ON SOME GENERALIZED RAPOPORT-ZINK SPACES

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Abstract. We enlarge the class of Rapoport-Zink spaces of Hodge type by modifying the centers of the associated p-adic reductive groups. These such-obtained Rapoport-Zink spaces are called of abelian type. The class of Rapoport-Zink spaces of abelian type is strictly larger than the class of Rapoport-Zink spaces of Hodge type, but the two type spaces are closely related as having isomorphic connected components. The rigid analytic generic fibers of formal Rapoport-Zink spaces of abelian type can be viewed as moduli spaces of local G-shtukas in mixed characteristic in the sense of Scholze.

We prove that Shimura varieties of abelian type can be uniformized by the associated Rapoport-Zink spaces of abelian type. As an application, we deduce a Rapoport-Zink type unifmormization for the supersingular locus of the moduli space of polarized K3 surfaces in mixed characteristic. Moreover, we show that the Artin invariants of supersingular K3 surfaces are related to some purely local invariants.

Contents

1. Introduction

The theory of Rapoport-Zink spaces finds its origin in the work of Drinfeld in [9]. Let E be a finite extension of $\mathbb{Q}_p$, and let Ω_E^d be the complement of all E-rational hyperplanes in the p-adic projective space $\mathbb{P}^{d-1}$ over E. In [9] Drinfeld interpreted this rigid-analytic space Ω_E^d as the generic fibre of a formal scheme over O_E parametrizing certain p-divisible groups. He used this formal moduli scheme to p-adically uniformize certain Shimura curves and to construct étale coverings of Ω_E^d. In their foundational and seminal work [37], Rapoport and Zink generalized greatly the construction of Drinfeld by introducing general formal moduli spaces of p-divisible groups with EL/PEL structures, and proved these spaces $\breve{\mathcal{M}}$ can be used to uniformize certain pieces of general PEL type Shimura varieties. Moreover, Rapoport and Zink constructed étale coverings $\mathcal{M}_K$ of the generic fibers of these formal moduli spaces, and realized these rigid analytic spaces as étale coverings of more general non-archimedean period domains. Besides these importances in arithemetic geometry and p-adic Hodge theory, it was conjectured by Kottwitz that the ℓ-adic cohomology of these

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Rapoport-Zink spaces $\mathcal{M}_K$ realizes the local Langlands correspondence for the related local reductive group G, cf. [33] section 5.

Recently, in [26] Kim has constructed more general formal moduli spaces of p-divisible groups with additional structures. (Here and throughout the rest of this introduction we assume $p > 2$.) These formal schemes

$$\breve{\mathcal{M}}$$

are called of Rapoport-Zink spaces of Hodge type, associated to unramified local Shimura data of Hodge type $(G, [b], \{\mu\})$ (see below). The additional structures on p-divisible groups are given by the so called crystalline Tate tensors, cf. [26] Definition 4.6, generalizing the EL/PEL structures introduced by Rapoport-Zink (in the unramified case). Kim also constructed a tower $(\mathcal{M}_K)_K$ of rigid analytic spaces (as usual, $K \subset G(\mathbb{Q}_p)$ runs through open compact subgroups of $G(\mathbb{Q}_p)$), when passing to the generic fibers of these formal moduli schemes. These Rapoport-Zink spaces of Hodge type appear as local analogues of the recent work of Kisin [28] on integral canonical models of Shimura varieties of Hodge type. In [27] Kim has proved his Rapoport-Zink spaces of Hodge type can be used to uniformize certain pieces of Shimura varieties of Hodge type. If the unramified local Shimura datum of Hodge type comes from a Shimura datum of Hodge type, Howard and Pappas has given another (global) construction of the associated Hodge type Rapoport-Zink spaces. We refer to [22] for more details.

In this note, we show that we can in fact go ahead one step further: we will construct some (slightly) more general formal and rigid analytic Rapoport-Zink spaces, and we will show that these spaces can be used to uniformize (pieces of) Kisin's integral canonical models Shimura varieties of abelian type, cf. [28]. Moreover, we will give some interesting applications to the moduli spaces of K3 surfaces in mixed characteristic.

There are several motivations for our work here. In our previous work [44], we constructed perfectoid Shimura varieties of abelian type. One of the main motivations for this work is to study the local geometric structures of these perfectoid Shimura varieties, and to study the local geometric structures of Kisin's integral models of Shimura varieties of abelian type [28]. Another motivation is the recent developments in the theory of local Shimura varieties. In [38], Rapoport-Viehmann conjectured the existence of a rigid analytic tower

$$(\mathcal{M}_K)_K$$

associated to a local Shimura datum $(G, [b], \{\mu\})$, where[1]

- G is a reductive group over $\mathbb{Q}_p$,
- $\{\mu\}$ is a conjugacy class of minuscule cocharacters $\mu : \mathbb{G}_m \to G_{\overline{\mathbb{Q}}_p}$,
- $[b]$ is a σ-conjugacy class in the Kottwitz set $B(G, \mu)$ (see [6] 2.3.4 for example)

These conjectural local Shimura varieties are intended to be generalizations of Rapoport-Zink spaces, and there should be a theory in the local situation as good as the classical theory of Shimura varieties ([8]). Recently, using the theory of perfectoid spaces ([41]), and the developments of p-adic Hodge theory due to Fargues, Fargues-Fontaine, and Kedalaya-Liu [12, 17, 25], Scholze has almost given a solution for Rapoport-Viehmann's conjecture by constructing moduli of local G-shtukas in mixed characteristic (cf. [42])

$$(\mathrm{Sht}_K)_K$$

as some reasonable geometric objects, cf. [42]. These geometric objects are called diamonds there, a generalization of perfectoid spaces and analytic adic spaces. In fact, along the way of construction, we get an infinite level moduli space Sht_∞, such that as diamonds we have $\mathrm{Sht}_\infty = \varprojlim_K \mathrm{Sht}_K$.

[1] Here we have followed[38] to write a local Shimura datum as $(G, [b], \{\mu\})$.

In fact, Scholze proved more: one can allow the conjugacy class of cocharacters $\{\mu\}$ non minuscule, contrary to the original requirement of Rapoport-Viehmann in [38], and in fact one can allow several $\{\mu\}$'s. Thus this theory is the mixed characteristic analogue of the theory of moduli of shtukas in the function fields case ([48]).

Although Scholze's method makes a great success, it is purely generic: a priori, one has no information on reduction mod p. In the case of EL/PEL Rapoport-Zink spaces $\mathcal{M}_K$, Scholze proved the associated diamonds $\mathcal{M}_K^\diamond$ are isomorphic to his moduli spaces of local G-shtukas Sht_K. From the point of view of moduli, this means that one can switch p-divisible groups with additional structures to local G-shtukas. Thus, in these classical cases, one gets formal integral structures and can talk about reduction mod p. Assume that G is unramified over $\mathbb{Q}_p$. Using Dieudonné theory, one can prove the special fibers of formal Rapoport-Zink spaces (of EL/PEL/Hodge type) are isomorphic to the related affine Deligne-Lusztig varieties

$$X_\mu^G(b) := \{g \in G(L)/G(W) \,|\, g^{-1}b\sigma(g) \in G(W)\mu(p)G(W)\},$$

where $W = W(\overline{\mathbb{F}}_p), L = W_{\mathbb{Q}}$. These objects are defined purely group theoretically, and thus make sense for arbitrary $(G, [b], \{\mu\})$ (as in the case of Scholze's moduli of local G-shtukas). These affine Deligne-Lusztig varieties play a crucial role in understanding the reduction mod p of Shimura varieties, cf. [34].

In this paper, we introduce a class of local Shimura datum $(G, [b], \{\mu\})$, the so called unramified local Shimura datum of abelian type, and we construct a formal scheme $\breve{\mathcal{M}}$, and a tower of rigid analytic spaces $(\mathcal{M}_K)_{K\subset G(\mathbb{Z}_p)}$ such that

- the reduced special fiber $\mathcal{M}_{red}(\overline{\mathbb{F}}_p) \simeq X_\mu^G(b)$;
- the rigid analytic (adic) generic fiber $\breve{\mathcal{M}}_\eta^{ad} = \mathcal{M}_{G(\mathbb{Z}_p)}$;
- the associated diamonds $\mathcal{M}_K^\diamond \simeq \mathrm{Sht}_K$.

Moreover, we can prove that there exists a preperfectoid space $\mathcal{M}_\infty$ over L such that

$$\mathcal{M}_\infty \sim \varprojlim_K \mathcal{M}_K.$$

This class of unramified local Shimura datum of abelian type is strictly larger than the class of unramified local Shimura datum of Hodge type. Thus, among all local Shimrua data, we find a class which is as large as possible such that

- there exists a formal integral model $\breve{\mathcal{M}}$, such that $\breve{\mathcal{M}}_\eta^{ad,\diamond} \simeq \mathrm{Sht}_{G(\mathbb{Z}_p)}, \quad \mathcal{M}_{red}(\overline{\mathbb{F}}_p) \simeq X_\mu^G(b)$;
- there exists a preperfectoid space $\mathcal{M}_\infty$, such that $\mathcal{M}_\infty^\diamond \simeq \mathrm{Sht}_\infty$.

We remark that the above two additional structures are known in the global situation of Shimura varieties of abelian type by [28, 45]. They are not known for general local Shimura data (or local shtuka data).

A local Shimura datum $(G, [b], \{\mu\})$ is called of unramified Hodge type, if G is unramified, and there exists an embedding $(G, [b], \{\mu\}) \hookrightarrow (\mathrm{GL}(V), [b'], \{\mu'\})$ of local Shimura data. A local Shimura datum $(G, [b], \{\mu\})$ is called of unramified abelian type, if there exists a unramified local Shimura datum of Hodge type $(G_1, [b_1], \{\mu_1\})$ such that we have an isomorphism of the associated adjoint local Shimura datum $(G^{ad}, [b^{ad}], \{\mu^{ad}\}) \simeq (G_1^{ad}, [b_1^{ad}], \{\mu_1^{ad}\})$. This is the local analogue of a Shimura datum of abelian type[2]. Our first main theorem is as follows. See Theorem 4.7, Proposition 4.13, Corollary 5.24.

[2]More precisely, our local Shimura data of abelian type are the local analogues of Shimura data of preabelian type.

Theorem 1.1. *Let $(G,[b],\{\mu\})$ be a unramified local Shimura datum of abelian type. Then there exists a formal scheme $\breve{\mathcal{M}}(G,b,\mu)$, which is formally smooth, formally locally of finite over W, such that*

$$\mathcal{M}(G,b,\mu)_{red}^{perf}=X_\mu^G(b).$$

Here $\mathcal{M}(G,b,\mu)_{red}^{perf}$ is the perfection of the reduced special fiber $\mathcal{M}(G,b,\mu)_{red}$, and $X_\mu^G(b)$ is considered as a perfect scheme by [52]. The formal scheme $\breve{\mathcal{M}}(G,b,\mu)$ is equipped with a transitive action of $J_b(\mathbb{Q}_p)$, compatible with the action of $J_b(\mathbb{Q}_p)$ on $X_\mu^G(b)$. Moreover, there exist a tower of rigid analytic spaces $(\mathcal{M}_K)$ and a preperfectoid space $\mathcal{M}_\infty$ such that

1. $\breve{\mathcal{M}}_\eta^{ad}=\mathcal{M}_{G(\mathbb{Z}_p)}$,
2. $\mathcal{M}_\infty\sim\varprojlim_K\mathcal{M}_K$,
3. $\mathcal{M}_K^\diamond\simeq\mathrm{Sht}_K$,
4. *there exists a compatible system of étale morphism $\pi_{dR}:\mathcal{M}_K\to\mathscr{F}\ell_{G,\mu}^{adm}$,*
5. *there exists a Hodge-Tate period morphism $\pi_{HT}:\mathcal{M}_\infty\to\mathscr{F}\ell_{G,-\mu}$.*

Here $\mathscr{F}\ell_{G,\mu}^{adm}$ is an open subspace of the p-adic flag variety $\mathscr{F}\ell_{G,\mu}$ associated to $(G,\{\mu\})$, $\mathscr{F}\ell_{G,-\mu}$ is the p-adic flag variety associated to $(G,\{\mu^{-1}\})$.

The construction of $\breve{\mathcal{M}}(G,b,\mu)$ associated to $(G,[b],\{\mu\})$ as above is based on the following observations. Take any unramified local Shimura datum of Hodge type $(G_1,[b_1],\{\mu_1\})$ as above. We have the associated formal Raoport-Zink space $\breve{\mathcal{M}}(G_1,b_1,\mu_1)$ constructed by Kim [26], by patching together Faltings's construction of deformation ring of p-divisible groups with crystalline Tate tensors with Artin's criterion for algebraic spaces. The special fiber of $\breve{\mathcal{M}}(G_1,b_1,\mu_1)$ is isomorphic to the affine Deligne-Lusztig variety $X_{\mu_1}^{G_1}(b_1)$. For any local Shimura datum $(G,[b],\{\mu\})$, we have a $J_b(\mathbb{Q}_p)$-equivariant map

$$\omega_G:X_\mu^G(b)\longrightarrow c_{b,\mu}\pi_1(G)^\Gamma,$$

which factors through the set of connected components $\pi_0(X_\mu^G(b))$. Here $\pi_1(G)$ is the algebraic fundamental algebraic group of G and $\Gamma=Gal(\overline{\mathbb{Q}}_p|\mathbb{Q}_p)$. See subsection 2.2 for the construction of this map and the element $c_{b,\mu}\in\pi_1(G)$. Moreover, by [6] Theorem 1.2, $J_b(\mathbb{Q}_p)$ acts transitively on $\pi_0(X_\mu^G(b))$. For any local Shimura datum $(G,[b],\{\mu\})$, by [6] Corollary 2.4.2, we have a cartesian diagram

$$\begin{array}{ccc} X_\mu^G(b) & \longrightarrow & X_{\mu^{ad}}^{G_1^{ad}}(b^{ad}) \\ \downarrow & & \downarrow \\ c_{b,\mu}\pi_1(G)^\Gamma & \longrightarrow & c_{b^{ad},\mu^{ad}}\pi_1(G^{ad})^\Gamma. \end{array}$$

In particular we apply the above diagram to $(G,[b],\{\mu\})$ and $(G_1,[b_1],\{\mu_1\})$ as above. Let $X_{\mu_1}^{G_1}(b_1)^+\subset X_{\mu_1}^{G_1}(b_1)$ be a fixed choice of non empty fiber of the map $\omega_{G_1}:X_{\mu_1}^{G_1}(b_1)\to c_{b_1,\mu_1}\pi_1(G_1)^\Gamma$. This is isomorphic to the corresponding local piece for $X_\mu^G(b)$. Let

$$\breve{\mathcal{M}}(G_1,b_1,\mu_1)^+\subset\breve{\mathcal{M}}(G_1,b_1,\mu_1)$$

be the open and closed subspace corresponding to $X_{\mu_1}^{G_1}(b_1)^+$. As $X_\mu^G(b)=J_b(\mathbb{Q}_p)X_\mu^G(b)^+$, we get the formal scheme $\breve{\mathcal{M}}(G,b,\mu)$ with special fiber $X_\mu^G(b)$. By construction, this formal scheme does not depend on the choice of the Hodge type local Shimura datum $(G_1,[b_1],\{\mu_1\})$. The other properties can be proved similarly.

We note that the above construction is simpler than the corresponding global situation, cf. [45, 28], where one has to make a quotient on each geometric connected component of Shimura varieties of Hodge type.

If the unramified local Shimrua datum of abelian type comes from a Shimura datum of abelian type (G, X), we can prove the following uniformization theorem. Let $K^p \subset G(\mathbb{A}_f^p)$ be a fixed sufficiently small open compact subgroup. Consider S_K, the Kisin integral canonical model over O_E of the Shimura variety Sh_K with $K = G(\mathbb{Z}_p)K^p$. Let

$$\phi : \mathfrak{Q} \to \mathfrak{G}_G$$

be a Langlands-Rapoport parameter with $b = b(\phi)$, with the associated reductive group I_ϕ over $\mathbb{Q}$. In section 6 we will construct a subspace $\mathcal{Z}_{\phi,K^p} \subset \overline{S}_K$, such that the formal completion of S_K along $\mathcal{Z}_{\phi,K^p}$ can be defined. The following theorem was prove by Rapoport and Zink in the PEL type case ([37]), and by Kim in the Hodge type case ([27], see also [22]). It can be viewed as the geometric version of the Langlands-Rapoport description for the underling $\overline{\mathbb{F}}_p$-points, cf. [29]. In fact, it was pointed out in the introduction of [38] that the works of Kisin [28, 29] should yield new Rapoport-Zink spaces (comp. [22]). Here we construct these spaces locally, and show that they admit global application (comp. [38] Remark 5.9). See Theorems 6.7 and 6.13.

Theorem 1.2. *We have an isomorphism of formal schemes over W*

$$\Theta : \coprod_{[\phi],\phi^{ad}=\phi_0} I_\phi(\mathbb{Q}) \setminus \breve{\mathcal{M}} \times G(\mathbb{A}_f^p)/K^p \xrightarrow{\sim} \coprod_{[\phi],\phi^{ad}=\phi_0} \widehat{S}_K \times \mathrm{Spf}W_{/\mathcal{Z}_{\phi,K^p}},$$

where $\phi_0 : \mathfrak{Q} \to \mathfrak{G}_{G^{ad}}$ is a fixed admissible morphism such that $b = b(\phi), \forall\, \phi, \phi^{ad} = \phi_0$. When b is basic, we have $\mathcal{Z}_{\phi,K^p} = \overline{S}_K^b$, and the above isomorphism reduces to

$$\Theta : I_\phi(\mathbb{Q}) \setminus \breve{\mathcal{M}} \times G(\mathbb{A}_f^p)/K^p \xrightarrow{\sim} \widehat{S}_K \times \mathrm{Spf}W_{/\overline{S}_K^b}.$$

Unsurprisingly, we apply the tricks of Kisin as in [29] to deduce the theorem from the Hodge type case. One can also deduce rigid analytic and perfectoid versions of the above uniformization theorem.

We consider the examples of basic GSpin and special orthogonal groups Rapoport-Zink spaces. Let $\breve{\mathcal{M}}_1 = \breve{\mathcal{M}}(\mathrm{GSpin}, b, \mu), \breve{\mathcal{M}} = \breve{\mathcal{M}}(\mathrm{SO}, b', \mu')$ be the associated Rapoport-Zink spaces, where $\mathrm{GSpin} = \mathrm{GSpin}(V, Q), \mathrm{SO} = \mathrm{SO}(V, Q)$ are unramified GSpin and special orthogonal groups associated to a quadratic space (V, Q) over $\mathbb{Q}_p$, with $dimV = n + 2$ for some integer $n \geq 1$. By considering the G-zips associated to the universal p-divisible group with crystalline Tate tensors on the reduced special fiber $\mathcal{M}_{1red}$ of $\breve{\mathcal{M}}_1$, we can define an Ekedahl-Oort stratification on $\mathcal{M}_{1red}$, which is the local analogue of the Ekedahl-Oort stratification for Shimura varieties of Hodge type, cf. [51]. The index set of this stratification is a subset ${}^J\mathcal{W}^b$ of the Weyl group of G_1 (for a fixed choice of maximal torus), which is then in bijection with some set of integers. For each $w \in {}^J\mathcal{W}^b$, we have the associated Ekedahl-Oort stratum $\mathcal{M}_{1w}$ of $\mathcal{M}_{1red}$. As $\breve{\mathcal{M}} \simeq \breve{\mathcal{M}}_1/p^{\mathbb{Z}}$, we get an induced Ekedahl-Oort stratification of $\mathcal{M}_{red}$. On the other hand, in [22], Howard and Pappas introduced another stratification for the reduced special fiber $\mathcal{M}_{1red}$:

$$\mathcal{M}_{1red} = \coprod_{\Lambda} \mathcal{M}_{1\Lambda}^{\circ},$$

where Λ runs through the set of vertex lattices. The following theorem is proved in subsection 7.2: see Theorem 7.4 and Corollary 7.5 for more precise statements.

Theorem 1.3. *Each Ekedahl-Oort stratum $\mathcal{M}_{1w}$ of $\mathcal{M}_{1red}$ is some (disjoint) union of Howard-Pappas strata.*

Similar result holds for $\mathcal{M}_{red}$.

For a similar result in the case of the basic unitary group $GU(1, n-1)$ Rapoport-Zink space, see [47] Theorem D.

Specializing further to the case of K3 surfaces, we have some interesting applications. Take an integer $d \geq 1$ such that $p \nmid 2d$. Let $\mathrm{M}_{2d,K}$ be the moduli spaces of K3 surfaces $f : X \to S$ together with a primitive polarization ξ of degree $2d$ and a K-level structure over $\mathbb{Z}_p$. Recall that by the global integral Torelli theorem (cf. [32] Corollary 5.15), the integral Kuga-Satake period map

$$\iota : \mathrm{M}_{2d,K} \longrightarrow S_K$$

is an open immersion, where S_K is the integral canonical model of the Shimura variety Sh_K for $G = \mathrm{SO}(2, 19)$, see subsection 7.3 for more details. Here, we assume in fact that $K_p = G(\mathbb{Z}_p)$ is the hyperspecial subgroup to have good reduction. Let X be a supersingular K3 surface over $\overline{\mathbb{F}}_p$, then the discriminant of its Néron-Severi lattice is equal to

$$-p^{2\sigma_0(X)}$$

for some integer $1 \leq \sigma_0(X) \leq 10$. The integer $\sigma_0(X)$ is called the Artin invariant of X. The following corollary is a consequence of the above theorems.

Corollary 1.4 (Corollaries 7.12 and 7.14). (1) *Let ϕ and $\mathcal{Z}_{\phi,K^p}$ be as in the above Theorem 1.2, and let J_ϕ be the pullback of $\mathcal{Z}_{\phi,K^p}$ under the open immersion $\overline{\mathrm{M}}_{2d,K} \hookrightarrow \overline{S}_K$ of special fibers. Then we have the following identity*

$$\widehat{\mathrm{M}}_{2d,K/J_\phi} = \coprod_{j \in I} \breve{\mathcal{N}}/\Gamma_j,$$

where $\breve{\mathcal{N}} \subset \breve{\mathcal{M}}(G, b, \mu)$ is an open subspace, $\Gamma_j \subset J_b(\mathbb{Q}_p)$ are some discrete subgroups. If moreover $b = b_0$ is basic, then $J_\phi = \overline{\mathrm{M}}_{2d,K}^{ss}$ and the above disjoint union is finite.

(2) *Let $x \in \overline{\mathrm{M}}_{2d,K}^{ss}(\overline{\mathbb{F}}_p)$ be a point, and X_x the associated supersingular K3 surface over $\overline{\mathbb{F}}_p$. Then we have the identity between the Artin invariant $\sigma_0(X_x)$ and the type $t(\Lambda_x)$:*

$$\sigma_0(X_x) = \frac{t(\Lambda_x)}{2},$$

where Λ_x is the vertex lattice attached to the special lattice associated to (X_x, ξ_x), cf. subsection 7.5.

We briefly describe the structure of this article. In section 2, we review some basics about affine Deligne-Lusztig varieties which will be used later. In section 3, we first recall the Rapoport-Viehmann conjecture on the theory of local Shimura varieties, then we concentrate on the case of unramified local Shimura datum of Hodge type, and review the construction of Kim [26] on the associated Rapoport-Zink spaces of Hodge type. In section 4, we introduce unramified local Shimura datum of abelian type, and construct the associated formal and rigid analytic Rapoport-Zink spaces. Section 5 is devoted as a review the general framework of moduli of local G-shtukas in mixed characteristic due to Scholze, to give a moduli interpretation of the generic fibers of our Rapoport-Zink spaces of abelian type. In section 6, we turn to the global situation of Shimura varieties of abelian type, and prove a Rapoport-Zink type uniformization theorem in this setting. In section 7, we work on the examples of basic GSpin and special orthogonal groups Rapoport-Zink spaces, and then more specially on the case of moduli spaces of K3 surfaces. Finally, we investigate p-adic period domains in the basic orthogonal case in the appendix following Fargues.

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2. Affine Deligne-Lusztig varieties in mixed characteristic

In this section, we recall some basic facts about affine Deligne-Lusztig varieties in mixed characteristic, which will be used later.

Fix a prime p. Let G be reductive group over $\mathbb{Q}_p$, which we assume to be unramified. Fix $T \subset B$ a maximal split torus inside a Borel subgroup of G. Let $W = W(\overline{\mathbb{F}}_p)$ be the ring of Witt vectors, and $L = W_{\mathbb{Q}}$. Denote σ as the Frobenius on L and W.

2.1. **Affine Deligne-Lusztig varieties.** For $b \in G(L)$ and a conjugacy class $\{\mu\}$ of cocharacters $\mu : \mathbb{G}_m \to G_{\overline{\mathbb{Q}}_p}$, we define the affine Deligne-Lusztig sets

$$X_\mu^G(b) = \{g \in G(L)/G(W) \,|\, g^{-1}b\sigma(g) \in G(W)\mu(p)G(W)\},$$

and

$$X_{\leq\mu}^G(b) = \{g \in G(L)/G(W) \,|\, g^{-1}b\sigma(g) \in \bigcup_{\mu' \leq \mu} G(W)\mu'(p)G(W)\}.$$

Here, for dominant elements $\mu, \mu' \in X_*(T)$, we say that $\mu' \leq \mu$ if $\mu - \mu'$ is a non-negative integral linear combination of positive coroots. The isomorphism classes of both $X_\mu^G(b)$ and $X_{\leq\mu}^G(b)$ depend only on the σ-conjugacy class $[b]$ of b, and they are non empty if and only if $[b] \in B(G, \mu)$. Here $B(G, \mu)$ is the Kottwitz subset inside $B(G)$, the set of all σ-conjugacy classes in $G(L)$. We assume $[b] \in B(G, \mu)$ from now on. The triple $(G, [b], \{\mu\})$ will be called a local Shimura datum in the next section, cf. Definition 3.1. By construction, we have $X_\mu^G(b) \subset X_{\leq\mu}^G(b)$. When $\{\mu\}$ is minuscule, we have $X_{\leq\mu}^G(b) = X_\mu^G(b)$.

By the recent work of Zhu [52] and Bhatt-Schoze [1], there exist perfect scheme structures on the sets $X_\mu^G(b)$ and $X_{\leq\mu}^G(b)$. More precisely, $X_\mu^G(b)$ and $X_{\leq\mu}^G(b)$ are the sets of $\overline{\mathbb{F}}_p$-points of some perfect schemes over $\overline{\mathbb{F}}_p$, which are locally closed subschemes of the Witt vector affine Grassmannian Gr_G (cf. [52, 1]). It will be useful to briefly recall the related moduli interpretation. Denote $\mathcal{E}_0$ the trivial G-torsor on W. For any perfect $\overline{\mathbb{F}}_p$-algebra R, we have (cf. [52]1.2 and 3.1) $Gr_G(R) = \{(\mathcal{E}, \beta)\}/\simeq$, where

- $\mathcal{E}$ is a G-torsor in $W(R)$,
- $\beta : \mathcal{E}[1/p] \simeq \mathcal{E}_0[1/p]$ is a trivialization,

and

$$X_{\leq\mu}^G(b)(R) = \{(\mathcal{E}, \beta) \in Gr_G(R) | \mathrm{Inv}_x(\beta^{-1}b\sigma(\beta)) \leq \mu, \forall x \in \mathrm{Spec}R\},$$
$$X_\mu^G(b)(R) = \{(\mathcal{E}, \beta) \in Gr_G(R) | \mathrm{Inv}_x(\beta^{-1}b\sigma(\beta)) = \mu, \forall x \in \mathrm{Spec}R\},$$

where Inv_x is the relative position at x. By abuse of notation, we denote also by $X_{\leq\mu}^G(b)$ and $X_\mu^G(b)$ the associated perfect schemes. By construction, $X_\mu^G(b) \subset X_{\leq\mu}^G(b)$ is an open subscheme.

Lemma 2.1. *Let $(G_1, [b_1], \{\mu_1\}) \to (G_2, [b_2], \{\mu_2\})$ be a morphism (cf. Definition 3.3) It induces a natural map*

$$X_{\leq\mu_1}^{G_1}(b_1) \to X_{\leq\mu_2}^{G_2}(b_2).$$

If $G_1 \to G_2$ is a closed immersion, the above map is a closed immersion.

Proof. The first statement is clear. For the second statement, see [26] Lemma 2.5.4 (1) and [22] 2.4.4. □

2.2. **Connected components.** In [6] 2.3.5, Chen, Kisin and Viehmann introduce a notion of connected components for the affine Deligne-Lusztig sets $X^G_{\leq\mu}(b)$ by some ad hoc methods, since the algebro-geometric structure on $X^G_{\leq\mu}(b)$ has not been known by then. We denote by $\pi_0(X^G_{\leq\mu}(b)$ the set of connected components defined by Chen-Kisin-Viehmann in such a way. By resorting on the perfect scheme structure, we have a naturally defined notion of connected components for $X^G_{\leq\mu}(b)$. It is conjectured that the two definitions coincide, cf. [52] Remark 3.2 and [6] 2.3.5. This is known for the case of unramified EL/PEL Rapoport-Zink spaces, cf. [6] Theorem 5.1.5.

Let $\pi_1(G)$ be the quotient[3] of $X_*(T)$ by the coroot lattice of G. There is the Kottwitz homomorphism

$$\omega_G : G(L) \to \pi_1(G)$$

for which an element $g \in G(W)\mu(p)G(W) \subset G(L)$ is sent to the class of μ. Recall that for our pair (b,μ) we assume that $[b] \in B(G,\mu)$. Then there is an element $c_{b,\mu} \in \pi_1(G)$ such that $\omega_G(b) - \mu = (1-\sigma)(c_{b,\mu})$. The $\pi_1(G)^\Gamma$-coset of $c_{b,\mu}$ is uniquely determined. Here and the following, $\Gamma = \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ is the local Galois group. As ω_G is trivial on $G(W)$, when restricting to $X^G_{\leq\mu}(b) \subset G(L)/G(W)$, [6] 2.3, we have a $J_b(\mathbb{Q}_p)$-equivariant map

$$\omega_G : X^G_{\leq\mu}(b) \longrightarrow c_{b,\mu}\pi_1(G)^\Gamma,$$

which factors through $\pi_0(X^G_{\leq\mu}(b))$, cf. [6] 2.3. Thus we get a commutative diagram

$$\begin{array}{ccc} X^G_{\leq\mu}(b) & & \\ \downarrow & \searrow^{\omega_G} & \\ \pi_0(X^G_{\leq\mu}(b)) & \longrightarrow & c_{b,\mu}\pi_1(G)^\Gamma. \end{array}$$

Therefore, the non empty fibers of the map $\omega_G : X^G_{\leq\mu}(b) \to c_{b,\mu}\pi_1(G)^\Gamma$ are unions of connected components of $X^G_{\leq\mu}(b)$. Moreover, all non empty fibers are isomorphic to each other under the transition induced by the action of $J_b(\mathbb{Q}_p)$. Recall the following main theorem of [6].

Theorem 2.2 ([6] Theorems 1.2 and 1.1)**.** *Assume that μ is minuscule.*

1. *$J_b(\mathbb{Q}_p)$ acts transitively on $\pi_0(X^G_\mu(b))$.*
2. *Assume that G^{ad} is simple, and (μ,b) is Hodge-Newton indecomposable in G. Then ω_G induces a bijection*

$$\pi_0(X^G_\mu(b)) \simeq c_{b,\mu}\pi_1(G)^\Gamma$$

unless $[b] = [\mu(p)]$ with μ central, in which case

$$X^G_\mu(b) \simeq G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$$

is discrete.

