L_p MINKOWSKI VALUATIONS ON POLYTOPES

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ABSTRACT. For $1 \leq p < \infty$, Ludwig, Haberl and Parapatits classified L_p Minkowski valuations intertwining the special linear group with additional conditions such as homogeneity and continuity. In this paper,a complete classification of L_p Minkowski valuations intertwining the special linear group on polytopes without any additional conditions is established for $p \geq 1$ including $p = \infty$. For $n = 3$ and $p = 1$, there exist valuations not mentioned before.

1. Introductions

Let $\mathcal{K}_o^n$ be the set of convex bodies (i.e., compact convex sets) in $\mathbb{R}^n$ containing the origin, $\mathcal{P}_o^n$ the set of polytopes in $\mathbb{R}^n$ containing the origin and $\mathcal{T}_o^n$ the set of simplices in $\mathbb{R}^n$ containing the origin as one of their vertices.

For $1 \leq p \leq \infty$ and $K, L \in \mathcal{K}_o^n$, the L_p *Minkowski sum* of K and L is defined by its support function as

$$h_{K+_pL}(x) = (h_K(x)^p + h_L(x)^p)^{1/p}, \ x \in \mathbb{R}^n. \tag{1.1}$$

Here h_K is the support function of K; see Section 2. When $p = \infty$, the definition (1.1) should be interpreted as $h_{K+_\infty L}(x) = h_K(x) \vee h_L(x)$, the maximum of $h_K(x)$ and $h_L(x)$. When $p = 1$, the definition (1.1) gives the ordinary Minkowski addition.

An L_p *Minkowski valuation* is a function $Z : \mathcal{P}_o^n \to \mathcal{K}_o^n$ such that

$$Z(K \cup L) +_p Z(K \cap L) = ZK +_p ZL, \tag{1.2}$$

whenever $K, L, K \cup L, K \cap L \in \mathcal{P}_o^n$. In some cases, we will just consider valuations defined on $\mathcal{T}_o^n$ that means (1.2) holds whenever $K, L, K \cup L, K \cap L \in \mathcal{T}_o^n$.

For $1 \leq p < \infty$, Ludwig [8], Haberl [3] and Parapatits [20], [21] classified L_p Minkowski valuations intertwining the special linear group, $\mathrm{SL}(n)$, with some additional conditions such as homogeneity and continuity.

A map Z from $\mathcal{K}_o^n$ to the power set of $\mathbb{R}^n$ is called $\mathrm{SL}(n)$ contravariant if

$$Z(\phi K) = \phi^{-t} ZK$$

for any $K \in \mathcal{K}_o^n$ and any $\phi \in \mathrm{SL}(n)$. The map Z is called $\mathrm{SL}(n)$ covariant if

$$Z(\phi K) = \phi ZK$$

for any $K \in \mathcal{K}_o^n$ and any $\phi \in \mathrm{SL}(n)$. Notice that $\{o\}$ is the only subset of $\mathbb{R}^n$ invariant under all $\mathrm{SL}(n)$ transforms. Thus if Z is $\mathrm{SL}(n)$ contravariant (or covariant), then

$$Z\{o\} = \{o\}. \tag{1.3}$$

Generalizing results for homogeneous or translation invariant valuations by Ludwig [6,8], Haberl [3] and Parapatits [20], [21] established the following classification theorem.

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Theorem 1.1 (Haberl [3] and Parapatits [20]). *Let $n \geq 3$. A map $Z : \mathcal{P}_o^n \to \mathcal{K}_o^n$ is an $\mathrm{SL}(n)$ contravariant Minkowski valuation if and only if there exist constants $c_1, c_2, c_3 \in \mathbb{R}$ with $c_1 \geq 0$ and $c_1 + c_2 + c_3 \geq 0$ such that*

$$ZP = c_1 \Pi P + c_2 \Pi_o P + c_3 \Pi_o(-P)$$

for every $P \in \mathcal{P}_o^n$.

For $1 < p < \infty$, a map $Z : \mathcal{P}_o^n \to \mathcal{K}_o^n$ is an $\mathrm{SL}(n)$ contravariant L_p Minkowski valuation if and only if there exist constants $c_1, c_2 \geq 0$ such that

$$ZP = c_1 \hat{\Pi}_p^+ P +_p c_2 \hat{\Pi}_p^- P$$

for every $P \in \mathcal{P}_o^n$.

Here Π is the classical *projection body*, while $\hat{\Pi}_p^+$ and $\hat{\Pi}_p^-$ are the *asymmetric L_p projection bodies* first defined in [8]; see Section 2. Π_o is a valuation defined by $h_{\Pi_o P} = h_{\Pi P} - h_{\hat{\Pi}^+ P}$.

Theorem 1.2 (Haberl [3] and Parapatits [21]). *Let $n \geq 3$, $1 \leq p < \infty$ and $\{e_i\}_{i=1}^n$ be the standard basis of $\mathbb{R}^n$. A map $Z : \mathcal{P}_o^n \to \mathcal{K}_o^n$ is an $\mathrm{SL}(n)$ covariant L_p Minkowski valuation which is continuous at the line segment $[o, e_1]$ if and only if there exist constants $c_1, \dots, c_4 \geq 0$ such that*

$$ZP = c_1 M_p^+ P +_p c_2 M_p^- P +_p c_3 P +_p c_4(-P)$$

for every $P \in \mathcal{P}_o^n$.

Here M_p^+, M_p^- are the *asymmetric L_p moment bodies* first defined in [8]; see Section 2.

Haberl and Schuster [5] established affine isoperimetric inequalities for asymmetric L_p projection bodies and asymmetric L_p moment bodies. For other results on L_p Minkowski valuations, see [1, 2, 7, 9, 10, 19, 22, 24–29]. L_p projection bodies and L_p moment bodies ($1 < p < \infty$) were first studied in [14] as part of L_p Brunn-Minkowski theory developed by Lutwak, Yang, and Zhang, and many others; see [4, 12, 13, 15–18].

As first result of this paper, we establish a classification of L_∞ Minkowski valuations. We remark that the L_∞ sum of $K, L \in \mathcal{K}^n$ is equal to its convex hull, $[K, L]$.

Theorem 1.3. *Let $n \geq 3$. A map $Z : \mathcal{P}_o^n \to \mathcal{K}_o^n$ is an $\mathrm{SL}(n)$ contravariant L_∞ Minkowski valuation if and only if there exist constants $c_1, c_2 \geq 0$ such that*

$$ZP = c_1 \hat{\Pi}_\infty^+ P +_\infty c_2 \hat{\Pi}_\infty^- P$$

for every $P \in \mathcal{P}_o^n$.

The *asymmetric L_∞ projection body* $\hat{\Pi}_\infty^+ : \mathcal{P}_o^n \to \mathcal{K}_o^n$ is defined by

$$\hat{\Pi}_\infty^+ P = \left[o, \frac{u_i}{h_P(u_i)} : u_i \in \mathcal{N}(P) \setminus \mathcal{N}_o(P) \right],$$

and

$$\hat{\Pi}_\infty^- P = -\hat{\Pi}_\infty^+ P.$$

Here $\mathcal{N}(P)$ is the set of outer unit normals to facets (that is $n-1$ dimensional faces) of P and $\mathcal{N}_o(P)$ is the set of outer unit normals to facets of P which contain the origin. Both $\hat{\Pi}_\infty^+$ and $\hat{\Pi}_\infty^-$ are the limits of $\hat{\Pi}_p^+$ and $\hat{\Pi}_p^-$ as $p \to \infty$. So they are clearly L_∞ Minkowski valuations. Also, $\hat{\Pi}_\infty^+$ is an extension of the polarity. Indeed, if a convex body K contains the origin in its interior, then $\hat{\Pi}_\infty^+ K = K^*$, the polar body of K. All the details can be found in Section 2.

If a valuation Z_p is an L_p Minkowski valuation, then the limit $\lim_{p\to\infty} Z_p$ is an L_∞ Minkowski valuation. But there could be more L_∞ Minkowski valuations than the limits of L_p cases. Indeed, Theorem 1.4 shows that there are additional examples.

Theorem 1.4. *Let $n \geq 3$. A map $Z : \mathcal{P}_o^n \to \mathcal{K}_o^n$ is an $\mathrm{SL}(n)$ covariant L_∞ Minkowski valuation if and only if there exist constants $0 \leq a_1 \leq \cdots \leq a_n$, $0 \leq b_1 \leq \cdots \leq b_n$ such that*

$$ZP = a_d P +_\infty (-b_d P)$$

for every d-dimensional convex polytope $P \in \mathcal{P}_o^n$, $1 \leq d \leq n$, while $Z\{o\} = \{o\}$.

If $\dim P = n$, then $\lim_{p\to\infty} M_p^+ P = P$ and $\lim_{p\to\infty} M_p^- P = -P$. If $\dim P < n$, $\lim_{p\to\infty} M_p^+ P = \{o\}$ and $\lim_{p\to\infty} M_p^- P = \{o\}$. This is the reason that $\lim_{p\to\infty} M_p^+$ and $\lim_{p\to\infty} M_p^-$ do not show up in Theorem 1.4; see Section 2 for details. In Theorem 1.4, we do not have any continuity assumptions. It inspires us to also find a classification result for $\mathrm{SL}(n)$ covariant L_p Minkowski valuations without any continuity assumptions for finite p.

Theorem 1.5. *Let $n \geq 3$ and $1 < p < \infty$. A map $Z : \mathcal{P}_o^n \to \mathcal{K}_o^n$ is an $\mathrm{SL}(n)$ covariant L_p Minkowski valuation if and only if there exist constants $c_1, \dots, c_4 \geq 0$ such that*

$$ZP = c_1 M_p^+ P +_p c_2 M_p^- P +_p c_3 P +_p c_4(-P)$$

for every $P \in \mathcal{P}_o^n$.

Theorem 1.6. *Let $n \geq 4$. A map $Z : \mathcal{P}_o^n \to \mathcal{K}_o^n$ is an $\mathrm{SL}(n)$ covariant Minkowski valuation if and only if there exist constants $c_1, \dots, c_4 \geq 0$ such that*

$$ZP = c_1 M^+ P + c_2 M^- P + c_3 P + c_4(-P)$$

for every $P \in \mathcal{P}_o^n$.

Theorem 1.7. *A map $Z : \mathcal{P}_o^3 \to \mathcal{K}_o^3$ is an $\mathrm{SL}(n)$ covariant Minkowski valuation if and only if there exist constants $a_1, a_2, b_1, b_2, c_1, c_2 \geq 0$ satisfying $a_1 \leq a_2$, $b_1 \leq b_2$, $a_2 - a_1 \leq b_2$ and $b_2 - b_1 \leq a_2$ such that*

$$ZP = c_1 M^+ P + c_2 M^- P + D_{a_1,a_2,b_1,b_2} P$$

for every $P \in \mathcal{P}_o^3$.

The convex body $D_{a_1,a_2,b_1,b_2} P$ is a generalization of the difference body. We remark that it was omitted in the classifcation by Ludwig [8, Theorem 1]. Denote by $\mathcal{E}_o(P)$ the set of edges of P that contain the origin and by $\mathcal{F}_o(P)$ the set of 2-dimensional faces of P that contain the origin. For $P \in \mathcal{P}_o^3$,

$$\begin{aligned} h_{D_{a_1,a_2,b_1,b_2} P} = a_1 h_P &+ (a_2 - a_1) \sum_{F \in \mathcal{F}_o(P)} h_F - (a_2 - a_1) \sum_{E \in \mathcal{E}_o(P)} h_E \\ &+ b_1 h_{-P} + (b_2 - b_1) \sum_{F \in \mathcal{F}_o(P)} h_{-F} - (b_2 - b_1) \sum_{E \in \mathcal{E}_o(P)} h_{-E} \end{aligned}$$

if $\dim P = 3$;

$$\begin{aligned} h_{D_{a_1,a_2,b_1,b_2} P} = (2a_2 - a_1) h_P &- (a_2 - a_1) \sum_{E \in \mathcal{E}_o(P)} h_E \\ &+ (2b_2 - b_1) h_{-P} - (b_2 - b_1) \sum_{E \in \mathcal{E}_o(P)} h_{-E} \end{aligned}$$

if $\dim P = 2$; and

$$h_{D_{a_1,a_2,b_1,b_2}P} = a_1 h_P + b_1 h_{-P}$$

if $\dim P = 1$. That $h_{D_{a_1,a_2,b_1,b_2}P}$ is a support function is guaranteed by the conditions on a_1, a_2, b_1, b_2.

Theorem 1.5, 1.6 and 1.7 are based on the classification of function-valued valuations (Lemma 5.2). The map $Z : \mathcal{P}_o^n \to \mathcal{K}_o^n$ is an L_p Minkowski valuation if and only if $\Phi : P \mapsto h_{ZP}^p$ is a function-valued valuation; see Section 5 for more details. There exist additional *complicated* function-valued valuations ($P \mapsto \Phi_{p;a_1,a_2}P + \Phi_{p;b_1,b_2}(-P)$; see the definition in Section 5) if we do not assume continuity like Haberl [3] and Parapatits [21] did. However, in generally, they are not L_p Minkowski valuations for $p > 1$. For $p = 1$, $h_{D_{a_1,a_2,b_1,b_2}P} = \Phi_{1;a_1,a_2}P + \Phi_{1;b_1,b_2}(-P)$ for $\dim P \leq 3$. For $n \geq 4$, $P \mapsto \Phi_{1;a_1,a_2}P + \Phi_{1;b_1,b_2}(-P)$ is also a function-valued valuation on $\mathcal{P}_o^n$. But the example used for $n \geq 4$ and $p = 1$ in Lemma 5.8 shows that $\Phi_{1;a_1,a_2}[-e_1, e_1, e_2, e_3, e_4] + \Phi_{1;b_1,b_2}(-[-e_1, e_1, e_2, e_3, e_4])$ is not a support function. That means D_{a_1,a_2,b_1,b_2} even cannot be extended to simplices that contain the origin in one of their edges for dimension greater than or equal to 4. However, Theorem 5.11 shows that D_{a_1,a_2,b_1,b_2} can be extended to a valuation on $\mathcal{T}_o^n$ also for $n \geq 4$.

2. Preliminaries and Notation

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and $\{e_i\}_{i=1}^n$ its standard basis. For $1 \leq d \leq n-1$, we will also use $\mathbb{R}^d$ to denote the linear space spanned by $\{e_1, \dots, e_d\}$. The usual scalar product of two vectors $x, y \in \mathbb{R}^n$ shall be denoted by $x \cdot y$. The convex hull of a set $A \subset \mathbb{R}^n$ is denoted by $[A]$.

Let $a, b \in \mathbb{R}$. We write $a \vee b := \max\{a, b\}$.

Let $\mathcal{K}^n$ be the set of convex bodies in $\mathbb{R}^n$. For $K \in \mathcal{K}^n$, $\operatorname{relint} K$, $\operatorname{relbd} K$, K^c and $\operatorname{lin} K$ denote the relative interior, the relative boundary, the relative complement with respect to the affine hull of K, and the linear hull of K, respectively. We mention that $\operatorname{relint} K \neq \emptyset$ if $K \neq \emptyset$.

Let $Gr(n, j)$ be the set of j-dimensional linear subspaces in $\mathbb{R}^n$. For $x \in \mathbb{R}^n$, $A \subset \mathbb{R}^n$, $V \in Gr(n, j)$, let $x|V$ be the orthogonal projection of x onto V and $A|V = \{x|V : x \in A\}$. We also write $x|K$ for the orthogonal projection of x onto the linear hull of $K \in \mathcal{K}_o^n$.

