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Calculus of Variations

Order property and modulus of continuity of weak KAM solutions

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Abstract For the Hamilton–Jacobi equation $H(x, \partial_x u + c) = \alpha(c)$ with $x \in \mathbb{T}^2$, it is shown in this paper that, for all $c \in \alpha^{-1}(E)$ with $E > \min \alpha$, the elementary weak KAM solutions can be parameterized so that they are $\frac{1}{3}$-Hölder continuous in C^0-topology.

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1 Introduction

In this paper, we study the weak KAM solutions of the Hamilton–Jacobi equation

$$H(x, \partial_x u) = E, \qquad x \in \mathbb{T}^2 \tag{1.1}$$

where H is a Tonelli Hamiltonian (see below), E is larger than the minimum of the α-function determined by H (see below).

1.1 Preliminaries on Mather theory

Let M be a closed manifold. A Hamiltonian $H \in C^2(T^*M \times \mathbb{T}, \mathbb{R})$ is called Tonelli if the Hessian matrix $\partial^2_{yy} H$ is positive definite everywhere, $H/\|y\| \to \infty$ as $\|y\| \to \infty$ and the Hamiltonian flow is complete. For autonomous Hamiltonian, the completeness is automatically satisfied, since each orbit entirely stays in certain compact energy level set. A Tonelli Hamiltonian is uniquely related to a Tonelli Lagrangian by the Legendre transformation

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$$L(x,\dot{x},t)=\max_{y}\{\langle\dot{x},y\rangle-H(x,y,t)\}.$$

The Mather theory is established for Tonelli Lagrangian, see [15–17].

We next define the minimal measure. We notice that, $\forall$ C^1 curve $\gamma\colon\mathbb{R}\to M$ with period k, there is a unique probability measure μ_γ on $TM\times\mathbb{T}$ so that the following holds

$$\int_{TM\times\mathbb{T}} f\,d\mu_\gamma=\frac{1}{k}\int_0^k f(d\gamma(s),s)\,ds$$

for each $f\in C^0(TM\times\mathbb{T},\mathbb{R})$, where we use the notation $d\gamma=(\gamma,\dot{\gamma})$. Let

$$\mathfrak{H}^*=\{\mu_\gamma|\ \gamma\in C^1(\mathbb{R},M)\ \text{ is periodic of } k\in\mathbb{Z}^+\}.$$

The set $\mathfrak{H}$ of holonomic probability measures is the closure of $\mathfrak{H}^*$ in the vector space of continuous linear functionals. Given a cohomology class $c\in H^1(M,\mathbb{R})$, the Lagrange action $A_c(\mu)$ over each $\mu\in\mathfrak{H}$ is defined as follows

$$A_c(\mu)=\int(L-\eta_c)d\mu$$

where $\eta_c(x,\dot{x})=\langle\eta_c(x),\dot{x}\rangle$ stands for a Lagrange multiplier, while $\langle\eta_c(x),dx\rangle$ denotes a closed 1-form so that $[\langle\eta_c(x),dx\rangle]=c$.

It is proved in [15,17] that for each class c there exists at least one probability measure μ_c minimizing the action over $\mathfrak{H}$

$$A_c(\mu_c)=\inf_{\mu\in\mathfrak{H}}\int(L-\eta_c)d\mu,$$

which is invariant for the Lagrange flow ϕ_L^t, called c-minimal measure. The α-function is defined as $\alpha(c)=-A_c(\mu_c):H^1(M,\mathbb{R})\to\mathbb{R}$. It is convex, finite everywhere with super-linear growth. Its Legendre dual $\beta:H_1(M,\mathbb{R})\to\mathbb{R}$ is called β-function

$$\beta(\omega)=\max_{c}\{\langle\omega,c\rangle-\alpha(c)\}.$$

It is also convex, finite everywhere with super-linear growth. The Fenchel–Legendre transformation $\mathscr{L}_\beta\colon H_1(M,\mathbb{R})\to H^1(M,\mathbb{R})$ is defined as follows

$$c\in\mathscr{L}_\beta(\omega)\quad\Longleftrightarrow\quad\alpha(c)+\beta(\omega)=\langle c,\omega\rangle.$$

Let $\mathfrak{H}_c\subset\mathfrak{H}$ be the set of c-minimal measures, the *Mather set* is defined as

$$\tilde{\mathcal{M}}(c)=\bigcup_{\mu_c\in\mathfrak{H}_c}\text{supp}\mu_c.$$

Let $\mathscr{L}_L\colon TM\times\mathbb{T}\to T^*M\times\mathbb{T}$ be the map such that $(x,\dot{x},t)\to(x,y=\partial_{\dot{x}}L(x,\dot{x},t),t)$. We also call the set $\mathscr{L}_L(\tilde{\mathcal{M}}(c))$ the Mather set for c. For autonomous system, we skip the component of time t, namely, $\mathscr{L}_L(\tilde{\mathcal{M}}(c))\subset T^*M$.

