DAR'S CONJECTURE AND THE LOG-BRUNN-MINKOSKI INEQUALITY

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ABSTRACT. In 1999, Dar conjectured if there is a stronger version of the celebrated Brunn-Minkowski inequality. However, as pointed out by Campi, Gardner, and Gronchi in 2011, this problem seems to be open even for planar o-symmetric convex bodies. In this paper, we give a positive answer to Dar's conjecture for all planar convex bodies. We also give the equality condition of this stronger inequality.

For planar o-symmetric convex bodies, the log-Brunn-Minkowski inequality was established by Böröczky, Lutwak, Yang, and Zhang in 2012. It is stronger than the classical Brunn-Minkowski inequality, for planar o-symmetric convex bodies. Gaoyong Zhang asked if there is a general version of this inequality. Fortunately, the solution of Dar's conjecture, especially, the definition of "dilation position", inspires us to obtain a general version of the log-Brunn-Minkowski inequality. As expected, this inequality implies the classical Brunn-Minkowski inequality for all planar convex bodies.

1. INTRODUCTION

Let $\mathcal{K}^n$ be the class of convex bodies (compact, convex sets with non-empty interiors) in Euclidean n-space $\mathbb{R}^n$, and let $\mathcal{K}^n_o$ be the class of members of $\mathcal{K}^n$ containing o (the origin) in their interiors. The classical Brunn-Minkowski inequality (see, e.g., [17, 18, 25, 33]) states that

$$|K+L|^{\frac{1}{n}} \geq |K|^{\frac{1}{n}} + |L|^{\frac{1}{n}}, \tag{1.1}$$

with equality if and only if K and L are homothetic. Here $K, L \in \mathcal{K}^n$, $|\cdot|$ denotes the n-dimensional Lebesgue measure, $K+L$ denotes the Minkowski sum of K and L:

$$K+L = \{x+y : x \in K \text{ and } y \in L\}.$$

In his survey article, Gardner [17] summarized the history of the Brunn-Minkowski inequality and some applications in many other fields. For recent related work about this inequality, see e.g., [5, 13–15, 19, 30, 37].

In 1999, Dar [9] conjectured that

$$|K+L|^{\frac{1}{n}} \geq M(K,L)^{\frac{1}{n}} + \frac{|K|^{\frac{1}{n}}|L|^{\frac{1}{n}}}{M(K,L)^{\frac{1}{n}}}, \tag{1.2}$$

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for convex bodies K and L. Here $M(K, L)$ is defined by

$$M(K, L) = \max_{x \in \mathbb{R}^n} |K \cap (x + L)|.$$

Dar's conjecture has a close relationship with the stability of the Brunn-Minkowski inequality, and plays an important role in asymptotic geometric analysis. The stability estimates are actually strong forms of Brunn-Minkowski inequality in special circumstances. Original works about this issue are due to Diskant, Groemer, and Schneider referred in [10,12,22–24,33]. Dar [9] pointed out that the "weak estimates" about the "geometric Banach-Mazur distance" cannot be essentially improved. In fact, this might be why Dar proposed his conjecture (1.2).

Figalli, Maggi, and Pratelli [13,14] tackled the stability problem for convex bodies with a more natural distance, i.e., "relative asymmetry" (which has a close relationship with the functional $M(K, L)$), by using mass transportation approach. Using the same distance as in [13, 14], Segal [34] improved the constants that appeared in the stability versions in these inequalities for convex bodies. He also showed in [34, Page 391] that Dar's conjecture (1.2) will lead to a stronger stability version of Brunn-Minkowski inequality for convex bodies.

Dar [9] showed that (1.2) implies (1.1) for convex bodies. He also proved (1.2) in some special cases, such as:

(1) K is unconditional with respect a basis $\{e_i\}_{i=1}^n$ and $L = TK$, where T is linear and diagonal with respect to the same basis;

(2) K and L are ellipsoids;

(3) $K \subset \mathbb{R}^2$ is a parallelogram and L is a planar symmetric convex body;

(4) K is a simplex and $L = -K$.

In their article, Campi, Gardner, and Gronchi [8, Page 1208] described this as "a fascinating conjecture". However, they also pointed out that Dar's conjecture "seems to be open even for planar o-symmetric bodies". Besides, the equality condition of (1.2) is also unknown.

In this paper, we prove that the inequality (1.2) holds for all planar convex bodies, and we also give the equality condition.

Theorem 1. *Let K, L be planar convex bodies. Then, we have*

$$|K + L|^{\frac{1}{2}} \geq M(K, L)^{\frac{1}{2}} + \frac{|K|^{\frac{1}{2}} |L|^{\frac{1}{2}}}{M(K, L)^{\frac{1}{2}}}. \tag{1.3}$$

Equality holds if and only if one of the following conditions holds:

(i) K and L are parallelograms with parallel sides, and $|K| = |L|$;

(ii) K and L are homothetic.

In our proof of Theorem 1, the definition of "dilation position" plays a key role. It makes us be able to do a further study on the other stronger version of (1.1), i.e., the log-Brunn-Minkowski inequality.

The log-Brunn-Minkowski inequality for planar o-symmetric (symmetry with respect to the origin) convex bodies was established by Böröczky, Lutwak, Yang, and Zhang [5]. It states that:

If K and L are o-symmetric convex bodies in the plane, then for all real $\lambda \in [0,1]$,

$$|(1-\lambda)\cdot K +_o \lambda \cdot L| \geq |K|^{1-\lambda}|L|^{\lambda}. \tag{1.4}$$

When $\lambda \in (0,1)$, equality in (1.4) holds if and only if K and L are dilates or K and L are parallelograms with parallel sides. Here h_K and h_L are support functions (see Section 2 for the definition); $(1-\lambda)\cdot K +_o \lambda \cdot L$ is the *geometric Minkowski combination* of K and L, which is defined in [5] for $K, L \in \mathcal{K}^n_o$ as the Aleksandrov body (see, e.g., [1]) associated with the function $h_K^{1-\lambda}h_L^{\lambda}$.

For o-symmetric convex bodies K and L, Böröczky, Lutwak, Yang, and Zhang [5] also established the following log-Minkowski inequality:

$$\int_{S^1} \log \frac{h_L}{h_K} dV_K \geq \frac{|K|}{2} \log \frac{|L|}{|K|}. \tag{1.5}$$

Equality holds if and only if K and L are dilates or K and L are parallelograms with parallel sides.

On one hand, we observe that the equality condition of (1.3) is similar to (1.4) and (1.5), equivalently to say, the uniqueness of the logarithmic Minkowski problem, see [5, 6, 35, 36] for details. We study the relationship between Dar's conjecture and the log-Brunn-Minkowski inequality in Section 4.

On the other hand, it is nature to ask if there is a general version of (1.4) for planar convex bodies that are not o-symmetric. Although there is a counterexample shown in [5] that: let K be an o-centered cube, and L be a translate of K, then (1.4) cannot hold; however, there exists a translate of K, say, $\overline{K}$, such that $\overline{K}$ and L satisfy (1.4). Here we only require that $\overline{K}$ and L are at a "dilation position" (see the definition below).

The following Problem was proposed by Professor Gaoyong Zhang when he was visiting Shanghai University in 2013.

Problem 1. *Let $K, L \in \mathcal{K}^2$. Is there a "good" position of the origin o, such that K and an "appropriate" translate of L satisfy (1.4)?*

The following Theorem 2 is an answer to Problem 1. Before this, we give the definition of the so-called dilation position.

Let $K, L \in \mathcal{K}^n$. We say K and L are at a *dilation position*, if $o \in K \cap L$, and

$$r(K,L)L \subset K \subset R(K,L)L. \tag{1.6}$$

Here $r(K,L)$ and $R(K,L)$ are *relative inradius* and *relative outradius* (e.g., see [5, 11, 21, 32]) of K with respect to L, i.e.,

$$r(K,L) = \max\{t > 0 : x + tL \subset K \text{ and } x \in \mathbb{R}^n\},$$
$$R(K,L) = \min\{t > 0 : K \subset x + tL \text{ and } x \in \mathbb{R}^n\}.$$

It is clear that

$$r(K,L) = 1/R(L,K). \tag{1.7}$$

By the definition, it is clear that two o-symmetric convex bodies are always at a dilation position. Therefore, Theorem 2 and Theorem 3 below are extensions of (1.4) and (1.5).

When K and L are at a dilation position, by Lemma 2.1, o may be in $\partial K \cap \partial L$. Therefore, we should extend the definition of "geometric Minkowski combination" slightly. Let $K, L \in \mathcal{K}^n$ with $o \in K \cap L$. The *geometric Minkowski combination* of K and L is defined as follows:

$$(1-\lambda)\cdot K +_o \lambda \cdot L := \bigcap_{u \in S^{n-1}} \{x \in \mathbb{R}^n : x \cdot u \le h_K(u)^{1-\lambda} h_L(u)^{\lambda}\}, \tag{1.8}$$

for $\lambda \in (0,1)$; $(1-\lambda)\cdot K +_o \lambda \cdot L := K$ for $\lambda = 0$; and $(1-\lambda)\cdot K +_o \lambda \cdot L := L$ for $\lambda = 1$.

Lemma 2.2 shows that $(1-\lambda)\cdot K +_o \lambda \cdot L$ defined by (1.8) is always a convex body, as long as K and L are at a dilation position. The following is the general *log-Brunn-Minkowski inequality* for planar convex bodies.

Theorem 2. *Let $K, L \in \mathcal{K}^2$ with $o \in K \cap L$. If K and L are at a dilation position, then for all real $\lambda \in [0,1]$,*

$$|(1-\lambda)\cdot K +_o \lambda \cdot L| \ge |K|^{1-\lambda}|L|^{\lambda}. \tag{1.9}$$

When $\lambda \in (0,1)$, equality in the inequality holds if and only if K and L are dilates or K and L are parallelograms with parallel sides.

The following is the general *log-Minkowski inequality* for planar convex bodies.

Theorem 3. *Let $K, L \in \mathcal{K}^2$ with $o \in K \cap L$. If K and L are at a dilation position, then*

$$\int_{S^1} \log \frac{h_L}{h_K} dV_K \ge \frac{|K|}{2} \log \frac{|L|}{|K|}. \tag{1.10}$$

Equality holds if and only if K and L are dilates or K and L are parallelograms with parallel sides.

Here V_K denotes the cone-volume measure (see Section 2 for its definition). It can be seen from (1.6) that $\{h_K = 0\} = \{h_L = 0\}$. The integral in (1.10) should be understood to be taken on S^1 except the set $\{h_K = 0\}$, which is of measure 0, with respect to the measure V_K.

For o-symmetric convex bodies in the plane, it has been shown in [5] that the log-Brunn-Minkowski inequality (1.4) is stronger than the classical Brunn-Minkowski inequality (1.1). In this paper, by Lemma 2.1 and Theorem 2, together with the fact $(1-\lambda)\cdot K +_o \lambda \cdot L \subset (1-\lambda)K + \lambda L$, we see that (1.9) implies the classical Brunn-Minkowski inequality (1.1) for all planar convex bodies.

In [5], the proofs of (1.4) and (1.5) use the o-symmetry in several crucial ways. However, in the general case, our proofs require new approaches. First, we prove (1.10) for bodies in $\mathcal{K}_o^2$ under the assumption that the cone-volume measure of a body satisfies the strict subspace concentration inequality. See Section 2 for the definition and the development history of the subspace concentration condition. Then, by establishing 2 approximation lemmas, we show that (1.10) does not require the subspace concentration condition, and it holds even for the case that o is in the boundary. That is to say, the definition of "dilation position" is natural.

This paper is organized as follows. Section 2 contains the basic notation and definitions, and some basic properties of dilation position. Section 3 proves Dar's conjecture of dimension 2, and gives the equation condition. In Section 4, we show a connection between Dar's conjecture and the log-Brunn-Minkowski inequality. In Section 5, we show some properties of dilation position, and prove the equivalence of the log-Brunn-Minkowski inequality (1.9) and the log-Minkowski inequality (1.10). Section 6 proves a version of the log-Minkowski inequality (1.10) under an assumption. In the final Section 7, we establish 2 approximation lemmas, and thereby prove Theorems 2 and 3.

2. Preliminaries

In this section, we collect some basic notation and definitions about convex bodies, and we show some basic properties of the dilation position. Good general references for the theory of convex bodies are the books of Gardner [18], Gruber [25], Leichtweiss [29], and Schneider [33].

Denote by B^n the unit ball in $\mathbb{R}^n$. By $\text{int}A$, $\text{cl}A$ and ∂A we denote, respectively, the interior, closure and boundary of $A \subset \mathbb{R}^n$.

Suppose $A_1, A_2, ..., A_k \subset \mathbb{R}^n$ are compact. Denote by $[A_1, A_2, ..., A_k]$ the convex hull of $A_1 \cup A_2 \cup ... \cup A_k$. When $A_i = \{x_i\}$ is a single point set, we will usually write $[A_1, A_2, ..., x_i, ..., A_k]$ rather than $[A_1, A_2, ..., \{x_i\}, ..., A_k]$. Thus, for distinctive points x_1 and x_2, $[x_1, x_2]$ is a line segment. We also denote by $l(x_1x_2)$ the line through the points x_1, x_2.

The scalar product "$\cdot$" in $\mathbb{R}^n$ will often be used to describe hyperplanes and half-spaces. A *hyperplane* can be written in the form

$$H_{u,\alpha} = \{x \in \mathbb{R}^n : x \cdot u = \alpha\}.$$

The hyperplane $H_{u,\alpha}$ bounds the two closed *half-spaces*

$$H^-_{u,\alpha} = \{x \in \mathbb{R}^n : x \cdot u \leq \alpha\},$$

$$H^+_{u,\alpha} = \{x \in \mathbb{R}^n : x \cdot u \geq \alpha\}.$$

Especially, a hyperplane in $\mathbb{R}^2$ is just a line. Similarly, $l_{u,\alpha}$ denotes a line. We also denote by l^- and l^+ two closed half-spaces bounded by the line l. Then, $l^-_{u,\alpha}$ and $l^+_{u,\alpha}$ are two closed half-spaces bounded by $l_{u,\alpha}$; $l(x_1x_2)^-$ and $l(x_1x_2)^+$ are two closed half-spaces bounded by $l(x_1x_2)$.

Let $A \subset \mathbb{R}^2$ be a subset and l a line. We say that l *supports* A *at* x if $x \in A \cap l$ and either $A \subset l^+$ or $A \subset l^-$. We call l a *support line* of A at x. In this paper, if l is a support line of a planar convex body K, we always assume $K \subset l^-$.

The *support function* $h_K : \mathbb{R}^n \to \mathbb{R}$ of a compact convex set $K \subset \mathbb{R}^n$ is defined, for $x \in \mathbb{R}^n$, by

$$h_K(x) = \max\{x \cdot y : y \in K\}.$$

We shall use d_H to denote the *Hausdorff metric* on $\mathcal{K}^n$. If $K, L \in \mathcal{K}^n$, the Hausdorff distance $d_H(K, L)$ is defined by

$$d_H(K, L) = \min\{\alpha : K \subset L + \alpha B^n \text{ and } L \subset K + \alpha B^n\},$$

or equivalently,

$$d_H(K, L) = \max_{u \in S^{n-1}} |h_K(u) - h_L(u)|.$$

Let $K \in \mathcal{K}^n$. The *surface area measure* $S_K(\cdot)$ of K is a Borel measure on S^{n-1} defined for a Borel set $\omega \subset S^{n-1}$ by

$$S_K(\omega) = \mathcal{H}^{n-1}(\nu_K^{-1}(\omega)),$$

where $\nu_K : \partial' K \to S^{n-1}$ is the Gauss map of K, defined on $\partial' K$, the set of points of ∂K that have a unique outer unit normal, and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure.

Let $K \in \mathcal{K}^n$ with $o \in K$. The *cone-volume measure* V_K of K is a Borel measure on S^{n-1} defined by

$$dV_K = \frac{1}{n} h_K dS_K.$$

We shall collect the notion of *subspace concentration condition*, which is defined in [6]. It limits how concentrated a measure can be in a subspace.

A finite Borel measure μ on S^{n-1} is said to satisfy the *subspace concentration inequality* if, for every subspace ξ of $\mathbb{R}^n$, such that $0 < \dim\xi < n$,

$$\mu(\xi \cap S^{n-1}) \leq \frac{1}{n}\mu(S^{n-1})\dim\xi. \tag{2.1}$$

The measure is said to satisfy the *subspace concentration condition* if in addition to satisfying the subspace concentration inequality (2.1), whenever

$$\mu(\xi \cap S^{n-1}) = \frac{1}{n}\mu(S^{n-1})\dim\xi,$$

for some subspace ξ, then there exists a subspace ξ', which is complementary to ξ in $\mathbb{R}^n$, so that also

$$\mu(\xi' \cap S^{n-1}) = \frac{1}{n}\mu(S^{n-1})\dim\xi',$$

or equivalently so that μ is concentrated on $S^{n-1} \cap (\xi \cup \xi')$.

The measure μ on S^{n-1} is said to satisfy the *strict subspace concentration inequality* if the inequality in (2.1) is strict for each subspace $\xi \subset \mathbb{R}^n$, such that $0 < \dim\xi < n$.

It was first proved by He, Leng, and Li [27] that the cone-volume measures of o-symmetric polytopes in $\mathbb{R}^n$ satisfy the subspace concentration inequality (2.1), see Xiong [38] for an alternate proof. Böröczky, Lutwak, Yang, and Zhang [6] proved that the subspace concentration condition is both necessary and sufficient for the existence of a solution to the even logarithmic Minkowski problem. Recently, Henk and Linke [28] proved that polytopes in $\mathbb{R}^n$ with centroid at o satisfy the subspace concentration condition.

