathbb{F}[[x, y]]$ and $M_{\eta^\vee}$ is flat over $E_{\eta^\vee}$.*

Proof. By [24, Prop. 7.1], we have a natural isomorphism $E_{\eta^\vee} \cong \mathrm{End}_{\mathfrak{C}(T)}((\mathrm{Inj}_T\eta)^\vee)$ and the latter ring is isomorphic to $\mathbb{F}[[x, y]]$ by [24, Cor. 7.2].

By [24, §3.2], $(\mathrm{Inj}_T\eta)^\vee$ is isomorphic to the universal deformation of the T-representation $\eta^\vee$ (with fixed central character), with $E_{\eta^\vee}$ being the universal deformation ring. In particular, it is flat over $E_{\eta^\vee}$. The result follows from this and the definition of $M_{\eta^\vee}$. □

Let $P = P_{\pi^\vee} = (\mathrm{Inj}_G\pi)^\vee$ and $E = E_{\pi^\vee}$.

Lemma 3.18. *The natural quotient morphisms $P \twoheadrightarrow M_{\eta^\vee} \twoheadrightarrow \pi_\infty(r, \lambda, \chi)^\vee$ induce surjective ring morphisms*

$$E \twoheadrightarrow E_{\eta^\vee} \twoheadrightarrow \mathrm{End}_{\mathfrak{C}}(\pi_\infty(r, \lambda, \chi)^\vee) \cong \mathbb{F}[[S]].$$

Proof. For the first surjection, see [24, Prop. 7.1]. Since $\mathrm{End}_G(\pi(r, \lambda, \chi)) = \mathbb{F}$, the dual version of (3.1) implies $\mathrm{End}_{\mathfrak{C}}(\pi_\infty(r, \lambda, \chi)^\vee) \cong \mathbb{F}[[S]]$. By Proposition 3.9(i)-(ii), the quotient map $\theta^\vee : P \to \pi_\infty(r, \lambda, \chi)^\vee$ induces a surjection $E \twoheadrightarrow \mathbb{F}[[S]]$. □

Proposition 3.19. *Let $M \in \mathfrak{C}$ be a coadmissible quotient of $M_{\eta^\vee}$. Then $\delta(M) \leq 2$.*

Proof. Since M is coadmissible while $M_{\eta^\vee}$ is not by Corollary 3.16, the kernel of $M_{\eta^\vee} \twoheadrightarrow M$ is non-zero; denote it by N. We claim that $\mathrm{Hom}_{\mathfrak{C}}(M_{\eta^\vee}, N) \neq 0$. For this it suffices to prove $\mathrm{Hom}_{\mathfrak{C}}(P, N) \neq 0$, because any morphism $P \to N$ must factor through $P \twoheadrightarrow M_{\eta^\vee} \to N$, see [24, Prop. 7.1(iii)]. Assume $\mathrm{Hom}_{\mathfrak{C}}(P, N) = 0$ for a contradiction. Then $\pi^\vee$ (recall $\pi := \mathrm{soc}_G \pi_\eta$) does not occur in N. This is impossible unless π_η is reducible, i.e. $\pi_\eta \cong \pi(p-1, 1)$ up to twist. Assuming this, we have $\pi = \mathbf{1}_G$ and all irreducible subquotients of N are isomorphic to $\mathrm{Sp}^\vee$. In particular, we obtain $\mathrm{Hom}_K(N, (\mathrm{Sym}^0\mathbb{F}^2)^\vee) = 0$. However, this would imply an isomorphism

$$\mathrm{Hom}_K(M_{\eta^\vee}, (\mathrm{Sym}^0\mathbb{F}^2)^\vee)^\vee \cong \mathrm{Hom}_K(M, (\mathrm{Sym}^0\mathbb{F}^2)^\vee)^\vee$$

which contradicts the coadmissibility of M.

The claim implies the existence of $f \in E_{\eta^\vee}$ which annihilates M. Since $E_{\eta^\vee} \cong \mathbb{F}[[x, y]]$ is a Cohen-Macaulay integral domain of Krull dimension 2, we may find $g \in E_{\eta^\vee}$ such that f, g is a system of parameters of $E_{\eta^\vee}$. Since $M_{\eta^\vee}/(f, g)$ is of finite length, so is $M/(f, g)M = M/gM$. Theorem 3.1 implies that $\delta(M/gM) \leq 1$ and we conclude by Proposition 2.3. □

3.5. **Coadmissible quotients.** Keep the notation in the previous subsection. Let $M \in \mathfrak{C}$ be a coadmissible quotient of $P = P_{\pi^\vee}$. We set $\mathrm{m}(M) := \mathrm{Hom}_{\mathfrak{C}}(P, M)$ which is a finitely generated E-module. There is a natural morphism

$$\mathrm{ev} : \mathrm{m}(M) \otimes_E P \to M \tag{3.5}$$

which is surjective by [24, Lem. 2.10]. Remark that we should have written $\mathrm{m}(M)\widehat{\otimes}_E P$ in (3.5), where $\widehat{\otimes}$ means taking completed tensor product. But since $\mathrm{m}(M)$ is finitely generated over E, the completed and usual tensor product coincide, see the discussion before [25, Lem. 2.1].

Proposition 3.20. *Let $M \in \mathfrak{C}$ be a coadmissible quotient of $P = P_{\pi^\vee}$. The following statements hold.*

(i) $\mathrm{m}(M) \otimes_E P$ *is coadmissible.*

(ii) If M is torsion over Λ, then so is $\mathrm{m}(M) \otimes_E P$.

Proof. Let Ker be the kernel of (3.5). By [24, Lem. 2.9] we have

$$\mathrm{Hom}_{\mathfrak{C}}(P, \mathrm{m}(M) \otimes_E P) \cong \mathrm{m}(M),$$

so $\mathrm{Hom}_{\mathfrak{C}}(P, \mathrm{Ker}) = 0$ because P is projective in $\mathfrak{C}$. This implies that Ker does not admit $\pi^\vee$ as a subquotient. In particular, if $\mathfrak{B}$ is of type (I) and (III) of Theorem 3.4, then $\mathrm{Ker} = 0$ and ev is an isomorphism, so both the assertions are trivial. In the rest of the proof, we assume $\mathfrak{B}$ is of type (II) or (IV).

(i) We need to check that for any irreducible $\sigma \in \mathrm{Rep}_{\mathbb{F}}(K)$, $\mathrm{Hom}_K(\mathrm{m}(M) \otimes_E P, \sigma^\vee)$ is finite dimensional over $\mathbb{F}$. By [25, Prop. 2.4] we have a natural isomorphism of compact E-modules:

$$\mathrm{Hom}_K(\mathrm{m}(M) \otimes_E P, \sigma^\vee)^\vee \cong \mathrm{m}(M) \otimes_E \mathrm{Hom}_K(P, \sigma^\vee)^\vee. \tag{3.6}$$

Therefore it is enough to consider those σ such that $\mathrm{Hom}_K(P, \sigma^\vee) \neq 0$. By Corollary 3.10, these are exactly the weights in $\mathcal{D}(\pi)$ if $\pi \notin \{\mathbf{1}_G, \mathrm{Sp}\}$ up to twist, and are $\{\mathrm{Sym}^0\mathbb{F}^2, \mathrm{Sym}^{p-1}\mathbb{F}^2\}$ if $\pi \in \{\mathbf{1}_G, \mathrm{Sp}\}$.

Assume $\pi \notin \{\mathbf{1}_G, \mathrm{Sp}\}$ up to twist. Lemma 3.8 implies that $\mathrm{Hom}_K(\mathrm{Ker}, \sigma^\vee) = 0$ for $\sigma \in \mathcal{D}(\pi)$ because $\pi^\vee$ does not occur in Ker. Hence, we obtain an isomorphism

$$\mathrm{Hom}_K(\mathrm{m}(M) \otimes_E P, \sigma^\vee)^\vee \cong \mathrm{Hom}_K(M, \sigma^\vee)^\vee.$$

Since M is coadmissible, they are finite dimensional.

Assume $\pi = \mathbf{1}_G$. The above argument (using Lemma 3.8) shows

$$\mathrm{Hom}_K(\mathrm{Ker}, (\mathrm{Sym}^0\mathbb{F}^2)^\vee) = 0,$$

so $\mathrm{Hom}_K(\mathrm{m}(M) \otimes_E P, (\mathrm{Sym}^0\mathbb{F}^2)^\vee)$ is finite dimensional as before. We are left to treat the case $\sigma = \mathrm{Sym}^{p-1}\mathbb{F}^2$. However, Proposition 3.9(iv) implies that the E-modules $\mathrm{Hom}_K(P, \sigma^\vee)^\vee$, with $\sigma \in \{\mathrm{Sym}^0\mathbb{F}^2, \mathrm{Sym}^{p-1}\mathbb{F}^2\}$, are naturally isomorphic. So we deduce the result from (3.6). The proof in the case $\pi = \mathrm{Sp}$ is similar.

(ii) It is equivalent to show that Ker is a torsion Λ-module. Since the case of type (II) is similar and simpler, we assume in the rest that $\mathfrak{B}$ is of type (IV), so that $\mathfrak{B}$ consists of three irreducible objects and we let π_1, π_2 be the two other than π. Since Ker is coadmissible by (i) and does not admit $\pi^\vee$ as a subquotient, we can find $s_1, s_2 \geq 0$ and a surjection

$$P_{\pi_1^\vee}^{\oplus s_1} \bigoplus P_{\pi_2^\vee}^{\oplus s_2} \twoheadrightarrow \mathrm{Ker}.$$

Let Q_1 (resp. Q_2) be the maximal quotient of $P_{\pi_1^\vee}$ (resp. $P_{\pi_2^\vee}$) none of whose irreducible subquotients is isomorphic to $\pi^\vee$. Then the above surjection must factor through $Q_1^{\oplus s_1} \oplus Q_2^{\oplus s_2} \twoheadrightarrow \mathrm{Ker}$. Hence, it is enough to show that any coadmissible quotient of Q_1 (resp. Q_2) is torsion. This follows from the results in [24, §10] as we explain below. Up to twist we may assume $\mathfrak{B} = \{\mathbf{1}_G, \mathrm{Sp}, \pi_\alpha\}$.

Let us first assume $\pi = \pi_\alpha$, so that up to order $\pi_1 = \mathbf{1}_G$ and $\pi_2 = \mathrm{Sp}$. We have the following exact sequences

$$0 \to P_{\pi_\alpha^\vee} \to P_{\mathbf{1}_G^\vee} \to M_{\mathbf{1}_T^\vee} \to 0, \tag{3.7}$$

$$P_{\pi_\alpha^\vee}^{\oplus 2} \to P_{\mathrm{Sp}^\vee} \to M_{\mathbf{1}_T^\vee,0} \to 0, \tag{3.8}$$

see [24, (234),(236)], where $M_{\mathbf{1}_T^\vee,0}$ is a submodule of $M_{\mathbf{1}_T^\vee}$ defined by (233) in *loc. cit.* Combining this with Proposition 3.19 implies the assertion.

If $\pi = \mathbf{1}_G$, then (up to order) $\pi_1 = \mathrm{Sp}$ and $\pi_2 = \pi_\alpha$. The assertion for Q_1 follows from the exact sequence (see [24, (179)])

$$P_{\mathbf{1}_G^\vee}^{\oplus 2} \to P_{\mathrm{Sp}^\vee} \to \mathrm{Sp}^\vee \to 0. \tag{3.9}$$

The assertion for Q_2 follows from (3.9) together with the following one

$$0 \to P_{\mathrm{Sp}^\vee} \to P_{\pi_\alpha^\vee} \to M_{\alpha^\vee} \to 0 \tag{3.10}$$

given in [24, (235)]. A similar argument works in the case $\pi = \mathrm{Sp}$. □

Remark 3.21. *(i) The above proof shows that in any case Q_i has a finite filtration (in fact of length ≤ 2) with graded pieces being subquotients of $M_{\eta^\vee}$.*

(ii) If $\pi = \mathbf{1}_G$ and if we set $\pi_1 = \mathrm{Sp}$, $\pi_2 = \pi_\alpha$, then we have the following description of Q_i: $Q_1 = \mathrm{Sp}^\vee$ and Q_2 splits into

$$0 \to \mathrm{Sp}^\vee \to Q_2 \to M_{\alpha^\vee} \to 0.$$

We record the following consequence of the above proof.

Corollary 3.22. *Keep the notation in Proposition 3.20. If $\pi \notin \{\mathbf{1}_G, \mathrm{Sp}\}$ up to twist, then*

$$\mathrm{Hom}_K(\mathrm{m}(M) \otimes_E P, \sigma^\vee) \cong \mathrm{Hom}_K(M, \sigma^\vee).$$

If $\pi \in \{\mathbf{1}_G, \mathrm{Sp}\}$, then for $\sigma \in \{\mathrm{Sym}^0\mathbb{F}^2, \mathrm{Sym}^{p-1}\mathbb{F}^2\}$,

$$\dim_{\mathbb{F}} \mathrm{Hom}_K\left(\mathrm{m}(M) \otimes_E P, \sigma^\vee\right) = \max_{\sigma}\{\dim_{\mathbb{F}} \mathrm{Hom}_K(M, \sigma^\vee)\}.$$

Theorem 3.23. *Let $\pi \in \mathrm{Rep}_{\mathbb{F}}(G)$ be irreducible and $M \in \mathfrak{C}$ be a coadmissible quotient of $P = P_{\pi^\vee}$. There exists $f \in R$ such that f annihilates M and P/fP is a finite free Λ-module.*

Proof. By Proposition 3.20, we may assume $M = \mathrm{m}(M) \otimes_E P$. The quotient map $P \twoheadrightarrow M$ induces a surjective map $E \twoheadrightarrow \mathrm{m}(M)$, that is $\mathrm{m}(M)$ is a cyclic E-module. Let $\mathfrak{a}$ denote the annihilator.

Let $\widetilde{\sigma} = \oplus_\sigma \sigma$ where the sum is taken for all the irreducible $\sigma \in \mathrm{Rep}_{\mathbb{F}}(K)$ such that $\mathrm{Hom}_K(P, \sigma^\vee) \neq 0$. By [24, Prop. 2.4], we have

$$\mathrm{Hom}_K(M, \widetilde{\sigma}^\vee)^\vee \cong \mathrm{m}(M) \otimes_E \mathrm{Hom}_K(P, \widetilde{\sigma}^\vee)^\vee. \tag{3.11}$$

Since $\mathrm{Hom}_K(P, \widetilde{\sigma}^\vee)^\vee$ is a Cohen-Macaulay E-module of dimension 1 by Corollary 3.11 and $\mathrm{Hom}_K(M, \widetilde{\sigma}^\vee)^\vee$ is a finite dimensional quotient (as a vector space over $\mathbb{F}$), there exists $f \in \mathfrak{a}$ which is regular for $\mathrm{Hom}_K(P, \widetilde{\sigma}^\vee)^\vee$ by [7, Thm. 2.1.2(b)].

If $\mathfrak{B}$ is of type (III), the above argument only gives an element $f \in E$ while we need an element in R. However, Corollary 3.27 below shows that $ff^* \in R$ and verifies the required condition. □

3.6. **Blocks of type (III).** In this subsection, we assume $\mathfrak{B}$ is of type (III), that is, $\mathfrak{B} = \{\pi\}$ with $\pi \cong \mathrm{Ind}_B^G \chi \otimes \chi\omega^{-1}$. Let $P = P_{\pi^\vee}$ and $E = \mathrm{End}_{\mathfrak{C}}(P)$. Then E is non-commutative. Let $R = Z(E)$ be the center of E. After twisting we assume $\pi \cong \mathrm{Ind}_B^G \mathbf{1} \otimes \omega^{-1}$ and that the central character of P is ω (being the one of $\pi^\vee$).

The goal of this subsection is to explain how to pass from E to R, hence complete the proof of Theorem 3.23. To this aim, we need pass to Galois side via a functor of Colmez. We first introduce some notation.

- Let $G_{\mathbb{Q}_p} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, $\mathcal{G}$ be the maximal pro-p quotient of $G_{\mathbb{Q}_p}$ and $\mathcal{G}^{\mathrm{ab}}$ the maximal abelian quotient of $\mathcal{G}$. Then
$$G_{\mathbb{Q}_p}^{\mathrm{ab}} \cong \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p) \times \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p) \cong \mathbb{Z}_p^\times \times \widehat{\mathbb{Z}}$$
$$\mathcal{G}^{\mathrm{ab}} \cong (1 + p\mathbb{Z}_p) \times \mathbb{Z}_p.$$
Here μ_{p^∞} is the group of p-power order roots of unity in $\overline{\mathbb{Q}}_p$ and $\mathbb{Q}_p^{\mathrm{ur}}$ is the maximal unramified extension of $\mathbb{Q}_p$. We choose a pair of generators $\bar{\gamma}, \bar{\delta}$ of $\mathcal{G}^{\mathrm{ab}}$ such that $\bar{\gamma} \mapsto (1+p, 0)$ and $\bar{\delta} \mapsto (1,1)$. Then $\mathcal{G}$ is a free pro-p group generated by 2 elements γ, δ which lift respectively $\bar{\gamma}, \bar{\delta}$. See [26, §2] for details.
- Let $R^{\mathrm{ps},\mathbf{1}}$ denote the universal deformation ring over $\mathcal{O}$ (recall $\mathcal{O} := W(\mathbb{F})$) that parameterizes all two-dimensional pseudo-characters of $\mathcal{G}$ lifting the trace of the trivial $\mathbb{F}$-representation and having determinant equal to $\mathbf{1}$. For our purpose, we only need to consider $\overline{R}^{\mathrm{ps},\mathbf{1}} := R^{\mathrm{ps},\mathbf{1}} \otimes_{\mathcal{O}} \mathbb{F}$. Let $T : \mathcal{G} \to \overline{R}^{\mathrm{ps},\mathbf{1}}$ be the associated universal pseudo-character.
- Colmez [11] has defined an exact and covariant functor $\mathbb{V}$ from the category of smooth, finite length representations of G on $\mathbb{F}$-vector spaces with a central character to the category of continuous finite length representations of $G_{\mathbb{Q}_p}$ on $\mathbb{F}$-vector spaces. We will use a modified version as in [24, §5.7], denoted by $\check{\mathbb{V}}$, which applies to objects in $\mathfrak{C}$.