Assume that μ is minuscule. By (1) of the above theorem, all non empty fibers of $\omega_G : X^G_{\leq\mu}(b) \longrightarrow c_{b,\mu}\pi_1(G)^\Gamma$ are isomorphic to each other under the transition induced by the action of $J_b(\mathbb{Q}_p)$. Fix a point $x_0 \in \mathrm{Im}(\omega_G : X^G_\mu(b) \to c_{b,\mu}\pi_1(G)^\Gamma)$ (Soon we will show that ω_G is surjective). Let

$$X^G_\mu(b)^+ \subset X^G_\mu(b)$$

be the fiber of ω_G over x_0. By (1) of the above theorem, we have the equality

$$X^G_\mu(b) = J_b(\mathbb{Q}_p)X^G_\mu(b)^+.$$

[3] We note that $\pi_1(G)$ is finite if G is semisimple.

In the following, we will not need to work on each connected component of $X_\mu^G(b)$. The subspace $X_\mu^G(b)^+$ and the equality above will be all what we need.

Now let μ be arbitrary. As in [29] 1.2.20 and 1.2.15, if we take $\upsilon = \sigma(\mu)$, we have a natural bijection

$$X_\upsilon^G(b) \xrightarrow{\sim} X_\mu^G(b), \quad g \mapsto \sigma^{-1}(b^{-1}g),$$

and similarly

$$X_{\leq\upsilon}^G(b) \xrightarrow{\sim} X_{\leq\mu}^G(b).$$

For $X_{\leq\upsilon}^G(b)$, as we can choose $b \in [b]$ such that $b \in G(W)\sigma(\mu(p))G(W)$, we have $1 \in X_{\leq\upsilon}^G(b)$ and thus $c_{b,\upsilon} = 1$ (we note that the element $c_{b,\mu}$ can be defined for arbitrary μ). Therefore, we may assume the element $c_{b,\mu} = 1$ by working on $X_{\leq\upsilon}^G(b)$ in the following.

Lemma 2.3. (1) *The restriction of $\omega_G : G(L) \to \pi_1(G)$ to $G(\mathbb{Q}_p)$ induces a surjective map*

$$\omega_G : G(\mathbb{Q}_p) \to \pi_1(G)^\Gamma.$$

(2) *The map $J_b(\mathbb{Q}_p) \to \pi_1(G)^\Gamma$ is surjective.*

Proof. For (1): this is contained in Lemma 1.2.3 of [29].

For (2): in the case that $(G, [b], \{\mu\})$ comes from a Hodge type Shimura datum $(\mathbb{G}, X)$ unramified at p (and $Z_\mathbb{G}$ is a torus), see Lemma 4.6.4 of [29]. The arguments there work also in the general case. □

With the above convention, we have

Proposition 2.4. *The map*

$$\omega_G : X_{\leq\mu}^G(b) \longrightarrow \pi_1(G)^\Gamma$$

is surjective. In particular we get a surjection

$$\pi_0(X_{\leq\mu}^G(b)) \twoheadrightarrow \pi_1(G)^\Gamma.$$

Proof. By Lemma 2.3.6 of [6], the map ω_G is compatible with the $J_b(\mathbb{Q}_p)$-actions on both sides. By construction, $J_b(\mathbb{Q}_p)$ acts on $\pi_1(G)^\Gamma$ by left multiplication via the map $J_b(\mathbb{Q}_p) \to \pi_1(G)^\Gamma$, which is surjective by (2) of Lemma 2.3. Thus $\omega_G : X_{\leq\mu}^G(b) \to \pi_1(G)^\Gamma$ is surjective. □

We continue to assume that μ can be arbitrary.

Proposition 2.5. *Let $(G_1, [b_1], \{\mu_1\}) \to (G_2, [b_2], \{\mu_2\})$ be a morphism. If $G_2 = G_1/Z$ for some central group $Z \subset Z_{G_1}$, we have the following cartesian diagram*

$$\begin{array}{ccc} X_{\leq\mu_1}^{G_1}(b_1) & \longrightarrow & X_{\leq\mu_2}^{G_2}(b_2) \\ \Big\downarrow{\scriptstyle \omega_{G_1}} & & \Big\downarrow{\scriptstyle \omega_{G_2}} \\ c_{b_1,\mu_1}\pi_1(G_1)^\Gamma & \longrightarrow & c_{b_2,\mu_2}\pi_1(G_2)^\Gamma. \end{array}$$

Proof. This is contained in [6] Corollary 2.4.2. □

Let the notations be as in the above proposition. Combined with Proposition 2.4, we get

Corollary 2.6. *Let $x_1 \in c_{b_1,\mu_1}\pi_1(G_1)^\Gamma$ be a point and $x_2 \in c_{b_2,\mu_2}\pi_1(G_2)^\Gamma$ be its image under $c_{b_1,\mu_1}\pi_1(G_1)^\Gamma \to c_{b_2,\mu_2}\pi_1(G_2)^\Gamma$. Let $X_{\leq\mu_1}^{G_1}(b_1)^+$ and $X_{\leq\mu_2}^{G_2}(b_2)^+$ be the fibers of ω_{G_1} and ω_{G_2} at x_1 and x_2 respectively, which are non empty by Proposition 2.4. Then the map $X_{\leq\mu_1}^{G_1}(b_1) \to X_{\leq\mu_2}^{G_2}(b_2)$ induces a bijection*

$$X_{\leq\mu_1}^{G_1}(b_1)^+ \xrightarrow{\sim} X_{\leq\mu_2}^{G_2}(b_2)^+.$$

We still keep the above notations.

Lemma 2.7. *If $\pi_1(G_1)^\Gamma \to \pi_1(G_2)^\Gamma$ is surjective, then the map $X^{G_1}_{\leq\mu_1}(b_1) \to X^{G_2}_{\leq\mu_2}(b_2)$ induces an isomorphism*

$$X^{G_1}_{\leq\mu_1}(b_1)/Z(\mathbb{Q}_p) \simeq X^{G_2}_{\leq\mu_2}(b_2).$$

Proof. This is implied by the proof of [6] Corollaries 2.4.2 and 2.4.3: under the assumption that $\pi_1(G_1)^\Gamma \to \pi_1(G_2)^\Gamma$ is surjective, all fibers of $X^{G_1}_{\leq\mu_1}(b_1) \to X^{G_2}_{\leq\mu_2}(b_2)$ are torsors under $X_*(Z)^\Gamma$. The group $Z(\mathbb{Q}_p)$ acts on $X^{G_1}_{\leq\mu_1}(b_1)$ via the natural map $Z(\mathbb{Q}_p) \to X_*(Z)^\Gamma$. □

3. Rapoport-Zink spaces of Hodge type

Following Rapoport-Viehmann, we first review the general conjecture on the theory of local Shimura varieties in [38]. Then we concentrate on the Hodge type case, cf. [26, 22].

3.1. **Local Shimura data and local Shimura varieties.**

Recall the following definition of Rapoport-Viehmann.

Definition 3.1 ([[38] Definition 5.1)**.** *A local Shimura datum over $\mathbb{Q}_p$ is a triple $(G, [b], \{\mu\})$ where*

- *G is a reductive group over $\mathbb{Q}_p$,*
- *$[b] \in B(G)$ is a σ-conjugacy class,*
- *$\{\mu\}$ is a conjugacy class of cocharacters $\mu : \mathbb{G}_m \to G_{\overline{\mathbb{Q}}_p}$,*

such that the following conditions are satisfied

1. *$[b] \in B(G, \mu)$,*
2. *$\{\mu\}$ is minuscule.*

Associated to a local Shimura datum, we have

- the reflex field $E = E(G, \{\mu\})$, which is the field of definition of $\{\mu\}$ inside the fixed algebraic closure $\overline{\mathbb{Q}}_p$,
- the reductive group J_b over $\mathbb{Q}_p$, for $b \in [b]$, which up to isomorphism only depends on $[b]$.

In fact, if G is unramified, we have also (cf. the last section)

- the affine Dellign-Lusztig variety $X^G_\mu(b)$ over $\overline{\mathbb{F}}_p$ (which will be expected to be the special fiber of some formal integral model of the following local Shimura variety $\mathcal{M}_{G(\mathbb{Z}_p)}$, cf. Conjecture 3.2).

Let $(G, [b], \{\mu\})$ be a local Shimura datum, with local reflex field E. Let $\breve{E}$ be the completion of the maximal unramified extension of E. We have the following conjecture ([38] 5.1):

Conjecture 3.2 (Rapoport-Viehmann)**.** *There is a tower of rigid analytic spaces over Sp$\breve{E}$,*

$$(\mathcal{M}_K)_K,$$

where K runs through all open compact subgroups of $G(\mathbb{Q}_p)$, with the following properties:

1. *the group $J_b(\mathbb{Q}_p)$ acts on each space $\mathcal{M}_K$,*
2. *the group $G(\mathbb{Q}_p)$ acts on the tower $(\mathcal{M}_K)_{K\subset G(\mathbb{Q}_p)}$ as Hecke correspondences,*
3. *the tower is equipped with a Weil descent datum over E,*
4. *there exists a compatible system of étale and partially proper period maps*

$$\pi_K : \mathcal{M}_K \to \mathscr{F}\ell^{wa}_{G,\mu}$$

which is equivariant for the action of $J_b(\mathbb{Q}_p)$, where $\mathscr{F}\ell^{wa}_{G,\mu} \subset \mathscr{F}\ell_{G,\mu}$ is the weakly admissible open subspace defined in [37] *1.35 and* [7] *Definition 9.5.4.*

In fact, in [38] 5.1 there is a more precise statement on the point (4) of the conjecture. In particular, there should be an open subspace

$$\mathscr{F}\ell_{G,\mu}^{adm} \subset \mathscr{F}\ell_{G,\mu}^{wa},$$

which should be the image of the period maps π_K for all K. We refer to [38] 5.1 and Proposition 5.3 there for more details. This conjecture has been known for the local Shimura data which arise from local EL/PEL data ([37]), and the unramified local Shimura datum of Hodge type ([26]). In both cases, these spaces $\mathcal{M}_K$ are finite étale covers of the rigid analytic generic fibers of some formal schemes $\breve{\mathcal{M}}$ over $\mathrm{Spf}O_{\breve{E}}$, which are formal moduli spaces of p-divisible groups with some additional structures. The special fibers of these formal schemes $\breve{\mathcal{M}}$ are the affine Deligne-Lusztig varieties which we introduced in the last section. In section 5 we will give a partial solution of the above conjecture by applying Scholze's ideas and methods in [42].

It will be useful to make a definition for morphisms of local Shimura data.

Definition 3.3. *Let $(G_1, [b_1], \{\mu_1\}), (G_2, [b_2], \{\mu_2\})$ be two local Shimura data. A morphism*

$$(G_1, [b_1], \{\mu_1\}) \to (G_2, [b_2], \{\mu_2\})$$

is a hommorphism of algebraic groups $f : G_1 \to G_2$ sending $([b_1], \{\mu_1\})$ to $([b_2], \{\mu_2\})$.

If $(G_1, [b_1], \{\mu_1\}) \to (G_2, [b_2], \{\mu_2\})$ is a morphism of local Shimura data, then it is conjectured ([38] Proposition 5.3 (iv)) that for any open compact subgroups $K_1 \subset G_1(\mathbb{Q}_p), K_2 \subset G_2(\mathbb{Q}_p)$ with $f(K_1) \subset K_2$, there exists a morphism of the associated local Shimura varieties

$$\mathcal{M}(G_1, b, \mu)_{K_1} \longrightarrow \mathcal{M}(G_2, b_2, \mu_2)_{K_2} \times \mathrm{Sp}\breve{E}_1,$$

and when $G_1 \to G_2$ is a closed immersion these are closed embeddings for $K_1 = K_2 \cap G_1(\mathbb{Q}_p)$.

3.2. **Local Shimura data of Hodge type.** Now we recall the definition of a special class of local Shimura data: those of Hodge type (cf. [38] Remark 5.4 (i)):

Definition 3.4. *A local Shimura datum $(G, [b], \{\mu\})$ is called of Hodge type, if there exists an embedding $f : G \hookrightarrow \mathrm{GL}(V)$ and a local Shimura datum $(\mathrm{GL}(V), [b'], \{\mu'\})$, such that $[b], \{\mu\}$ are mapped to $[b'], \{\mu'\}$ under f.*

If G is moreover unramified, by [28] Lemma 2.3.1, we can find some $\mathbb{Z}_p$-lattice $V_{\mathbb{Z}_p} \subset V$ such that $G \hookrightarrow \mathrm{GL}(V)$ is induced by an embedding $G_{\mathbb{Z}_p} \hookrightarrow \mathrm{GL}(V_{\mathbb{Z}_p})$.

Definition 3.5. *A local Shimura datum of Hodge type $(G, [b], \{\mu\})$ is called unramified, if G is unramified.*

We note that for a unramified local Shimura datum of Hodge type $(G, [b], \{\mu\})$, the local reflex field E is a unramified extension of $\mathbb{Q}_p$. Thus $\breve{E} = L, O_{\breve{E}} = W$ where as before $W = W(\overline{\mathbb{F}}_p), L = W_{\mathbb{Q}}$.

Remark 3.6. *The above definition of unramified local Shimura data of Hodge type is more general than that in [22] Definition 2.3.3. Moreover, for a unramified local Shimura datum of Hodge type $(G, [b], \{\mu\})$ in the sense of [22], one always has $Z(G) \supset \mathbb{G}_m$.*

We want to classify local Shimura data of Hodge type. Let $(G, [b], \{\mu\})$ be a given local Shimura datum. Take any faithful representation V of G over $\mathbb{Q}_p$, so that we get an embedding $f : G \hookrightarrow \mathrm{GL}(V)$. Therefore we get a conjugacy class $\{\mu'\}$ of cocharacters $\mu' = f \circ \mu : \mathbb{G}_m \to \mathrm{GL}(V)_{\overline{\mathbb{Q}}_p}$. Let $N(G)$ be the set of Newton points of G, cf. [36] 1.7. Recall that the maps

$$\nu_G : B(G) \to N(G), \quad \kappa_G : B(G) \to \pi_1(G)_\Gamma$$

are functorial in G, cf. [36] 1.9 and 1.15. We get in particular a map

$$B(G,\mu)\to B(\mathrm{GL}(V),\mu').$$

Let $[b']\in B(\mathrm{GL}(V),\mu')$ be the image of $[b]$ under this map. The triple $(\mathrm{GL}(V),[b'],\{\mu'\})$ is a local Shimura datum if and only if $\{\mu'\}$ is minuscule. In which case $(G,[b],\{\mu\})$ is of Hodge type. Write

$$V_{\overline{\mathbb{Q}}_p}=\bigoplus_{\lambda\in X^*(T)_+}V_\lambda$$

as the decomposition of the faithful representation $V_{\overline{\mathbb{Q}}_p}$ of $G_{\overline{\mathbb{Q}}_p}$ into irreducible sub representations V_λ with highest weights λ. There are only finitely many λ such that $V_\lambda\neq 0$. Then $\{\mu'\}$ is minuscule if and only if

$$|\langle\mu',w\lambda\rangle-\langle\mu',\lambda\rangle|\leq 1,$$

for any λ such that $V_\lambda\neq 0$ and any $w\in\mathcal{W}$. Here $\mathcal{W}$ is the Weyl group of G.

Lemma 3.7. *Given a local Shimura datum $(G,[b],\{\mu\})$, to check whether it is of Hodge type, we can only check for finitely many faithful representations V of G over $\mathbb{Q}_p$, for any $w\in\mathcal{W}$, and any $\lambda\in X^*(T)_+$ such that $V_\lambda\neq 0$,*

$$|\langle\mu',w\lambda\rangle-\langle\mu',\lambda\rangle|\leq 1.$$

Proof. We need only to prove the finiteness. This comes from the fact that there are only finitely many weights λ satisfying the equality condition above. □

The following examples of local Shimura datum of Hodge type are standard.

Example 3.8. (1) *Let $(G,[b],\{\mu\})$ be a local Shimura datum which comes from a local EL/PEL data, then it is of Hodge type.*

(2) *Let (G,X) be a Shimura datum of Hodge type, i.e. there exists some embedding into the Siegel Shimura datum $(G,X)\hookrightarrow(\mathrm{GSp},S^{\pm})$. Let μ be the cocharacter associated to X. Take any $[b]\in B(G_{\mathbb{Q}_p},\mu)$. Then the local Shimura datum $(G_{\mathbb{Q}_p},[b],\{\mu\})$ is of Hodge type.*

Here is an example of non Hodge type local Shimura datum.

Example 3.9 (See [38] Example 5.5)**.** *Let $G=\mathrm{PGL}_n$, μ be any non trivial minuscule cocharacter, and $[b]\in B(G,\mu)$ be arbitrary. Then the local Shimura datum $(G,[b],\{\mu\})$ is not of Hodge type.*

3.3. **Rapoport-Zink spaces of Hodge type.** Throughout the rest of this section, we assume that $p>2$. Let $(G,[b],\{\mu\})$ be a unramified local Shimura datum of Hodge type. Kim ([26]) constructs a formal moduli scheme $\breve{\mathcal{M}}=\breve{\mathcal{M}}(G,b,\mu)$ over $\mathrm{Spf}W$ of p-divisible groups with Tate tensors. We review the related constructions in this subsection. By abuse of notation, we write also G as the associated reductive group scheme over $\mathbb{Z}_p$. Then there exists a faithful representation

$$\rho:G\to\mathrm{GL}(\Lambda),$$

such that the induced cocharacter $\mu'=\rho_{\overline{\mathbb{Q}}_p}\mu:\mathbb{G}_m\to\mathrm{GL}(\Lambda\otimes\overline{\mathbb{Q}}_p)$ is minuscule. Let $\Lambda^\vee$ be the dual lattice, and $\Lambda^\otimes$ be the tensor algebra of $\Lambda\oplus\Lambda^\vee$. By Proposition 1.3.2 of [28], there exists a finite collection of tensors $\{s_\alpha\in\Lambda^\otimes\}_{\alpha\in I}$ such that $\rho:G\subset\mathrm{GL}(\Lambda)$ is the schematic stabilizer of (s_α).

Let Nilp_W be the category of W-algebras on which p is locally nilpotent. Denote by Nilp_W^{sm} the full subcategory of Nilp_W consisting of formally smooth formally finitely generated W/p^m-algebras for $m\geq 1$. We use the following version of Rapoport-Zink functor, cf. [52] Definition 3.8, which is equivalent to Definition 4.6 of [26].

Definition 3.10. *The Rapoport-Zink space associated to the unramified local Shimura datum of Hodge type is the functor* $\breve{\mathcal{M}}$ *on* Nilp_W^{sm} *defined by* $\breve{\mathcal{M}}(R) = \{(X, (t_\alpha)_{\alpha\in I}, \rho)\}/\simeq$ *where*

- X *is a p-divisible group on* $\mathrm{Spec}R$*,*
- $(t_\alpha)_{\alpha\in I}$ *is a collection of cystalline Tate tensors of* X*,*
- $\rho : X_0 \otimes R/J \to X \otimes R/J$ *is a quasi-isogeny which sends* $s_\alpha \otimes 1$ *to* t_α *for* $\alpha \in I$*, where* J *is some ideal of definition of* R*,*

such that the following condition holds:
the R-scheme

$$\mathrm{Isom}\Big((\mathcal{D}(X)_R, (t_\alpha), \mathrm{Fil}^\bullet(\mathcal{D}(X)_R)), (\Lambda \otimes R, (s_\alpha \otimes 1), \mathrm{Fil}^\bullet\Lambda \otimes R)\Big)$$

that classifies the isomorphisms between locally free sheaves $\mathcal{D}(X)_R$ *and* $\Lambda \otimes R$ *on* $\mathrm{Spec}R$ *preserving the tensors and the filtrations is a* $P_\mu \otimes R$*-torsor.*

Theorem 3.11 ([26] Theorem 4.9.1)**.** *The functor* $\breve{\mathcal{M}}$ *is represented by a separated formal scheme, formally smooth and locally formally of finite type over* W

In the classical EL/PEL case (and with ramification), see [37] Theorem 3.25. In [26] 4.7, the unramified local EL/PEL data are explained as special examples of unramified Hodge type data. See also [22] Theorem 3.2.1 for the case that $(G, [b], \{\mu\})$ comes from a Shimura datum of Hodge type.

Let $\mathcal{M}$ be the rigid analytic generic fiber over L of the formal scheme $\breve{\mathcal{M}}$. In [26], Kim explained how to construct a tower of rigid analytic spaces

$$(\mathcal{M}_K)_{K\subset G(\mathbb{Z}_p)}$$

that satisfies the list of properties in Conjecture 3.2. Moreover, $\mathcal{M}_{G(\mathbb{Z}_p)} = \mathcal{M}$, and $\mathcal{M}_K \to \mathcal{M}$ is finite étale for any open compact subgroup $K \subset G(\mathbb{Z}_p)$. In particular, for unramified local Shimura datum of Hodge type, the Conjecture 3.2 is true.

Let $\mathcal{M}_{red}$ be the reduced special fiber over $\overline{\mathbb{F}}_p$ of $\breve{\mathcal{M}}$. Recall that in section 2 attached to $(G, [b], \{\mu\})$, we introduced the affine Deligne-Lusztig variety $X_\mu^G(b)$ over $\overline{\mathbb{F}}_p$, viewed as a perfect scheme. The relation between $\mathcal{M}_{red}$ and $X_\mu^G(b)$ is as follows.

Proposition 3.12 ([52] Proposition 3.11)**.** $X_\mu^G(b)$ *is the perfection* $\mathcal{M}_{red}^{perf}$ *of* $\mathcal{M}_{red}$*.*

If $(G, [b], \{\mu\}) \hookrightarrow (\mathrm{GL}_n, [b'], \{\mu'\})$ is an embedding of unramified local Shimura data of Hodge type, by construction, we have the following embeddings

$$\breve{\mathcal{M}}(G, b, \mu) \hookrightarrow \breve{\mathcal{M}}(\mathrm{GL}_n, b', \mu'), \quad X_\mu^G(b) \hookrightarrow X_{\mu'}^{\mathrm{GL}_n}(\mu'),$$

which are compatible in the sense of the above proposition.

3.4. **Connected components.** Let the notations be as above. The map

$$\omega_G : X_\mu^G(b) \longrightarrow c_{b,\mu}\pi_1(G)^\Gamma$$

is in fact induced by a map

$$\omega_G : \breve{\mathcal{M}} \to c_{b,\mu}\pi_1(G)^\Gamma.$$

Let $G^{der} \subset G$ be the derived subgroup, and G^{ab} the abelian quotient G/G^{der}. Consider the exact sequence

$$1 \to G^{der} \to G \to G^{ab} \to 1,$$

which induces a map

$$c_{b,\mu}\pi_1(G)^\Gamma \to c_{b,\mu}\pi_1(G^{ab})^\Gamma = c_{b,\mu}X_*(G^{ab})^\Gamma,$$

where $X_*(G^{ab})$ is the cocharacter group of the torus G^{ab} over $\overline{\mathbb{Q}}_p$. Let $X^*_{\mathbb{Q}_p}(G)$ be the group of $\mathbb{Q}_p$-rational characters of G. Then we have

$$X^*_{\mathbb{Q}_p}(G) = X^*(G^{ab})^\Gamma.$$

The Γ-equivariant pairing $X_*(G^{ab}) \times X^*(G^{ab}) \to \mathbb{Z}$ then induces a map

$$c_{b,\mu}X_*(G^{ab})^\Gamma \to \mathrm{Hom}(X^*(G^{ab})^\Gamma, \mathbb{Z}) = \mathrm{Hom}(X^*_{\mathbb{Q}_p}(G), \mathbb{Z}).$$

In summary, we get a map by considering the composition

$$\varkappa_{\breve{\mathcal{M}}} : \breve{\mathcal{M}} \to c_{b,\mu}\pi_1(G)^\Gamma \to c_{b,\mu}X_*(G^{ab})^\Gamma \to \mathrm{Hom}(X^*_{\mathbb{Q}_p}(G), \mathbb{Z}).$$

In the EL/PEL case, this is just the map constructed in [37] 3.52. (See also [6] 5.1.3.)

If $(G, [b], \{\mu\}) \hookrightarrow (\mathrm{GL}_n, [b'], \{\mu'\})$ is an embedding of unramified local Shimura data of Hodge type, we get the following commutative diagram

$$\begin{array}{ccc} X^G_\mu(b) & \longrightarrow & X^{\mathrm{GL}_n}_{\mu'}(b') \\ \downarrow & & \downarrow \\ c_{b,\mu}\pi_1(G)^\Gamma & \longrightarrow & c_{b',\mu'}\pi_1(\mathrm{GL}_n)^\Gamma. \end{array}$$

Moreover, we know $\pi_1(\mathrm{GL}_n)^\Gamma = \pi_1(\mathrm{GL}_n) \simeq \mathbb{Z}$.

Since by Proposition 3.12 $X^G_\mu(b)$ is the perfection $\mathcal{M}^{perf}_{red}$ of $\mathcal{M}_{red}$, we have the isomorphism between the sets of connected components

$$\pi_0(\mathcal{M}_{red}) \simeq \pi_0^{perf}(X^G_\mu(b)).$$

Here $\pi_0^{perf}(X^G_\mu(b))$ denotes the set of connected components of the perfect scheme $X^G_\mu(b)$. On the other hand, we have also the set of connected components $\pi_0(X^G_\mu(b))$ defined in [6].

Proposition 3.13. *Let $(G, [b], \{\mu\})$ be the unramified local Shimura datum of Hodge type as above. There is a bijection*

$$\pi_0(\mathcal{M}_{red}) \simeq \pi_0(X^G_\mu(b)).$$

Proof. See the Remark 3.2 of [52]. □

Let $\pi_0(\breve{\mathcal{M}})$ be the set of connected components of the formal scheme $\breve{\mathcal{M}}$, which is the same as $\pi_0(\mathcal{M}_{red})$. On the other hand, we have also the set of connected components $\pi_0(\mathcal{M})$ of the generic fiber $\mathcal{M}$. As $\breve{\mathcal{M}}$ is formally smooth and in particular normal, by [24] Theorem 7.4.1, we have a bijection

$$\pi_0(\mathcal{M}_{red}) \simeq \pi_0(\mathcal{M}).$$

One can also consider the set of connected components $\pi_0(\mathcal{M}_K)$ for the finite étale cover $\mathcal{M}_K$ of $\mathcal{M}$. In [38], Rapoport and Viehmann made a conjecture on $\pi_0(\mathcal{M}_K \times \mathbb{C}_p)$ under the assumption that G^{der} is *simply connected*. We refer to [38] Conjecture 4.26 for the precise statement on the existence of a determinant morphism for the tower $(\mathcal{M}_K)_K$. This conjecture is known in the unramified simple EL/PEL case, cf. Theorem 6.3.1 of [5] (see also [6] Theorem 5.1.10 and Remark 5.1.11). It will be interesting to consider the more general Hodge type case studied here.

Fix a point $x_0 \in c_{b,\mu}\pi_1(G)^\Gamma$. Let $\mathcal{M}^+_{red} \subset \mathcal{M}_{red}$ be the fiber of ω_G over x_0. Then $\mathcal{M}^+_{red}$ is some union of connected components of $\mathcal{M}_{red}$. Let $\breve{\mathcal{M}}^+ \subset \breve{\mathcal{M}}$ be the associated sub formal scheme, with generic fiber $\mathcal{M}^+$. For any open compact subgroup $K \subset G(\mathbb{Q}_p)$, let $\mathcal{M}^+_K \subset \mathcal{M}_K$ be the pullback of $\mathcal{M}^+ \subset \mathcal{M}$. We get a tower

$$(\mathcal{M}^+_K)_{K \subset G(\mathbb{Z}_p)}.$$

We have the equalities

$$\breve{\mathcal{M}} = J_b(\mathbb{Q}_p)\breve{\mathcal{M}}^+, \quad \mathcal{M}_{red} = J_b(\mathbb{Q}_p)\mathcal{M}_{red}^+, \quad \mathcal{M} = J_b(\mathbb{Q}_p)\mathcal{M}^+$$

and

$$\mathcal{M}_K = J_b(\mathbb{Q}_p)\mathcal{M}_K^+.$$

4. Rapoport-Zink spaces of abelian type

We enlarge the class of Rapoport-Zink spaces of Hodge type in this section. They are constructed locally from Rapoport-Zink spaces of Hodge type. Throughout this section we assume $p > 2$.

4.1. **Local Shimura data of abelian type.** Let $(G, [b], \{\mu\})$ be a local Shimrua datum. Consider the natural projection $G \to G^{ad}$ from G to its associated adjoint group. We get induced $[b^{ad}], \{\mu^{ad}\}$, so that

$$(G^{ad}, [b^{ad}], \{\mu^{ad}\})$$

is also a local Shimura datum and $(G, [b], \{\mu\}) \to (G^{ad}, [b^{ad}], \{\mu^{ad}\})$ is a morphism of local Shimura data. We introduce the local analogue of a Shimura datum of abelian type (more precisely, of preabelian type) as follows.

Definition 4.1. *A local Shimura datum $(G, [b], \{\mu\})$ is called of abelian type, if there exists a local Shimura datum of Hodge type $(G_1, [b_1], \{\mu_1\})$ such that we have an isomorphism of the associated adjoint local Shimura data $(G^{ad}, [b^{ad}], \{\mu^{ad}\}) \simeq (G_1^{ad}, [b_1^{ad}], \{\mu_1^{ad}\})$.*

Thus any local Shimura datum of Hodge type is also of abelian type. The later class is strictly larger.

Example 4.2. *Let $G = \mathrm{PGL}_n$. Consider a nontrivial minuscule cocharacter $\mu_1 : \mathbb{G}_m \to \mathrm{GL}_n$ and $b_1 \in B(\mathrm{GL}_n, \mu_1)$. Take $\mu = \mu_1^{ad}, b = b_1^{ad}$. Then $(G, [b], \{\mu\})$ is of abelian type, but not of Hodge type, cf. Example 3.9.*

Remark 4.3. *By the classification of Shimura data of abelian type in* [8]*, we know that the class of simple factors of G appearing in local Shimura data of abelian type at least contains local reductive groups of types A, B, C, D, cf. Example 4.5.*

4.2. **The associated Rapoport-Zink spaces.** To construct Rapoport-Zink spaces, we need the following unramified assumption.

Definition 4.4. *A local Shimura datum of abelian type $(G, [b], \{\mu\})$ is called unramified, if G is unramified and there exists a unramified local Shimura datum of Hodge type $(G_1, [b_1], \{\mu_1\})$ such that $(G^{ad}, [b^{ad}], \{\mu^{ad}\}) \simeq (G_1^{ad}, [b_1^{ad}], \{\mu_1^{ad}\})$.*

Example 4.5. *Let (G, X) be a Shimura datum of abelian type such that G is unramified at p. Take any $[b] \in B(G, \mu)$, the associated triple $(G, [b], \{\mu\})$ is a unramified local Shimura datum of abelian type.*

Lemma 4.6. *Let $(G, [b], \{\mu\})$ be a unramified local Shimura datum of abelian type. Consider the associated adjoint local Shimura datum $(G^{ad}, [b^{ad}], \{\mu^{ad}\})$. We have the following isomorphism of reductive groups over $\mathbb{Q}_p$*

$$J_b^{ad} \simeq J_{b^{ad}}.$$

Proof. This follows from the definitions of J_b and $J_{b^{ad}}$. □

Theorem 4.7. *Let $(G, [b], \{\mu\})$ be a unramified local Shimura datum of abelian type. Then there exists a formal scheme $\breve{\mathcal{M}}(G, b, \mu)$, which is formally smooth, formally locally of finite over W, such that*

$$\mathcal{M}(G, b, \mu)_{red}^{perf} = X_\mu^G(b).$$

The formal scheme $\breve{\mathcal{M}}(G, b, \mu)$ is equipped with a transitive action of $J_b(\mathbb{Q}_p)$, compatible with the action of $J_b(\mathbb{Q}_p)$ on $X_\mu^G(b)$.