Suppose $K, L \subset \mathbb{R}^n$ are convex bodies. The *mixed volume* $V_1(K,L)$ of K, L is defined by

$$V_1(K,L) = \frac{1}{n}\lim_{\epsilon\to 0^+}\frac{|K+\epsilon L|-|K|}{\epsilon} = \frac{1}{n}\int_{S^{n-1}} h_L(u)dS_K(u). \tag{2.2}$$

When $n=2$, it is clear that $V_1(K,L) = V_1(L,K)$, and we will write $V(K,L)$ rather than $V_1(K,L)$.

Let K be a convex body in $\mathbb{R}^n$. For $x \in K$, the *extended radial function* $\rho_K(x,z)$ of K is defined by

$$\rho_K(x,z) = \max\{\lambda \ge 0 : x+\lambda z \in K\} \quad \text{for } z \in \mathbb{R}^n \backslash \{0\}.$$

Note that x could be in the boundary of K. Generally, see [20] for the definition of extended radial function of star-shaped set with respect to the point x.

Now we show some basic properties of the dilation position. Let $K, L \in \mathcal{K}^n$ with $o \in K \cap L$. K and L are at a dilation position if they satisfy (1.6). Note that:

(1) dilation position may not be unique, i.e., if K and L are at a dilation position, then a translate of K and a translate of L may also be at a dilation position (e.g., K, L are parallelograms with parallel sides and centered at o);

(2) if K and L are at a dilation position, then K and a dilation of L are also at a dilation position;

(3) for arbitrary convex bodies K and L, they may not be at a dilation position, however, the following is true.

Lemma 2.1. *Let $K, L \in \mathcal{K}^n$.*
(i) There is a translate of L, say $\overline{L}$, and a translate of K, say $\overline{K}$, so that $\overline{K}$ and $\overline{L}$ are at a dilation position.
(ii) If K and L are at a dilation position, then $o \in \text{int}(K\cap L) \cup (\partial K \cap \partial L)$.

Proof. Set $R = R(K,L)$ and $r = r(K,L)$.
(i) If K and L are homothetic, then $R = r$, and there exists a point $t_0 \in \mathbb{R}^n$ such that

$$K = rL + t_0.$$

Choose a $p_0 \in L$. Let $\overline{L} = L - p_0$, and let $\overline{K} = r\overline{L}$. Then we are done.

Assume K and L are not homothetic, then $R > r$. There are points $t_1, t_2 \in \mathbb{R}^n$ so that

$$L_r := rL + t_1 \subset K \subset RL + t_2 =: L_R.$$

Let t' be given by

$$t' := \frac{R}{R-r}t_1 - \frac{r}{R-r}t_2.$$

Let

$$\overline{L} = \frac{1}{r}(L_r - t') \quad \text{and} \quad \overline{K} = K - t'.$$

By a direct computation, we see that $\overline{K}$ and $\overline{L}$ are at a dilation position.

(ii) By the definition, $o \in K \cap L$ and $rL \subset K \subset RL$. Then, there does not exist this case: $o \in \partial K$ but $o \in \text{int}L$. Otherwise, there is a $\delta > 0$ such that $\delta B^n \subset L \subset \frac{1}{r}K$. It follows that $o \in \mathrm{int}K$, a contradiction. Similarly, there does not exist this case: $o \in \partial L$ but $o \in \mathrm{int}K$. Therefore, either $o \in \partial K \cap \partial L$ or $o \in \mathrm{int}K \cap \mathrm{int}L = \mathrm{int}(K \cap L)$. $\square$

Let $K \in \mathcal{K}_o^n$ and $L \in \mathcal{K}^n$. If K and L are at a dilation position, then, by Lemma 2.1, $o \in \mathrm{int}L$, and

$$R(K,L) = \max_{u \in S^{n-1}} \frac{h_K(u)}{h_L(u)} = \max_{u \in S^{n-1}} \frac{\rho_K(u)}{\rho_L(u)}, \tag{2.3}$$

and

$$r(K,L) = \min_{u \in S^{n-1}} \frac{h_K(u)}{h_L(u)} = \min_{u \in S^{n-1}} \frac{\rho_K(u)}{\rho_L(u)}. \tag{2.4}$$

The following lemma shows that $(1-\lambda)\cdot K +_o \lambda \cdot L$ is well defined for $K, L \in \mathcal{K}^n$ that at a dilation position.

Lemma 2.2. *Let $K, L \in \mathcal{K}^n$ with $o \in K \cap L$. Suppose K and L are at a dilation position. Then, for all real $\lambda \in [0,1]$, the geometric Minkowski combination of K and L defined by (1.8) is a convex body.*

Moreover, $(1-\lambda)\cdot K +_o \lambda \cdot L \to K$ as $\lambda \to 0$, and $(1-\lambda)\cdot K +_o \lambda \cdot L \to L$ as $\lambda \to 1$, with respect to the Hausdorff measure.

Proof. Set $r = r(K,L)$, and $R = R(K,L)$. Since $(1-\lambda)\cdot K +_o \lambda \cdot L$ is defined by the intersection of closed and convex sets, it is also closed and convex. It remains to show that $(1-\lambda)\cdot K +_o \lambda \cdot L$ is bounded, and has interior points. Since $rL \subset K \subset RL$, we have $rh_K(u) \le h_L(u) \le Rh_K(u)$ for all $u \in S^{n-1}$. It follows that $r^\lambda h_K(u) \le h_K(u)^{1-\lambda} h_L(u)^\lambda \le R^\lambda h_K(u)$ for all $u \in S^{n-1}$ and $\lambda \in (0,1)$. This and the fact $K = \bigcap_{u \in S^{n-1}} \{x \in \mathbb{R}^n : x \cdot u \le h_K(u)\}$ show that

$$r^\lambda K \subset (1-\lambda)\cdot K +_o \lambda \cdot L \subset R^\lambda K. \tag{2.5}$$

Since $(1-\lambda)\cdot K +_o \lambda \cdot L := K$ for $\lambda = 0$, and $(1-\lambda)\cdot K +_o \lambda \cdot L := L$ for $\lambda = 1$, hence (2.5) holds even for $\lambda \in \{0,1\}$. Therefore $(1-\lambda)\cdot K +_o \lambda \cdot L$ is bounded, and has interior points, for all $\lambda \in [0,1]$.

From (2.5), it is easy to see that $(1-\lambda)\cdot K +_o \lambda \cdot L \to K$ as $\lambda \to 0$. In a similar way, it follows that $(1-\lambda)\cdot K +_o \lambda \cdot L \to L$ as $\lambda \to 1$. $\square$

In this paper, we shall make use of the overgraph and undergraph functions. Let $K \in \mathcal{K}^n$. For $u \in S^{n-1}$, denote by K_u the image of the orthogonal projection of K onto $u^\perp$. Define the *overgraph function* $f(K;x)$ and *undergraph function* $g(K;x)$ of K as follows:

$$K = \{x + tu : -g(K;x) \le t \le f(K;x) \text{ for } x \in K_u\}. \tag{2.6}$$

Then, $f(K;x)$ and $g(K;x)$ are concave on K_u.

3. Proof of Dar's conjecture of dimension 2

In order to prove Theorem 1, we need 7 Lemmas. The relative Bonnesen inequality plays an important role, it states that:

Lemma 3.1. [5] *If $K, L \in \mathcal{K}^2$, then for $r(K,L) \le t \le R(K,L)$,*

$$|K| - 2tV(K,L) + t^2|L| \le 0. \tag{3.1}$$

The inequality is strict whenever $r(K,L) < t < R(K,L)$. When $t = r(K,L)$, equality will occur in (3.1) if and only if K is the Minkowski sum of a dilation of L and a line segment. When $t = R(K,L)$, equality will occur in (3.1) if and only if L is the Minkowski sum of a dilation of K and a line segment.

Bonnesen [4] proved this inequality for $L = B^2$; the proof for the relative case was established by Blaschke [2] and may also be found in [16]. We refer to [5, Lemma 4.1] for a detailed proof. Further study of Bonnesen-type inequalities can be seen in [3, 12, 21, 32].

Lemma 3.2. *Let $K, L \in \mathcal{K}^2$. Suppose that $K \cap L$ has nonempty interior. Then, the set $\partial K \backslash \partial L$ is the union of at most countably many disjoint connected open subsets (with respect to the relative topology in ∂K) of ∂K.*

Proof. Let $p_o \in \mathrm{int}(K \cap L)$. Then, we have

$$\partial K \backslash \partial L = \{x \in \partial K : \rho_L(p_o, x - p_o) \neq 1\}. \tag{3.2}$$

Suppose there exists a point $x_0 \in \partial K \cap \partial L$. Without loss of generality, we assume $\frac{x_0 - p_o}{\|x_0 - p_o\|} = (1, 0)$, where $\|\cdot\|$ denotes the Euclidean norm, and $(\cos\theta, \sin\theta)$ denotes the coordinate of a unit vector. Note that the map $\theta \mapsto (\cos\theta, \sin\theta)$ is a homeomorphism from $[0, 2\pi)$ to S^1. We define the function $g_K(\theta)$ of a planar convex body K by

$$g_K(\theta) := \rho_K(p_o, (\cos\theta, \sin\theta)) \quad \text{for} \quad \theta \in [0, 2\pi). \tag{3.3}$$

It is clear that the map $\theta \mapsto p_o + g_K(\theta)(\cos\theta, \sin\theta)$ is a homeomorphism from $[0, 2\pi)$ to ∂K, and $\theta \mapsto p_o + g_L(\theta)(\cos\theta, \sin\theta)$ is a homeomorphism from $[0, 2\pi)$ to ∂L.

Notice that the set $\{\theta \in [0, 2\pi) : g_K(\theta) \neq g_L(\theta)\}$ is open on $\mathbb{R}^1$, because $g_K(0) = g_L(0)$. By the structure of open sets on a line, the set $\{\theta \in [0, 2\pi) : g_K(\theta) \neq g_L(\theta)\}$ is the union of at most countably many disjoint open intervals (α_i, β_i). It is also easy to see that $g_K(\alpha_i) = g_L(\alpha_i)$ and $g_K(\beta_i) = g_L(\beta_i)$.

Note that $\{x \in \partial K : \rho_L(p_o, x - p_o) \neq 1\}$ is just the image set

$$\{p_o + g_K(\theta)(\cos\theta, \sin\theta) : g_K(\theta) \neq g_L(\theta) \text{ and } \theta \in [0, 2\pi)\},$$

and the map $\theta \mapsto g_K(\theta)(\cos\theta, \sin\theta)$ is a homeomorphism from $[0, 2\pi)$ to ∂K. Thus, we complete the proof of this lemma. □

Now we give the definition of an arc. Let $K, L \in \mathcal{K}^2$. Suppose that $K \cap L$ has nonempty interior. From Lemma 3.2, we have

$$\partial K \backslash \partial L = \bigcup_{i \in I} (\widetilde{a_i b_i})_K.$$

Here I contains at most countably many elements, $(\widetilde{a_i b_i})_K$ are disjoint connected open subsets (with respect to the relative topology in ∂K) of ∂K, and $a_i, b_i \in \partial K \cap \partial L$ are endpoints of $(\widetilde{a_i b_i})_K$. Note $a_i, b_i \notin (\widetilde{a_i b_i})_K$. We call $(\widetilde{a_i b_i})_K$ *an arc on ∂K with respect to L* (or simply *arc*), for $i \in I$. The arc $(\widetilde{a_i b_i})_K$ is precisely the boundary part of K

from a_i to b_i counterclockwise. In addition, $(\widetilde{a_ib_i})_L$, $i \in I$, are precisely all the arcs on ∂L with respect to K.

Note that $\partial K\backslash\partial L$ is the union of $(\partial K\backslash L)$ and $(\partial K\cap \mathrm{int}L)$, two open subsets (with respect to the relative topology in ∂K) of ∂K. Thus, an arc $(\widetilde{a_ib_i})_K$ is either contained in $(\partial K\backslash L)$ or contained in $(\partial K\cap \mathrm{int}L)$.

Lemma 3.3. *Let $K, L \in \mathcal{K}^2$. Suppose $L_r = r(K,L)L + t_1 \subset K$ and $L_r \neq K$, where $t_1 \in \mathbb{R}^2$. Let $(\widetilde{ab})_K \subset \partial K\backslash L_r$ be an arc on ∂K with respect to L_r. Suppose that $(\widetilde{ab})_K$ is contained in $l^-(ab)$. Then, the arc $(\widetilde{ab})_K$ satisfies the following property* $(\mathbf{P})$:
$(\mathbf{P})$: *there are two support lines: l_1 support K at a, and l_2 support K at b, such that $l_1 \cap l_2 \subset \mathrm{int}l^-(ab)$.*

Proof. At the beginning, we will give an equivalent statement of property $(\mathbf{P})$.

Set $u = (a-b)/\|a-b\|$. Let v be the unit vector orthogonal to u and such that $l^+(ab) = l^+_{v,\alpha}$ for some α. Choose a Cartesian system, such that v is the positive direction of e_1-axis, and u is the positive direction of e_2-axis. Without loss of generality, suppose $v = (1,0)$ and $u = (0,1)$.

For this u, let the overgraph and undergraph functions $f(K;x)$, $g(K;x)$ and $f(L_r;x)$, $g(L_r;x)$ be defined by (2.6). Let $[s_K, t_K]$ denote the projection of K on e_1-axis, and let $[s_{L_r}, t_{L_r}]$ denote the projection of L_r on e_1-axis. Then $f(K;x)$ and $g(K;x)$ are concave on $[s_K, t_K]$, and $f(L_r;x)$, $g(L_r;x)$ are concave on $[s_{L_r}, t_{L_r}]$. Here x should be understood as a coordinate as well as a point on the e_1-axis.

Denote by $f'_-(K;\cdot)$ and $g'_-(K;\cdot)$ the left derivatives of $f(K;\cdot)$ and $g(K;\cdot)$.

Note the following facts:

(1) if $l_1//l_2$ (i.e., with opposite outer normal vectors), then $l_1\cap l_2 = \emptyset \subset \mathrm{int}l^-(ab)$;

(2) a line l is tangent to the graph of $f(K;x)$ at $(0, f(K;0))$ if and only if l supports K at a, and a line l' is tangent to the graph of $-g(K;x)$ at $(0,-g(K;0))$ if and only if l' supports K at b;

(3) let λ_1 be the slope of a tangent line of the graph of $f(K;x)$ at $(0, f(K;0))$, and λ_2 be the slope of a tangent line of the graph of $-g(K;x)$ at $(0,-g(K;0))$, then

$$f'_+(K;0) \le \lambda_1 \le f'_-(K;0), \qquad -g'_-(K;0) \le \lambda_2 \le -g'_+(K;0).$$

From the facts above it is easy to see that property $(\mathbf{P})$ is equivalent to

$$f'_-(K;0) + g'_-(K;0) \ge 0. \tag{3.4}$$

To prove this lemma, we suppose the contrary, i.e.,

$$f'_-(K;0) + g'_-(K;0) < 0.$$

Since $f'_-(K;x)$ and $g'_-(K;x)$ are left-continuous, there exists a constant $\delta > 0$, such that

$$f'_-(K;x) + g'_-(K;x) < 0,$$

for all $x \in [-\delta, 0]$. Let $c_1 = f'_-(K;-\delta) + g'_-(K;-\delta) < 0$, $c_2 = f'_-(K;-\delta)$, and $c_3 = g'_-(K;-\delta)$. Then, by the concavity of $f(K;\cdot)$ and $g(K;\cdot)$, we have

$$f(K;x-\epsilon) \ge f(K;x) - c_2\epsilon, \quad \text{and} \quad g(K;x-\epsilon) \ge g(K;x) - c_3\epsilon, \tag{3.5}$$

for all $x \in [-\frac{\delta}{2}, t_{L_r}]$ and $\epsilon < \frac{\delta}{2}$.

Since $\widetilde{(ab)}_K \subset \partial K \backslash L$, and the functions $f(K;\cdot)$, $g(K;\cdot)$, $f(L_r;\cdot)$, and $g(L_r;\cdot)$ are continuous, there exist $m_1, m_2 > 0$ such that

$$f(K; x-\epsilon) - f(L_r; x) \geq m_1 > 0, \quad \text{and} \quad g(K; x-\epsilon) - g(L_r; x) \geq m_1 > 0, \tag{3.6}$$

for all $x \in [s_{L_r}, -\frac{\delta}{2}]$ and $\epsilon < m_2$. Note: $\widetilde{(ab)}_K \subset \partial K \backslash L$ implies $s_{L_r} > s_K$, and hence we can choose m_2 sufficiently small so that $x - \epsilon \in [s_K, t_K]$.

Note that $c_2 + c_3 = c_1 < 0$. For $0 < \epsilon < \min\{\frac{\delta}{2}, \frac{m_1}{|c_3| - c_1}, m_2\}$, let η be such that $0 < \eta < -c_1\epsilon$. Then, by (3.5) and (3.6), we have

$$f(K; x-\epsilon) \geq f(K;x) - c_1\epsilon + c_3\epsilon > f(K;x) + c_3\epsilon + \eta \geq f(L_r;x) + c_3\epsilon + \eta,$$
$$g(K; x-\epsilon) \geq g(K;x) - c_3\epsilon > g(L_r;x) - c_3\epsilon - \eta,$$

for all $x \in [-\frac{1}{2}\delta, t_{L_r}]$; and

$$f(K; x-\epsilon) \geq f(L_r;x) + c_3\epsilon + m_1 - c_3\epsilon > f(L_r;x) + c_3\epsilon + \eta,$$
$$g(K; x-\epsilon) \geq g(L_r;x) - c_3\epsilon + m_1 + c_3\epsilon > g(L_r;x) - c_3\epsilon - \eta,$$

for all $x \in [s_{L_r}, -\frac{\delta}{2}]$.