The *support function* of a convex body K is defined by

$$h_K(x) = \max\{x \cdot y : y \in K\}$$

for any $x \in \mathbb{R}^n$. The support function is sublinear, i.e., it is homogeneous,

$$h_K(\lambda x) = \lambda h_K(x)$$

for any $x \in \mathbb{R}^n$, $\lambda \geq 0$, and subadditive,

$$h_K(x + y) \leq h_K(x) + h_K(y)$$

for any $x, y \in \mathbb{R}^n$. The support function is also continuous on $\mathbb{R}^n$ by its convexity. A convex body is uniquely determined by its support function, and for any sublinear function h, there exists a convex body K such that $h_K = h$. It is easy to see that

$$h_{\lambda K} = \lambda h_K \tag{2.1}$$

for any $\lambda \geq 0$ and $K \in \mathcal{K}^n$. Also,

$$h_{\phi K}(x) = h_K(\phi^t x)$$

for $x \in \mathbb{R}^n$, $\phi \in \mathrm{GL}(n)$ and $K \in \mathcal{K}^n$.

For $K, L \in \mathcal{K}^n$, if $K \cup L$ is convex, then

$$h_{K\cup L} = \max\{h_K, h_L\},\ h_{K\cap L} = \min\{h_K, h_L\}.$$

Hence the identity map is an L_p Minkowski valuation on $\mathcal{K}^n$ (or on $\mathcal{P}_o^n$).

The *face* of $K \in \mathcal{K}^n$ with normal vector $u \in S^{n-1}$ is $F(K,u) = \{y \in K : y \cdot u = h_K(u)\}$.

A *hyperplane* H through the origin with a normal vector u is defined by $\{x \in \mathbb{R}^n : x{\cdot}u = 0\}$. Furthermore define $H^- := \{x \in \mathbb{R}^n : x{\cdot}u \le 0\}$ and $H^+ := \{x \in \mathbb{R}^n : x{\cdot}u \ge 0\}$. For $0 < \lambda < 1$, let H_λ be the hyperplane through the origin with normal vector $(1-\lambda)e_1 - \lambda e_2$.

The following $\mathrm{SL}(n)$ transforms $\phi_1, \phi_2, \phi_3, \phi_4$ depending on λ, $0 < \lambda < 1$, will be useful.

$$\phi_1 e_1 = \lambda e_1 + (1-\lambda)e_2,\ \phi_1 e_2 = e_2,\ \phi_1 e_n = \frac{1}{\lambda}e_n,\ \phi_1 e_i = e_i,\ \text{for } 3 \le i \le n-1,$$

$$\phi_2 e_1 = e_1,\ \phi_2 e_2 = \lambda e_1 + (1-\lambda)e_2,\ \phi_2 e_n = \frac{1}{1-\lambda}e_n,\ \phi_2 e_i = e_i,\ \text{for } 3 \le i \le n-1,$$

$$\phi_3 e_1 = (\frac{1}{\lambda})^{1/n}(\lambda e_1 + (1-\lambda)e_2),\ \phi_3 e_2 = (\frac{1}{\lambda})^{1/n}e_2,\ \phi_3 e_i = (\frac{1}{\lambda})^{1/n}e_i,\ \text{for } 3 \le i \le n,$$

and

$$\phi_4 e_1 = (\frac{1}{1-\lambda})^{1/n}e_1,\ \phi_4 e_2 = (\frac{1}{1-\lambda})^{1/n}(\lambda e_1 + (1-\lambda)e_2), \phi_4 e_i = (\frac{1}{1-\lambda})^{1/n}e_i,\ \text{for } 3 \le i \le n.$$

For $1 \le d \le n$, let $T^d = [o, e_1, e_2, e_3, \dots, e_d]$ and $\hat{T}^{d-1} = [o, e_1, e_3 \dots, e_d]$. Hence, for $s > 0$, $sT^d \cap H_\lambda^- = \phi_1 sT^d$, $sT^d \cap H_\lambda^+ = \phi_2 sT^d$ and $sT^d \cap H_\lambda = \phi_1 s\hat{T}^{d-1}$ for $2 \le d \le n-1$. Also, $sT^n \cap H_\lambda^- = \phi_3 \lambda^{1/n} sT^n$, $sT^n \cap H_\lambda^+ = \phi_4 (1-\lambda)^{1/n} sT^n$ and $sT^n \cap H_\lambda = \phi_1 \lambda^{1/n} s\hat{T}^{n-1}$.

The *asymmetric L_p moment body* of a star body K is defined by

$$h_{M_p^+ K}(x) = \left(\int_K (\max\{x \cdot y, 0\})^p dy\right)^{1/p},\ x \in \mathbb{R}^n$$

and

$$h_{M_p^- K}(x) = \left(\int_K (\max\{-x \cdot y, 0\})^p dy\right)^{1/p},\ x \in \mathbb{R}^n.$$

Both M_p^+, M_p^- are $\mathrm{SL}(n)$ covariant L_p Minkowski valuations. Positive combinations of M_p^+ and M_p^- were first characterized as $(\frac{n}{p}+1)$-homogeneous and $\mathrm{SL}(n)$ covariant L_p Minkowski valuations by Ludwig [8]. Also see Theorem 1.2. For $\dim K = n$,

$$h_{M_\infty^+ K}(x) = \lim_{p\to\infty} h_{M_p^+ K}(x) = \max_{y\in K}\{x \cdot y\} = h_K(x),\ x \in \mathbb{R}^n$$

and for $\dim K < n$, $M_\infty^+ K = \{o\}$.

The *projection body* of $K \in \mathcal{K}^n$ is defined by

$$h_{\Pi K}(x) = \frac{1}{2}\int_{S^{n-1}} |x \cdot u| dS_K(u),\ x \in \mathbb{R}^n,$$

where S_K is the *surface area measure* of K. For a Borel set $\omega \subset S^{n-1}$, $S_K(\omega)$ is the $(n-1)$-Hausdorff measure of $\{x \in \mathrm{bd}\, K : \nu_K(x) \in \omega\}$, where $\nu_K(x)$ are outer normal vectors to K at x.

The *cone-volume measure* of $K \in \mathcal{K}_o^n$ is defined by $dv_K(u) = h_K(u)dS_K(u)$. The *asymmetric L_p projection body* of $P \in \mathcal{P}_o^n$ is defined by

$$h_{\hat{\Pi}_p^+ P}(x) = \left(\int_{S^{n-1}\setminus\mathcal{N}_o(P)} (\frac{\max\{x\cdot u, 0\}}{h_P(u)})^p dv_P(u)\right)^{1/p}$$

for any $x \in \mathbb{R}^n$ and

$$h_{\hat{\Pi}_p^- P}(x) = \left(\int_{S^{n-1}\setminus\mathcal{N}_o(P)} (\frac{\max\{-x\cdot u, 0\}}{h_P(u)})^p dv_P(u)\right)^{1/p} = h_{\hat{\Pi}_p^+ P}(-x)$$

for any $x \in \mathbb{R}^n$. Positive combinations of $\hat{\Pi}_p^+$ and $\hat{\Pi}_p^-$ were first characterized as $(\frac{n}{p}-1)$-homogeneous, $\mathrm{SL}(n)$ contravariant L_p Minkowski valuations by Ludwig [8]. Also see Theorem 1.1. For $p=1$, Π_o defined by $h_{\Pi_o P} = h_{\Pi P} - h_{\hat{\Pi}^+ P}$ is an additional valuation.

When $p \to \infty$, we have

$$\lim_{p\to\infty} h_{\hat{\Pi}_p^+ P}(x) = \max_{u_i \in \mathcal{N}(P)\setminus\mathcal{N}_o(P)} \{\frac{x\cdot u_i}{h_P(u_i)}, 0\} = h_{\hat{\Pi}_\infty^+ P}(x).$$

Hence $\hat{\Pi}_\infty^+$ is a (-1)-homogeneous, $\mathrm{SL}(n)$ contravariant L_∞ Minkowski valuation. For $K \in \mathcal{K}^n$ containing the origin in its interior,

$$\lim_{p\to\infty} h_{\hat{\Pi}_p^+ K}(x) = \lim_{p\to\infty}\left(\int_{S^{n-1}} (\frac{\max\{x\cdot u, 0\}}{h_K(u)})^p dv_K(u)\right)^{1/p} = \operatorname*{ess\,sup}_{u\in S^{n-1}} \frac{x\cdot u}{h_K(u)}.$$

Here the essential supremum is with respect to the cone-volume measure. We have

$$\frac{x\cdot u}{h_K(u)} = \frac{1}{\rho_K(x)}\frac{\rho_K(x)x\cdot u}{h_K(u)} \le \frac{1}{\rho_K(x)}\frac{\rho_K(x)x\cdot u}{\rho_K(x)x\cdot u} = \frac{1}{\rho_K(x)},$$

where equality holds when $h_K(u) = \rho_K(x)x\cdot u$. Here $\rho_K(x) := \max\{\lambda > 0 : \lambda x \in K\}$ is the radial function of K. Also since there exists a normal vector u at $\rho_K(x)x$ such that $u \in \operatorname{supp} v_K$, the support set of v_K, and $u \mapsto \frac{x\cdot u}{h_K(u)}$ is continuous,

$$h_{\hat{\Pi}_\infty^+ K}(x) = \operatorname*{ess\,sup}_{u\in S^{n-1}} \frac{x\cdot u}{h_K(u)} = \frac{1}{\rho_K(x)} = h_{K^*}(x).$$

The following lemma will be used to classify L_∞ Minkowski valuations. It is an L_∞ version of the Cauchy functional equation.

Lemma 2.1. *If a function $f:(0,\infty)\to[0,\infty)$ satisfies*

$$f(x+y)\vee a = f(x)\vee f(y), \tag{2.2}$$

for any $x, y > 0$, where $a \ge 0$ is a constant, then

$$f(z) = f(1) \ge a$$

for any $z > 0$.

Proof. For $x = y = 1$ in (2.2), we directly get $f(1) \ge a$. We will prove $f(z) = f(1)$ in two steps.

Step ①: Let k be an integer. We will show, by induction, that

$$f(2^k) = f(1). \tag{2.3}$$

The case $k = 0$ is trivial. Taking $x = y = 2^k$ in (2.2), we get

$$f(2^{k+1}) \vee a = f(2^k) \vee f(2^k). \tag{2.4}$$

for any integer k. Hence

$$a \leq f(2^k)$$

for any k. For $k \geq 1$, assume that (2.3) holds for $k-1$. By (2.4), if $a < f(1)$, we have

$$f(2^k) = f(2^{k-1}) = f(1);$$

if $a = f(1)$, we have

$$f(2^k) \leq f(2^{k-1}) = f(1) = a \leq f(2^k).$$

Thus, (2.3) holds for $k \geq 1$.

For $k \leq -1$, assume that (2.3) holds for $k+1$. Since (2.4) and $a \leq f(1)$, we have

$$f(2^k) = f(2^{k+1}) = f(1).$$

Thus we obtain that (2.3) holds for any integer k.

Step ②: Let $z > 0$. There exists an integer k such that $2^k \leq z < 2^{k+1}$. Taking $x+y = 2^{k+1}$, $x = z$ in (2.2), we obtain that

$$f(2^{k+1}) \vee a = f(z) \vee f(2^{k+1} - z).$$

Since $a \leq f(1)$ and $f(2^{k+1}) = f(1)$ (step ①), we have

$$f(z) \leq f(1). \tag{2.5}$$

for any $z > 0$.

We assume $z \neq 2^k$. If $a < f(1)$, taking $x + y = z$, $x = 2^k$ in (2.2), we obtain that

$$f(z) \vee a = f(2^k) \vee f(z - 2^k).$$

By (2.5), $f(z - 2^k) \leq f(1)$. Also since $f(2^k) = f(1)$ from step ①, we have

$$f(z) = f(1).$$

If $a = f(1)$, taking $x = y = z$ in (2.2), we get

$$f(2z) \vee a = f(z) \vee f(z).$$

Then, we have

$$f(1) = a \leq f(z) \leq f(1).$$

The proof is complete. □

The following statements will be used to determine L_∞ Minkowski valuations by their values on $\mathcal{T}_o^n$.

Define $\mathcal{P}_1 := \mathcal{T}_o^n$ and $\mathcal{P}_i := \mathcal{P}_{i-1} \cup \{P_1 \cup P_2 \in \mathcal{P}_o^n : P_1, P_2 \in \mathcal{P}_{i-1} \text{ with disjoint relative interiors}\}$ recursively. Note that for any $P \in \mathcal{P}_o^n$, there exists an i such that $P \in \mathcal{P}_i$.

Let $H \subset \mathbb{R}^n$ be a hyperplane through the origin. For any $P \in \mathcal{P}_i$, $i \geq 1$, we also have

$$P \cap H \in \mathcal{P}_i. \tag{2.6}$$

Indeed, for any $T \in \mathcal{T}_o^n$, we have $T \cap H \in \mathcal{T}_o^n$. Assume that for any $P \in \mathcal{P}_{i-1}$, $i \geq 2$, we have $P \cap H \in \mathcal{P}_{i-1}$. Then for any $P = P_1 \cup P_2$, where $P_1, P_2 \in \mathcal{P}_{i-1}$ have disjoint relative interiors, we have

$$P \cap H = (P_1 \cap H) \cup (P_2 \cap H).$$

If $P_1 \cap H$ and $P_2 \cap H$ have disjoint relative interiors, then $P \cap H \in \mathcal{P}_i$. Otherwise, only two possibilities could happen: $(P_1 \cap H) \subset (P_2 \cap H)$ and $(P_2 \cap H) \subset (P_1 \cap H)$. For both possibilities, we have $P \cap H \in \mathcal{P}_{i-1} \subset \mathcal{P}_i$.

3. SL(n) contravariant L_∞ Minkowski valuations

In this section, we first show that any $\mathrm{SL}(n)$ contravariant L_∞ Minkowski valuation on $\mathcal{T}_o^n$ vanishes on lower dimensional simplices in $\mathcal{T}_o^n$.

Lemma 3.1. *Let $n \geq 3$. If $Z : \mathcal{T}_o^n \to \mathcal{K}_o^n$ be an $\mathrm{SL}(n)$ contravariant L_∞ Minkowski valuation, then $ZT = \{o\}$ for any $T \in \mathcal{T}_o^n$ satisfying $\dim T < n$.*

Proof. Let $T \in \mathcal{T}_o^n$ and $\dim T = d < n$. We can assume (w.l.o.g.) that the linear hull of T is $\mathrm{lin}\{e_1, \dots, e_d\}$, the linear space spanned by $\{e_1, \dots, e_d\}$. Let $\phi := \begin{bmatrix} I & A \\ 0 & B \end{bmatrix} \in \mathrm{SL}(n)$, where $I \in \mathbb{R}^{d\times d}$ is the identity matrix, $A \in \mathbb{R}^{d\times(n-d)}$ is an arbitrary matrix, $B \in \mathbb{R}^{(n-d)\times(n-d)}$ is a matrix with $\det B = 1$, $0 \in \mathbb{R}^{(n-d)\times d}$ is the zero matrix. Also let $x = \begin{pmatrix} x' \\ x'' \end{pmatrix} \in \mathbb{R}^{d\times(n-d)}$ and $x'' \neq 0$. Then $\phi T = T$, and with the $\mathrm{SL}(n)$ contravariance of Z, we have

$$h_{ZT}(x) = h_{Z\phi T}(x) = h_{ZT}(\phi^{-1}x) = h_{ZT}\begin{pmatrix} x' - AB^{-1}x'' \\ B^{-1}x'' \end{pmatrix}.$$

For $d \leq n-2$, we can choose an suitable matrix B such that $B^{-1}x''$ be any nonzero vector on $\mathrm{lin}\{e_{d+1}, \dots, e_n\}$. After fixing B we can also choose an suitable matrix A such that $x' - AB^{-1}x''$ is any vector in $\mathrm{lin}\{e_1, \dots, e_d\}$. So $h_{ZT}(x)$ is constant on a dense set of $\mathbb{R}^n$. By the continuity of the support function, we get $h_{ZT}(x) = 0$.