The Hamilton–Jacobi equation

$$H(x,\partial_x u+c)=\alpha(c)\tag{1.2}$$

was studied in [12] where the existence of viscosity solutions was established for continuous Hamiltonians periodic in x and coercive in p. When the Hamiltonian is Tonelli, the connection of the viscosity solution theory to the Aubry–Mather theory was shown in [7,8]. For each $c\in H^1(M,\mathbb{R})$, the equation admits viscosity solutions $u_c^+,u_c^-:M\to\mathbb{R}$. They are Lipschitz function and determine asymptotic orbit in the following sense. If $u^\pm$ is differentiable at x, the initial condition $(x, \partial u_c^{\pm}(x))$ determines an orbit $(x(t), y(t))$ of the Hamiltonian flow that approaches the Mather set as $t \to \pm\infty$. In other words, the weak KAM solution $u_c^{\pm}$ defines the stable (unstable) set of the Mather set

$$W_c^s = \bigcup_{x \in M} \Big\{x, \frac{\partial u_c^+(x)}{\partial x}\Big\}, \qquad W_c^u = \bigcup_{x \in M} \Big\{x, \frac{\partial u_c^-(x)}{\partial x}\Big\}.$$

Up to a constant, there is a unique pair of weak KAM solutions if there exists only one Aubry class to be defined next. To define the Aubry set for the class c, let

$$[A_c(\gamma)|_{[t,t']}] = \int_t^{t'} \Big(L(d\gamma(t), t) - \eta_c(d\gamma(t))\Big)dt + \alpha(c)(t' - t),$$

if $\gamma\colon [t, t'] \to M$ is an absolutely continuous curve. For time-T-periodic Lagrangian we say that $(x, t) \in \mathcal{A}(c)$ the *Aubry set* if there exists a sequence of closed curve $\gamma_{k_i}\colon [t, t+k_i T] \to M$ such that $\gamma_{k_i}(t) = \gamma_{k_i}(t + k_i T) = x$ and $[A_c(\gamma_{k_i})] \to 0$ as $\mathbb{Z} \ni k_i \to \infty$. For autonomous system one skips the time component. Curves in the Aubry set $\mathcal{A}(c)$ are called *c-static curves.*

To define Aubry class, let

$$h_c((x, \tau), (x', \tau')) = \inf_{\substack{\xi \in C^1, \xi(\tau)=x \\ \xi(\tau')=x'}} [A_c(\xi)|_{[\tau,\tau']}],$$

and for time-T-periodic Lagrangian one defines

$$h_c^{\infty}((x, t), (x', t')) = \liminf_{\substack{\tau=t \mod T \\ \tau'=t' \mod T \\ \tau'-\tau\to\infty}} h_c((x, \tau), (x', \tau')).$$

If the Lagrangian is autonomous, one has

$$h_c^{\infty}(x, x') = \lim_{\tau'-\tau\to\infty} h_c((x, \tau), (x', \tau')).$$

With h_c^{∞} Mather introduced a pseudo metric on the Aubry set

$$d_c((x, t), (x', t')) = h_c^{\infty}((x, t), (x', t')) + h_c^{\infty}((x', t'), (x, t)).$$

Two points (x, t) and (x', t') are said to be in one *Aubry class* if $d_c((x, t), (x', t')) = 0$.