Note that $f(K; x-\epsilon) = f(K+\epsilon v; x)$, $g(K; x-\epsilon) = g(K+\epsilon v; x)$, $f(L_r;x) + c_3\epsilon + \eta = f(L_r + (c_3\epsilon+\eta)u; x)$, and $g(L_r;x) - c_3\epsilon - \eta = g(L_r + (c_3\epsilon+\eta)u; x)$. Then, the body $L_1 = L_r - \epsilon v + (c_3\epsilon + \eta)u$ is contained in the interior of K. This leads to a contradiction, since a larger homothetic copy of L_1 will be also contained in K.

Therefore, we get (3.4). This means that the arc $\widetilde{(ab)}_K$ satisfies property (**P**). □

Lemma 3.4. *Let $K, L \in \mathcal{K}^2$ satisfy $o \in K \cap L$ and $r(K,L) < 1$. Suppose K and L are at a dilation position. Let $\widetilde{(ab)}_K \subset \partial K \backslash L$ be an arc on ∂K with respect to L. Suppose that $\widetilde{(ab)}_K$ is contained in $l^-(ab)$. Then, $\widetilde{(ab)}_K$ satisfies property* (**P**).

Proof. Since K and L are at a dilation position, hence

$$L_r := r(K,L)L \subset K.$$

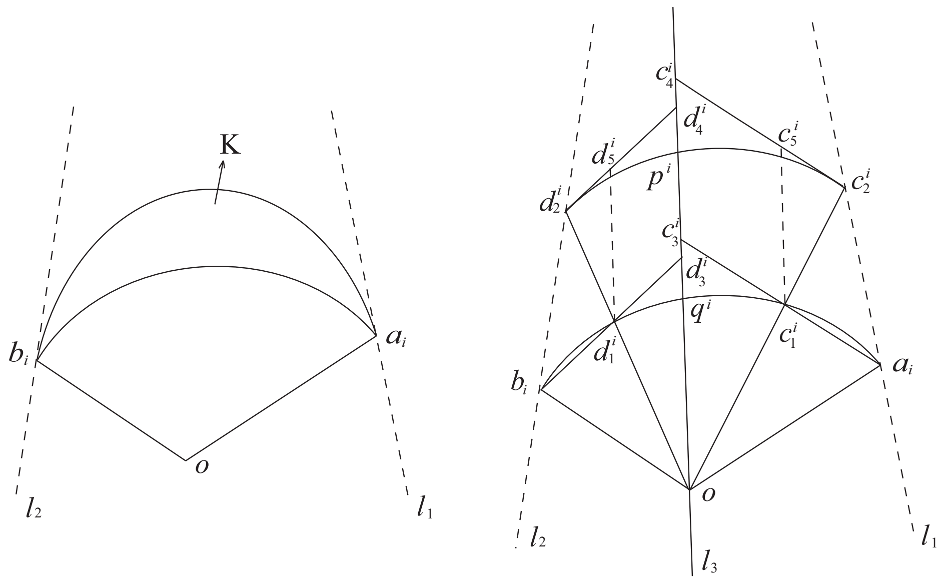
By the assumptions $\widetilde{(ab)}_K \subset \partial K \backslash L$ and $r(K,L) < 1$, we see that $\widetilde{(ab)}_K \subset \partial K \backslash L_r$. Since $\widetilde{(ab)}_K$ is a connected open subset (with respect to the relative topology in ∂K) of ∂K, there exists an arc $\widetilde{A}$ on ∂K with respect to L_r, such that $\widetilde{(ab)}_K \subset \widetilde{A}$. By Lemma 3.3, $\widetilde{A}$ satisfies property (**P**). It follows from the convexity of K that the arc $\widetilde{(ab)}_K$ must satisfy property (**P**), too. □

Let $K, L \in \mathcal{K}^2$ with $o \in K \cap L$. Suppose K and L are at a dilation position. Then $K \cap L$ has nonempty interior. Denote the arcs on ∂K with respect to L by $\widetilde{(a_i b_i)}_K$, $i \in I$, where I contains at most countably many elements. For $i \in I$, we define the *branch B_i^K of K with respect to the arc $\widetilde{(a_i b_i)}_K$* by

$$B_i^K := \{\lambda x : x \in \mathrm{cl}\widetilde{(ab)}_K \text{ and } 0 \leq \lambda \leq \rho_K(x)\}.$$

We also define $C(K,L)$ by

$$C(K,L) := \{\lambda x : x \in \partial K \cap \partial L \text{ and } 0 \leq \lambda \leq \rho_K(x)\}. \tag{3.7}$$



(a) (b)

Figure 1. The Branches B_i^K and B_i^L in Lemma 3.5.

Since I contains at most countably many elements, we have

$$\sum_{i\in I} |B_i^K| + |C(K,L)| = |K|, \tag{3.8}$$

and

$$\sum_{i\in I} |B_i^L| + |C(K,L)| = |L|. \tag{3.9}$$

The following lemma is crucial in the proof of Theorem 1.

Lemma 3.5. *Let $K, L \in \mathcal{K}^2$ satisfy $o \in K \cap L$ and $r(K,L) < 1 < R(K,L)$. Suppose K and L are at a dilation position. Let $(\widetilde{a_ib_i})_K \subset \partial K \backslash L$ and $(\widetilde{a_ib_i})_L \subset \partial L \cap \mathrm{int}K$ be arcs. Denote by B_i^K the branch of K with respect to $(\widetilde{a_ib_i})_K$, and by B_i^L the branch of L with respect to $(\widetilde{a_ib_i})_L$. Then, we have*

$$(2R(K,L) - 1)|B_i^L| \geq |B_i^K|. \tag{3.10}$$

*Suppose $(\widetilde{a_ib_i})_K \subset l^-(a_ib_i)$. If equality holds in (3.10), then there are parallel (i.e., with opposite outer normal vectors) support lines of K at a_i and b_i, and there are no other support lines of K at a_i and b_i satisfying property (**P**).*

Proof. Set $R = R(K,L)$. Suppose that $(\widetilde{a_ib_i})_K$ is contained in $l^-(a_ib_i)$. Then, by Lemma 3.4, there are two support lines: l_1 support K at a_i, and l_2 support K at b_i, such that $l_1 \cap l_2 \subset \mathrm{int}l^-(a_ib_i)$. The lines l_1 and l_2 are either parallel or meeting at a point $s \in l^-(a_ib_i)$. Let l_3 be such that $o \in l_3$ and $l_3//l_1//l_2$ in the first case, and let $l_3 = l(os)$ in the second case. See Figure 1 for details. Note: our proof is feasible even for the case $\angle a_iob_i \geq \pi$.

Suppose $K \subset l_1^- \cap l_2^-$. Set $\widetilde{c_2^i d_2^i} = (R \cdot (\widetilde{a_i b_i})_L) \cap l_1^- \cap l_2^-$, where $c_2^i \in l_1, d_2^i \in l_2$. Let $\widetilde{c_1^i d_1^i} = \frac{1}{R}\widetilde{c_2^i d_2^i}$, then $\widetilde{c_1^i d_1^i} \subset (\widetilde{a_i b_i})_L$. Define $E(\widetilde{c_2^i d_2^i})$ and $E(\widetilde{c_1^i d_1^i})$ as follows:

$$E(\widetilde{c_2^i d_2^i}) = \{\lambda x : x \in \mathrm{cl}\widetilde{c_2^i d_2^i} \text{ and } \lambda \in [0,1]\},$$

$$E(\widetilde{c_1^i d_1^i}) = \{\lambda x : x \in \mathrm{cl}\widetilde{c_1^i d_1^i} \text{ and } \lambda \in [0,1]\}.$$

There are points $c_4^i, d_4^i, c_3^i, d_3^i \in l_3$ such that $c_1^i \in (a_i, c_3^i)$, $d_1^i \in (b_i, d_3^i)$, $[c_2^i, c_4^i]//[c_1^i, c_3^i]$, and $[d_2^i, d_4^i]//[d_1^i, d_3^i]$. By the convexity of K and L, it is clear that

$$R \cdot E(\widetilde{c_1^i d_1^i}) = E(\widetilde{c_2^i d_2^i}); \tag{3.11}$$

$$R \cdot [o, c_1^i, c_3^i] = [o, c_2^i, c_4^i], \quad R \cdot [o, d_1^i, d_3^i] = [o, d_2^i, d_4^i]; \tag{3.12}$$

$$E(\widetilde{c_1^i d_1^i}) \cup [o, a_i, c_1^i] \cup [o, b_i, d_1^i] \subset B_i^L; \tag{3.13}$$

$$B_i^K \subset E(\widetilde{c_2^i d_2^i}) \cup [o, a_i, c_2^i] \cup [o, b_i, d_2^i]; \tag{3.14}$$

$$E(\widetilde{c_1^i d_1^i}) \subset [o, c_1^i, c_3^i] \cup [o, d_1^i, d_3^i]; \tag{3.15}$$

$$E(\widetilde{c_2^i d_2^i}) \subset [o, c_2^i, c_4^i] \cup [o, d_2^i, d_4^i]. \tag{3.16}$$

Set $V_1 = |[o, a_i, c_1^i]|$, $V_2 = |[o, b_i, d_1^i]|$, $V_3 = |E(\widetilde{c_1^i d_1^i})|$. By (3.11), (3.12), (3.13), and (3.14), to prove (3.10), it suffices to prove

$$(2R-1)(V_1 + V_2 + V_3) \geq R^2 V_3 + |[o, a_i, c_2^i]| + |[o, b_i, d_2^i]|.$$

Since $\|c_2^i\| = R\|c_1^i\|$ and $\|d_2^i\| = R\|d_1^i\|$, it suffices to show

$$(R-1)(V_1 + V_2) \geq (R-1)^2 V_3. \tag{3.17}$$

Let $c_5^i \in [c_2^i, c_4^i]$ and $d_5^i \in [d_2^i, d_4^i]$ be such that $[c_1^i, c_5^i]//[d_1^i, d_5^i]//l_3$. $(R-1)V_1$ is just the area of $[a_i, c_1^i, c_2^i]$, $(R-1)V_2$ is the area of $[b_i, d_1^i, d_2^i]$. By (3.15) and (3.16), $(R-1)^2 V_3$ is less than or equal to the sum of $|[c_2^i, c_1^i, c_5^i]|$ and $|[d_2^i, d_1^i, d_5^i]|$. Recall that l_1, l_2 and l_3 are either parallel or meeting at a common point $s \in l^-(a_i b_i)$. Thus, we will deduce that

$$|[c_2^i, c_1^i, c_5^i]| \leq |[c_1^i, a_i, c_2^i]|, \tag{3.18}$$

and

$$|[d_2^i, d_1^i, d_5^i]| \leq |[d_1^i, b_i, d_2^i]|. \tag{3.19}$$

In fact, if $l_1//l_2//l_3$, then $[a_i, c_2^i, c_5^i, c_1^i]$ and $[b_i, d_2^i, d_5^i, d_1^i]$ are parallelograms, and equalities hold in (3.18) and (3.19). If l_1, l_2 and l_3 meet at an $s \in l^-(a_i b_i)$, then $\|c_2^i - c_5^i\| < \|a_i - c_1^i\|$ and $\|d_2^i - d_5^i\| < \|b_i - d_1^i\|$, and the inequalities (3.18) and (3.19) are strict.

Thus, (3.17) holds, and (3.10) is established. If equality holds in (3.10), then (3.18) and (3.19) must be equalities, which implies $l_3//l_1$ and $l_3//l_2$, and there are no other support lines of K at a_i and b_i satisfying property (**P**). Therefore, we complete the proof of this lemma. □

To establish the equality condition, we need the following 2 lemmas.

Lemma 3.6. *Let $K \in \mathcal{K}^2$. Suppose $a_1, a_2, a_3, a_4 \in \partial K$ are distinctive, and they locate counterclockwise on ∂K. If there is a pair of parallel support lines (i.e., with opposite outer normal vectors) of K at a_1, a_2, and there is a pair of parallel support lines of K at a_3, a_4, then $[a_1, a_4]$, $[a_2, a_3] \subset \partial K$.*

Proof. Denote the outer normal vectors of these support lines of K at a_1, a_2, a_3, a_4 by $(\cos\theta_1, \sin\theta_1)$, $(\cos\theta_2, \sin\theta_2)$, $(\cos\theta_3, \sin\theta_3)$, $(\cos\theta_4, \sin\theta_4)$ respectively.

Since a_1, a_2, b_1, b_2 are distinctive and locate counterclockwise on the boundary of the planar convex body K, we can assume

$$0 \le \theta_1 \le \theta_2 \le \theta_3 \le \theta_4 \le 2\pi. \tag{3.20}$$

Since $(\cos\theta_1, \sin\theta_1)$ and $(\cos\theta_2, \sin\theta_2)$ are opposite, we have $\theta_2 = \pi + \theta_1$. Similarly, $\theta_4 = \pi + \theta_3$. Therefore, the inequality (3.20) becomes

$$0 \le \theta_1 \le \pi + \theta_1 \le \theta_3 \le \pi + \theta_3 \le 2\pi,$$

which implies

$$\theta_1 = 0, \quad \theta_2 = \pi = \theta_3, \quad \theta_4 = 2\pi.$$

Thus, by the convexity of K, we get the desired result. □

Lemma 3.7. [31] *Let K_1 and K_2 be two convex bodies in $\mathbb{R}^n$ and $u \in S^{n-1}$. For $y \in P_u(K_i), i = 1, 2,$ write*

$$\phi_i^+(y) = \max\{t : tu + y \in K_i\},$$
$$\phi_i^-(y) = \min\{t : tu + y \in K_i\},$$

and

$$f(r) = |K_1 \cap (ru + K_2)|,$$

where $P_u(K_i)$ denotes the projection of K_i onto $u^\perp$. Then, we have

$$f'_+(0) = \mathcal{H}^{n-1}(C_u^+(1,2)) - \mathcal{H}^{n-1}(C_u^-(2,1)),$$
$$f'_-(0) = \mathcal{H}^{n-1}(C_u^-(1,2)) - \mathcal{H}^{n-1}(C_u^+(2,1)),$$

where

$$C_u^+(1,2) = P_u(K_1 \cap K_2) \cap \{\phi_1^+ > \phi_2^+ \ge \phi_1^- > \phi_2^-\},$$
$$C_u^-(1,2) = P_u(K_1 \cap K_2) \cap \{\phi_1^+ \ge \phi_2^+ > \phi_1^- \ge \phi_2^-\},$$

and $C_u^\pm(2,1)$ are defined analogously.

Proof of Theorem 1. Set $R_1 = R(K, L)$, and $R_2 = R(L, K)$. If $R_1 \le 1$ or $R_2 \le 1$, then $M(K, L) = \min\{|K|, |L|\}$, and (1.3) is just the classical Brunn-Minkowski inequality (1.1). In this case, equality holds in (1.3) if and only if the condition (ii) holds.

In the following, we may assume $R_1, R_2 > 1$. We claim that either $R_1 M(K, L) \ge |K|$ or $R_2 M(K, L) \ge |L|$.

By Lemma 2.1, we can assume without loss of generality that K and L are at a dilation position. Denote the arcs on ∂K with respect to L by $(\widetilde{a_i b_i})_K$, $i \in I$, where I contains at most countably many elements. Denote the branches of K with respect to the arc $(\widetilde{a_i b_i})_K$ by B_i^K, and the branches of L with respect to the arc $(\widetilde{a_i b_i})_L$ by

B_i^L. Let $C(K,L)$ be defined by (3.7). Note that $(\widetilde{a_ib_i})_K \subset \partial K \backslash L$ is equivalent to $B_i^L \subset B_i^K$. We define index sets I_1 and I_2 as follows:

$$I_1 = \{i : B_i^L \subset B_i^K\};$$
$$I_2 = \{j : B_j^K \subset B_j^L\}.$$

By Lemma 3.5, we have

$$(2R_1 - 1)\sum_{i\in I_1} |B_i^L| \geq \sum_{i\in I_1} |B_i^K|,$$

and

$$(2R_2 - 1)\sum_{j\in I_2} |B_j^K| \geq \sum_{j\in I_2} |B_j^L|.$$

If $\sum\limits_{i\in I_1} |B_i^L| \geq \sum\limits_{j\in I_2} |B_j^K|$, then

$$R_2\sum_{j\in I_2} |B_j^K| + (R_2 - 1)\sum_{i\in I_1} |B_i^L| \geq \sum_{j\in I_2} |B_j^L|. \tag{3.21}$$

Since

$$|K\cap L| = \sum_{i\in I_1} |B_i^L| + \sum_{j\in I_2} |B_j^K| + |C(K,L)|,$$

by (3.9), (3.21), and $R_2 > 1$, we get

$$R_2 M(K,L) \geq R_2|K\cap L| = R_2(\sum_{i\in I_1} |B_i^L| + \sum_{j\in I_2} |B_j^K| + |C(K,L)|) \geq |L|.$$

In a similar way, if $\sum\limits_{j\in I_2} |B_j^K| \geq \sum\limits_{i\in I_1} |B_i^L|$, then we get

$$R_1 M(K,L) \geq |K|.$$

Therefore, we have proved either $R_1M(K,L) \geq |K|$ or $R_2M(K,L) \geq |L|$. Note that $r(K,L) = \frac{1}{R_2}$, and we have assumed $R_1, R_2 > 1$. Then, either $\frac{|K|}{M(K,L)} \in [r(K,L), R(K,L)]$ or $\frac{M(K,L)}{|L|} \in [r(K,L), R(K,L)]$. Substituting $t = \frac{|K|}{M(K,L)}$ or $t = \frac{M(K,L)}{|L|}$ in (3.1), we obtain

$$2V(K,L) \geq M(K,L) + \frac{|K||L|}{M(K,L)}. \tag{3.22}$$

By the arithmetic-geometric mean inequality, we have

$$|K| + |L| \geq 2|K|^{\frac{1}{2}}|L|^{\frac{1}{2}}. \tag{3.23}$$

This together with (3.22) and the fact $|K + L| = |K| + 2V(K,L) + |L|$, give (1.3).