For $d = n-1$, $B = 1$. We can choose an suitable A such that $x' - AB^{-1}x'' = 0$, and then $h_{ZT}(x) = h_{ZT}(x_n e_n)$, where x_n is the n-th coordinate of x. By the $\mathrm{SL}(n)$ contravariance of Z, we only need to show that $h_{Z(sT^{n-1})}(x) = h_{Z(sT^{n-1})}(x_n e_n) = 0$ for any $s > 0$.

For $0 < \lambda < 1$, define H_λ and $\phi_1, \phi_2 \in \mathrm{SL}(n)$ as in Section 2. Since Z is a valuation,

$$h_{Z(sT^{n-1})}(e_n) \vee h_{Z(sT^{n-1}\cap H_\lambda)}(e_n) = h_{Z(sT^{n-1}\cap H_\lambda^-)}(e_n) \vee h_{Z(sT^{n-1}\cap H_\lambda^+)}(e_n).$$

From the conclusion above for $d = n-2$, we get

$$h_{Z(sT^{n-1})}(e_n) = h_{Z(sT^{n-1}\cap H_\lambda^-)}(e_n) \vee h_{Z(sT^{n-1}\cap H_\lambda^+)}(e_n).$$

Also by the $\mathrm{SL}(n)$ contravariance of Z, we obtain

$$\begin{aligned} h_{Z(sT^{n-1})}(e_n) &= h_{Z(\phi_1 sT^{n-1})}(e_n) \vee h_{Z(\phi_2 sT^{n-1})}(e_n) \\ &= h_{Z(sT^{n-1})}(\phi_1^{-1}e_n) \vee h_{Z(sT^{n-1})}(\phi_2^{-1}e_n) \\ &= h_{Z(sT^{n-1})}(\lambda e_n) \vee h_{Z(sT^{n-1})}((1-\lambda)e_n). \end{aligned}$$

If $h_{Z(sT^{n-1})}(e_n) \neq 0$, we get

$$\lambda \vee (1-\lambda) = 1$$

for any $0 < \lambda < 1$. This is a contradiction. Hence, $h_{Z(sT^{n-1})}(e_n) = 0$ for any $s > 0$. □

The following lemma establishes a homogeneity property.

Lemma 3.2. *Let $n \ge 3$. If $Z : \mathcal{P}_o^n \to \mathcal{K}_o^n$ is an* $\mathrm{SL}(n)$ *contravariant L_∞ Minkowski valuation, then*

$$h_{Z(sT^n)}(\pm e_i) = s h_{ZT^n}(\pm e_i), \ 1 \le i \le n, \tag{3.1}$$

for any $s > 0$.

Proof. Since Z is $\mathrm{SL}(n)$ contravariant, we only need to show that (3.1) holds for $i = n$.

Define H_λ and $\phi_3, \phi_4 \in \mathrm{SL}(n)$ as in Section 2. Since Z is a valuation,

$$h_{Z(sT^n)}(x) \vee h_{Z(sT^n \cap H_\lambda)}(x) = h_{Z(sT^n \cap H_\lambda^-)}(x) \vee h_{Z(sT^n \cap H_\lambda^+)}(x),$$

for any $x \in \mathbb{R}^n$, $s > 0$. By Lemma 3.1, $h_{Z(sT^n \cap H_\lambda)}(x) = 0$. Thus,

$$h_{Z(sT^n)}(x) = h_{Z(sT^n \cap H_\lambda^-)}(x) \vee h_{Z(sT^n \cap H_\lambda^+)}(x).$$

Note that $sT^n \cap H_\lambda^- = \phi_3 \lambda^{1/n} sT^n$, $sT^n \cap H_\lambda^+ = \phi_4 (1-\lambda)^{1/n} sT^n$. Since Z is $\mathrm{SL}(n)$ contravariant, we have

$$\begin{aligned} h_{Z(sT^n)}(x) &= h_{Z(\phi_3 \lambda^{1/n} sT^n)}(x) \vee h_{Z(\phi_4 (1-\lambda)^{1/n} sT^n)}(x) \\ &= h_{Z(\lambda^{1/n} sT^n)}(\phi_3^{-1} x) \vee h_{Z((1-\lambda)^{1/n} sT^n)}(\phi_4^{-1} x), \end{aligned} \tag{3.2}$$

where $x = (x_1, \dots, x_n)^t$, $\phi_3^{-1} x = \lambda^{1/n}(\frac{1}{\lambda} x_1, \frac{\lambda-1}{\lambda} x_1 + x_2, x_3, \dots, x_n)^t$ and $\phi_4^{-1} x = (1-\lambda)^{1/n}(x_1 - \frac{\lambda}{1-\lambda} x_2, \frac{1}{1-\lambda} x_2, x_3, \dots, x_n)^t$. If we choose $x = e_n$, then

$$h_{Z(sT^n)}(e_n) = h_{\lambda^{1/n} Z(\lambda^{1/n} sT^n)}(e_n) \vee h_{(1-\lambda)^{1/n} Z((1-\lambda)^{1/n} sT^n)}(e_n)$$

for any $0 < \lambda < 1$ and $s > 0$. Taking $\lambda = \frac{\lambda_1}{\lambda_2}$, $0 < \lambda_1 < \lambda_2$ and then taking $s = \lambda_2^{1/n}$, with (2.1), we get

$$h_{\lambda_2^{1/n} Z(\lambda_2^{1/n} T^n)}(e_n) = h_{\lambda_1^{1/n} Z(\lambda_1^{1/n} T^n)}(e_n) \vee h_{(\lambda_2-\lambda_1)^{1/n} Z((\lambda_2-\lambda_1)^{1/n} T^n)}(e_n) \tag{3.3}$$

for any $0 < \lambda_1 < \lambda_2$.

Let $f(\lambda) = h_{\lambda^{1/n} Z(\lambda^{1/n} T^n)}(e_n)$, $\lambda > 0$. Hence f satisfies the condition in Lemma 2.1. Thus we have

$$h_{\lambda^{1/n} Z(\lambda^{1/n} T^n)}(e_n) = h_{ZT^n}(e_n).$$

This shows $h_{Z(sT^n)}(e_n) = s h_{ZT^n}(e_n)$ for any $s > 0$. Similarly, $h_{Z(sT^n)}(-e_n) = s h_{ZT^n}(-e_n)$ for any $s > 0$.

□

Proof of Theorem 1.3. In Section 2, we have already shown that $\hat{\Pi}_\infty^+$ and $\hat{\Pi}_\infty^-$ are $\mathrm{SL}(n)$ contravariant L_∞ Minkowski valuations. Hence $c_1 \hat{\Pi}_\infty^+ P +_\infty c_2 \hat{\Pi}_\infty^- P$ is an $\mathrm{SL}(n)$ contravariant L_∞ Minkowski valuation.

Now we need to show that if $Z : \mathcal{P}_o^n \to \mathcal{K}_o^n$ is an $\mathrm{SL}(n)$ contravariant L_∞ Minkowski valuation, then there exists constants $c_1, c_2 \ge 0$ such that

$$ZP = c_1 \hat{\Pi}_\infty^+ P +_\infty c_2 \hat{\Pi}_\infty^- P \tag{3.4}$$

for any $P \in \mathcal{P}_o^n$.

Let $c_1 = h_{Z(sT^n)}(e_1)$ and $c_2 = h_{Z(sT^n)}(-e_1)$. We first want to show that

$$Z(sT^n) = [-c_2 s(e_1 + \cdots + e_n), c_1 s(e_1 + \cdots + e_n)] = c_1 \hat{\Pi}_\infty^+(sT^n) +_\infty c_2 \hat{\Pi}_\infty^-(sT^n)$$

for any $s > 0$. The second equality follows directly from the definitions of $\hat{\Pi}_\infty^+$ and $\hat{\Pi}_\infty^-$.

We will show that the orthogonal projection of $Z(sT^n)$ onto any plane spanned by $\{e_i, e_j\}$, $1 \le i < j \le n$ is the segment $[-c_2 s(e_i + e_j), c_1 s(e_i + e_j)]$. By the $\mathrm{SL}(n)$ contravariance of Z, we only need to show that $Z(sT^n)|\mathbb{R}^2$ has the desired result. Since

$$h_{Z(sT^n)}(x|\mathbb{R}^2) = h_{(Z(sT^n))|\mathbb{R}^2}(x),$$

we only need to consider $h_{Z(sT^n)}(\alpha e_1 + \beta e_2)$. Also since the support function is continuous, we will further assume that α, β are not zero.

If α, β have the same sign, taking $x = \alpha e_1 + \beta e_2$, $\lambda = \frac{\alpha}{\alpha+\beta}$ in (3.2), with (2.1), we obtain that

$$h_{Z(sT^n)}(\alpha e_1 + \beta e_2) = h_{\lambda^{1/n} Z(\lambda^{1/n} sT^n)}((\alpha+\beta)e_1) \vee h_{(1-\lambda)^{1/n} Z((1-\lambda)^{1/n} sT^n)}((\alpha+\beta)e_2).$$

Combined with the Lemma 3.2, we get

$$h_{Z(sT^n)}(\alpha e_1 + \beta e_2) = h_{Z(sT^n)}((\alpha+\beta)e_1) \vee h_{Z(sT^n)}((\alpha+\beta)e_2).$$

If $\alpha, \beta > 0$, we get

$$h_{Z(sT^n)}(\alpha e_1 + \beta e_2) = c_1 s(\alpha+\beta) = h_{[-c_2 s(e_1+e_2), c_1 s(e_1+e_2)]}(\alpha e_1 + \beta e_2).$$

If $\alpha, \beta < 0$, we get

$$h_{Z(sT^n)}(\alpha e_1 + \beta e_2) = -c_2 s(\alpha+\beta) = h_{[-c_2 s(e_1+e_2), c_1 s(e_1+e_2)]}(\alpha e_1 + \beta e_2).$$

If $\alpha > -\beta > 0$ or $-\alpha > \beta > 0$, taking $x = (\alpha+\beta)e_1$, $\lambda = \frac{\alpha+\beta}{\alpha}$, $s = \lambda^{-1/n} s$ in (3.2), with (2.1), we obtain

$$h_{\lambda^{-1/n} Z(\lambda^{-1/n} sT^n)}((\alpha+\beta)e_1) = h_{Z(sT^n)}(\alpha e_1 + \beta e_2) \vee h_{(\frac{1}{\lambda}-1)^{1/n} Z((\frac{1}{\lambda}-1)^{1/n} sT^n)}((\alpha+\beta)e_1).$$

Combined with Lemma 3.2, we get

$$\begin{aligned} h_{Z(sT^n)}(\alpha e_1 + \beta e_2) &\le h_{\lambda^{-1/n} Z(\lambda^{-1/n} sT^n)}((\alpha+\beta)e_1) \\ &= h_{Z(sT^n)}((\alpha+\beta)e_1) \\ &= h_{[-c_2 s(e_1+e_2), c_1 s(e_1+e_2)]}(\alpha e_1 + \beta e_2). \end{aligned}$$

If $\beta > -\alpha > 0$ or $-\beta > \alpha > 0$, taking $x = (\alpha+\beta)e_2$, $\lambda = -\frac{\alpha}{\beta}$, $s = (1-\lambda)^{-1/n} s$ in (3.2), we obtain

$$\begin{aligned} &h_{(1-\lambda)^{-1/n} Z((1-\lambda)^{-1/n} sT^n)}((\alpha+\beta)e_2) \\ &= h_{(1-\lambda)^{-1/n}\lambda^{1/n} Z((1-\lambda)^{-1/n}\lambda^{1/n} sT^n)}((\alpha+\beta)e_2) \vee h_{Z(sT^n)}(\alpha e_1 + \beta e_2). \end{aligned}$$

Similarly, we get

$$h_{Z(sT^n)}(\alpha e_1 + \beta e_2) \le h_{[-c_2 s(e_1+e_2), c_1 s(e_1+e_2)]}(\alpha e_1 + \beta e_2).$$

Combined, we get

$$h_{(Z(sT^n))|\mathbb{R}^2}(x) = h_{Z(sT^n)}(x) \le h_{[-c_2 s(e_1+e_2), c_1 s(e_1+e_2)]}(x)$$

for an arbitrary $x \in \mathbb{R}^2$ by the continuity of the support function. Hence we get that $(Z(sT^n))|\mathbb{R}^2 \subset [-c_2 s(e_1+e_2), c_1 s(e_1+e_2)]$. Since $(Z(sT^n))|\mathbb{R}^2$ is convex, there exist real a, b with $-c_2 \le a \le b \le c_1$ such that $(Z(sT^n))|\mathbb{R}^2 = [as(e_1+e_2), bs(e_1+e_2)]$. However, $h_{(Z(sT^n))|\mathbb{R}^2}(e_1) = h_{Z(sT^n)}(e_1) = c_1 s$ and $h_{(Z(sT^n))|\mathbb{R}^2}(-e_1) = h_{Z(sT^n)}(-e_1) = c_2 s$ show that $a = -c_2$, $b = c_1$. Hence, $(Z(sT^n))|\mathbb{R}^2 = [-c_2 s(e_1+e_2), c_1 s(e_1+e_2)]$.

Since the orthogonal projection of $Z(sT^n)$ onto any plane spanned by $\{e_i, e_j\}$, $1 \le i < j \le n$ is the segment $[-c_2 s(e_i+e_j), c_1 s(e_i+e_j)]$, we obtain that $Z(sT^n) = [-c_2 s(e_1 + \cdots + e_n), c_1 s(e_1 + \cdots + e_n)]$.

By the SL(n) contravariance of Z, (3.4) holds true for every simplex in $\mathcal{T}_o^n$. Assume that (3.4) holds on $\mathcal{P}_{i-1}$, $i \geq 2$. Let $P = P_1 \cup P_2 \in \mathcal{P}_i$, where $P_1, P_2 \in \mathcal{P}_{i-1}$ have disjoint relative interiors. We can assume $P \neq P_1$ and $P \neq P_2$. Set $d = \dim P_1 = \dim P_2$, $\dim(P_1 \cap P_2) = d-1$. By (2.6), we have $P_1 \cap P_2 \in \mathcal{P}_{i-1}$. Hence,

$$h_{Z(P_1 \cap P_2)} = 0 \leq h_{ZP_i}$$

for $i = 1, 2$. Therefore $Z(P_1 \cup P_2)$ is uniquely determined by $h_{Z(P_1 \cup P_2)} = h_{ZP_1} \vee h_{ZP_2}$. Thus (3.4) holds on $\mathcal{P}_i$. For any $P \in \mathcal{P}_o^n$, there exists i such that $P \in \mathcal{P}_i$. Thus (3.4) holds on $\mathcal{P}_o^n$. □

4. SL(n) covariant L_∞ Minkowski valuations

We will use the following lemma by Ludwig [8] and Haberl [3] for maps to $\mathcal{K}_o^n$ and Parapatits [21] for maps to $C_p(\mathbb{R}^n)$, the set of p-homogenous continuous functions on $\mathbb{R}^n$. (Maps to $C_p(\mathbb{R}^n)$ are considered in Section 5.)

Lemma 4.1. *Let $n \geq 2$. If a map $\Phi : \mathcal{P}_o^n \to C_p(\mathbb{R}^n)$ is SL(n) covariant, then*

$$\Phi(P)(x) = \Phi(P)(x|P),\ x \in \mathbb{R}^n$$

for any $P \in \mathcal{P}_o^n$. In particular, if a map $Z : \mathcal{P}_o^n \to \mathcal{K}_o^n$ is SL(n) covariant (hence $P \mapsto h_{ZP}$ is also SL(n) covariant), then $ZP \subset \operatorname{lin} P$, and

$$h_{ZP}(x) = h_{ZP}(x|P),\ x \in \mathbb{R}^n$$

for any $P \in \mathcal{P}_o^n$.

The following Lemma determines the constants in Theorem 1.4 and establishes a homogeneity property of SL(n) covariant L_∞ Minkowski valuations.