Following [24, (145)], we let (note that in *loc. cit.* the ring is defined over $\mathcal{O}$ and is denoted by R)

$$R' := (\overline{R}^{\mathrm{ps},\mathbf{1}} \widehat{\otimes} \mathbb{F}[[\mathcal{G}]])/\mathbf{J}$$

where $\mathbf{J}$ is the closure of the ideal generated by $g^2 - T(g)g + 1$ for all $g \in G_{\mathbb{Q}_p}$. One may show that the center of R' is equal to $\overline{R}^{\mathrm{ps},\mathbf{1}}$ and the natural morphism

$$\varphi : \mathbb{F}[[\mathcal{G}]] \to R' \tag{3.12}$$

is surjective; see [24, (150), Cor. 9.24].

Note that $\overline{R}^{\mathrm{ps},\mathbf{1}}$ is isomorphic to $\mathbb{F}[[t_1,t_2,t_3]]$. Set

$$u := \gamma - 1 - t_1, \quad v := \delta - 1 - t_2. \tag{3.13}$$

By [24, Cor. 9.25], R' is a free $\overline{R}^{\mathrm{ps},\mathbf{1}}$-module of rank 4 with a basis given by $\{1,u,v,t'\}$ where $t' := uv - vu$. Using [24, (160)], one checks that $R't' = t'R'$.

Lemma 3.24. *The kernel of $R' \twoheadrightarrow R'^{\mathrm{ab}}$ is generated by t' and R'^{ab} is isomorphic to $\mathbb{F}[[u,v]]$.*

Proof. It follows from Lemma [24, Lem. 9.3] and the proof of [24, Cor. 9.27]. □

On the other hand, R' is equipped with an involution $*$ by letting $g^* := g^{-1}$. By [24, Lem. 9.14], we know

$$\overline{R}^{\mathrm{ps},\mathbf{1}} = Z(R') = \{a \in R' : \ a = a^*\}.$$

We also have (see [24, (160)]):

$$u^* = -u, \quad v^* = -v, \quad t'^* = -t'$$

which determine explicitly $*$. Lemma 3.24 implies that the kernel of $R' \twoheadrightarrow \mathbb{F}[[u,v]]$ is stable under $*$.

Now we explain the relation between R' and E. As explained in [24, §9.1], the functor $\check{\mathbb{V}}$ induces a natural transformation $\mathrm{Def}_{\pi^\vee} \to \mathrm{Def}_{\check{\mathbb{V}}(\pi^\vee)}$ between certain deformation functors of $\pi^\vee$ and of $\check{\mathbb{V}}(\pi^\vee)$ respectively, hence induces a morphism

$$\varphi_{\check{\mathbb{V}}} : \mathbb{F}[[\mathcal{G}]]^{op} \to E \tag{3.14}$$

which can be shown to be surjective. By [24, Cor. 9.27], (3.12) induces an isomorphism $\varphi : E \xrightarrow{\sim} R'^{op}$, hence an isomorphism $E^{\mathrm{ab}} \cong \mathbb{F}[[u,v]]$ using Lemma 3.24.

Recall from Lemma 3.18 that we have a surjective morphism $q' : E^{\mathrm{ab}} \twoheadrightarrow \mathbb{F}[[S]]$.

Lemma 3.25. *The composite morphism $\mathbb{F}[[u,v]] \overset{\varphi^{-1}}{\cong} E^{\mathrm{ab}} \xrightarrow{q'} \mathbb{F}[[S]]$ is obtained by modulo v.*

Proof. It suffices to prove that the image of v in $\mathbb{F}[[S]]$ is zero. Choose a parameter triple (r,λ,χ) such that $\pi \cong \pi(r,\lambda,\chi)$. Then $\mathbb{F}[[S]]$ is identified with $\mathrm{End}_{\mathfrak{C}}(\pi_\infty(r,\lambda,\chi)^\vee)$. So we are reduced to prove v annihilates $\check{\mathbb{V}}(\pi_\infty(r,\lambda,\chi)^\vee)$.

It is proved in [17, 1.5.9] and reformulated in [16, Prop. 2.11] that

$$\check{\mathbb{V}}(\pi_\infty(r,\lambda,\chi)^\vee) \cong \mu_{S+\lambda}^{-1}$$

where $\mu_{S+\lambda} : \mathcal{G} \to \mathbb{F}[[S]]^\times$ is the unramified character sending geometric Frobenii to $S+\lambda$. In particular, $\mu_{S+\lambda}(\delta) = 1$ by our choice. Since $t_2 = \frac{\delta+\delta^{-1}}{2} - 1$, and $v = \delta - 1 - t_2$ by definition, we obtain the result. □

We finally obtain the following result which completes the proof of Theorem 3.23.

Proposition 3.26. *The kernel of the quotient morphism $E \to \mathbb{F}[[S]]$ is stable under the involution $*$.*

Corollary 3.27. *Let $f \in E$. Then $ff^* \in R$. If P/fP is coadmissible, so are $P/(f^*)P$ and $P/(ff^*)P$. In particular, $ff^* \neq 0$.*

Proof. Let $\sigma \in \mathrm{Rep}_{\mathbb{F}}(K)$ be a weight such that $\mathrm{Hom}_K(P,\sigma^\vee) \neq 0$. By Corollary 3.10, this implies $\sigma \in \mathcal{D}(\pi)$. The exact sequence $P \to P \to P/fP \to 0$ induces

$$\mathrm{Hom}_K(P,\sigma^\vee)^\vee \xrightarrow{f} \mathrm{Hom}_K(P,\sigma^\vee)^\vee \to \mathrm{Hom}_K(P/fP,\sigma^\vee)^\vee \to 0$$

Since P/fP is coadmissible by assumption, $\mathrm{Hom}_K(P/fP,\sigma^\vee)$ is finite dimensional. By identifying $\mathrm{End}_E(\mathrm{Hom}_K(P,\sigma^\vee)^\vee)$ with $\mathbb{F}[[S]]$, we deduce that f is non-zero in $\mathbb{F}[[S]]$. Proposition 3.26 implies that the image of f^* in $\mathbb{F}[[S]]$ is also non-zero. The result follows from this. □

We also note the following result. Recall that R denotes the center of E.

Proposition 3.28. *Up to a change of variable S, the image of $R \twoheadrightarrow \mathbb{F}[[S]]$ is equal to $\mathbb{F}[[S^2]]$.*

Proof. By Lemma 3.25, $E^{\mathrm{ab}} \twoheadrightarrow \mathbb{F}[[S]]$ induces an isomorphism $\mathbb{F}[[u]] \cong \mathbb{F}[[S]]$. We choose S to be the image of u.

Via the isomorphism $\varphi : E \cong R'^{op}$, we are reduced to determine the image of

$$\iota : Z(R') \hookrightarrow R' \cong E^{op} \twoheadrightarrow \mathbb{F}[[S]].$$

By [24, (159)], we have $u^2 \in Z(R')$, so $\mathbb{F}[[S^2]]$ is contained in $\mathrm{Im}(\iota)$. On the other hand, we have $S \notin \mathrm{Im}(\iota)$. Indeed, if $S = \iota(x)$ with $x \in Z(R')$, then $x - u$ lies in the kernel of $R' \twoheadrightarrow \mathbb{F}[[S]]$. However, by Proposition 3.26 this kernel is stable under the involution $*$, so the image of $(x-u)^*$ in $\mathbb{F}[[S]]$ is also zero. This gives a contradiction because $(x-u)^* = x+u$ is sent to $2S \neq 0$. □

3.7. **Blocks of type (IV).** In this subsection, we complement some results in the work of Paškūnas [24, 25] when $\mathfrak{B}$ is of type (IV). The notation here are the same as in the previous subsections. In particular, $\pi \in \mathrm{Rep}_{\mathbb{F}}(G)$ is irreducible of type (IV), and $P_{\pi^\vee}$ is a projective envelope of $\pi^\vee$ in $\mathfrak{C}$ and $E_{\pi^\vee} = \mathrm{End}_{\mathfrak{C}}(P_{\pi^\vee})$. Note that the rings $E_{\pi^\vee}$ are naturally isomorphic (to $\mathbb{F}[[x,y,z,w]]/(xw-yz)$) for any $\pi \in \mathfrak{B}$ (see [24, §10]), so the subscript will be omitted in the rest (while the one of $P_{\pi^\vee}$ will be kept). Up to twist, we may assume $\mathfrak{B} = \{\mathbf{1}_G, \mathrm{Sp}, \pi_\alpha\}$.

3.7.1. $\mathbb{F} \otimes_E P_{\pi^\vee}$. Our first aim is to determine $\mathbb{F} \otimes_E P_{\pi^\vee}$ for $\pi \in \mathfrak{B}$. For $\pi_1, \pi_2 \in \mathrm{Rep}^{\mathrm{l,fin}}_{\mathbb{F}}(G)$, we will write (following [24, §10])

$$e^1(\pi_1, \pi_2) := \dim_{\mathbb{F}} \mathrm{Ext}^1_G(\pi_1, \pi_2).$$

For convenience of the reader, we recall the list of $e^1(\pi_1, \pi_2)$ for $\pi_1, \pi_2 \in \mathfrak{B}$, see [24, §10.1]:

$$\begin{aligned} e^1(\mathbf{1}_G, \mathbf{1}_G) &= 0, \quad & e^1(\mathrm{Sp}, \mathbf{1}_G) &= 1, \quad & e^1(\pi_\alpha, \mathbf{1}_G) &= 1, \\ e^1(\mathbf{1}_G, \mathrm{Sp}) &= 2, \quad & e^1(\mathrm{Sp}, \mathrm{Sp}) &= 0, \quad & e^1(\pi_\alpha, \mathrm{Sp}) &= 0, \\ e^1(\mathbf{1}_G, \pi_\alpha) &= 0, \quad & e^1(\mathrm{Sp}, \pi_\alpha) &= 1, \quad & e^1(\pi_\alpha, \pi_\alpha) &= 2. \end{aligned}$$

We deduce that there exists a unique (up to isomorphism) non-split sequence

$$0 \to \mathbf{1}_G \to \kappa \to \pi_\alpha \to 0. \tag{3.15}$$

Also, let τ_1 be the universal extension of $\mathbf{1}_G^{\oplus 2}$ by Sp, i.e. we have

$$0 \to \mathrm{Sp} \to \tau_1 \to \mathbf{1}_G^{\oplus 2} \to 0 \tag{3.16}$$

with $\mathrm{soc}_G \tau_1 = \mathrm{Sp}$.

Lemma 3.29. *We have*

$$e^1(\mathrm{Sp}, \kappa) = 2, \quad e^1(\pi_\alpha, \tau_1) = 2, \quad e^1(\tau_1, \pi_\alpha) = 1.$$

Proof. See [24, Lem. 10.18] for the first equality, [24, Lem. 10.12] for the second, [24, (187)] and the argument before it for the third. □

Proposition 3.30. *Let $\pi \in \mathfrak{B}$ and set $Q_{\pi^\vee} = \mathbb{F} \otimes_E P_{\pi^\vee}$. In the following statements, the existence of the extensions is guaranteed by Lemma 3.29.*

(i) *If $\pi = \mathbf{1}_G$, $Q_{\mathbf{1}_G^\vee}$ is isomorphic to the universal extension of $\kappa^\vee$ by $(\mathrm{Sp}^\vee)^{\oplus 2}$:*

$$0 \to (\mathrm{Sp}^\vee)^{\oplus 2} \to Q_{\mathbf{1}_G^\vee} \to \kappa^\vee \to 0. \tag{3.17}$$

(ii) *If $\pi = \mathrm{Sp}$, $Q_{\mathrm{Sp}^\vee}$ is isomorphic to the universal extension of $\tau_1^\vee$ by $(\pi_\alpha^\vee)^{\oplus 2}$:*

$$0 \to (\pi_\alpha^\vee)^{\oplus 2} \to Q_{\mathrm{Sp}^\vee} \to \tau_1^\vee \to 0. \tag{3.18}$$

(iii) *If* $\pi=\pi_\alpha$, $Q_{\pi_\alpha^\vee}$ *is isomorphic to the unique non-split extension of* $\pi_\alpha^\vee$ *by* $\tau_1^\vee$*:*

$$0\to\tau_1^\vee\to Q_{\pi_\alpha^\vee}\to\pi_\alpha^\vee\to 0. \tag{3.19}$$

Proof. By definition, $\mathbb{F}\otimes_E P_{\pi^\vee}$ is characterized as the maximal quotient of P which contains $\pi^\vee$ with multiplicity one. We need to check that if π' is irreducible such that $\mathrm{Ext}^1_G(\pi',(Q_{\pi^\vee})^\vee)\neq 0$, then $\pi'\cong\pi$. Proposition 3.3 implies that we may assume $\pi'\in\mathfrak{B}$.

(i) Write (in this proof) τ for the dual of the extension (3.17); we need to check

$$\mathrm{Ext}^1_G(\mathrm{Sp},\tau)=0=\mathrm{Ext}^1_G(\pi_\alpha,\tau).$$

Since $e^1(\mathrm{Sp},\mathrm{Sp})=0$, the first equality follows from the construction of τ. The second is clear since $e^1(\pi_\alpha,\mathrm{Sp})=0$ (see the formulae recalled above) and $e^1(\pi_\alpha,\kappa)=0$ by [24, (194)].

(ii) is proved in [16, Lem. 4.4, (19)].

(iii) is proved in [25, Prop. 6.1, (35)]. □

3.7.2. $\mathrm{Tor}_i^E(\mathbb{F},P_{\pi^\vee})$. Recall that if $\eta:T\to\mathbb{F}^\times$ is a smooth character, we let $\pi_\eta=\mathrm{Ind}_B^G\,\eta$ and

$$\Pi_\eta=\mathrm{Ind}_B^G\,\mathrm{Inj}_T\eta,\quad M_{\eta^\vee}=(\Pi_\eta)^\vee,\quad E_{\eta^\vee}=\mathrm{End}_{\mathfrak{C}}(M_{\eta^\vee})$$

where $\mathrm{Inj}_T\eta$ denotes an injective envelope of η in $\mathrm{Rep}_{\mathbb{F}}(T)$. In the rest we only consider $\eta\in\{\mathbf{1}_T,\alpha\}$. By Lemma 3.18 there is a natural surjection $q:E\twoheadrightarrow E_{\eta^\vee}$ induced by $P_{\pi_\eta^\vee}\twoheadrightarrow M_{\eta^\vee}$.

Lemma 3.31. *In the isomorphism* $E\cong\mathbb{F}[[x,y,z,w]]/(xw-yz)$*, we may choose the variables such that* $q:E\twoheadrightarrow E_{\eta^\vee}$ *is given by modulo* (z,w)*.*

Proof. First, via Colmez's functor we may identity E with the special fiber of a certain universal Galois pseudo-deformation ring over $\mathcal{O}:=W(\mathbb{F})$, see [24, Thm. 10.71]. This ring is denoted by R^ψ in *loc. cit.* and we write $\overline{R}^\psi$ for its special fiber. Let $\mathfrak{r}$ denote the reducible locus of R^ψ (see [24, Cor. B.6] for its definition) and $\bar{\mathfrak{r}}$ its image in $\overline{R}^\psi$. Then by [24, Cor. B.5, B.6], $\overline{R}^\psi$ is isomorphic to $\mathbb{F}[[c_0,c_1,d_0,d_1]]/(c_0d_1+c_1d_0)$ and $\bar{\mathfrak{r}}=(c_0,c_1)$. On the other hand, via the natural isomorphism $E\cong\overline{R}^\psi$, $\ker(q)$ is identified with $\bar{\mathfrak{r}}$ and $E_{\eta^\vee}$ with $\overline{R}^\psi/\bar{\mathfrak{r}}$, see [24, Lem. 10.80]. This gives the result up to a change of variables. Note that the choice we make is not the one in [24, Lem. 10.93]. □

Lemma 3.32. *We have*

$$\mathrm{Tor}_1^E(\mathbb{F},M_{\eta^\vee})\cong(\pi_\eta^\vee)^{\oplus 2},\quad \mathrm{Tor}_2^E(\mathbb{F},M_{\eta^\vee})\cong\pi_\eta^\vee,\quad \mathrm{Tor}_i^E(\mathbb{F},M_{\eta^\vee})=0,\ \forall i\geq 3.$$

Proof. By Lemma 3.31, we have a resolution of $E_{\eta^\vee}$ by free E-modules:

$$0\to E\overset{(-z,w)}{\to}E^{\oplus 2}\overset{\binom{w}{z}}{\to}E\to E_{\eta^\vee}\to 0.$$

We deduce that

$$\mathrm{Tor}_1^E(E_{\eta^\vee},M_{\eta^\vee})\cong M_{\eta^\vee}^{\oplus 2},\quad \mathrm{Tor}_2^E(E_{\eta^\vee},M_{\eta^\vee})\cong M_{\eta^\vee},\quad \mathrm{Tor}_i^E(E_{\eta^\vee},M_{\eta^\vee})=0,\ \forall i\geq 3.$$

Because $M_{\eta^\vee}$ is a flat $E_{\eta^\vee}$-module by Lemma 3.17, the base change spectral sequence gives the result. □

Proposition 3.33. *We have*

$\mathrm{Tor}_i^E(\mathbb{F},P_{\pi^\vee})$	$i=1$	$i=2$	$i\geq 3$
$\pi=\mathbf{1}_G$	$\mathrm{Sp}^\vee\oplus\mathrm{Sp}^\vee$	$\mathrm{Sp}^\vee$	0
$\pi=\mathrm{Sp}$	$\kappa^\vee$	0	0
$\pi=\pi_\alpha$	$\mathbf{1}_G^\vee$	0	0

Proof. We first observe the following facts:

(a) $\mathrm{SL}_2(\mathbb{Q}_p)$ acts trivially on $\mathrm{Tor}_i^E(\mathbb{F}, P_{\pi_\alpha^\vee})$ for $i \geq 1$. Indeed, [24, Cor. 10.43] states this for $i = 1$ but the proof works for all $i \geq 1$. This implies that $\mathrm{Tor}_i^E(\mathbb{F}, P_{\pi_\alpha^\vee})$ is isomorphic to a finite direct sum of $\mathbf{1}_G^\vee$.