Now we turn to the equality condition. Note that we have assumed $R_1, R_2 > 1$, which implies K and L are not homethetic. When K and L satisfy condition (i) in Theorem 1, it is easy to verify that equality holds in (1.3).

Conversely, suppose equality holds in (1.3). Since (1.3) is established by using (3.23) and (3.1), then $|K| = |L|$, and either $|K|/M(K,L) = R_1$ or $|L|/M(K,L) = R_2$. From the proof above, this implies that equality holds in (3.10) for all branches B_i^K and B_j^L, $i \in I_1$ and $j \in I_2$. By Lemma 3.5, there are parallel support lines of K

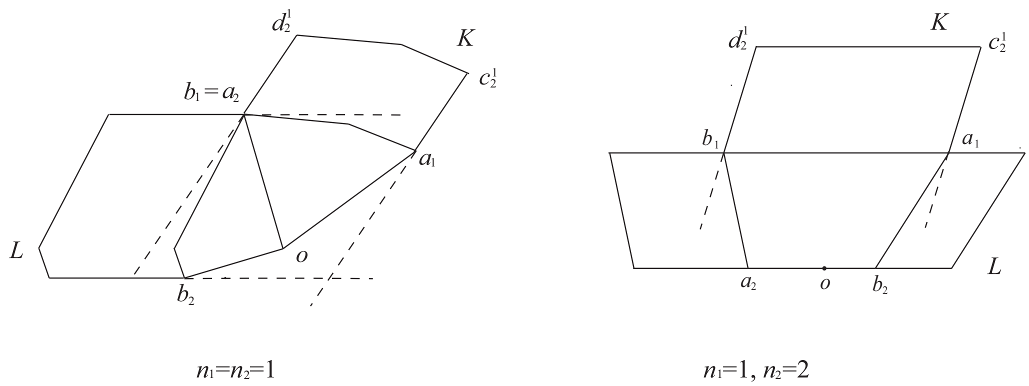


Figure 2. The cases $n_1 = n_2 = 1$ and $n_1 = 1, n_2 = 2$.

at a_i, b_i for $i \in I_1$, and parallel support lines of L at a_j, b_j for $j \in I_2$. Moreover, $M(K, L) = |K \cap L|$ and $|C(K, L)| = 0$.

Let n_1 be the number (the finiteness can be seen in the following) of the arcs contained in $\partial K \backslash L$, and n_2 be the number of arcs contained in $\partial L \backslash K$. Then, $n_1, n_2 \geq 1$. Otherwise, we will get $L \subset K$ or $K \subset L$, a contradiction. If $n_1 \geq 3$, then it follows from $n_2 \geq 1$ that there are arcs $(\widetilde{a_1b_1})_K$ and $(\widetilde{a_2b_2})_K$ contained in $\partial K \backslash L$, so that a_1, b_1, a_2, b_2 are distinctive. Suppose a_1, b_1, a_2, b_2 locate counterclockwise on ∂K. Then, it follows from Lemma 3.6 that $[b_1, a_2], [a_1, b_2] \subset \partial K$. Then, there will not be any other arcs contained in $\partial K \backslash L$ except $(\widetilde{a_1b_1})_K$ and $(\widetilde{a_2b_2})_K$, a contradiction. So $n_1 \leq 2$. In a similar way, we deduce $1 \leq n_2 \leq 2$, too. Therefore, there are only 4 cases: $n_1 = n_2 = 2$; $n_1 = n_2 = 1$; $n_1 = 1$ and $n_2 = 2$; $n_1 = 2$ and $n_2 = 1$.

When $n_1 = n_2 = 2$, suppose

$$(\widetilde{a_1b_1})_K, (\widetilde{a_2b_2})_K \subset \partial K \backslash L, \quad \text{and} \quad (\widetilde{a_3b_3})_L, (\widetilde{a_4b_4})_L \subset \partial L \backslash K.$$

If a_1, b_1, a_2, b_2 are not distinctive, assume $b_1 = a_2, a_1 \neq b_2$. By the necessary condition of Lemma 3.5 for $(\widetilde{a_1b_1})_K, (\widetilde{a_2b_2})_K$ respectively, it must be the case that: there is a unique support line of K at $b_1 = a_2$, say l_1, and there is a common support line of K through a_1 and b_2 parallel to l_1. Then, $[a_1, b_2] \subset \partial K$, and there are no more than 1 arc contained in $\partial L \backslash K$, a contradiction. Thus, a_1, b_1, a_2, b_2 are distinctive, and a_3, b_3, a_4, b_4 are distinctive too. From Lemma 3.6, it follows that $K \cap L$ is a parallelogram, and the arcs $(\widetilde{a_1b_1})_L, (\widetilde{a_2b_2})_L, (\widetilde{a_3b_3})_K, (\widetilde{a_4b_4})_K$ are all line segments. Furthermore, equality in (3.17) implies that (3.14) and (3.16) are also equalities for $i = 1, 2, 3, 4$. Then, K and L must be parallelograms with parallel sides in this case.

When $n_1 = n_2 = 1$, suppose $(\widetilde{a_1b_1})_K \subset \partial K \backslash L$ and $(\widetilde{a_2b_2})_L \subset \partial L \backslash K$. Then, we have

$$\partial K \backslash ((\widetilde{a_1b_1})_K \cup (\widetilde{a_2b_2})_K) \subset \partial K \cap \partial L.$$

These two arcs must have a common endpoint, and we suppose $b_1 = a_2$. Otherwise, we will get $|C(K, L)| > 0$, a contradiction. $|C(K, L)| = 0$ also implies that $[o, a_1]$, $[o, b_2] \subset \partial K \cap \partial L$. Consider the branch B_1^K, and use the same notation c_2^1, d_2^1 as in Lemma 3.5 (let $i = 1$). Equality in (3.17) implies that (3.13), (3.14), (3.15) and (3.16) are all equalities. Then, $(\widetilde{a_1b_1})_L$ is either a line segment or the union of two line segments, and so is $(\widetilde{a_2b_2})_K$. Let $v_1 = (a_1 - c_2^1)/\|a_1 - c_2^1\|$, and substitute $K_1 = L$, $K_2 = K$ into Lemma 3.7. The existence of parallel support lines at a_1, b_1 implies $P_{v_1}(K \cap L) = P_{v_1}[a_1, b_1]$. Since $[o, b_2] \subset \partial K \cap \partial L$, we have $P_{v_1}[a_1, b_1] \cap \{\phi_2^- \geq \phi_1^-\} = \emptyset$. Thus, $\mathcal{H}^1(C_{v_1}^-(2,1)) = 0$. Since $M(K, L) = |K \cap L|$, by Lemma 3.7, it must be $\mathcal{H}^1(C_{v_1}^+(1,2)) = 0$. Note that $\{\phi_2^+ \geq \phi_1^- > \phi_2^-\} = P_{v_1}[a_1, b_1]$. Then, $C_{v_1}^+(1,2) = P_{v_1}[a_1, b_1] \cap \{\phi_1^+ > \phi_2^+\}$, and hence $\mathcal{H}^1(C_{v_1}^+(1,2)) = 0$ if and only if $(\widetilde{a_2b_2})_K$ is the line segment $[a_2, b_2]$. Similarly, we deduce that $(\widetilde{a_1b_1})_L$ is the line segment $[a_1, b_1]$. If $[o, a_1]$ is not parallel to $[a_2, b_2]$, let $v_2 = -a_1/\|a_1\|$. By a direct computation as above, we will get $\mathcal{H}^1(C_{v_2}^-(2,1)) = 0$, and $\mathcal{H}^1(C_{v_2}^+(1,2)) > 0$, which is contradict to $M(K, L) = |K \cap L|$. Thus, $[o, a_1]//[a_2, b_2]$. Similarly, $[o, b_2]//[a_1, b_1]$. Therefore, $K \cap L$ is a parallelogram. Equality in (3.17) implies that (3.14) and (3.16) are also equalities for $i = 1, 2$. Therefore, K and L are parallelograms with parallel sides in this case.

When $n_1 = 1, n_2 = 2$, suppose the 3 arcs are $(\widetilde{a_1b_1})_K \subset \partial K \backslash L$, and $(\widetilde{b_1a_2})_L$, $(\widetilde{b_2a_1})_L \subset \partial L \backslash K$. If a_1, b_1, a_2, b_2 are not distinctive, assume without loss of generality that $a_2 = b_2, a_1 \neq b_1$. By the necessary condition of Lemma 3.5 (consider the 2 branches of L), it must be the case that: there is a unique support line of K at $a_2 = b_2$, say l', there is a common support line of K through a_1 and b_1 parallel to l'. Then, $[a_1, b_1]$, $[a_2, b_2] \subset \partial L$, here $[b_1, a_2]$ should be seen as a degenerate line segment. If a_1, b_1, a_2, b_2 are distinctive, by Lemma 3.6, $[a_1, b_1]$, $[a_2, b_2] \subset \partial L$, and $(\widetilde{a_1b_1})_L$ is a line segment. Consider the branch B_1^K, and use the same notation c_2^1, d_2^1 as in Lemma 3.5 (for $i = 1$). Let $v_3 = (a_1 - c_2^1)/\|a_1 - c_2^1\|$. By computing $\mathcal{H}^1(C_{v_3}^+(1,2))$, $\mathcal{H}^1(C_{v_3}^-(2,1))$, and using Lemma 3.7, a similar way as in the case $n_1 = n_2 = 1$, we deduce that K and L must be parallelograms with parallel sides.

The case $n_1 = 2, n_2 = 1$ is similar to the case $n_1 = 1, n_2 = 2$. Therefore, we complete the proof of Theorem 1. □

4. Connection of Dar's conjecture and the log-Brunn-Minkowski inequality

From Theorem 1, we find that the equality condition of Dar's conjecture coincides with the log-Brunn-Minkowski inequality's. Actually, we have the following proposition.

Proposition 4.1. *Let $K, L \in \mathcal{K}^2$ with $o \in K \cap L$ and $|K| = |L|$. If K and L are at a dilation position, then $V_K = V_L$ if and only if*

$$|K + L|^{\frac{1}{2}} = M(K, L)^{\frac{1}{2}} + \frac{|K|^{\frac{1}{2}}|L|^{\frac{1}{2}}}{M(K, L)^{\frac{1}{2}}}. \tag{4.1}$$

We will show this by establishing Lemma 4.2, which is an extension of [5, Lemma 5.1]. However, the equality case needs different steps.

Lemma 4.2. *Let $K, L \in \mathcal{K}^2$ with $o \in K \cap L$. Suppose K and L are at a dilation position. Then,*

$$\int_{S^1} \frac{h_K}{h_L} dV_K \le \frac{1}{2} \cdot \frac{|K|}{|L|} \int_{S^1} h_L dS_K, \tag{4.2}$$

with equality if and only if K and L are dilates, or K and L are parallelograms with parallel sides.

Note that the set $\{h_K = 0\} = \{h_L = 0\}$ is of measure 0, with respect to the measure V_K. Thus, the integral in (4.2) is well-defined.

Proof. By Lemma 2.1, either $o \in \mathrm{int}(K \cap L)$ or $o \in \partial K \cap \partial L$. We will consider these cases simultaneously.

Since $r(K, L)L \subset K \subset R(K, L)L$, we see that $h_K(u) = 0$ if and only if $h_L(u) = 0$. Define the set ω by

$$\omega := \{u \in S^1 : h_K(u) = 0\} = \{u \in S^1 : h_L(u) = 0\}.$$

Then, we have

$$r(K, L) \le \frac{h_K(u)}{h_L(u)} \le R(K, L),$$

for all $u \in S^1 \backslash \omega$. Thus, by Lemma 3.1, for $u \in S^1 \backslash \omega$, we get

$$|K| - 2\frac{h_K(u)}{h_L(u)} V(K, L) + \Big(\frac{h_K(u)}{h_L(u)}\Big)^2 |L| \le 0.$$

Integrating both sides of this, with respect to the measure $h_L dS_K$, noticing that the set ω is of measure 0 (whenever the respective measure is $h_L dS_K$ or dV_K), we obtain

$$\begin{aligned} 0 &\ge \int_{S^1} \Big(|K| - 2\frac{h_K(u)}{h_L(u)} V(K, L) + \Big(\frac{h_K(u)}{h_L(u)}\Big)^2 |L|\Big) h_L(u) dS_K(u) \\ &= -2|K|V(K, L) + 2|L| \int_{S^1} \frac{h_K(u)}{h_L(u)} dV_K(u), \end{aligned}$$

which implies (4.2).

If K and L are dilates or parallelograms with parallel sides, then it is easy to see that equality holds in (4.2).

Now suppose equality holds in (4.2). Then,

$$|K| - 2\frac{h_K(u)}{h_L(u)} V(K, L) + \Big(\frac{h_K(u)}{h_L(u)}\Big)^2 |L| = 0, \quad \text{for all } u \in \mathrm{supp} S_K \backslash \omega. \tag{4.3}$$

By Lemma 3.1, we have

$$\frac{h_K(u)}{h_L(u)} \in \{r(K, L), R(K, L)\} \quad \text{for all } u \in \mathrm{supp} S_K \backslash \omega. \tag{4.4}$$

Since K is a convex body, ω must be contained in an open subset of a half-sphere, and $\mathrm{supp} S_K$ cannot be concentrated on a half-sphere. Then $\mathrm{supp} S_K \backslash \omega \neq \emptyset$. Without loss of generality, we may assume that there exists a $u_0 \in \mathrm{supp} S_K \backslash \omega$, such that $h_K(u_0) = r(K, L) h_L(u_0)$. From (4.3) and the equality conditions of Lemma 3.1 we conclude that K must be a dilation of the Minkowski sum of L and a line segment.

Thus, $K = sL + I_1$ with $s > 0$, where I_1 is a line segment. In fact, it can be seen from the proof of Lemma 3.1 or from the following discussion that $s = r(K, L)$.

If $s > r(K, L)$, then a larger homothetic copy of L will be contained in K, a contradiction. Thus, $s \leq r(K, L)$. Since $r(K, L)L \subset K$, we have

$$sh_L(u) + h_{I_1}(u) = h_K(u) \geq r(K, L)h_L(u) \quad \text{for all } u \in S^1.$$

Then,

$$h_{I_1}(u) \geq (r(K, L) - s)h_L(u) \geq 0 \quad \text{for all } u \in S^1.$$

Thus, $o \in I_1$. If $s < r(K, L)$, then $h_{I_1} > 0$, which is impossible because I_1 is a line segment. Therefore,

$$h_K(u) = r(K, L)h_L(u) + h_{I_1}(u) \quad \text{for all } u \in S^1, \tag{4.5}$$

with $o \in I_1$.

Note that K and L are dilates if and only if $I_1 = \{o\}$. Suppose K and L are not dilates, then I_1 is nondegenerate. From (4.5) and $o \in I_1$, it follows that the set $\Omega := \{u \in S^1 : h_K(u) = r(K, L)h_L(u)\}$ is contained in a half-sphere. Since K has interior points, $\text{supp}S_K$ cannot be concentrated on a half-sphere. This, together with the fact that Ω is contained in a half-sphere, proves that $\text{supp}S_K \backslash \Omega$ must contain at least one unit vector u_1. Then, from (4.4) and the fact $\omega \subset \Omega$, we conclude that $h_K(u_1)/h_L(u_1) = R(K, L)$. By the same argument above (4.5) we deduce that

$$L = \frac{1}{R(K, L)}K + I_2,$$

with $o \in I_2$. This together with (4.5) implies that

$$K = \frac{r(K, L)}{R(K, L)}K + r(K, L)I_2 + I_1.$$

Note that K and L are not dilates if and only if $r(K, L)/R(K, L) < 1$. Thus, we have

$$K = \frac{1}{1 - r(K, L)/R(K, L)}\Big(r(K, L)I_2 + I_1\Big),$$

which implies that K is a parallelogram with sides parallel to I_1 and I_2. Similarly, we have

$$L = \frac{1}{1 - r(K, L)/R(K, L)}\Big(\frac{1}{R(K, L)}I_1 + I_2\Big),$$

which implies that L is also a parallelogram with sides parallel to I_1 and I_2. □

Proof of Proposition 4.1. Suppose K and L are at a dilation position, and $|K| = |L|$. Note that the set $\{h_K = 0\} = \{h_L = 0\}$ is of measure 0, whenever the respective measure is V_K or V_L. If $V_K = V_L$, then, by Lemma 4.2, we have

$$\begin{aligned}\frac{1}{2}\int_{S^1} h_L dS_K &= \int_{S^1} \frac{h_L}{h_K} dV_K \\ &= \int_{S^1} \frac{h_L}{h_K} dV_L\end{aligned}$$

$$\leq \frac{1}{2}\int_{S^1} h_K dS_L$$
$$= \frac{1}{2}\int_{S^1} h_L dS_K.$$

Then, there is equality in (4.2), and hence $K = L$ or K and L are parallelograms with parallel sides and $|K| = |L|$. This implies (4.1).