1.2 Elementary weak KAM solutions

If two or more Aubry classes exist, there are infinitely many weak KAM solutions, among which we are interested in so-called *elementary weak KAM solution*, obtained from the function h_c^{∞}. Indeed, treated as the function of (x, t), the function $h_c^{\infty}((x, t), (x', t'))$ is a weak KAM solution that determines orbits approaching the Aubry set as the time approaches infinity, treated as the function of (x', t'), the function $h_c^{\infty}((x, t), (x', t'))$ is a weak KAM solution that determines orbits approaching the Aubry set as the time approaches minus infinity. Let (x, t) range over an Aubry class, denoted by $\mathcal{A}_{c,i}$ one has a decomposition

$$h_c^{\infty}((x, t), (x', t')) = u_{c,i}^-(x', t') - u_{c,i}^+(x, t), \qquad \forall\, (x', t') \in M \times \mathbb{T},$$

where $u_{c,i}^+$ is a constant, and $u_{c,i}^-$ is called elementary weak KAM solution with respect to $\mathcal{A}_{c,i}$. Similarly, let (x', t') range over an Aubry class, one obtains an elementary weak KAM solution $u_{c,i}^-$. Again, for autonomous system, one skips the time component.

In this paper, we study the special case $M = \mathbb{T}^2$. We next define the globally elementary weak KAM solutions on $\mathbb{R}^2$, the universal covering space of $\mathbb{T}^2$. The well-definedness of these objects will be shown in Sect. 2. For $c \in \alpha^{-1}(E)$ with $E > \min \alpha$, we shall show later in Lemma 2.2 that if $\omega_1, \omega_2 \in \mathscr{L}_\beta^{-1}(c)$, then $\exists\, \nu > 0$ such that $\omega_1 = \nu\omega_2$. So we are only concerned about the direction of rotation vectors. A rotation vector $\omega \in \mathbb{R}^2$ is called irrational if there does not exist real number $\nu \neq 0$ such that $\nu\omega \in \mathbb{Z}^2$.

Definition 1.1 For each rotation vector $\omega \in \mathscr{L}_\beta^{-1}(\alpha^{-1}(E))$,

(1) if $\mathscr{L}_\beta(\omega) \cap \alpha^{-1}(E)$ is a singleton $\{c\}$, then we define *globally elementary weak KAM solution on* $\mathbb{R}^2$

$$U_c^{\pm}(x) = \bar{u}_c^{\pm}(x) + \langle c, x\rangle,$$

where $\bar{u}_c^{\pm}(x)$ is the lift of the elementary weak KAM solution $u_c^{\pm}(x)$ to $\mathbb{R}^2$ satisfying $\bar{u}_c^{\pm}(0) = 0$. This includes all the cases of irrational ω.

(2) If $\mathscr{L}_\beta(\omega) \cap \alpha^{-1}(E)$ is a line segment with endpoints c^l and c^r, for $c \in \{c^l, c^r\}$, the weak KAM solution $u_c^{\pm}$ is uniquely determined if we normalize $u_c^{\pm}(0) = 0$. We define

$$U_{c^l}^{\pm}(x) := u_{c^l}^{\pm}(x) + \langle c^l, x\rangle, \quad U_{c^r}^{\pm}(x) := u_{c^r}^{\pm}(x) + \langle c^r, x\rangle.$$

Remark 1.1 For $E > \min \alpha$, the set $\{c : \alpha(c) \le E\}$ is a convex set containing 0 if we assume $\alpha(0) = \min \alpha$ by adding a closed 1-form to the Lagrangian. In this case, each point $c \in \alpha^{-1}(E)$ can be identified as a point $\frac{c}{\|c\|}$ on the unit circle on $\mathbb{S}^1$, and the line segment in case (2) above can be identified as an interval on $\mathbb{S}^1$. The superscripts "l" and "r" means "left" and "right" for an observer at the center.

1.3 The main result

We are going to establish certain modulus of continuity of elementary weak KAM solution defined on the universal covering space, instead of the original manifold. The main result is the following

Theorem 1.1 *Let $\mathcal{E}_E$ be the set of extremal points of the convex set $\cup_{E' \le E}\{\alpha^{-1}(E')\}$, $E > \min \alpha$. For given bounded domain $\Omega \subset \mathbb{R}^2$, there exists a constant $C(\Omega, H)$ depending only on Ω and the Tonelli Hamiltonian H, and a one-to-one parametrization of the elementary weak KAM solutions of cohomology classes in $\mathcal{E}_E$ by a number $\sigma \in \Sigma \subset [0, 1]$, such that we have the following Hölder regularity: $\forall\, \sigma \in \Sigma, \forall\, c \in c(\Sigma) = \mathcal{E}_E$,*

$$\|U_{c(\sigma)}^{\pm} - U_{c(\sigma')}^{\pm}\|_{C^0(\Omega)} \le C(\Omega, H)(\|c(\sigma) - c(\sigma')\| + |\sigma - \sigma'|^{\frac{1}{3}}),$$

where C is a constant depending only on the Tonelli Hamiltonian H.