(b) $\mathbf{1}_G^\vee$ does not occur in $\mathrm{Tor}_i^E(\mathbb{F}, P_{\mathbf{1}_G^\vee})$ for $i \geq 1$; this is a special case of Lemma 3.6.

Recall the following exact sequences

$$0 \to P_{\pi_\alpha^\vee} \to P_{\mathbf{1}_G^\vee} \to M_{\mathbf{1}_T^\vee} \to 0, \tag{3.20}$$

$$0 \to P_{\mathrm{Sp}^\vee} \to P_{\pi_\alpha^\vee} \to M_{\alpha^\vee} \to 0, \tag{3.21}$$

see (3.7), (3.10). From (3.20) and Lemma 3.32, we obtain a long exact sequence

$$\cdots \to \pi_{\mathbf{1}_T}^\vee \to \mathrm{Tor}_1^E(\mathbb{F}, P_{\pi_\alpha^\vee}) \to \mathrm{Tor}_1^E(\mathbb{F}, P_{\mathbf{1}_G^\vee}) \to (\pi_{\mathbf{1}_T}^\vee)^{\oplus 2} \to Q_{\pi_\alpha^\vee} \to Q_{\mathbf{1}_G^\vee} \to \pi_{\mathbf{1}_T}^\vee \to 0.$$

From (a), (b), we deduce

$$0 \to \mathrm{Tor}_1^E(\mathbb{F}, P_{\mathbf{1}_G^\vee}) \to (\pi_{\mathbf{1}_T}^\vee)^{\oplus 2} \to Q_{\pi_\alpha^\vee} \to Q_{\mathbf{1}_G^\vee} \to \pi_{\mathbf{1}_T}^\vee \to 0, \tag{3.22}$$

$$0 \to \mathrm{Tor}_2^E(\mathbb{F}, P_{\mathbf{1}_G^\vee}) \to \pi_{\mathbf{1}_T}^\vee \to \mathrm{Tor}_1^E(\mathbb{F}, P_{\pi_\alpha^\vee}) \to 0, \tag{3.23}$$

$$\mathrm{Tor}_i^E(\mathbb{F}, P_{\pi_\alpha^\vee}) = 0, \quad i \geq 2$$
$$\mathrm{Tor}_i^E(\mathbb{F}, P_{\mathbf{1}_G^\vee}) = 0, \quad i \geq 3.$$

Using Proposition 3.30, (3.22) implies $\mathrm{Tor}_1^E(\mathbb{F}, P_{\mathbf{1}_G^\vee}) \cong (\mathrm{Sp}^\vee)^{\oplus 2}$, while (3.23) implies

$$\mathrm{Tor}_1^E(\mathbb{F}, P_{\pi_\alpha^\vee}) \cong \mathbf{1}_G^\vee, \quad \mathrm{Tor}_2^E(\mathbb{F}, P_{\mathbf{1}_G^\vee}) \cong \mathrm{Sp}^\vee.$$

Similarly, using Lemma 3.32 the sequence (3.21) induces

$$\cdots \to \pi_\alpha^\vee \to \mathrm{Tor}_1^E(\mathbb{F}, P_{\mathrm{Sp}^\vee}) \to \mathrm{Tor}_1^E(\mathbb{F}, P_{\pi_\alpha^\vee}) \to (\pi_\alpha^\vee)^{\oplus 2} \to Q_{\mathrm{Sp}^\vee} \to Q_{\pi_\alpha^\vee} \to \pi_\alpha^\vee \to 0.$$

Using Proposition 3.30 and what has been proved, we deduce the result for $\mathrm{Tor}_i^E(\mathbb{F}, P_{\mathrm{Sp}^\vee})$. □

Proposition 3.34. *Let* m *be an* E*-module of finite length. Then* $\mathrm{m} \otimes_E P_{\mathbf{1}_G^\vee}$ *is a Cohen-Macaulay* Λ*-module of codimension 2.*

More generally, if $N \subset \mathrm{m} \otimes_E P_{\mathbf{1}_G^\vee}$ *is a subobject in* $\mathfrak{C}$ *which is a finite direct sum of* $\mathrm{Sp}^\vee$*, then* $(\mathrm{m} \otimes_E P_{\mathbf{1}_G^\vee})/N$ *is a Cohen-Macaulay* Λ*-module of codimension 2.*

Proof. (i) We prove this by induction on the length of m. We may enlarge $\mathbb{F}$ so that irreducible subquotients of m (as E-modules) are isomorphic to $\mathbb{F}$. If $\mathrm{m} = \mathbb{F}$, we need to prove $Q_{\mathbf{1}_G^\vee}$ is Cohen-Macaulay of codimension 2. By Proposition 3.30(i) and Theorem 3.2, it is enough to prove $\kappa^\vee$ is Cohen-Macaulay. This follows from [18, Prop. 5.7], which says that $\kappa^\vee$ is isomorphic to $\mathrm{Ext}^2_\Lambda(\mathrm{Sp}^\vee, \Lambda)$ up to twist (the latter module is naturally equipped with an action of G). Alternatively, we may apply [24, Lem. 10.23] which says that $\kappa^\vee$ has projective dimension 2, hence is Cohen-Macaulay.

If the length of m is ≥ 2 then let $\mathrm{m}_1 \subsetneq \mathrm{m}$ be a submodule of length 1 and let $\mathrm{m}_2 := \mathrm{m}/\mathrm{m}_1$. We then obtain a long exact sequence

$$\mathrm{Tor}_1^E(\mathrm{m}_2, P_{\mathbf{1}_G^\vee}) \xrightarrow{\partial} \mathbb{F} \otimes_E P_{\mathbf{1}_G^\vee} \to \mathrm{m} \otimes_E P_{\mathbf{1}_G^\vee} \to \mathrm{m}_2 \otimes_E P_{\mathbf{1}_G^\vee} \to 0.$$

From Proposition 3.33 and Proposition 3.30(i), we deduce that the cokernel of ∂ has the shape

$$0 \to (\mathrm{Sp}^\vee)^{\oplus m} \to \mathrm{Coker}(\partial) \to \kappa^\vee \to 0 \tag{3.24}$$

with $m \in \{0, 1, 2\}$. Since both $\mathrm{Sp}^\vee$ and $\kappa^\vee$ are Cohen-Macaulay of codimension 2, we obtain the result.

(ii) Let $N \subset \mathrm{m} \otimes_E P_{\mathbf{1}_G^\vee}$ be as in the statement and M be the corresponding quotient. If $\mathrm{m} = \mathbb{F}$, then M has the shape as in (3.24), and we conclude in the same way. The general case is proved by induction in a similar way as in (i). □

4. Main result

We keep the notation of Section 3. For $n \geq 1$, let

$$K_n = \begin{pmatrix} 1+p^n\mathbb{Z}_p & p^n\mathbb{Z}_p \\ p^n\mathbb{Z}_p & 1+p^n\mathbb{Z}_p \end{pmatrix}, \quad T_1(p^n) = \begin{pmatrix} 1+p\mathbb{Z}_p & p^n\mathbb{Z}_p \\ p^n\mathbb{Z}_p & 1+p\mathbb{Z}_p \end{pmatrix}.$$

Recall that $\Lambda := \mathbb{F}[[K_1/Z_1]]$.

The main result of this section is as follows.

Theorem 4.1. *Let $\Pi \in \mathrm{Rep}_{\mathbb{F}}(G)$. Assume that Π is admissible and that $\Pi^\vee$ is torsion as a Λ-module. Then for any $i \geq 0$, we have*

$$\dim_{\mathbb{F}} H^i(T_1(p^n)/Z_1, \Pi) \ll np^n.$$

Remark 4.2. *Note that $|K_1 : T_1(p^n)| = p^{2(n-1)}$. Hence a trivial upper bound for the dimension of $H^i(T_1(p^n)/Z_1, \Pi)$ is given by $c\cdot p^{2n}$ for some constant $c > 0$. In* [19, Prop. 4]*, Marshall has improved this bound to be $\ll p^{\frac{4n}{3}}$ (for a general finitely generated torsion Λ-module which need not carry a compatible action of G). In Section 6, we use Theorem 4.1 to improve some results in* [19].

The rest of the whole section is devoted to the proof of Theorem 4.1 (and its extension to $\mathrm{SL}_2(\mathbb{Q}_p)$). The proof is divided into several steps, the first of which is the following.

Lemma 4.3. *In Theorem 4.1, we may assume that Π is indecomposable and has an irreducible G-socle.*

Proof. Let S be the G-socle of Π. Since Π is admissible, S decomposes as a finite direct sum $\oplus_{i=1}^r \pi_i$ with π_i irreducible. For each i, we let $\mathrm{Inj}_G\pi_i$ be an injective envelope of π_i in $\mathrm{Rep}_{\mathbb{F}}(G)$. The inclusion $\pi_1 \hookrightarrow \Pi$ extends to a G-equivariant morphism $\alpha_1 : \Pi \to \mathrm{Inj}_G\pi_1$. It is clear that $\mathrm{Ker}(\alpha_1)$ has G-socle isomorphic to $\oplus_{i=2}^r \pi_i$ and $\mathrm{Im}(\alpha_1)$ has G-socle π_1. Continuing this with $\mathrm{Ker}(\pi_1)$, we get a finite filtration of Π such that each graded piece, say $\mathrm{gr}^i(\Pi)$, has an irreducible G-socle. Since $\Pi^\vee$ is torsion as a Λ-module if and only if each $(\mathrm{gr}^i(\Pi))^\vee$ is, we obtain the result. □

The plan of the proof of Theorem 4.1 is as follows: in §4.1, we prove a bound of the dimension of $\Pi^{T_1(p^n)}$ for Π of finite length; in §4.2 we prove a lemma which allows to control the dimension of invariants from that of representations of lower canonical dimension; we combine these results to prove the theorem for $i = 0$ in §4.3 and for $i \geq 1$ in §4.4.

4.1. Irreducible representations.

The following control theorem will play a key role in the proof of Theorem 4.1.

Theorem 4.4. *Let $\Pi \in \mathrm{Rep}_{\mathbb{F}}(G)$ be of finite length. Then*

$$\dim_{\mathbb{F}} \Pi^{T_1(p^n)} \ll n.$$

It is clear that we may assume Π is irreducible in Theorem 4.4. Further, by the recall in §3.1, up to twist it is enough to prove the following

Theorem 4.5. *For any $0 \leq r \leq p-1$ and $\lambda \in \mathbb{F}$, we have*

$$\dim_{\mathbb{F}} \pi(r, \lambda, 1)^{T_1(p^n)} \ll n.$$

We need some preparation to prove Theorem 4.5. To begin with, we establish a double coset decomposition formula in K. Let

$$K_0(p^n) = \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p^n\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & 1+p\mathbb{Z}_p \end{pmatrix}.$$

Lemma 4.6. *For any $n \geq 1$, we have*

$$|K_0(p^n)\backslash K/H| = (2n-1)(p-1)+2. \tag{4.1}$$

Proof. Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K$. We have the following facts:

(i) if $A \in K_0(p)$, i.e. $c \in p\mathbb{Z}_p$, we have two subcases:
- if $c \in p^n\mathbb{Z}_p$, then $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K_0(p^n)$;
- if $c \in p\mathbb{Z}_p \backslash p^n\mathbb{Z}_p$ (so $n \geq 2$), write $c = up^k$ with $u \in \mathbb{Z}_p^\times$ and $1 \leq k \leq n-1$, then
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d^{-1}(ad-bc) & ubd^{-1} \\ 0 & u \end{pmatrix} \begin{pmatrix} 1 & 0 \\ p^k & [\lambda] \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}$$
where $\lambda := \overline{u^{-1}d} \in \mathbb{F}_p^\times$ and $t := \frac{u^{-1}d}{[\lambda]} \in 1+p\mathbb{Z}_p$.

We deduce that
$$K_0(p) = K_0(p^n) \bigcup \Bigg(\bigcup_{1\leq k \leq n-1, \lambda \in \mathbb{F}_p^\times} K_0(p^n) \begin{pmatrix} 1 & 0 \\ p^k & [\lambda] \end{pmatrix} H \Bigg).$$
It is easy to check that this is a disjoint union, hence the cardinality of $K_0(p^n)\backslash K_0(p)/H$ is $1+(n-1)(p-1)$.

(ii) if $A \notin K_0(p)$, i.e. $c \in \mathbb{Z}_p^\times$, we still have two cases:
- if $d \in \mathbb{Z}_p^\times$, then
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -[\lambda]d^{-1}(ad-bc) & a \\ 0 & c \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & [\lambda] \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}$$
where $\lambda := \overline{c^{-1}d} \in \mathbb{F}_p^\times$ and $t = \frac{c^{-1}d}{[\lambda]} \in 1+p\mathbb{Z}_p$;
- if $d \in p\mathbb{Z}_p$, then
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & a \\ d & c \end{pmatrix} \in K_0(p).$$

Combining (i) and (ii) the cardinality of $K_0(p^n)\backslash K/H$ is equal to
$$[1+(n-1)(p-1)] + [(p-1)+1+(n-1)(p-1)] = (2n-1)(p-1)+2.$$
□

Proposition 4.7. *Let $n \geq 1$ and σ be a smooth $\mathbb{F}$-representation of $K_0(p^n)$ of finite dimension d. Let V be a quotient K-representation of $\mathrm{Ind}_{K_0(p^n)}^K \sigma$, then $\dim_{\mathbb{F}} V^H \leq 2dpn$.*

Proof. Let W be the corresponding kernel so that we have an exact sequence
$$0 \to W \to \mathrm{Ind}_{K_0(p^n)}^K \sigma \to V \to 0.$$
Taking H-invariants, it induces
$$0 \to W^H \to \left(\mathrm{Ind}_{K_0(p^n)}^K \sigma\right)^H \to V^H \xrightarrow{\partial} H^1(H, W),$$
hence an equality of dimensions
$$\dim_{\mathbb{F}} W^H + \dim_{\mathbb{F}} V^H = \dim_{\mathbb{F}} \left(\mathrm{Ind}_{K_0(p^n)}^K \sigma\right)^H + \dim_{\mathbb{F}} \mathrm{Im}(\partial). \tag{4.2}$$
Now note that $H \cong 1+p\mathbb{Z}_p \cong \mathbb{Z}_p$ is a pro-p group of cohomological dimension 1, so by Lemma 4.8 below we have
$$\dim_{\mathbb{F}} W^H = \dim_{\mathbb{F}} H^1(H, W) \geq \dim_{\mathbb{F}} \mathrm{Im}(\partial),$$
hence by (4.2)
$$\dim_{\mathbb{F}} V^H \leq \dim_{\mathbb{F}} \left(\mathrm{Ind}_{K_0(p^n)}^K \sigma\right)^H.$$
We are thus reduced to prove the proposition in the special case $V = \mathrm{Ind}_{K_0(p^n)}^K \sigma$. Using [2, Lemma 3], it is easy to see that any irreducible smooth $\mathbb{F}$-representation of $K_0(p^n)$ is one-dimensional, so there exists a filtration of σ by sub-representations, of length d, such that all

graded pieces are one-dimensional. Hence, we may assume $d = 1$, in which case the result follows from Lemma 4.6. □

Lemma 4.8. *Let W be a finite dimensional $\mathbb{F}$-representation of $\mathbb{Z}_p$, then*

$$\dim_{\mathbb{F}} H^1(\mathbb{Z}_p, W) = \dim_{\mathbb{F}} H^0(\mathbb{Z}_p, W). \tag{4.3}$$

Proof. This is clear if $\dim_{\mathbb{F}} W = 1$ because then W must be the trivial representation of $\mathbb{Z}_p$ so that

$$H^1(\mathbb{Z}_p, W) \cong \mathrm{Hom}(\mathbb{Z}_p, \mathbb{F})$$

is of dimension 1. The general case is proved by induction on $\dim_{\mathbb{F}} W$ using the fact that $H^2(\mathbb{Z}_p, *) = 0$ and that W always contains a one-dimensional sub-representation. □

Remark 4.9. *In the proof of Proposition 4.7, we crucially used the fact that H has cohomological dimension 1. This fact, very special to the group $\mathrm{GL}_2(\mathbb{Q}_p)$, is also used in [4] and [23] (but for the unipotent subgroup of $B(\mathbb{Z}_p)$).*

4.1.1. *Supersingular case.* We give the proof of Theorem 4.5 when π is supersingular, i.e. $\pi = \pi(r, 0, 1)$ for some $0 \leq r \leq p-1$. Since we have a G-equivariant isomorphism ([4, Thm. 1.3])

$$\pi(r, 0, 1) \cong \pi(p-1-r, 0, \omega^r)$$

we may assume $r > 0$ in the following.