Lemma 4.2. *Let $n \geq 3$. If $Z : \mathcal{P}_o^n \to \mathcal{K}_o^n$ is an SL(n) covariant L_∞ Minkowski valuation, then*

$$h_{Z(sT^d)}(\pm e_1) = s h_{ZT^d}(\pm e_1) \tag{4.1}$$

for $1 \leq d \leq n$ and $s > 0$, while

$$h_{ZT^1}(\pm e_1) \leq \cdots \leq h_{ZT^n}(\pm e_1). \tag{4.2}$$

Proof. Let $a_d := h_{ZT^d}(e_1)$, $b_d := h_{ZT^d}(-e_1)$ for $1 \leq d \leq n$.

If $d \leq n-1$, it is easy to see that $Z(sT^d) = sZT^d$ by the SL(n) covariance of Z. Hence (4.1) holds for $d \leq n-1$.

For $0 < \lambda < 1$, define H_λ, ϕ_1, ϕ_2, ϕ_3, and ϕ_4 as in Section 2. Since Z is an L_∞ Minkowski valuation,

$$h_{Z(sT^d)}(x) \vee h_{Z(sT^d \cap H_\lambda)}(x) = h_{Z(sT^d \cap H_\lambda^-)}(x) \vee h_{Z(sT^d \cap H_\lambda^+)}(x),\ x \in \mathbb{R}^n \tag{4.3}$$

for any $s > 0$.

For $2 \leq d \leq n-1$, since Z is SL(n) covariant, we obtain

$$h_{ZT^d}(x) \vee h_{Z\hat{T}^{d-1}}(\phi_1^t x) = h_{ZT^d}(\phi_1^t x) \vee h_{ZT^d}(\phi_2^t x), \tag{4.4}$$

where $x = (x_1, \ldots, x_n)^t \in \mathbb{R}^n$, $\phi_1^t x = (\lambda x_1 + (1-\lambda)x_2, x_2, x_3, \ldots, x_{n-1}, \frac{1}{\lambda}x_n)^t$ and $\phi_2^t x = (x_1, \lambda x_1 + (1-\lambda)x_2, x_3, \ldots, x_{n-1}, \frac{1}{\lambda}x_n)^t$. Taking $x = e_1$, $s = 1$ in (4.4), we get

$$h_{ZT^d}(e_1) \vee h_{Z\hat{T}^{d-1}}(\lambda e_1) = h_{ZT^d}(\lambda e_1) \vee h_{ZT^d}(e_1 + \lambda e_2).$$

Also since support functions are homogeneous and continuous, and $h_{Z\hat{T}^{d-1}}(e_1) = a_{d-1}$ by the $\mathrm{SL}(n)$ covariance of Z,

$$a_d \vee (\lambda a_{d-1}) = (\lambda a_d) \vee h_{ZT^d}(e_1 + \lambda e_2) \tag{4.5}$$

holds for $0 \le \lambda \le 1$.

We need to show that $a_{d-1} \le a_d$. Indeed, if we assume $a_d < a_{d-1}$, then there exists $0 \le \lambda_0 < 1$ such that $a_{d-1}\lambda_0 = a_d$. Taking $\lambda_0 \le \lambda \le 1$ in (4.5), we get $h_{ZT^d}(e_1+\lambda e_2) = a_{d-1}\lambda$. However, choosing $\lambda_0 \le \lambda_1 < \lambda_2 \le 1$, by the sublinearity of the support function, we have

$$\begin{aligned} a_{d-1}\lambda_2 = h_{ZT^d}(e_1 + \lambda_2 e_2) &\le h_{ZT^d}(e_1 + \lambda_1 e_2) + h_{ZT^d}((\lambda_2 - \lambda_1)e_2) \\ &= a_{d-1}\lambda_1 + a_d(\lambda_2 - \lambda_1), \end{aligned}$$

which is a contradiction to the assumption.

Similarly, taking $x = -e_1$ in (4.4), we get $b_{d-1} \le b_d$.

If $d = n$, define $\phi_3, \phi_4 \in \mathrm{SL}(n)$ as in Section 2. Since Z is $\mathrm{SL}(n)$ covariant, (4.3) shows that

$$h_{Z(sT^n)}(x) \vee h_{Z(\lambda^{1/n}s\hat{T}^{n-1})}(\phi_3^t x) = h_{Z(\lambda^{1/n}sT^n)}(\phi_3^t x) \vee h_{Z((1-\lambda)^{1/n}sT^n)}(\phi_4^t x), \tag{4.6}$$

where $x = (x_1, \dots, x_n)^t$, $\phi_3^t x = \lambda^{-1/n}(\lambda x_1 + (1-\lambda)x_2, x_2, x_3, \dots, x_n)^t$ and $\phi_4^t x = (1-\lambda)^{-1/n}(x_1, \lambda x_1 + (1-\lambda)x_2, x_3, \dots, x_n)^t$. So if we choose $x = e_n$ in (4.6), we have

$$h_{Z(sT^n)}(e_n) \vee h_{\lambda^{-1/n}Z(\lambda^{1/n}s\hat{T}^{n-1})}(e_n) = h_{\lambda^{-1/n}Z(\lambda^{1/n}sT^n)}(e_n) \vee h_{(1-\lambda)^{-1/n}Z((1-\lambda)^{1/n}sT^n)}(e_n) \tag{4.7}$$

for $0 < \lambda < 1$, $s > 0$. Since (4.1) holds for $d \le n-1$ and Z is $\mathrm{SL}(n)$ covariant, we have $h_{\lambda^{-1/n}Z(\lambda^{1/n}s\hat{T}^{n-1})}(e_n) = a_{n-1}$. Combining it with (2.1), taking $\lambda = \frac{\lambda_1}{\lambda_2}$, $0 < \lambda_1 < \lambda_2$ and $s = \lambda_2^{1/n}$ in (4.7), we get

$$h_{\lambda_2^{-1/n}Z(\lambda_2^{1/n}T^n)}(e_n) \vee a_{n-1} = h_{\lambda_1^{-1/n}Z(\lambda_1^{1/n}T^n)}(e_n) \vee h_{(\lambda_2-\lambda_1)^{-1/n}Z((\lambda_2-\lambda_1)^{1/n}T^n)}(e_n) \tag{4.8}$$

for $0 < \lambda_1 < \lambda_2$.

Let $f(\lambda) = h_{\lambda^{-1/n}Z(\lambda^{1/n}T^n)}(e_n)$, $\lambda > 0$. Hence f satisfies the condition in Lemma 2.1. Thus we have $h_{\lambda^{-1/n}Z(\lambda^{1/n}T^n)}(e_n) = h_{ZT^n}(e_n) \ge a_{n-1}$. Combined with the $\mathrm{SL}(n)$ covariance of Z, we have

$$h_{Z(sT^n)}(e_1) = s h_{ZT^n}(e_1), h_{ZT^n}(e_1) \ge a_{n-1} = h_{ZT^{n-1}}(e_1).$$

Similarly, taking $x = -e_1$ in (4.6), we get

$$h_{Z(sT^n)}(-e_1) = s h_{ZT^n}(-e_1), h_{ZT^n}(-e_1) \ge h_{ZT^{n-1}}(-e_1).$$

□

Proof of Theorem 1.4. It is easy to see that the identity map and the reflection map are $\mathrm{SL}(n)$ covariant L_∞ Minkowski valuations. Hence

$$ZP = a_d P +_\infty (-b_d P) = [a_d P, -b_d P] \tag{4.9}$$

is also an $\mathrm{SL}(n)$ covariant L_∞ Minkowski valuation.

Now we will show that if Z is an $\mathrm{SL}(n)$ covariant L_∞ Minkowski valuation, then (4.9) holds. We will first show that (4.9) holds for simplices sT^d, $d \le n$, $s > 0$. We will prove the result by induction on the dimension d. $Z\{o\} = \{o\}$ has been shown in (1.3). Set $a_d := h_{ZT^d}(e_1)$ and $b_d := h_{ZT^d}(-e_1)$. Lemma 4.2 shows that

$$0 \le a_1 \le \dots \le a_n,\ 0 \le b_1 \le \dots \le b_n.$$

If $d = 1$, by the $\mathrm{SL}(n)$ covariance of Z, we have $Z[0, se_1] = sZ[0, e_1]$ for any $s > 0$. By Lemma 4.1, we get that $Z[0, e_1] = [-b_1, a_1]$. The case $d = 1$ is done.

Assume that (4.9) holds true for dimension $d-1$, $2 \le d \le n$. We want to show that (4.9) also holds true for dimension d.

We will show by induction on the number m of coordinates of x not equal to zero that

$$h_{Z(sT^d)}(x) = h_{[a_d sT^d, -b_d sT^d]}(x). \tag{4.10}$$

For $m = 1$, (4.10) holds true by (4.1), the $\mathrm{SL}(n)$ covariance of Z and the homogeneity of the support function. Assume that (4.10) holds true for $m-1$. We need to show that (4.10) also holds true for m. By the $\mathrm{SL}(n)$ covariance of Z, we can assume w.l.o.g. that $x = x_1e_1 + \cdots + x_me_m$, $x_1, \ldots, x_m \neq 0$.

Note that (4.4) is a special form of (4.6) for dimension $d \le n-1$ since $Z(sT^d) = sZT^d$ for any $s > 0$. We will use (4.6) to get the value of h_{ZT^d} not just for $d = n$ but also for $d \le n-1$.

For $x_1 > x_2 > 0$ or $0 > x_2 > x_1$, taking $x = x_1e_1 + x_3e_3 + \cdots + x_me_m$, $\lambda = \frac{x_2}{x_1}$, $s = (1-\lambda)^{-1/d}s$ in (4.6), by (2.1), we get

$$\begin{aligned}
&h_{(1-\lambda)^{1/d}Z((1-\lambda)^{-1/d}sT^d)}(x_1e_1 + x_3e_3 + \cdots + x_me_m)\\
&\qquad\qquad \vee h_{(1-\lambda)^{1/d}\lambda^{-1/d}Z((1-\lambda)^{-1/d}\lambda^{1/d}s\hat{T}^{d-1})}(x_2e_1 + x_3e_3 + \cdots + x_me_m)\\
=&h_{(1-\lambda)^{1/d}\lambda^{-1/d}Z((1-\lambda)^{-1/d}\lambda^{1/d}sT^d)}(x_2e_1 + x_3e_3 + \cdots + x_me_m)\\
&\qquad\qquad \vee h_{Z(sT^d)}(x_1e_1 + \cdots + x_me_m).
\end{aligned} \tag{4.11}$$

Since $a_{d-1} \le a_d$, $b_{d-1} \le b_d$, $|x_2| < |x_1|$, combining the induction assumption with the $\mathrm{SL}(n)$ covariance of Z, we have

$$\begin{aligned}
&h_{(1-\lambda)^{1/d}Z((1-\lambda)^{-1/d}sT^d)}(x_1e_1 + x_3e_3 + \cdots + x_me_m)\\
&\qquad = \max\{a_dsx_i, -b_dsx_i : 1 \le i \le m \text{ and } i \neq 2\}\\
&\qquad \ge \max\{a_{d-1}sx_i, -b_{d-1}sx_i : 2 \le i \le m\}\\
&\qquad = h_{(1-\lambda)^{1/d}\lambda^{-1/d}Z((1-\lambda)^{-1/d}\lambda^{1/d}s\hat{T}^{d-1})}(x_2e_1 + x_3e_3 + \cdots + x_me_m).
\end{aligned}$$

It follows from (4.11) that

$$\begin{aligned}
&h_{Z(sT^d)}(x_1e_1 + \cdots + x_me_m)\\
&\qquad \le h_{(1-\lambda)^{1/d}Z((1-\lambda)^{-1/d}sT^d)}(x_1e_1 + x_3e_3 + \cdots + x_me_m)\\
&\qquad = \max\{a_dsx_i, -b_dsx_i : 1 \le i \le m\}.
\end{aligned} \tag{4.12}$$

For $x_2 > x_1 > 0$ or $0 > x_1 > x_2$, taking $x = x_2e_2 + x_3e_3 + \cdots + x_me_m$, $1 - \lambda = \frac{x_1}{x_2}$, $s = \lambda^{-1/d}s$ in (4.6), by (2.1), we get

$$\begin{aligned}
&h_{\lambda^{1/d}Z(\lambda^{-1/d}sT^d)}(x_2e_2 + x_3e_3 + \cdots + x_me_m)\\
&\qquad\qquad \vee h_{Z(s\hat{T}^{d-1})}(x_1e_1 + \cdots + x_me_m)\\
=&h_{Z(sT^d)}(x_1e_1 + \cdots + x_me_m)\\
&\qquad\qquad \vee h_{(1-\lambda)^{-1/d}\lambda^{1/d}Z((1-\lambda)^{1/d}\lambda^{-1/d}sT^d)}(x_1e_2 + x_3e_3 + \cdots + x_me_m).
\end{aligned} \tag{4.13}$$

Similarly to the case $|x_2| < |x_1|$, since

$$h_{\lambda^{1/d}Z(\lambda^{-1/d}sT^d)}(x_2e_2 + x_3e_3 + \cdots + x_me_m) \ge h_{Z(s\hat{T}^{d-1})}(x_1e_1 + \cdots + x_me_m),$$

we get

$$h_{Z(sT^d)}(x_1e_1 + \cdots + x_me_m)$$

$$\begin{aligned}&\le h_{\lambda^{1/d}Z(\lambda^{-1/d}sT^d)}(x_2e_1+x_3e_3+\cdots+x_me_m)\\&=\max\{a_dsx_i,-b_dsx_i:1\le i\le m\}.\end{aligned}\tag{4.14}$$

For $x_1>0>x_2$ or $x_2>0>x_1$, taking $0<\lambda=\frac{x_2}{x_2-x_1}<1$ and $x=x_1e_1+\cdots+x_me_m$ in (4.6), we get

$$\begin{aligned}&h_{Z(sT^d)}(x_1e_1+\cdots+x_me_m)\\&\qquad\qquad\vee h_{\lambda^{-1/d}Z(\lambda^{1/d}s\hat{T}^{d-1})}(x_2e_2+x_3e_3+\cdots+x_me_m)\\=&h_{\lambda^{-1/d}Z(\lambda^{1/d}sT^d)}(x_2e_2+x_3e_3+\cdots+x_me_m)\\&\qquad\qquad\vee h_{(1-\lambda)^{-1/d}Z((1-\lambda)^{1/d}sT^d)}(x_1e_1+x_3e_3+\cdots+x_me_m).\end{aligned}\tag{4.15}$$

Combined with the induction assumption and the $\mathrm{SL}(n)$ covariance of Z, we have

$$h_{Z(sT^d)}(x_1e_1+\cdots+x_me_m)\le\max\{a_dsx_i,-b_dsx_i:1\le i\le m\}.\tag{4.16}$$

Combining (4.12), (4.14) and (4.16) with the continuity of the support function, we get

$$\begin{aligned}h_{Z(sT^d)|\mathbb{R}^m}(x_1e_1+\cdots+x_me_m)&=h_{Z(sT^d)}(x_1e_1+\cdots+x_me_m)\\&\le h_{[a_dsT^d,-b_dsT^d]}(x_1e_1+\cdots+x_me_m)\\&=h_{[a_dsT^m,-b_dsT^m]}(x_1e_1+\cdots+x_me_m)\end{aligned}$$

for any $x_1,\ldots,x_m\in\mathbb{R}$. Thus, $Z(sT^d)|\mathbb{R}^m\subset[a_dsT^m,-b_dsT^m]$. For any $y\in[a_dsT^m,-b_dsT^m]$ with $y\ne a_dse_1$, we have $y\cdot e_1<a_ds$, and also $h_{Z(sT^d)|\mathbb{R}^m}(e_1)=h_{Z(sT^d)}(e_1)=a_ds$. Thus, we obtain $a_dse_1\in Z(sT^d)|\mathbb{R}^m$. Similarly, $a_dse_i,-b_dse_i\in Z(sT^d)|\mathbb{R}^m$, $1\le i\le m$. Hence, we have

$$\begin{aligned}[a_dsT^m,-b_dsT^m]&=s[a_de_1,\ldots,a_de_m,-b_de_1,\ldots,-b_de_m]\\&\subset Z(sT^d)|\mathbb{R}^m\subset[a_dsT^m,-b_dsT^m].\end{aligned}$$

That means

$$h_{Z(sT^d)}(x_1e_1+\cdots+x_me_m)=h_{[a_dsT^d,-b_dsT^d]}(x_1e_1+\cdots+x_me_m)$$

for any $x_1,\ldots,x_m\in\mathbb{R}$. The induction is complete.