By Theorem 1 and $|K| = |L|$, if (4.1) holds, then either K and L are parallelograms with parallel sides or $K = L$, which implies $V_K = V_L$. □

There might be a direct proof of this equivalence that without the help of Lemma 4.2, and then it might be a new approach to consider the uniqueness of the logarithmic Minkowski problem.

5. Properties of dilation position and equivalence of (1.9) and (1.10)

In this section, we prove several properties of dilation position, and show the equivalence of (1.9) and (1.10).

The following lemma is a useful tool when dealing with the dilation position.

Lemma 5.1. *Let $K, L \in \mathcal{K}^n$.*
(i) Suppose $r(K,L)L$ is the biggest homothetic copy of L contained in K. Then, there are $u_1, u_2, ..., u_{n+1} \in S^{n-1}$ with $o \in [u_1, u_2, ..., u_{n+1}]$, such that

$$h_K(u_i) = r(K,L)h_L(u_i),$$

for $i = 1, 2, ..., n+1$. Here $u_1, u_2, ..., u_{n+1}$ may be distinctive or not. Furthermore, there are $x_i \in \partial K \cap \partial\big(r(K,L)L\big)$, so that

$$h_K(u_i) = r(K,L)h_L(u_i) = x_i \cdot u_i,$$

for $i = 1, 2, ..., n+1$.

(ii) Suppose there is an $s > 0$ so that $sL \subset K$, and there are $u_1, u_2, ..., u_{n+1} \in S^{n-1}$ with $o \in [u_1, u_2, ..., u_{n+1}]$, such that

$$h_K(u_i) = sh_L(u_i),$$

for $i = 1, 2, ..., n+1$. Then, sL is the biggest homothetic copy of L contained in K, i.e., $s = r(K,L)$.

Proof. (i) See [33, Page 414], it is easy to conclude that $o \in \text{conv}\{u \in S^{n-1} : h_K(u) = r(K,L)h_L(u)\}$. Then, by Carathéodory's theorem (see [33, Theorem 1.1.4]), o is the convex combination of $n+1$ or fewer points of the set $\{u \in S^{n-1} : h_K(u) = r(K,L)h_L(u)\}$, which implies the first result.

Then, there are $x_1, x_2, ..., x_{n+1} \in \partial\big(r(K,L)L\big)$, so that

$$x_i \cdot u_i = r(K,L)h_L(u_i),$$

for $i = 1, 2, ..., n+1$. Since $r(K, L)L \subset K$, we see that $x_i \in K$. This and $x_i \cdot u_i = h_K(u_i)$ imply $x_i \in \partial K$.

(ii) Suppose $sL \subset K$, and there are $u_1, u_2, ..., u_{n+1} \in S^{n-1}$ with $o \in [u_1, u_2, ..., u_{n+1}]$, such that

$$h_K(u_i) = sh_L(u_i),$$

for $i = 1, 2, ..., n+1$. If there is an $s' > s$, and a $t \in \mathbb{R}^n$, so that $s'L + t \subset K$, then

$$s'h_L(u_i) + t \cdot u_i \leq h_K(u_i) = sh_L(u_i), \tag{5.1}$$

for $i = 1, 2, ..., n+1$. Since $o \in [u_1, u_2, ..., u_{n+1}]$, there are $\lambda_i \in [0, 1]$, so that $\sum_{i=1}^{n+1} \lambda_i = 1$ and

$$\sum_{i=1}^{n+1} \lambda_i u_i = 0.$$

This and (5.1) together with the sub-additivity of support function give

$$0 = (s' - s)h_L(0) \leq (s' - s) \sum_{i=1}^{n+1} \lambda_i h_L(u_i) \leq -t \cdot \sum_{i=1}^{n+1} \lambda_i u_i = 0.$$

Then, $h_L(u_i) = 0$ for $i = 1, 2, ..., n+1$. Since $o \in [u_1, u_2, ..., u_{n+1}]$, L will not contain an interior point, a contradiction.

Thus, sL is the biggest homothetic copy of L contained in K. □

The next lemma is important.

Lemma 5.2. *Let $K, L \in \mathcal{K}^n$ with $o \in K \cap L$. Suppose K and L are at a dilation position. If $s > 0$, then K and $L + sK$ are also at a dilation position.*

Proof. Set $r = r(K, L)$ and $R = R(K, L)$. Since K and L are at a dilation position. we have

$$rL \subset K \subset RL.$$

Then it is trivial that

$$\frac{r}{1 + sr}(L + sK) \subset K \subset \frac{R}{1 + sR}(L + sK).$$

We remain to show that $r(K, L + sK) = r/(1 + sr)$ and $R(K, L + sK) = R/(1 + sR)$. If there is a bigger homothetic copy of $L + sK$ contained in K, i.e.,

$$r'(L + sK) + t_0 \subset K,$$

with $r' > r/(1 + sr)$ and $t_0 \in \mathbb{R}^n$, then

$$r'h_L(u) + t_0 \cdot u \leq (1 - sr')h_K(u) \qquad \text{for all} \quad u \in S^{n-1}.$$

The case $1 - sr' \leq 0$ is impossible, because that $h_K \geq 0$, L has an interior point and t_0 is a fixed point. So $1 - sr' > 0$. Then a bigger homothetic copy of L will be contained in K, because $r'/(1 - sr') > r$. This is a contradiction. Thus $r(K, L + sK) = r/(1 + sr)$.

If there is a smaller homothetic copy of $L + sK$ containing K, i.e.,

$$r'(L + sK) + t_0 \subset K,$$

with $0 < R' < R/(1 + sR)$ and $t_1 \in \mathbb{R}^n$, then

$$R' h_L(u) + t_1 \cdot u \geq (1 - sR')h_K(u) \qquad \text{for all} \quad u \in S^{n-1}.$$

Since $0 < R' < R/(1 + sR)$, we have $(1 - sR')R > R' > 0$, and hence $(1 - sR') > 0$. Then $R'/(1 - sR') < R$, and $(R'/(1 - sR'))L + (1/(1 - sR'))t_1$ contains K, a contradiction. Therefore, $R(K, L + sK) = R/(1 + sR)$, and we complete the proof of this lemma. □

The following lemma is needed, and natural.

Lemma 5.3. *Let $K, L \in \mathcal{K}^n$ with $o \in K \cap L$. If K and L are at a dilation position, then $(1 - \lambda) \cdot K +_o \lambda \cdot L$ and K are also at a dilation position, for each $\lambda \in [0, 1]$.*

Proof. Set $r = r(K, L)$, $R = R(K, L)$, and $Q_\lambda = (1-\lambda)\cdot K +_o \lambda \cdot L$. By lemma 5.1, there are $u_1, u_2, ..., u_{n+1} \in S^{n-1}$ with $o \in [u_1, u_2, ..., u_{n+1}]$, and there are $x_1, x_2, ..., x_{n+1} \in \partial K \cap \partial\big(r(K, L)L\big)$, so that

$$x_i \cdot u_i = r h_L(u_i),$$

for $i = 1, 2, ..., n + 1$.

Since $rL \subset K \subset RL$, we have

$$R^{-\lambda} h_K \leq h_K^{1-\lambda} h_L^{\lambda} \leq r^{-\lambda} h_K.$$

By the definition (1.8),

$$Q_\lambda = \bigcap_{u \in S^{n-1}} \{x \in \mathbb{R}^n : x \cdot u \leq h_K(u)^{1-\lambda} h_L(u)^{\lambda}\},$$

we have

$$R^{-\lambda} K \subset Q_\lambda \subset r^{-\lambda} K. \tag{5.2}$$

Now $x_i \in K \cap \big(r(K, L)L\big)$ implies $r^{-\lambda} x_i \cdot u \leq h_K(u)^{1-\lambda} h_L(u)^{\lambda}$, which means $r^{-\lambda} x_i \in Q_\lambda$ for $i = 1, 2, ..., n + 1$. This and the fact $x_i \cdot u_i = r h_L(u_i) = h_K(u_i)$ give

$$h_{Q_\lambda}(u_i) \geq r^{-\lambda} x_i \cdot u_i = h_K(u_i)^{1-\lambda} h_L(u_i)^{\lambda} \geq h_{Q_\lambda}(u_i),$$

for $i = 1, 2, ..., n + 1$. Since $o \in [u_1, u_2, ..., u_{n+1}]$, by Lemma 5.1, we see that $r^{\lambda} Q_\lambda$ is the biggest homothetic copy of Q_λ contained in K.

Similarly, noticing that $R^{-1}K$ is the biggest homothetic copy of K contained in L, we deduce that $R^{-\lambda}K$ is the biggest homothetic copy of K contained in Q_λ. Therefore, $(1 - \lambda) \cdot K +_o \lambda \cdot L$ and K are also at a dilation position. □

Lemma 5.4. *Let $K, L \in \mathcal{K}^n$ with $o \in K \cap L$. If K and L are at a dilation position, then*

$$\lim_{\lambda \to 0^+} \frac{|(1 - \lambda) \cdot K +_o \lambda \cdot L| - |K|}{\lambda} = n \int_{S^{n-1}} \log \frac{h_L}{h_K} dV_K.$$

Note: the set $\{h_K = 0\} = \{h_L = 0\}$ is of measure 0, with respect to the measure V_K. The proof of this lemma is just an examination of the proof of [26, Lemma 1], as long as $(1-\lambda)\cdot K +_o \lambda \cdot L \to K$ as $\lambda \to 0$, which is guaranteed by Lemma 2.2. So we omit it here.

The following lemma shows the equivalence of the log-Brunn-Minkowski inequality (1.9) and the log-Minkowski inequality (1.10).

Lemma 5.5. *The log-Brunn-Minkowski inequality (1.9) and the log-Minkowski inequality (1.10) are equivalent.*

With the aid of Lemmas 5.3 and 5.4, we are able to use the idea in [5] to prove this lemma. For the sake of completeness we present the proof here.

Proof. Let $K, L \in \mathcal{K}^2$ with $o \in K \cap L$, and suppose K and L are at a dilation position. Set $Q_\lambda = (1-\lambda)\cdot K +_o \lambda \cdot L$, for $\lambda \in [0,1]$.

First suppose that we have the log-Minkowski inequality (1.10) for each pair of convex bodies that at a dilation position. Now Lemma 5.3 tells us Q_λ and K are at a dilation position, and Q_λ and L are also at a dilation position. Then the set $\{h_K = 0\} = \{h_L = 0\} = \{h_{Q_\lambda} = 0\}$ is of measure 0, with respect to V_{Q_λ}. By this, and the fact that $h_{Q_\lambda} = h_K^{1-\lambda} h_L^\lambda$ a.e. with respect to S_{Q_λ}, we know that $h_{Q_\lambda} = h_K^{1-\lambda} h_L^\lambda$ a.e. with respect to V_{Q_λ}. Then, we have

$$\begin{aligned} 0 &= \frac{1}{|Q_\lambda|}\int_{S^1} \log \frac{h_K^{1-\lambda} h_L^\lambda}{h_{Q_\lambda}} dV_{Q_\lambda} \\ &= (1-\lambda)\frac{1}{|Q_\lambda|}\int_{S^1} \log \frac{h_K}{h_{Q_\lambda}} dV_{Q_\lambda} + \lambda \frac{1}{|Q_\lambda|}\int_{S^1} \log \frac{h_L}{h_{Q_\lambda}} dV_{Q_\lambda} \\ &\geq (1-\lambda)\frac{1}{2}\log\frac{|K|}{|Q_\lambda|} + \lambda \frac{1}{2}\log \frac{|L|}{|Q_\lambda|} \qquad (5.3) \\ &= \frac{1}{2}\log \frac{|K|^{1-\lambda}|L|^\lambda}{|Q_\lambda|}. \end{aligned}$$

This gives the log-Brunn-Minkowski inequality (1.9).

Suppose now that we have the log-Brunn-Minkowski inequality (1.9) for K, L and $\lambda \in [0,1]$. Lemma 5.4 shows

$$\lim_{\lambda \to 0^+} \frac{|Q_\lambda| - |K|}{\lambda} = 2\int_{S^{n-1}} \log \frac{h_L}{h_K} dV_K. \tag{5.4}$$

The log-Brunn-Minkowski inequality (1.9) says that $\lambda \mapsto \log |Q_\lambda|$ is a concave function, and hence

$$\lim_{\lambda \to 0^+} \frac{\log |Q_\lambda| - \log |K|}{\lambda} \geq \log |Q_1| - \log |Q_0| = \log |L| - \log |K|.$$

This and (5.4) yield the log-Minkowski inequality (1.10). $\square$

6. The general log-Minkowski inequality under an assumption

From now on, we shall make use of the notations of R_K, r_K, and $\mathcal{F}_K$. Let $K \in \mathcal{K}^2$ with $o \in K$. We always set

$$R_K = \max_{u\in S^1} h_K(u) \quad \text{and} \quad r_K = \min_{u\in S^1} h_K(u).$$

In addition, suppose $|K| = 1$. Define $\mathcal{F}_K$ as a set of planar convex bodies by

$$\mathcal{F}_K := \{Q \in \mathcal{K}^2 : \quad Q \text{ and } K \text{ are at a dilation position, and } |Q| = 1\}$$

Consider the minimization problem,

$$\inf \int_{S^1} \log h_Q dV_K, \qquad Q \in \mathcal{F}_K. \tag{6.1}$$

By the argument after Theorem 2, even for the case $o \in \partial K \cap \partial L$, the integral in (6.1) is well defined.

Our main purpose in this section is to establish the following version of the general log-Minkowski inequality.

Theorem 6.1. *Let $K \in \mathcal{K}_o^2$. Suppose the cone-volume measure V_K satisfies the strict subspace concentration inequality. If K and L are at a dilation position, then*

$$\int_{S^1} \log \frac{h_L}{h_K} dV_K \geq \frac{|K|}{2} \log \frac{|L|}{|K|}. \tag{6.2}$$

Equality holds if and only if K and L are dilates.

The following lemma shows that the set $\mathcal{F}_K$ is closed.

Lemma 6.2. *Let $K \in \mathcal{K}^2$ with $o \in K$. Suppose $L_k \in \mathcal{F}_K$, and $L_k \to L_0$ with respect to the Hausdorff distance as $k \to \infty$. Then $L_0 \in \mathcal{F}_K$.*

Proof. Since $|L_k| = 1$, and the volume is continuous with respect to the Hausdorff distance, we have $|L_0| = 1$. It remains to prove that K and L_0 are at a dilation position.

By Lemma 5.1, there are 3 sequences of vectors $\{u_{i,k}\} \subset S^1$, $i = 1, 2, 3$, such that $o \in [u_{1,k}, u_{2,k}, u_{3,k}]$, and

$$R(L_k, K)h_K(u_{i,k}) = h_{L_k}(u_{i,k}), \tag{6.3}$$

for $i = 1, 2, 3$ and $k \in \mathbb{N}$. Since S^1 is compact, and a subsequence of $\{L_k\}$ is always convergent, we may assume that

$$\lim_{k\to\infty} u_{i,k} =: u_i \in S^1, \tag{6.4}$$

for $i = 1, 2, 3$. Then $o \in [u_1, u_2, u_3]$. This and the fact that K contains an interior point show that there is a u_i satisfying $h_K(u_i) \neq 0$. We may assume

$$h_K(u_1) \neq 0. \tag{6.5}$$

This and (6.4) imply $h_K(u_{1,k}) \neq 0$ for sufficiently large k.

Now, from (6.3), (6.5) and $L_k \to L_0$, it follows that $\{R(L_k, K)\}$ converges to an $R_0 > 0$, and

$$R_0 := \lim_{k\to\infty} R(L_k, K) = \lim_{k\to\infty} \frac{h_{L_k}(u_{1,k})}{h_K(u_{1,k})} = \frac{h_{L_0}(u_1)}{h_K(u_1)}.$$

Then, $L_k \subset R(L_k, K)K$ and (6.3) show that

$$L_0 \subset R_0 K,$$

and

$$R_0 h_K(u_i) = h_{L_0}(u_i),$$

for $i = 1, 2, 3$. Since $o \in [u_1, u_2, u_3]$, it follows from Lemma 5.1 that $(1/R_0)L_0$ is the biggest homothetic copy of L_0 contained in K.

Similarly, we have that $\{r(L_k, K)\}$ converges to a number $r_0 > 0$, and $r_0 K$ is the biggest homothetic copy of K contained in L_0. Therefore, K and L_0 are at a dilation position. □

Lemma 6.3. *Let $K \in \mathcal{K}^2$ with $|K| = 1$ and $o \in K$. If $L_0 \in \mathcal{F}_K$ is a minimizer of the problem (6.1), then either $L_0 = K$ or L_0 and K are parallelograms with parallel sides.*

Proof. By Lemma 5.2, $(L_0 + sK)/|L_0 + sK|^{\frac{1}{2}} \in \mathcal{F}_K$, and by the assumption that L_0 is a minimizer of the problem (6.1), we have

$$\int_{S^1} \log \frac{h_{L_0} + s h_K}{|L_0 + sK|^{\frac{1}{2}}} dV_K - \int_{S^1} \log h_{L_0} dV_K \geq 0,$$

for all $s > 0$. Then, recalling $|K| = 1$, we have

$$\int_{S^1} [\log(h_{L_0} + s h_K) - \log h_{L_0}] dV_K \geq \frac{1}{2} \log |L_0 + sK| \tag{6.6}$$

It is clear that $(\log(h_{L_0} + s h_K) - \log h_{L_0})/s \to h_K/h_{L_0}$ a.e. as $s \to 0^+$, and $|(\log(h_{L_0} + s h_K) - \log h_{L_0})/s|$ is dominated by h_K/h_{L_0}, which is integrable S^1 with respect to the measure V_K. By the dominated convergence theorem, we know that the right derivative of the left sides of (6.6) equals

$$\int_{S^1} \frac{h_K}{h_{L_0}} dV_K.$$

One the other hand, by (2.2), we have

$$\lim_{s\to 0^+} \frac{|L_0 + sK| - |L_0|}{s} = 2V(L_0, K) = 2V(K, L_0) = \int_{S^1} h_{L_0} dS_K.$$

Thus, by taking right derivative of both sides of (6.6) at $s = 0$, we have

$$\int_{S^1} \frac{h_K}{h_{L_0}} dV_K \geq \frac{1}{2} \int_{S^1} h_{L_0} dS_K.$$

But Lemma 4.2 tells us

$$\int_{S^1} \frac{h_K}{h_{L_0}} dV_K \leq \frac{1}{2} \int_{S^1} h_{L_0} dS_K.$$

Then, equality holds in (4.2). This and $|K| = |L_0| = 1$ show that either $L_0 = K$ or L_0 and K are parallelograms with parallel sides. □

Lemma 6.4. *Let $K \in \mathcal{K}_o^2$. Suppose K and L are at a dilation position. Then, there exists a $u_1 \in S^1$ such that*

$$r(L, K) = \frac{h_L(u_1)}{h_K(u_1)} \leq \frac{h_L(-u_1)}{h_K(-u_1)} \leq 2c_K \cdot r(L, K). \tag{6.7}$$

Here $c_K = R_K/r_K$ is a constant depends only on K.