Set $\sigma := \mathrm{Sym}^r\mathbb{F}^2$ and for $n \geq 1$ denote by σ_n the following representation of $K_0(p^n)$:

$$\sigma_n\left(\begin{pmatrix} a & b \\ p^n c & d \end{pmatrix}\right) := \sigma\left(\begin{pmatrix} d & c \\ p^n b & a \end{pmatrix}\right).$$

Let $R_0 := \sigma$ and $R_n := \mathrm{Ind}_{K_0(p^n)}^K \sigma_n$ for $n \geq 1$. It is easy to see that

$$\dim_{\mathbb{F}} R_0 = (r+1), \quad \dim_{\mathbb{F}} R_n = (r+1)(p+1)p^{n-1}, \ \forall n \geq 1. \tag{4.4}$$

Moreover, the following properties hold (see [5, §4]):

(i) $\text{c-Ind}_{KZ}^G \sigma|_K \cong \oplus_{n\geq 0} R_n$;
(ii) the Hecke operator $T|_{R_n} : R_n \to R_{n+1} \oplus R_{n-1}$ is the sum of a K-equivariant injection $T^+ : R_n \hookrightarrow R_{n+1}$ and (for $n \geq 1$) a K-equivariant surjection $T^- : R_n \twoheadrightarrow R_{n-1}$.
(iii) we have an isomorphism of K-representations

$$\pi(r, 0, 1) \cong \big(\varinjlim_{n \text{ even}} R_0 \oplus_{R_1} \oplus R_2 \oplus \cdots \oplus R_n\big) \oplus \big(\varinjlim_{n \text{ odd}} R_1/R_0 \oplus_{R_2} R_3 \oplus \cdots \oplus R_n\big). \tag{4.5}$$

Denote by π_0 and π_1 the two direct summands of π in (4.5). For all $n \geq 0$, we let $\overline{R}_n$ denote the image of $R_n \to \pi(r, 0, 1)$. Then $\overline{R}_n \subset \pi_0$ if n is even, and $\overline{R}_n \subset \pi_1$ if n is odd.

Lemma 4.10. *For all $n \geq 0$, we have $\overline{R}_n \subset \overline{R}_{n+2}$ and $\dim_{\mathbb{F}} \overline{R}_n = (r+1)p^n$.*

Proof. The inclusion $\overline{R}_n \subset \overline{R}_{n+2}$ follows from (4.5). Moreover, (4.5) shows that if n is even, then

$$\dim_{\mathbb{F}} \overline{R}_n = \sum_{k=0}^{n} (-1)^k \dim_{\mathbb{F}} R_k,$$

and if n is odd,

$$\dim_{\mathbb{F}} \overline{R}_n = \sum_{k=0}^{n} (-1)^{k+1} \dim_{\mathbb{F}} R_k.$$

The result then follows from (4.4). □

At this point, we need the following result of Morra. Recall that $\pi = \pi_0 \oplus \pi_1$ as K-representations.

Theorem 4.11. *Let $n \geq 1$. For $i \in \{0,1\}$, the dimension of K_n-invariants of π_i satisfies*

$$\dim_{\mathbb{F}} \pi_i^{K_n} \leq (p+1)p^{n-1}.$$

Moreover, π_i is (nearly) uniserial in the following sense: if W_1, W_2 are two K-stable subspaces of π_i such that

$$\dim_{\mathbb{F}} W_2 - \dim_{\mathbb{F}} W_1 \geq p,$$

then $W_1 \subset W_2$.

Proof. See [22, Cor. 4.14, 4.15] for the dimension formula. Note that the formula in *loc. cit.* is for the dimension of $\pi_0^{K_n} \oplus \pi_1^{K_n}$. The second statement follows from [21, Thm. 1.1] which describes the K-socle filtration of π_i. To explain this, fix $i \in \{0,1\}$. By [21, Thm. 1.1], π_i admits a filtration $\mathrm{Fil}^k\pi_i$, $k \geq 0$ such that

$$\mathrm{Fil}^0\pi_i = 0, \quad \mathrm{Fil}^1\pi_i = \mathrm{soc}_K\pi_i, \quad \mathrm{Fil}^{k+1}/\mathrm{Fil}^k\pi_i \cong \mathrm{Ind}_{B(\mathbb{F}_p)}^{\mathrm{GL}_2(\mathbb{F}_p)} \chi_k, \ \forall k \geq 2,$$

for suitable characters $\chi_k : B(\mathbb{F}_p) \to \mathbb{F}^\times$. In particular, the graded pieces have dimension $p+1$ except for the first. Moreover, the filtration satisfies the property that for any K-stable subspace $W \subset \pi_i$ and any $k \leq k'$, the condition $\dim_{\mathbb{F}} \mathrm{Fil}^k\pi_i \leq \dim_{\mathbb{F}} W \leq \dim \mathrm{Fil}^{k'}\pi_i$ implies $\mathrm{Fil}^k\pi_i \subset W \subset \mathrm{Fil}^{k'}\pi_i$.

Now, for the given W_1 let k_1 be the smallest index such that $W_1 \subset \mathrm{Fil}^{k_1}\pi_i$; then $\dim_{\mathbb{F}} \mathrm{Fil}^{k_1}\pi_i - \dim_{\mathbb{F}} W_1 \leq p$. The assumption then implies that W_2 contains $\mathrm{Fil}^{k_1}\pi_i$, proving the result. □

Corollary 4.12. *We have $\pi^{K_n} \subset \overline{R}_n \oplus \overline{R}_{n+1}$.*

Proof. We have assumed $r \geq 1$, so by Lemma 4.10 we get for $n \geq 1$:

$$\dim_{\mathbb{F}} \overline{R}_n \geq 2p^n \geq (p+1)p^{n-1} + p \geq \dim_{\mathbb{F}} \pi_i^{K_n} + p.$$

By the (nearly) uniserial property of π_i, this implies $\pi_0^{K_n} \subset \overline{R}_n$ if n is even, while $\pi_1^{K_n} \subset \overline{R}_n$ if n is odd. Putting them together, we obtain the result. □

Proof of Theorem 4.5 when $\lambda = 0$. Since $T_1(p^n)$ contains K_n, we have an inclusion $\pi^{T_1(p^n)} \subset \pi^{K_n}$, so Corollary 4.12 implies

$$\pi^{T_1(p^n)} \subset (\overline{R}_n)^{T_1(p^n)} \oplus (\overline{R}_{n+1})^{T_1(p^n)} \subset (\overline{R}_n)^H \oplus (\overline{R}_{n+1})^H.$$

Noting that $\dim_{\mathbb{F}} \sigma \leq p$, we obtain by Proposition 4.7

$$\dim_{\mathbb{F}} \pi^{T_1(p^n)} \leq \dim_{\mathbb{F}}(\overline{R}_n)^H + \dim_{\mathbb{F}}(\overline{R}_{n+1})^H \leq 4p^2 n,$$

hence the result. □

4.1.2. *Non-supersingular case.* Assume from now on $\lambda \neq 0$. We define the subspaces R_n ($n \geq 0$) of $\text{c-Ind}_{KZ}^G\sigma$ as above. We still have the properties (i) and (ii) recalled there. The only difference, also the key difference with the supersingular case, is that the induced morphisms $R_n \to \pi(r,\lambda,1)$ are all injective (because $\lambda \neq 0$). Moreover, if we write $\overline{R}_n$ for the image of R_n in $\pi(r,\lambda,1)$, then $\overline{R}_n \subset \overline{R}_{n+1}$ and

$$\pi(r,\lambda,1) = \varinjlim_{n\geq 0} \overline{R}_n.$$

Proposition 4.13. *Let $n \geq 1$, we have an inclusion $\pi(r,\lambda,1)^{K_n} \subset \overline{R}_n$.*

Proof. By [21, Thm. 1.2], $\pi(r,\lambda,1)$ satisfies a (nearly) uniserial property as in the supersingular case. Moreover, we have (see [22, §5]) $\dim_{\mathbb{F}} \pi(r,\lambda,1)^{K_n} = (p+1)p^{n-1}$ while

$$\dim_{\mathbb{F}} \overline{R}_n = \dim_{\mathbb{F}} R_n = (r+1)(p+1)p^{n-1}.$$

We then conclude as in the supersingular case. □

Proof of Theorem 4.5 when $\lambda \neq 0$. Since $T_1(p^n)$ contains K_n, we obtain by Proposition 4.13

$$\pi^{T_1(p^n)} \subset (\overline{R}_n)^{T_1(p^n)} \subset (\overline{R}_n)^H.$$

The result then follows from Proposition 4.7. □

We close this subsection by the following consequence.

Corollary 4.14. *Let $M \in \mathfrak{C}$ be coadmissible. If $\delta(M) \leq 1$, then $\dim_{\mathbb{F}} M_{T_1(p^n)} \ll n$.*

Proof. A direct consequence of Theorem 3.1 and Theorem 4.4. □

4.2. **Key lemma.** In this subsection, we prove a lemma which can be viewed as an analogue of Proposition 2.3. These two results will allow us to relate the canonical dimension of a coadmissible module $M \in \mathfrak{C}$ and the $\mathbb{F}$-dimension of $T_1(p^n)$-coinvariants of M.

Lemma 4.15. *Let M be a finitely generated Λ-module and $\phi \in \mathrm{End}_{\Lambda}(M)$. Assume that $\bigcap_{n\geq 1} \phi^n(M) = 0$.*

(i) *If ϕ is nilpotent, then $\dim_{\mathbb{F}} M_{T_1(p^n)} \sim \dim_{\mathbb{F}} \big(M/\phi(M)\big)_{T_1(p^n)}$.*
(ii) *If ϕ is not nilpotent, then for $k_0 \gg 1$,*

$$\dim_{\mathbb{F}} M_{T_1(p^n)} \ll \max\left\{ \dim_{\mathbb{F}} \big(M/\phi(M)\big)_{T_1(p^n)}, p^n \dim_{\mathbb{F}} \big(\phi^{k_0}(M)/\phi^{k_0+1}(M)\big)_{T_1(p^n)} \right\}.$$

In any case, we have $\dim_{\mathbb{F}} M_{T_1(p^n)} \ll p^n \dim_{\mathbb{F}}((M)/\phi(M))_{T_1(p^n)}$.

Proof. (i) If ϕ is nilpotent, M admits a finite filtration by $\phi^k(M)$, for $k \leq k_0$ where $k_0 \gg 1$ is such that $\phi^{k_0} = 0$. For any $k \geq 1$, ϕ induces a surjective morphism $M/\phi(M) \twoheadrightarrow \phi^k(M)/\phi^{k+1}(M)$, giving the result.

(ii) By Lemma 2.4, there exists $k_0 \gg 0$ such that $\phi : \phi^{k_0}(M) \to \phi^{k_0}(M)$ is injective. Using the short exact sequence $0 \to \phi^{k_0}(M) \to M \to M/\phi^{k_0}(M) \to 0$ and applying (i) to $M/\phi^{k_0}(M)$, we are reduced to prove

$$\dim_{\mathbb{F}} \phi^{k_0}(M)_{T_1(p^n)} \ll p^n \dim_{\mathbb{F}} \big(\phi^{k_0}(M)/\phi^{k_0+1}(M)\big)_{T_1(p^n)}.$$

That is, by replacing M by $\phi^{k_0}(M)$, we may assume ϕ is injective in the rest.

Set $Q := M/\phi(M)$ so that we have a short exact sequence $0 \to M \overset{\phi}{\to} M \to Q \to 0$. Let J denote the maximal ideal of Λ. Since M/JM is finite dimensional and $M = \varprojlim_k M/\phi^k(M)$ by (a), we may choose $k_1 \gg 0$ such that the composite morphism $M \overset{\phi^{k_1}}{\to} M \to M/JM$ is zero. Replacing ϕ by ϕ^{k_1} and Q by $M/\phi^{k_1}(M)$, we may assume $\phi(M) \subset JM$. Since ϕ is G-equivariant, we obtain inductively

$$\phi^k(J^s M) \subset J^{k+s} M, \quad \forall k, s \geq 1.$$

Letting $Q_k := M/\phi^k(M)$, the short exact sequence $0 \to M \overset{\phi^k}{\to} M \to Q_k \to 0$ then induces by modulo J^{k+1}:

$$M/JM \overset{\phi^k}{\to} M/J^{k+1}M \to Q_k/J^{k+1}Q_k \to 0.$$

If I is another (two-sided) ideal of Λ containing J^{k+1}, then we obtain by modulo I again:

$$M/(J+I)M \to M/IM \to Q_k/IQ_k \to 0.$$

Since $\dim_{\mathbb{F}} M/(J+I)M$ is bounded by $c_0 := \dim_{\mathbb{F}} M/JM$ which depends only on M, and since Q_k is a successive extension of Q (k times), we obtain the following inequality:

$$\dim_{\mathbb{F}} M/IM \leq \dim_{\mathbb{F}} Q_k/IQ_k + c_0 \leq k \dim_{\mathbb{F}} Q/IQ + c_0. \tag{4.6}$$

We specialize the above inequality to our situation. Recall that Λ is topologically generated by three elements, say z_1, z_2, z_3, such that every element of Λ can be uniquely expressed as a sum over multi-indices $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$:

$$x = \sum_{\alpha} \lambda_\alpha z^\alpha, \quad z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3}.$$

Moreover, $z^\alpha z^\beta = z^{\alpha+\beta}$ up to terms of degree $> |\alpha| + |\beta|$, see [19, Thm. 10]. The ideal J is simply spanned by the set of elements z^α with $|\alpha| > 0$. Let I_n denote the two-sided ideal of Λ generated by the maximal ideal of $\mathbb{F}[[T_1(p^n)/Z_1]]$. Then it is easy to see that J^{3p^n} is contained in I_n. Applying (4.6) to $I = I_n$, we obtain

$$\dim_{\mathbb{F}} M_{T_1(p^n)} = \dim_{\mathbb{F}} M/I_n M \leq (3p^n - 1) \cdot \dim_{\mathbb{F}} Q/I_n Q + c_0,$$

giving the result. □

Remark 4.16. *In the proof of Lemma 4.15, it is crucial that we are working with $T_1(p^n)$ instead of $K_1(p^{2n})$ (this group will show up in §6 for application), although they are (up to finite order) conjugate to each other in $\mathrm{GL}_2(\mathbb{Q}_p)$. We have learnt this trick of "averaging" from* [19] *(used in a different manner there).*

4.3. **The proof in degree 0.**

Proof of Theorem 4.1 for $i = 0$. Let $M := \Pi^\vee \in \mathfrak{C}$. Then the Pontryagin dual induces natural isomorphisms for $i \geq 0$

$$\big(H^i(T_1(p^n)/Z_1, \Pi)\big)^\vee \cong H_i(T_1(p^n)/Z_1, M).$$

So we could instead work with M.

By Lemma 4.3, we may assume M is a quotient of $P_{\pi^\vee}$ for some irreducible $\pi \in \mathrm{Rep}_{\mathbb{F}}(G)$. Write $P = P_{\pi^\vee}$, $E = E_{\pi^\vee}$ and $R = Z(E)$ in the following.

By Theorem 3.23 there exists a regular element $f \in \mathfrak{m}_R$ (where $\mathfrak{m}_R$ denotes the maximal ideal of R) such that:

(a) P/fP is a finitely generated free Λ-module;
(b) f annihilates M, i.e. M is a quotient of P/fP.

Since R is a Cohen-Macaulay integral domain of Krull dimension 3, we can find $g, h \in \mathfrak{m}_R$ such that f, g, h form a regular sequence. As a consequence, f, g, h is a system of parameters for R. We deduce from Proposition 3.7 that $P/(f, g, h)P$ is of finite length in $\mathfrak{C}$, hence so is $M/(g, h)M$ (noting that f annihilates M). Theorem 4.4 then implies that

$$\dim_{\mathbb{F}} \big(M/(g, h)M\big)_{T_1(p^n)} \ll n. \tag{4.7}$$

By assumption M is torsion, so $\delta(M) \leq 2$. Consider the endomorphism $g : M \to M$ and the cokernel M/gM. By Proposition 2.3 there are two possibilities:

- g is nilpotent on M. In this case, Lemma 4.15 implies

$$\dim_{\mathbb{F}} M_{T_1(p^n)} \sim \dim_{\mathbb{F}} (M/gM)_{T_1(p^n)} \ll p^n \dim_{\mathbb{F}} (M/(g, h)M)_{T_1(p^n)} \ll np^n.$$

- g is not nilpotent and for $k_0 \gg 1$, $\delta\big(g^{k_0}(M)/g^{k_0+1}(M)\big) \leq 1$. In this case, we have $\dim_{\mathbb{F}}(g^{k_0} M)_{T_1(p^n)} \ll np^n$ by Lemma 4.15 and Theorem 4.4. It suffices to show the same estimation for $M/g^{k_0}M$. But $M/(g^{k_0}, h)M$ is of finite length in $\mathfrak{C}$ because $M/(g, h)M$ is, so the result follows again by Lemma 4.15 and Theorem 4.4.