The modulus continuity of elementary weak KAM solution plays a key role in the study of global dynamics. By applying the modulus continuity of Peierls's barrier in [13], Mather proved that for a twist map an invariant circle with Liouville rotation number can be destructed by arbitrarily C^∞-small perturbation [14]. Another case is about Arnold diffusion in *a priori* unstable system. With the modulus of continuity of barrier function one obtains the genericity of diffusion orbits [4,5,20].

It is natural to ask if similar Hölder regularity results hold in higher dimensional cases. The proof of the main theorem relies crucially on the order property (Proposition 4.3) coming from the low dimensionality. We consider our result as an early step in understanding the regularity problem of all weak KAM solutions on arbitrary manifold M. However, to study manifolds with dimension greater than 2, some new ideas are needed. In the presence of the normally hyperbolic invariant manifold, some partial result can be obtained. See Sect. 6.

The paper is organized as follows.

- In Sect. 2, we study globally elementary weak KAM solutions on $\mathbb{R}^2$.
- In Sect. 3, we use method of viscosity solutions to study the structure of the zero set of the difference of two globally elementary weak KAM solutions.
- In Sect. 4, we study the order property of the globally elementary weak KAM solutions.
- In Sect. 5, we prove the main theorem.
- In Sect. 6, we generalize the regularity result in the main theorem to the higher dimensional case, with the help of the normally hyperbolic invariant manifold structure.
- In "Appendix A", we prove a technical lemma used in Sect. 3.
- In "Appendix B", we give further structure of the weak KAM solutions in the rational case.

2 Elementary weak KAM on universal covering space

In this section we study the relation of the weak KAM solutions on a manifold M, a finite covering space $\bar{M}$ and its universal covering space. Let $\mathcal{A}(c, \bar{M})$ denote the Aubry set and let $\bar{\mathcal{A}}(c, M)$ denote the lift of $\mathcal{A}(c, M)$ to the covering space, then

Lemma 2.1 $\bar{\mathcal{A}}(c, M) = \mathcal{A}(c, \bar{M})$.

Proof For each point $x \in \mathcal{A}(c, M)$, by definition, there exists a sequence of closed curves $\gamma_k : [0, T_k] \to M$ such that $\gamma_k(0) = \gamma_k(T_k) = x$ and $[A_c(\gamma_k)] \to 0$ as $T_k \to \infty$. If the lift of γ_k consists of several curves $\bar{\gamma}_{k,1}, \ldots, \bar{\gamma}_{k,j}$, either each curve is still a closed curve or the conjunction $\bar{\gamma}_{k,1} * \cdots * \bar{\gamma}_{k,j}$ is a closed curve. Clearly, $[A_c(\bar{\gamma}_{k,1} * \cdots * \bar{\gamma}_{k,j})] = j[A_c(\gamma_k)] \to 0$ as $T_k \to \infty$. This proves $\bar{\mathcal{A}}(c, M) \subseteq \mathcal{A}(c, \bar{M})$.

Given a point $\bar{x} \in \mathcal{A}(c, \bar{M})$, there exists a sequence of closed curves $\{\bar{\gamma}_k : [0, T_k] \to \bar{M}\}$ such that $\bar{\gamma}_k(0) = \bar{\gamma}_k(T_k) = \bar{x}$ and $[A_c(\bar{\gamma}_k)] \to 0$ as $T_k \to \infty$. Let γ_k be the projection of $\bar{\gamma}_k$ to M, one has $[A_c(\bar{\gamma}_k)] = [A_c(\gamma_k)]$. It implies $\bar{\mathcal{A}}(c, M) \supseteq \mathcal{A}(c, \bar{M})$. □

Let us focus on 2-torus $\mathbb{T}^2$.