Proof. By Lemma 5.1, there are unit vectors v_1, v_2, v_3 (may be distinctive or not) such that $o \in [v_1, v_2, v_3]$ and

$$\frac{h_L(v_1)}{h_K(v_1)} = \frac{h_L(v_2)}{h_K(v_2)} = \frac{h_L(v_3)}{h_K(v_3)} = r(L, K).$$

Then, there are $\lambda_1, \lambda_2, \lambda_3 \in [0, 1]$, so that $\lambda_1 + \lambda_2 + \lambda_3 = 1$ and $\sum_{i=1}^{3} \lambda_i v_i = 0$. Then, there exists a $\lambda_i \geq \frac{1}{3}$, say, λ_1. It follows that

$$\frac{\lambda_2}{\lambda_1} + \frac{\lambda_3}{\lambda_1} = \frac{1 - \lambda_1}{\lambda_1} \leq 2. \tag{6.8}$$

We may write

$$-v_1 = \frac{\lambda_2}{\lambda_1} v_2 + \frac{\lambda_3}{\lambda_1} v_3.$$

Then, by the sub-additivity of support function and (6.8), we have

$$\begin{aligned}\frac{h_L(-v_1)}{h_K(-v1)} &\leq \frac{\frac{\lambda_2}{\lambda_1} h_L(v_2) + \frac{\lambda_3}{\lambda_1} h_L(v_3)}{r_K} \\ &= r(L, K) \frac{\frac{\lambda_2}{\lambda_1} h_K(v_2) + \frac{\lambda_3}{\lambda_1} h_K(v_3)}{r_K} \\ &\leq 2c_K \cdot r(L, K).\end{aligned}$$

Let $u_1 = v_1$ and we are done. □

Lemma 6.5. *Let $K \in \mathcal{K}_o^2$. Suppose its cone-volume measure V_K satisfies the strict subspace concentration inequality. Let $\{L_k\}$ be a sequence of planar convex bodies in $\mathcal{F}_K$. If $\{L_k\}$ is not bounded, then the sequence*

$$\int_{S^1} \log h_{L_k} dV_K$$

is not bounded from above.

Proof. Since $K \in \mathcal{K}_o^2$ is fixed, from (2.3) and (2.4), and the facts that $\{L_k\}$ is unbounded and $|L_k| = 1$, it is easy to see that

$$\liminf_{k \to \infty} r(L_k, K) = 0, \quad \text{and} \quad \limsup_{k \to \infty} R(L_k, K) = +\infty.$$

Therefore, there is a subsequence (also denoted by $\{L_k\}$) satisfying

$$\lim_{k\to\infty} r(L_k, K) = 0, \quad \text{and} \quad \lim_{k\to\infty} R(L_k, K) = +\infty. \tag{6.9}$$

By Lemma 6.4, there is a sequence $\{u_{1,k}\} \subset S^1$ satisfying

$$r(L_k, K) = \frac{h_{L_k}(u_{1,k})}{h_K(u_{1,k})} \le \frac{h_{L_k}(-u_{1,k})}{h_K(-u_{1,k})} \le 2c_K \cdot r(L_k, K), \tag{6.10}$$

where $c_K = R_K/r_K$ depends only on the convex body K. For each $u_{1,k} \in S^1$, denote by $u_{2,k} \in S^1$ the unit vector that rotates $u_{1,k}$ clockwise by $90°$.

Since S^1 is compact, $\{u_{1,k}\}$ has a convergent subsequence. Thus, we may assume $\{u_{1,k}\}$ itself is convergent, and

$$\lim_{k\to\infty} u_{1,k} = u_1 \in S^1. \tag{6.11}$$

Since a subsequence of $\{L_k\}$ will also satisfies (6.9), we have found a subsequence satisfying all of (6.9), (6.10), and (6.11). It follows that $\lim_{k\to\infty} u_{2,k} = u_2 \in S^1$, where u_2 is the unit vector that rotates u_1 clockwise by $90°$.

Set $h_{\pm 1,k} = h_{L_k}(\pm u_{1,k})$, and $h_{\pm 2,k} = h_{L_k}(\pm u_{2,k})$. Clearly, (6.10) implies that

$$\min\{h_{1,k}, h_{-1,k}\} \ge c_0(h_{1,k} + h_{-1,k}), \tag{6.12}$$

where c_0 is a constant depends only on the convex body K.

Then, by (6.10) and (6.9), we have

$$\lim_{k\to\infty} h_{1,k} = \lim_{k\to\infty} h_{-1,k} = 0. \tag{6.13}$$

From Lemma 5.1 we know that there are unit vectors $v_{1,k}$ and $v_{2,k}$ such that

$$\frac{h_{L_k}(v_{1,k})}{h_K(v_{1,k})} = \frac{h_{L_k}(v_{2,k})}{h_K(v_{2,k})} = R(L_k, K),$$

with $v_{1,k} \cdot u_{2,k} \ge 0$ and $v_{2,k} \cdot u_{2,k} \le 0$. This implies

$$\lim_{k\to\infty} h_{L_k}(v_{1,k}) = \lim_{k\to\infty} R(L_k, K) h_K(v_{1,k}) \ge r_K \lim_{k\to\infty} R(L_k, K) = +\infty.$$

From (6.9) and (6.10) we conclude that $v_{1,k} \cdot u_{2,k} > 0$ and $v_{2,k} \cdot u_{2,k} < 0$ for all sufficiently large k. If $v_{1,k} \cdot u_{1,k} \ge 0$, then we write $v_{1,k} = (v_{1,k} \cdot u_{1,k})u_{1,k} + (v_{1,k} \cdot u_{2,k})u_{2,k}$. By the subadditivity of support function, we have

$$h_{2,k} = h_{L_k}(u_{2,k}) \ge \frac{h_{L_k}(v_{1,k}) - (v_{1,k} \cdot u_{1,k})h_{L_k}(u_{1,k})}{v_{1,k} \cdot u_{2,k}} \ge \frac{1}{2} h_{L_k}(v_{1,k}),$$

for all sufficiently large k. The last inequality is because $\lim_{k\to\infty} h_{L_k}(u_{1,k}) = 0$ and $\lim_{k\to\infty} h_{L_k}(v_{1,k}) = +\infty$. Then, we have

$$h_{2,k} \ge \frac{1}{2} h_{L_k}(v_{1,k}) = \frac{1}{2} R(L_k, K) h_K(v_{1,k}) \ge \frac{1}{2} r_K R(L_k, K), \tag{6.14}$$

for all sufficiently large k. Similarly, when $v_{1,k} \cdot (-u_{1,k}) \ge 0$, we also have (6.14).

In a similar way, we have

$$h_{-2,k} \geq \frac{1}{2}h_{L_k}(v_{2,k}) = \frac{1}{2}R(L_k, K)h_K(v_{2,k}) \geq \frac{1}{2}r_K R(L_k, K), \tag{6.15}$$

for all sufficiently large k, and then

$$\lim_{k\to\infty} h_{2,k} = \lim_{k\to\infty} h_{-2,k} = +\infty.$$

From (2.3), (6.14) and (6.15), it is obvious that for all sufficiently large k,

$$\min\{h_{2,k}, h_{-2,k}\} \geq c_1(h_{2,k} + h_{-2,k}), \tag{6.16}$$

where c_1 is a constant depends only on the convex body K.

For $\delta \in (0, \frac{2}{5})$, let U_δ be the neighborhood of $\{\pm u_1\}$ on S^1, that is,

$$U_\delta := \{u \in S^1 : |u \cdot u_1| > 1 - \delta\}.$$

Let

$$V_\delta^1 := \{u \in S^1 : |u \cdot u_1| \leq 1 - \delta \quad \text{and} \quad u \cdot u_2 \geq 0\},$$

and

$$V_\delta^2 := \{u \in S^1 : |u \cdot u_1| \leq 1 - \delta \quad \text{and} \quad u \cdot u_2 \leq 0\}.$$

Then, $V_\delta := V_\delta^1 \cup V_\delta^2$ is the complement of U_δ.

Since V_K satisfies the strict subspace concentration inequality, $V_K(\{\pm u_1\}) < \frac{1}{2}$. When δ is decreasing, the U_δ are also decreasing (with respect to set inclusion) and have a limit of $\{\pm u_1\}$,

$$\lim_{\delta\to 0^+} V_K(U_\delta) = V_K(\{\pm u_1\}).$$

Then, there is a $\delta_0 \in (0, \frac{2}{5})$ such that

$$V_K(U_{\delta_0}) < \frac{1}{2},$$

and then,

$$V_K(V_{\delta_0}) = V_K(S^1) - V_K(U_{\delta_0}) > \frac{1}{2}.$$

By (6.11), we have $|u_{i,k} - u_i| < \delta_0$ for all sufficiently large k, where $i = 1, 2$. Note that $|u \cdot u_1|^2 + |u \cdot u_2|^2 = 1$. Thus, for $u \in V_{\delta_0}^1$, we have

$$u \cdot u_2 \geq (1 - (1 - \delta_0)^2)^{\frac{1}{2}} > 2\delta_0,$$

where the last inequality follows from the fact that $\delta_0 < \frac{2}{5}$. This shows that

$$\begin{aligned} u \cdot u_{2,k} &= u \cdot u_2 - u \cdot (u_2 - u_{2,k}) \\ &\geq u \cdot u_2 - |u_{2,k} - u_2| \\ &\geq 2\delta_0 - \delta_0 \\ &= \delta_0, \end{aligned} \tag{6.17}$$

for all sufficiently large k. For $u \in V_{\delta_0}^2$, we have

$$u \cdot (-u_2) \geq (1 - (1 - \delta_0)^2)^{\frac{1}{2}} > 2\delta_0,$$

which shows that

$$\begin{aligned} u\cdot(-u_{2,k}) &= u\cdot(-u_2) - u\cdot(u_{2,k}-u_2)\\ &\geq u\cdot(-u_2) - |u_{2,k}-u_2|\\ &\geq 2\delta_0 - \delta_0\\ &= \delta_0, \end{aligned} \tag{6.18}$$

for all sufficiently large k.

By (6.10), for $u\in U_{\delta_0}$ and sufficiently large k, we have

$$h_{L_k}(u) \geq \frac{1}{c_K} h_{L_k}(u_{1,k})\frac{h_K(u)}{h_K(u_{1,k})} \geq \frac{1}{c_K^2}\min\{h_{1,k}, h_{-1,k}\}. \tag{6.19}$$

Let $x_k\in L_k$ and $y_k\in L_k$ be such that

$$h_{L_k}(u_{2,k}) = x_k\cdot u_{2,k}, \text{ and } h_{L_k}(-u_{2,k}) = y_k\cdot(-u_{2,k}).$$

By (6.17) and (6.13), for $u\in V^1_{\delta_0}$ and sufficiently large k, we have

$$\begin{aligned} h_{L_k}(u) &\geq x_k\cdot\Big((u\cdot u_{2,k})u_{2,k} + (u\cdot u_{1,k})u_{1,k}\Big)\\ &\geq \delta_0 h_{L_k}(u_{2,k}) - \max\{h_{1,k}, h_{-1,k}\}\\ &\geq \frac{\delta_0}{2} h_{2,k}\\ &\geq \frac{\delta_0}{2}\min\{h_{2,k}, h_{-2,k}\}. \end{aligned} \tag{6.20}$$

By (6.18) and (6.13), for $u\in V^2_{\delta_0}$ and sufficiently large k, we have

$$\begin{aligned} h_{L_k}(u) &\geq y_k\cdot\Big((u\cdot(-u_{2,k}))(-u_{2,k}) + (u\cdot u_{1,k})u_{1,k}\Big)\\ &\geq \delta_0 h_{L_k}(-u_{2,k}) - \max\{h_{1,k}, h_{-1,k}\}\\ &\geq \frac{\delta_0}{2} h_{-2,k}\\ &\geq \frac{\delta_0}{2}\min\{h_{2,k}, h_{-2,k}\}. \end{aligned} \tag{6.21}$$

Therefore, by (6.19), (6.20), (6.21), and then (6.12), (6.16), we have

$$\begin{aligned} \int_{S^1}\log h_{L_k}(u)dV_K(u) &\geq V_K(U_{\delta_0})\log(\frac{1}{c_K^2}\min\{h_{1,k}, h_{-1,k}\}) + V_K(V_{\delta_0})\log(\frac{\delta_0}{2}\min\{h_{2,k}, h_{-2,k}\})\\ &\geq V_K(U_{\delta_0})\log(\frac{c_0}{c_K^2}(h_{1,k}+h_{-1,k})) + V_K(V_{\delta_0})\log(\frac{\delta_0}{2}(h_{2,k}+h_{-2,k}))\\ &\geq (V_K(V_{\delta_0}) - V_K(U_{\delta_0}))\log(h_{2,k}+h_{-2,k})\\ &\qquad + V_K(U_{\delta_0})\log[(h_{1,k}+h_{-1,k})(h_{2,k}+h_{-2,k})]\\ &\qquad + V_K(U_{\delta_0})\log(\frac{c_0}{c_K^2}) + V_K(V_{\delta_0})\log(\frac{\delta_0}{2}). \end{aligned} \tag{6.22}$$

Since L_k is contained in the parallelogram

$$\bigcap_{i=1,2}\{x : x\cdot u_{i,k} \le h_{i,k},\ \text{and } x\cdot u_{-i,k} \le h_{-i,k}\},$$

we deduce

$$(h_{1,k}+h_{-1,k})(h_{2,k}+h_{-2,k}) \ge |L_k| = 1.$$

This and (6.22) together with the fact $\lim_{k\to\infty}(h_{2,k}+h_{-2,k}) = +\infty$ imply the desired result. □

Proof of Theorem 6.1. Firstly, assume $|K| = |L| = 1$. Let $\{L_k\}$ be a minimizing sequence of the minimization problem (6.1), i.e., a sequence of bodies in $\mathcal{F}_K$ so that $\int_{S^1}\log h_{L_k} dV_K$ tends to the infimum (which may be $-\infty$).

By Lemma 6.5, $\{L_k\}$ is bounded, since otherwise $\int_{S^1}\log h_{L_k} dV_K$ will be unbounded from above, which is contradict to the fact that $\{L_k\}$ is a minimizing sequence. Then there is a subsequence of $\{L_k\}$ converging to L_0, and Lemma 6.2 implies $L_0 \in \mathcal{F}_K$. Thus, L_0 is a minimizer of the problem (6.1). The fact that V_K satisfies the strict subspace concentration inequality implies that K is not a parallelogram. Thus, by Lemma 6.3, we deduce $L_0 = K$. Then, we have

$$\int_{S^1}\log\frac{h_L}{h_K}dV_K \ge \int_{S^1}\log\frac{h_K}{h_K}dV_K = 0.$$

Secondly, for arbitrary K and L, notice that $V_{K/|K|^{\frac{1}{2}}} = V_K/|K|$ and $L/|L|^{\frac{1}{2}} \in \mathcal{F}_{K/|K|^{\frac{1}{2}}}$. By the argument above, we have

$$\int_{S^1}\log\frac{h_{L/|L|^{\frac{1}{2}}}}{h_{K/|K|^{\frac{1}{2}}}}dV_{K/|K|^{\frac{1}{2}}} \ge 0,$$

which implies the inequality (6.2).

If K and L are dilates, then it is easy to see that the equality in (6.2) holds.

If there is equality in (6.2), then the convex body $L/|L|^{\frac{1}{2}}$ must be a minimizer of the problem (6.1) for $K/|K|^{\frac{1}{2}}$. From Lemma 6.3 and the fact K is not a parallelogram, it follows immediately that K and L are dilates. □

For the case that K is a parallelogram (not necessary o-symmetric) with o in its interior, we can also use similar method and consider several cases to prove that the inequality (6.2) holds. However, such a proof will be complicated, and it can be replaced by the approximation lemmas in the next section. So we omit it.

7. Approximation process

We say a convex body K is *strictly convex*, if its boundary does not contain a line segment. If $K \in \mathcal{K}_o^2$ is strictly convex, then it is easy to see that its cone-volume measure V_K always satisfies the strictly subspace concentration inequality.