□

4.4. **Higher homological degrees.** In [19] or [8], once a bound is obtained for torsion Λ-modules, the extension to higher homological degrees is rather easy using the trivial dimension formula for free Λ-modules. However, it is much subtler in our situation, because to be able to apply Theorem 4.1 we need guarantee in each step that the module $M \in \mathfrak{C}$ in consideration can be split into two parts in $\mathfrak{C}$, not just as Λ-modules. To overcome this difficulty, we prove the following result.

Proposition 4.17. *Fix π as above. Let $M \in \mathfrak{C}$ be a non-zero coadmissible quotient of P. Assume that M is torsion-free as a Λ-module.*

(i) *If $\mathfrak{B}$ is not of type (IV), then M is a free Λ-module.*
(ii) *If $\pi \cong \chi \circ \det$, then M is a free Λ-module and the kernel of the evaluation morphism* (3.5) *has finite length.*

Proof. Let $f, g, h \in \mathfrak{m}_R$ be constructed in the last proof. In particular, f annihilates M and P/fP is a free Λ-module.

We claim that $g : M \to M$ is injective. Otherwise, the submodule $M[g]$ killed by g would be torsion by Proposition 2.3, contradicting the assumption. Similarly, if $h : M/gM \to M/gM$ were not injective, the pre-image of $(M/gM)[h]$ in M would be torsion. Therefore, g, h is an M-sequence.

Since Λ is a local ring, M is free over Λ if and only if the projective dimension $\mathrm{pd}(M) = 0$. Since $j(M) = 0$ by assumption, it suffices to prove M is Cohen-Macaulay. Since g, h is an M-sequence, we are left to show $M/(g, h)M$ is Cohen-Macaulay (of codimension 2) by Lemma 2.5. If $\mathfrak{B}$ is not of type (IV), each object in $\mathfrak{B}$ is Cohen-Macaulay by Theorem 3.2, hence so is $M/(g, h)M$ which is of finite length. This proves (i).

(ii) Up to twist we may assume $\pi = \mathbf{1}_G$. Let $\mathrm{m} = \mathrm{m}(M) := \mathrm{Hom}_{\mathfrak{C}}(P, M)$ and let Ker be the kernel of $\mathrm{ev} : \mathrm{m} \otimes_E P \twoheadrightarrow M$. Remark 3.21 implies that Ker splits into two parts

$$0 \to \mathrm{Ker}_1 \to \mathrm{Ker} \to \mathrm{Ker}_2 \to 0$$

where Ker_1 is a *finite* direct sum of $\mathrm{Sp}^\vee$ and Ker_2 has all its irreducible subquotients isomorphic to $\pi_\alpha^\vee$. Since g, h is an M-sequence and $\mathrm{Hom}_{\mathfrak{C}}(P, -)$ is exact, g, h is also an m-sequence. Consider the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ker} & \longrightarrow & \mathrm{m} \otimes_E P & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & \mathrm{Ker} & \longrightarrow & \mathrm{m} \otimes_E P & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

which implies by the Snake lemma that $\mathrm{Ker}[g] \cong (\mathrm{m} \otimes_E P)[g]$ and

$$0 \to \mathrm{Ker}/g\,\mathrm{Ker} \to (\mathrm{m}/g) \otimes_E P \to M/gM \to 0. \tag{4.8}$$

Since $(\mathrm{m} \otimes_E P)[g]$ is a quotient of $\mathrm{Tor}_1^E(\mathrm{m}/g, P)$, it contains only $\mathrm{Sp}^\vee$ as subquotients by Proposition 3.33. Hence the same is true for $\mathrm{Ker}[g]$. We deduce that[8] $\mathrm{Ker}_1 \supseteq \mathrm{Ker}[g]$ and $g : \mathrm{Ker}_2 \to \mathrm{Ker}_2$ is injective. Moreover, we have

$$0 \to \mathrm{Ker}_1/g\,\mathrm{Ker}_1 \to \mathrm{Ker}/g\,\mathrm{Ker} \to \mathrm{Ker}_2/g\,\mathrm{Ker}_2 \to 0.$$

We claim that $\mathrm{Ker}_2 = 0$. Since $g : \mathrm{Ker}_2 \to \mathrm{Ker}_2$ is injective, it is equivalent to show $\mathrm{Ker}_2/g\,\mathrm{Ker}_2 = 0$ by Nakayama's lemma. But if not, the same argument with h applied to (4.8) shows that $h : \mathrm{Ker}_2/g\,\mathrm{Ker}_2 \to \mathrm{Ker}_2/g\,\mathrm{Ker}_2$ is injective and $\mathrm{Ker}_2/(g, h)\,\mathrm{Ker}_2$ is non-zero by Nakayama's lemma again. This shows that g, h is a Ker_2-sequence, so $\delta(\mathrm{Ker}_2) = 3$ by Proposition 2.3 and Theorem 3.1. This is impossible because Ker_2 is torsion by Proposition 3.20.

[8] In fact we have equality $\mathrm{Ker}_1 = \mathrm{Ker}[g]$, because Ker_1 is a semi-simple module, hence is annihilated by g. But we don't need this stronger fact.

The claim implies an exact sequence

$$0 \to \mathrm{Ker}_1 / (g,h)\,\mathrm{Ker}_1 \to (\mathrm{m}/(g,h)) \otimes_E P \to M/(g,h)M \to 0.$$

Since $\mathrm{Ker}_1 \cong (\mathrm{Sp}^\vee)^m$ for some $m \geq 0$ and since $\mathrm{m}/(g,h)$ has finite length, we conclude by Proposition 3.34 as in (i). $\square$

The next result implies in particular Theorem 4.1.

Theorem 4.18. *Let $M \in \mathfrak{C}$ and assume it is finitely generated of rank s as a Λ-module. Then for some constant $c > 0$ depending on M,*

$$\dim_{\mathbb{F}} M_{T_1(p^n)} \leq s|K_1 : T_1(p^n)| + c \cdot np^n \tag{4.9}$$

$$\dim_{\mathbb{F}} H_i(T_1(p^n)/Z_1, M) \leq c \cdot np^n, \quad i \geq 1. \tag{4.10}$$

Proof. (i) The proof of (4.9) is similar to the proof of (26) in [19, §3.2] but using Proposition 4.17 as the main input. Since the rank function is additive with respect to short exact sequences, using Lemma 4.3 we may assume M is a quotient of $P_{\pi^\vee}$ with $\pi^\vee \in \mathfrak{C}$ irreducible. The proof proceeds by induction on s. If $s = 0$, then it follows from Theorem 4.1. Assume $s \geq 1$ and (4.9) holds for objects of rank $\leq s-1$.

Let M_{tor} be the torsion part of M (as a Λ-module) and M_{tf} be the quotient M/M_{tor}. Then M_{tor} is stable under the action of G, so that Theorem 4.1 applies to M_{tor} (it is also finitely generated over Λ because M is and Λ is noetherian). So by Theorem 4.1, we may assume M is torsion-free of rank s.

If up to twist $\pi \notin \{\mathrm{Sp}, \pi_\alpha\}$, then M is already a free Λ-module by Proposition 4.17, so the result is obvious. Assume $\pi = \mathrm{Sp}$. We must have $\mathrm{Hom}_{\mathfrak{C}}(P_{\mathbf{1}_G^\vee}, M) \neq 0$, otherwise M would be torsion by (3.9). Choose a non-zero morphism $P_{\mathbf{1}_G^\vee} \to M$ with M' being its image and $M'' := M/M'$. Since M is torsion-free, so is M', hence M' is actually a free Λ-module by Proposition 4.17. Since the rank of M'' is $\leq s-1$, we conclude by induction. The case $\pi = \pi_\alpha$ is proven in a similar way.

(ii) To prove (4.10), again we may assume M is a quotient of $P_{\pi^\vee}$ for some irreducible $\pi^\vee \in \mathfrak{C}$, and further a quotient of $\overline{P}$ where $\overline{P}$ is a finite free Λ-module by Theorem 3.23.[9] Letting N be the kernel, we have a short exact sequence $0 \to N \to \overline{P} \to M \to 0$ which induces

$$0 \to H_1(T_1(p^n)/Z_1, M) \to N_{T_1(p^n)} \to \overline{P}_{T_1(p^n)} \to M_{T_1(p^n)} \to 0 \tag{4.11}$$

$$H_i(T_1(p^n)/Z_1, M) \cong H_{i-1}(T_1(p^n)/Z_1, N), \quad i \geq 2. \tag{4.12}$$

From (4.11) and using (4.9) we obtain for some constant $c > 0$

$$\dim_{\mathbb{F}} H_1(T_1(p^n)/Z_1, M) + \dim_{\mathbb{F}} \overline{P}_{T_1(p^n)} \leq (s+s')|K_1 : K_1(p^n)| + c \cdot np^n$$

where s' denotes the Λ-rank of N. Since $\overline{P}$ is free of rank $s+s'$, we have

$$\dim_{\mathbb{F}} \overline{P}_{T_1(p^n)} = (s+s')|K_1 : T_1(p^n)|,$$

proving the result for $i = 1$. Finally, the estimation for higher i follows from (4.12) by induction. $\square$

[9] Alternatively we may also apply [6, Cor. 9.11].

4.5. $\mathrm{GL}_2(\mathbb{Q}_p)$ **vs** $\mathrm{SL}_2(\mathbb{Q}_p)$**.** For the application in Section 6, we need to consider smooth admissible $\mathbb{F}$-representations of $\mathrm{SL}_2(\mathbb{Q}_p)$ and their Pontryagin duals. It is easy to translate the results above to $\mathrm{SL}_2(\mathbb{Q}_p)$ case. If H is a subgroup of $\mathrm{GL}_2(\mathbb{Q}_p)$, we denote by H' the intersection $H \cap \mathrm{SL}_2(\mathbb{Q}_p)$.

Lemma 4.19. *Let Π be a smooth admissible $\mathbb{F}$-representation of $\mathrm{SL}_2(\mathbb{Q}_p)$. Then for all $i \geq 0$,*

$$\dim_{\mathbb{F}} H^i(T_1(p^n)', \Pi) \ll np^n.$$

Proof. Write Id for the identity matrix of $\mathrm{SL}_2(\mathbb{Q}_p)$. The center of $\mathrm{SL}_2(\mathbb{Q}_p)$ is $\{\pm\mathrm{Id}\}$ and we have an isomorphism $\mathrm{SL}_2(\mathbb{Q}_p)/\{\pm\mathrm{Id}\} \cong \mathrm{GL}_2(\mathbb{Q}_p)/Z$. Depending on the action of $-\mathrm{Id}$, Π decomposes as $\Pi^+ \oplus \Pi^-$, where $-\mathrm{Id}$ acts on $\Pi^\pm$ via ± 1. Up to twist, we only need treat Π^+. But then we may view Π^+ as a representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ with a trivial central character and conclude by Theorem 4.1. □

5. Generalization

For application in §6, we need generalize Theorem 4.1 to representations of a finite product of $\mathrm{GL}_2(\mathbb{Q}_p)$. Although the main result (Theorem 5.1) we will prove below is similar to [19, Prop. 4], the proof is quite different. The reason is that to carry out the inductive step as in [19, §3.1], one need a stronger statement than Theorem 4.1; cf. [19, Prop. 7]. Instead, we use a direct generalization of the proof of Theorem 4.1, at the cost of obtaining a weaker result.[10] See also Remark 6.2.

We let $G = \mathrm{GL}_2(\mathbb{Q}_p)$, $K = \mathrm{GL}_2(\mathbb{Z}_p)$ and other subgroups of G are defined as in the previous sections. Given $r \geq 1$, we let

$$\mathcal{G} = \prod_{i=1}^{r} G, \quad \mathcal{K} = \prod_{i=1}^{r} K, \quad \mathcal{K}_1 = \prod_{i=1}^{r} K_1$$

$$\mathcal{Z}_1 = \prod_{i=1}^{r} Z_1, \quad \Lambda = \mathbb{F}[[\mathcal{K}_1/\mathcal{Z}_1]] \cong \widehat{\otimes}_{i=1}^{r}\mathbb{F}[[K_1/Z_1]].$$

That is, $\mathcal{G}$ is a product of r copies of G, and so on. If $\mathbf{n} = (n_1, ..., n_r) \in (\mathbb{Z}_{\geq 1})^r$, let

$$\mathcal{T}_1(p^{\mathbf{n}}) = \prod_{i=1}^{r} T_1(p^{n_i}), \quad \mathcal{K}_{\mathbf{n}} = \prod_{i=1}^{r} K_{n_i}.$$

The aim of this section is to prove the following result.

Theorem 5.1. *Let $M \in \mathfrak{C}(\mathcal{G})$ and assume it is finitely generated of rank s as a Λ-module. Then for some constant $c > 0$ depending on M,*

$$\dim_{\mathbb{F}} M_{\mathcal{T}_1(p^{\mathbf{n}})} \leq s|\mathcal{K}_1 : \mathcal{T}_1(p^{\mathbf{n}})| + c \cdot \kappa(\mathbf{n})^r p^{(2r-1)\kappa(\mathbf{n})}$$

$$\dim_{\mathbb{F}} H_i(\mathcal{T}_1(p^{\mathbf{n}})/\mathcal{Z}_1, M) \leq c \cdot \kappa(\mathbf{n})^r p^{(2r-1)\kappa(\mathbf{n})}, \quad \forall i \geq 1.$$

where $\kappa(\mathbf{n}) := \max_i\{n_i\}$.

In §5.1 we establish some results generalizing the case $r = 1$. We give the proof of Theorem 5.1 in §5.2, and in §5.3 translate it into a form adapted for application.

5.1. **Preliminaries.**

[10] we mean the appearance of $\kappa(\mathbf{n})$ in Theorem 5.1

5.1.1. *Irreducible representations.* For $1 \le i \le r$, let $\pi_i \in \mathfrak{C}(G)$ be (absolutely) irreducible.

Lemma 5.2. *(i) The tensor product $\pi_1 \otimes \cdots \otimes \pi_r$ is an irreducible admissible representation of $\mathcal{G}$ and each irreducible admissible representation of $\mathcal{G}$ is of this form.*

(ii) Let $\pi = \pi_1 \otimes \cdots \otimes \pi_r$ be as in (i). Then $\delta_\Lambda(\pi^\vee)$ is equal to the cardinality of $i \in \{1, ..., r\}$ such that π_i is infinite dimensional. Moreover, $\pi^\vee$ is a Cohen-Macaulay Λ-module.

(iii) Let $\pi = \otimes_{i=1}^r \pi_i$ and $\pi' = \otimes_{i=1}^r \pi_i'$ be irreducible representations of $\mathcal{G}$. Then $\pi \sim \pi'$ (i.e. in the same block) if and only if $\pi_i \sim \pi_i'$ for all i.

Proof. (i) it is standard; see [29, Prop. 9.2] for a proof.

(ii) is a direct consequence of Theorems 3.1 and 3.2.

(iii) It follows from the fact that: $\mathrm{Ext}^1_{\mathcal{G}}(\pi, \pi') \ne 0$ if and only if there exists $1 \le i \le r$ such that $\mathrm{Ext}^1_G(\pi_i, \pi_i') \ne 0$ and $\pi_j \cong \pi_j'$ for all $j \ne i$. □

Let $P_{\pi_i^\vee}$ be a projective envelope of $\pi_i^\vee$ in $\mathfrak{C}(G)$ and set $E_{\pi_i^\vee} = \mathrm{End}_{\mathfrak{C}(G)}(P_{\pi_i^\vee})$. Write

$$P = \widehat{\otimes}_{i=1}^r P_{\pi_i^\vee}, \quad E = \widehat{\otimes}_{i=1}^r E_{\pi_i^\vee},$$

where $\widehat{\otimes}$ denotes the completed tensor product (always over $\mathbb{F}$). Let $R = Z(E)$ be the center of E.

Lemma 5.3. *(i) P is a projective envelope of $\pi_1^\vee \otimes \cdots \otimes \pi_r^\vee$ in $\mathfrak{C}(\mathcal{G})$ and $\mathrm{End}_{\mathfrak{C}(\mathcal{G})}(P) \cong E$.*

(ii) $R = Z(E)$ is a complete local noetherian integral domain, and is Cohen-Macaulay of Krull dimension $3r$.

Proof. (i) is obvious and (ii) follows from Theorem 3.5. □

Lemma 5.4. *(i) $\mathbb{F} \otimes_E P$ (resp. $\mathbb{F} \otimes_R P$) has finite length in $\mathfrak{C}(\mathcal{G})$.*

(ii) If $\pi_i \notin \{\mathrm{Sp}, \pi_\alpha\}$ (up to twist) for any i, then $\mathbb{F} \otimes_E P$ (resp. $\mathbb{F} \otimes_R P$) is Cohen-Macaulay.