Proof. Take any unramified local Shimura datum of Hodge type $(G_1, [b_1], \{\mu_1\})$ as in Definition 4.4. Consider the associated formal Rapoport-Zink space $\breve{\mathcal{M}}(G_1, b_1, \mu_1)$ over Spf$W$. Then its reduced special fiber $\mathcal{M}(G_1, b_1, \mu_1)_{red}$ satisfies

$$\mathcal{M}(G_1, b_1, \mu_1)_{red}^{perf} \simeq X_{\mu_1}^{G_1}(b_1).$$

Recall that we have following cartesian diagram (cf. Proposition 2.5)

$$\begin{array}{ccc} X_{\mu_1}^{G_1}(b_1) & \longrightarrow & X_{\mu_1^{ad}}^{G_1^{ad}}(b_1^{ad}) \\ \downarrow{\scriptstyle \omega_{G_1}} & & \downarrow{\scriptstyle \omega_{G_1^{ad}}} \\ c_{b_1,\mu_1}\pi_1(G_1)^\Gamma & \longrightarrow & c_{b_1^{ad},\mu_1^{ad}}\pi_1(G_1^{ad})^\Gamma. \end{array}$$

Let $X_{\mu_1}^{G_1}(b_1)^+ \subset X_{\mu_1}^{G_1}(b_1)$ be the fiber over c_{b_1,μ_1} under the map $\omega_{G_1} : X_{\mu_1}^{G_1}(b_1) \to c_{b_1,\mu_1}\pi_1(G_1)^\Gamma$. Let $\breve{\mathcal{M}}(G_1, b_1, \mu_1)^+$ be the corresponding formal sub scheme of $\breve{\mathcal{M}}(G_1, b_1, \mu_1)$. On the other hand, we can consider also the fiber $X_\mu^G(b)^+ \subset X_\mu^G(b)$ over $c_{b,\mu}$ under $\omega_G : X_\mu^G(b) \to c_{b,\mu}\pi_1(G)^\Gamma$. Then as

$$X_{\mu_1}^{G_1}(b_1)^+ \simeq X_\mu^G(b)^+,$$

we set

$$\breve{\mathcal{M}}(G, b, \mu)^+ := \breve{\mathcal{M}}(G_1, b_1, \mu_1)^+,$$

which is a formal model of $X_\mu^G(b)^+$. By Theorem 2.2 (1), we have

$$X_\mu^G(b) = J_b(\mathbb{Q}_p)X_\mu^G(b)^+.$$

Therefore, there exists a formal scheme

$$\breve{\mathcal{M}}(G, b, \mu)$$

equipped with an action of $J_b(\mathbb{Q}_p)$, such that

$$\breve{\mathcal{M}}(G, b, \mu) = J_b(\mathbb{Q}_p)\breve{\mathcal{M}}(G, b, \mu)^+,$$
$$\mathcal{M}(G, b, \mu)_{red}^{perf} = X_\mu^G(b),$$

and the induced action of $J_b(\mathbb{Q}_p)$ on $\mathcal{M}(G, b, \mu)_{red}$ is compatible with that on $X_\mu^G(b)$ under the above identification. In fact, we can take

$$\breve{\mathcal{M}}(G, b, \mu) = [J_b(\mathbb{Q}_p) \times \breve{\mathcal{M}}(G, b, \mu)^+]/J_b(\mathbb{Q}_p)^+ \simeq \coprod_{J_b(\mathbb{Q}_p)/J_b(\mathbb{Q}_p)^+} \breve{\mathcal{M}}(G, b, \mu)^+,$$

where $J_b(\mathbb{Q}_p)^+ \subset J_b(\mathbb{Q}_p)$ is the stabilizer of $X_\mu^G(b)^+$ under the action of $J_b(\mathbb{Q}_p)$ on $X_\mu^G(b)$.

The above construction does not depend on the choice of the unramified local Shimura datum of Hodge type $(G_1, [b_1], \{\mu_1\})$ as in the statement of the theorem, since if $(G_2, [b_2], \{\mu_2\})$ is another such one, then we have a canonical isomorphism

$$\breve{\mathcal{M}}(G_1, b_1, \mu_1)^+ \simeq \breve{\mathcal{M}}(G_2, b_2, \mu_2)^+.$$

This follows from the bijection $X_{\mu_1}^{G_1}(b_1)^+ \simeq X_{\mu_2}^{G_2}(b_2)^+$, the isomorphism of deformation rings $R_{G_1,x_1} \simeq R_{G_2,x_2}$, where $X_{\mu_1}^{G_1}(b_1)^+ \ni x_1 \mapsto x_2 \in X_{\mu_2}^{G_2}(b_2)^+$, cf. [28] 1.5.4 (from the description there, R_G depends only on the adjoint group G^{ad}), and the constructions in section 6 of [26].

□

Let $(G, [b], \{\mu\})$ be a unramified local Shimura datum of abelian type. Take an embedding $G \hookrightarrow \mathrm{GL}_n$. Then we get an induced triple $(\mathrm{GL}_n, [b'], \{\mu'\})$. If $(G, [b], \{\mu\})$ is not of Hodge type, then $\{\mu'\}$ is not minuscule. In any case, we have the embedding

$$\mathcal{M}(G, b, \mu)_{red}^{perf} \simeq X_\mu^G(b) \hookrightarrow X_{\leq \mu'}^{\mathrm{GL}_n}(b').$$

4.3. **A moduli interpretation.** Let $(G, [b], \{\mu\})$ be as in Theorem 4.7. Then by construction, locally the formal scheme $\breve{\mathcal{M}}(G, b, \mu)$ admits a moduli interpretation. More precisely, take $(G_1, [b_1], \{\mu_1\})$ as in Theorem 4.7. Then the formal scheme $\breve{\mathcal{M}}(G_1, b_1, \mu_1)$ is a moduli space of p-divisible groups with Tate tensors. In particular, $\breve{\mathcal{M}}(G, b, \mu)^+$ is a moduli space of p-divisible groups with Tate tensors such that under the map ω_{G_1} the image is fixed.

Suppose now that there exists a triple $(G_1, [b_1], \{\mu_1\})$ as in Theorem 4.7 such that the map

$$\pi_1(G_1)^\Gamma \to \pi_1(G_1^{ad})^\Gamma$$

is surjective. Then the formal scheme $\breve{\mathcal{M}}(G, b, \mu)$ admits a global moduli interpretation as follows. Let Z_{G_1} be the center of G_1.

Proposition 4.8. *Under the above setup,*

(1) *we have an isomorphism of formal schemes*

$$\breve{\mathcal{M}}(G_1^{ad}, b_1^{ad}, \mu_1^{ad}) \simeq \breve{\mathcal{M}}(G_1, b_1, \mu_1)/X_*(Z_{G_1})^\Gamma.$$

(2) $\breve{\mathcal{M}}(G, b, \mu)$ *is the pullback of* $\breve{\mathcal{M}}(G_1, b_1, \mu_1)/X_*(Z_{G_1})^\Gamma$ *under the morphism* $\pi_1(G)^\Gamma \to \pi_1(G^{ad})^\Gamma$.

Proof. We have the following cartesian diagrams

$$\begin{array}{ccccc} X_{\mu_1}^{G_1}(b_1) & \longrightarrow & X_{\mu_1^{ad}}^{G_1^{ad}}(b_1^{ad}) & \longleftarrow & X_\mu^G(b) \\ \downarrow & & \downarrow & & \downarrow \\ c_{b_1,\mu_1}\pi_1(G_1)^\Gamma & \longrightarrow & c_{b_1^{ad},\mu_1^{ad}}\pi_1(G_1^{ad})^\Gamma & \longleftarrow & c_{b,\mu}\pi_1(G)^\Gamma \end{array}$$

All the vertical maps are surjective by Proposition 2.4. The assertions follow by the assumption $\pi_1(G)^\Gamma \to \pi_1(G_1)^\Gamma$ is surjective. □

Example 4.9. *Consider Example 4.2 again. As the exact sequence* $1 \to \mathbb{G}_m \to \mathrm{GL}_n \to \mathrm{PGL}_n \to 1$ *induces a surjection*

$$\pi_1(\mathrm{GL}_n)^\Gamma = \pi_1(\mathrm{GL}_n) \to \pi_1(\mathrm{PGL}_n)^\Gamma,$$

we have

$$\breve{\mathcal{M}}(\mathrm{PGL}_n, b, \mu) \simeq \breve{\mathcal{M}}(\mathrm{GL}_n, b_1, \mu_1)/p^{\mathbb{Z}}.$$

Another example will be given in section 7.

By construction, both the above local moduli interpretation in the general case, and the global moduli interpretation in the above Proposition 4.8 are not canonical. Moreover, the formal scheme $\breve{\mathcal{M}}(G, b, \mu)$ associated to a unramified local Shimura datum of abelian type but not of Hodge type does not admit a moduli interpretation by p-divisible groups.

Nevertheless, when passing to the generic fibers, they are indeed canonical moduli spaces of some objects (local G-shtukas in the sense of Scholze): see the next section.

4.4. **Generic fibers and local Shimura varieties of abelian type.** Let $(G,[b],\{\mu\})$ and $\breve{\mathcal{M}} = \breve{\mathcal{M}}(G,b,\mu)$ be as in Theorem 4.7. We consider the rigid analytic fiber $\mathcal{M} = \mathcal{M}(G,b,\mu)$ as an adic space over Spa$L$. For any open compact subgroup $K \subset G(\mathbb{Z}_p)$, we construct a finite étale cover $\mathcal{M}_K$ of $\mathcal{M}$ as follows.

First, assume that $K = K_n$ for some $n \geq 1$, where $K_n = \ker\Big(G(\mathbb{Z}_p) \to G(\mathbb{Z}_p/p^n\mathbb{Z}_p)\Big)$. On the component $\mathcal{M}^+ = \breve{\mathcal{M}}(G,b,\mu)^{+,ad}$, we can construct a finite étale cover $\mathcal{M}_n^+$ by taking some unramified loal Shimura datum of Hodge type $(G_1,[b_1],\{\mu_1\})$ and using the moduli interpretation of $\mathcal{M}(G_1,b_1,\mu_1)$. As we can do this on each connected component of $\mathcal{M}$, we get a finite étale cover $\mathcal{M}_n$ of $\mathcal{M}$ by taking the disjoint union. Or equivalently, we can take

$$\mathcal{M}_n = [J_b(\mathbb{Q}_p) \times \mathcal{M}_n^+]/J_b(\mathbb{Q}_p)^+.$$

In this way we get a tower

$$(\mathcal{M}_n)_n,$$

on which $G(\mathbb{Z}_p)$ acts. Set $\mathcal{M}_0 = \mathcal{M}$. The action of $G(\mathbb{Z}_p)$ on $\mathcal{M}_n$ factors through $G(\mathbb{Z}_p)/K_n = G(\mathbb{Z}_p/p^n\mathbb{Z}_p)$. Now let $K \subset G(\mathbb{Z}_p)$ be arbitrary. Take some sufficiently large n such that $K_n \subset K$. Set

$$\mathcal{M}_K = \mathcal{M}_n/K.$$

Then $\mathcal{M}_K$ is a finite étale cover of $\mathcal{M}$, and it does not depend on the choice of n. When $K \subset G(\mathbb{Z}_p)$ is normal, this cover is a Galois cover with Galois group $G(\mathbb{Z}_p)/K$. For any $g \in G(\mathbb{Q}_p)$ and any open compact subgroup $K \subset G(\mathbb{Z}_p)$, we have a natural isomorphism

$$\mathcal{M}_K \stackrel{\sim}{\longrightarrow} \mathcal{M}_{gKg^{-1}}.$$

As a result, the group $G(\mathbb{Q}_p)$ acts on the tower

$$(\mathcal{M}_K)_{K \subset G(\mathbb{Z}_p)}$$

by Hecke correspondences.

As before, for any open compact $K \subset G(\mathbb{Z}_p)$, let $\mathcal{M}_K^+ \subset \mathcal{M}_K$ be the pullback of $\mathcal{M}^+ \subset \mathcal{M}$. In this way we get a sub tower $(\mathcal{M}_K^+)_K \subset (\mathcal{M}_K)_K$. Let $G(\mathbb{Q}_p)^+ \subset G(\mathbb{Q}_p)$ be the subgroup which is the stabilizer of the sub tower

$$(\mathcal{M}_K^+)_K \subset (\mathcal{M}_K)_K.$$

By Lemma 2.3 (1) the map

$$\omega_G : G(\mathbb{Q}_p) \to \pi_1(\mathbb{Q}_p)^\Gamma$$

is surjective. By construction we have

$$\omega_G : G(\mathbb{Q}_p)/G(\mathbb{Q}_p)^+ \stackrel{\sim}{\longrightarrow} \pi_1(\mathbb{Q}_p)^\Gamma,$$

and

$$\mathcal{M}_K = J_b(\mathbb{Q}_p)\mathcal{M}_K^+, \quad (\mathcal{M}_K)_K = G(\mathbb{Q}_p)(\mathcal{M}_K^+)_K.$$

Let $(G,[b],\{\mu\})$ be a unramified local Shimura datum of abelian type. Take any $(G_1,[b_1],\{\mu_1\})$ as in Theorem 4.7. Then we have the canonical identification of the associated p-adic flag varieties over L

$$\mathscr{F}\ell_{G,\mu} = G/P_\mu = \mathscr{F}\ell_{G_1,\mu_1} = G_1/P_{\mu_1}.$$

We will simply write them as $\mathscr{F}\ell_\mu$. By [26] 7.5, we have a period map

$$\pi_{G_1,dR} : \mathcal{M}(G_1,b_1,\mu_1) \to \mathscr{F}\ell_\mu,$$

which is $J_{b_1}(\mathbb{Q}_p)$-equivalent. Restricting this map to $\mathcal{M}(G_1,b_1,\mu_1)^+=\mathcal{M}^+$, we get a map

$$\pi_{dR}^+:\mathcal{M}^+=\mathcal{M}(G_1,b_1,\mu_1)^+\to \mathscr{F}\ell_\mu.$$

Then applying the group action of $J_b(\mathbb{Q}_p)$, we can define a $J_b(\mathbb{Q}_p)$-equivalent period map for $\mathcal{M}$

$$\pi_{G,dR}:\mathcal{M}=\mathcal{M}(G,b,\mu)\to \mathscr{F}\ell_\mu.$$

Let $\mathscr{F}\ell_{G_1,\mu_1}^{adm}\subset\mathscr{F}\ell_\mu$ and $\mathscr{F}\ell_{G,\mu}^{adm}\subset\mathscr{F}\ell_\mu$ be the images of the above period maps $\pi_{G_1,dR}$ and $\pi_{G,dR}$ respectively, which are open subsapces of $\mathscr{F}\ell_\mu$. Let $\mathscr{F}\ell_{G_1,\mu_1}^{adm,+}\subset\mathscr{F}\ell_{G_1,\mu_1}^{adm}$ be the image of π_{dR}^+.

For any open compact subgroup $K\subset G(\mathbb{Z}_p)$, we have the finite étale map $\mathcal{M}_K^+=\mathcal{M}(G_1,b_1,\mu_1)_K^+\to\mathcal{M}^+=\mathcal{M}(G_1,b_1,\mu_1)^+$, thus we get a morphism

$$\mathcal{M}_K^+\to\mathscr{F}\ell_\mu$$

with image $\mathscr{F}\ell_{G_1,\mu_1}^{adm,+}$. From this we can define a $J_b(\mathbb{Q}_p)$-equivalent period map for $\mathcal{M}_K$

$$\pi_{G,dR}:\mathcal{M}_K\to\mathscr{F}\ell_\mu.$$

When K varies, these period maps are compatible with the Hecke action of $G(\mathbb{Q}_p)$ on $(\mathcal{M}_K)_K$. Thus we may think that there exists a $G(\mathbb{Q}_p)$-invariant map $(\mathcal{M}_K)_K\to\mathscr{F}\ell_\mu$.

Recall that we also have $\mathscr{F}\ell_{G_1,\mu_1}^{wa}$ and $\mathscr{F}\ell_{G,\mu}^{wa}$. By construction, we have $\mathscr{F}\ell_{G_1,\mu_1}^{adm}\subset\mathscr{F}\ell_{G_1,\mu_1}^{wa}$, and similarly $\mathscr{F}\ell_{G,\mu}^{adm}\subset\mathscr{F}\ell_{G,\mu}^{wa}$.

Lemma 4.10. *We have*

$$\mathscr{F}\ell_{G_1,\mu_1}^{wa}=\mathscr{F}\ell_{G,\mu}^{wa},\quad \mathscr{F}\ell_{G_1,\mu_1}^{adm}=\mathscr{F}\ell_{G,\mu}^{adm}.$$

Proof. The equality $\mathscr{F}\ell_{G_1,\mu_1}^{wa}=\mathscr{F}\ell_{G,\mu}^{wa}$ follows by [7] Proposition 9.5.3 (iv).

Since $(\mathcal{M}(G_1,b_1,\mu_1)_K)_K=G_1(\mathbb{Q}_p)(\mathcal{M}(G_1,b_1,\mu_1)_K^+)_K$ and the map $\mathcal{M}(G_1,b_1,\mu_1)_K\to\mathscr{F}\ell_{G_1,\mu_1}^{adm}$ is $G_1(\mathbb{Q}_p)$-invariant, we get

$$\mathscr{F}\ell_{G_1,\mu_1}^{adm}=\mathscr{F}\ell_{G_1,\mu_1}^{adm,+}.$$

We have also $(\mathcal{M}(G,b,\mu)_K)_K=G(\mathbb{Q}_p)(\mathcal{M}(G,b,\mu)_K^+)_K$, and by our construction the map $(\mathcal{M}(G,b,\mu)_K)_K\to\mathscr{F}\ell_\mu$ is $G(\mathbb{Q}_p)$-invariant, we get also

$$\mathscr{F}\ell_{G,\mu}^{adm}=\mathscr{F}\ell_{G_1,\mu_1}^{adm,+}.$$

□

Remark 4.11. (1) *There are several descriptions of the subspace $\mathscr{F}\ell_{G,\mu}^{adm}\subset\mathscr{F}\ell_{G,\mu}$: in* [21] *(and* [42]*) it is described using the Robba rings; in* [11] *it is described using the crystalline period ring B_{cris}; and in* [43, 35] *it is described using the Fargues-Fontaine curve. See also Proposition 5.13.*

(2) *We always have $\mathscr{F}\ell_{G,\mu}^{adm}\subset\mathscr{F}\ell_{G,\mu}^{wa}$. In* [35] *Question A. 20, Rapoport asked that when is $\mathscr{F}\ell_{G,\mu}^{adm}=\mathscr{F}\ell_{G,\mu}^{wa}$? Fargues conjectures that this holds true if and only if $(G,\{\mu\})$ is fully Hodge-Newton decomposable in the sense of* [19] *Definition 2.1 (2), cf.* [19] *Theorem B and Conjecture 0.1. In the appendix we will see that $\mathscr{F}\ell_{G,\mu}^{adm}=\mathscr{F}\ell_{G,\mu}^{wa}$ in the case b is basic and G is the special orthogonal group.*

Recall that by Lemma 2.3 (1) the map

$$\omega_G:G(\mathbb{Q}_p)/G(\mathbb{Z}_p)\to\pi_1(G)^\Gamma$$

is surjective.

Lemma 4.12. (1) *The following diagram is cartesian:*

$$\begin{array}{ccc} G(\mathbb{Q}_p)/G(\mathbb{Z}_p) & \xrightarrow{\omega_G} & \pi_1(G)^\Gamma \\ \downarrow & & \downarrow \\ G^{ad}(\mathbb{Q}_p)/G^{ad}(\mathbb{Z}_p) & \xrightarrow{\omega_{G^{ad}}} & \pi_1(G^{ad})^\Gamma. \end{array}$$

(2) *In particular, for G and G_1 as above we have $G(\mathbb{Q}_p)^+ \simeq G_1(\mathbb{Q}_p)^+$.*

Proof. Note that non empty fibers of both vertical maps are torsors under $X_*(Z_G)^\Gamma$. By [29] Lemma 1.2.4, if $g^{ad} \in G^{ad}(\mathbb{Q}_p)/G^{ad}(\mathbb{Z}_p)$ and $\omega_{G^{ad}}(g^{ad})$ lifts to an element of $\pi_1(G)^\Gamma$, then g^{ad} lies in the image of $G(\mathbb{Q}_p)/G(\mathbb{Z}_p) \to G^{ad}(\mathbb{Q}_p)/G^{ad}(\mathbb{Z}_p)$. Therefore the above diagram is cartesian.

In particular, we have the bijection $G(\mathbb{Q}_p)^+ \simeq G_1(\mathbb{Q}_p)^+$ from (1) for G and G_1 as above. □

By construction, we have

Proposition 4.13. *There exists a $\mathbb{Q}_p$-G-local system $\mathbb{V}$ on $\mathscr{F}\ell^{adm}_{G,\mu}$ such that $\mathcal{M}(R, R^+)$ is the set of $\mathbb{Z}_p$-lattices in $\mathbb{V}_{\mathrm{Spa}(R,R^+)}$. In particular, there exists a $\mathbb{Z}_p$-G-local system $\mathbb{L}$ on $\mathcal{M}$, and the tower $(\mathcal{M}_K)_{K\subset G(\mathbb{Z}_p)}$ is obtained by trivializing $\mathbb{L}$.*

Proof. Under the identity $\mathscr{F}\ell^{adm}_{G_1,\mu_1} = \mathscr{F}\ell^{adm}_{G,\mu}$, we have a $\mathbb{Q}_p - G_1$-local system $\mathbb{V}_1$ on $\mathscr{F}\ell^{adm}_{G,\mu}$ such that $\mathcal{M}_1(R, R^+)$ is the set of $\mathbb{Z}_p$-lattices in $\mathbb{V}_{1\mathrm{Spa}(R,R^+)}$. The tower $(\mathcal{M}_{1K})_{K\subset G_1(\mathbb{Z}_p)}$ is the geometric realization of $\mathbb{Q}_p - G_1$-local system $\mathbb{V}_1$ on $\mathscr{F}\ell^{adm}_{G,\mu}$. Fix any geometric point $\overline{x} \to \mathscr{F}\ell^{adm}_{G,\mu}$, and let

$$\rho_{\mathbb{V}_1,\overline{x}} : \pi_1(\mathscr{F}\ell^{adm}_{G,\mu}, \overline{x}) \to G_1(\mathbb{Q}_p)$$

be the p-adic representation of the (de Jong's) fundamental group $\pi_1(\mathscr{F}\ell^{adm}_{G,\mu}, \overline{x})$ corresponding to $\mathbb{V}_1$, cf. [23] Theorem 4.2. Then $\pi_1(\mathscr{F}\ell^{adm}_{G,\mu}, \overline{x})$ acts on $G_1(\mathbb{Q}_p)$ through $\rho_{\mathbb{V}_1,\overline{x}}$. The group $J_{b_1}(\mathbb{Q}_p)$ acts on $G_1(\mathbb{Q}_p)$ as $\mathbb{V}_1$ on $\mathscr{F}\ell^{adm}_{G,\mu}$ is $J_{b_1}(\mathbb{Q}_p)$-equivariant.

Fix a point $x_0 \in \pi_1(G_1)^\Gamma$. Then we have the associated $\breve{\mathcal{M}}_1^+$ and $(\mathcal{M}^+_{1K})_K$. The tower $(\mathcal{M}^+_{1K})_K$ defines a subset $G_1(\mathbb{Q}_p)^+ \subset G_1(\mathbb{Q}_p)$. By Lemma 4.12 (2), we have $G(\mathbb{Q}_p)^+ \simeq G_1(\mathbb{Q}_p)^+$. Therefore, we can define an action of $\pi_1(\mathscr{F}\ell^{adm}_{G,\mu}, \overline{x})$ on $G(\mathbb{Q}_p)$, which commutes with the natural action of $J_b(\mathbb{Q}_p)$. Thus we get a p-adic representation

$$\rho_{\overline{x}} : \pi_1(\mathscr{F}\ell^{adm}_{G,\mu}, \overline{x}) \to G(\mathbb{Q}_p),$$

which defines the desired $\mathbb{Q}_p$-G-local system $\mathbb{V}$ on $\mathscr{F}\ell^{adm}_{G,\mu}$. □

Let $(G, [b], \{\mu\})$ be a unramified local Shimura datum of abelian type. For each open compact subgroup $K \subset G(\mathbb{Q}_p)$, we get the associated Rapoport-Zink space

$$\mathcal{M}_K \simeq \coprod_{\pi_1(G)^\Gamma} \mathcal{M}^+_K.$$

Let Δ_G be the image of $\pi_1(G)^\Gamma \to \pi_1(G^{ad})^\Gamma$. This is a finite group. We have an exact sequence

$$1 \to X_*(Z_G)^\Gamma \to \pi_1(G)^\Gamma \to \Delta_G \to 1.$$

We have the Hecke action of $G(\mathbb{Q}_p)$ on the tower $(\mathcal{M}_K)_K$. The Hecke action of the central subgroup $Z_G(\mathbb{Q}_p) \subset G(\mathbb{Q}_p)$ stabilizes each $\mathcal{M}_K$. This action of $Z_G(\mathbb{Q}_p)$ is the same of that induced from $J_b(\mathbb{Q}_p)$ when we view $Z_G(\mathbb{Q}_p) \subset J_b(\mathbb{Q}_p)$. This action on

$$\begin{aligned}\mathcal{M}_K &\simeq \coprod_{\pi_1(G)^\Gamma} \mathcal{M}_K^+ \\ &= \coprod_{\Delta_G} \coprod_{X_*(Z_G)^\Gamma} \mathcal{M}_K^+\end{aligned}$$

is through the map $Z_G(\mathbb{Q}_p) \to X_*(Z_G)^\Gamma$ and the injection $X_*(Z_G)^\Gamma \to \pi_1(G)^\Gamma$.

In summary, the tower $(\mathcal{M}_K)_{K\subset G(\mathbb{Z}_p)}$ associated to a unramified local Shimura datum of abelian type can be viewed as the local Shimura varieties thought of in Conjecture 3.2. In the next section, we will put these spaces in a more general framework to get some moduli interpretation for each $\mathcal{M}_K$.

4.5. **Infinite level and the Hodge-Tate period map.** Let $(G, [b], \{\mu\})$ be a unramified local Shimura datum of abelian type, and $(\mathcal{M}_K)_K$ be associated tower of Rapoport-Zink spaces of abelian type. Let $\mathscr{F}\ell_{G,-\mu}$ be the p-adic flag variety over L associated to $(G, \{\mu^{-1}\})$.

Proposition 4.14. *There exists a preperfectoid space $\mathcal{M}_\infty$ over L such that*

$$\mathcal{M}_\infty \sim \varprojlim_K \mathcal{M}_K.$$

Moreover, there exists a Hodge-Tate period map

$$\pi_{HT} : \mathcal{M}_\infty \to \mathscr{F}\ell_{G,-\mu},$$

which agrees with the period map previously defined in the EL/PEL cases in [43, 4].

Proof. If $(G, [b], \{\mu\})$ is of Hodge type, the existence of the preperfectoid space $\mathcal{M}_\infty$ over L such that $\mathcal{M}_\infty \sim \varprojlim_K \mathcal{M}_K$ is proved in [26] Proposition 7.6.1. Fix an embedding $(G, [b], \{\mu\}) \hookrightarrow (\mathrm{GL}_n, [b'], \{\mu'\})$ with $\{\mu'\}$ minuscule. We have the associated preperfeoid space $\mathcal{M}(\mathrm{GL}_n, b', \mu')_\infty$ over L such that $\mathcal{M}(\mathrm{GL}_n, b', \mu')_\infty \sim \varprojlim_{K'} \mathcal{M}(\mathrm{GL}_n, b', \mu')_{K'}$. The Hodge-Tate period map

$$\pi_{HT} : \mathcal{M}(\mathrm{GL}_n, b', \mu')_\infty \to \mathscr{F}\ell_{\mathrm{GL}_n,-\mu'}$$

is defined in [43] 7.1. Arguing as [4] section 2, we get that the composition

$$\mathcal{M}_\infty \hookrightarrow \mathcal{M}(\mathrm{GL}_n, b', \mu')_\infty \to \mathscr{F}\ell_{\mathrm{GL}_n,-\mu'}$$

factors through $\mathscr{F}\ell_{G,-\mu}$. In particular we get

$$\pi_{HT} : \mathcal{M}_\infty \to \mathscr{F}\ell_{G,-\mu}.$$

Now assume that we are in the general case. As $J_b(\mathbb{Q}_p)$ acts on $|\mathcal{M}_\infty| := \varprojlim_K |\mathcal{M}_K|$, it suffices to prove that there exist a preperfectoid space $\mathcal{M}_\infty^+$ over L such that

$$\mathcal{M}_\infty^+ \sim \varprojlim_K \mathcal{M}_K^+,$$

and a Hodge-Tate period map

$$\pi_{HT}^+ : \mathcal{M}_\infty^+ \to \mathscr{F}_{G,-\mu}.$$

This follows from the Hodge type case. ☐

The following corollary is clear now.

Corollary 4.15. *There exists a sub preperfectoid space $\mathcal{M}_\infty^+ \subset \mathcal{M}_\infty$ over L, which is stable under $G(\mathbb{Q}_p)^+$, such that*

$$\mathcal{M}_\infty^+ \sim \varprojlim_K \mathcal{M}_K^+, \quad \mathcal{M}_\infty = G(\mathbb{Q}_p)\mathcal{M}_\infty^+.$$

5. Generic fibers of Rapoport-Zink spaces as moduli of local G-shtukas

In this section, we work purely on generic fibers. We want to explain that the generic fibers of the formal schemes $\breve{\mathcal{M}}(G, b, \mu)$, associated to unramified local Shimura data of abelian type $(G, [b], \{\mu\})$, can be viewed as moduli spaces fo local G-shtukas in mixed characteristic in the sense of Scholze, cf. [42]. We will work in the more general context of Conjecture 3.2. The first few subsections are denoted as some review of works of Fargues [14, 17] and Scholze [42]. The reader familiar with these can go directly to subsection 5.5. We do not claim any originality here.

5.1. **The Fargues-Fontaine curve and G-bundles.** The Fargues-Fontaine curve $X_{E,F}$ is associated to a datum (E, F), where E is a local field with finite residue field $\mathbb{F}_q$ and $F|\mathbb{F}_q$ is a perfectoid field of characteristic p. For our purpose, we set $E = \mathbb{Q}_p$, and denote simply $X_{\mathbb{Q}_p,F}$ as X_F. It has several incarnations.

5.1.1. *The adic curve.* The adic curve X_F admits the following adic uniformization

$$X_F = Y_F/\varphi^{\mathbb{Z}},$$

where $Y_F = \mathrm{Spa}(W(\mathcal{O}_F)) \setminus V(p[\varpi_F])$, with $\varpi_F \in F$ satisfying $0 < |\varpi_F| < 1$. The action of the Frobenius φ on the Witt vectors is given by

$$\varphi(\sum_n [x_n]p^n) = \sum_n [x_n^p]p^n, \quad \forall \sum_n [x_n]p^n \in W(\mathcal{O}_F).$$

It induces a totally discontinuous action on Y_F.