Given a pair of convex bodies that are at a dilation position. The main goal of this section is to construct a new pair of convex bodies, so that one of them is strictly convex, and that they satisfy some other desired properties. Before this, we give a lemma concerning concave functions. A concave function f is called *strictly concave* on an interval $[a,b]$, if

$$f((1-t)x+ty) > (1-t)f(x)+tf(y),$$

for $t \in (0,1)$, $x, y \in [a,b]$, and $x \neq y$.

Lemma 7.1. *Let f_1, f_2 be nonnegative concave functions defined on $[b_1, b_2]$.*
(i) Suppose $f_2 > f_1$ on $[b_1,b_2]$. Then, there is a strictly concave function g defined on $[b_1,b_2]$ so that $g(b_1) = f_1(b_1)$, $g(b_2) = f_2(b_2)$, and $f_1 \leq g \leq f_2$.
(ii) Let $b_1 < b_0 < b_2$. Suppose $f_2 > f_1$ on $[b_1,b_2]$, and

$$f_2(x) > \frac{b_2-x}{b_2-b_0} f_2(b_0) + \frac{x-b_0}{b_2-b_0} f_2(b_2), \tag{7.1}$$

for $b_0 < x < b_2$. Then, there is a strictly concave function g defined on $[b_1,b_2]$ so that $g(b_1) = f_1(b_1)$, $g(b_0) = f_2(b_0)$, $g(b_2) = f_2(b_2)$, and $f_1 \leq g \leq f_2$.

Proof. (i) Since f_1 is concave and $f_2(b_2) > f_1(b_2)$, there exists a $x_0 \in [b_1, b_2)$ such that the line through the point $(b_2, f_2(b_2))$ tangent the graph of $f_1(x)$ at $(x_0, f_1(x_0))$. Let

$$\overline{f}_1(x) = \begin{cases} f_1(x), & x \in [b_1, x_0], \\ f_1(x_0) + \frac{f_2(b_2)-f_1(x_0)}{b_2-x_0}(x-x_0), & x \in [x_0, b_2]. \end{cases}$$

Then, it is clear that $\overline{f}_1(x)$ is concave on $[b_1,b_2]$, $\overline{f}_1(b_1) = f_1(b_1)$, $\overline{f}_1(b_2) = f_2(b_2)$, and $\overline{f}_1 < f_2$ on $[b_1, b_2)$.

Since f_2 is concave, its left derivative f_2^l is decreasing, and it satisfies

$$f_2^l(b_2) \leq \frac{f_2(b_2)-f_2(x_0)}{b_2-x_0} < \frac{f_2(b_2)-f_1(x_0)}{b_2-x_0}.$$

From the fact that f_2^l is left-continuous, it follows that there exists an η with $0 < \eta < \min\{\frac{b_2-b_1}{2}, b_2-x_0\}$, such that

$$f_2^l(x) < \frac{f_2(b_2)-f_1(x_0)}{b_2-x_0}$$

for all $x \in [b_2-\eta, b_2]$. Thus, when

$$0 < c < \frac{1}{b_2-b_1}\Big(\frac{f_2(b_2)-f_1(x_0)}{b_2-x_0} - f_2^l(b_2-\eta)\Big), \quad \text{and} \quad c < \frac{4}{(b_2-b_1)^2} \min_{x\in[b_1,b_2-\eta]} \big(f_2(x) - \overline{f}_1(x)\big),$$

we have

$$\frac{f_2(b_2)-f_1(x_0)}{b_2-x_0} - c\cdot(2x-b_2-b_1) > f_2^l(x) \tag{7.2}$$

for all $x \in [b_2-\eta, b_2]$, and

$$c\cdot\frac{(b_2-b_1)^2}{4} < f_2(x) - \overline{f}_1(x) \tag{7.3}$$

for all $x \in [b_1, b_2-\eta]$.

Let $g(x) = \overline{f}_1(x) + c\cdot[\frac{(b_2-b_1)^2}{4} - (x-\frac{b_2+b_1}{2})^2]$. Then (7.2) and the fact $g(x) = g_1(b_2) - \int_x^a g_1^l(t)dt$ imply that $g < f_2$ on $[b_2-\eta, b_2)$; (7.3) shows that $g_1 < f_2$

on $[b_1, b_2 - \eta]$. Therefore, $g(x)$ is a strictly concave function on $[b_1, b_2]$ satisfying $g(b_1) = f_1(b_1)$, $g(b_2) = f_2(b_2)$, and $f_1 \leq g \leq f_2$ on $[b_1, b_2]$.

(ii) By the argument in (i), there is a strictly concave function g_1 on $[b_1, b_0]$, with $g_1(b_1) = f_1(b_1)$, $g_1(b_0) = f_2(b_0)$, and $f_1 \leq g_1 \leq f_2$ on $[b_1, b_0]$.

Set $a_0 := (b_0 + b_2)/2$. Define a function $\overline{f}_2$ on $[b_0, b_2]$ by

$$\overline{f}_2(x) = \begin{cases} \frac{a_0 - x}{a_0 - b_0} f(b_0) + \frac{x - b_0}{b_0 - a_0} f(a_0), & x \in [b_0, a_0], \\ \frac{b_2 - x}{b_2 - a_0} f(a_0) + \frac{x - a_0}{b_2 - a_0} f(b_2), & x \in [a_0, b_2]. \end{cases}$$

Then, $\overline{f}_2 \leq f_2$ on $[b_0, b_2]$.

For $x \in [b_0, b_2]$, let

$$G(x) = \frac{b_2 - x}{b_2 - b_0} f(b_0) + \frac{x - b_0}{b_2 - b_0} f(b_2) + c_0 \cdot [\frac{(b_2 - b_0)^2}{4} - (x - a_0)^2]$$

From (7.1), it follows that $\frac{b_2 - x}{b_2 - b_0} f(b_0) + \frac{x - b_0}{b_2 - b_0} f(b_2) < \overline{f}_2(x)$ for $x \in (b_0, b_2)$. It is easy to choose a sufficiently small and positive constant c_0, so that $G(x) \leq \overline{f}_2 \leq f_2$ on $[b_0, b_2]$. Let

$$g(x) = \begin{cases} g_1(x), & x \in [b_1, b_0], \\ G(x), & x \in [b_0, b_2]. \end{cases}$$

Then, $g(x)$ is the desired function.

To prove that g is strictly concave, suppose $x \in [b_1, b_0]$, $y \in [b_0, b_2]$, and $t \in [0, 1]$. We may assume $(1 - t)x + ty \in [b_1, b_0]$, since the case $(1 - t)x + ty \in [b_0, b_2]$ is similar. Then, there exists a $t \leq t_0 \leq 1$, so that $b_0 = (1 - t_0)x + t_0 y$. It follows that $(1 - t)x + ty = ((t_0 - t)/t_0)x + (t/t_0)b_0$. Then, we have

$$\begin{aligned} g((1 - t)x + ty) &= g_1\big(\frac{t_0 - t}{t_0} x + \frac{t}{t_0} b_0\big) \\ &> (1 - \frac{t}{t_0}) g_1(x) + \frac{t}{t_0} f_2(b_0) \\ &\geq (1 - \frac{t}{t_0}) g_1(x) + \frac{t}{t_0} [(1 - t_0) f_2(x) + t_0 f_2(y)] \\ &> (1 - \frac{t}{t_0}) g_1(x) + (\frac{t}{t_0} - t) g(x) + t g(y) \\ &= (1 - t) g(x) + t g(y). \end{aligned}$$

□

Let $u \in S^1$. We shall make use of the notion of exposed face (also called support set) $F(K, u)$ of a convex body K. That is,

$$F(K, u) := K \cap \{x \in \mathbb{R}^2 : x \cdot u = h_K(u)\}.$$

Lemma 7.2. *Let $K, L \in \mathcal{K}_o^2$. Suppose K and L are not dilates, and they are at a dilation position. Then, for each $\epsilon > 0$, there are convex bodies $K_\epsilon, L_\epsilon \in \mathcal{K}_o^2$ so that K_ϵ is strictly convex,*

$$d_H(K_\epsilon, K),\ d_H(L_\epsilon, L) < c_1 \epsilon,$$

and K_ϵ and L_ϵ are at a dilation position. Here c_1 is a constant depends only on K and L.

Proof. Set $r = r(L, K), R = R(L, K)$, and $B = B^2$. Since K and L are not dilates, we have $r < R$. By Lemma 5.1, there are $u_i, v_i \in S^1$, $x_i \in \partial K \cap \partial(\frac{1}{r}L)$ and $y_i \in \partial K \cap \partial(\frac{1}{R}L)$, so that $o \in [u_1, u_2, u_3] \cap [v_1, v_2, v_3]$,

$$h_L(u_i) = rh_K(u_i) = rx_i \cdot u_i, \quad \text{and} \quad h_L(v_i) = Rh_K(v_i) = Ry_i \cdot v_i, \tag{7.4}$$

for $i = 1, 2, 3$. Here u_1, u_2, u_3 may be distinctive or not, and so is the triple v_1, v_2, v_3. We will give 3 reasonable assumptions.

(A1) Assume $y_j \notin F(K, v_i)$ for $y_i \neq y_j$.

Otherwise, suppose $y_2 \in F(K, v_1)$ with $y_1 \neq y_2$. This and the fact $o \in [v_1, v_2, v_3]$ imply that $-v_3$ must be a normal vector at y_2. Letting $y_1' = y_2$ and $v_1' = v_2' = -v_3$, we will consider the points $\{y_1', y_2, y_3\}$ and vectors $\{v_1', v_2', v_3\}$. Note: there does not exist the case that $y_1 = y_3 \in F(K, v_1)$ and $v_1 = v_2 = -v_3$, since otherwise K will not contain an interior point. Therefore, the assumption (**A1**) is reasonable.

(A2) Similarly, assume $x_j \notin F(K, u_i)$ for $x_i \neq x_j$.

(A3) Suppose $\{i, j, k\} = \{1, 2, 3\}$. If $y_i = y_j$, assume $v_i = v_j = -v_k$; if $x_i = x_j$, assume $u_i = u_j = -u_k$.

Otherwise, suppose $y_1 = y_2$, and $v_1 \neq v_2$. Since $o \in [v_1, v_2, v_3]$, then $-v_3$ must be a normal vector at y_1. Letting $v_1' = v_2' = -v_3$, we will consider the points $\{y_1, y_2, y_3\}$ and vectors $\{v_1', v_2', v_3\}$. Clearly, the assumption **(A1)** will be preserved. The discussion for x_i is similar.

Next, we use 2 procedures to construct the desired bodies. In fact, Procedure 1 is to make the new body K_ϵ^1 satisfy that $F(K_\epsilon^1, v_i)$ contains only one point, for $i = 1, 2, 3$.

Procedure 1. Let K_ϵ^1 and L_ϵ^1 be defined by

$$K_\epsilon^1 = [K, (1+\epsilon)y_1, (1+\epsilon)y_2, (1+\epsilon)y_3], \tag{7.5}$$

and

$$L_\epsilon^1 = L \cap \frac{R}{1+\epsilon} K_\epsilon^1. \tag{7.6}$$

Thus, for $\epsilon < \frac{R}{r} - 1$, we have

$$rK_\epsilon^1 \subset L_\epsilon^1 \subset \frac{R}{1+\epsilon} K_\epsilon^1.$$

It can be seen that $(1+\epsilon)y_i \in \partial K_\epsilon^1 \cap \partial(\frac{1+\epsilon}{R} L_\epsilon^1)$, and $x_i \in \partial K_\epsilon^1 \cap \partial(\frac{1}{r} L_\epsilon^1)$. Since $K_\epsilon^1 \subset (1+\epsilon)K$, we see that

$$h_K(v_i) \leq (1+\epsilon)h_K(v_i) = (1+\epsilon)y_i \cdot v_i.$$

This, together with $(1+\epsilon)y_i \in K_\epsilon^1$, gives

$$h_{K_\epsilon^1}(v_i) = (1+\epsilon)y_i \cdot v_i.$$

Then, by $\frac{1+\epsilon}{R}L_\epsilon^1 \subset K_\epsilon^1$ and $(1+\epsilon)y_i \in \frac{1+\epsilon}{R}L_\epsilon^1$, we deduce that

$$\frac{1+\epsilon}{R}h_{L_\epsilon^1}(v_i) = (1+\epsilon)y_i \cdot v_i = h_{K_\epsilon^1}(v_i).$$

Since $o \in [v_1, v_2, v_3]$, by $\frac{1+\epsilon}{R}L_\epsilon^1 \subset K_\epsilon^1$ and Lemma 5.1, we know that $\frac{1+\epsilon}{R}L_\epsilon^1$ is the biggest homothetic copy of L_ϵ^1 contained in K_ϵ^1.

Recall that for $\epsilon < \frac{R}{r} - 1$, we have $x_i \in K_\epsilon^1 \subset \frac{1}{r}L_\epsilon^1 \subset \frac{1}{r}L$. Thus

$$x_i \cdot u_i \leq h_{K_\epsilon^1}(u_i) \leq \frac{1}{r}h_{L_\epsilon^1}(u_i) \leq \frac{1}{r}h_L(u_i) = x_i \cdot u_i.$$

It follows that

$$rh_{K_\epsilon^1}(u_i) = h_L(u_i).$$

Thus, by $o \in [u_1, u_2, u_3]$, $rK_\epsilon^1 \subset L_\epsilon^1$ and Lemma 5.1, we know that rK_ϵ^1 is the biggest homothetic copy of K_ϵ^1 contained in L_ϵ^1. Therefore K_ϵ^1 and L_ϵ^1 are at a dilation position, for $\epsilon < \frac{R}{r} - 1$.

From (7.5), it follows that

$$K \subset K_\epsilon^1 \subset (1+\epsilon)K \subset K + \epsilon R_K B. \tag{7.7}$$

(7.5) and (7.6) give

$$L \subset (1+\epsilon)(L \cap \frac{R}{1+\epsilon}K_\epsilon) = (1+\epsilon)L_\epsilon^1 \subset L_\epsilon^1 + \epsilon L \subset L_\epsilon^1 + \epsilon R_L B. \tag{7.8}$$

Now (7.7) implies $d_H(K_\epsilon^1, K) < R_K\epsilon$, and (7.8) implies $d_H(L_\epsilon^1, L) < R_L\epsilon$. Therefore, we have

$$d_H(K_\epsilon^1, K) < c_2\epsilon, \quad \text{and} \quad d_H(L_\epsilon^1, L) < c_2\epsilon, \tag{7.9}$$

where $c_2 = \max\{R_K, R_L\}$.

By (7.5), a point p in K_ϵ^1 can be written as

$$p = \sum_{i=1}^k \lambda_i z_i,$$

with $\lambda_i \in [0,1]$, $\sum_{i=1}^k \lambda_i = 1$, and $z_i \in K \cup (1+\epsilon)\{y_1, y_2, y_3\}$. Since we have assumed $y_i \cdot v_j < h_K(v_j)$ for $y_i \neq y_j$, it follows that $F(K_\epsilon^1, v_i) = \{(1+\epsilon)y_i\}$ for $i = 1, 2, 3$.

Procedure 2. Set $R_\epsilon = R/(1+\epsilon)$. From Procedure 1, we see that $R_\epsilon = R(L_\epsilon^1, K_\epsilon^1)$, and $r = r(L_\epsilon^1, K_\epsilon^1)$. Let $K_\epsilon^2 = K_\epsilon^1 + \frac{\epsilon}{R_\epsilon}B$, $K_\epsilon^3 = K_\epsilon^1 + \frac{\epsilon}{r}B$, and $L_\epsilon = L_\epsilon^1 + \epsilon B$. For $i = 1, 2, 3$, define H_i^- by

$$H_i^- = \{x \in \mathbb{R}^2 : x \cdot v_i \leq (1+\epsilon)y_i \cdot v_i + \frac{\epsilon}{R_\epsilon}\}.$$

Then $(1+\epsilon)y_i + \frac{\epsilon}{R_\epsilon}v_i$ is the unique point in $F(K_\epsilon^2, v_i)$, for $i = 1, 2, 3$.

Since $F(K_\epsilon^1, v_i) = \{(1+\epsilon)y_i\}$, we know that $x_j \cdot v_i < h_{K_\epsilon^1}(v_i)$. Let

$$\eta = \min_{i,j\in\{1,2,3\}} \frac{h_K(v_i) - x_j \cdot v_i}{1/r - 1/R}.$$

When $\epsilon < \eta$, we have

$$x_i + \frac{1}{r}\epsilon u_i \in \partial K_\epsilon^3 \cap \operatorname{int}(H_1^- \cap H_2^- \cap H_3^-),$$

for $i = 1, 2, 3$. Now we are able to construct the desired convex body.

Notice the assumption **(A3)**. When ϵ is sufficiently small, the half-spaces H_1^-, H_2^-, H_3^- divide ∂K_ϵ^3 into 3 parts (may be distinctive or not), and we denote them by ∂_{12}, ∂_{23} and ∂_{31}. Here $\partial_{ij} = \partial K_\epsilon^3 \cap H_i^- \cap H_j^-$. Clearly, when ϵ is sufficiently small, ∂_{ij} has nonempty relative interior.

In fact, we only need to construct the boundary parts of the new convex body. Our aim is to get new boundary parts ∂'_{ij} satisfying:

(B1) the support line that supports K_ϵ^3 at $x_i + \frac{\epsilon}{r}u_i$ also supports $\partial'_{12} \cup \partial'_{23} \cup \partial'_{31}$ at $x_i + \frac{\epsilon}{r}u_i$;
(B2) the support line that supports K_ϵ^2 at $(1+\epsilon)y_i + \frac{\epsilon}{R_\epsilon}v_i$ also supports $\partial'_{12} \cup \partial'_{23} \cup \partial'_{31}$ at $(1+\epsilon)y_i + \frac{\epsilon}{R_\epsilon}v_i$;
(B3) $\partial'_{12} \cup \partial'_{23} \cup \partial'_{31}$ is the boundary of the new convex body $K_\epsilon = [\partial'_{12}, \partial'_{23}, \partial'_{31}]$, and K_ϵ is strictly convex.