Proof. Both assertions follow from Proposition 3.7. □

Lemma 5.5. *Assume $\delta_\Lambda(\pi^\vee) = r$. Let $M \in \mathfrak{C}$ be a non-zero coadmissible quotient of $P = P_{\pi^\vee}$. Then M has finite length if and only if $\delta_\Lambda(M) = r$.*

Proof. Since any irreducible object in $\mathfrak{C}(\mathcal{G})$ has canonical dimension $\le r$ and $\pi^\vee$ occurs as the G-cosocle of M, the "only if" part is obvious.

Assume $\delta_\Lambda(M) = r$. Consider the natural filtration of M given by $\mathfrak{m}_E^i M$, $i \ge 0$. Let n_i denote the $\mathbb{F}$-dimension of $\mathfrak{m}_E^i/\mathfrak{m}_E^{i+1}$. Then each of the grade pieces $\mathfrak{m}_E^i M/\mathfrak{m}_E^{i+1} M$ is a quotient of $(M/\mathfrak{m}_E M)^{\oplus n_i}$, hence of finite length by Lemma 5.4(i). Since the cosocle of $M/\mathfrak{m}_E M$ is isomorphic to $\pi^\vee$, we see that if $\mathfrak{m}_E^i M/\mathfrak{m}_E^{i+1} M \ne 0$ then it has canonical dimension r. However, as in the proof of Theorem 3.1, there are only a finite number of irreducible subquotients of M which have canonical dimension r. This shows that $\mathfrak{m}_E^i M/\mathfrak{m}_E^{i+1} M = 0$ for $i \gg 0$, hence $\mathfrak{m}_E^i M = 0$ by Nakayama's lemma. □

Remark 5.6. *In general, it is not true that $\delta_\Lambda(M) \le r$ implies M has finite length. For example, if $r \ge 2$, $M = \widehat{\otimes}_i M_i$ with $\delta_{\mathbb{F}[[K_1/Z_1]]}(M_1) = 2$ and $M_i = \mathbf{1}_G^\vee$ for $2 \le i \le r$, then $\delta_\Lambda(M) = 2$ but M is not of finite length.*

The next result is a weaker generalization of Theorem 3.1.

Lemma 5.7. *If $M \in \mathfrak{C}(\mathcal{G})$ has finite length, then $\dim_{\mathbb{F}} M_{\mathcal{I}_1(p^{\mathbf{n}})} \ll \prod_{i=1}^r n_i$.*

Proof. We may assume M is irreducible so $M \cong \otimes_{i=1}^r \pi_i^\vee$ with π_i irreducible. The result then follows from the case $r = 1$, see Theorem 4.4. □

5.1.2. *Serre weights.* Similar to Lemma 5.2, an irreducible representation of $\mathcal{K}$ is of the form $\sigma = \otimes_{i=1}^r \sigma_i$ with each $\sigma_i \in \mathrm{Rep}_{\mathbb{F}}(K)$ irreducible. We have the obvious notion of Serre weights for $\pi = \otimes_{i=1}^r \pi_i$. Clearly, $\sigma \in \mathcal{D}(\pi)$ if and only if $\sigma_i \in \mathcal{D}(\pi_i)$ for each i. The following lemma is a direct generalization of Corollary 3.11.

Lemma 5.8. *Let $\sigma = \otimes_{i=1}^r \sigma_i \in \mathrm{Rep}_{\mathbb{F}}(\mathcal{K})$ be irreducible. Whenever non-zero, $\mathrm{Hom}_{\mathcal{K}}(P, \sigma^\vee)^\vee$ is a Cohen-Macaulay E-module of Krull dimension r. Moreover, if $\sigma = \otimes_i \sigma_i$ and $\sigma' = \otimes_i \sigma_i'$ are two weights such that $\sigma_i = \sigma_i'$ whenever π_i is supersingular, then*

$$\mathrm{Hom}_{\mathcal{K}}(P, \sigma^\vee)^\vee \cong \mathrm{Hom}_{\mathcal{K}}(P, \sigma'^\vee)^\vee$$

as E-modules when they are both non-zero.

Lemma 5.9. *Let $M \in \mathfrak{C}(\mathcal{G})$ be a coadmissible quotient of P. Then $\mathrm{Hom}_{\mathfrak{C}(\mathcal{G})}(P, M) \otimes_E P$ is also coadmissible.*

Proof. Write $\mathrm{m} = \mathrm{Hom}_{\mathfrak{C}(\mathcal{G})}(P, M)$ and let Ker be the kernel of the surjective morphism

$$\mathrm{ev} : \mathrm{m} \otimes_E P \twoheadrightarrow M.$$

As in the proof of Proposition 3.20, we have $\mathrm{Hom}_{\mathfrak{C}(\mathcal{G})}(P, \mathrm{Ker}) = 0$, that is $\pi^\vee$ does not occur in Ker.

We need to show that $\mathrm{Hom}_{\mathcal{K}}(\mathrm{m} \otimes_E P, \sigma^\vee)$ is finite dimensional for any irreducible $\sigma \in \mathrm{Rep}_{\mathbb{F}}(\mathcal{K})$. Since

$$\mathrm{Hom}_{\mathcal{K}}(\mathrm{m} \otimes_E P, \sigma^\vee)^\vee \cong \mathrm{m} \otimes_E \mathrm{Hom}_{\mathcal{K}}(P, \sigma^\vee)^\vee$$

it suffices to consider those σ such that $\mathrm{Hom}_{\mathcal{K}}(P, \sigma^\vee)^\vee \neq 0$, or equivalently $\mathrm{Hom}_K(P_{\pi_i^\vee}, \sigma_i^\vee)^\vee \neq 0$ for all i. The rest of the proof is identical to the proof of Proposition 3.20, using Lemma 5.8. □

Proposition 5.10. *Let $M \in \mathfrak{C}(\mathcal{G})$ be a coadmissible quotient of $P = P_{\pi^\vee}$. Then there exist $f_1, ..., f_r \in \mathfrak{m}_R$ such that*

(i) *$f_1, ..., f_r$ is an R-sequence and also a P-sequence;*
(ii) *M is a quotient of $P/(f_1, ..., f_r)P$;*
(iii) *$P/(f_1, ..., f_r)P$ is a finite free Λ-module.*

Proof. By Lemma 5.9(i), we may assume $M = \mathrm{m} \otimes_E P$, where $\mathrm{m} = \mathrm{Hom}_{\mathfrak{C}(\mathcal{G})}(P, M)$. The projectivity of P implies that m is a cyclic E-module; let $\mathfrak{a}$ be the annihilator.

Let $\widetilde{\sigma} = \oplus_\sigma \sigma$ where the sum is taken for all the irreducible $\sigma \in \mathrm{Rep}_{\mathbb{F}}(\mathcal{K})$ such that $\mathrm{Hom}_{\mathcal{K}}(P, \sigma^\vee) \neq 0$. Then we have

$$\mathrm{Hom}_{\mathcal{K}}(M, \widetilde{\sigma}^\vee)^\vee \cong \mathrm{m} \otimes_E \mathrm{Hom}_{\mathcal{K}}(P, \widetilde{\sigma}^\vee)^\vee.$$

By Lemma 5.8, $\mathrm{Hom}_{\mathcal{K}}(P, \widetilde{\sigma}^\vee)^\vee$ is a Cohen-Macaulay E-module of dimension s. Since M is coadmissible, $\mathrm{Hom}_{\mathcal{K}}(M, \widetilde{\sigma}^\vee)^\vee$ is finite dimensional. So we may find $f_1, ..., f_r \in \mathfrak{a}$ which form a regular sequence for $\mathrm{Hom}_{\mathcal{K}}(P, \widetilde{\sigma}^\vee)^\vee$. As in the proof of Theorem 3.23 we may modify f_i so that they all lie in $\mathfrak{a} \cap R$. Using repeatedly Proposition 3.13 we obtain the condition (iii) and that $f_1, ..., f_r$ is a P-sequence.

We are left to check that $f_1, ..., f_r$ is an R-sequence. When $r = 1$, this is trivial because $f_1 \neq 0$ and R is an integral domain. In general case, we first observe that for any $1 \leq i \leq r$ there is a natural isomorphism

$$E/(f_1, ..., f_i) \cong \mathrm{End}_{\mathfrak{C}(\mathcal{G})}(P/(f_1, ..., f_i)). \tag{5.1}$$

Indeed, since $f_1, ..., f_r$ lie in the center of E, the proof of [15, Lem. 7.11] shows that any morphism $P \to P/(f_1, ..., f_{i-1})$ factors through $P/(f_1, ..., f_{i-1}) \to P/(f_1, ..., f_{i-1})$. Since P is projective, the exact sequence

$$0 \to P/(f_1, ..., f_{i-1}) \xrightarrow{f_i} P/(f_1, ..., f_{i-1}) \to P/(f_1, ..., f_i) \to 0$$

induces an isomorphism $\mathrm{End}_{\mathfrak{C}(\mathcal{G})}\big(P/(f_1,...,f_{i-1})\big)/f_i \cong \mathrm{End}_{\mathfrak{C}(\mathcal{G})}(P/(f_1,...,f_i))$. An obvious induction then implies (5.1). In particular, $E/(f_1,...,f_i)$ acts faithfully on $P/(f_1,...,f_i)$. This implies (i) because $f_1,...,f_r$ is a P-sequence. □

Corollary 5.11. *Let $M \in \mathfrak{C}(\mathcal{G})$ be coadmissible. Then there exists a resolution of M*

$$P_\bullet \to M \to 0$$

where $P_i \in \mathfrak{C}(\mathcal{G})$ are finite free Λ-modules.

Proof. Since M is coadmissible, the proof of Lemma 4.3 implies that it admits a finite filtration such that each of the graded pieces is a (coadmissible) quotient of $P_{\pi^\vee}$ for some irreducible $\pi \in \mathrm{Rep}_{\mathbb{F}}(\mathcal{G})$. By the horseshoe lemma in homological algebra, we may assume M is just a coadmissible quotient of $P_{\pi^\vee}$, in which case we conclude (inductively) by Proposition 5.10. □

Remark 5.12. *The above result can be viewed as a generalization (in a weak form) of the construction of Breuil-Paškūnas* [6]. *Note that when $r \geq 2$ it is not clear (to the author) how to generalize the construction to $\mathcal{G}$.*

5.1.3. *Generalized key lemma.* The next lemma generalizes the key Lemma 4.15. It is where the quantity $\kappa(\mathbf{n}) = \max_i\{n_i\}$ comes.

Lemma 5.13. *Let M be a finitely generated* $\max_i\{n_i\}$*Lambda-module and $\phi \in \mathrm{End}_\Lambda(M)$. Assume that $\bigcap_{n\geq 1}\phi^n(M) = 0$.*

(i) *If ϕ is nilpotent, then $\dim_{\mathbb{F}} M_{\mathcal{I}_1(p^{\mathbf{n}})} \sim \dim_{\mathbb{F}}\big(M/\phi(M)\big)_{\mathcal{I}_1(p^{\mathbf{n}})}$.*
(ii) *If ϕ is not nilpotent, then for $k_0 \gg 1$,*

$$\dim_{\mathbb{F}} M_{\mathcal{I}_1(p^{\mathbf{n}})} \ll \max\left\{\dim_{\mathbb{F}}\big(M/\phi(M)\big)_{\mathcal{I}_1(p^{\mathbf{n}})}, p^{\kappa(\mathbf{n})}\dim_{\mathbb{F}}\big(\phi^{k_0}(M)/\phi^{k_0+1}(M)\big)_{\mathcal{I}_1(p^{\mathbf{n}})}\right\}$$

where $\kappa(\mathbf{n}) := \max_i\{n_i\}$.

Proof. The proof is identical to that of Lemma 4.15, except that we need $k \geq 3p^{\kappa(\mathbf{n})}$ to guarantee that J^k is contained in the two-sided ideal of Λ generated by the maximal ideal of $\mathbb{F}[[\mathcal{I}_1(p^{\mathbf{n}})/\mathcal{Z}_1]]$. Here J denotes the maximal ideal of Λ. □

5.1.4. *Principal series.*

Definition 5.14. *Let S be a subset of $\{1,...,r\}$. Given irreducible $\pi_i \in \mathrm{Rep}_{\mathbb{F}}(G)$ for each $i \notin S$ and a character $\eta_i \in \mathrm{Rep}_{\mathbb{F}}(T)$ for each $i \in S$, we define $P(\pi_i,\eta_i,S) \in \mathfrak{C}(\mathcal{G})$ by*

$$P(\pi_i,\eta_i,S) = \big(\widehat{\otimes}_{i\notin S}P_{\pi_i^\vee}\big)\widehat{\otimes}\big(\widehat{\otimes}_{i\in S}M_{\eta_i^\vee}\big).$$

Denote by E_S the ring $\mathrm{End}_{\mathfrak{C}(\mathcal{G})}(P(\pi_i,\eta_i,S))$. Then

$$E_S \cong \big(\widehat{\otimes}_{i\notin S}E_{\pi_i^\vee}\big)\widehat{\otimes}\big(\widehat{\otimes}_{i\in S}E_{\eta_i^\vee}\big).$$

If $i \in S$, let $\pi_i := \mathrm{soc}_G\pi_{\eta_i}$ and let $P_{\pi_i^\vee}$ be a projective envelope of π_i so that $M_{\eta_i^\vee}$ becomes naturally a quotient of $P_{\pi_i^\vee}$. Set $P = \widehat{\otimes}_i P_{\pi_i^\vee}$, $E = \widehat{\otimes}_i E_{\pi_i^\vee}$ and $R = Z(E)$. Then E_S is naturally a quotient of E. Let R_S be the image of R in E_S (note that R_S might be smaller that the center of E_S.) Then R_S is Cohen-Macaulay of Krull dimension $3r - |S|$. Moreover, if $\sigma = \otimes_i\sigma_i \in \mathcal{D}(\otimes_i\pi_i)$, then

$$\mathrm{Hom}_{\mathcal{K}}(P,\sigma^\vee)^\vee \cong \mathrm{Hom}_{\mathcal{K}}(P(\pi_i,\eta_i,S),\sigma^\vee)^\vee$$

which is a cyclic E_S-module and a Cohen-Macaulay E_S-module of Krull dimension r by Corollary 3.11.

Lemma 5.15. *With the above notation, if $f_1,...,f_r \in R$ is a regular sequence for $\mathrm{Hom}_{\mathcal{K}}(P,\sigma^\vee)^\vee$, then they are R_S-regular.*

Proof. Although it is possible to prove the result using a similar argument as in Proposition 5.10, we instead do it via a commutative algebra argument based on the following observation. Let $J'_\sigma \subset R_S$ be the annihilator of $\mathrm{Hom}_{\mathcal{K}}(P,\sigma^\vee)^\vee$ (viewed as an R_S-module). Then J'_σ is a prime ideal of height $2r-|S|$. If we can find a sequence of elements $g_1,...,g_{2r-|S|}\in J'_\sigma$ such that

$$J'_\sigma/(g_1,...,g_{2r-|S|})$$

has finite length, then we are done. Indeed, the latter condition implies that (we still use f_i to denote its image in R_S)

$$g_1,...,g_{2r-|S|},f_1,...,f_r$$

form a system of parameters of R_S. Since R_S is Cohen-Macaulay of dimension $3r-|S|$, the sequence is in particular R_S-regular, hence $f_1,...,f_r$ is also R_S-regular. Finally, it is easy to construct such elements g_i by Lemma 3.12. □

Proposition 5.16. *Let M be a coadmissible quotient of $P(\pi_i,\eta_i,S)$. Then M is torsion over Λ and*

$$\dim_{\mathbb{F}} M_{\mathcal{I}_1(p^{\mathbf{n}})} \ll \kappa(\mathbf{n})^r p^{(2r-|S|)\kappa(\mathbf{n})}.$$

Proof. We may view M as a coadmissible quotient of P, hence by Proposition 5.10 we find an R-sequence $f_1,...,f_r$ which annihilate M. As in the proof of Lemma 5.15, we may complete $f_1,...,f_r$ by $g_1,...,g_{2r-|S|}$ to obtain a system of parameters of R_S. Since $M/(g_1,...,g_{2r-|S|})$ has finite length, we conclude by Lemma 5.13 and Lemma 5.7. □

Lemma 5.17. *Let $M\in\mathfrak{C}(\mathcal{G})$ be a coadmissible quotient of P. If M is torsion, then so is $\mathrm{Hom}_{\mathfrak{C}(\mathcal{G})}(P,M)\otimes_E P$.*

Proof. Let Ker be the kernel of the natural surjection

$$\mathrm{Hom}_{\mathfrak{C}(\mathcal{G})}(P,M)\otimes_E P\twoheadrightarrow M.$$

Then $\mathrm{Hom}_{\mathfrak{C}(\mathcal{G})}(P,\mathrm{Ker})=0$, i.e. $\pi^\vee$ does not occur in Ker. If all the $\mathfrak{B}_i$ are of type (I) or (III), then this implies $\mathrm{Ker}=0$. Let S be the indices i such that $\mathfrak{B}_i$ is of type (II) or (IV).

As in the proof of Proposition 3.20(ii), it is enough to show the following:

Claim: For $i\in S$, let $\pi'_i\in\mathfrak{B}_i$ be distinct with π_i and let Q'_i be the maximal quotient of $P_{\pi_i'^\vee}$ none of whose irreducible subquotients is isomorphic to $\pi_i^\vee$. Then

$$Q'_i\widehat{\otimes}\big(\widehat{\otimes}_{j\in S,j\neq i}P_{\pi_j'^\vee}\big)\widehat{\otimes}\big(\widehat{\otimes}_{j\notin S}P_{\pi_j^\vee}\big)$$

is torsion.