By the $\mathrm{SL}(n)$ covariance of Z, (4.9) holds true for any simplex in $\mathcal{T}_o^n$. Assume that (4.9) holds on $\mathcal{P}_{i-1}$, $i\ge 2$. Let $P=P_1\cup P_2\in\mathcal{P}_i$, where $P_1,P_2\in\mathcal{P}_{i-1}$ have disjoint relative interiors. We can assume $P\ne P_1$ and $P\ne P_2$. Set $d=\dim P_1=\dim P_2$, $\dim(P_1\cap P_2)=d-1$. By (2.6), we have $P_1\cap P_2\in\mathcal{P}_{i-1}$. Hence,

$$h_{Z(P_1\cap P_2)}=h_{[a_{d-1}(P_1\cap P_2),-b_{d-1}(P_1\cap P_2)]}\le h_{[a_{d-1}P_i,-b_{d-1}P_i]}\le h_{[a_dP_i,-b_dP_i]}=h_{ZP_i}$$

for $i=1,2$. Therefore

$$h_{Z(P_1\cup P_2)}=h_{ZP_1}\vee h_{ZP_2}=h_{[a_d(P_1\cup P_2),-b_d(P_1\cup P_2)]}.$$

Thus (4.9) holds on $\mathcal{P}_i$. For any $P\in\mathcal{P}_o^n$, there exists i such that $P\in\mathcal{P}_i$. Thus (4.9) holds on $\mathcal{P}_o^n$. $\square$

5. SL(n) COVARIANT L_p MINKOWSKI VALUATIONS AND FUNCTION-VALUED VALUATIONS

First, let us consider function-valued valuations as Parapatits did in [20,21]. Let $1 \le p < \infty$ throughout this section if there are no further remarks. The function $f : \mathbb{R}^n \to \mathbb{R}$ is *p-homogenous* if

$$f(\lambda x) = \lambda^p f(x), \ x \in \mathbb{R}^n$$

for any $\lambda \ge 0$. Let $C_p(\mathbb{R}^n)$ be the set of p-homogenous continuous functions on $\mathbb{R}^n$. We call $\Phi : \mathcal{P}_o^n \to C_p(\mathbb{R}^n)$ a *valuation* if

$$\Phi(K \cup L) + \Phi(K \cap L) = \Phi(K) + \Phi(L)$$

whenever $K \cup L, K \cap L, K, L \in \mathcal{P}_o^n$. Here the addition is the ordinary addition of functions.

We call $\Phi : \mathcal{P}_o^n \to C_p(\mathbb{R}^n)$ is SL(n) (or GL(n)) *covariant* if

$$\Phi(\phi K)(x) = \Phi(K)(\phi^t x)$$

for any $K \in \mathcal{P}_o^n$ and any $\phi \in \mathrm{SL}(n)$ (or GL(n)).

The map $Z : \mathcal{P}_o^n \to \mathcal{K}_o^n$ is an SL(n) (or GL(n)) covariant L_p Minkowski valuation if and only if $\Phi : P \mapsto h_{ZP}^p$ is an SL(n) (or GL(n)) covariant valuation.

Lemma 5.1 (Haberl [3] and Parapatits [21])**.** *Let $n \ge 3$ and Φ map $\mathcal{P}_o^n$ to $C_p(\mathbb{R}^n)$. Assume further that, for every $y \in \mathbb{R}^n$, the function $s \mapsto \Phi(sT^n)(y)$ is bounded from below on some non-empty open interval $I_y \subset (0, +\infty)$. Also assume that Φ is continuous at the interval $[o, e_1]$. Then Φ is an SL(n) covariant valuation if and only if there exist constants $c_1, c_2, c_3, c_4 \in \mathbb{R}$ such that*

$$\Phi P = c_1 h_{M_p^+ P}^p + c_2 h_{M_p^- P}^p + c_3 h_P^p + c_4 h_{-P}^p$$

for every $P \in \mathcal{P}_o^n$.

In [3], Haberl just considered the valuation $P \mapsto h_{ZP}$, where Z is a Minkowski valuation. Hence he has the restrictions that $c_1, c_2, c_3, c_4 \ge 0$. However, his method also can be used to get this Lemma for $p = 1$. This also works for Lemma 5.7 below.

We remove the assumption that Φ is continuous at the interval $[o, e_1]$ and get the following result.

Lemma 5.2. *Let $n \ge 3$ and Φ map $\mathcal{P}_o^n$ to $C_p(\mathbb{R}^n)$. Assume further that, for every $y \in \mathbb{R}^n$, the function $s \mapsto \Phi(sT^n)(y)$ is bounded from below on some non-empty open interval $I_y \subset (0, +\infty)$. Then Φ is an SL(n) covariant valuation if and only if there exist constants $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}$ such that*

$$\Phi P = c_1 h_{M_p^+ P}^p + c_2 h_{M_p^- P}^p + \Phi_{p;a_1,a_2} P + \Phi_{p;b_1,b_2}(-P)$$

for every $P \in \mathcal{P}_o^n$, where $\Phi_{p;a_1,a_2}$ is defined as follows.

For $1 \le j \le \dim P - 1$, let $\mathcal{F}_{j,o}(P)$ denote the set of j-dimensional faces of $P \in \mathcal{P}_o^n$ that contain the origin. Let $a_1, a_2 \in \mathbb{R}$. For $P \in \mathcal{P}_o^n$, define $\Phi_{p;a_1,a_2}(P)$ by

$$\Phi_{p;a_1,a_2} P = a_1 h_P^p + (a_2 - a_1) \sum_{1 \le j \le \dim P - 1} (-1)^j \sum_{F \in \mathcal{F}_{j,o}(P)} h_F^p$$

if $\dim P$ is odd; and

$$\Phi_{p;a_1,a_2} P = (2a_2 - a_1) h_P^p + (a_2 - a_1) \sum_{1 \le j \le \dim P - 1} (-1)^j \sum_{F \in \mathcal{F}_{j,o}(P)} h_F^p$$

if $\dim P$ is even.

For $1 < p < \infty$, $n \geq 3$ and $p = 1$, $n \geq 4$, if we further assume that ΦP is non-negative and $(\Phi P)^{1/p}$ is sublinear for every $P \in \mathcal{P}_o^n$, then we obtain Theorem 5.3 which is equivalent to Theorem 1.5 and Theorem 1.6.

Theorem 5.3. *Let $n \geq 3$, $1 < p < \infty$ or $n \geq 4$, $p = 1$, and Φ map $\mathcal{P}_o^n$ to $C_p(\mathbb{R}^n)$. Assume further that ΦP is non-negative and $(\Phi P)^{1/p}$ is sublinear for every $P \in \mathcal{P}_o^n$. Then Φ is an $\mathrm{SL}(n)$ covariant valuation if and only if there exist constants $a_1, b_1, c_1, c_2 \geq 0$ such that*

$$\Phi P = c_1 h^p_{M_p^+ P} + c_2 h^p_{M_p^- P} + a_1 h_P^p + b_1 h_{-P}^p$$

for every $P \in \mathcal{P}_o^n$.

Now we begin to prove Lemma 5.2 and Theorem 5.3.

The inclusion-exclusion principle states that a function-valued valuation Φ satisfies

$$\Phi(T_1 \cup \dots \cup T_m) = \sum_i \Phi(T_i) - \sum_{i<j} \Phi(T_i \cap T_j) + \dots$$

for any $T_1, \dots, T_m, T_1 \cup \dots \cup T_m \in \mathcal{T}_o^n$. In particular, $\Phi(T_1 \cup \dots \cup T_m)$ does not dependent on the choice of $T_1, \dots, T_m$; see Ludwig and Reitzner [11].

Proof of Lemma 5.2. For $a_1, a_2 \in \mathbb{R}$, we first need to show that $\Phi_{p;a_1,a_2}$ is a valuation.

Lemma 5.4. *For $a_1, a_2 \in \mathbb{R}$, $\Phi_{p;a_1,a_2}$ is a $\mathrm{GL}(n)$ covariant valuation.*

Proof. It is easy to see from the definition that $\Phi_{p;a_1,a_2}$ is $\mathrm{GL}(n)$ covariant. Next, we prove that $\Phi_{p;a_1,a_2}$ is a valuation.

Let $K, L \in \mathcal{P}_o^n$, $K \neq L$. To show that

$$\Phi_{p;a_1,a_2}(K \cup L) + \Phi_{p;a_1,a_2}(K \cap L) = \Phi_{p;a_1,a_2}(K) + \Phi_{p;a_1,a_2}(L) \tag{5.1}$$

whenever $K \cup L$ is convex, we can assume that $\dim K = \dim L = \dim(K \cup L)$, denoted by d. Otherwise (5.1) holds trivially since $K \subset L$ or $L \subset K$. Hence, we only need to consider the following four cases:

(i) $o \in \operatorname{relint} K \cap \operatorname{relint} L$;
(ii) $o \in \operatorname{relint} K$, and $o \in \operatorname{relbd} L$;
(iii) $o \in \operatorname{relbd} K \cap \operatorname{relbd} L$ and $\dim(K \cap L) = d$;
(iv) $o \in \operatorname{relbd} K \cap \operatorname{relbd} L$ and $\dim(K \cap L) = d - 1$.

First we notice that the map $P \mapsto h_P, P \in \mathcal{P}_o^n$ is a valuation. Hence (5.1) holds true for the case (i). Also, for case (ii), (iii), we only need to consider the faces containing the origin.

For the case (ii), since $K \cup L$ is convex, we have $\bigcup_{1 \leq j \leq d-1} \mathcal{F}_{j,o}(K \cap L) = \bigcup_{1 \leq j \leq d-1} \mathcal{F}_{j,o}(L)$. Hence (5.1) also holds true.

We will denote the elements of $\mathcal{F}_{j,o}(K)$ by F_K^j, and the elements of $\mathcal{F}_{j,o}(L)$ by F_L^j.

Now we deal with the case (iii). For $1 \leq j \leq d-1$, since $K \cup L$ is convex, we can separate $\mathcal{F}_{j,o}(K)$ and $\mathcal{F}_{j,o}(L)$ into five disjoint parts, respectively:

$$\mathcal{F}_{j,o}(K) = \mathcal{A}_K^j \cup \mathcal{B}_K^j \cup \mathcal{C}_K^j \cup \mathcal{D}_K^j \cup \mathcal{G}_K^j, \tag{5.2}$$

where

$$\begin{aligned}
\mathcal{A}_K^j &= \{F_K^j : F_K^j \cap \operatorname{relint} L \neq \emptyset\}, \\
\mathcal{B}_K^j &= \{F_K^j : F_K^j \cap L^c \neq \emptyset, \nexists F_L^j \text{ s.t. } F_L^j \subset \operatorname{lin} F_K^j\}, \\
\mathcal{C}_K^j &= \{F_K^j : \exists F_L^j \neq F_K^j, \exists H \in Gr(n,j) \text{ s.t. } F_L^j \cup F_K^j \subset H\},
\end{aligned}$$

$$\mathcal{D}_K^j = \{F_K^j : \exists F_L^j = F_K^j\},$$
$$\mathcal{G}_K^j = \{F_K^j : \exists F_L^i, i > j \text{ s.t. } F_K^j \subset \operatorname{relint} F_L^i\};$$

and

$$\mathcal{F}_{j,o}(L) = \mathcal{A}_L^j \cup \mathcal{B}_L^j \cup \mathcal{C}_L^j \cup \mathcal{D}_L^j \cup \mathcal{G}_L^j, \tag{5.3}$$

where

$$\mathcal{A}_L^j = \{F_L^j : F_L^j \cap \operatorname{relint} K \neq \emptyset\},$$
$$\mathcal{B}_L^j = \{F_L^j : F_L^j \cap K^c \neq \emptyset, \nexists F_K^j \text{ s.t. } F_K^j \subset \operatorname{lin} F_L^j\},$$
$$\mathcal{C}_L^j = \{F_L^j : \exists F_K^j \neq F_L^j, \exists H \in Gr(n,j) \text{ s.t. } F_K^j \cup F_L^j \subset H\},$$
$$\mathcal{D}_L^j = \{F_L^j : \exists F_K^j = F_L^j\},$$
$$\mathcal{G}_L^j = \{F_L^j : \exists F_K^i, i > j \text{ s.t. } F_L^j \subset \operatorname{relint} F_K^i\}.$$

Set $\mathcal{D}^j := \mathcal{D}_K^j = \mathcal{D}_L^j$. Since $\operatorname{relbd}(K \cup L) = (\operatorname{relbd} K \cap L^c) \cup (\operatorname{relbd} L \cap K^c) \cup (\operatorname{relbd} K \cap \operatorname{relbd} L)$, $\operatorname{relbd}(K \cap L) = (\operatorname{relbd} K \cap \operatorname{relint} L) \cup (\operatorname{relbd} L \cap \operatorname{relint} K) \cup (\operatorname{relbd} K \cap \operatorname{relbd} L)$ and $K \cup L$ is convex, we have

$$\mathcal{F}_{j,o}(K \cup L) = \mathcal{B}_K^j \cup \mathcal{B}_L^j \cup \mathcal{M}^j \cup (\mathcal{D}^j \cap \mathcal{F}_{j,o}(K \cup L)), \tag{5.4}$$

where

$$\mathcal{M}^j = \{F_K^j \cup F_L^j : F_K^j \in \mathcal{C}_K^j, F_L^j \in \mathcal{C}_L^j, \exists H \in Gr(n,j), F_K^j \cup F_L^j \subset H\};$$

and

$$\mathcal{F}_{j,o}(K \cap L) = \mathcal{A}_K^j \cup \mathcal{A}_L^j \cup (\mathcal{N}^j \cap \mathcal{F}_{j,o}(K \cap L)) \cup \mathcal{D}^j \cup \mathcal{G}_K^j \cup \mathcal{G}_L^j, \tag{5.5}$$

where

$$\mathcal{N}^j = \{F_K^j \cap F_L^j : F_K^j \in \mathcal{C}_K^j, F_L^j \in \mathcal{C}_L^j, \exists H \in Gr(n,j), F_K^j \cup F_L^j \subset H\}.$$

Combining (5.2), (5.3), (5.4), (5.5) with the definition of $\Phi_{p;a_1,a_2}$, if

$$\begin{aligned}
&\sum_{1\le j\le d-1} (-1)^j \sum_{F_K^j \in \mathcal{C}_K^j} h^p_{F_K^j} + \sum_{1\le j\le d-1} (-1)^j \sum_{F_L^j \in \mathcal{C}_L^j} h^p_{F_L^j} + 2\sum_{1\le j\le d-1} (-1)^j \sum_{F \in \mathcal{D}^j} h^p_F \\
&= \sum_{1\le j\le d-1} (-1)^j \sum_{F_{K\cup L}^j \in \mathcal{M}^j} h^p_{F_{K\cup L}^j} + \sum_{1\le j\le d-1} (-1)^j \sum_{F_{K\cup L}^j \in (\mathcal{D}^j \cap \mathcal{F}_{j,o}(K\cup L))} h^p_{F_{K\cup L}^j} \\
&\quad + \sum_{1\le j\le d-1} (-1)^j \sum_{F_{K\cap L}^j \in (\mathcal{N}^j \cap \mathcal{F}_{j,o}(K\cup L))} h^p_{F_{K\cap L}^j} + \sum_{1\le j\le d-1} (-1)^j \sum_{F \in \mathcal{D}^j} h^p_F,
\end{aligned} \tag{5.6}$$

then (5.1) holds true.