Lemma 2.2 *For $c \in \alpha^{-1}(E)$ with $E > \min \alpha$, the Fenchel-Legendre dual $\omega(c) = \mathscr{L}_\beta^{-1}(c)$ is either a non-zero vector or a radial line segment which does not contain 0.*

Proof It is known that $\mathscr{L}_\beta^{-1}(c)$ is a convex set. Suppose that $\omega(c)$ contains two extremal points $\omega_1 \neq \omega_2$, because of the special topology of $\mathbb{T}^2$, certain nonzero number $\xi \in \mathbb{R}$ exists so that $\omega_1 = \xi\omega_2$. Otherwise, there would be two intersecting c-minimal curves, which violates Mather's graph theorem. □

To study the elementary weak KAM in the universal covering space, let us consider a finite covering space $\bar{\pi}_k$: $k\mathbb{T}^2 = \{x \in \mathbb{R}^2 : x_i \bmod k_i\} \to \mathbb{T}^2$ where $k = (k_1, k_2) \in \mathbb{Z}^2$ with $k_1, k_2 > 0$. Let $u^\pm_{c,k}$ denote the elementary weak KAM solution if the covering space $k\mathbb{T}^2$ is treated as the configuration manifold, and $u^\pm_{c,k}$ is treated as a function defined on $\mathbb{R}^2$, k_i-periodic in x_i for $i = 1, 2$. For the configuration manifold $k\mathbb{T}^2$ with $\min\{k_1, k_2\} \to \infty$, the number of ergodic minimal measures may increase, depending on whether $\omega(c)$ is irrational. By definition of the elementary weak KAM solutions in Sect. 1.2, the elementary weak KAM solutions for the covering spaces can be defined explicitly as follows. Given an Aubry class $\mathcal{A}(c, k\mathbb{T})$, we have the associated elementary weak KAM solution

$$u_{c,k}(x) = \inf_{\xi, t<0} \left\{ \int_t^0 L(\dot{\xi}(t), \xi(t)) + \langle c, \dot{\xi}(t) \rangle \, dt \mid \xi(0) = x, \ \xi(t) \in x_0 + k\mathbb{Z}^2 \right\}$$

where x_0 is any point in $\mathcal{A}(c, k\mathbb{T})$.

In the following two subsections, we show that the globally elementary weak KAM solutions on $\mathbb{R}^2$ in Definition 1.1 are well-defined. Note that instead of defining the globally elementary weak KAM solutions directly on $\mathbb{R}^2$, we define them by lifting the weak KAM solutions on $\mathbb{T}^2$.

2.1 Uniqueness of elementary weak KAM solutions: the irrational case

Proposition 2.3 *For $c \in H^1(\mathbb{T}^2, \mathbb{R})$ with irrational rotation vector $\omega(c)$, there is a unique ergodic c-minimal measure with respect to $k\mathbb{T}^2$, $k \in \mathbb{Z}^2$ with $k = (k_1, k_2)$ and $k_1 > 0$, $k_2 > 0$. For different covering manifolds $k\mathbb{T}^2$, $k'\mathbb{T}^2$, if we think the elementary weak KAM solutions $u^{\pm}_{c,k}$, $u^{\pm}_{c,k'}$ as functions defined on $\mathbb{R}^2$, then one has $u^{\pm}_{c,k} = u^{\pm}_{c,k'}$ up to an additive constant.*

Proof By definition, each curve in the lift of a c-static curve to the finite covering manifold is obviously c-static. Let γ be a curve lying in $\mathcal{A}(c, \mathbb{T}^2)$, and $\bar{\gamma}_k$ be a curve in the lift of γ. Then one has

$$u^{\pm}_c(\gamma(t')) - u^{\pm}_c(\gamma(t)) = u^{\pm}_{c,k}(\bar{\gamma}_k(t')) - u^{\pm}_{c,k}(\bar{\gamma}_k(t)), \qquad \forall\, t' \geq t. \tag{2.1}$$

It follows that, restricted on the Aubry set, one has $u^{\pm}_{c,k} = u^{\pm}_{c,k'}$.