By Remark 3.21, each Q'_i has a finite filtration with graded pieces being subquotients of $M_{\eta^\vee}$. The claim then follows from Proposition 5.16. □

5.1.5. *Torsion free vs free.* We fix an irreducible representation $\pi=\otimes_i\pi_i\in\mathrm{Rep}_{\mathbb{F}}(\mathcal{G})$ and let $\mathfrak{B}$ (resp. $\mathfrak{B}_i$) be the block of π (resp. π_i). Let $P=\widehat{\otimes}_iP_{\pi_i^\vee}$, $E=\widehat{\otimes}_iE_{\pi_i^\vee}$.

In the rest, we make the following assumption:

(**H**) if $\mathfrak{B}_i$ is of type (IV), then $\pi_i\cong\mathbf{1}_G$.

Definition 5.18. *Let $M\in\mathfrak{C}(\mathcal{G})^{\mathfrak{B}}$. We say M is* of Cohen-Macaulay type *(**abbr.** CM-type) if M admits an exhaustive decreasing filtration*

$$M=\mathrm{Fil}^0M\supsetneq\mathrm{Fil}^1M\supsetneq\cdots\supsetneq\mathrm{Fil}^kM\supsetneq\cdots$$

such that each of the graded pieces is of the shape $\otimes_{i=1}^r\lambda_i$, where $\lambda_i\in\mathfrak{C}(G)^{\mathfrak{B}_i}$ and

- *either λ_i is irreducible, not of type (IV),*
- *or $\lambda_i\in\{\mathrm{Sp}^\vee,\kappa^\vee\}$, where we recall that κ is defined in* (3.15).

By definition, if M is CM-type, then $M=\varprojlim_k M_k$ with M_k being quotients of finite length and CM-type.

Lemma 5.19. *Let $\lambda=\otimes_{i=1}^r\lambda_i$ be a graded piece as in Definition 5.18. Let $S_\kappa\subset\{1,...,r\}$ be the set of indices i such that $\lambda_i=\kappa^\vee$. Then the G-socle (resp. G-cosocle) of λ is*

$$\big(\otimes_{i\notin S_\kappa}\lambda_i\big)\otimes\big(\otimes_{i\in S_\kappa}\pi_\alpha^\vee\big),\quad (\text{resp. } \big(\otimes_{i\notin S_\kappa}\lambda_i\big)\otimes\big(\otimes_{i\in S_\kappa}\mathbf{1}_G^\vee\big)).$$

Proof. This is clear from the definition of κ. □

Lemma 5.20. *Let $M\in\mathfrak{C}(\mathcal{G})^{\mathfrak{B}}$ be non-zero, of finite length and CM-type. Then as a Λ-module M is Cohen-Macaulay of projective dimension $2r$.*

Proof. We may assume M is of the form $\otimes_{i=1}^r\lambda_i$ with notation in Definition 5.18, in which case the result follows from Theorem 3.2 and the fact that $\kappa^\vee$ is Cohen-Macaulay, see the proof of Proposition 3.34. □

Lemma 5.21. *Let $M\in\mathfrak{C}(\mathcal{G})^{\mathfrak{B}}$ be of finite length and CM-type, and let M' be any submodule of M. Let $h_{\mathbf{1}}(M')$ (resp. $h_\alpha(M')$) denote the number of Jordan-Hölder factors of M' which are of the form $\otimes_i\pi_i'^\vee$ with $\pi_i'\notin\{\pi_\alpha\}$ (resp. $\pi_i'\notin\{\mathbf{1}_G\}$). Then $h_\alpha(M')\geq h_{\mathbf{1}}(M')$. Moreover, the equality holds if and only if M' is CM-type, in which case M/M' is CM-type too.*

Proof. Take a CM-type filtration $\{\mathrm{Fil}^kM,\ k\geq 0\}$ of M as in Definition 5.18. It is enough to check the inequality for each graded piece $\otimes_i\lambda_i$. This follows from Lemma 5.19. Moreover, if M' is CM-type, then the equality holds.

Now assume $h_\alpha(M')=h_{\mathbf{1}}(M')$. We need to show M' is CM-type. Consider the induced filtration $\mathrm{Fil}^kM':=M'\cap\mathrm{Fil}^kM$ of M'. Each graded piece gr^kM' is a submodule of gr^kM. Since $h_\alpha(M')=h_{\mathbf{1}}(M')$, Lemma 5.19 shows that either $\mathrm{gr}^kM'=0$ or $\mathrm{gr}^kM'=\mathrm{gr}^kM$. That is, $\{\mathrm{Fil}^kM',\ k\geq 0\}$ can be refined to be a CM-type filtration of M' and the quotient filtration of M/M' is also CM-type. □

Lemma 5.22. *(i) Being of CM type is stable under extensions.*

(ii) Let $M,M'\in\mathfrak{C}(\mathcal{G})^{\mathfrak{B}}$ be CM-type and $\phi:M\to M'$ be a morphism in $\mathfrak{C}(\mathcal{G})$. Then $\mathrm{Ker}(\phi)$*,* $\mathrm{Im}(\phi)$*,* $\mathrm{Coker}(\phi)$ *are CM-type.*

Proof. (i) is obvious.

(ii) We may choose CM-type quotients of finite length M_i (resp. M_i') of M (resp. M') such that $M=\varprojlim_i M_i$ (resp. $M'=\varprojlim_i M_i'$) and that ϕ is induced from compatible morphisms $M_i\to M_i'$. Hence it suffices to prove the result for M,M' both of finite length. Consider the exact sequence $0\to\mathrm{Ker}(\phi)\to M\to\mathrm{Im}(\phi)\to 0$. By Lemma 5.21, we have

$$h_\alpha(\mathrm{Im}(\phi))\geq h_{\mathbf{1}}(\mathrm{Im}(\phi)),\quad h_\alpha(M)=h_{\mathbf{1}}(M),\quad h_\alpha(\mathrm{Ker}(\phi))\geq h_{\mathbf{1}}(\mathrm{Ker}(\phi)),$$

hence $h_\alpha(\mathrm{Im}(\phi))=h_{\mathbf{1}}(\mathrm{Im}(\phi))$ and $h_\alpha(\mathrm{Ker}(\phi))=h_{\mathbf{1}}(\mathrm{Ker}(\phi))$. Applying the same lemma gives the result. □

Lemma 5.23. *Let $\pi\in\mathrm{Rep}_{\mathbb{F}}(\mathcal{G})$ be irreducible satisfying (**H**) and m be a finitely generated E-module. Then for any $i\geq 0$, $\mathrm{Tor}_i^E(\mathrm{m},P_{\pi^\vee})$ is CM-type.*

Proof. We first treat the case when m is of finite length. By induction on the length of m and using Lemma 5.22, it suffices to show $\mathrm{Tor}_i^E(\mathbb{F},P)$ are CM-type. This follows from the Künneth formula and Propositions 3.30 and 3.33 (under the assumption (**H**)).

For general m, we note that $\mathrm{m}\cong\varprojlim_k \mathrm{m}/\mathfrak{m}_E^k$ where $\mathfrak{m}_E$ is the maximal ideal of E. Therefore, $\mathrm{m}\otimes_E P$ is isomorphic to $\varprojlim_k\big((\mathrm{m}/\mathfrak{m}_E^k)\otimes_E P\big)$, hence is CM-type. The result for $i\geq 1$ follows from a dimension-shifting argument. □

Proposition 5.24. *Let $\pi = \otimes_i \pi_i$ be irreducible satisfying (**H**). Let $M \in \mathfrak{C}$ be a non-zero coadmissible quotient of P. If M is torsion-free as a Λ-module then M is free.*

Proof. The proof is similar to that of Proposition 4.17. By assumption $j_\Lambda(M) = 0$, so to prove M is free is equivalent to prove M is a Cohen-Macaulay Λ-module. Since M is coadmissible, there exist $f_1, ..., f_r \in \mathfrak{m}_R$ as in Proposition 5.10. We complete them by $g_1, ..., g_{2r} \in \mathfrak{m}_R$ to obtain a system of parameters. As in the proof of Proposition 4.17, $g_1, ..., g_{2r}$ form a regular sequence for M, so by Lemma 2.5 we are left to show $M/(g_1, ..., g_{2r})M$ is Cohen-Macaulay, or even $M/(g_1^{k_1}, ..., g_{2r}^{k_{2r}})M$ is Cohen-Macaulay for some $k_i \geq 1$.

Letting $\mathrm{m} := \mathrm{Hom}_{\mathfrak{C}(\mathcal{G})}(P, M)$, we have an exact sequence induced by the evaluation morphism (with Ker being the kernel)

$$0 \to \mathrm{Ker} \to \mathrm{m}\otimes_E P \overset{\mathrm{ev}}{\to} M \to 0. \tag{5.2}$$

By Lemma 5.17, Ker is a torsion Λ-module.

Consider the action of g_1 on Ker. Let $\mathrm{Ker}[g_1^\infty]$ be the submodule of elements on which g_1 acts nilpotently; this is an object in $\mathfrak{C}(\mathcal{G})$. Since Ker is coadmissible, so is $\mathrm{Ker}[g_1^\infty]$, so we may find $k_1 \gg 0$ such that

$$\mathrm{Ker}[g_1^\infty] = \mathrm{Ker}[g_1^{k_1}].$$

Consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ker} & \longrightarrow & \mathrm{m}\otimes_E P & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle g_1^{k_1}} & & \downarrow{\scriptstyle g_1^{k_1}} & & \downarrow{\scriptstyle g_1^{k_1}} & & \\ 0 & \longrightarrow & \mathrm{Ker} & \longrightarrow & \mathrm{m}\otimes_E P & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

which implies by the Snake lemma that $\mathrm{Ker}[g_1^{k_1}] \cong (\mathrm{m}\otimes_E P)[g_1^{k_1}]$ and an exact sequence

$$0 \to \mathrm{Ker}/g_1^{k_1}\mathrm{Ker} \to (\mathrm{m}/g_1^{k_1})\otimes_E P \to M/g_1^{k_1}M \to 0. \tag{5.3}$$

Observe that $\mathrm{m}\otimes_E P$ is CM-type by Lemma 5.23. Hence so is $(\mathrm{m}\otimes_E P)[g_1^{k_1}]$ by Lemma 5.22. We have the following two possibilities:

(1) g_1 is nilpotent on Ker, i.e. $g_1^{k_1} = 0$ on Ker. Then $\mathrm{Ker} = \mathrm{Ker}[g_1^{k_1}]$ is CM-type, so (5.3) implies that $M/g_1^{k_1}M$ is CM-type using Lemma 5.22 and Lemma 5.23. By Lemma 5.22 again, $M/(g_1^{k_1}, g_2, ..., g_{2r})M$ is also CM-type. However, $M/(g_1^{k_1}, ..., g_{2r})M$ is of finite length because $g_1^{k_1}, ..., g_{2r}$ form an M-sequence, so it is a Cohen-Macaulay Λ-module by Lemma 5.20. As remarked above, we deduce that M is a free Λ-module by Lemma 2.5.

(2) g_1 is not nilpotent on Ker. In this case, the induced morphism

$$g_1^{k_1} : \mathrm{Ker}/\mathrm{Ker}[g_1^{k_1}] \to \mathrm{Ker}/\mathrm{Ker}[g_1^{k_1}]$$

is injective and we let Ker_1 be its cokernel. Similarly, let Σ_1 be the cokernel of

$$g_1^{k_1} : \mathrm{m}\otimes_E P/(\mathrm{m}\otimes_E P)[g_1^{k_1}] \to \mathrm{m}\otimes_E P/(\mathrm{m}\otimes_E P)[g_1^{k_1}].$$

Then we have the following facts:
- $\delta_\Lambda(\mathrm{Ker}_1) \leq \delta_\Lambda(\mathrm{Ker}) - 1$,
- Σ_1 is CM-type by Lemma 5.22,
- setting $M_1 := M/g_1^{k_1}M$, (5.3) induces an exact sequence

$$0 \to \mathrm{Ker}_1 \to \Sigma_1 \to M_1 \to 0.$$

Recall that it is enough to show M_1 is a Cohen-Macaulay Λ-module. Continue the above argument, that is, consider the action of g_2 on Ker_1, and so on. Then, provided that g_i does not act nilpotently on Ker_{i-1}, we obtain an exact sequence

$$0 \to \mathrm{Ker}_i \to \Sigma_i \to M_i \to 0$$

with $\delta_\Lambda(\mathrm{Ker}_i) \le \delta_\Lambda(\mathrm{Ker}_{i-1}) - 1$. However, since Ker is torsion, we eventually arrive at some i such that g_i acts nilpotently on Ker_{i-1} and conclude as in (1).

□

5.2. **The proof.**

Theorem 5.25. *Let $\pi = \otimes_{i=1}^r \pi_i \in \mathrm{Rep}_{\mathbb{F}}(\mathcal{G})$ be irreducible with $\delta_\Lambda(\pi^\vee) = r$. Let $P = P_{\pi^\vee}$ and $M \in \mathfrak{C}(\mathcal{G})$ be a non-zero coadmissible quotient of P. Then*

$$\dim_{\mathbb{F}} M_{\mathcal{T}_1(p^{\mathbf{n}})} \ll \kappa(\mathbf{n})^r p^{(\delta_\Lambda(M)-r)\kappa(\mathbf{n})}.$$

Proof. Being non-zero, M admits $\pi^\vee$ as its G-cosocle, so we have $\delta_\Lambda(M) \ge r$. We do induction on $\delta_\Lambda(M)$. If $\delta_\Lambda(M) = r$, then Lemma 5.5 implies that M has finite length and the result follows from Lemma 5.7. Assume the result is proven for all M of canonical dimension $\le t$ and treat the case $\delta_\Lambda(M) = t+1$ below.

We know that R is a Cohen-Macaulay local ring of Krull dimension $3r$ by Theorem 3.5. Since M is coadmissible, there exist $f_1, ..., f_r \in \mathfrak{m}_R$ as in Proposition 5.10. We complete them by $g_1, ..., g_{2r} \in \mathfrak{m}_R$ to obtain a system of parameters.

For $i \ge 1$, set $M_i := M/(g_1, ..., g_i)M$ and consider the endomorphism $g_i : M_{i-1} \to M_{i-1}$, where $M_0 := M$. If $g_1 : M \to M$ is nilpotent, then by Proposition 2.3 and Lemma 5.13 we have

$$\delta_\Lambda(M) = \delta_\Lambda(M_1), \quad \dim_{\mathbb{F}} M_{\mathcal{T}_1(p^{\mathbf{n}})} \sim \dim_{\mathbb{F}}(M_1)_{\mathcal{T}_1(p^{\mathbf{n}})}$$

so it suffices to prove the result for M_1. For this reason we assume, without loss of generality, that we are in the following setting: there exist $g_1, ..., g_{r'} \in R$ (with $1 \le r' \le 2r$) such that $M/(g_1, ..., g_{r'})M \in \mathfrak{C}(\mathcal{G})$ is of finite length and that g_1 is *not* nilpotent on M.

By Proposition 2.3, there exists $k_0 \gg 1$ such that

$$\delta_\Lambda(M) = \max\{\delta_\Lambda(M_1), \delta_\Lambda(g_1^{k_0}M/g_1^{k_0+1}M) + 1\}.$$

In particular, $\delta_\Lambda(g_1^{k_0}M/g_1^{k_0+1}M) \le \delta_\Lambda(M) - 1 = t$. The inductive hypothesis then implies that $\dim_{\mathbb{F}}(g_1^{k_0}M/g_1^{k_0+1}M)_{\mathcal{T}_1(p^{\mathbf{n}})} \ll \kappa(\mathbf{n})^r p^{(t-r)\kappa(\mathbf{n})}$, and by Lemma 5.13

$$\dim_{\mathbb{F}}(g_1^{k_0}M)_{\mathcal{T}_1(p^{\mathbf{n}})} \ll \kappa(\mathbf{n})^r p^{(t+1-r)\kappa(\mathbf{n})}.$$

Therefore, to prove the result it suffices to show

$$(M_1)_{\mathcal{T}_1(p^{\mathbf{n}})} \ll \kappa(\mathbf{n})^r p^{(t+1-r)\kappa(\mathbf{n})}.$$

If $\delta_\Lambda(M_1) = t$, it follows from the inductive hypothesis. If $\delta_\Lambda(M_1) = t+1$, we consider the action of $g_2, ..., g_{r'}$ on M_1 and conclude by an induction on r' (i.e. the length of the sequence). □

Remark 5.26. *If we let $M = P/(f_1, ..., f_r, g_1, ..., g_{r'})P$ with $1 \le r' \le 2r$, then $\delta_\Lambda(M) = 3r - r'$ (for this we don't need assume $\delta_\Lambda(\pi^\vee) = r$ because $\delta_\Lambda(\mathbb{F} \otimes_R P)$ is always equal to r). Assume moreover that $\mathbf{n}$ is parallel, i.e. $\mathbf{n} = (n, ..., n)$. The bound established in Theorem 5.25 is tight in the sense that we have the following lower bound:*

$$\dim_{\mathbb{F}} M_{\mathcal{T}_1(p^{\mathbf{n}})} \gg_\epsilon p^{(\delta(M)-r)n-\epsilon} = p^{(2r-r')n-\epsilon}, \quad \forall \epsilon > 0.$$

Indeed, otherwise Lemma 5.13 would imply for some $\epsilon_0 > 0$

$$\dim_{\mathbb{F}} \big(P/(f_1, ..., f_r)P\big)_{\mathcal{T}_1(p^{\mathbf{n}})} \ll p^{2r-\epsilon_0},$$

which is not the case because $P/(f_1, ..., f_r)P$ is a finite free Λ-module.