Suppose now that F is algebraically closed. Then there is a unique non analytic point $x_k \in \mathrm{Spa}(W(\mathcal{O}_F))$. Set $\mathcal{Y} = \mathcal{Y}_F = \mathrm{Spa}(W(\mathcal{O}_F)) \setminus \{x_k\}$. There exists a surjective continuous map $\kappa : \mathcal{Y} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ defined by

$$\kappa(x) = \frac{\log |[\varpi_F](\widetilde{x})|}{\log |p(\widetilde{x})|},$$

where $\widetilde{x}$ is the unique maximal generalization of x, cf. [42] 12.2. For any $I \subset \mathbb{R}_{\geq 0} \cup \{\infty\}$, we denote $\mathcal{Y}_I = \kappa^{-1}(I)$. Then $Y := Y_F = \mathcal{Y}_{(0,\infty)}$.

Let $I \subset [0, \infty]$ be an interval of the form $[r, \infty)$ or $[r, \infty]$. Recall that a ϕ-module over $\mathcal{Y}_I$ is a pair $(\mathcal{E}, \phi_{\mathcal{E}})$, where $\mathcal{E}$ is a vector bundle over $\mathcal{Y}_I$ and $\phi_E : \phi^*\mathcal{E}|_{\mathcal{Y}_I} \to \mathcal{E}$ is an isomorphism, cf. [42] Definition 13.2.1. It follows that ϕ-modules over $\mathcal{Y}_{(0,\infty)}$ are the same as vector bundles over $X := X_F$.

5.1.2. *The algebraic curve.* There is natural line bundle $\mathcal{O}(1)$ on X_F, corresponding to the ϕ-module on $\mathcal{Y}_{(0,\infty)}$ whose underlying line bundle is trivial and for which ϕ is $p^{-1}\varphi$. Set $\mathcal{O}(n) = \mathcal{O}(1)^{\otimes n}$, and

$$P = \bigoplus_{n \geq 0} H^0(X, \mathcal{O}(n)).$$

We have

$$H^0(X, \mathcal{O}(n)) = \mathcal{O}(Y)^{\varphi = p^n}.$$

Let

$$X^{sch} = \mathrm{Proj}(P).$$

By [17], this is a one dimensional noetherian regular scheme over $\mathbb{Q}_p$. There exists a morphism of ringed spaces

$$X \to X^{sch},$$

and X may be viewed as the analytification of X^{sch} in some generalized sense.

Remark 5.1. *Using the theory of diamond developed in* [42]*, the curve admits yet another version: the diamond curve*

$$X^\diamond = (\mathrm{Spa}(F) \times \mathrm{Spa}(\mathbb{Q}_p)^\diamond)/\varphi^{\mathbb{Z}},$$

where $\varphi = Frob_F \times Id$*. We will not use this version in the following.*

Let $\mathrm{Bun}_{X^{sch}}$ and Bun_X be the categories of vector bundles on X^{sch} and X respectively. The morphism $X \longrightarrow X^{sch}$ induces a GAGA functor

$$\mathrm{Bun}_{X^{sch}} \longrightarrow \mathrm{Bun}_X.$$

Theorem 5.2 ([25, 13])**.** *The GAGA functor induces an equivalence of categories* $\mathrm{Bun}_{X^{sch}} \xrightarrow{\sim} \mathrm{Bun}_X$*.*

There is another way to describe vector bundles on X. Consider the Robba ring

$$\widetilde{\mathcal{R}}_F = \varinjlim_r H^0(\mathcal{Y}_{(0,r]}, \mathcal{O}_{\mathcal{Y}_{(0,r]}}).$$

The Frobenius ϕ induces an action on $\widetilde{\mathcal{R}}_F$. Recall a ϕ-module over $\widetilde{\mathcal{R}}_F$ is a finite free $\widetilde{\mathcal{R}}_F$-module M equipped with a ϕ-linear automorphism.

Theorem 5.3 ([25], Theorem 6.3.12)**.** *There is an equivalence of categories*

$$\mathrm{Bun}_X \simeq \{\phi - \textit{modules over} \widetilde{\mathcal{R}}_F\}.$$

The idea for the proof is that any ϕ-module over $\widetilde{\mathcal{R}}_F$ is defined over $\widetilde{\mathcal{R}}_F^r := H^0(\mathcal{Y}_{(0,r]}, \mathcal{O}_{\mathcal{Y}_{(0,r]}})$ for some r small enough. This can be spread to a ϕ-module over $Y_F = \mathcal{Y}_{(0,\infty)}$ via pullback under Frobenius. Giving a ϕ-module over $\mathcal{Y}_{(0,\infty)}$ is the same giving a vector bundle over X_F by the uniformization $X_F = \mathcal{Y}_{(0,\infty)}/\varphi^{\mathbb{Z}}$.

Let $\varphi - \mathrm{Mod}_L$ be the category of F-isocystals over $\overline{\mathbb{F}}_p$, where as before $L = W(\overline{\mathbb{F}}_p)_{\mathbb{Q}}$. For any $(D, \varphi) \in \varphi - \mathrm{Mod}_L$, we can construct a vector bundle $\mathcal{E}(D, \varphi)$ on X^{sch} by

$$\mathcal{E}(D,\varphi) = \mathrm{Proj}(\bigoplus_{n\geq 0}(D \otimes_L \mathcal{O}(Y))^{\varphi\otimes\varphi = p^n}).$$

Theorem 5.4 ([17])**.** *The functor* $\mathcal{E}(-) : \varphi - \mathrm{Mod}_L \to \mathrm{Bun}_{X^{sch}}$ *is essentially surjective.*

Therefore, the composite $\mathcal{E}(-) : \varphi - \mathrm{Mod}_L \to \mathrm{Bun}_{X^{sch}} \longrightarrow \mathrm{Bun}_X$ is also essentially surjective.

Let G be a reductive group over $\mathbb{Q}_p$. We have the following three equivalent definitions of a G-bundle on X (or equivalently on X^{sch}):

(1) an exact tensor functor $\mathrm{Rep}G \to \mathrm{Bun}_X$, where $\mathrm{Rep}G$ is the category of rational algebraic representations of G,
(2) a G-torsor on X locally trivial for the étale topology,
(3) a vector bundle $\mathcal{E}$ of rank n together with a locally direct factor sub line bundle of $\rho_*\mathcal{E}$, where we take an embedding $G \subset \mathrm{GL}_n$ and a representation $\rho : \mathrm{GL}_n \to \mathrm{GL}(W)$ with a line $D \subset W$ such that G is the stabilizer of D inside GL_n.

Recall that an F-isocrystal with G-structure over $\overline{\mathbb{F}}_p$ is an exact tensor functor

$$\mathrm{Rep}G \to \varphi - \mathrm{Mod}_L.$$

If $b \in G(L)$, it then defines an F-isocystal with a G-structure

$$M_b : \mathrm{Rep}G \to \varphi - \mathrm{Mod}_L$$
$$(V, \rho) \mapsto (V_L, \rho(b)\sigma).$$

Its isomorphism class only depends on the σ-conjugacy class $[b] \in B(G)$ of b. Conversely, any F-isocrystal with G-structure arises in this way. Thus $B(G)$ is the set of isomorphism classes of F-isocrystals with G-structure, cf. [36] Remarks 3.4 (i). For $b \in G(L)$, let $\mathcal{E}_b$ be the composition of the above functor M_b and

$$\mathcal{E}(-) : \varphi - \mathrm{Mod}_L \to \mathrm{Bun}_{X^{sch}} \simeq \mathrm{Bun}_X.$$

In this way, the set $B(G)$ also classifies G-bundles on X. In fact, we have

Theorem 5.5 ([14])**.** *Assume that F is algebraically closed. Then there is a bijection of sets*

$$B(G) \xrightarrow{\sim} H^1_{et}(X, G)$$
$$[b] \mapsto [\mathcal{E}_b].$$

Let (R, R^+) be a perfectoid affinoid $\overline{\mathbb{F}}_p$-algebra, and $S = \mathrm{Spa}(R, R^+)$ be the associated perfectoid space. We have an adic space

$$X_S = Y_S/\varphi^{\mathbb{Z}},$$

with $Y_S = Y_{R,R^+} = \mathrm{Spa}(A, A^+) \setminus V(p[\varpi_R])$, where

$$A = W(R^\circ) = \{\sum_{n\geq 0}[x_n]p^n | x_n \in R^\circ\}, \quad A^+ = \{\sum_{n\geq 0}[x_n]p^n \in A | x_0 \in R^+\}.$$

The adic space X_S is the relative version of the Fargues-Fontaine curve. We can also define the scheme

$$X_S^{sch} = \mathrm{Proj}\Big(\bigoplus_{d\geq 0} H^0(X_S, \mathcal{O}_{X_S}(d))\Big).$$

Then there exists a map of locally ringed spaces $X_S \to X_S^{sch}$. We can define vector bundles on X_S, X_S^{sch} as above, and the relative Robba ring $\widetilde{\mathcal{R}}_R$. Moreover, we have

Theorem 5.6 ([13, 25])**.**

$$Bun_{X_S^{sch}} \simeq Bun_{X_S} \simeq \{\phi - \text{modules over} \widetilde{\mathcal{R}}_R\}.$$

Let G be a reductive group over $\mathbb{Q}_p$. Let $S = \mathrm{Spa}(R, R^+)$ be a affinoid perfectoid space over $\mathbb{F}_p$, and ϖ be a pseudo-uniformizer of R. We denote

$$\mathcal{Y}_{[0,\infty)}(R, R^+) = \mathrm{Spa} W(R^+) \setminus \{[\varpi] = 0\}.$$

Then we have a continuous map

$$\kappa : \mathcal{Y}_{[0,\infty)}(R, R^+) \to [0, \infty),$$

the relative version of the map defined in the last subsection. With the same notation there, we have

$$Y_S = \mathcal{Y}_{(0,\infty)}(R, R^+).$$

Then as above we can define G-bundles on X_S, $Y_S = \mathcal{Y}_{(0,\infty)}(R, R^+)$ and $\mathcal{Y}_{[0,\infty)}(R, R^+)$.

If we start with a perfectoid space S over $\mathbb{Q}_p$, then there exits a canonical closed embedding

$$x_S : S \hookrightarrow Y_{S^\flat}$$

which in turn induces a closed embedding

$$x_S : S \hookrightarrow X_{S^\flat},$$

cf. [14] 1.4. Thus we can view S as a Cartier divisor on $X_{S^\flat}$. If $S = \mathrm{Spa}(R, R^+)$ is perfectoid affinoid over $\mathbb{Q}_p$, by [14] 1.6 we have a corresponding Cartier divisor D on $X_{S^\flat}^{sch}$. The formal completion of $X_{S^\flat}^{sch}$ along D is

$$\mathrm{Spf} B^+_{dR,R},$$

cf. Proposition 1.33 of [14].

5.2. **Local G-shtukas in mixed characteristic.** Let the notations be as above. Recall that the following three sets are naturally identified (cf. [42] Proposition 11.3.1):

- sections of $\mathcal{Y}_{[0,\infty)}(R,R^+)^\diamond \to S^\diamond$,
- morphisms $S^\diamond \to \mathrm{Spa}(\mathbb{Z}_p)^\diamond$,
- untilts $S^\sharp$ of S.

In which case, there exists a closed embedding $S^\sharp \hookrightarrow \mathcal{Y}_{[0,\infty)}(R,R^+)$.

Definition 5.7 ([42] Definition 11.4.1)**.** *A local G-shtuka over S with one paw $x : S^\sharp \to \mathcal{Y}_{[0,\infty)}(R,R^+)$ is a pair $(\mathcal{E},\phi_{\mathcal{E}})$, where*

- *$\mathcal{E}$ is a G-bundle over $\mathcal{Y}_{[0,\infty)}(R,R^+)$,*
- *$\phi_{\mathcal{E}}$ is an isomorphism $\phi_{\mathcal{E}} : \phi^*\mathcal{E} \to \mathcal{E}$ over $\mathcal{Y}_{[0,\infty)}(R,R^+) \setminus \Gamma_x$, such that along Γ_x it is meromorphic. Here Γ_x is the image of x.*

One can then generalize the above definition to define a local G-shtuka over a general perfectoid space over $\mathbb{F}_p$.

Let C be a complete algebraically closed extension of $\mathbb{Q}_p$. We have the associated de Rham period ring $B^+_{dR} := B^+_{dR,C}$ with a fixed uniformizer $\xi \in B^+_{dR}$. Let $B^+_{dR} = B^+_{dR}[\frac{1}{\xi}], A_{\mathrm{inf}} = W(\mathcal{O}_{C^\flat})$. We have the following various descriptions of local G-shtukas with one paw at C, in the case $G = \mathrm{GL}_n$.

Theorem 5.8 ([42] Proposition 20.1.1; see also [16])**.** *The following categories are equivalent.*

1. *Shtukas over $\mathrm{Spa}(C^\flat,\mathcal{O}_{C^\flat})$ with one paw at C.*
2. *Pairs (T,Ξ), where T is a finite free $\mathbb{Z}_p$-module, and $\Xi \subset T \otimes B_{dR}$ is a B^+_{dR}-lattice.*
3. *Breuil-Kisin modules over A_{inf}.*
4. *Quadruples $(\mathcal{F},\mathcal{F}',\beta,T)$, where $\mathcal{F}$ and $\mathcal{F}'$ are vector bundles on the Fargues-Fontaine curve $X = X_{C^\flat}$, and $\beta : \mathcal{F}|_{X\setminus\{\infty\}} \overset{\sim}{\to} \mathcal{F}'_{X\setminus\{\infty\}}$ is an isomorphism, where $\mathcal{F}$ is trivial, and $T \subset H^0(X,\mathcal{F})$ is a $\mathbb{Z}_p$-lattice.*

If the paw is minuscule, i.e. we have

$$\xi(T \otimes_{\mathbb{Z}_p} B^+_{dR}) \subset \Xi \subset T \otimes_{\mathbb{Z}_p} B^+_{dR},$$

then these categories are equivalent to the category of p-divisible groups over $\mathcal{O}_C$.

5.3. **Moduli of local G-shtukas in mixed characteristic.**

Definition 5.9. (1) *A local shtuka datum is a triple $(G,[b],\{\mu\})$, where*

- *G is a reductive group over $\mathbb{Q}_p$,*
- *$\{\mu\}$ is a conjugacy class of cocharacters $\mu : \mathbb{G}_m \to G_{\overline{\mathbb{Q}}_p}$ over $\overline{\mathbb{Q}}_p$,*
- *$[b] \in B(G,\mu) \subset B(G)$.*

(2) *Let $(G_1,[b_1],\{\mu_1\}),(G_2,[b_2],\{\mu_2\})$ be two local shtuka data. A morphism*

$$(G_1,[b_1],\{\mu_1\}) \to (G_2,[b_2],\{\mu_2\})$$

is a hommorphism $f : G_1 \to G_2$ of algebraic groups sending $([b_1],\{\mu_1\})$ to $([b_2],\{\mu_2\})$.

Remark 5.10. (1) *By definition, a local Shimura datum $(G,[b],\{\mu\})$ is a local shtuka datum with $\{\mu\}$ minuscule.*

(2) *In [42], several $\{\mu\}$'s can be allowed, as in the classical function field case, cf. [48].*

(3) *In particular, if $(G,[b],\{\mu\})$ is a local shtuka datum, and $G \to G'$ is a hommorphism of reductive groups over $\mathbb{Q}_p$, we get the induced $[b'],\{\mu'\}$ such that $(G',[b'],\{\mu'\})$ is also a local shtuka datum.*

Let $(G, [b], \{\mu\})$ be a local shtuka datum. As before, we have the associated local reflex field E, and the reductive group J_b over $\mathbb{Q}_p$. Let F be an algebraically closed perfectoid field of characteristic p. By Theorem 5.5 we have a G-bundle on X_F, which is the same as a ϕ-module $(\mathcal{E}_b, \phi_{\mathcal{E}_b})$ on Y_F. We define a functor on the category of perfectoid affinoid algebras over $\overline{\mathbb{F}}_p$ as follows.

Definition 5.11 ([42] Definition 19.3.3)**.** *Let (R, R^+) be a perfectoid affinoid $\overline{\mathbb{F}}_p$-algebra together with a map $x : \mathrm{Spa}(R, R^+)^\diamond \to \mathrm{Spa}(\breve{E})^\diamond$. Let $\mathrm{Sht}(G, b, \mu) \to \mathrm{Spa}(\breve{E})^\diamond$ be the functor such that for any $\big((R, R^+), x\big)$,*

$$\mathrm{Sht}(G, b, \mu)\big((R, R^+), x\big) = \{\big((\mathcal{E}, \phi_{\mathcal{E}}), \iota\big)\}/\simeq$$

where

- *$(\mathcal{E}, \phi_{\mathcal{E}})$ is a G-shtuka over $\mathcal{Y}_{[0,\infty)}(R, R^+)$ with one paw at x, such that $(\mathcal{E}, \phi_{\mathcal{E}})$ is bounded by $\{\mu\}$.*
- *$\iota : (\mathcal{E}, \phi_{\mathcal{E}})|_{[\rho,\infty)} \xrightarrow{\sim} (\mathcal{E}_b, \phi_{\mathcal{E}_b})|_{[\rho,\infty)}$ is an isomorphism for some sufficiently large ρ.*

The main theorem of [42] is

Theorem 5.12 (Scholze, [42] Theorem 20.3.1)**.** *The functor $\mathrm{Sht}(G, b, \mu)$ is represented by a diamond over $\mathrm{Spa}(\breve{E})^\diamond$.*

(In [42] the theorem is proved for the case $G = \mathrm{GL}_n$, but one sees immediately that the proof given there works also for the general case.)

Assume that G is unramified from now on. Scholze's theorem above in fact tells us more information. More precisely, we get a tower of diamonds

$$\Big(\mathrm{Sht}(G, b, \mu)_K\Big)_{K \subset G(\mathbb{Q}_p)}$$

indexed by open compact subgroups $K \subset G(\mathbb{Q}_p)$ with $\mathrm{Sht}(G, b, \mu)_{G(\mathbb{Z}_p)} = \mathrm{Sht}(G, b, \mu)$, and the group $G(\mathbb{Q}_p)$ acts on this tower $\Big(\mathrm{Sht}(G, b, \mu)_K\Big)_{K \subset G(\mathbb{Q}_p)}$ by Hecke correspondences. Let (R, R^+) be a perfectoid affinoid $\overline{\mathbb{F}}_p$-algebra together with a map $x : \mathrm{Spa}(R, R^+)^\diamond \to \mathrm{Spa}(\breve{E})^\diamond$. Then

$$\mathrm{Sht}(G, b, \mu)_K\big((R, R^+), x\big) = \{\big((\mathcal{E}, \phi_{\mathcal{E}}), \iota, \alpha\big)\}/\simeq$$

where

- $(\mathcal{E}, \phi_{\mathcal{E}})$ is a G-shtuka over $\mathcal{Y}_{[0,\infty)}(R, R^+)$ with one paw at x, such that $(\mathcal{E}, \phi_{\mathcal{E}})$ is bounded by $\{\mu\}$.
- $\iota : (\mathcal{E}, \phi_{\mathcal{E}})|_{[\rho,\infty)} \xrightarrow{\sim} (\mathcal{E}_b, \phi_{\mathcal{E}_b})|_{[\rho,\infty)}$ is an isomorphism for some sufficiently large ρ.
- α is a K-orbit of an isomorphism $\mathbb{L}(\mathcal{E}, \phi_{\mathcal{E}}) \simeq \mathbb{L}_0$, where $\mathbb{L}(\mathcal{E}, \phi_{\mathcal{E}})$ is the G-local system associated to $(\mathcal{E}, \phi_{\mathcal{E}})$ (see below), $\mathbb{L}_0$ is the trivial G-local system over $\mathcal{Y}_{[0,\infty)}(R, R^+)$.

As

$$J_b(\mathbb{Q}_p) \subset Aut(\mathcal{E}_b, \phi_{\mathcal{E}_b}),$$

cf. [12] 2.5, $J_b(\mathbb{Q}_p)$ acts each $\mathrm{Sht}(G, b, \mu)_K$ by modifying ι, and these actions are compatible when K varies. When the context is clear, we will simply denote $\mathrm{Sht}(G, b, \mu)_K$ by Sht_K.

Consider the B_{dR}^+-affine Grassmannian $Gr_G^{B_{dR}^+}$ over $\mathbb{Q}_p$. This is the functor associating to any perfectoid affinoid $\mathbb{Q}_p$-algebra (R, R^+) the set

$$Gr_G^{B_{dR}^+}(R, R^+) = \{(\mathcal{E}, \beta)\}/\simeq$$

where $\mathcal{E}$ is a G-torsor over Spec$B^+_{dR,R}$, and β is a trivialization of $\mathcal{E}\otimes_{B^+_{dR,R}} B_{dR,R}$. One can check that $Gr_G^{B^+_{dR}}$ is the étale sheaf associated to the presheaf

$$(R,R^+)\mapsto G(B_{dR,R})/G(B^+_{dR,R}).$$

Consider the case (K,K^+) with K an algebraically closed perfectoid field. Then we have the Cartan decomposition

$$G(B_{dR,K})=\coprod_{\mu\in X_*(T)_+} G(B^+_{dR,K})\mu(\xi)G(B^+_{dR,K}),$$

where $T\subset B\subset G$ is a fixed choice of maximal torus inside a Borel subgroup B of G, and $X_*(T)_+\subset X_*(T)$ is the associated set of dominant cocharacters. Fix a conjugacy class of cocharacters $\{\mu\}$ with the dominant representative μ. Let E be the field of definition of $\{\mu\}$. Consider $Gr^{B^+_{dR}}_{G,\leq\mu}\subset Gr_G^{B^+_{dR}}\otimes E$ the sub functor such that

$$Gr^{B^+_{dR}}_{G,\leq\mu}(R,R^+)=\{(\mathcal{E},\xi)\in Gr_G^{B^+_{dR}}(R,R^+)|\mathrm{Inv}(\mathcal{E}_x,\mathcal{E}_{0x})\leq\mu,\,\forall x\in\mathrm{Spa}(R,R^+)\}.$$

This is the analogue of the classical Schubert variety associated to $\{\mu\}$ in the setting of B^+_{dR}-affine Grassmannian $Gr_G^{B^+_{dR}}$. There is an action of $J_b(\mathbb{Q}_p)$ on $Gr^{B^+_{dR}}_{G,\leq\mu}$. By abuse of notation, we still denote $Gr^{B^+_{dR}}_{G,\leq\mu}\to\mathrm{Spa}(\breve{E})^\diamond$ the sheaf base changed over $\mathrm{Spa}(\breve{E})^\diamond$. By Theorem 21.3.6 of [42], this is a diamond.

There exists an étale morphism of diamonds over $\mathrm{Spa}(\breve{E})^\diamond$ (cf. [42] 20.4)

$$\pi_{dR}:\mathrm{Sht}(G,b,\mu)\to Gr^{B^+_{dR}}_{G,\leq\mu}.$$

Let

$$Gr^{B^+_{dR},adm}_{G,\leq\mu}\subset Gr^{B^+_{dR}}_{G,\leq\mu}$$

be the image of π_{dR}. This is an open sub-diamond, and we call it the admissible locus. We have the following description of the admissible locus.

Proposition 5.13 ([42] 20.5, [25])**.** *Let* $(\mathcal{E},\beta)\in Gr^{B^+_{dR}}_{G,\leq\mu}(R,R^+)$. *Then*

$$(\mathcal{E},\beta)\in Gr^{B^+_{dR},adm}_{G,\leq\mu}(R,R^+)$$

if and only if one of the following equivalent conditions holds: for any representation $V\in$ Rep$G$, *with the associated vector bundle* $(\mathcal{E}_V,\beta_V)$,

(1) $\forall x\in\mathrm{Spa}(R,R^+)$ *the vector bundle* $\mathcal{E}_{V,x}$ *is semi-stable of slope 0;*
(2) ϕ*-module of* $\mathcal{E}_V$ *is trivial;*
(3) $\mathcal{E}_V$ *extends to a* ϕ*-module over* $\widetilde{\mathcal{R}}^{int}_R$, *where* $\widetilde{\mathcal{R}}^{int}_R=\varinjlim_r H^0(\mathcal{Y}_{[0,r]},\mathcal{O}_{\mathcal{Y}})$ *is the integral Robba ring.*

The action of $J_b(\mathbb{Q}_p)$ on $Gr^{B^+_{dR}}_{G,\leq\mu}$ stabilizes the open sub diamond $Gr^{B^+_{dR},adm}_{G,\leq\mu}$. The period morphism

$$\pi_{dR}:\mathrm{Sht}(G,b,\mu)\to Gr^{B^+_{dR},adm}_{G,\leq\mu}$$

is then $J_b(\mathbb{Q}_p)$-equivariant.

We have the following definition of local systems with additional structures on the diamond $Gr^{B^+_{dR},adm}_{G,\leq\mu}$, similar to the classical situation.

Definition 5.14. *Let X be a diamond, and G be a reductive group over $\mathbb{Q}_p$. Denote by* $\mathrm{Rep}G$ *the category of rational representations of G, and $\mathbb{Q}_p - Loc_X$ the category of $\mathbb{Q}_p$-local systems on X. Then a $\mathbb{Q}_p$-G-local system on X is a tensor functor* $\mathrm{Rep}G \to \mathbb{Q}_p - Loc_X$. *If G is moreover unramified, then we can define similarly $\mathbb{Z}_p$-G-local systems on X.*

By [25] Corollary 8.7.10, there exists a $J_b(\mathbb{Q}_p)$-equivariant $\mathbb{Q}_p$-G-local system $\mathbb{V}$ over $Gr_{G,\leq\mu}^{B_{dR}^+,adm}$, which realizes $\mathrm{Sht}(G,b,\mu)$ as the functor of the set of $\mathbb{Z}_p$-lattices in $\mathbb{V}$. In particular, there exists a $\mathbb{Z}_p$-G-local system $\mathbb{L}$ over $\mathrm{Sht}(G,b,\mu)$, and the cover

$$\pi_K : \mathrm{Sht}(G,b,\mu)_K \to \mathrm{Sht}(G,b,\mu)$$

is obtained by trivializing K-level structures, which is finite étale. By trivializing all of $\mathbb{L}$ we get a pro-étale cover

$$\pi_\infty : \mathrm{Sht}(G,b,\mu)_\infty \to \mathrm{Sht}(G,b,\mu).$$

We have the following moduli interpretation for $\mathrm{Sht}(G,b,\mu)_\infty$. Let (R,R^+) be a perfectoid affinoid $\overline{\mathbb{F}}_p$-algebra together with a map $x : \mathrm{Spa}(R,R^+)^\diamond \to \mathrm{Spa}(\breve{E})^\diamond$. Then

$$\mathrm{Sht}(G,b,\mu)_K\big((R,R^+),x\big) = \{\big((\mathcal{E},\phi_\mathcal{E}),\iota,\alpha\big)\}/\simeq$$

where

- $(\mathcal{E},\phi_\mathcal{E})$ is a G-shtuka over $\mathcal{Y}_{[0,\infty)}(R,R^+)$ with one paw at x, such that $(\mathcal{E},\phi_\mathcal{E})$ is bounded by $\{\mu\}$.
- $\iota : (\mathcal{E},\phi_\mathcal{E})|_{[\rho,\infty)} \xrightarrow{\sim} (\mathcal{E}_b,\phi_{\mathcal{E}_b})|_{[\rho,\infty)}$ is an isomorphism for some sufficiently large ρ.
- $\alpha : \mathbb{L}(\mathcal{E},\phi_\mathcal{E}) \simeq \mathbb{L}_0$ is an isomorphism, where as before $\mathbb{L}_0$ is the trivial G-local system over $\mathcal{Y}_{[0,\infty)}(R,R^+)$.

By construction, we have an isomorphism of diamonds over $\mathrm{Spa}(\breve{E})^\diamond$

$$\mathrm{Sht}(G,b,\mu)_\infty/K \simeq \mathrm{Sht}(G,b,\mu)_K, \quad \mathrm{Sht}(G,b,\mu)_\infty = \varprojlim_K \mathrm{Sht}(G,b,\mu)_K.$$

Question 5.15. *For any open compact subgroup $K \subset G(\mathbb{Q}_p)$, we know that the fibers of*

$$\mathrm{Sht}(G,b,\mu)_K(C,\mathcal{O}_C) \to Gr_{G,\leq\mu}^{B_{dR}^+,adm}(C,\mathcal{O}_C)$$

are in bijection with $G(\mathbb{Q}_p)/K$. Is it possible to define a notion of étale fundamental group for the diamond $Gr_{G,\leq\mu}^{B_{dR}^+,adm}$ as [23], so that the $\mathbb{Q}_p$-G-local system $\mathbb{V}$ on $Gr_{G,\leq\mu}^{B_{dR}^+,adm}$ can be described in term of a collection of representations

$$\pi_1(Gr_{G,\leq\mu}^{B_{dR}^+,adm},\overline{x}) \longrightarrow G(\mathbb{Q}_p),$$

for the geometric point $\overline{x}$ runs through each connected component of $Gr_{G,\leq\mu}^{B_{dR}^+,adm}$?

At the infinite level, there exists a Hodge-Tate period map (cf. [12] p.38; see also [20] Theorem 5.4)

$$\pi_{HT} : \mathrm{Sht}(G,b,\mu)_\infty \longrightarrow Gr_{G,\leq-\mu}^{B_{dR}^+},$$

where $Gr_{G,\leq-\mu}^{B_{dR}^+} \subset Gr_G^{B_{dR}^+} \otimes E$ is the Schubert diamond associated to $\{\mu^{-1}\}$. We can make a little precise on the image of π_{HT}. By [4] Corollary 3.5.2 there is a natural map

$$\mathcal{E} : Gr_G^{B_{dR}^+}(R,R^+) \to Bun_{G,X_{R^\flat,R^{+\flat}}}.$$

Take $(R, R^+) = (C, \mathcal{O}_C)$ with $C|\mathbb{Q}_p$ complete and algebraically closed. Then we get a map $b(\cdot) : Gr_G^{B_{dR}^+}(C, \mathcal{O}_C) \to B(G)$. By [4] Proposition 3.5.3, when restricting to $x \in Gr_{G,\leq -\mu}^{B_{dR}^+}(C, \mathcal{O}_C)$, one has

$$b(x) \in B(G, \mu).$$

Then for any $b \in B(G, \mu)$, we get a locally closed sub diamond

$$Gr_{G,\leq -\mu}^{B_{dR}^+, b} \subset Gr_{G,\leq -\mu}^{B_{dR}^+},$$

such that the underling topological space $|Gr_{G,\leq -\mu}^{B_{dR}^+, b}|$ is the fiber over b under the above map $b(\cdot)$. One has

$$\pi_{HT} : \mathrm{Sht}(G, b, \mu)_\infty(C, \mathcal{O}_C) \longrightarrow Gr_{G,\leq -\mu}^{B_{dR}^+, b}(C, \mathcal{O}_C),$$

for any $(C, \mathcal{O}_C)$ with $C|\mathbb{Q}_p$ complete and algebraically closed.