Let $x \in \mathbb{T}^2$ where u^-_c is differentiable, there exists a unique curve $\gamma^-_x \colon (-\infty, 0] \to \mathbb{T}^2$ such that $\gamma^-_x(0) = x$ and

$$u^-_c(x) = u^-_c(\gamma^-_x(-t)) + [A_c(\gamma^-_x|_{[-t,0]})], \qquad \forall\, t \in [0, \infty).$$

Let $\bar{x} \in k\mathbb{T}^2$ be a point in the lift of x, $u^-_{c,k}$ also determines a minimal curve $\bar{\gamma}_{\bar{x},k}$ such that $\bar{\gamma}_{\bar{x},k}(0) = \bar{x}$ and

$$u^-_{c,k}(\bar{x}) = u^-_{c,k}(\bar{\gamma}_{\bar{x},k}(-t)) + [A_c(\bar{\gamma}_{\bar{x},k}|_{[-t,0]})], \qquad \forall\, t \in [0, \infty).$$

We claim that $\bar{\pi}_k \bar{\gamma}_{\bar{x},k} = \gamma^-_x$. Indeed, let $\bar{\gamma}^-_x$ be the lift of γ^-_x to $k\mathbb{T}^2$ such that $\bar{\gamma}^-_x(0) = \bar{x}$, then there exist two sequences of times $\{t_\ell, t'_\ell\}$ such that $|\bar{\gamma}^-_x(-t_\ell) - \bar{\gamma}^-_{\bar{x},k}(-t'_\ell)| \to 0$ when $t_\ell, t'_\ell \to \infty$, because there is only one ergodic minimal measure. If $\bar{\pi}_k \bar{\gamma}^-_{\bar{x},k} \neq \gamma^-_x$, one would have $[A_c(\bar{\gamma}^-_{\bar{x},k}|_{[-t'_\ell,0]})] > [A_c(\bar{\gamma}^-_x|_{[-t_\ell,0]})]$ for large ℓ. It follows from (2.1) that $u^-_c(\gamma^-_x(-t_\ell)) > u^-_{c,k}(\bar{\gamma}^-_{\bar{x},k}(-t'_\ell))$ for large ℓ, but it is absurd because, regarded as a function defined on $\mathbb{R}^2$, we have $u^-_c = u^-_{c,k}$ when they are restricted on the lift of the Aubry set. □

2.2 Uniqueness of elementary weak KAM solutions: the rational case

Assume that $\omega \in \mathscr{L}_\beta(\alpha^{-1}(E))$ is rational. Then there are two cases, either $\mathscr{L}^{-1}_\beta(\omega) \cap \alpha^{-1}(E)$ contains a single point or an interval I_ω. The first case occurs if and only if the torus is foliated by periodic orbits with the rotation vector ω. Up to a coordinate transformation on $\mathbb{T}^2$, we assume $\omega = (0, \hat{\omega})$.

Lemma 2.4 *Suppose ω is rational. Let c be $\mathscr{L}^{-1}_\beta(\omega)$ when $\mathscr{L}^{-1}_\beta(\omega)$ is a single point, or be either end point when $\mathscr{L}^{-1}_\beta(\omega)$ is an interval. In either case, there is only one Aubry class, each connected component of the set $\mathbb{T}^2 \backslash (\mathcal{A}(c) + \delta)$ is contractible, where $\mathcal{A}(c) + \delta = \{x \in \mathbb{T}^2 : \mathrm{dist}(x, \mathcal{A}(c)) < \delta\}$ and $\delta > 0$ can be arbitrarily small.*

Proof In this case, the Mather set consists of periodic curves $\{\gamma_{\omega,\lambda}\}$ sharing the same homology type $(0, 1)$, and remains the same for all $c \in I_\omega$. Next, we consider the weak KAM

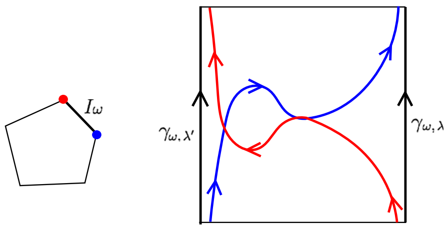

Fig. 1 The blue curve is in $\mathcal{A}(c)$ for c at one end point of I_ω, the red curve is in $\mathcal{A}(c')$ for c' at another end point of I_ω

solution for the class c an end point of I_ω. In this case, the Aubry set properly contains the Mather set as well as some static curve approaching periodic curves when $t \to \pm\infty$. Indeed, the complementary part of the Mather set is made up of annuli, each annulus is crossed by at least one static curve (Fig. 1). □

Lemma 2.5 *Assume $c \in \partial I_\omega$ and ω is rational. Let $\bar{\pi}_k$: $k\mathbb{T}^2 \to \mathbb{T}^2$ is a covering space, let $\bar{x}, \bar{x}' \in k\mathbb{T}^2$ be the points such that $\bar{\pi}_k\bar{x} = \bar{\pi}_k\bar{x}' = x \in \mathcal{M}(c)$. Then $h_c^\infty(\bar{x}, \bar{x}') = 0$.*