We are ready to prove Theorem 5.1 whose statement we recall.

Theorem 5.27. *Let $M \in \mathfrak{C}(\mathcal{G})$ and assume it is finitely generated of rank s as a Λ-module. Then for some constant $c > 0$ depending on M,*

$$\dim_{\mathbb{F}} M_{\mathcal{T}_1(p^{\mathbf{n}})} \leq s|\mathcal{K}_1 : \mathcal{T}_1(p^{\mathbf{n}})| + c \cdot \kappa(\mathbf{n})^r p^{(2r-1)\kappa(\mathbf{n})} \tag{5.4}$$

$$\dim_{\mathbb{F}} H_i(\mathcal{T}_1(p^{\mathbf{n}})/\mathcal{Z}_1, M) \ll \kappa(\mathbf{n})^r p^{(2r-1)\kappa(\mathbf{n})}. \tag{5.5}$$

where $\kappa(\mathbf{n}) := \max_i\{n_i\}$.

Proof. (i) We first assume M is torsion. Since $\dim \mathcal{G} = 3r$, we get $\delta_\Lambda(M) \leq 3r - 1$. As in Lemma 4.3, we may assume M is a (coadmissible) quotient of $P = P_{\pi^\vee}$ for some irreducible $\pi \in \mathrm{Rep}_{\mathbb{F}}(\mathcal{G})$. If $\delta_\Lambda(\pi^\vee) = r$, then the result is a special case of Theorem 5.25.

Assume in the rest $\delta_\Lambda(\pi^\vee) < r$. This amounts to saying that the subset $S \subset \{1, ..., r\}$ of indices i such that π_i is one-dimensional is non-empty; indeed, we have $\delta_\Lambda(\pi^\vee) = r - |S|$ by Lemma 5.2(ii). Up to twist by a suitable central character, we may assume $\pi_i = \mathbf{1}_G$ for $i \in S$. We do induction on $|S|$. Assume $1 \in S$ without loss of generality. Define $P' = \widehat{\otimes}_i P'_i$ by

$$P'_1 = P_{\pi_\alpha^\vee} \ \text{ and } \ P'_i = P_{\pi_i^\vee}, \ i \neq 1.$$

Then we have a natural inclusion $P' \hookrightarrow P$ by (3.7), with quotient isomorphic to $P(\pi_i, \alpha, \{1\})$, see Definition 5.14. This allows to divide M into two parts: a submodule M' being image of P' and the corresponding quotient $M'' = M/M'$. We obtain the required bound by Theorem 5.25, Proposition 5.16 and the inductive hypothesis.

Now we prove (5.4) in general. We may assume M is torsion-free. Let $S' \subset \Sigma$ be the set of indices i such that $\pi_i \in \{\mathrm{Sp}, \pi_\alpha\}$. We do induction on the quantity $s + |S'|$. The case when $S' = \emptyset$ follows from Proposition 5.24 which implies that M is a free Λ-module. If $S' \neq \emptyset$, say $1 \in S'$ without loss of generality, then we may divide M into

$$0 \to M' \to M \to M'' \to 0$$

in such a way that

- M' is a quotient of $P' = \widehat{\otimes}_i P'_i$ where
$$P'_1 = P_{\mathbf{1}_G^\vee} \ \text{ and } \ P'_i = P_{\pi_i^\vee}, \ i \neq 1,$$
and the inductive hypothesis implies that (5.4) holds for M';
- M'' has rank $< s$, hence also verifies (5.4) by induction (remark that M'' is not necessarily torsion-free, in which case we split it as $0 \to M''_{\mathrm{tor}} \to M'' \to M''_{\mathrm{tf}} \to 0$ and apply the inductive hypothesis to M''_{tf}).

This completes the proof of (5.4).

(ii) Finally, using Corollary 5.11, (5.5) is proved by the same argument as in Theorem 4.18. □

5.3. **Change of groups.** We keep the notation in the previous subsection. For $n \geq 1$, let

$$K_1(p^n) := K_1 \cap K_0(p^n K_1(p^n)) = \begin{pmatrix} 1 + p\mathbb{Z}_p & p\mathbb{Z}_p \\ p^n\mathbb{Z}_p & 1 + p\mathbb{Z}_p \end{pmatrix}.$$

These groups are closely related to the $T_1(p^n)$ in the sense that letting $D = \left(\begin{smallmatrix} 1 & 0 \\ 0 & p^{\lfloor n/2 \rfloor} \end{smallmatrix}\right)$ and $n' = \lfloor n/2 \rfloor + 1$, we have

$$D^{-1}K_1(p^n)D < T_1(p^{n'}), \quad |T_1(p^{n'}) : D^{-1}K_1(p^n)D| \leq p, \quad |n' - n/2| \leq 1. \tag{5.6}$$

On the other hand, using (essentially) the fact $K_1/K_1(p^n) \cong p\mathbb{Z}/p^n\mathbb{Z}$, Marshall proved the following interesting result ([19, Cor. 14]).

Lemma 5.28. *Let $L \subset \mathbb{F}[K_1/K_1(p^n)]$ be a submodule of dimension d, and let the base p expansion of d be written as*

$$d = \sum_{i=1}^{l} p^{\alpha(i)}$$

where $\alpha(i)$ is a non-increasing sequence of non-negative integers. Then there exists a filtration $0 = L_0 \subset \cdots \subset L_l = L$ of L by submodules L_i such that $L_i/L_{i-1} \cong \mathbb{F}[K_1/K_1(p^{\alpha(i)+1})]$.

If $\mathbf{n} = (n_1, ..., n_r) \in (\mathbb{Z}_{\geq 1})^r$, let

$$\mathcal{K}_1(p^{\mathbf{n}}) = \prod_{i=1}^{r} K_1(p^{n_i}).$$

Theorem 5.29. *Let $M \in \mathfrak{C}(\mathcal{G})$ be coadmissible and torsion, and let L be any sub-representation of $\mathbb{F}[\mathcal{K}_1/\mathcal{K}_1(p^{\mathbf{n}})]$ which factorizes as $\otimes_{i=1}^r L_i$ with $L_i \subset \mathbb{F}[K_1/K_1(p^{n_i})]$. Then for $i \geq 0$ we have*

$$\dim_{\mathbb{F}} H_i(\mathcal{K}_1/\mathcal{Z}_1, M \otimes L) \ll \kappa(\mathbf{n})^{2r} p^{(r-\frac{1}{2})\kappa(\mathbf{n})}.$$

In particular, if $\mathbf{n} = (n, ..., n)$ is parallel, then

$$\dim_{\mathbb{F}} H_i(\mathcal{K}_1/\mathcal{Z}_1, M \otimes L) \ll n^{2r} p^{(r-\frac{1}{2})n}.$$

Proof. The proof goes as in that of [19, Lem. 19]. We explain this briefly. First, if $L = \mathbb{F}[\mathcal{K}_1/\mathcal{K}_1(p^{\mathbf{m}})]$ for some $\mathbf{m} \in (\mathbb{Z}_{\geq 1})^r$, we apply Shapiro's lemma to obtain

$$H_i(\mathcal{K}_1/\mathcal{Z}_1, M \otimes L) \cong H_i(\mathcal{K}_1(p^{\mathbf{m}})/\mathcal{Z}_1, M). \tag{5.7}$$

Using a diagonal element of $\mathcal{G}$, precisely

$$D = \left(\begin{pmatrix} 1 & 0 \\ 0 & p^{\lfloor m_1 \rfloor/2} \end{pmatrix}, \dots, \begin{pmatrix} 1 & 0 \\ 0 & p^{\lfloor m_r/2 \rfloor} \end{pmatrix} \right)$$

we obtain by (5.6) that for some $\mathbf{m'}$:

$$D^{-1}\mathcal{K}_1(p^{\mathbf{m}})D \leq \mathcal{T}_1(p^{\mathbf{m'}}), \quad |\mathcal{T}_1(p^{\mathbf{m'}}) : D^{-1}\mathcal{K}_1(p^{\mathbf{m}})D| \leq p^r, \quad |m'_i - m_i/2| \leq 1.$$

Since M carries a compatible action of $\mathcal{G}$, we have natural isomorphisms

$$H_i(\mathcal{K}_1(p^{\mathbf{m}})/\mathcal{Z}_1, M) \cong H_i(D^{-1}\mathcal{K}_1(p^{\mathbf{m}})D/\mathcal{Z}_1, M).$$

Hence we deduce from [19, Lem. 20] that

$$\dim_{\mathbb{F}} H_i(\mathcal{K}_1(p^{\mathbf{m}})/\mathcal{Z}_1, M) \leq p^r \dim_{\mathbb{F}} H_i(\mathcal{T}_1(p^{\mathbf{m'}})/\mathcal{Z}_1, M)$$

and the result follows from Theorem 5.1.

For general L, Lemma 5.28 provides a filtration of L

$$L = F_0 \supset F_1 \supset \cdots$$

such that every quotient F_i/F_{i+1} is isomorphic to $\mathbb{F}[\mathcal{K}_1/\mathcal{K}_1(p^{\mathbf{m}})]$ for some $\mathbf{m} \leq \mathbf{n}$ and each isomorphism class of quotient occurs at most p^r times. We then deduce from the first case that

$$\begin{aligned} \dim_{\mathbb{F}} H_i(\mathcal{K}_1, M \otimes L) &\leq p^r \sum_{\mathbf{m} \leq \mathbf{n}} \dim_{\mathbb{F}} H_i(\mathcal{K}_1(p^{\mathbf{m}})/\mathcal{Z}_1, M) \\ &\ll \sum_{\mathbf{m} \leq \mathbf{n}} \kappa(\mathbf{m})^r p^{(r-\frac{1}{2})\kappa(\mathbf{m})} \\ &\ll \kappa(\mathbf{n})^{2r} p^{(r-\frac{1}{2})\kappa(\mathbf{n})}. \end{aligned}$$

Here we use the fact that the cardinality of the set $\{\mathbf{m} : \mathbf{m} \leq \mathbf{n}\}$ is $\prod_{i=1}^r n_i$, hence bounded by $\kappa(\mathbf{n})^r$. □

6. Application

Let F be a number field of degree r, and r_1 (resp. $2r_2$) be the number of real (resp. complex) embeddings. Let $F_\infty = F \otimes_{\mathbb{Q}} \mathbb{R}$, so that $\mathrm{SL}_2(F_\infty) = \mathrm{SL}_2(\mathbb{R})^{r_1} \times \mathrm{SL}_2(\mathbb{C})^{r_2}$. Let K_∞ be the standard maximal compact subgroup of $\mathrm{SL}_2(F_\infty)$.

Let $\{\sigma_1, ..., \sigma_r\}$ be the set of complex embeddings of F and let $\mathbf{d} \in (\mathbb{Z}_{\geq 1})^r$ be an r-tuple indexed by the σ_i such that $d_i = d_j$ when σ_i and σ_j are complex conjugates. Let $W_{\mathbf{d}}$ be the representation of $\mathrm{SL}_2(F_\infty)$ obtained by forming the tensor product

$$\Big(\bigotimes_{\sigma_i \text{ real}} \mathrm{Sym}^{d_i}\mathbb{C}^2\Big) \bigotimes \Big(\bigotimes_{\{\sigma_i,\sigma_j\} \text{ complex}} \mathrm{Sym}^{d_i}\mathbb{C}^2 \otimes \overline{\mathrm{Sym}}^{d_j}\mathbb{C}^2\Big).$$

Theorem 6.1. *Let $Y = \mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A})/K_f K_\infty$ for some compact open subgroup $K_f \subset \mathrm{SL}_2(\mathbb{A}_f)$. If F is not totally real and $\mathbf{d} = (d_1, ..., d_r)$ as above, then*

$$\dim_{\mathbb{C}} H_i(Y, W_{\mathbf{d}}) \ll_\epsilon \kappa(\mathbf{d})^{r-1/2+\epsilon}$$

where $\kappa(\mathbf{d}) = \max_i\{d_i\}$.

Proof. The proof follows closely the one presented in [19]. We content ourselves with briefly explaining the main ingredients. Below we abuse the notation by letting the same letters to denote subgroups of SL_2 obtained by intersection from GL_2.

(1) By [19, Lem. 18], there exists a p-adic local system $V_{\mathbf{d}}$ defined over $\mathcal{O} = W(\mathbb{F})$, such that

$$\dim_{\mathbb{C}} H_i(Y, W_{\mathbf{d}}) = \dim_L H_i(Y, V_{\mathbf{d}}).$$

This need to choose a bijection between the set of complex places and p-adic places of F.

(2) Emerton's theory of complete homology gives a bound ([19, §6, (34),(35)])

$$\dim H_q(Y, V_{\mathbf{d}}) \leq \sum_{i+j=q} \dim H_i(\mathcal{K}_1/\mathcal{Z}_1, \widetilde{H}_{j,\mathbb{Q}_p} \otimes V_{\mathbf{d}})$$

where $\widetilde{H}_j$ is the j-th completed homology of Emerton with (trivial) coefficients in $\mathcal{O}$, and $\widetilde{H}_{j,\mathbb{Q}_p} = \widetilde{H}_j \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Note that $\widetilde{H}_j$ is a coadmissible module over $\mathcal{O}[[\mathcal{K}_1]]$ and carries a natural compatible action of $\prod_{i=1}^r \mathrm{SL}_2(\mathbb{Q}_p)$.

(3) Let $\mathbf{n} = (n_1, ..., n_r)$ where n_i is the smallest integer such that $p^{n_i} \geq d_i$ (resp. $p^{n_i} \geq d_i/2$) if σ_i is real (resp. complex). By [19, Lem. 17] we may choose lattice $\mathcal{V}_{d_i} \subset V_{d_i}$ such that $\mathcal{V}_{d_i}/p \subset \mathbb{F}[[K_1/K_1(p^{k_i})]]$. Let $L_{\mathbf{d}}$ be the reduction mod p of $\otimes_{i=1}^r \mathcal{V}_{d_i}$.

(4) Let M_j be the reduction modulo p of the image of $\widetilde{H}_j \to \widetilde{H}_{j,\mathbb{Q}_p}$. We then have

$$\dim_L H_i(\mathcal{K}_1/\mathcal{Z}_1, \widetilde{H}_{j,\mathbb{Q}_p} \otimes V_{\mathbf{d}}) \leq \dim_{\mathbb{F}} H_i(\mathcal{K}_1/\mathcal{Z}_1, M_j \otimes L_{\mathbf{d}}).$$

(5) Because $\mathrm{SL}_2(\mathbb{C})$ does not admit discrete series, the assumption that F is not totally real implies that $\widetilde{H}_{j,\mathbb{Q}_p}$ is a torsion $\mathcal{O}[[\mathcal{K}_1]] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$-module, see [8, Thm. 3.4]. So by Lemma 2.6, M_j is a torsion Λ-module. Therefore our Theorem 5.29 applies, via Lemma 4.19, and shows that

$$\dim_{\mathbb{F}} H_i(\mathcal{K}_1, M_j \otimes L_{\mathbf{d}}) \ll \kappa(\mathbf{n})^{2r} p^{(r-\frac{1}{2})\kappa(\mathbf{n})} \ll_\epsilon \kappa(\mathbf{d})^{r-\frac{1}{2}+\epsilon}.$$

□

Remark 6.2. *In* [19]*, Marshall considered a more general setting, allowing a subset of the weights d_i to be fixed and letting the others vary. We have restricted ourselves for two reasons. On the one hand, Theorem 6.1 provides interesting bounds only when* all *weights vary in a* uniform *way (because of the appearance of $\kappa(\mathbf{d})$), for example, when $\mathbf{d}$ is parallel. On the other hand, this already includes the most interesting cases: for example, when F is imaginary quadratic, we do have $d_1 = d_2$.*

Now we change our notation. Let Z_∞ be the centre of $\mathrm{GL}_2(F_\infty)$, K_f be a compact open subgroup of $\mathrm{GL}_2(\mathbb{A}_f)$ and let

$$X = \mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A})/K_f Z_\infty.$$

If $\mathbf{d} = (d_1, ..., d_{r_1+r_2})$ is an $(r_1 + r_2)$-tuple of positive even integers, let $S_{\mathbf{d}}(K_f)$ denote the space of cusp forms on X which are of cohomological type with weight $\mathbf{d}$. Then using the Eichler-Shimura isomorphism, see [19, §2.1], Theorem 6.1 can be restated as follows.

Theorem 6.3. *If F is not totally real then for any fixed K_f and $\mathbf{d} = (d_1, ..., d_{r_1+r_2})$ as above, we have*

$$\dim_{\mathbb{C}} S_{\mathbf{d}}(K_f) \ll_\epsilon \kappa(\mathbf{d})^{r-1/2+\epsilon}.$$

In particular, when $\mathbf{d} = (d, ..., d)$ is parallel, we obtain

$$\dim_{\mathbb{C}} S_{\mathbf{d}}(K_f) \ll_\epsilon d^{r-1/2+\epsilon}$$

which strengthens Corollary 2 of [19] by a power $d^{1/6}$.

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