Without loss of generality, we study with ∂_{12}. Define the body K_ϵ^4 by

$$K_\epsilon^4 := K_\epsilon^3 \cap H_1^- \cap H_2^- \cap H_3^-.$$

There may be the following 3 cases.

Case 1. ∂_{12} contains precisely 1 point in $\{x_1 + \frac{\epsilon}{r}u_1, x_2 + \frac{\epsilon}{r}u_2, x_3 + \frac{\epsilon}{r}u_3\}$.

Case 2. ∂_{12} contains precisely 2 points in $\{x_1 + \frac{\epsilon}{r}u_1, x_2 + \frac{\epsilon}{r}u_2, x_3 + \frac{\epsilon}{r}u_3\}$.

Case 3. ∂_{12} does not contain a point in $\{x_1 + \frac{\epsilon}{r}u_1, x_2 + \frac{\epsilon}{r}u_2, x_3 + \frac{\epsilon}{r}u_3\}$.

In Case 1, assume $x_1 + \frac{\epsilon}{r}u_1 \in \partial_{12}$. Denote by v'_1 the unit vector perpendicular to v_1 such that

$$(x_1 + \frac{\epsilon}{r}u_1) \cdot v'_1 > (1+\epsilon)y_1 \cdot v'_1.$$

For the direction v'_1, consider the overgraph functions $f(K_\epsilon^2; \cdot)$ and $f(K_\epsilon^4; \cdot)$. Denote by p_1 the projection of $(1+\epsilon)y_1 + \frac{\epsilon}{R_\epsilon}v_1$ on $l(ov_1)$ (the line through o and v_1). Denote by p_2 the projection of $x_1 + \frac{\epsilon}{r}u_1$ on $l(ov_1)$. Since $o \in [v_1, v_2, v_3]$, by the definition of K_ϵ^2 and K_ϵ^3, we see that $f(K_\epsilon^2; \cdot) < f(K_\epsilon^3; \cdot)$ on $[p_1, p_2]$. Then, it follows immediately from (i) of Lemma 7.1 that there is a boundary part ∂_{12}^l through $(1+\epsilon)y_1 + \frac{\epsilon}{R_\epsilon}v_1$ and $x_1 + \frac{\epsilon}{r}u_1$. In a similarly, we get a boundary part ∂_{12}^r through $x_1 + \frac{\epsilon}{r}u_1$ and $(1+\epsilon)y_2 + \frac{\epsilon}{R_\epsilon}v_2$. Then, $\partial'_{12} = \partial_{12}^l \cup \partial_{12}^r$ is the desired boundary part.

In Case 2, assume $x_1+\frac{\epsilon}{r}u_1, x_2+\frac{\epsilon}{r}u_2 \in \partial_{12}$. Consider the overgragh functions of K_ϵ^2 and K_ϵ^4 with respect to the direction v_1', where v_1' is the same as in Case 1. The assumption **(A2)** implies that $[x_1+\frac{\epsilon}{r}u_1, x_2+\frac{\epsilon}{r}u_2] \nsubseteq \partial K_\epsilon^3$. Then the functions $f(K_\epsilon^2;\cdot)$ and $f(K_\epsilon^4;\cdot)$ satisfy all the conditions in (ii) of Lemma 7.1. Then there is a boundary part ∂_{12}^l through $(1+\epsilon)y_1+\frac{\epsilon}{R_\epsilon}v_1, x_1+\frac{\epsilon}{r}u_1$ and $x_2+\frac{\epsilon}{r}u_2, (1+\epsilon)y_2+\frac{\epsilon}{R_\epsilon}v_2$. Similar to Case 1, we get a boundary part ∂_{12}^r through $x_2+\frac{\epsilon}{r}u_2$ and $(1+\epsilon)y_2+\frac{\epsilon}{R_\epsilon}v_2$. Then, $\partial_{12}' = \partial_{12}^l \cup \partial_{12}^r$ is the desired boundary part.

In Case 3, choose a point $z_0 \in \partial_{ij}\backslash\{(1+\epsilon)y_i+\frac{\epsilon}{R_\epsilon}v_i, (1+\epsilon)y_j+\frac{\epsilon}{R_\epsilon}v_j\}$. By using the same method as in Case 1, we get a desired boundary part through $(1+\epsilon)y_1+\frac{\epsilon}{R_\epsilon}v_1, z_0, (1+\epsilon)y_2+\frac{\epsilon}{R_\epsilon}v_2$.

Then, we get the boundary parts $\partial_{12}', \partial_{23}', \partial_{31}'$. From our construction, it is obvious that they satisfy **(B1)**, **(B2)** and **(B3)**. Recall that $K_\epsilon = [\partial_{12}', \partial_{23}', \partial_{31}']$ and $L_\epsilon = L_\epsilon^1 + \epsilon B$. Now **(B1)**, **(B2)**, and Lemma 5.1 guarantee that K_ϵ and L_ϵ are at a dilation position. By (7.9), we have

$$d_H(L_\epsilon, L) \leq d_H(L_\epsilon, L_\epsilon^1) + d_H(L_\epsilon^1, L) < (1+c_2)\epsilon.$$

It is also easy to see that

$$K_\epsilon^2 \subset K_\epsilon \subset K_\epsilon^3.$$

From this, $K_\epsilon^2 = K_\epsilon^1 + \frac{\epsilon}{R_\epsilon}B$, $K_\epsilon^3 = K_\epsilon^1 + \frac{\epsilon}{r}B$, and (7.9), we deduce

$$d_H(K_\epsilon, K) \leq d_H(K_\epsilon, K_\epsilon^1) + d_H(K_\epsilon^1, K) < (\frac{1}{r}+c_2)\epsilon.$$

Then we finished the proof of this lemma, provided $c_1 = \max\{1, \frac{1}{r}\} + c_2$. □

Lemma 7.3. *Let K, L be planar convex bodies with $o \in \partial K \cap \partial L$. Suppose K and L are not dilates, and they are at a dilation position. Then, for each $\epsilon > 0$, there are convex bodies $K_\epsilon, L_\epsilon \in \mathcal{K}_o^2$ so that K_ϵ and L_ϵ are at a dilation position, and*

$$d_H(K_\epsilon, K),\ d_H(L_\epsilon, L) < \epsilon.$$

Proof. Set $r = r(L,K), R = R(L,K)$, and $B = B^2$. By Lemma 5.1, there are $u_i, v_i \in S^1$, $x_i \in \partial K \cap \partial(\frac{1}{r}L)$ and $y_i \in \partial K \cap \partial(\frac{1}{R}L)$, so that $o \in [u_1,u_2,u_3] \cap [v_1,v_2,v_3]$,

$$h_L(u_i) = rh_K(u_i) = rx_i \cdot u_i, \quad \text{and} \quad h_L(v_i) = Rh_K(v_i) = Ry_i \cdot v_i, \tag{7.10}$$

for $i = 1,2,3$. Here u_1,u_2,u_3 may be distinctive or not, and so is the triple v_1,v_2,v_3.

We shall use the same assumptions **(A1)**, **(A2)** and **(A3)** as in the proof of Lemma 7.2, with the same reason. In addition, we should give the following assumption.

(A4) If $o \neq x_i$, assume $o \notin F(K,u_i)$; if $o \neq y_i$, assume $o \notin F(K,v_i)$.

Otherwise, suppose $o \neq x_1$ and $o \in F(K,u_1)$ (the discussion of the case $o \neq y_i$ is similar). Since $L \subset RK$, and $o \in \partial(RK) \cap \partial L$, we see that u_1 is also a normal vector of L at o. Then, we can replace x_1 by o. That is, consider $\{o, x_2, x_3\}$ with normal vectors $\{u_1,u_2,u_3\}$. Thus, this assumption is reasonable.

We will consider 2 cases.

Case 1. $o \notin \{x_1,x_2,x_3\} \cap \{y_1,y_2,y_3\}$.

If $o \notin \{y_1, y_2, y_3\}$, then, by **(A4)**, we deduce $h_K(v_i) > 0$, for $i = 1, 2, 3$. This makes us be able to use Procedure 1 and Procedure 2 in the proof of Lemma 7.2 directly to construct the desired bodies K_ϵ and L_ϵ. It is just an examination of the method there, so we omit it. After Procedure 2, it is clear that the resulting bodies K_ϵ and L_ϵ satisfy $\frac{1}{R}\epsilon B \subset K_\epsilon$ and $\epsilon B \subset L_\epsilon$, thus they contain o in their interiors.

If $o \notin \{x_1, x_2, x_3\}$, then, by **(A4)**, we deduce $h_L(u_i) > 0$, for $i = 1, 2, 3$. By changing the position of K and L, we can also use Procedure 1 and Procedure 2 in the proof of Lemma 7.2 to construct the desire bodies.

Case 2. $o \in \{x_1, x_2, x_3\} \cap \{y_1, y_2, y_3\}$. Assume without loss of generality that

$$x_1 = y_1 = o. \tag{7.11}$$

Write $u^\theta = (\cos\theta, \sin\theta)$, for $\theta \in [-\pi, \pi]$. Define the half-spaces H_θ^- by

$$H_\theta^- := \{x : x \cdot u^\theta \le 0\},$$

and denote its boundary by H_θ. Without loss of generality, assume $v_1 = u^0$. There are θ_1, θ_2 with $\theta_1 \le 0 \le \theta_2$, so that θ_1 is the minimum in $[-\pi, \pi]$ so that $K \subset H_{\theta_1}^-$, and θ_2 is the maximum in $[-\pi, \pi]$ so that $K \subset H_{\theta_2}^-$.

In addition to **(A3)**, it will be convenient to assume that

$$o \ne x_2 \qquad \text{and} \qquad o \ne x_3.$$

Let $\delta > 0$. Define K_δ and L_δ as follows.

If $\{x_2, x_3, y_2, y_3\} \cap H_{\theta_1} \ne \emptyset$, let $K_\delta^1 = K$ and $L_\delta^1 = L$; if $\{x_2, x_3, y_2, y_3\} \cap H_{\theta_1} = \emptyset$, let $K_\delta^1 = K \cap H_{\theta_1-\delta}^-$ and $L_\delta^1 = L \cap H_{\theta_1-\delta}^-$.

If $\{x_2, x_3, y_2, y_3\} \cap H_{\theta_2} \ne \emptyset$, let $K_\delta = K_\delta^1$ and $L_\delta = L_\delta^1$; if $\{x_2, x_3, y_2, y_3\} \cap H_{\theta_2} = \emptyset$, let $K_\delta = K_\delta^1 \cap H_{\theta_2+\delta}^-$ and $L_\delta = L_\delta^1 \cap H_{\theta_2+\delta}^-$.

Then, for sufficiently small $\delta > 0$, the points o, x_2, x_3 are also in $K_\delta \cap (\frac{1}{r}L_\delta)$, and the points o, y_2, y_3 are also in $K_\delta \cap (\frac{1}{R}L_\delta)$. Clearly, $\lim_{\delta\to 0^+} K_\delta = K$ and $\lim_{\delta\to 0^+} L_\delta = L$.

Furthermore, by the definition of K_δ and L_δ, and **(A4)**, there are 2 distinctive facet (1-dimensional face) containing o. Thus, there are points $z_\delta^1, z_\delta^2 \notin \{o, y_2, y_3\}$, so that

$$z_\delta^1 \in \partial K_\delta \cap \partial(\frac{1}{R}L_\delta) \cap (H_{\theta_1-\delta} \cup H_{\theta_1}),$$

and

$$z_\delta^2 \in \partial K_\delta \cap \partial(\frac{1}{R}L_\delta) \cap (H_{\theta_2+\delta} \cup H_{\theta_2}).$$

Let v_δ^i be a unit normal vector at z_δ^i, for $i = 1, 2$. Then, v_1 is a positive combination of v_δ^1 and v_δ^2.

Now $o \in [v_\delta^1, v_\delta^2, v_2, v_3]$, by Carathéodory's theorem, there are 3 or fewer members of them containing o in their convex hull. Denote them by v_1', v_2', v_3', and denote the corresponding boundary points by y_1', y_2', y_3'. Then, $o \notin \{y_1', y_2', y_3'\}$. We can assume that $\{y_1', y_2', y_3'\}$ and $\{v_1', v_2', v_3'\}$ satisfies **(A1)**, and $o \notin \{y_1', y_2', y_3'\}$ will be preserved. Then, by using Procedure 1 and Procedure 2 in the proof of Lemma 7.2 for K_δ and L_δ, we get the desired convex bodies. After Procedure 2, it is clear that the resulting bodies K_ϵ and L_ϵ satisfy $\frac{1}{R}\epsilon B \subset K_\epsilon$ and $\epsilon B \subset L_\epsilon$, thus they contain o in their interiors. □

Lemma 7.4. *Let $K, L \in \mathcal{K}_o^2$. If K and L are at a dilation position, then*

$$\int_{S^1} \log \frac{h_L}{h_K} dV_K \geq \frac{|K|}{2} \log \frac{|L|}{|K|} \tag{7.12}$$

Equality holds if and only if K and L are dilates or K and L are parallelograms with parallel sides.

Proof. First, suppose K and L are not dilates. By Lemma 7.2, there are $K_i \in \mathcal{K}_o^2$, and $L_i \in \mathcal{K}_o^2$, such that K_i are strictly convex, $K_i \to K$ and $L_i \to L$, and K_i and L_i are at a dilation position. Since K_i are strictly convex, the cone-volume measures V_{K_i} satisfy the strictly subspace concentration inequality. Thus, by Theorem 6.1, we have

$$\int_{S^1} \log \frac{h_{L_i}}{h_{K_i}} dV_{K_i} \geq \frac{|K_i|}{2} \log \frac{|L_i|}{|K_i|}.$$

Since $K_i \to K$ and $L_i \to L$, it follows that the functions $\log \frac{h_{L_i}}{h_{K_i}}$ converge to $\log \frac{h_L}{h_K}$ uniformly on S^1, and the cone-volume measures V_{K_i} converge weakly to V_K, then (7.12) follows.

Next, if K and L are dilates or parallelogram with parallel sides, then it is easy to see that the equality in (7.12) holds.

Finally, suppose equality holds in (7.12). Then the convex body $L/|L|^{\frac{1}{2}}$ must be a minimizer of the problem (6.1) for $K/|K|^{\frac{1}{2}}$. It follows immediately from Lemma 6.3 that K and L are dilates or parallelograms with parallel sides. □

Proof of Theorem 3. If $K, L \in \mathcal{K}_o^2$, then Theorem 3 follows immediately from Lemma 7.4. Suppose $o \in \partial K \cap \partial L$, and K and L are not dilates. By Lemma 7.3, there are convex bodies $K_i, L_i \in \mathcal{K}_o^2$, such that $K_i \to K$ and $L_i \to L$, and K_i and L_i are at a dilation position. By Lemma 7.4, and $dV_{K_i} = \frac{1}{2} h_{K_i} dS_{K_i}$, we have

$$\frac{1}{2} \int_{S^1} \big(\log \frac{h_{L_i}}{h_{K_i}}\big) h_{K_i} dS_{K_i} \geq \frac{|K_i|}{2} \log \frac{|L_i|}{|K_i|}.$$

Since $r(K, L)L \subset K \subset R(K, L)L$, we see that $h_K(u) = 0$ if and only if $h_L(u) = 0$. Define the set ω by

$$\omega := \{u \in S^1 : h_K(u) = 0\} = \{u \in S^1 : h_L(u) = 0\},$$

and define $(\log \frac{h_L(u)}{h_K(u)}) h_K(u) = 0$ for $u \in \omega$. Then, it is easy to see from $r(L, K) \leq \frac{h_L}{h_K} \leq R(L, K)$ that the function $(\log \frac{h_L}{h_K}) h_K$ is well-defined and continuous on S^1.

Since $K_i \to K$ and $L_i \to L$, by using the same method as in the proof of Lemma 6.2, we deduce that $r(L_i, K_i) \to r(L, K)$ and $R(L_i, K_i) \to R(L, K)$. Then, $\log \frac{h_{L_i}}{h_{K_i}}$ are uniformly bounded. This, together with the fact that $h_{K_i} \to h_K$ uniformly, shows that $(\log \frac{h_{L_i}}{h_{K_i}}) h_{K_i} \to (\log \frac{h_L}{h_K}) h_K$ uniformly on S^1. The fact $K_i \to K$ also implies that the surface area measures S_{K_i} converge to S_K weakly. From these, and the continuity of Lebesgue measure, (1.10) follows.

If K and L are dilates or parallelograms with parallel sides, then it is easy to see that the equality in (1.10) holds.

Suppose equality holds in (1.10). Then the convex body $L/|L|^{\frac{1}{2}}$ must be a minimizer of the problem (6.1) for $K/|K|^{\frac{1}{2}}$. It follows immediately from Lemma 6.3 that K and L are dilates or parallelograms with parallel sides. . □

Proof of Theorem 2. By Lemma 5.5 and Theorem 3, the inequality (1.9) holds. Suppose $\lambda \in (0, 1)$. If K and L are dilates, or they are parallelograms with parallel sides, then it is clear that the equality in (1.9) holds.

If equality holds in (1.9), then, by the proof of Lemma 5.5, equality in (5.3) holds. From the equality condition for the log-Minkowski inequality (1.10), it follows that either $(1-\lambda)\cdot K +_o \lambda\cdot L$, K and L are dilates, or they are parallelograms with parallel sides. □

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