Let $F_K^j \in \mathcal{C}_K^j, F_L^j \in \mathcal{C}_L^j$ and $F_K^j \cup F_L^j$ lie in the same j-dimensional plane. Since $F_K^j \cup F_L^j$ is convex, $h^p_{F_K^j \cup F_L^j} + h^p_{F_K^j \cap F_L^j} = h^p_{F_K^j} + h^p_{F_L^j}$. Thus

$$\begin{aligned}
&\sum_{1\le j\le d-1} (-1)^j \sum_{F_K^j \in \mathcal{C}_K^j} h^p_{F_K^j} + \sum_{1\le j\le d-1} (-1)^j \sum_{F_L^j \in \mathcal{C}_L^j} h^p_{F_L^j} \\
&= \sum_{1\le j\le d-1} (-1)^j \sum_{F_{K\cup L}^j \in \mathcal{M}^j} h^p_{F_{K\cup L}^j} + \sum_{1\le j\le d-1} (-1)^j \sum_{F_{K\cap L}^j \in (\mathcal{N}^j \cap \mathcal{F}_{j,o}(K\cup L))} h^p_{F_{K\cap L}^j}
\end{aligned}$$

$$+\sum_{1\le j\le d-1}(-1)^j \sum_{F_K^j\cap F_L^j\in(\mathcal{N}^j\setminus\mathcal{F}_{j,o}(K\cup L))} h^p_{F_K^j\cap F_L^j}. \tag{5.7}$$

Let $F_K^j \cap F_L^j \in \mathcal{N}^j \setminus \mathcal{F}_{j,o}(K\cap L)$. Hence $F_K^j\cap F_L^j$ is a $(j-1)$-face of both K and L that contains the origin. Also $F_K^j\cap F_L^j$ is not a $(j-1)$-face of $K\cup L$. Hence $F_K^j\cap F_L^j \in \mathcal{D}^{j-1}\setminus\mathcal{F}_{j-1,o}(K\cup L)$. That means $\mathcal{N}^j\setminus\mathcal{F}_{j,o}(K\cap L)\subset\mathcal{D}^{j-1}\setminus\mathcal{F}_{j-1,o}(K\cup L)$. On the other hand, $\mathcal{D}^{j-1}\setminus\mathcal{F}_{j-1,o}(K\cup L)\subset\mathcal{N}^j\setminus\mathcal{F}_{j,o}(K\cap L)$. Indeed, for $F\in\mathcal{D}^{j-1}\setminus\mathcal{F}_{j-1,o}(K\cup L)$, there exist an $i\ge j$ such that $F\subset \operatorname{relint} F^i_{K\cup L}$. Then $i=j$ since otherwise F will be contained in the relative interior of an $(i-1)$-face of K which is a contradiction for the fact that F is a $(j-1)$-face of K. Hence

$$\mathcal{D}^{j-1}\setminus\mathcal{F}_{j-1,o}(K\cup L)=\mathcal{N}^j\setminus\mathcal{F}_{j,o}(K\cap L). \tag{5.8}$$

Combining (5.7) with (5.8), (5.6) holds true since

$$\begin{aligned}0=&\sum_{1\le j\le d-1}(-1)^j\sum_{F\in\mathcal{D}^j}h_F^p-\sum_{1\le j\le d-2}(-1)^j\Bigg(\sum_{F\in(\mathcal{D}^j\setminus\mathcal{F}_{j,o}(K\cup L))}h_F^p+\sum_{F\in(\mathcal{D}^j\cap\mathcal{F}_{j,o}(K\cup L))}h_F^p\Bigg)\\&-(-1)^{d-1}\sum_{F\in(\mathcal{D}^{d-1}\cap\mathcal{F}_{d-1,o}(K\cup L))}h_F^p\end{aligned}$$

(since $\mathcal{D}^{d-1}\cap\mathcal{F}_{d-1,o}(K\cup L)=\mathcal{D}^{d-1}$ or $\mathcal{D}^{d-1}\cap\mathcal{F}_{d-1,o}(K\cup L)=\emptyset$).

For case (iv), set $M=K\cup L$. There exists a hyperplane H through the origin such that $K=M\cap H^+$, $L=M\cap H^-$ and $K\cap L=M\cap H$. Note that $\dim M=d$, $\dim(M\cap H)=d-1$ and $M\cap H$ is a $(d-1)$-face of $M\cap H^+$ and $M\cap H^-$, respectively. For $1\le j\le d-1$, it is easy to see that

$$\mathcal{F}_{j,o}(M)=\{F^j_{(M\cap H^+)}\in\mathcal{F}_{j,o}(M)\}\cup\{F^j_{(M\cap H^-)}\in\mathcal{F}_{j,o}(M)\}\cup\{F^j_{(M\cap H^+)}\cup F^j_{(M\cap H^-)}\in\mathcal{F}_{j,o}(M)\}$$

and

$$\mathcal{F}_{j-1,o}(M\cap H)=\{F^j_{(M\cap H^+)}\cap F^j_{(M\cap H^-)} : F^j_{(M\cap H^+)}\cup F^j_{(M\cap H^-)}\in\mathcal{F}_{j,o}(M)\}.$$

For $F^j_{(M\cap H^+)}\cup F^j_{(M\cap H^-)}\in\mathcal{F}_{j,o}(M)$, since

$$h^p_{F^j_{(M\cap H^+)}\cup F^j_{(M\cap H^-)}}+h^p_{F^j_{(M\cap H^+)}\cap F^j_{(M\cap H^-)}}=h^p_{F^j_{(M\cap H^+)}}+h^p_{F^j_{(M\cap H^-)}},$$

we can check step by step that

$$\begin{aligned}&\sum_{1\le j\le d-1}(-1)^j\sum_{F^j_{(M\cap H^+)}\in(\mathcal{F}_{j,o}(M\cap H^+)\setminus\{M\cap H\})}h^p_{F^j_{(M\cap H^+)}}\\&\qquad+\sum_{1\le j\le d-1}(-1)^j\sum_{F^j_{(M\cap H^-)}\in(\mathcal{F}_{j,o}(M\cap H^-)\setminus\{M\cap H\})}h^p_{F^j_{(M\cap H^-)}}\\&=\sum_{1\le j\le d-1}(-1)^j\sum_{F^j_M\in\mathcal{F}_{j,o}(M)}h^p_{F^j_M}+\sum_{1\le j\le d-2}(-1)^j\sum_{F^j_{M\cap H}\in\mathcal{F}_{j,o}(M\cap H)}h^p_{F^j_{(M\cap H)}}.\end{aligned}$$

Now we only need to show that

$$\big(a_1h^p_{M\cap H^+}+(a_2-a_1)h^p_{M\cap H}\big)+\big(a_1h^p_{M\cap H^-}+(a_2-a_1)h^p_{M\cap H}\big)=a_1h^p_M+(2a_2-a_1)h^p_{M\cap H} \tag{5.9}$$

if d is odd, and

$$\begin{aligned}&\big((2a_2-a_1)h^p_{M\cap H^+}-(a_2-a_1)h^p_{M\cap H}\big)+\big((2a_2-a_1)h^p_{M\cap H^-}-(a_2-a_1)h^p_{M\cap H}\big)\\&\qquad=(2a_2-a_1)h^p_M+a_1h^p_{M\cap H}\end{aligned}\tag{5.10}$$

if d is even. Indeed, (5.9) and (5.10) hold true since $h^p_{M\cap H^+}+h^p_{M\cap H^-}=h^p_M+h^p_{M\cap H}$. □

For $a\in\mathbb{R}$, we write a^p for $\operatorname{sgn}(a)|a|^p$, where $\operatorname{sgn}(a)=1$ if $a\ge 0$, $\operatorname{sgn}(a)=-1$ if $a<0$.

Proposition 5.5. *Let $0\le m\le n$ and $v_0\in\mathbb{R}^n$ be such that $o\in\operatorname{relint}\,[v_0,e_1,\dots,e_m]$ and let $x=(x_1,\dots,x_d)^t\in\mathbb{R}^d$. Set $\alpha_1=\max\{v_0\cdot x,x_1,\dots,x_m\}$, $\alpha_2=\min\{v_0\cdot x,x_1,\dots,x_m\}$, $\beta_1=\max\{x_{m+1},\dots,x_d\}$ and $\beta_2=\min\{x_{m+1},\dots,x_d\}$. Then*

$$\begin{aligned}&\Phi_{p;a_1,a_2}([v_0,e_1,\dots,e_d])(x)\\&=a_2\max\{\alpha_1^p,\beta_1^p\}+(a_2-a_1)(-1)^{m+1}\max\{\alpha_1^p,\beta_2^p\}+(a_2-a_1)(-1)^m\alpha_1^p,\\&\Phi_{p;b_1,b_2}(-[v_0,e_1,\dots,e_d])(x)\\&=b_2\max\{-\alpha_2^p,-\beta_2^p\}+(b_2-b_1)(-1)^{m+1}\max\{-\alpha_2^p,-\beta_1^p\}+(b_2-b_1)(-1)^m(-\alpha_2^p).\end{aligned}\tag{5.11}$$

Especially, for $m=0$ and $v_0=o$,

$$\begin{aligned}\Phi_{p;a_1,a_2}(T^d)(x)&=a_2\max\{\beta_1^p,0\}-(a_2-a_1)\max\{\beta_2^p,0\},\\\Phi_{p;b_1,b_2}(-T^d)(x)&=b_2\max\{-\beta_2^p,0\}-(b_2-b_1)\max\{-\beta_1^p,0\}.\end{aligned}\tag{5.12}$$

Moreover,

$$\begin{aligned}\Phi_{p;a_1,a_2}(T^d)(e_1)+\Phi_{p;b_1,b_2}(-T^d)(e_1)&=a_2,\\\Phi_{p;a_1,a_2}(T^d)(-e_1)+\Phi_{p;b_1,b_2}(-T^d)(-e_1)&=b_2\end{aligned}\tag{5.13}$$

for $d\ge 2$, and

$$\begin{aligned}\Phi_{p;a_1,a_2}(T^1)(e_1)+\Phi_{p;b_1,b_2}(-T^1)(e_1)&=a_1,\\\Phi_{p;a_1,a_2}(T^1)(-e_1)+\Phi_{p;b_1,b_2}(-T^1)(-e_1)&=-b_1\end{aligned}\tag{5.14}$$

for $d=1$.

Proof. We will use the following basic equalities for binomial coefficients.

$$\sum_{m+1\le j\le d-1}(-1)^j\binom{d-m-1}{j-m-1}=(-1)^{d-1},\tag{5.15}$$

$$\sum_{m+1\le j\le d-i+m+1}(-1)^j\binom{d-i}{j-m-1}=0,\ \ m+2\le i\le d-1.\tag{5.16}$$

Since $[v_0,e_1,\dots,e_d]$ is invariant under permutations of $\{e_{m+1},\dots,e_d\}$ and $\Phi_{p;a_1,a_2}$ is $\mathrm{GL}(n)$ covariant, we can assume w.l.o.g. that $x_{m+1}\ge\cdots\ge x_d$. For $j<m$, $\mathcal{F}_{j,o}([v_0,e_1,\dots,e_d])=\emptyset$. For $j=m$, $\mathcal{F}_{j,o}([v_0,e_1,\dots,e_d])=\{[v_0,e_1,\dots,e_m]\}$. For $m+1\le j\le d-1$,

$$\mathcal{F}_{j,o}([v_0,e_1,\dots,e_d])=\big\{[v_0,e_1,\dots,e_m,e_{\sigma_{m+1}},\dots,e_{\sigma_j}]:\{\sigma_{m+1},\dots,\sigma_j\}\subset\{m+1,\dots,d\}\big\},$$

and

$$\sum_{F\in\mathcal{F}_{j,o}([v_0,e_1,\dots,e_d])}h^p_F(x)=\Bigg(\binom{d-m-1}{j-m-1}\max\{\alpha_1^p,x^p_{m+1}\}+\binom{d-m-2}{j-m-1}\max\{\alpha_1^p,x^p_{m+2}\}$$

$$+\cdots+\begin{pmatrix} j-m-1 \\ j-m-1 \end{pmatrix}\max\{\alpha_1^p,x_{d-j+m+1}^p\}\Big).$$

Hence, the definition of $\Phi_{p;a_1,a_2}$, (5.15) and (5.16) show that

$$\Phi_{p;a_1,a_2}([v_0,e_1,\dots,e_d])(x)=a_2\max\{\alpha_1^p,x_{m+1}^p\}+(a_2-a_1)(-1)^{m+1}\max\{\alpha_1^p,x_d^p\}$$
$$+(a_2-a_1)(-1)^m\alpha_1^p.$$

Then the second equation of (5.11) follows from

$$\Phi_{p;b_1,b_2}(-[v_0,e_1,\dots,e_d])(x)=\Phi_{p;b_1,b_2}([v_0,e_1,\dots,e_d])(-x).$$

For $m=0$ and $v_0=o$, we have $\alpha_1=\alpha_2=0$. Hence (5.12) holds true.
(5.13) and (5.14) follow directly from (5.12). □

Second, we give a lemma on lower dimensional polytopes.

Lemma 5.6. *Let $n\ge 3$. If $\Phi:\mathcal{P}_o^n\to C_p(\mathbb{R}^n)$ is an $\mathrm{SL}(n)$ covariant valuation, then there exist constants $a_1,a_2,b_1,b_2\in\mathbb{R}$ such that*

$$\Phi P=\Phi_{p;a_1,a_2}P+\Phi_{p;b_1,b_2}(-P)$$

for every $P\in\mathcal{P}_o^n$ with $\dim P\le n-1$.

Proof. By the $\mathrm{SL}(n)$ covariance of Φ, Lemma 4.1 and the inclusion-exclusion principle, we only need to show that

$$\Phi T^d(x)=\Phi_{p;a_1,a_2}T^d(x)+\Phi_{p;b_1,b_2}(-T^d)(x),\ \ x\in\mathbb{R}^d \tag{5.17}$$

for $d\le n-1$.

Set $a_d=\Phi(T^d)(e_1)$ and $b_d=\Phi(T^d)(-e_1)$ for $d\le n-1$.

For $0<\lambda<1$, define H_λ, ϕ_1,ϕ_2 as in Section 2. For $d\le n-1$, since Φ is a valuation, we get that

$$\Phi(T^d)+\Phi(T^d\cap H_\lambda)=\Phi(T^d\cap H_\lambda^-)+\Phi(T^d\cap H_\lambda^+).$$

Also since Φ is $\mathrm{SL}(n)$ covariant,

$$\Phi(T^d)(x)+\Phi(\hat{T}^{d-1})(\phi_1^tx)=\Phi(T^d)(\phi_1^tx)+\Phi(T^d)(\phi_2^tx), \tag{5.18}$$

where $x=(x_1,\dots,x_n)^t$, $\phi_1^tx=(\lambda x_1+(1-\lambda)x_2,x_2,x_3,\dots,x_{n-1},\frac{1}{\lambda}x_n)^t$ and $\phi_2^tx=(x_1,\lambda x_1+(1-\lambda)x_2,x_3,\dots,x_{n-1},\frac{1}{\lambda}x_n)^t$.