In summary, we get two period morphisms

$$\begin{array}{ccc} & \mathrm{Sht}(G, b, \mu)_\infty & \\ {}^{\pi_{dR}}\swarrow & & \searrow^{\pi_{HT}} \\ Gr_{G,\leq \mu}^{B_{dR}^+, adm} & & Gr_{G,\leq -\mu}^{B_{dR}^+, b}, \end{array}$$

and the period morphism π_{dR} factors through $\mathrm{Sht}(G, b, \mu)$.

Remark 5.16. (1) *In* [12] *8.2.1, there is an alternative construction of the diamond* $\mathrm{Sht}(G, b, \mu)_\infty$.

(2) *It is natural to ask whether* $\mathrm{Sht}(G, b, \mu)_\infty$ *is representable by a perfectoid space. We will show that this is the case if* $(G, [b], \{\mu\})$ *is a unramified local Shimura datum of abelian type, cf. Corollary 5.24.*

By construction, a morphism $(G_1, [b_1], \{\mu_1\}) \to (G_2, [b_2], \{\mu_2\})$ of local shtuka data induces a morphism of diamonds

$$\mathrm{Sht}(G_1, b_1, \mu_1) \to \mathrm{Sht}(G_2, b_2, \mu_2).$$

More generally, we have morphisms

$$\mathrm{Sht}(G_1, b_1, \mu_1)_{K_1} \to \mathrm{Sht}(G_2, b_2, \mu_2)_{K_2}$$

if K_1 is mapped into K_2 under $G_1 \to G_2$.

The above functoriality enables us to apply the Tannakian formalism. As before, we assume that G is unramified over $\mathbb{Q}_p$. Consider now an embedding $G \hookrightarrow \mathrm{GL}_n$, then $([b], \{\mu\})$ induces $([b'], \{\mu'\})$, so that $(\mathrm{GL}_n, [b'], \{\mu'\})$ forms a local shtuka datum, and we get a morphism of local shtuka data $(G, [b], \{\mu\}) \to (\mathrm{GL}_n, [b'], \{\mu'\})$. The following proposition is the local analogue of Deligne's theorem for Shimura varieties.

Proposition 5.17. *In the above setting, for any* $K \subset G(\mathbb{Z}_p)$*, there exists a* $K' \subset \mathrm{GL}_n(\mathbb{Z}_p)$ *such that there exists a natural closed embedding of diamonds*

$$\mathrm{Sht}(G, b, \mu)_K \hookrightarrow \mathrm{Sht}(\mathrm{GL}_n, b', \mu')_{K'}.$$

The induced embedding of diamonds

$$\mathrm{Sht}(G, b, \mu)_\infty \hookrightarrow \mathrm{Sht}(\mathrm{GL}_n, b', \mu')_\infty$$

is compatible with the de Rham and Hodge-Tate period morphisms on both sides.

Proof. It suffices to prove that we have a closed embedding of diamonds

$$\mathrm{Sht}(G,b,\mu)_\infty \hookrightarrow \mathrm{Sht}(\mathrm{GL}_n,b',\mu')_\infty.$$

This is clear from the construction above, since we have a closed embedding

$$Gr_{G,\leq\mu}^{B_{dR}^+,adm} \hookrightarrow Gr_{\mathrm{GL}_n,\leq\mu'}^{B_{dR}^+,adm},$$

and the following diagram on de Rham period maps is cartesian

$$\begin{array}{ccc} \mathrm{Sht}(G,b,\mu)_\infty & \hookrightarrow & \mathrm{Sht}(\mathrm{GL}_n,b',\mu')_\infty \\ \downarrow{\scriptstyle \pi_{dR}} & & \downarrow{\scriptstyle \pi_{dR}} \\ Gr_{G,\leq\mu}^{B_{dR}^+,adm} & \hookrightarrow & Gr_{\mathrm{GL}_n,\leq\mu'}^{B_{dR}^+,adm}. \end{array}$$

Moreover, we have also the following cartesian diagram on Hodge-Tate period maps

$$\begin{array}{ccc} \mathrm{Sht}(G,b,\mu)_\infty & \hookrightarrow & \mathrm{Sht}(\mathrm{GL}_n,b',\mu')_\infty \\ \downarrow{\scriptstyle \pi_{HT}} & & \downarrow{\scriptstyle \pi_{HT}} \\ Gr_{G,\leq-\mu}^{B_{dR}^+,b} & \hookrightarrow & Gr_{\mathrm{GL}_n,\leq-\mu'}^{B_{dR}^+,b'}. \end{array}$$

□

5.4. **Moduli of local G-shtukas and affine Deligne-Lusztig varieties.** Let $(G,[b],\{\mu\})$ be a local shtuka datum. We want to compare the moduli space of local G-shtukas $\mathrm{Sht}(G,b,\mu)$ and affine Deligne-Lusztig variety $X_{\leq\mu}^G$ associated to $(G,[b],\{\mu\})$.

Let $(C,\mathcal{O}_C)$ be an affinoid perfectoid field of characteristic p with a untilt $C^\sharp$ of C. Let k be the residue field of $\mathcal{O}_C$. We have a $J_b(\mathbb{Q}_p)$-equivariant morphism of sets

$$sp = sp_{(G,b,\mu)} : \mathrm{Sht}(G,b,\mu)(C,\mathcal{O}_C) \to X_{\leq\mu}^G(b)(k).$$

Indeed, consider first the case $G=\mathrm{GL}_n$, we have

$$\mathrm{Sht}(G,b,\mu)(C,\mathcal{O}_C) = \{\big((\mathcal{E},\phi_{\mathcal{E}}),\iota\big)\}/\simeq$$

with $\big((\mathcal{E},\phi_{\mathcal{E}}),\iota\big)$ a shtuka over $\mathrm{Spa}(C,\mathcal{O}_C)$ with one paw at $C^\sharp$. By Theorem 5.8, there exists a Breuil-Kisin module (M,ϕ) over $A_{\mathrm{inf}}=W(\mathcal{O}_C)$. Let

$$(M\otimes_{A_{\mathrm{inf}}} W(k),\phi)$$

be the associated Dieudonné module. This defines a point in $X_{\leq\mu}^G(b)(k)$. This construction is compatible with the $J_b(\mathbb{Q}_p)$ actions on both sides. For the general case, we apply the Tannakian formalism: take any embedding $(G,[b],\{\mu\})\to(\mathrm{GL}_n,[b'],\{\mu'\})$, then we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Sht}(G,b,\mu)(C,\mathcal{O}_C) & \hookrightarrow & \mathrm{Sht}(\mathrm{GL}_n,b',\mu')(C,\mathcal{O}_C) \\ \downarrow{\scriptstyle sp_{G,b.\mu}} & & \downarrow{\scriptstyle sp_{\mathrm{GL}_n,b',\mu'}} \\ X_{\leq\mu}^G(b)(k) & \hookrightarrow & X_{\leq\mu'}^{\mathrm{GL}_n}(b')(k). \end{array}$$

Remark 5.18. *It would be nice to have a morphism of sheaves $sp = sp_{(G,b,\mu)} : \mathrm{Sht}(G,b,\mu)\to X_{\leq\mu}^G(b)$ which realizes the above map on the level of sets of points. If $\mathrm{Sht}(G,b,\mu)$ comes from the generic fiber of a Rapoport-Zink space, then it is clear how to define sp: it is just the usual specialization map from the generic fiber to the special fiber.*

Recall that we have the map $\omega_G : G(L)/G(W) \to \pi_1(G)$. Restricting to $X^G_{\leq\mu}(b)$, it gives

$$\omega_G : X^G_{\leq\mu}(b) \to c_{b,\mu}\pi_1(G)^\Gamma.$$

Lemma 5.19. *There is a map*

$$G(\mathbb{Q}_p)/G(\mathbb{Z}_p) \to X^G_{\leq\mu}(b), \quad g \mapsto g_0,$$

such that $\omega_G(g) = \omega_G(g_0)$.

Proof. Fix any point $x \in \mathrm{Sht}(G, b, \mu)(C, \mathcal{O}_C)$. Then we have an injection

$$G(\mathbb{Q}_p)/G(\mathbb{Z}_p) \to \mathrm{Sht}(G, b, \mu)(C, \mathcal{O}_C)$$

which identifies $G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$ with the Hecke orbit of x. The composite map

$$G(\mathbb{Q}_p)/G(\mathbb{Z}_p) \to \mathrm{Sht}(G, b, \mu)(C, \mathcal{O}_C) \to X^G_{\leq\mu}(b)$$

gives the desired map. The second assertion follows by the same argument as in the proof of Lemma 1.2.18 of [29], by applying Theorem 5.8 (and Tannakian formalism) instead of subsection 1.1 of [29]. □

Remark 5.20. *Consider the composite map* $G(\mathbb{Q}_p)/G(\mathbb{Z}_p) \to X^G_{\leq\mu}(b) \to \pi_1(G)^\Gamma$. *Then this is surjective by Lemma 2.3 (1). In* [29] *Proposition 1.2.23, Kisin proved a stronger result: the map*

$$G(\mathbb{Q}_p)/G(\mathbb{Z}_p) \to \pi_0(X^G_{\leq\mu}(b))$$

is surjective if $(G, [b], \{\mu\})$ *is a unramified local Shimura datum of Hodge type.*

The following is an analogue of Lemma 2.4.1 and Corollary 2.4.2 of [6].

Proposition 5.21. *Let* $Z \subset Z_G$ *be a central sub group and* $G' = G/Z$, *with the induced* $[b']$ *and* $\{\mu'\}$ *such that* $(G', [b'], \{\mu'\})$ *is a local shtuka datum. Then we have a cartesian diagram*

$$\begin{array}{ccc} \mathrm{Sht}(G, b, \mu)(C, \mathcal{O}_C) & \longrightarrow & \mathrm{Sht}(G', b', \mu')(C, \mathcal{O}_C) \\ \downarrow & & \downarrow \\ X^G_{\leq\mu}(b) & \longrightarrow & X^{G'}_{\leq\mu'}(b'). \end{array}$$

Proof. By the above proposition, we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{Sht}(G, b, \mu)(C, \mathcal{O}_C) & \longrightarrow & \mathrm{Sht}(G', b', \mu')(C, \mathcal{O}_C) \\ & {\scriptstyle \pi_{G,dR}} \searrow \quad \swarrow {\scriptstyle \pi_{G',dR}} & \\ & Gr^{B^+_{dR}, adm}_{G,\leq\mu}(C, \mathcal{O}_C). & \end{array}$$

For any point $x \in Gr^{B^+_{dR}, adm}_{G,\leq\mu}(C, \mathcal{O}_C)$, the above horizontal map induces a map on fibers

$$G(\mathbb{Q}_p)/G(\mathbb{Z}_p) \to G'(\mathbb{Q}_p)/G'(\mathbb{Z}_p),$$

thus it suffices to show that the following diagram is cartesian

$$\begin{array}{ccc} G(\mathbb{Q}_p)/G(\mathbb{Z}_p) & \longrightarrow & G'(\mathbb{Q}_p)/G'(\mathbb{Z}_p) \\ \downarrow & & \downarrow \\ X^G_{\leq\mu}(b) & \longrightarrow & X^{G'}_{\leq\mu'}(b'), \end{array}$$

where the vertical maps are those constructed in Lemma 5.19. Consider the following diagram

$$\begin{array}{ccc} G(\mathbb{Q}_p)/G(\mathbb{Z}_p) & \longrightarrow & G'(\mathbb{Q}_p)/G'(\mathbb{Z}_p) \\ \downarrow & & \downarrow \\ X^G_{\leq\mu}(b) & \longrightarrow & X^{G'}_{\leq\mu'}(b') \\ \downarrow \omega_G & & \downarrow \omega_{G'} \\ \pi_1(G)^\Gamma & \longrightarrow & \pi_1(G')^\Gamma. \end{array}$$

We know that the lower square is cartesian, cf. Proposition 2.5, and by Lemma 4.12

$$\begin{array}{ccc} G(\mathbb{Q}_p)/G(\mathbb{Z}_p) & \longrightarrow & G'(\mathbb{Q}_p)/G'(\mathbb{Z}_p) \\ \downarrow & & \downarrow \\ \pi_1(G)^\Gamma & \longrightarrow & \pi_1(G')^\Gamma \end{array}$$

is also cartesian. Therefore the upper square is cartesian. □

5.5. **Local Shimura varieties as moduli of local G-shtukas.** The following strengthened version of Theorem 5.12, which may be viewed as a partial solution of Conjecture 3.2 (as we do not give information on the desired Weil descent datum), is implied by the results in [4, 42]. Recall that by [42] Proposition 10.2.8, there is a fully faithful functor $X \mapsto X^\diamond$ from the category of normal rigid analytic spaces over k to the category of diamonds over $\mathrm{Spa}(k)^\diamond$ for any non-archimedean field k of characteristic 0.

Theorem 5.22. *Let $(G, [b], \{\mu\})$ be a local Shimura datum. Then there exists a tower of rigid analytic spaces over Sp$\breve{E}$*

$$(\mathcal{M}_K)_{K\subset G(\mathbb{Q}_p)},$$

where K runs through all open compact subgroups of $G(\mathbb{Q}_p)$, with the following properties:

(1) *the group $J_b(\mathbb{Q}_p)$ acts on each space $\mathcal{M}_K$,*
(2) *the group $G(\mathbb{Q}_p)$ acts on the tower $(\mathcal{M}_K)_{K\subset G(\mathbb{Q}_p)}$ as Hecke correspondences,*
(3) *there exists a compatible system of étale and partially proper period maps*

$$\pi_K : \mathcal{M}_K \to \mathscr{F}\ell^{adm}_{G,\mu}$$

which is equivariant for the action of $J_b(\mathbb{Q}_p)$, where $\mathscr{F}\ell^{adm}_{G,\mu} \subset \mathscr{F}\ell_{G,\mu}$ is an open subspace,

(4) *for any K, we have an isomorphism of diamonds $\mathcal{M}_K^\diamond \simeq \mathrm{Sht}_K$.*

Proof. Let $G \hookrightarrow \mathrm{GL}_n$ be an embedding and $(\mathrm{GL}_n, [b'], \{\mu'\})$ be the induced local shtuka datum. Then we have an embedding of B^+_{dR}-affine Grassmannians

$$Gr_G^{B^+_{dR}} \hookrightarrow Gr_{\mathrm{GL}_n}^{B^+_{dR}},$$

inducing a commutative diagram

$$\begin{array}{ccc} Gr^{B^+_{dR}}_{G,\mu} & \hookrightarrow & Gr^{B^+_{dR}}_{\mathrm{GL}_n,\mu'} \\ \downarrow & & \downarrow \\ \mathscr{F}\ell_{G,\mu} & \hookrightarrow & \mathscr{F}\ell_{\mathrm{GL}_n,\mu'}, \end{array}$$

where the vertical maps are the Bialynicki-Birula morphisms, cf. [4] Proposition 3.4.3. Since μ is minuscule, the left vertical map is an isomorphism, cf. [4] Theorem 3.4.5. Thus we get a commutative diagram

$$\begin{array}{ccc} Gr_{G,\mu}^{B_{dR}^+,adm} & \hookrightarrow & Gr_{\mathrm{GL}_n,\mu'}^{B_{dR}^+,adm} . \\ \simeq \uparrow & \nearrow & \\ \mathscr{F}\ell_{G,\mu}^{adm,\diamond} & & \end{array}$$

The tower $(\mathrm{Sht}_K)_K$ is constructed out of a $\mathbb{Q}_p - G$-local system $\mathbb{V}$ over $Gr_{G,\leq\mu}^{B_{dR}^+,adm}$, which realizes $\mathrm{Sht}(G,b,\mu)$ as the functor of the set of $\mathbb{Z}_p$-lattices in $\mathbb{V}$. Since $Gr_{G,\leq\mu}^{B_{dR}^+,adm} \simeq \mathscr{F}\ell_{G,\mu}^{adm,\diamond}$, there exists a corresponding $\mathbb{Q}_p$-G-local system over $\mathscr{F}\ell_{G,\mu}^{adm}$ which we still denote by $\mathbb{V}$. Therefore we get a tower of rigid analytic spaces $(\mathcal{M}_K)_K$ with the properties listed as in the theorem. □

Remark 5.23. *(Compare Remark 5.16 (2)) In the above situation, it is natural to conjecture that there exists a preperfectoid space $\mathcal{M}_\infty$ over $\breve{E}$ such that $\mathcal{M}_\infty \sim \varprojlim_K \mathcal{M}_K$ and $\mathcal{M}_\infty^\diamond = \mathrm{Sht}_\infty$. In the following we will see that this is true if $(G,[b],\{\mu\})$ is unramified of abelian type. This is the local analogue of the fact that Shimura varieties of abelian type with infinite level at p are perfectoid, cf.* [44].

We return to Rapoport-Zink spaces of abelian type.

Corollary 5.24. *Let $(G,[b],\{\mu\})$ be a unramified local Shimura datum of abelian type. For any open compact subgroup $K \subset G(\mathbb{Z}_p)$, let $\mathcal{M}_K$ and $\mathcal{M}'_K$ be the rigid analytic spaces over $\breve{E}$ constructed in subsection 4.4 and Theorem 5.22 respectively. Then we have an isomorphism of rigid analytic spaces over $\breve{E}$*

$$\mathcal{M}_K \simeq \mathcal{M}'_K.$$

In particular, we get isomorphisms of diamonds over $\mathrm{Spa}(\breve{E})^\diamond$

$$\mathcal{M}_K^\diamond \simeq \mathrm{Sht}_K,$$

and

$$\mathcal{M}_\infty^\diamond \simeq \mathrm{Sht}_\infty,$$

with compatible period morphisms on both sides.

Proof. We first prove the case $(G,[b],\{\mu\})$ is of Hodge type. This follows exactly as the proof of [42] Theorem 19.4.5. Moreover, we have the following cartesian diagram

$$\begin{array}{ccc} \mathcal{M}(G,b,\mu)_K^\diamond & \xrightarrow{\sim} & \mathrm{Sht}(G,b,\mu)_K \\ \downarrow & & \downarrow \\ \mathcal{M}(\mathrm{GL}_n,b',\mu')_{K'}^\diamond & \xrightarrow{\sim} & \mathrm{Sht}(\mathrm{GL}_n,b',\mu')_{K'} \end{array}$$

Now assume that $(G,[b],\{\mu\})$ is of abelian type. We can apply Proposition 4.13, and compare the construction of $\mathcal{M}(G,b,\mu)_K$ with that of $\mathrm{Sht}(G,b,\mu)_K$. Here we use the fact that the categories of étale $\mathbb{Z}_p$-local systems and $\mathbb{Q}_p$-local systems on an adic space X are eauivalent to the corresponding categories on the pro-étale site $X_{\text{proét}}$, cf. [25] Lemma 9.1.11. □

Let $(G, [b], \{\mu\})$ be a local Shimura datum. By Theorem 5.22, there exists a tower of local Shimura varieties $\Big(\mathcal{M}(G, b, \mu)_K\Big)_K$ over $\mathrm{Sp}\breve{E}$ as conjectured by Rapoport-Viehmann. Take an embedding $G \hookrightarrow \mathrm{GL}_n$. Then we get an induced triple $(\mathrm{GL}_n, [b'], \{\mu'\})$, which is a local shtuka datum. The following corollary is now a consequence of Proposition 5.17 and Theorem 5.22.

Corollary 5.25. *For any $K \subset G(\mathbb{Z}_p)$, there exists a $K' \subset \mathrm{GL}_n(\mathbb{Z}_p)$ such that there exists a natural closed embedding of diamonds*

$$\mathcal{M}(G, b, \mu)_K^\diamond \hookrightarrow \mathrm{Sht}(\mathrm{GL}_n, b', \mu')_{K'}.$$

Remark 5.26. (1) *Let $(G, [b], \{\mu\})$ be a unramified local Shimura datum of Hodge type, with the associated Rapoport-Zink spaces $\mathcal{M}_K$ and the moduli spaces of local G-shtukas Sht_K. The isomorphism of diamonds over $\mathrm{Spa}(\breve{E})^\diamond$*

$$\mathcal{M}_K^\diamond \simeq \mathrm{Sht}_K$$

indicates the magic "switching p-divisible groups with additional structures to local G-shtukas".

(2) *If $(G, [b], \{\mu\})$ is a general local Shimura datum, e.g. a unramified local Shimura datum of abelian type but not of Hodge type, then we do not have p-divisible groups any more. However, via $\mathcal{M}_K^\diamond \simeq \mathrm{Sht}_K$, the local Shimura varieties $\mathcal{M}_K$ can be viewed as moduli of local G-shtukas.*

Remark 5.27. *We refer to* [38] *sections 6,7,8 and* [12] *section 7 for the discussions on the conjectures on the realizations of local Langlands correspondences and local Jacquet-Langlands correspondences in the ℓ-adic cohomology of the tower $(\mathcal{M}_K)_K$ or $(\mathrm{Sht}_K)_K$.*

6. Rapoport-Zink uniformization for Shimura varieties of abelian type

We return to the formal integral setting. As [37] chapter 6 and [27], we apply our construction of the formal schemes $\breve{\mathcal{M}}(G, b, \mu)$ to prove a uniformization theorem for Kisin's integral canonical models of Shimura varieties of abelian type [28]. Throughout this section, we assume $p > 2$.

6.1. **Integral canonical models for Shimura varieties of abelian type.** Let (G, X) be a Shimura datum of abelian type, i.e. there exists a Shimura datum of Hodge type (G_1, X_1) together with a central isogeny $G_1^{der} \to G^{der}$, such that it induces an isomorphism of the associated adjoint Shimura datum $(G_1^{ad}, X_1^{ad}) \simeq (G^{ad}, X^{ad})$. Fix a prime $p > 2$. Assume that G is unramified at p from now on. By Lemma 3.4.13 of [28], we can find a Shimura datum of Hodge type (G_1, X_1) satisfying the above and G_1 is unramified at p. Let E be the local reflex field of (G, X) for some place over p. In the following we will only consider the open compact subgroups $K \subset G(\mathbb{A}_f)$ in the form $K = K_pK^p$ with $K_p = G(\mathbb{Z}_p)$.

Theorem 6.1 ([28] Theorem 3.4.10, Corollary 3.4.14)**.** *With the above notation and assumption, for any sufficiently small open compact subgroup $K^p \subset G(\mathbb{A}_f^p)$, there exists an integral canonical smooth model*

$$S_K(G, X)$$

of $\mathrm{Sh}_K(G, X)$ over O_E. When K^p varies, the prime to p Hecke action on $\big(\mathrm{Sh}_K(G, X)\big)_K$ extends to

$$\big(S_K(G, X)\big)_K$$

It will be useful to review how these integral models are constructed, cf. [28] 2.3 and 3.4.

6.1.1. *Case (G, X) is of Hodge type.* Take an embedding of Shimura data $(G, X) \hookrightarrow (\mathrm{GSp}, S^{\pm})$. Let $K = K_pK^p \subset G(\mathbb{A}_f)$ be an open compact subgroup with $K_p = G(\mathbb{Z}_p)$. Take an open compact subgroup $K' = K'_pK'^p$ with $K'_p = \mathrm{GSp}(\mathbb{Z}_p)$, such that $K \subset K'$ and we have an closed immersion

$$\mathrm{Sh}_K(G, X) \hookrightarrow \mathrm{Sh}_{K'}(\mathrm{GSp}, S^{\pm})_E,$$

where E is the local reflex field for (G, X). For $\mathrm{Sh}_{K'}(\mathrm{GSp}, S^{\pm})$ we have the integral canonical model $S_{K'}(\mathrm{GSp}, S^{\pm})$. Consider the Zariski closure $S_K^-(G, X)$ of $\mathrm{Sh}_K(G, X)$ in $S_{K'}(\mathrm{GSp}, S^{\pm}) \times O_E$. Then $S_K(G, X)$ is defined as the normalization of $S_K^-(G, X)$. In particular we have a finite morphism

$$S_K(G, X) \to S_K^-(G, X) \subset S_{K'}(\mathrm{GSp}, S^{\pm})_{O_E}.$$

It will be useful to review some further structures for the integral canonical model $S_K(G, X)$. Let T be a scheme over O_E. Attached to each point $x \in S_K(G, X)(T)$ we have a triple

$$(A_x, \lambda_x, \varepsilon^p_{x,K}),$$

where (A_x, λ_x) is the polarized abelian scheme up to prime to p isogeny coming from pullback of the universal polarized abelian scheme over $S_{K'}(\mathrm{GSp}, S^{\pm})$, and

$$\varepsilon^p_{x,K} \in \Gamma(T, \mathrm{Isom}(V_{\mathbb{A}^p_f}, V^p(A_x)_{\mathbb{Q}})/K^p)$$

is the (promoted) K-level structure coming from the K'-level structure $\varepsilon^p_{x,K'}$ on A_x, cf. [28] 3.4.2. The triple $(A_x, \lambda_x, \varepsilon^p_{x,K'})$ can be viewed as the polarized abelian scheme with level structure attached to the T-point of $S_{K'}(\mathrm{GSp}, S^{\pm})$ induced by x. Let (s_α) be a finite collection of tensors which cut off the inclusion $G \subset \mathrm{GL}(V)$. As explained in 1.3.6 of [29], there exist de Rham tensors $s_{\alpha,dR,x}$ and ℓ-adic étale tensors $(s_{\alpha,l,x})_{l\neq p}$ on the first relative de Rham cohomology and the first ℓ-adic cohomology of A_x respectively. The level $\varepsilon^p_{x,K}$ takes s_α to $(s_{\alpha,l,x})_{l\neq p}$.

If $T = \mathrm{Spec} k$ where $k \subset \overline{\mathbb{F}}_p$ is a subfield containing the residue field k_E of O_E, then there exists cristalline Tate tensors $(s_{\alpha,0,x})$ on the first cristalline cohomology of A_x. If x is the specialization of a point $\widetilde{x}$ over F with $F|E$ an extension, then there exists p-adic étale tensors $(s_{\alpha,p,\widetilde{x}})$ on the first p-adic étale cohomology of $A_{\widetilde{x}}$, and $(s_{\alpha,0,x})$ and $(s_{\alpha,p,\widetilde{x}})$ are related by the p-adic comparison theorem, cf. Proposition 1.3.7 of [29]. By Corollary 1.3.11 of [29] the data

$$(A_x, \lambda_x, \varepsilon^p_{x,K}, (s_{\alpha,0,x}))$$

uniquely determines the point $x \in\in S_K(G, X)(k)$. Sometimes we will write $s_{\alpha,0,x}$ as $t_{\alpha,x}$ to be compatible with our previous notation on cristalline Tate tensors on p-divisible groups.

6.1.2. *Case (G, X) is of abelian type.* Take a unramified Shimura datum of Hodge type (G_1, X_1), together with a central isogeny $G_1^{der} \to G^{der}$, such that it induces an isomorphism of the associated adjoint Shimura datum $(G_1^{ad}, X_1^{ad}) \simeq (G^{ad}, X^{ad})$. Let $K = K_pK^p \subset G(\mathbb{A}_f)$ be an open compact subgroup with $K_p = G(\mathbb{Z}_p)$. The integral model $S_K(G, X)$ is constructed as the quotient

$$S_{K_p}(G, X)/K^p$$

of an integral model $S_{K_p}(G, X)$ of the pro-scheme

$$\mathrm{Sh}_{K_p}(G, X) = \varprojlim_{K^p} \mathrm{Sh}_{K_pK^p}(G, X).$$

The scheme $S_{K_p}(G, X)$ is constructed as follows. Consider the connected component

$$\mathrm{Sh}_{K_{1p}}(G_1, X_1)^+ = \varprojlim_{K_1^p} \mathrm{Sh}_{K_{1p}K_1^p}(G_1, X_1)^+$$

of $\mathrm{Sh}_{K_{1p}}(G_1,X_1)=\varprojlim_{K_1^p}\mathrm{Sh}_{K_{1p}K_1^p}(G_1,X_1)$. Let $S_{K_{1p}}(G_1,X_1)^+$ be its the Zariski closure in $S_{K_{1p}}(G_1,X_1)$ over $W=W(\overline{\mathbb{F}}_p)$. Write $Z=Z_G$. The above integral of $\mathrm{Sh}_{K_p}(G,X)$ is given by

$$S_{K_p}(G,X)=[\mathscr{A}(G_{\mathbb{Z}_{(p)}})\times S_{K_{1p}}(G_1,X_1)^+]/\mathscr{A}(G_{1\mathbb{Z}_{(p)}})^\circ,$$

where

$$\mathscr{A}(G_{\mathbb{Z}_{(p)}})=G(\mathbb{A}_f^p)/Z(\mathbb{Z}_{(p)})^- *_{G(\mathbb{Z}_{(p)})+/Z(\mathbb{Z}_{(p)})} G^{ad}(\mathbb{Z}_{(p)})^+$$

and

$$\mathscr{A}(G_{\mathbb{Z}_{(p)}})^\circ=G(\mathbb{Z}_{(p)})_+^-/Z(\mathbb{Z}_{(p)})^- *_{G(\mathbb{Z}_{(p)})+/Z(\mathbb{Z}_{(p)})} G^{ad}(\mathbb{Z}_{(p)})^+;$$

similarly we have $\mathscr{A}(G_{1\mathbb{Z}_{(p)}})$ and $\mathscr{A}(G_{1\mathbb{Z}_{(p)}})^\circ$, see [28] 3.3.2. The scheme $S_{K_p}(G,X)$ descends to O_E and gives the integral canonical model of $\mathrm{Sh}_{K_p}(G,X)=\varprojlim_{K^p}\mathrm{Sh}_{K_pK^p}(G,X)$, see the proof of loc. cit. Theorem 3.4.10.

6.2. **Newton stratification of the special fibers.** We keep the notations as above. We will work over $\overline{\mathbb{F}}_p$ in this subsection. By abuse of notation, denote the special fiber of $S_K=S_K(G,X)$ over $\overline{\mathbb{F}}_p$ by $\overline{S}_K$ for simplicity. In [46], we proved the following results.

Theorem 6.2. (1) *For any $b\in B(G,\mu)$, there exists a non empty locally closed subset $\overline{S}_K^b\subset\overline{S}_K$, which we view as a subscheme of $\overline{S}_K$ with its reduced structure, such that set theoritically we have*

$$\overline{S}_K=\coprod_{b\in B(G,\mu)}\overline{S}_K^b.$$

(2) *For any $b\in B(G,\mu)$, the Zariski closure of $\overline{S}_K^b$ in $\overline{S}_K$ is $\coprod_{b'\leq b}\overline{S}_K^{b'}$.*

For $b\in B(G,\mu)$, we call the subschemes $\overline{S}_K^b$ as the Newton strata of $\overline{S}_K$. If (G,X) is of Hodge type, then the existence of the Newton stratification is implied by [36], see also [50] 5.2.