For $3\le d\le n-1$, taking $x=e_d$ in (5.18), by Lemma 4.1 and the $\mathrm{SL}(n)$ covariance of Φ, we obtain that $a_d=a_{d-1}$. Thus, we have

$$a_{n-1}=\cdots=a_2. \tag{5.19}$$

Similarly, taking $x=-e_d$ in (5.18), we get

$$b_{n-1}=\cdots=b_2. \tag{5.20}$$

Now we will prove the desired result by induction on the dimension d. Proposition 5.5 and the p-homogeneity of ΦT^d, $\Phi_{p;a_1,a_2}T^d$ and $\Phi_{p;b_1,b_2}(-T^d)$ show that (5.17) holds true for $d=1$. Assume that (5.17) holds true for $d-1$. Then we will show that (5.17) holds true for d. We will prove this by induction on the number m of coordinates of x not equal to zero. By the $\mathrm{SL}(n)$ covariance of Φ, we can assume w.l.o.g. that $x=x_1e_1+\cdots+x_me_m$, $x_1,\dots,x_m\ne 0$.

Proposition 5.5, relations (5.19) and (5.20) show that (5.17) holds true for $m=1$. Assume that (5.17) holds true for $m-1$.

For $x_1 > x_2 > 0$ or $0 > x_2 > x_1$, taking $x = x_1e_1 + x_3e_3 + \cdots + x_me_m$, $\lambda = \frac{x_2}{x_1}$ in (5.18), we get

$$\begin{aligned}&\Phi(T^d)(x_1e_1 + x_3e_3 + \cdots + x_me_m) + \Phi(\hat{T}^{d-1})(x_2e_1 + x_3e_3 + \cdots + x_me_m)\\&= \Phi(T^d)(x_2e_1 + x_3e_3 + \cdots + x_me_m) + \Phi(T^d)(x_1e_1 + \cdots + x_me_m)\}.\end{aligned} \tag{5.21}$$

For $x_2 > x_1 > 0$ or $0 > x_1 > x_2$, taking $x = x_2e_2 + x_3e_3 + \cdots + x_me_m$, $1 - \lambda = \frac{x_1}{x_2}$, in (5.18), we get

$$\begin{aligned}&\Phi(T^d)(x_2e_2 + x_3e_3 + \cdots + x_me_m) + \Phi(\hat{T}^{d-1})(x_1e_1 + \cdots + x_me_m)\\&= \Phi(T^d)(x_1e_1 + \cdots + x_me_m) + \Phi(T^d)(x_1e_2 + x_3e_3 + \cdots + x_me_m).\end{aligned} \tag{5.22}$$

For $x_1 > 0 > x_2$ or $x_2 > 0 > x_1$, taking $0 < \lambda = \frac{x_2}{x_2 - x_1} < 1$ and $x = x_1e_1 + \cdots + x_me_m$ in (5.18), we get

$$\begin{aligned}&\Phi(T^d)(x_1e_1 + \cdots + x_me_m) + \Phi(\hat{T}^{d-1})(x_2e_2 + x_3e_3 + \cdots + x_me_m)\\&= \Phi(T^d)(x_2e_2 + x_3e_3 + \cdots + x_me_m) + \Phi(T^d)(x_1e_1 + x_3e_3 + \cdots + x_me_m).\end{aligned} \tag{5.23}$$

Combined with the $\mathrm{SL}(n)$ covariance of Φ, (5.21), (5.22) and (5.23) show that $\Phi(T^d)(x_1e_1 + \cdots + x_me_m)$ is uniquely determined by $\Phi(T^d)(y_1e_1 + \cdots + y_{m-1}e_{m-1})$, $y_1, \ldots, y_{m-1} \neq 0$, and $\Phi(T^{d-1})$. Since $\Phi_{p;a_1,a_2}(T^d) + \Phi_{p;b_1,b_2}(-T^d)$ also satisfies the equations (5.21), (5.22) and (5.23), we get that (5.17) holds true for m. The proof is complete. □

Finally, let $\Phi' P = \Phi P - \Phi_{p;a_1,a_2}P - \Phi_{p;b_1,b_2}(-P)$, $P \in \mathcal{P}_o^n$. Hence Φ' is a simple $\mathrm{SL}(n)$ covariant valuation. Here simple means that the valuation vanishes on lower dimensional bodies. Combined with the following classification of simple valuations by Haberl [3] and Parapatits [21], we finish the proof of Lemma 5.2.

Lemma 5.7 (Haberl [3] and Parapatits [21])**.** *Let $n \geq 3$ and $\Phi : \mathcal{P}_o^n \to C_p(\mathbb{R}^n)$ be a simple $\mathrm{SL}(n)$ covariant valuation. Assume further that, for every $y \in \mathbb{R}^n$, the function $s \mapsto \Phi(sT^n)(y)$ is bounded from below on some non-empty open interval $I_y \subset (0, +\infty)$. Then there exist constants $c_1, c_2 \in \mathbb{R}$ such that*

$$\Phi P = c_1 h^p_{M_p^+ P} + c_2 h^p_{M_p^- P}$$

for every $P \in \mathcal{P}_o^n$.

□

Proof of Theorem 5.3. For $a_1, b_1, c_1, c_2 \geq 0$, clearly $P \mapsto c_1 h^p_{M_p^+ P} + c_2 h^p_{M_p^- P} + a_1 h^p_P + b_1 h^p_{-P}$ is a valuation satisfying all conditions. Hence we only need to show the necessity.

Let Φ be a valuation satisfying all the conditions of Theorem 5.3. Since Φ also satisfies all the conditions of Lemma 5.2, there exist constants $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}$ such that

$$\Phi P = c_1 h^p_{M_p^+ P} + c_2 h^p_{M_p^- P} + \Phi_{p;a_1,a_2} P + \Phi_{p;b_1,b_2}(-P) \tag{5.24}$$

for every $P \in \mathcal{P}_o^n$. The main aim is to show that $a_1 = a_2$ and $b_1 = b_2$.

Lemma 5.8. *Let Φ satisfies (5.24). Assume that ΦP is non-negative and $(\Phi P)^{1/p}$ is a sublinear function for all $P \in \mathcal{P}_o^n$. Then $a_1, a_2, b_1, b_2, c_1, c_2 \geq 0$. Moreover, if $n \geq 3$, $p > 1$ or $n \geq 4$, $p = 1$, then*

$$a_1 = a_2,\ b_1 = b_2;$$

if $p = 1$ *and* $n = 3$, *then*

$$a_1 \le a_2,\ b_1 \le b_2,\ a_2 - a_1 \le b_2,\ b_2 - b_1 \le a_2.$$

Proof. From the definitions,

$$h^p_{\alpha M_p^+ P} = \alpha^{n+p} h^p_{M_p^+ P},\ h^p_{\alpha M_p^- P} = \alpha^{n+p} h^p_{M_p^- P}, \tag{5.25}$$

and

$$\Phi_{p;a_1,a_2}(\alpha P) = \alpha^p \Phi_{p;a_1,a_2} P,\ \Phi_{p;b_1,b_2}(-\alpha P) = \alpha^p \Phi_{p;b_1,b_2}(-P). \tag{5.26}$$

for $\alpha > 0$ and $P \in \mathcal{P}^n_o$. Also since $h^p_{M_p^+ T^n}(e_1) > 0$, $h^p_{M_p^- T^n}(e_1) = 0$, if $c_1 < 0$, then $\alpha^{-p}\Phi(\alpha T^n)(e_1) \to -\infty$ when $\alpha \to \infty$. It is a contradiction since $\Phi(\alpha T^n)(e_1) \ge 0$ for any $\alpha > 0$. Hence $c_1 \ge 0$. Similarly we get $c_2 \ge 0$.

Define $h(x) := \lim\limits_{\alpha \to 0^+} \alpha^{-1}(\Phi(\alpha T^3)(x))^{1/p}$, $x \in \mathbb{R}^3$. By (5.25) and (5.26), we have

$$0 \le h = (\Phi_{p;a_1,a_2} T^3 + \Phi_{p;b_1,b_2}(-T^3))^{1/p}. \tag{5.27}$$

Let $0 \le \mu \le \lambda \le 1$. By Proposition 5.5, we get that

$$h(e_1 + \lambda e_2 + \mu e_3) = (a_2 + \mu^p(a_1 - a_2))^{1/p}.$$

Especially,

$$0 \le h(e_1 + e_2 + e_3) = a_1^{1/p},$$

and

$$0 \le h(e_1 + e_2) = h(e_1 + e_3) = a_2^{1/p}.$$

On the other hand, h is sublinear since h is the limit of sublinear functions. Hence taking $\frac{1}{2} \le \lambda \le 1$, we get

$$\begin{aligned}(a_2 + (1-\lambda)^p(a_1 - a_2))^{1/p} &= h(e_1 + \lambda e_2 + (1-\lambda)e_3)\\ &\le h(\lambda e_1 + \lambda e_2) + h((1-\lambda)e_1 + (1-\lambda)e_3) = a_2^{1/p}.\end{aligned}$$

Then $a_1 \le a_2$.

Next we will prove $a_2 \le a_1$ for $n \ge 3$ and $p > 1$.

Since h is sublinear, it is also a support function of a convex body, denoted by $K \subset \mathbb{R}^3$. Let $x_1, x_2 \in \mathbb{R}$. By (5.27) and Proposition 5.5, we get that

$$\begin{aligned}h_{K|\mathbb{R}^2}(x_1e_1 + x_2e_2) &= h_K(x_1e_1 + x_2e_2)\\ &= (\Phi_{p;a_1,a_2}T^3(x_1e_1 + x_2e_2) + \Phi_{p;b_1,b_2}(-T^3)(x_1e_1 + x_2e_2))^{1/p}\\ &= (a_2 \max\{x_1^p, x_2^p, 0\} + b_2 \max\{-x_1^p, -x_2^p, 0\})^{1/p}\\ &= h_{a_2^{1/p}T^2 +_p (b_2)^{1/p}(-T^2)}(x_1e_1 + x_2e_2).\end{aligned}$$

Hence $K|\mathbb{R}^2 = a_2^{1/p}T^2 +_p a_2^{1/p}T^2$. If $a_2^{1/p}e_1 \notin K$, then K must contain a point $a_2^{1/p}e_1 + \alpha e_3$, $\alpha \ne 0$. However, by similar arguments, the orthogonal projection of K onto the linear space spanned by $\{e_1, e_3\}$ is $a_2^{1/p}[o, e_1, e_3] +_p (b_2)^{1/p}(-[o, e_1, e_3])$. This is a contradiction since $a_2^{1/p}e_1 + \alpha e_3 \notin a_2^{1/p}[o, e_1, e_3] +_p (b_2)^{1/p}(-[o, e_1, e_3])$ when $p > 1$. Hence $a_2^{1/p}e_1 \in K$. Together with Proposition 5.5, we have

$$a_2^{1/p} = a_2^{1/p}e_1 \cdot (e_1 + e_2 + e_3) \le h_K(e_1 + e_2 + e_3) = a_1^{1/p}.$$

For $n \geq 4$ and $p = 1$, we use $[-e_1, e_1, \ldots, e_4]$ to show that $a_2 \leq a_1$.
Setting $d = 4$, $m = 1$, $v_0 = -e_1$ in (5.11), we have

$$\begin{aligned}
&\Phi_{1;a_1,a_2}([-e_1, e_1, \ldots, e_4]) \begin{pmatrix} 1 \\ 3 \\ 3 \\ 2 \end{pmatrix} + \Phi_{1;b_1,b_2}(-[-e_1, e_1, \ldots, e_4]) \begin{pmatrix} 1 \\ 3 \\ 3 \\ 2 \end{pmatrix} \\
&= \Phi_{1;a_1,a_2}([-e_1, e_1, \ldots, e_4]) \begin{pmatrix} 1 \\ 3 \\ 2 \\ 3 \end{pmatrix} + \Phi_{1;b_1,b_2}(-[-e_1, e_1, \ldots, e_4]) \begin{pmatrix} 1 \\ 3 \\ 2 \\ 3 \end{pmatrix} \\
&= 3a_2 + 2(a_2 - a_1) - (a_2 - a_1) + b_2,
\end{aligned}$$

and

$$\begin{aligned}
&\Phi_{1;a_1,a_2}([-e_1, e_1, \ldots, e_4]) \begin{pmatrix} 2 \\ 6 \\ 5 \\ 5 \end{pmatrix} + \Phi_{1;b_1,b_2}(-[-e_1, e_1, \ldots, e_4]) \begin{pmatrix} 2 \\ 6 \\ 5 \\ 5 \end{pmatrix} \\
&= 6a_2 + 5(a_2 - a_1) - 2(a_2 - a_1) + 2b_2.
\end{aligned}$$

Also since $\Phi_{1;a_1,a_2}([-e_1, e_1, \ldots, e_4]) + \Phi_{1;b_1,b_2}(-[-e_1, e_1, \ldots, e_4])$ is sublinear, we have

$$5(a_2 - a_1) \leq 4(a_2 - a_1).$$

Hence $a_2 \leq a_1$.

The proof for the restrictions on b_1, b_2 is similar.

Finally, for $p = 1$, $n = 3$, since $h_{M_p^+ T^2} = h_{M_p^- T^2} = 0$, $\Phi_{p;a_1,a_2} T^2 + \Phi_{p;b_1,b_2}(-T^2)$ is sublinear. Also, for $i = 1, 2$, Proposition 5.5 shows that

$$\begin{aligned}
\Phi_{p;a_1,a_2} T^2(e_i) + \Phi_{p;b_1,b_2}(-T^2)(e_i) &= a_2, \\
\Phi_{p;a_1,a_2} T^2(-e_i) + \Phi_{p;b_1,b_2}(-T^2)(-e_i) &= b_2, \\
\Phi_{p;a_1,a_2} T^2(e_1 + e_2) + \Phi_{p;b_1,b_2}(-T^2)(e_1 + e_2) &= a_1, \\
\Phi_{p;a_1,a_2} T^2(-e_1 - e_2) + \Phi_{p;b_1,b_2}(-T^2)(-e_1 - e_2) &= b_1.
\end{aligned}$$

Hence

$$\begin{aligned}
a_2 &= \Phi_{p;a_1,a_2} T^2(e_1) + \Phi_{p;b_1,b_2}(-T^2)(e_1) \\
&\leq \Phi_{p;a_1,a_2} T^2(e_1 + e_2) + \Phi_{p;b_1,b_2}(-T^2)(e_1 + e_2) + \Phi_{p;a_1,a_2} T^2(-e_2) + \Phi_{p;b_1,b_2}(-T^2)(-e_2) \\
&= a_1 + b_2,
\end{aligned}$$

and

$$\begin{aligned}
b_2 &= \Phi_{p;a_1,a_2} T^2(-e_1) + \Phi_{p;b_1,b_2}(-T^2)(-e_1) \\
&\leq \Phi_{p;a_1,a_2} T^2(-e_1 - e_2) + \Phi_{p;b_1,b_2}(-T^2)(-e_1 - e_2) + \Phi_{p;a_1,a_2} T^2(e_2) + \Phi_{p;b_1,b_2}(-T^2)(e_2) \\
&= b_1 + a_2.
\end{aligned}$$

The proof is complete. □

Since $a_1 = a_2$ and $b_1 = b_2$, we get

$$\Phi_{p;a_1,a_2} P = a_1 h_P^p, \ \Phi_{p;b_1,b_2}(-P) = b_1 h_{-P}^p$$

for every $P \in \mathcal{P}_o^n$. Hence the proof is complete and the restrictions for a_1, b_1, c_1, c_2 are given by Lemma 5.8. □

Proof of Theorem 1.7. First we show that $\Phi_{1;a_1,a_2} P + \Phi_{1;b_1,b_2}(-P)$ for $\dim P \leq 3$ is a support function (under the restrictions on a_1, a_2, b_1, b_2). We will use following two lemmas.

Lemma 5.9. [23, Lemma 3.2.9] *Let $K, L \in \mathcal{K}^n$. If $L|V$ is a summand of $K|V$, for all 2-dimensional linear subspaces V in some dense subset of $Gr(n, 2)$, then L is a summand of K.*

Lemma 5.10. [23, Theorem 3.2.11] *Let $P, K \in \mathcal{K}^n$, where P is a polytope. Then P is a summand of K if and only if $F(K, u)$ contains a translate of $F(P, u)$ whenever $F(P, u)$ is an edge of P ($u \in S^{n-1}$).*

Now let $P \in \mathcal{P}_o^3$. If $o \in \operatorname{relint} P$, then there is nothing to prove. Assume $o \in \operatorname{relbd} P$.