For later use, we briefly review the construction of the Newton stratification. If (G,X) is of Hodge type, it is constructed by the associated p-divisible groups with Tate tensors. We now assume that (G,X) is of abelian type. In this case, let (G_1,X_1) be a unramified Shimura datum of Hodge type (G_1,X_1), together with a central isogeny $G_1^{der}\to G^{der}$, such that it induces an isomorphism of the associated adjoint Shimura datum $(G_1^{ad},X_1^{ad})\simeq(G^{ad},X^{ad})$. Then we have a canonical bijection $B(G_1,\mu_1)\simeq B(G,\mu)$. Consider the Newton stratification at level K_1^p,

$$\overline{S}_{K_{1p}K_1^p}(G_1,X_1)=\coprod_{b\in B(G_1,\mu_1)}\overline{S}_{K_{1p}K_1^p}(G_1,X_1)^b.$$

When the level K_1^p varies, the Newton stratifications are compatible. Therefore, we get a Newton stratification

$$\overline{S}_{K_{1p}}(G_1,X_1)=\coprod_{b\in B(G_1,\mu_1)}\overline{S}_{K_{1p}}(G_1,X_1)^b$$

by taking inverse limit over K_1^p. As [29] 3.5.8, consider

$$\pi(G_1):=G_1(\mathbb{Q})_+^-\setminus G_1(\mathbb{A}_f)/G_1(\mathbb{Z}_p)=G_1(\mathbb{Z}_{(p)})_+^-\setminus G_1(\mathbb{A}_f^p),$$

which is the set of geometric connected components of $S_{K_{1p}}(G_1,X_1)$. By [46],

$$\overline{S}_{K_{1p}}(G_1,X_1)^b\subset\overline{S}_{K_{1p}}(G_1,X_1)$$

is stable under the action of $\mathscr{A}(G_{1\mathbb{Z}_{(p)}})$, and we have a surjective $\mathscr{A}(G_{1\mathbb{Z}_{(p)}})$-equivariant map

$$\overline{S}_{K_{1p}}(G_1, X_1)^b \to \pi(G_1).$$

Let $\overline{S}_{K_{1p}}(G_1, X_1)^{b,+}$ be the pullback of $\overline{S}_{K_{1p}}(G_1, X_1)^b$ under the inclusion $\overline{S}_{K_{1p}}(G_1, X_1)^+ \hookrightarrow \overline{S}_{K_{1p}}(G_1, X_1)$. In other words, we consider the following commutative diagrams

$$\begin{array}{ccc} \overline{S}_{K_{1p}}(G_1, X_1)^{b,+} & \hookrightarrow & \overline{S}_{K_{1p}}(G_1, X_1)^+ \\ \downarrow & & \downarrow \\ \overline{S}_{K_{1p}}(G_1, X_1)^b & \hookrightarrow & \overline{S}_{K_{1p}}(G_1, X_1) \\ \downarrow & & \downarrow \\ \pi(G_1) & = & \pi(G_1), \end{array}$$

where the above diagram is cartesian. The stablizer of $\overline{S}_{K_{1p}}(G_1, X_1)^{b,+} \subset \overline{S}_{K_{1p}}(G_1, X_1)^b$ is $\mathscr{A}(G_{1\mathbb{Z}_{(p)}})^\circ$, and we have the identity

$$\overline{S}_{K_{1p}}(G_1, X_1)^b = [\mathscr{A}(G_{1\mathbb{Z}_{(p)}}) \times \overline{S}_{K_{1p}}(G_1, X_1)^{b,+}]/\mathscr{A}(G_{1\mathbb{Z}_{(p)}})^\circ.$$

For more details we refer to [46]. Now as

$$\overline{S}_{K_p}(G, X) = [\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times \overline{S}_{K_{1p}}(G_1, X_1)^+]/\mathscr{A}(G_{1\mathbb{Z}_{(p)}})^\circ,$$

we get the Newton stratification

$$\overline{S}_{K_p}(G, X) = \coprod_{b \in B(G,\mu)} \overline{S}_{K_p}(G, X)^b,$$

where for any $b \in B(G, \mu)$, the associated stratum

$$\overline{S}_{K_p}(G, X)^b = [\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times \overline{S}_{K_{1p}}(G_1, X_1)^{b,+}]/\mathscr{A}(G_{1\mathbb{Z}_{(p)}})^\circ \hookrightarrow \overline{S}_{K_p}(G, X).$$

For any sufficiently small open compact subgroup $K^p \subset G(\mathbb{A}_f^p)$, we define

$$\overline{S}_{K_pK^p}(G, X)^b = \overline{S}_{K_p}(G, X)^b/K^p.$$

Therefore we get the Newton stratification

$$\overline{S}_{K_pK^p}(G, X) = \coprod_{b \in B(G,\mu)} \overline{S}_{K_pK^p}(G, X)^b.$$

6.3. **Rapoport-Zink uniformization.** The notations will be the same as the previous subsection. Let $b \in B(G, \mu)$. We get a unramified local Shimura datum of abelian type $(G, b, \{\mu\})$, thus a formal scheme $\breve{\mathcal{M}} = \breve{\mathcal{M}}(G, b, \mu)$ over W. Fix a point $x \in \overline{S}_K^b(\overline{\mathbb{F}}_p)$.

6.3.1. *Case (G, X) is of Hodge type.* We want to construct a morphism of formal schemes

$$\Theta = \Theta_x : \breve{\mathcal{M}} \times G(\mathbb{A}_f^p)/K^p \longrightarrow \widehat{S}_K \times \mathrm{Spf}W,$$

where $\widehat{S}_K$ is the formal completion of S_K along its special fiber. The morphism Θ is constructed in [27] Proposition 4.3 and Corollary 4.3.2. Let $(A_x, (t_{\alpha,x}), \overline{\eta})$ be the abelian variety with additional structures attached to x, and let $I_\phi(\mathbb{Q})$ be the group of quasi-isogenies of A_x preserving $(t_{\alpha,x})$. Then $I_\phi(\mathbb{Q})$ is the group of $\mathbb{Q}$-points of a reductive group I_ϕ over $\mathbb{Q}$ (cf. [29] Corollary 2.3.1) which depends only on the isogeny class of x ([29] 1.4.14). In this case, Θ factors through the quotient by $I_\phi(\mathbb{Q})$

$$\Theta : I_\phi(\mathbb{Q}) \setminus \breve{\mathcal{M}} \times G(\mathbb{A}_f^p)/K^p \longrightarrow \widehat{S}_K \times \mathrm{Spf}W,$$

and the image $\mathcal{Z}_{\phi,K^p}$ is contained in the stratum $\overline{S}_K^b \times \mathrm{Spec}\overline{\mathbb{F}}_p$.

6.3.2. *Case (G,X) is of abelian type.* We first work on the level of sets. By [29] Theorem 4.6.7, we have the following bijection

$$\overline{S}_{K_p}(G,X)^b(\overline{\mathbb{F}}_p) \xrightarrow{\sim} \coprod_{[\phi],b(\phi)=b} S(G,\phi),$$

where

$$\phi : \mathfrak{Q} \to \mathfrak{G}_G$$

runs through the set the admissible morphisms of Galois gerbs, $[\phi]$ is the associated equivalence class, cf. [29] 3.3, and

$$S(G,\phi) = \varprojlim_{K^p} I_\phi(\mathbb{Q}) \setminus \mathcal{M}_{red}(\overline{\mathbb{F}}_p) \times G(\mathbb{A}_f^p)/K^p,$$

where $\mathcal{M}_{red}$ is the reduced special fiber of the Rapoport-Zink space $\breve{\mathcal{M}}$ associated to $(G_{\mathbb{Q}_p}, b(\phi), \mu)$.

Remark 6.3. *In* [29] *3.3, in fact one considers the set*

$$S(G,\phi) = \varprojlim_{K^p} I_\phi(\mathbb{Q}) \setminus X_p(\phi) \times X^p(\phi)/K^p,$$

where $X_p(\phi)$ and $X^p(\phi)$ are certain sets canonically associated to ϕ, such that (cf. Lemma 3.3.4 of [29]*)*

$$X_p(\phi) \simeq X_\mu^G(b) \simeq \mathcal{M}_{red}(\overline{\mathbb{F}}_p)$$

and $X^p(\phi)$ is a $G(\mathbb{A}_f^p)$-torsor.

Take a unramified Shimura datum of Hodge type (G_1, X_1), together with a central isogeny $G_1^{der} \to G^{der}$, such that it induces an isomorphism of the associated adjoint Shimura datum $(G_1^{ad}, X_1^{ad}) \simeq (G^{ad}, X^{ad})$. Let

$$\phi_1 : \mathfrak{Q} \to \mathfrak{G}_{G_1}$$

be an admissible morphism of Galois gerbs. We note that

$$\begin{aligned} S(G_1,\phi_1) &= \varprojlim_{K_1^p} I_{\phi_1}(\mathbb{Q}) \setminus \mathcal{M}_{1red}(\overline{\mathbb{F}}_p) \times G_1(\mathbb{A}_f^p)/K^p \\ &= I_{\phi_1}(\mathbb{Q}) \setminus \mathcal{M}_{1red}(\overline{\mathbb{F}}_p) \times G_1(\mathbb{A}_f^p), \end{aligned}$$

where $\mathcal{M}_{1red}$ is the reduced special fiber of the Rapoport-Zink space $\breve{\mathcal{M}}_1$ associated to $(G_{1\mathbb{Q}_p}, b(\phi_1), \mu_1)$.

Fix an admissible morphism $\phi_0 : \mathfrak{Q} \to \mathfrak{G}_{G^{ad}}$. Consider

$$S(G,\phi_0) = \coprod_{[\phi],\phi^{ad}=\phi_0} S(G,\phi) = \coprod_{[\phi],\phi^{ad}=\phi_0} \varprojlim_{K^p} I_\phi(\mathbb{Q}) \setminus \mathcal{M}_{red}(\overline{\mathbb{F}}_p) \times G(\mathbb{A}_f^p)/K^p.$$

By [29] Lemmas 3.7.2 and 3.7.4, there is an action of $\mathscr{A}(G_{\mathbb{Z}_{(p)}})$ on $S(G,\phi_0)$, together with a $\mathscr{A}(G_{\mathbb{Z}_{(p)}})$-equivariant surjective map

$$c_G : S(G,\phi_0) \to \pi(G).$$

Let ϕ_0 be such that $x \in S(G,\phi_0)$. Let (G_1, X_1) be a unramified Shimura datum of Hodge type (G_1, X_1), together with a central isogeny $G_1^{der} \to G^{der}$, such that it induces an isomorphism of the associated adjoint Shimura datum $(G_1^{ad}, X_1^{ad}) \simeq (G^{ad}, X^{ad})$. For the identity class $e \in \pi(G)$, consider the fiber

$$S(G,\phi_0)^+ = c_{G_1}^{-1}(e).$$

Similarly we have $S(G_1,\phi_0) = \coprod_{[\phi_1],\phi_1^{ad}=\phi_0} S(G_1,\phi_1)$ and $S(G_1,\phi_0)^+$.

Proposition 6.4. *We have the following isomorphism of sets with $\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times \langle\Phi\rangle$-action*

$$S(G, \phi_0) \simeq [\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times S(G_1, \phi_0)^+]/\mathscr{A}(G_{1\mathbb{Z}_{(p)}})^\circ.$$

Proof. This follows from Corollary 3.8.12 of [29]. □

Now we come back to Rapoport-Zink spaces. If $K_1^{p'} \subset K_1^p$ is another open compact subgroup of $G_1(\mathbb{A}_f^p)$, then we have the following commutative diagram

$$\begin{array}{ccc} I_{\phi_1}(\mathbb{Q}) \setminus \breve{\mathcal{M}}_1 \times G_1(\mathbb{A}_f^p)/K_1^{p'} & \longrightarrow & I_{\phi_1}(\mathbb{Q}) \setminus \breve{\mathcal{M}}_1 \times G_1(\mathbb{A}_f^p)/K_1^{p} \\ \downarrow \Theta_{1K_1^{p'}} & & \downarrow \Theta_{1K_1^p} \\ \widehat{S}_{K_{1p}K_1^{p'}}(G_1, X_1) \times \mathrm{Spf} W & \longrightarrow & \widehat{S}_{K_{1p}K_1^{p}}(G_1, X_1) \times \mathrm{Spf} W \end{array}$$

with horizontal maps finite. Therefore, if we set

$$\widehat{S}(G_1, \phi_1) := \varprojlim_{K_1^p} I_{\phi_1}(\mathbb{Q}) \setminus \breve{\mathcal{M}}_1 \times G_1(\mathbb{A}_f^p)/K_1^p,$$

then we get

$$\Theta_1 = \varprojlim_{K_1^p} \Theta_{1K_1^p} : \widehat{S}(G_1, \phi_1) \longrightarrow \varprojlim_{K_1^p} \widehat{S}_{K_{1p}K_1^p}(G_1, X_1) \times \mathrm{Spf} W,$$

where both limits are taken in the category of formal schemes.

Lemma 6.5. *Let $\widehat{S}_{K_{1p}}(G_1, X_1) \times \mathrm{Spf} W$ be the formal completion of $S_{K_{1p}}(G_1, X_1) \times \mathrm{Spec} W$ along its special fiber. Then we have a canonical isomorphism of formal schemes*

$$\varprojlim_{K_1^p} \widehat{S}_{K_{1p}K_1^p}(G_1, X_1) \times \mathrm{Spf} W = \widehat{S}_{K_{1p}}(G_1, X_1) \times \mathrm{Spf} W.$$

Proof. This follows from the definition of inverse limit of formal schemes. □

We have a surjective map

$$c_{G_1} : S_{K_{1p}}(G_1, X_1) \times \mathrm{Spec} W \longrightarrow \pi(G_1).$$

Consider the fiber over e of this map c_{G_1}, $S_{K_{1p}}(G_1, X_1)^+ \subset S_{K_{1p}}(G_1, X_1) \times \mathrm{Spec} W$, and let $\widehat{S}_{K_{1p}}(G_1, X_1)^+$ be formal completion of $S_{K_{1p}}(G_1, X_1)^+$ along its special fiber. Let

$$\widehat{S}(G_1, \phi_1)^+ := \Big(\varprojlim_{K_1^p} I_{\phi_1}(\mathbb{Q}) \setminus \breve{\mathcal{M}}_1 \times G_1(\mathbb{A}_f^p)/K_1^p\Big)^+ \longrightarrow \widehat{S}_{K_{1p}}(G_1, X_1)^+$$

be the pullback of

$$\Theta_1 : \widehat{S}(G_1, \phi_1) = \varprojlim_{K_1^p} I_{\phi_1}(\mathbb{Q}) \setminus \breve{\mathcal{M}}_1 \times G_1(\mathbb{A}_f^p)/K_1^p$$

under the inclusion

$$\widehat{S}_{K_{1p}}(G_1, X_1)^+ \hookrightarrow \widehat{S}_{K_{1p}}(G_1, X_1) \times \mathrm{Spf} W.$$

Θ_1^+ can be written as $\Theta_1^+ = \varprojlim_{K_1^p} \Theta_{1K_1^p}^+$ with

$$\begin{array}{ccc} \Big(I_{\phi_1}(\mathbb{Q}) \setminus \breve{\mathcal{M}}_1 \times G_1(\mathbb{A}_f^p)/K_1^p\Big)^+ & \longrightarrow & I_{\phi_1}(\mathbb{Q}) \setminus \breve{\mathcal{M}}_1 \times G_1(\mathbb{A}_f^p)/K_1^p \\ \downarrow \Theta_{1K_1^p}^+ & & \downarrow \Theta_{1K_1^p} \\ \widehat{S}_{K_{1p}K_1^p}(G_1, X_1)^+ & \longrightarrow & \widehat{S}_{K_{1p}K_1^p}(G_1, X_1). \end{array}$$

Define formal schemes

$$\widehat{S}(G_1,\phi_0)^+=\coprod_{[\phi_1],\phi_1^{ad}=\phi_0}\widehat{S}(G_1,\phi_1)^+=\coprod_{[\phi_1],\phi_1^{ad}=\phi_0}\Big(\varprojlim_{K_1^p} I_{\phi_1}(\mathbb{Q})\setminus\breve{\mathcal{M}}_1\times G_1(\mathbb{A}_f^p)/K_1^p\Big)^+$$

and

$$\widehat{S}(G,\phi_0)=\coprod_{[\phi],\phi^{ad}=\phi_0}\varprojlim_{K^p} I_\phi(\mathbb{Q})\setminus\breve{\mathcal{M}}\times G(\mathbb{A}_f^p)/K^p.$$

Proposition 6.6. *In the above situation, we have*

$$\widehat{S}(G,\phi_0)\simeq[\mathscr{A}(G_{\mathbb{Z}_{(p)}})\times\widehat{S}(G_1,\phi_0)^+]/\mathscr{A}(G_{1\mathbb{Z}_{(p)}})^\circ.$$

Proof. This is identical to the proof of Proposition 6.4. □

Let $\mathcal{Z}_{\phi_1,K_1^p}$ (resp. $\mathcal{Z}^+_{\phi_1,K_1^p}$) be the image of $\Theta_{1K_1^p}$ (resp. $\Theta^+_{1K_1^p}$). This exists a geometric structure on $\mathcal{Z}_{\phi_1,K_1^p}$ as follows. We can write

$$\mathcal{Z}_{\phi_1,K_1^p}=\bigcup_{j\in J_{K_1^p}}Z^j_{\phi_1,K_1^p},$$

where $J_{K_1^p}$ is the $I_{\phi_1}(\mathbb{Q})$-orbits of irreducible components of $\breve{\mathcal{M}}_1\times G_1(\mathbb{A}_f^p)/K^p$, and $Z^j_{\phi_1,K_1^p}$ is the image of the irreducible components under $\Theta_{1K_1^p}$ corresponding to j. For each $j\in J_{K_1^p}$, there exists only finitely many $j'\in J_{K_1^p}$ such that

$$Z^j_{\phi_1,K_1^p}\bigcap Z^{j'}_{\phi_1,K_1^p}\neq\emptyset.$$

Thus we get an induced geometric structure on $\mathcal{Z}^+_{\phi_1,K_1^p}$ as

$$\mathcal{Z}^+_{\phi_1,K_1^p}=\bigcup_{j\in J_{K_1^p}}Z^{j,+}_{\phi_1,K_1^p},$$

where $Z^{j,+}_{\phi_1,K_1^p}$ is the pullback of $Z^j_{\phi_1K_1^p}$ to $\widehat{S}_{K_{1p}K_1^p}(G_1,X_1)^+$. When K_1^p varies, $J_{K_1^p}$, $\mathcal{Z}_{\phi_1,K_1^p}$, and $\mathcal{Z}^+_{\phi_1,K_1^p}$ form inverse systems, and we set

$$\mathcal{Z}_{\phi_1}=\varprojlim_{K_1^p}\mathcal{Z}_{\phi_1,K_1^p},\quad \mathcal{Z}^+_{\phi_1}=\varprojlim_{K_1^p}\mathcal{Z}^+_{\phi_1,K_1^p}.$$

Let J_1 be the $I_{\phi_1}(\mathbb{Q})$-orbits of irreducible components of $\breve{\mathcal{M}}_1\times G_1(\mathbb{A}_f^p)$. For any $j\in J$, let $Z^j_{\phi_1}$ be the image of the irreducible components under Θ_1 corresponding to j, then we can write

$$\mathcal{Z}_{\phi_1}=\bigcup_{j\in J_1}Z^j_{\phi_1}$$

and

$$Z^j_{\phi_1}=\varprojlim_{K_1^p}Z^j_{\phi_1,K_1^p},$$

where $Z^j_{\phi_1K_1^p}$ is the image of the irreducible components corresponding to j under the composition

$$\breve{\mathcal{M}}_1\times G_1(\mathbb{A}_f^p)\to\breve{\mathcal{M}}_1\times G_1(\mathbb{A}_f^p)/K^p\to\widehat{S}_{K_{1p}K_1^p}(G_1,X_1).$$

Similarly for $\mathcal{Z}^+_{\phi_1}$. By the proof of Proposition 4.6.2 of [29], we have $\langle\Phi\rangle\times Z_1(\mathbb{Q}_p)\times\mathscr{A}(G_{1\mathbb{Z}_{(p)}})^{I_{\phi_1}}$-equivariant bijection of sets

$$\mathcal{Z}_{\phi_1}(\overline{\mathbb{F}}_p)\simeq S(G_1,\phi_1).$$

We have

$$[\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times \widehat{S}_{K_{1p}}(G_1, X_1)^+]/\mathscr{A}(G_{1\mathbb{Z}_{(p)}})^\circ = \widehat{S}_{K_p}(G, X) \times \mathrm{Spf}W.$$

Applying the functor $[\mathscr{A}(G_{\mathbb{Z}_{(p)}})^{I_\phi} \times -]/\mathscr{A}(G_{1\mathbb{Z}_{(p)}})^{I_{\phi_1},\circ}$ to $\mathcal{Z}^+_{\phi_1}$, cf. Remark 6.8, we get a subset $\mathcal{Z}_\phi \subset \widehat{S}_{K_p} \times \mathrm{Spf}W$. Let $\mathcal{Z}_{\phi,K^p}$ be the image of $\mathcal{Z}_\phi$ under the projection $\widehat{S}_{K_p} \times \mathrm{Spf}W \to \widehat{S}_{K_pK^p} \times \mathrm{Spf}W$. Then we can define the formal completion of $\widehat{S}_K \times \mathrm{Spf}W$ along $\mathcal{Z}_{\phi,K^p}$ as [37] chapter 6 and [27] Definition 4.6.

Theorem 6.7. *We have an isomorphism of formal schemes over W*

$$\Theta: \coprod_{[\phi],\phi^{ad}=\phi_0} I_\phi(\mathbb{Q}) \setminus \breve{\mathcal{M}} \times G(\mathbb{A}^p_f)/K^p \xrightarrow{\sim} \coprod_{[\phi],\phi^{ad}=\phi_0} \widehat{S}_K \times \mathrm{Spf}W_{/\mathcal{Z}_{\phi,K^p}}.$$

Proof. If (G, X) is of Hodge type, this is proved in [27] Theorem 4.7. Assume that we are in the general case. By the above notation, it suffices to prove that

$$\coprod_{[\phi],\phi^{ad}=\phi_0} \varprojlim_{K^p} I_\phi(\mathbb{Q}) \setminus \breve{\mathcal{M}} \times G(\mathbb{A}^p_f)/K^p \simeq [\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times \widehat{S}(G_1, \phi_0)^+]/\mathscr{A}(G_{1\mathbb{Z}_{(p)}})^\circ.$$

This is given by Proposition 6.6. □

Remark 6.8. *Denote by $G_1^{ad}(\mathbb{Z}_{(p)})^{+,I_{\phi_1}}$ the kernel of the composite of*

$$G_1^{ad}(\mathbb{Z}_{(p)})^+ \hookrightarrow G_1^{ad}(\mathbb{Z}_{(p)}) \to H^1(\mathbb{Q}, Z_1) \to H^1(\mathbb{Q}, I_{\phi_1}),$$

where Z_1 is the center of G_1. Similarly we define $G^{ad}(\mathbb{Z}_{(p)})^{+,I_\phi}$. Following [29] *4.3.4, we define*

$$\mathscr{A}(G_{1\mathbb{Z}_{(p)}})^{I_{\phi_1}} = G_1(\mathbb{A}^p_f)/Z_1(\mathbb{Z}_{(p)})^- *_{G_1(\mathbb{Z}_{(p)})+/Z_1(\mathbb{Z}_{(p)})} G_1^{ad}(\mathbb{Z}_{(p)})^{+,I_{\phi_1}}$$
$$\mathscr{A}(G_{1\mathbb{Z}_{(p)}})^{I_{\phi_1},\circ} = G_1(\mathbb{Z}_{(p)})^-_+/Z_1(\mathbb{Z}_{(p)})^- *_{G_1(\mathbb{Z}_{(p)})+/Z_1(\mathbb{Z}_{(p)})} G_1^{ad}(\mathbb{Z}_{(p)})^{+,I_{\phi_1}}.$$

Similarly we define $\mathscr{A}(G_{\mathbb{Z}_{(p)}})^{I_\phi}$ and $\mathscr{A}(G_{\mathbb{Z}_{(p)}})^{I_\phi,\circ}$. The group $\mathscr{A}(G_{1\mathbb{Z}_{(p)}})^{I_{\phi_1}}$ acts on $S(G_1, \phi_1)$, cf. [29] *Lemma 4.5.9. By construction, we have an $\mathscr{A}(G_{1\mathbb{Z}_{(p)}})^{I_{\phi_1}}$-equivariant map*

$$c_{G_1}: S(G_1, \phi_1) \to \pi(G_1),$$

which is surjective since $G_1(\mathbb{A}^p_f)$ (and thus $\mathscr{A}(G_{1\mathbb{Z}_{(p)}})^{I_{\phi_1}}$) acts transitively on $\pi(G_1)$. For the identity class $e \in \pi(G_1)$, consider the fiber

$$S(G_1, \phi_1)^+ = c^{-1}_{G_1}(e).$$

Then the stabilizer of $S(G_1, \phi_1)^+ \subset S(G_1, \phi_1)$ is

$$\mathscr{A}(G_{1\mathbb{Z}_{(p)}})^{I_{\phi_1},\circ} \subset \mathscr{A}(G_{1\mathbb{Z}_{(p)}})^{I_{\phi_1}}.$$

We have

$$S(G_1, \phi_1) = [\mathscr{A}(G_{1\mathbb{Z}_{(p)}})^{I_{\phi_1}} \times S(G_1, \phi_1)^+]/\mathscr{A}(G_{1\mathbb{Z}_{(p)}})^{I_{\phi_1},\circ}.$$

Take any $\phi_1: \mathfrak{Q} \to \mathfrak{G}_{G_1}$, such that

$$\phi^{ad} = \phi_1^{ad}: \mathfrak{Q} \to \mathfrak{G}_{G^{ad}}.$$

It should be possible that the strategy of [29] *3.8 enables us to prove the following refinement of Proposition 6.4*

$$S(G, \phi) = [\mathscr{A}(G_{1\mathbb{Z}_{(p)}})^{I_{\phi_1}} \times S(G_1, \phi_1)^+]/\mathscr{A}(G_{\mathbb{Z}_{(p)}})^{I_\phi,\circ}.$$

Once this is done, the same argument as above shows that there is an isomorphism of formal schemes over W

$$I_\phi(\mathbb{Q}) \setminus \breve{\mathcal{M}} \times G(\mathbb{A}^p_f)/K^p \xrightarrow{\sim} \widehat{S}_K \times \mathrm{Spf}W_{/\mathcal{Z}_{\phi,K^p}}.$$

Remark 6.9. *In the special cases of Shimura curves associated to quaternion algebras over totally real field, see* [3] *for a construction of the uniformization by Drinfeld spaces.*

Let $\mathrm{Sh}_K(\phi_0) = (\coprod_{[\phi],\phi^{ad}=\phi_0} \widehat{S}_K \times \mathrm{Spf} W_{/\mathcal{Z}_{\phi,K^p}})^{ad}_\eta$. We get a natural morphism of adic spaces $\mathrm{Sh}_K(\phi_0) \to \mathrm{Sh}_K^{ad}$. For any open compact subgroup $K_p' \subset G(\mathbb{Q}_p)$, let $\mathrm{Sh}_{K_p'K^p}(\phi_0) \to \mathrm{Sh}^{ad}_{K_p'K^p}$ be the pullback of $\mathrm{Sh}_K(\phi_0) \to \mathrm{Sh}_K^{ad}$ under the projection $\mathrm{Sh}^{ad}_{K_p'K^p} \to \mathrm{Sh}^{ad}_{K_pK^p}$. We get the following corollary from Theorem 6.7.

Corollary 6.10. *With the above notations,* Θ *induces an isomorphism of rigid analytic spaces over* $\breve{E}$

$$\Theta : \coprod_{[\phi],\phi^{ad}=\phi_0} I_\phi(\mathbb{Q}) \setminus \mathcal{M}_{K_p'} \times G(\mathbb{A}_f^p)/K^p \xrightarrow{\sim} \mathrm{Sh}_{K_p'K^p}(\phi_0).$$

Recall that in [44] we have proved that there exists a perfectoid space $\mathcal{S}_{K^p}$ over $\mathbb{C}_p$ such that

$$\mathcal{S}_{K^p} \sim \varprojlim_{K_p'} \mathrm{Sh}_{K_p'K^p}(G,X)^{ad}.$$

On the other hand, by Proposition 4.14, we get a perfectoid space $\mathcal{M}_\infty$ over $\mathbb{C}_p$ such that

$$\mathcal{M}_\infty \sim \varprojlim_{K_p'} \mathcal{M}_{K_p'}.$$

From the above Corollary 6.10 we get

Corollary 6.11. *There exists a perfectoid space* $\mathcal{S}_{K^p}(\phi_0)$ *together with a map* $\mathcal{S}_{K^p}(\phi_0) \to \mathcal{S}_{K^p}$*, such that*

$$\mathcal{S}_{K^p}(\phi_0) \simeq \coprod_{[\phi],\phi^{ad}=\phi_0} I_\phi(\mathbb{Q}) \setminus \mathcal{M}_\infty \times G(\mathbb{A}_f^p)/K^p.$$

Remark 6.12. *For the* $b \in B(G,\mu)$ *we fixed in this subsection, we can define the Newton stratum* $\mathcal{S}^b_{K^p} \subset \mathcal{S}_{K^p}$*, which is a locally closed subspace, cf.* [4] *subsection 4.3 or* [45]*. Then we have* $\mathcal{S}_{K^p}(\phi_0) \to \mathcal{S}_{K^p}$ *factors through* $\mathcal{S}^b_{K^p}$*. In the case that* b *is basic, we will have* $\mathcal{S}_{K^p}(\phi_0) = \mathcal{S}^b_{K^p}$*, cf. the next subsection. In the general case, the image of* $\mathcal{S}_{K^p}(\phi_0) \to \mathcal{S}^b_{K^p}$ *is a strict subsapce, and to under understand the whole stratum* $\mathcal{S}^b_{K^p}$*, one should introduce Igusa varieties, cf.* [4] *section 4 in the PEL case and* [45] *in the general case.*

6.4. **The case of basic strata.** Let the notations be as in the last subsection. Assume now that $b = b_0$ is the basic element. Note that there is only one ϕ such that $b(\phi) = b_0$.

Theorem 6.13. *In the setting above,* $\mathcal{Z}_{\phi,K^p} = \overline{S}^b_K$*. Thus we have an isomorphism*

$$\Theta : I_\phi(\mathbb{Q}) \setminus \breve{\mathcal{M}} \times G(\mathbb{A}_f^p)/K^p \xrightarrow{\sim} \widehat{S}_K \times \mathrm{Spf} W_{/\overline{S}^b_K}.$$

Proof. In the case that (G,X) is of Hodge type, this is proved in Theorem 4.11 of [27]. The general case follows from this by the construction. □

Corresponding to Corollaries 6.10 and 6.11, we have

Corollary 6.14. *For any open compact subgroup* $K_p' \subset G(\mathbb{Q}_p)$*,* Θ *induces an isomorphism of rigid analytic spaces over* $\breve{E}$

$$\Theta : I_\phi(\mathbb{Q}) \setminus \mathcal{M}_{K_p'} \times G(\mathbb{A}_f^p)/K^p \xrightarrow{\sim} \mathrm{Sh}^b_{K_p'K^p},$$

and an isomorphism of perfectoid spaces over $\mathbb{C}_p$

$$\Theta : I_\phi(\mathbb{Q}) \setminus \mathcal{M}_\infty \times G(\mathbb{A}_f^p)/K^p \xrightarrow{\sim} \mathcal{S}^b_{K^p}.$$

7. Application to moduli spaces of K3 surfaces in mixed characteristic

In this section, we will apply our constructions to K3 surfaces and their moduli in mixed characteristic. We will first study some Rapoport-Zink spaces of orthogonal type. Again, we will assume $p > 2$ in this section.

7.1. **Rapoport-Zink spaces of orthogonal type.** Let (L, Q) be a non degenerate self dual quadratic lattice of rank $n+2$ over $\mathbb{Z}_p$, where $n \geq 1$ is an integer. We write (V, Q) as the induced quadratic space over $\mathbb{Q}_p$. Let $G = \mathrm{SO}(V, Q), G_1 = \mathrm{GSpin}(V, Q)$ be the associated special orthogonal and spinor similitudes groups over $\mathbb{Q}_p$. By our assumption that L is self dual, both G and G_1 are unramified. We have an exact sequence of groups

$$1 \to \mathbb{G}_m \to G_1 \to G \to 1,$$

which is in fact defined over $\mathbb{Z}_p$.

As in [22] subsection 4.2, there is a natural choice of minuscule cocharacter μ_1 of G_1. Take any $[b_1] \in B(G_1, \mu_1)$. Then $(G_1, [b_1], \{\mu_1\})$ is a local Shimura datum of Hodge type. We get a local Shimura datum $(G, [b], \{\mu\})$ by taking $[b], \{\mu\}$ as the image of $[b_1], \{\mu_1\}$ under the map $G_1 \to G$. By construction $(G, [b], \{\mu\})$ is unramified of abelian type. We get the associated Rapoport-Zink spaces $\breve{\mathcal{M}}_1 = \breve{\mathcal{M}}(G_1, b_1, \mu_1)$ and $\breve{\mathcal{M}} = \breve{\mathcal{M}}(G, b, \mu)$.

Let X_0 be the p-divisible group over $\overline{\mathbb{F}}_p$ with (covariant) Dieudonné module $(C(V) \otimes W, b_1\sigma)$, where $C(V)$ is the Clifford algebra attached to V. Fix any $\delta \in C(V)^\times$ with $\delta^* = \delta$ where $*$ is the canonical involution on $C(V)$. Then $\psi_\delta(c_1, c_2) = \mathrm{Tr}(c_1 \delta c_2^*)$ is a perfect symplectic form on $C(V)$. Here $\mathrm{Tr}: C(V) \to \mathbb{Z}_p$ is the reduced trace map. The perfect symplectic form ψ_δ on $C(V)$ induces a principal polarization $\lambda_0 : X_0 \to X_0^\vee$. There exists a finite collection tensors $(s_\alpha)_{\alpha \in I}$ which includes ψ_δ, such that $G_1 \subset \mathrm{GL}(C(V))$ is cut out by $(s_\alpha)_{\alpha\in I}$. Recall that $\breve{\mathcal{M}}_1$ has the following moduli interpretation. For any $R \in \mathrm{Nilp}_W^{sm}$, $\breve{\mathcal{M}}_1(R) = \{(X, (t_\alpha)_{\alpha\in I}, \rho)\}/\simeq$, where

- X is a p-divisible group on $\mathrm{Spec} R$,
- $(t_\alpha)_{\alpha\in I}$ is a collection of cystalline Tate tensors of X,
- $\rho : X_0 \otimes R/J \to X \otimes R/J$ is a quasi-isogeny which sends $s_\alpha \otimes 1$ to t_α for $\alpha \in I$, where J is some ideal of definition of R,

such that the following condition holds:
the R-scheme

$$\mathrm{Isom}\Big((\mathcal{D}(X)_R, (t_\alpha), \mathrm{Fil}^\bullet(\mathcal{D}(X)_R)), (\Lambda \otimes R, (s_\alpha \otimes 1), \mathrm{Fil}^\bullet \Lambda \otimes R)\Big)$$

that classifies the isomorphisms between locally free sheaves $\mathcal{D}(X)_R$ and $\Lambda \otimes R$ on $\mathrm{Spec} R$ preserving the tensors and the filtrations is a $P_{\mu_1} \otimes R$-torsor.

The exact sequence $1 \to \mathbb{G}_m \to G_1 \to G \to 1$ induces a long exact sequence (cf. [2] Lemma 1.5)

$$1 \to \pi_1(\mathbb{G}_m)^\Gamma \to \pi_1(G_1)^\Gamma \to \pi_1(G)^\Gamma \to H^1(\Gamma, \pi_1(\mathbb{G}_m)) \to \cdots$$

We have the following isomorphisms

$$\pi_1(\mathbb{G}_m)^\Gamma = \pi_1(\mathbb{G}_m) \simeq X_*(\mathbb{G}_m) \simeq \mathbb{Z}.$$

Since $G_1^{der} = \mathrm{Spin}(V)$ is simply connected and we have the exact sequence

$$1 \to \mathrm{Spin}(V) \to \mathrm{GSpin}(V) \to \mathbb{G}_m \to 1,$$

we get ([2] 1.6)

$$\pi_1(G_1)^\Gamma = \pi_1(\mathbb{G}_m)^\Gamma \simeq \mathbb{Z}.$$

On the other hand, since

$$1 \to \mu_2 \to \mathrm{Spin}(V) \to \mathrm{SO}(V) \to 1$$

is exact, we get

$$\pi_1(G) = \mu_2(-1) = \mathbb{Z}/2\mathbb{Z}.$$

Lemma 7.1. *We have* $\pi_1(G)^\Gamma \simeq \mathbb{Z}/2\mathbb{Z}$ *and the map* $\pi_1(G_1)^\Gamma \to \pi_1(G)^\Gamma$ *is surjective.*

Proof. As $\mu_2 \subset \mathbb{G}_m$, and Γ acts trivially on the later, we get $\pi_1(G)^\Gamma = \mu_2(-1)^\Gamma = \mu_2(-1) = \mathbb{Z}/2\mathbb{Z}$. For the second assertion, note that

$$\pi_1(G_1)^\Gamma / 2\pi_1(G_1)^\Gamma = \mathbb{Z}/2\mathbb{Z} \subset \mathrm{Im}(\pi_1(G_1)^\Gamma \to \pi_1(G)^\Gamma).$$

Thus the image is $\pi_1(G)^\Gamma$. □

Corollary 7.2. *We have an isomorphism of formal schemes* $\breve{\mathcal{M}} \simeq \breve{\mathcal{M}}_1/p^{\mathbb{Z}}$.

Proof. By the above lemma $\pi_1(G_1)^\Gamma \to \pi_1(G)^\Gamma$ is surjective.Thus $\breve{\mathcal{M}} \simeq \breve{\mathcal{M}}_1/p^{\mathbb{Z}}$ as the proof of (1) of Proposition 4.8. □

7.2. **Ekedahl-Oort and Howard-Pappas stratifications of basic Rapoport-Zink spaces of orthogonal type.** Let the notations be as above. Assume that $[b_1]$ (thus $[b]$) is basic.

In [22], Howard and Pappas introduced a stratification[4] for the reduced special fiber $\mathcal{M}_{1red}$ of $\breve{\mathcal{M}}_1$:

$$\mathcal{M}_{1red} = \coprod_{\Lambda} \mathcal{M}^\circ_{1\Lambda},$$

where Λ runs through the set of vertex lattices. By definition (cf. [22] section 5), a vertex Λ lattice is a $\mathbb{Z}_p$-lattice in V_L^Φ, such that

$$p\Lambda \subset \Lambda^\vee \subset \Lambda \subset V_L^\Phi.$$

Here $L = W(\overline{\mathbb{F}}_p)_{\mathbb{Q}}, \Phi = b\sigma$ is the Frobenius, V_L^Φ admits a quadratic form induced from V_L, so that this quadratic space V_L^Φ has the same dimension and determinant as V, but has Hasse invariant -1. Associated to a vertex, we have the type

$$t(\Lambda) := \dim \Lambda/\Lambda^\vee,$$

which is an even integer, and $2 \leq t(\Lambda) \leq t_{max}$, where

$$t_{max} = \begin{cases} n+1, & n \quad \text{odd}, \\ n+2, & n \quad \text{even}, \quad \det V \neq (-1)^{\frac{n}{2}}, \\ n, & n \quad \text{even}, \quad \det V = (-1)^{\frac{n}{2}}. \end{cases}$$

Recall that we have the inclusion

$$V_L^\Phi \subset End(X_0)_{\mathbb{Q}},$$

so each vertex lattice $\Lambda \subset V_L^\Phi$ can be viewed as a set of self quasi-isogenies of X_0. For each vertex lattice Λ, the associated Howard-Pappas stratum

$$\mathcal{M}^\circ_{1\Lambda} \subset \mathcal{M}_{1red}$$

is the locus $(X, (t_\alpha), \rho)$ where

$$\rho \circ \Lambda^\vee \circ \rho \subset End(X)$$

and this does not hold for any smaller vertex lattice $\Lambda' \subsetneq \Lambda$. Let $\mathcal{M}_{1\Lambda} \subset \mathcal{M}_{1red}$ be its Zariski closure. In [22] 4.3.3 and 6.4.1, Howard and Pappas proved that there exists a decomposition

$$\mathcal{M}_{1red} = \coprod_{j \in \mathbb{Z}} \mathcal{M}_1^{(j)},$$

[4]In [22] 6.5 it is called the Bruhat-Tits stratification, and our $\mathcal{M}^\circ_{1\Lambda}$ is denoted as BT_Λ there.

such that each $\mathcal{M}_1^{(j)}$ is a connected component of $\mathcal{M}_{1red}$. Accordingly, we get a decomposition for each stratum

$$\mathcal{M}_{1\Lambda}^{\circ} = \coprod_{j\in\mathbb{Z}} \mathcal{M}_{1\Lambda}^{(j),\circ}.$$

By [22] Theorem 6.5.6, each connected stratum $\mathcal{M}_{1\Lambda}^{(j),\circ}$ is isomorphic to a Deligne-Lusztig variety $X_B(w)$ for the group $\mathrm{SO}(\Lambda_W/\Lambda_W^\vee)$.

As

$$\breve{\mathcal{M}} \simeq \breve{\mathcal{M}}_1/p^{\mathbb{Z}} \simeq \breve{\mathcal{M}}^{(0)} \coprod \breve{\mathcal{M}}^{(1)},$$

we get an induced Howard-Pappas stratification for $\mathcal{M}_{red}$

$$\mathcal{M}_{red} = \coprod_{\Lambda} \mathcal{M}_{\Lambda}^{\circ}.$$

In fact, in [22] sections 5 and 6 Howard and Pappas studied the geometric structures of $\mathcal{M}_{1red}$ by passing to the quotient space $\mathcal{M}_{red} = \mathcal{M}_{1red}/p^{\mathbb{Z}}$ first.

Recall that $W = W(\overline{\mathbb{F}}_p), L = W_{\mathbb{Q}}$. Following [22], we can describe the sets $\mathcal{M}_{red}(\overline{\mathbb{F}}_p), \mathcal{M}_{\Lambda}(\overline{\mathbb{F}}_p)$ and $\mathcal{M}_{\Lambda}^{\circ}(\overline{\mathbb{F}}_p)$ in terms of special lattices of V_L as follows. By definition ([22] Definition 5.2.1) a special lattice $\mathcal{L} \subset V_L$ is a self-dual W-lattice such that

$$(\mathcal{L} + \Phi_*(\mathcal{L}))/\mathcal{L} \simeq W/pW.$$

By Proposition 6.2.2 of [22], we have a bijection

$$\mathcal{M}_{red}(\overline{\mathbb{F}}_p) \simeq \{\text{special lattices}\, \mathcal{L} \subset V_L\}.$$

By loc. cit. 5.3.1 and Theorem 6.3.1 we have bijections

$$\begin{aligned} \mathcal{M}_{\Lambda}(\overline{\mathbb{F}}_p) &\simeq \{\text{Lagrangians}\, \mathcal{L} \subset \Omega : dim(\mathcal{L} + \Phi(\mathcal{L})) = d+1\} \\ &\simeq \{\text{special lattices}\, \mathcal{L} \subset V_L : \Lambda_W^\vee \subset \mathcal{L} \subset \Lambda_W\} \\ &= \{\text{special lattices}\, \mathcal{L} \subset V_L : \Lambda(\mathcal{L}) \subset \Lambda\}, \end{aligned}$$

where $\Omega = \Lambda_W/\Lambda_W^\vee, \Lambda(\mathcal{L}) = (\mathcal{L}^{(d)})^{\Phi}, d = \frac{t(\Lambda)}{2}$, and $\mathcal{L}^{(d)} = \mathcal{L} + \Phi(\mathcal{L}) + \cdots + \Phi^d(\mathcal{L})$. Under the above description, we have the bijection

$$\begin{aligned} \mathcal{M}_{\Lambda}^{\circ}(\overline{\mathbb{F}}_p) &\simeq \{\text{special lattices}\, \mathcal{L} \subset V_L : \mathcal{L}^{(d)} = \Lambda_W\} \\ &= \{\text{special lattices}\, \mathcal{L} \subset V_L : \Lambda(\mathcal{L}) = \Lambda\}. \end{aligned}$$

In fact the above descriptions are true for any finitely generated field extension $k'|\overline{\mathbb{F}}_p$ (cf. [22])

Let $G_1 - \mathrm{Zip}^{\mu_1}$ be the stack of G_1-zips of type μ_1. The universal p-divisible group with crystalline Tate tensors on $\breve{\mathcal{M}}_1$ defines a morphism

$$\zeta : \mathcal{M}_{1\overline{\mathbb{F}}_p} \longrightarrow G_1 - \mathrm{Zip}^{\mu_1},$$

where $\mathcal{M}_{1\overline{\mathbb{F}}_p}$ is the closed formal subscheme of $\breve{\mathcal{M}}_1$ defined by the p-adic ideal. This is the local analogue of the construction in [51] 3.1.

Proposition 7.3. *The above morphism ζ is formally smooth.*

Proof. By the construction of section 4 of [22], we can globalize the unramified local Shimura datum of Hodge type $(G_1, [b_1], \{\mu_1\})$, in the sense that we can find a GSpin Shimura datum (GSpin, X) such that $G_1 = \mathrm{GSpin}_{\mathbb{Q}_p}$, $\{\mu_1\}$ is associated to X and $[b_1] \in B(G_1, \mu_1)$ is basic.

Take any point $x \in \mathcal{M}_{1\overline{\mathbb{F}}_p}(\overline{\mathbb{F}}_p)$. Applying the uniformization morphism in the last subsection, we get a corresponding point $x' \in S_0(\overline{\mathbb{F}}_p)$ where S_0 is the special fiber of the integral GSpin Shimura variety with level $K^pG_1(\mathbb{Z}_p)$. Then we have a canonical isomorphism of complete local rings $\widehat{\mathcal{O}}_{\mathcal{M}_{1\overline{\mathbb{F}}_p},x} \simeq \widehat{\mathcal{O}}_{S_0,x'}$, cf. [27] Proposition 4.1.6. By [51] Theorem 3.1.2, there exists a smooth morphism $\zeta' : S_0 \to G_1 - \mathrm{Zip}^{\mu_1}$. By construction, ζ and ζ' are compatible under the uniformization morphism $\mathcal{M}_{1\overline{\mathbb{F}}_p} \to S_0$. Therefore, the smoothness of ζ' implies the formal smoothness of ζ. □

The underling set of geometric points of $G_1 - \mathrm{Zip}^{\mu_1}$ is in canonical bijection with a subset ${}^J\mathcal{W}$ of the Weyl group of G_1 (for a fixed choice of maximal torus). In fact we have isomorphisms of topological spaces

$$|G_1 - \mathrm{Zip}^{\mu_1}| \simeq |G - \mathrm{Zip}^{\mu}| \simeq {}^J\mathcal{W}.$$

Let ${}^J\mathcal{W}^b \subset {}^J\mathcal{W}$ be the subset defined by the image of ζ. For each $w \in {}^J\mathcal{W}^b$, we call

$$\mathcal{M}_{1w} := \zeta^{-1}(w)_{red}$$

the Ekedahl-Oort stratum of $\mathcal{M}_{1red}$ associated to w. We get a stratification

$$\mathcal{M}_{1red} = \coprod_{w \in {}^J\mathcal{W}^b} \mathcal{M}_{1w}.$$

We get also an induced Ekedahl-Oort stratification for $\mathcal{M}_{red}$.

Let $m \geq 1$ be such that $2m = n+1$ if n is odd, and $2m = n+2$ if n is even. Then there is a bijection (cf. [51] subsection 4.4)

$${}^J\mathcal{W} \xrightarrow{\sim} \begin{cases} \{0, 1, \cdots, 2m-1\}, & n = 2m-1 \quad \text{odd} \\ \{0, 1, \cdots, m-2, m-1, m-1', m, \cdots, 2m-2\}, & n = 2m-2 \quad \text{even} \end{cases}$$

induced by the length function $w \mapsto \ell(w)$, where we use the symbols $m-1', m-1$ to distinguish the two elements with the same length $m-1$. Under the above bijection, the subset ${}^J\mathcal{W}^b \subset {}^J\mathcal{W}$ can be described as

$${}^J\mathcal{W}^b \xrightarrow{\sim} \begin{cases} \{m, \cdots, 2m-1\}, & n = 2m-1 \quad \text{odd}, \\ \{m, \cdots, 2m-2\}, & n = 2m-2 \quad \text{even}, \quad \det V = (-1)^{\frac{n}{2}}, \\ \{m-1, m-1', m, \cdots, 2m-2\}, & n = 2m-2 \quad \text{even}, \quad \det V \neq (-1)^{\frac{n}{2}}. \end{cases}$$

For each $i \neq m-1'$ on the right hand side, we denote the corresponding element of the left hand side as w_i. The element corresponding to $m-1'$ will be denoted by w'_{m-1}.

We can describe the map $i \mapsto w_i$ in more details. Assume first that n is odd. The simple reflections are

$$\begin{cases} s_i = (i, i+1)(2m+1-i, 2m+2-i), & 1 \leq i \leq m-1 \\ s_m = (m, m+2), & i = m, \end{cases}$$

and we have

$$w_i = \begin{cases} s_1 \cdots s_i, & 0 \leq i \leq m \\ s_1 \cdots s_{m-1} s_m s_{m-1} \cdots s_{2m-i}, & m+1 \leq i \leq 2m-1. \end{cases}$$

Now assume that n is even. The simple reflections are

$$\begin{cases} s_i = (i, i+1)(2m-i, 2m+1-i), & 1 \leq i \leq m-1 \\ s_m = (m-1, m+1)(m, m+2), & i = m, \end{cases}$$

and we have

$$w_i = \begin{cases} s_1 \cdots s_i, & 0 \leq i \leq m \\ s_1 \cdots s_m s_{m-2} \cdots s_{2m-1-i}, & m+1 \leq i \leq 2m-2, \end{cases}$$

and

$$w'_{m-1} = s_1 \cdots s_{m-2} s_m.$$

Let $\overline{V} = L_W \otimes \overline{\mathbb{F}}_p$ be the quadratic space over $\overline{\mathbb{F}}_p$. For each $w_i \in {}^J\mathcal{W}$ we will attach to it an orthogonal F-zip (also called a $\mathrm{SO}(V)$-zip) as follows. Fix a basis $e_1, \dots, e_{n+2}$ of L such that the quadratic form Q has the form $x_1x_{n+2}+x_2x_{n+1}+\cdots+x_mx_{m+2}+x_{m+1}^2$ (cf. [51] the proof of Proposition 4.4.1). By abuse of notation we still denote by $e_1, \dots, e_{n+2}$ the induced basis of $(\overline{V}, Q)$. For each $w \in {}^J\mathcal{W}$, let M_w be the orthogonal F-zip $(\overline{V}, Q, C^\bullet, D_\bullet, \varphi_\bullet)$ where

- $C^\bullet$ is the descending filtration $\overline{V} \supset \langle e_2, e_3, \dots, e_{n+2}\rangle \supset \langle e_{n+2}\rangle \supset 0$, denoted by $C^0 \supset C^1 \supset C^2 \supset C^3$,
- $D_\bullet$ is the ascending filtration $0 \subset \langle w(e_1)\rangle \subset \langle w(e_1), w(e_2), \dots, w(e_{n+1})\rangle \subset \overline{V}$, denoted by $D_0 \subset D_1 \subset D_2 \subset D_3$,
- $\varphi_\bullet$ is the collections of isomorphisms $\varphi_0 : (C^0/C^1)^{(p)} \xrightarrow{\sim} D_1, \quad \varphi_1 : (C^1/C^2)^{(p)} \xrightarrow{\sim} D_2/D_1, \quad \varphi_2 : (C^2/C^3)^{(p)} \xrightarrow{\sim} D_3/D_2$.

We remark that the above construction is not the standard isomorphism ${}^J\mathcal{W} \simeq |G\!-\!\mathrm{Zip}^\mu|$ of Pink-Wedhorn-Ziegler (for example as in Theorem 3.1.5 of [51]): the standard association is the twist $w \mapsto M_{w_0w}$ of ours, where w_0 is the maximal length element of ${}^J\mathcal{W}$. In particular $\ell(w_0w) = n - \ell(w)$.

Theorem 7.4. *Each stratum $\mathcal{M}_{1w}$ is some union of Howard-Pappas strata of $\mathcal{M}_{1red}$.*

Proof. By the methods of [22], it suffices to prove the following assertion first. □

Corollary 7.5. *Each stratum $\mathcal{M}_w$ is some union of Howard-Pappas strata of $\mathcal{M}_{red}$.*

Proof. We first prove the equalities for the sets of k-points, where k is an algebraically closed field of characteristic p. This follows from [22] Theorem 6.5.6 and [18] Corollary 4.1.3.

Indeed, by [22] Theorem 6.5.6, we have an isomorphism

$$\mathcal{M}_\Lambda^\circ \simeq X_B(w^+) \coprod X_B(w^-),$$

where $X_B(w^+)$ and $X_B(w^-)$ are the Deligne-Lusztig varieties associated to the elements w^+ and w^- of W, the Weyl group of $\mathrm{SO}(\Omega)$, where as before $\Omega = \Lambda_W/\Lambda_W^\vee$. Write $w(\Lambda) = w^+$, and consider it as an element in $\mathcal{W}$ under the inclusion $W \hookrightarrow \mathcal{W}$. Then by [18] Corollary 4.1.3, we have

$$\mathcal{M}_w(k) = \coprod_{\Lambda, w(\Lambda)=w} \mathcal{M}_\Lambda^\circ(k).$$

To prove the identities on the level of schemes, we argue as in the proof of Corollary 4.10 of [47]. That is, it suffices to show that $\mathcal{M}_\Lambda^\circ$ is open and closed in $\mathcal{M}_w$. This follows from the facts that $\mathcal{M}_\Lambda^\circ$ is open in $\mathcal{M}_\Lambda$, $\mathcal{M}_\Lambda \cap \mathcal{M}_w = \mathcal{M}_\Lambda^\circ$, and the above identities on the level of points.

□

Consider the case $k = \overline{\mathbb{F}}_p$. For any vertex lattice Λ and any point $x \in \mathcal{M}_\Lambda^\circ(\overline{\mathbb{F}}_p)$, we have the associated special lattice $\mathcal{L}_x$. Reduction modulo p, we get an orthogonal F-zip M_x, which we write it as $M_{w_0w_x}$ attached to $w_0w_x \in {}^J\mathcal{W}^b$ for some $w_x \in {}^J\mathcal{W}^b$. Then by definition $x \in \mathcal{M}_{w_0w_x}$. By the above corollary, we have the equality

$$d - 1 = \ell(w_0w_x)$$

where $d = \frac{t(\Lambda)}{2}$. The following corollaries are coarser versions of Theorem 7.4 and Corollary 7.5. However, they are more explicit in terms of types.

Corollary 7.6. (1) *If n is odd, or n is even with* $\det(V) = (-1)^{\frac{n}{2}}$, *then we have the following identity*

$$\mathcal{M}_{1w_i} = \coprod_{\Lambda, t(\Lambda)=2(n-i+1)} \mathcal{M}_{1\Lambda}^\circ.$$

(2) *If* n *is even with* $\det(V) \neq (-1)^{\frac{n}{2}}$, *then*

(a) *if* $m \leq i \leq 2m-1$,

$$\mathcal{M}_{1w_i} = \coprod_{\Lambda, t(\Lambda)=2(n-i+1)} \mathcal{M}^{\circ}_{1\Lambda}.$$

(b) *if* $i = m-1$,

$$\mathcal{M}_{1w_{m-1}} \coprod \mathcal{M}_{1w'_{m-1}} = \coprod_{\Lambda, t(\Lambda)=2m} \mathcal{M}^{\circ}_{1\Lambda}.$$

Corollary 7.7. (1) *If* n *is odd, or* n *is even with* $\det(V) = (-1)^{\frac{n}{2}}$, *then we have the following identity*

$$\mathcal{M}_{w_i} = \coprod_{\Lambda, t(\Lambda)=2(n-i+1)} \mathcal{M}^{\circ}_{\Lambda}.$$

(2) *If* n *is even with* $\det(V) \neq (-1)^{\frac{n}{2}}$, *then*

(a) *if* $m \leq i \leq 2m-1$,

$$\mathcal{M}_{w_i} = \coprod_{\Lambda, t(\Lambda)=2(n-i+1)} \mathcal{M}^{\circ}_{\Lambda}.$$

(b) *if* $i = m-1$,

$$\mathcal{M}_{w_{m-1}} \coprod \mathcal{M}_{w'_{m-1}} = \coprod_{\Lambda, t(\Lambda)=2m} \mathcal{M}^{\circ}_{\Lambda}.$$

7.3. **Moduli spaces of polarized K3 surfaces with level structures and the integral Kuga-Satake map.** We will discuss the relation with moduli spaces of polarized K3 surfaces, cf. [32] sections 2 and 4, [39] section 6.

Let U be the hyperbolic lattice over $\mathbb{Z}$ of rank 2, and E_8 be the positive quadratic lattice associated to the Dynkin diagram of type E_8. Set $N = U^{\oplus 3} \oplus E_8^{\oplus 2}$, which is a self-dual lattice. Let $d \geq 1$ be an integer. Choose a basis e, f for the first copy of U in N and set

$$L_d = \langle e - df \rangle^{\perp} \subset N.$$

This is a quadratic lattice over $\mathbb{Z}$ of discriminant $2d$ and rank 21 (in [39] it is denoted by L_{2d}). Let $V_d = L_d \otimes \mathbb{Q}$ and $L_d^{\vee} \subset V_d$ be the dual lattice. Set

$$G = \mathrm{SO}(V_d),$$

which is isomorphic to the special orthogonal group over $\mathbb{Q}$ of signature $(2, 19)$. Let $K \subset G(\mathbb{A}_f)$ be an open compact subgroup which stabilizes $L_{d,\widehat{\mathbb{Z}}}$ and acts trivially on $L_d^{\vee}/L_d$. Such compact opens are called *admissible*. We fix a prime $p > 2$ such that $p \nmid d$ from now on. Then as L is self dual at p, the local reductive group $G_{\mathbb{Q}_p}$ is unramified. Let $K_p = G(\mathbb{Z}_p)$ be the hyperspecial group. We only consider open compact subgroups $K^p \subset G(\mathbb{A}_f^p)$ which is contained in the discriminant kernel of $L_{d,\widehat{\mathbb{Z}}^p}$ with finite index. In particular, $K = K_pK^p$ is admissible, cf. [39] 5.3. For the reductive group G, we have the associated Shimura varieties $\mathrm{Sh}_{K_pK^p}$, which are defined over $\mathbb{Q}$. By [28], there exists an integral smooth canonical model $S_{K_pK^p}$ of $\mathrm{Sh}_{K_pK^p}$ over $\mathbb{Z}_p$.

Let M_{2d} (resp. M_{2d}^*) be the moduli spaces of K3 surfaces $f : X \to S$ together with a primitive polarization ξ (resp. quasi-polarization) of degree $2d$ over $\mathbb{Z}_p$ (in [32] section 2, these spaces are denoted by M_{2d}° and M_{2d} respectively). These are Deligne-Mumford stacks of finite type over $\mathbb{Z}_p$. The natural map $\mathrm{M}_{2d} \to \mathrm{M}_{2d}^*$ is an open immersion. Moreover, M_{2d} is separated and smooth of dimension 19 over $\mathbb{Z}_p$, cf. [39] Theorem 4.3.3, Proposition 4.3.11 and [32] Proposition 2.2.

Let $(f : \mathcal{X} \to \mathrm{M}_{2d}, \xi)$ be the universal object over M_{2d}. For any prime ℓ, we consider the second relative étale cohomology H^2_ℓ of $\mathcal{X}$ over M_{2d}. This is a lisse $\mathbb{Z}_\ell$-sheaf of rank 22 equipped with a perfect symmetric Poincaré pairing $\langle , \rangle : H^2_\ell \times H^2_\ell \to \mathbb{Z}_\ell(-2)$. The ℓ-adic Chern class $\mathrm{ch}_\ell(\xi)$ of ξ is a global section of the Tate twist $H^2_\ell(1)$ that satisfies $\langle \mathrm{ch}_\ell(\xi), \mathrm{ch}_\ell(\xi)\rangle = 2d$. The product

$$H^2_{\widehat{\mathbb{Z}}} = \prod_\ell H^2_\ell$$

is a lisse $\widehat{\mathbb{Z}}$-sheaf, and the Chern classes of ξ can be put together to get the Chern class $\mathrm{ch}_{\widehat{\mathbb{Z}}}(\xi)$ in $H^2_{\widehat{\mathbb{Z}}}(1)$. Recall that we have the quadratic lattice N of rank 22 over $\mathbb{Z}$.

Definition 7.8. *Consider the étale sheaf over* M_{2d} *whose sections over any scheme* $T \to \mathrm{M}_{2d}$ *are given by*

$$I(T) = \{\eta : N \otimes \widehat{\mathbb{Z}} \xrightarrow{\sim} H^2_{\widehat{\mathbb{Z}},T}(1) \textit{ isometries, with } \eta(e - df) = \mathrm{ch}_{\widehat{\mathbb{Z}}}(\xi)\}.$$

Let $K = K_pK^p \subset K_{L_{\widehat{\mathbb{Z}}p}}$ *be an admissible open compact subgroup. Then* I *admits a natural action by the constant sheaf of groups* K. *A section* $\overline{\eta} \in H^0(T, I/K)$ *is called a* K*-level structure over* T *(in* [39] *5.3 it is called a full* K*-level structure).*

Let $\mathrm{M}_{2d,K}$ (resp. $\mathrm{M}^*_{2d,K}$) be the relative moduli problem over M_{2d} (resp. M^*_{2d}) which parametrizes K-level structures. For K^p (thus K) small enough, these are smooth algebraic spaces. Moreover, the maps

$$\mathrm{M}_{2d,K} \to \mathrm{M}_{2d}, \quad \mathrm{M}^*_{2d,K} \to \mathrm{M}^*_{2d}$$

are finite étale. For another admissible $K' = K_pK^{p'} \subset K = K_pK^p$, we have natural finite étale projections

$$\mathrm{M}_{2d,K'} \to \mathrm{M}_{2d,K}, \quad \mathrm{M}^*_{2d,K'} \to \mathrm{M}^*_{2d,K}$$

as algebraic spaces over $\mathrm{M}_{2d}, \mathrm{M}^*_{2d}$ respectively. When $K^{p'}$ is a normal subgroup of K^p, these projections are Galois with Galois group $K^p/K^{p'}$.