First let $\dim P = 3$. Notice that

$$\begin{aligned}\Phi_{1;a_1,a_2} P + \Phi_{1;b_1,b_2}(-P) = a_1 h_P &+ (a_2 - a_1) \sum_{F \in \mathcal{F}_o(P)} h_F - (a_2 - a_1) \sum_{E \in \mathcal{E}_o(P)} h_E \\ &+ b_1 h_{-P} + (b_2 - b_1) \sum_{F \in \mathcal{F}_o(P)} h_{-F} - (b_2 - b_1) \sum_{E \in \mathcal{E}_o(P)} h_{-E}\end{aligned}$$

is a support function if and only if $(a_2 - a_1) \sum_{E \in \mathcal{E}_o(P)} E + (b_2 - b_1) \sum_{E \in \mathcal{E}_o(P)} (-E) =: P_1$ is a summand of $a_1 P + (a_2 - a_1) \sum_{F \in \mathcal{F}_o(P)} F + b_1(-P) + (b_2 - b_1) \sum_{F \in \mathcal{F}_o(P)} (-F) =: P_2$. According to Lemma 5.9 and 5.10, it is sufficient to show that $F(P_2|V, u)$ contains a translate of $F(P_1|V, u)$ for all V in a dense set of $Gr(n, 2)$, whenever $F(P_1|V, u)$ is an edge of $P_1|V$. Here and in the following $u \in S^{n-1} \cap V$. Also we can assume that for different edges $E_1, E_2 \in \mathcal{E}_o(P)$, $E_1|V$ and $E_2|V$ does not lie on the same line.

Let m be the cardinality of the set $\mathcal{F}_o(P)$. Since the pointwise limit of a support function is a support function, it does not change the desired result. Thus we can assume that every face in $\mathcal{F}_o(P)$ has two edges containing the origin. Also every edge in $\mathcal{E}_o(P)$ belongs to two faces in $\mathcal{F}_o(P)$. Hence P also has m edges through the origin. Now we can write $\mathcal{F}_o(P) = \{F_i\}_{i=1}^m$ and $\mathcal{E}_o(P) = \{E_i\}_{i=1}^m$ such that $E_i \subset F_i \cap F_{i+1}$ for any $1 \leq i \leq n$. Here we set $F_{m+1} = F_1$.

Since

$$P_1|V = (a_2 - a_1) \sum_{i=1}^m E_i|V + (b_2 - b_1) \sum_{i=1}^m (-E_i|V),$$

$$P_2|V = a_1 P|V + (a_2 - a_1) \sum_{i=1}^m F_i|V + b_1(-P|V) + (b_2 - b_1) \sum_{i=1}^m (-F_i|V),$$

and $F(K + L, u) = F(K, u) + F(L, u)$ for $K, L \in \mathcal{K}^n$, we only need to show that if $F(E_i|V, u)$ is a non-degenerate interval (hence $F(E_i|V, u) = E_i|V$), then $F(P_2|V, u)$ contains a translate of $(a_2 - a_1)E_i|V + (b_2 - b_1)(-E_i|V)$. We need to deal with two cases:

(i) $E_i|V$ is contained in the boundary of $P|V$,

(ii) the relative interior of $E_i|V$ is contained in the relative interior of $P|V$.

In case (i), u is an outer normal vector of $P|V$ or an inner normal vector of $P|V$. If u is an outer normal vector of $P|V$, then $E_i|V$ is contained in $F(F_i|V, u)$, $F(F_{i+1}|V, u)$ and

$F(P|V,u)$. Hence $(a_2-a_1)F(F_i|V,u)+(a_2-a_1)F(F_{i+1}|V,u)+a_1F(P|V,u)$ contains a translate of $(a_2-a_1)E_i|V+(b_2-b_1)(-E_i|V)$ since $b_2-b_1\le a_2$. Also since $F(P_2|V,u)$ contains a translate of $(a_2-a_1)F(F_i|V,u)+(a_2-a_1)F(F_{i+1}|V,u)+a_1F(P|V,u)$, we have that $F(P_2|V,u)$ contains a translate of $(a_2-a_1)E_i|V+(b_2-b_1)(-E_i|V)$.
If u is an inner normal vector of $P|V$, then $E_i|V$ is contained in $-F(-F_i|V,u)$, $-F(-F_{i+1}|V,u)$ and $-F(-P|V,u)$. Similarly $F(P_2|V,u)$ contains a translate of $(b_2-b_1)F(-F_i|V,u)+(b_2-b_1)F(-F_{i+1}|V,u)+b_1F(-P|V,u)$ which contains a translate of $(a_2-a_1)E_i|V+(b_2-b_1)(-E_i|V)$ since $a_2-a_1\le b_2$.

In case (ii), $E_i|V$ is contained in $F(F_i|V,u)\cap F(F_{i+1}|V,-u)$ or $F(F_i|V,-u)\cap F(F_{i+1}|V,u)$. Hence $F(P_2|V,u)$ contains a translate of $(a_2-a_1)E_i|V+(b_2-b_1)(-E_i|V)$.

The proof for $\dim P=2$ is similar (and easier). For $\dim P=1$, there is nothing to prove.

Now we turn to the necessity. Since $P\mapsto h_{ZP}$ satisfies the conditions of Lemma 5.2, there exist constants $a_1,a_2,b_1,b_2,c_1,c_2\in\mathbb{R}$ such that

$$h_{ZP}=c_1h_{M^+P}+c_2h_{M^-P}+\Phi_{1;a_1,a_2}P+\Phi_{1;b_1,b_2}(-P)$$

for every $P\in\mathcal{P}_o^n$. The restrictions for a_1,a_2,b_1,b_2,c_1,c_2 are given by Lemma 5.8. □

If we just consider valuations defined on $\mathcal{T}_o^n$, then D_{a_1,a_2,b_1,b_2} is a valuation even for $n\ge 4$.

Theorem 5.11. *Let $n\ge 3$. The map $Z:\mathcal{T}_o^n\to\mathcal{K}_o^n$ is an $\mathrm{SL}(n)$ covariant Minkowski valuation if and only if there exist constants $a_1,a_2,b_1,b_2,c_1,c_2\ge 0$ satisfying $a_1\le a_2$, $b_1\le b_2$, $a_2-a_1\le b_2$ and $b_2-b_1\le a_2$ such that*

$$ZT=c_1M^+T+c_2M^-T+D_{a_1,a_2,b_1,b_2}T$$

for every $T\in\mathcal{T}_o^n$, where

$$D_{a_1,a_2,b_1,b_2}T=[a_2v_i-b_2v_j,\,a_2v_i-(a_2-a_1)v_j,\,(b_2-b_1)v_i-b_2v_j:1\le i,j\le d]$$

for $T=[o,v_1,\dots,v_d]$, $2\le d\le n$, and

$$D_{a_1,a_2,b_1,b_2}T=[-b_1v_1,a_1v_1]$$

for $T=[o,v_1]$. Here $o,v_1,\dots,v_d\in\mathbb{R}^n$ are affinely independent.

Proof. First, we show that the support function of $D_{a_1,a_2,b_1,b_2}T$ defined in this theorem is $\Phi_{1;a_1,a_2}T+\Phi_{1;b_1,b_2}(-T)$ if a_1,a_2,b_1,b_2 satisfy all the conditions. Since D_{a_1,a_2,b_1,b_2} and $\Phi_{1;a_1,a_2}$ are both $\mathrm{GL}(n)$ covariant, we only need to show that

$$h_{D_{a_1,a_2,b_1,b_2}T^d}(y)=\Phi_{1;a_1,a_2}T^d(y)+\Phi_{1;b_1,b_2}(-T^d)(y)$$

for $y\in\mathbb{R}^n$. But from the definition of D_{a_1,a_2,b_1,b_2} and from Lemma 4.1 and Lemma 5.4, we have

$$h_{D_{a_1,a_2,b_1,b_2}T^d}(y|\mathbb{R}^d)=h_{D_{a_1,a_2,b_1,b_2}T^d}(y),$$
$$\Phi_{1;a_1,a_2}T^d(y)+\Phi_{1;b_1,b_2}(-T^d)(y)=\Phi_{1;a_1,a_2}T^d(y|\mathbb{R}^d)+\Phi_{1;b_1,b_2}(-T^d)(y|\mathbb{R}^d).$$

Combined with the $\mathrm{GL}(n)$ covariance of $D_{a_1,a_2,b_1,b_2},\Phi_{1;a_1,a_2}$ again, we only need to show that

$$h_{D_{a_1,a_2,b_1,b_2}T^d}(x)=\Phi_{1;a_1,a_2}T^d(x)+\Phi_{1;b_1,b_2}(-T^d)(x) \tag{5.28}$$

for $x=(x_1,\dots,x_d)^t\in\mathbb{R}^d$ with $x_1\ge\dots\ge x_d$. A simple calculation shows that

$$\begin{aligned}h_{D_{a_1,a_2,b_1,b_2}T^d}(x)&=\max_{1\le i,j\le d}\{a_2x_i-b_2x_j,\,a_2x_i-(a_2-a_1)x_j,\,(b_2-b_1)x_i-b_2x_j\}\\&=\max\{a_2x_1-b_2x_d,\,a_2x_1-(a_2-a_1)x_d,\,(b_2-b_1)x_1-b_2x_d\}.\end{aligned} \tag{5.29}$$

Also Proposition 5.5 shows that

$$\begin{aligned}&\Phi_{1;a_1,a_2}(T^d)(x)+\Phi_{1;b_1,b_2}(-T^d)(x)\\=&a_2\max\{x_1,0\}-(a_2-a_1)\max\{x_d,0\}+b_2\max\{-x_d,0\}-(b_2-b_1)\max\{-x_1,0\}.\end{aligned} \tag{5.30}$$

For all the three cases $0\ge x_1\ge x_d$, $x_1\ge 0\ge x_d$ and $x_1\ge x_d\ge 0$, the right side of (5.29) and (5.30) is equal. Hence, (5.28) holds true.

Since $T\mapsto c_1h_{M+T}+c_2h_{M-T}+\Phi_{1;a_1,a_2}T+\Phi_{1;b_1,b_2}(-T)$ is a valuation so is $T\mapsto c_1M^+T+c_2M^-T+D_{a_1,a_2,b_1,b_2}T$. The proof of the sufficient part is complete.

Next we turn to the necessity. Since $T\mapsto h_{ZT}$ satisfies the conditions of Lemma 5.2, there exist constants $a_1,a_2,b_1,b_2,c_1,c_2\in\mathbb{R}$ such that

$$h_{ZT}=c_1h_{M+T}+c_2h_{M-T}+\Phi_{1;a_1,a_2}T+\Phi_{1;b_1,b_2}(-T)$$

for every $T\in\mathcal{T}^n_o$ (Although the domain of the valuation is just $\mathcal{T}^n_o$ not $\mathcal{P}^n_o$, we still can get this result from the proof of Lemma 5.2). The restrictions on a_1,a_2,b_1,b_2,c_1,c_2 are given by Lemma 5.8. □

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References

[1] *J. Abardia*, Minkowski valuations in a 2-dimensional complex vector space, Int. Math. Res. Not. **2015** (2015), 1247–1262.
[2] *J. Abardia* and *A. Bernig*, Projection bodies in complex vector spaces, Adv. Math. **227** (2011), no. 2, 830–846.
[3] *C. Haberl*, Minkowski valuations intertwining with the special linear group, J. Eur. Math. Soc. **14** (2012), no. 5, 1565–1597.
[4] *C. Haberl* and *F. Schuster*, Asymmetric affine L_p Sobolev inequalities, J. Funct. Anal. **257** (2009), no. 3, 641–658.
[5] *C. Haberl* and *F. Schuster*, General L_p affine isoperimetric inequalities, J. Differential Geom. **83** (2009), no. 1, 1–26.
[6] *M. Ludwig*, Projection bodies and valuations, Adv. Math. **172** (2002), no. 2, 158–168.
[7] *M. Ludwig*, Ellipsoids and matrix-valued valuations, Duke Math. J. **119** (2003), no. 1, 159–188.
[8] *M. Ludwig*, Minkowski valuations, Trans. Amer. Math. Soc. **357** (2005), no. 10, 4191–4213.
[9] *M. Ludwig*, Minkowski areas and valuations, J. Differential Geom. **86** (2010), no. 1, 133–161.
[10] *M. Ludwig*, Valuations on Sobolev spaces, Amer. J. Math. **134** (2012), 824–842.
[11] *M. Ludwig* and *M. Reitzner*, Elementary moves on triangulations, Discrete Comput. Geom. **35** (2006), no. 4, 527–536.
[12] *E. Lutwak*, The Brunn-Minkowski-Firey theory I: Mixed volumes and the Minkowski problem, J. Differential Geom. **38** (1993), no. 1, 131–150.
[13] *E. Lutwak*, The Brunn-Minkowski-Firey theory II: Affine and geominimal surface area, Adv. Math. **118** (1996), no. 2, 244–294.
[14] *E. Lutwak*, *D. Yang* and *G. Zhang*, L_p affine isoperimetric inequalities, J. Differential Geom. **56** (2000), no. 1, 111–132.
[15] *E. Lutwak*, *D. Yang* and *G. Zhang*, A new ellipsoid associated with convex bodies, Duke. Math. J. **104** (2000), no. 3, 375–390.
[16] *E. Lutwak*, *D. Yang* and *G. Zhang*, Sharp affine L_p Sobolev inequalities, J. Differential Geom. **62** (2002), no. 1, 17–38.

[17] *E. Lutwak*, *D. Yang* and *G. Zhang*, On the L_p-Minkowski problem, Tran. Amer. Math. Soc. **356** (2004), no. 11, 4359–4370.

[18] *E. Lutwak*, *D. Yang* and *G. Zhang*, L_p John ellipsoids, Proc. London Math. Soc. **90** (2005), no. 2, 497–520.

[19] *M. Ober*, L_p-Minkowski valuations on L^q-spaces, J. Math. Anal. Appl. **414** (2014), no. 1, 68–87.

[20] *L. Parapatits*, SL(n)-contravariant L_p-Minkowski valuations, Trans. Amer. Math. Soc. **366** (2014), no. 3, 1195–1211.

[21] *L. Parapatits*, SL(n)-covariant L_p-Minkowski valuations, J. London Math. Soc. **89** (2014), no. 2, 397–414.

[22] *L. Parapatits* and *F. Schuster*, The Steiner formula for Minkowski valuations, Adv. Math. **230** (2012), no. 3, 978–994.

[23] *R. Schneider*, Convex Bodies: The Brunn-Minkowski Theory, Cambridge University Press, Cambridge, 2014, 2nd edition.

[24] *R. Schneider* and *F. Schuster*, Rotation equivariant Minkowski valuations, Int. Math. Res. Not. **Art. ID 72894** (2006), 1–20.

[25] *F. Schuster*, Valuations and Busemann-Petty type problems, Adv. Math. **219** (2008), no. 1, 344–368.

[26] *F. Schuster*, Crofton measures and Minkowski valuations, Duke Math. J. **154** (2010), no. 1, 1–30.

[27] *F. Schuster* and *T. Wannerer*, *GL*(n) contravariant Minkowski valuations, Trans. Amer. Math. Soc. **364** (2012), no. 2, 815–826.

[28] *A. Tsang*, Minkowski valuations on L^p-spaces, Trans. Amer. Math. Soc. **364** (2012), no. 12, 6159–6186.

[29] *T. Wannerer*, GL(n) equivariant Minkowski valuations, Indiana Univ. Math. J. **60** (2011), 1655–1672.

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