For any prime ℓ, we have the primitive cohomology sheaf

$$P_\ell = \langle \mathrm{ch}_\ell(\xi)\rangle^\perp \subset H^2_\ell.$$

Let H^2_B and H^2_{dR} be the second relative Betti and de Rham cohomology respectively of the universal K3 surface $\mathcal{X} \to \mathrm{M}^*_{2d,K,\mathbb{C}}$. We have also the primitive cohomology sheaves

$$P_B = \langle \mathrm{ch}_B(\xi)\rangle^\perp \subset H^2_B, \quad P_{dR} = \langle \mathrm{ch}_{dR}(\xi)\rangle^\perp \subset H^2_{dR}.$$

Consider $\widetilde{\mathrm{M}}^*_{2d,K} \to \mathrm{M}^*_{2d,K}$, the two-fold finite étale cover parameterizing isometric trivializations $\det(L_d) \otimes \mathbb{Z}_2 \xrightarrow{\sim} \det(P_2)$ of the determinant of the primitive 2-adic cohomology of the universal quasi-polarized K3 surface. We can identify $\widetilde{\mathrm{M}}^*_{2d,K}$ with the the space of isometric trivializations $\det(L_d) \xrightarrow{\sim} \det(P_B)$ of the determinant of the primitive Betti cohomology. There is a Hodge-de Rham filtration $F^\bullet P_{dR}$ on P_{dR}, and we have a natural isometric trivialization $\eta : \mathrm{disc}(L_d) \xrightarrow{\sim} \mathrm{disc}(P_B)$ and the the tautological trivialization $\beta : \det(L_d) \xrightarrow{\sim} \det(P_B)$. The tuple $(P_B, F^\bullet P_{dR}, \eta, \beta)$ gives rise to a natural period map

$$\widetilde{\mathrm{M}}^*_{2d,K,\mathbb{C}} \to \mathrm{Sh}_{K,\mathbb{C}},$$

cf. [32] Propositions 4.2 and 3.3. There is a section map $\mathrm{M}_{2d,K,\mathbb{C}} \subset \mathrm{M}^*_{2d,K,\mathbb{C}} \to \widetilde{\mathrm{M}}^*_{2d,K,\mathbb{C}}$, whose composition with the above period map gives us the Kuga-Satake period map

$$\iota_{\mathbb{C}} : \mathrm{M}_{2d,K,\mathbb{C}} \longrightarrow \mathrm{Sh}_{K,\mathbb{C}}.$$

It follows from [40] Theorem 3.9.1, this map is defined over $\mathbb{Q}$. Therefore we get the map over $\mathbb{Q}_p$

$$\iota_{\mathbb{Q}_p} : \mathrm{M}_{2d,K,\mathbb{Q}_p} \longrightarrow \mathrm{Sh}_{K,\mathbb{Q}_p}.$$

As S_K is the integral canonical model of Sh_K, by extension property of S_K, the Kuga-Satake map extends to a map over $\mathbb{Z}_p$

$$\iota : \mathrm{M}_{2d,K} \longrightarrow S_K.$$

Theorem 7.9 ([32] Corollary 5.15)**.** *The integral Kuga-Satake period map*

$$\iota : \mathrm{M}_{2d,K} \longrightarrow S_K$$

is an open immersion.

When $K_1^p \subset K^p$ is another open compact subgroup, we note that the following diagram is cartesian:

$$\begin{array}{ccc} \mathrm{M}_{2d,K_1} & \longrightarrow & S_{K_1} \\ \downarrow & & \downarrow \\ \mathrm{M}_{2d,K} & \longrightarrow & S_K. \end{array}$$

As a corollary, we see that for K^p small enough, $\mathrm{M}_{2d,K}$ is a scheme.

7.4. **Newton and Ekedahl-Oort stratifications of the moduli spaces of K3 surfaces.** Let $\overline{\mathrm{M}}_{2d,K}$ be the special fiber of $\mathrm{M}_{2d,K}$, which can be viewed as an open subspace of the special fiber $\overline{S}_K$ of S_K. For the good reduction of Shimura varieties of abelian type, in [46] we have introduced the Newton and Ekedahl-Oort stratifications for the special fibers. In subsection 6.2 we have seen the Newton stratification. In the cases of GSpin and SO Shimura varieties, we can compare the Newton and Ekedahl-Oort stratifications as follows. These are in the list of Shimura varieties of coxeter type studied in [18] (comp. [19]).

Theorem 7.10 ([46])**.** *Assume that n is odd.*

(1) *We have*

$$\overline{S}_K = \coprod_{b\in B(G,\mu)} \overline{S}_K^b, \quad \overline{S}_K = \coprod_{w\in {}^J\mathcal{W}} \overline{S}_K^w,$$

with each stratum in the two stratifications non empty.

(2) *Let b_0 be the unique basic element in $B(G,\mu)$. We have*
  - *for $b \neq b_0$, there exist a unique $w_b \in^J \mathcal{W}$ such that $\overline{S}_K^b = \overline{S}_K^w$*
  - *for b_0, $\overline{S}_K^{b_0} = \coprod_{w\in {}^J\mathcal{W}, w\neq w_b, \forall b\neq b_0} \overline{S}_K^w$*

Note that we denoted the subset $\{w \in {}^J\mathcal{W}, w \neq w_b, \forall b \neq b_0\}$ as ${}^J\mathcal{W}^{b_0}$ in subsection 7.2. We return to the case $n = 19$. Consider the Kuga-Satake map

$$\bar{\iota} : \overline{\mathrm{M}}_{2d,K} \hookrightarrow \overline{S}_K,$$

which is an open immersion by Theorem 7.9. The above stratifications of $\overline{S}_K$ in turn induce stratifications of $\overline{\mathrm{M}}_{2d,K}$

$$\overline{\mathrm{M}}_{2d,K} = \coprod_{b\in B(G,\mu)} \overline{\mathrm{M}}_{2d,K}^b, \quad \overline{\mathrm{M}}_{2d,K} = \coprod_{w\in {}^J\mathcal{W}} \overline{\mathrm{M}}_{2d,K}^w,$$

where $\overline{\mathrm{M}}_{2d,K}^b$ and $\overline{\mathrm{M}}_{2d,K}^w$ are the pullbacks of the corresponding strata $\overline{S}_K^b$ and $\overline{S}_K^w$ under the open immersion $\bar{\iota} : \overline{\mathrm{M}}_{2d,K} \hookrightarrow \overline{S}_K$. We have the similar relation

- for $b \neq b_0$, there exists a unique $w_b \in {}^J\mathcal{W}$ such that $\overline{\mathrm{M}}_{2d,K}^b = \overline{\mathrm{M}}_{2d,K}^w$,

- for b_0, $\overline{\mathrm{M}}_{2d,K}^{b_0} = \coprod_{w\in {}^J\mathcal{W}^{b_0}} \overline{\mathrm{M}}_{2d,K}^{w}$. We will also write $\overline{\mathrm{M}}_{2d,K}^{b_0}$ as $\overline{\mathrm{M}}_{2d,K}^{ss}$ to indicate that it is the supersingular locus of $\overline{\mathrm{M}}_{2d,K}$.

We will investigate these stratifications in some more classical terms.

7.4.1. *Newton stratification vs. height stratification.* Let X be a K3 surface over a field k of characteristic p. Consider the functor on local Artinian k-algebras with residue field k defined by

$$\Phi^2_{X/k} : (Art/k) \to (\text{Abelian groups})$$
$$R \mapsto \ker\big(H^2_{et}(X \times \mathrm{Spec}R, \mathbb{G}_m) \to H^2_{et}(X, \mathbb{G}_m)\big).$$

It is pro-representable by a one-dimensional formal group $\widehat{\mathrm{Br}}(X)$, the so called formal Brauer group. The height h of this formal Brauer group of the K3 surface X satisfies $1 \leq h \leq 10$ or $h = \infty$.

The Newton slopes of the F-crystal $H^2_{cris}(X/W)$ are equal to $(1-\frac{1}{h}, 1, 1+\frac{1}{h})$. Thus the set $B(G,\mu)$ is in bijection with the set $\{1,\dots,10,\infty\}$. The basic element b_0 corresponds to ∞. We write $B(G,\mu) = \{b_1,\dots,b_{10}, b_{11} = b_0\}$. The Newton stratification of $\overline{\mathrm{M}}_{2d,K}$ is just the classical height stratification.

7.4.2. *Ekedahl-Oort stratification vs. Artin invariant stratification.* Thanks to the recent proof of the Tate conjecture for K3 surfaces, we know that for a K3 surface X over $\overline{\mathbb{F}}_p$, $h = \infty$ if and only if its Picard rank $\rho = 22$, i.e. it is Artin supersingular if and only if it is Shioda supersingular, cf. [31] Theorem 2.3. We simply call X supersingular in this case. Let X be a supersingular K3 surface over $\overline{\mathbb{F}}_p$, then the discriminant of its Néron-Severi lattice is equal to

$$-p^{2\sigma_0(X)}$$

for some integer $1 \leq \sigma_0(X) \leq 10$. The integer $\sigma_0(X)$ is called the Artin invariant of X.

By [10], we have an explicit description of the set ${}^J\mathcal{W}$ as

$$\{w_1,\dots,w_{20}\},$$

with w_i corresponds to b_i for $1 \leq i \leq 10$, and for $11 \leq i \leq 20$ the elements w_i are basic. The K3 surfaces in the stratum $\overline{\mathrm{M}}_{2d,K}^{w_i}$ have Artin invariant $21 - i$. In particular, we note that the index i in the description of the set ${}^J\mathcal{W}$ in subsection 7.2 (where $0 \leq i \leq 19$ in our case) is shifted to $i+1$ here.

7.5. **Rapoport-Zink type uniformization and Artin invariants.** Let $\widehat{\mathrm{M}}_{2d,K}$ and $\widehat{S}_K$ be the formal completion of $\mathrm{M}_{2d,K}$ and S_K along their special fibers respectively. Then the integral Kuga-Satake period map in Theorem 7.9 induces an open immersion of formal schemes

$$\widehat{\iota} : \widehat{\mathrm{M}}_{2d,K} \to \widehat{S}_K.$$

Let $x_0 \in \overline{\mathrm{M}}_{2d,K}$ be any point in the special fiber $\overline{\mathrm{M}}_{2d,K}$ of $\mathrm{M}_{2d,K}$, and $x = \iota(x_0)$ be its image. Let $b \in B(G,\mu)$ be the Newton point associated to x and consider the corresponding formal Rapoport-Zink space $\breve{\mathcal{M}} = \breve{\mathcal{M}}_b$ for the group $\mathrm{SO}(V)$. The choice of the point x determines a morphism of formal schemes

$$\Theta_x : \breve{\mathcal{M}} \to \widehat{S}_K.$$

Denote by $\breve{\mathcal{N}}$ the pullback of $\breve{\mathcal{M}}$ under $\widehat{\iota}: \widehat{\mathrm{M}}_{2d,K} \to \widehat{S}_K$. In other words, we get a cartesian diagram

$$\begin{array}{ccc} \breve{\mathcal{N}} & \longrightarrow & \breve{\mathcal{M}} \\ \downarrow & & \downarrow \Theta_x \\ \widehat{\mathrm{M}}_{2d,K} & \xrightarrow{\widehat{\iota}} & \widehat{S}_K, \end{array}$$

with the upper horizontal map $\breve{\mathcal{N}} \to \breve{\mathcal{M}}$ is an open immersion. By the moduli description of $\breve{\mathcal{M}}$, we get the following description of $\breve{\mathcal{N}}$: for any $R \in \mathrm{Nilp}_W^{sm}$,

$$\breve{\mathcal{N}}(R) = \{(X, (t_\alpha), \overline{\rho}) \in \breve{\mathcal{M}}(R)\}$$

where

- $(X, (t_\alpha), \rho) \in \breve{\mathcal{M}}_1(R)$, with $X = KS(Y)[p^\infty]$, where Y is a K3 surface over R,
- $\overline{\rho}$ is a $p^{\mathbb{Z}}$-orbit of ρ.

In particular, $\breve{\mathcal{N}}$ is stable under the action of $J_b(\mathbb{Q}_p)$ on $\breve{\mathcal{M}}$.

Remark 7.11. *By construction, we have an open subspace $\breve{\mathcal{N}}_1 \subset \breve{\mathcal{M}}_1$, such that for any $R \in \mathrm{Nilp}_W^{sm}$,*

$$\breve{\mathcal{N}}_1(R) = \{(X, \rho, (t_\alpha))\}$$

with $(X, (t_\alpha), \rho) \in \breve{\mathcal{M}}_1(R)$ as above. The space $\breve{\mathcal{N}}$ is given by $\breve{\mathcal{N}} = \breve{\mathcal{N}}_1/p^{\mathbb{Z}}$. On the level of affine Deligne-Lusztig varieties, we get subsets

$$\mathcal{N}_{red}(\overline{\mathbb{F}}_p) \subset \mathcal{M}_{red}(\overline{\mathbb{F}}_p) = X_\mu^G(b), \quad \mathcal{N}_{1red}(\overline{\mathbb{F}}_p) \subset \mathcal{M}_{1red}(\overline{\mathbb{F}}_p) = X_{\mu_1}^{G_1}(b_1).$$

In the case that b is basic, it will be interesting to describe the above subsets by special lattices as in [22] *section 5.*

We can apply the Rapoport-Zink uniformization theorem for S_K to deduce a similar uniformization for $\mathrm{M}_{2d,K}$. Recall that as $dimV = 21$ is odd, the group $G = \mathrm{SO}(V)$ is adjoint.

Corollary 7.12. *Let J_ϕ be the pullback of $\mathcal{Z}_{\phi,K^p}$ under the open immersion $\overline{\iota}: \overline{\mathrm{M}}_{2d,K} \hookrightarrow \overline{S}_K$. Then we have the following identity*

$$\widehat{\mathrm{M}}_{2d,K/J_\phi} = \coprod_{j\in I} \breve{\mathcal{N}}/\Gamma_j,$$

where $\Gamma_j \subset J_b(\mathbb{Q}_p)$ are some discrete subgroups (constructed as usual from the uniformization theorem of the last section). If moreover $b = b_0$ is basic, then $J_\phi = \overline{\mathrm{M}}_{2d,K}^{ss}$ and the above disjoint union is finite.

Remark 7.13. *If the open compact subgroup $K = K_pK^p \subset G(\mathbb{A}_f)$ $(K_p = G(\mathbb{Z}_p))$ is the image of some open compact subgroup $K_1 = K_{1p}K_1^p \subset G_1(\mathbb{A}_f)$ $(K_{1p} = G_1(\mathbb{Z}_p))$, then it will be much easier to prove the uniformization theorem for S_K: one can work directly on the finite level and take a finite étale quotient from the corresponding Rapoport-Zink uniformization for G_1, cf.* [44] *section 4 for example.*

Assume that $b = b_0$ is basic. Let $\mathcal{N}_{red}$ be the reduced special fiber of $\breve{\mathcal{N}}$. Then the Howard-Pappas stratification of the reduced special fiber $\mathcal{M}_{red}$ of $\breve{\mathcal{M}}$ induces a similar stratification of the open subspace $\mathcal{N}_{red}$:

$$\mathcal{N}_{red} = \coprod_{\Lambda} \mathcal{N}_\Lambda^\circ,$$

where $\mathcal{N}_\Lambda^\circ \subset \mathcal{N}_{red}$ is the pullback of the stratum $\mathcal{M}_\Lambda^\circ \subset \mathcal{M}_{red}$. For each $w_i \in {}^J\mathcal{W}^b$, consider the corresponding Ekedahl-Oort stratum

$$\mathcal{M}_{w_i} = \coprod_{\Lambda, t(\Lambda)=2(21-i)} \mathcal{M}_\Lambda^\circ, \quad \mathcal{N}_{w_i} = \coprod_{\Lambda, t(\Lambda)=2(21-i)} \mathcal{N}_\Lambda^\circ.$$

For each $11 \leq i \leq 20$, the image of $\mathcal{N}_{w_i}$ under the uniformization morphism gives us the corresponding Ekedahl-Oort stratum $\overline{\mathrm{M}}_{2d,K}^{w_i}$ in supersingular locus.

For $(X, \xi) \in \overline{\mathrm{M}}_{2d,K}^{ss}(\overline{\mathbb{F}}_p)$, consider

$$L = \langle \mathrm{ch}_{cris}(\xi) \rangle^\perp \subset H^2_{cris}(X/W).$$

This is a special lattice in the sense of Definition 5.2.1 of [22]. Then we can apply Proposition 5.2.2 of loc. cit. to produce a vertex lattice $\Lambda(L)$. For any integer $r \geq 0$ define

$$L^{(r)} = L + \Phi(L) + \cdots + \Phi^r(L).$$

Then there is a unique integer $1 \leq d \leq 10$ such that

$$L = L^{(0)} \subsetneq L^{(1)} \subsetneq \cdots \subsetneq L^{(d)} = L^{(d+1)}.$$

The vertex lattice $\Lambda(L)$ is defined by

$$\Lambda(L) = (L^{(d)})^\Phi.$$

It has type $2d$ and $\Lambda(L)^\vee = L^\Phi$. The following corollary follows from the above uniformization and Corollary 7.5.

Corollary 7.14. *Under the uniformization identity*

$$\overline{\mathrm{M}}_{2d,K}^{ss} = \coprod_{j \in I} \mathcal{N}_{red}/\Gamma_j,$$

the Ekedahl-Oort stratum $\overline{\mathrm{M}}_{2d,K}^{w_i}$ for each $11 \leq i \leq 20$ is the image of $\mathcal{N}_{w_i}$. In particular, if $x \in \overline{\mathrm{M}}_{2d,K}^{ss}(\overline{\mathbb{F}}_p)$, let X_x be the associated supersingular K3 surface over $\overline{\mathbb{F}}_p$, then we have the identity between the Artin invariant $\sigma_0(X_x)$ and the type $t(\Lambda_x)$

$$\sigma_0(X_x) = \frac{t(\Lambda_x)}{2},$$

where $\Lambda_x = \Lambda(L_x)$ is the vertex lattice attached to the special lattice associated to (X_x, ξ_x) as above.

Appendix A. Admissibility and weakly admissibility in the basic orthogonal case

In this appendix, we investigate the p-adic period domains $\mathscr{F}\ell_{G,\mu}^{adm}$ and $\mathscr{F}\ell_{G,\mu}^{wa}$ in the case b is basic and $G = \mathrm{SO}$. All the following materials are taken from [15]. We thank Fargues sincerely for kindly allowing us to include it here.

Let $V = \mathbb{Q}_p^n$ equipped with the quadratic form Q with matrix $\begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$. Let $G = \mathrm{SO}(V, Q)$ and consider the minuscule cocharacter $\mu : \mathbb{G}_m \to G_{\overline{\mathbb{Q}}_p}$ given by $\mu(z) = \mathrm{diag}(z, 1, \cdots, 1, z^{-1})$. Then the basic class in $B(G, \mu)$ is $[b] = [1]$ and thus $J_b = G$. One checks easily that any non basic Newton polygon has a non trivial contact point with the Hodge polygon, i.e. $(G, \{\mu\})$ is fully Hodge-Newton decomposable in the sense of [19] Definition 2.1.

For simplicity, we write $\mathscr{F}\ell = \mathscr{F}\ell_{G,\mu}$ as the p-adic flag variety, $\mathscr{F}\ell^{wa} = \mathscr{F}\ell^{wa}_{G,\mu}$, and $\mathscr{F}\ell^{adm} = \mathscr{F}\ell^{adm}_{G,\mu}$. We first describe the weakly admissible locus $\mathscr{F}\ell^{wa}$. The associated isocrystal is $\breve{\mathbb{Q}}_p^n$ with Frobenius $\sigma^{\oplus n}$. The sub-isocrystals are in bijection with the sub $\mathbb{Q}_p$-vector space of V. Let C be a complete and algebraically closed extension of $\breve{\mathbb{Q}}_p$. Then we have

$$\mathscr{F}\ell(C, \mathcal{O}_C) = \{\text{Lagrangian lines}\, D \subset V_C\}.$$

It follows that $\mathscr{F}\ell \subset \mathbb{P}^n_{\breve{\mathbb{Q}}_p}$ is the quadric defined by the equation $\sum_{i=1}^n x_i x_{n-i+1} = 0$. Let $\mathbb{Q}_p^{[\frac{n}{2}]} \oplus (0) \subset V$ be a Lagrangian subspace with associated parabolic subgroup $P \subset G$. For any line $D \in \mathscr{F}\ell(C, \mathcal{O}_C)$ we attach to it the following Hodge filtration

$$0 \subset \mathrm{Fil}^1 = D \subset \mathrm{Fil}^0 = D^\perp \subset \mathrm{Fil}^{-1} = V_C.$$

Then

$$\mathscr{F}\ell^{wa}(C, \mathcal{O}_C) = \{D \in \mathscr{F}\ell(C, \mathcal{O}_C) |\, D \cap W_C = 0, \forall\, \text{totally isotropic subspace}\, W \subset V\}.$$

Therefore, we get

Proposition A.1.
$$\mathscr{F}\ell^{wa} = \mathscr{F}\ell \setminus G(\mathbb{Q}_p) S^{ad},$$
where S^{ad} is the adic space associated to the Schubert variety attached to P (S is defined by the locus $x_{[\frac{n}{2}]+1} = \cdots = x_n = 0$ inside $\mathscr{F}\ell$).

Now we look at the admissible locus $\mathscr{F}\ell^{adm}$ (cf. [35] Definition A.6). We have the following

Claim A.2. $\mathscr{F}\ell^{adm} = \mathscr{F}\ell^{wa}$.

Proof. For any point $x \in \mathscr{F}\ell^{wa}(C, \mathcal{O}_C)$, let $\mathcal{E}_x$ be the associated modification of $\mathcal{O}_X^n$ such that the relative position of $(B_{dR}^+)^n$ and $\widehat{\mathcal{E}}_{x,\infty}$ is bounded by μ. Here X is the Fargues-Fontaine curve over $\mathbb{Q}_p$ associated to the perfectoid field $C^\flat$, and $\infty = x_C \in X$ is the point defined by C. We need to show this weakly admissible modification is in fact an admissible modification (i.e. $\mathcal{E}_x$ is semi-stable).

By [35] Proposition A. 9, we have either

$$\mathcal{E}_x \simeq \mathcal{O}_X(\frac{1}{r}) \oplus \mathcal{O}_X^{n-2r} \oplus \mathcal{O}_X(-\frac{1}{r})$$

for some integer $1 \leq r \leq [\frac{n}{2}]$, or

$$\mathcal{E}_x \simeq \mathcal{O}_X^n.$$

The second case is admissible. Thus we suppose that we are in the first case. The perfect quadratic form on $\mathcal{E}_x$ is such that for any $\lambda \in \mathbb{Q}$, we have $(\mathcal{E}_x^{\geq \lambda})^\perp = \mathcal{E}_x^{>-\lambda}$, where $\mathcal{E}_x^\lambda \subset \mathcal{E}_x$ is a step in the Harder-Narasimhan filtration of $\mathcal{E}_x$. Therefore, we get

$$\mathcal{O}_X(\frac{1}{r})^\perp = \mathcal{O}_X(\frac{1}{r}) \oplus \mathcal{O}_X^{n-2r}$$

and $\mathcal{O}_X(\frac{1}{r})$ is totally isotropic. It follows that there exists a unique sub vector bundle $\mathscr{F} \subset \mathcal{O}_X^n$ which is a locally direct summand, such that the modification $\mathcal{E}_x|_{X\setminus\infty} \xrightarrow{\sim} \mathcal{O}^n_{X\setminus\infty}$ induces a modification

$$\mathcal{O}_X(\frac{1}{r})|_{X\setminus\infty} \xrightarrow{\sim} \mathscr{F}|_{X\setminus\infty}.$$

In particular, $\mathscr{F}$ is totally isotropic in $\mathcal{O}_X^n$. Such a modification is necessarily of one of the following types:

(1) $(-1, 0, \ldots, 0)$,

(2) $(0, \ldots, 0, 1)$,
(3) $(0, \ldots, 0)$.

Indeed, it suffices to look at for all the sub B_{dR}-vector spaces E of B_{dR}^n, the relative positions of the lattices $E \cap (B_{dR}^+)^n$ and $E \cap \langle te_1, e_2, \ldots, e_{n-1}, t^{-1}e_n \rangle$, where $e_1, \ldots, e_n$ is a basis of V. As $\mathcal{O}_X^n$ is semi-stable, we have $\deg(\mathscr{F}) \leq 0$. By looking at the above three cases, we get that $\mathscr{F}$ is a degree -1 modification of $\mathcal{O}_X(\frac{1}{r})$. Thus,

$$\mathscr{F} \simeq \mathcal{O}_X^r,$$

that is $\mathscr{F} = W \otimes \mathcal{O}_X$ for some totally isotropic subspace $W \subset \mathbb{Q}_p^n$ of dimension r. This implies that our modification $\mathcal{E}_x|_{X \setminus \infty} \xrightarrow{\sim} \mathcal{O}_{X \setminus \infty}^n$ is admissible. □

References

[1] B. Bhatt, P. Scholze, *Projectivity of the Witt vector affine Grassmannian*, Preprint, arXiv:1507.06490, to appear in Invent. Math.

[2] M. Borovoi. *Abelian Galois cohomology of reductive groups*, Mem. Amer. Math. Soc., 132(626):viii+50, 1998.

[3] J.-F. Boutot, T. Zink, *The p-adic uniformization of Shimura curves*, Preprint, available at https://www.math.uni-bielefeld.de/∼zink/p-adicuni.ps

[4] A. Caraiani, P. Scholze, *On the generic part of the cohomology of compact unitary Shimura varieties*, Preprint, arXiv:1511.02418.

[5] M. Chen, *Composantes connexes géométriques d'espaces de modules de groupes p-divisibles*, Ann. Sci. École Norm. Sup. 2014 (4), 723-764.

[6] M. Chen, M. Kisin, and E. Viehmann, *Connected components of affine Deligne- Lusztig varieties in mixed characteristic*, Compositio. Math. 151 (2015), 1697-1762.

[7] J.-F. Dat, S. Orlik, and M. Rapoport, *Period domains over finite and p- adic fields*, Cambridge Tracts in Mathematics, vol. 183, Cambridge University Press, Cambridge, 2010.

[8] P. Deligne, *Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques*, in "Automorphic forms, representations and L-functions" (Corvallis 1977), Proc. Sympos. Pure Math. XXXIII, Amer. Math. Soc., 247-289, 1979.

[9] V. G. Drinfeld, *Coverings of p-adic symmetric regions*, Functional Anal. Appi.10, 29-40 (1976).

[10] T. Ekedahl, G. van der Geer, *Cycle classes on the moduli of K3 surfaces in positive characteristic*, Sel. Math. New Ser. 2014

[11] G. Faltings, *Coverings of p-adic period domains*, J. Reine Angew. Math. 643 (2010), 111-139.

[12] L. Fargues, *Geometrization of the local Langlands correspondence: an overview*, Preprint.

[13] L. Fargues, *Quelques résultats et conjectures concernant la courbe*, in "De la géométrie algébrique aux formes automorphes (I) - (Une collection d'articles en l'honneur du soixantième anniversaire de Gérard Laumon)", vol. 369, Astérisque 2015.

[14] L. Fargues, *G-torseurs en théorie de Hodge p-adique*, Preprint.

[15] L. Fargues, *Lettre à Rapoport.*

[16] L. Fargues, *p-adic Twistors and Shtukas*, lecture notes.

[17] L. Fargues, J.-M. Fontaine, *Courbes et fibrés vectoriels en théorie de Hodge p-adique*, Preprint.

[18] U. Görtz, X. He, *Basic loci in Shimura varieties of coxeter type*, Cambridge Journal of Mathematics 3 (2015), 323-353.

[19] U. Görtz, X. He, S. Nie, *Fully Hodge-Newton decomposable Shimura varieties*, Preprint, arXiv: 1610.05381.

[20] D. Hansen, *Period morphisms and variation of p-adic Hodge structure (preliminary draft)*, Preprint, available at http://www.math.columbia.edu/∼hansen/periodmapmod.pdf

[21] U. Hartl, *On a conjecture of Rapoport and Zink*, Invent. Math. (2013) 193, 627-696.

[22] B. Howard, G. Pappas, *Rapoport-Zink spaces for spinor groups*, Preprint, arXiv: 1509.03914.

[23] A. J. de Jong, *Étale fundamental groups of non-Archimedean analytic spaces*, Compositio Math. 97 (1995), 89-118.

[24] A. J. de Jong, *Crystalline Dieudonné module theory via formal and rigid geometry*, Pub. math. IHES 82 (1995), 5-96.

[25] K. S. Kedlaya, R. Liu, *Relative p-adic Hodge theory: Foundations*, Astérisque 371, Soc. Math. France, 2015.

[26] W. Kim, *Rapoport-Zink spaces of Hodge type*, Preprint, arXiv: 1308.5537.

[27] W. Kim, *Rapoport-Zink uniformization of Hodge type Shimura varieties*, Preprint.

[28] M. Kisin, *Integral models for Shimura varieties of abelian type*, J. Amer. Math. Soc. 23 (2010), 967-1012.

[29] M. Kisin, *Mod p points on Shimura varieties of abelian type*, Preprint, to appear in J. Amer. Math. Soc.

[30] V. Lafforgue, *Introduction to chtoucas for reductive groups and to the global langlands parameterization*, Preprint.

[31] C. Liedtke, *Supersingular K3 surfaces are unifrational*, Invent. Math. (2015) 200, 979-1014.

[32] K. Madapusi Pera, *The Tate conjecture for K3 surfaces in odd characteristic*, Invent. Math. (2015) 201, 625-668.

[33] M. Rapoport, *Non-archimedean period domains*, In Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), pages 423-434.

[34] M. Rapoport, *A guide to the reduction modulo p of Shimura varieties*, in "Automorphic forms I", Astérisque 298 (2005), 271-318.

[35] M. Rapoport, *Accessible and weakly accessible period domains*, Appendix of *On the p-adic cohomology of the Lubin-Tate tower* by Scholze, Preprint.

[36] M. Rapoport, M. Richartz, *On the classification and specialization of F- isocrystals with additional structure*, Compositio Math. 103 (1996), no. 2, 153181.

[37] M. Rapoport, T. Zink, *Period spaces for p-divisible groups*, Ann. of Math. Stud. 141, Princetion Univ. Press, 1996.

[38] M. Rapoport, E. Viehmann, *Towards a theory of local Shimura varieties*, Münster J. of Math. 7 (2014), 273-326.

[39] J. Rizov, *Moduli stacks of polarized K3 surfaces in mixed characteristic*, Serdica Math. J. 32 (2006), no. 2-3, 131-178.

[40] J. Rizov, *Kuga-Satake abelian varieties of K3 surfaces in mixed characteristic*, J. reine angew. Math. 648 (2010), 13-67.

[41] P. Scholze, *Perfectoid spaces*, Publ. Math. IHES 116 (2012), no.1, 245-313.

[42] P. Scholze, *Lectures on p-adic geometry*, notes taken by J. Weinstein, Fall 2014.

[43] P. Scholze, J. Weinstein, *Moduli of p-divisible groups*, Cambridge Journal of Mathematics 1 (2013), 145-237.

[44] X. Shen, *Perfectoid Shimura varieties of abelian type*, Preprint, to appear in IMRN.

[45] X. Shen, *Geometric structures of perfectoid Shimura varieties*, in preparation.

[46] X. Shen, C. Zhang, *Stratifications of Shimura varieties of abelian type*, in preparation.

[47] I. Vollaard, T. Wedhorn, *The supersingular locus of the Shimura variety of $GU(1, n-1)$, II*, Invent. Math. 184 (2011), 591-627.

[48] Y. Varshavsky, *Moduli spaces of principal F-bundles*, Selecta Math. (N.S.) 10 (2004), no. 1, 131-166.

[49] T. Wedhorn, *Ekedahl-Oort strata of Shimura varieties*, lecture notes in a Summer School on Shimura varieties, 2016, NCTS, Taiwan.

[50] D. Wortmann, *The μ-ordinary locus for Shimura varieties of Hodge type*, Preprint, arXiv: 1310.6444.

[51] C. Zhang, *Ekedahl-Oort strata for good reductions of Shimura varieties of Hodge type*, Preprint, arXiv: 1312.4869.

[52] X. Zhu, *Affine Grassmannians and the geometric Satake in mixed characteristic*, Preprint, arXiv:1407.8519

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