Proof For $c \in I_\omega$, there is a periodic curve η which entirely lies in the Mather set for c. If the lemma does not hold, there would be some $D > 0$ such that for any closed curve ξ: $[0, T] \to \mathbb{T}^2$ with $[\xi] \neq \ell\omega\ \forall\ \ell \in \mathbb{Z}$, one has $[A_c(\xi)] \geq D$, where

$$[A_c(\xi)] = \int_0^T (L(\dot{\xi}(s), \xi(s)) - \langle c, \dot{\xi}\rangle + \alpha(c))ds.$$

For $c' \in \alpha^{-1}(E)\backslash I_\omega$ with irrational rotation vector, we pick up a curve lying in $\mathcal{A}(c')$. This curve γ intersects the the curve η infinitely many times. Let $\cdots < t_i < t_{i+1} < \cdots$ be a sequence of time when the curve γ intersects the curve η. Let $\gamma_i = \gamma|_{[t_i,t_{i+1}]}$ be a segment of γ, η_i be a segment of η that connects $\gamma(t_{i+1})$ to $\gamma(t_i)$. Since the curve $\eta_i * \gamma_i$ is closed and $[\eta_i * \gamma_i] \neq \ell\omega\ \forall\ \ell \in \mathbb{Z}$ one has

$$[A_c(\gamma_i)] + [A_c(\eta_i)] \geq D.$$

Since the curve c' lies in the Mather set for c', there exists some large integer K such that $\gamma(t_K)$ is sufficiently close to $\gamma(t_0)$. Let $\bar{\gamma}$ the the lift of γ to the universal covering space $\mathbb{R}^2$, we have

$$\begin{aligned}\sum_{i=0}^{K-1}\Big([A_{c'}(\gamma_i)] + [A_c(\eta_i)]\Big) &= \sum_{i=0}^{K-1}\Big([A_c(\gamma_i)] + [A_c(\eta_i)] + \langle c' - c, \bar{\gamma}(t_{i+1}) - \bar{\gamma}(t_i)\rangle\Big)\\ &\geq K(D - |\langle c' - c, \bar{\gamma}(t_{i+1}) - \bar{\gamma}(t_i)\rangle|).\end{aligned}$$

By the construction, both $|\sum_{i=0}^{K-1}[A_{c'}(\gamma_i)]|$ and $|\sum_{i=0}^{K-1}[A_c(\eta_i)]|$ are sufficiently small and the term $D - |\langle c' - c, \bar{\gamma}(t_{i+1}) - \bar{\gamma}(t_i)\rangle| \geq \frac{D}{2}$ if c' is sufficiently close to c. Therefore, the absurdity of above inequality implies the lemma. □

Therefore, similar to Proposition 2.3, one has the following proposition.

Proposition 2.6 *Assume $c \in \partial I_\omega$ with rational rotation vector ω where $I_\omega = \mathscr{L}_\beta(\omega)$ is a line segment. For different covering manifolds $k\mathbb{T}^2, k'\mathbb{T}^2$, if we think the elementary weak KAM solutions $u^\pm_{c,k}, u^\pm_{c,k'}$ as the functions defined on $\mathbb{R}^2$, then $u^\pm_{c,k} = u^\pm_{c,k'}$ up to an additive constant.*

Proof Since ω is rational, the minimal measure is supported on periodic orbits. Pick up such a circle γ and consider its lift to $k\mathbb{T}^2$, denoted by $\{\bar{\gamma}_i\}$ with i modulo some $i_0 \in \mathbb{N}$. Let $\mathbb{A}_i$ be an annulus bounded by $\bar{\gamma}_i$ and $\bar{\gamma}_{i+1}$ and no other $\bar{\gamma}_j$ lying inside of $\mathbb{A}_i$. Once $u^{\pm}_{c,k}$ is well defined on some $\mathbb{A}_i$, one obtain its value on its other copies in the covering space. Indeed, for any $\bar{x} \in \mathbb{A}_j$, let $\bar{x}' \in \mathbb{A}_i$ such that $\bar{\pi}_k\bar{x} = \bar{\pi}_k\bar{x}'$. Because of Lemma 2.5, one has $u^{\pm}_{c,k}(\bar{x}) = u^{\pm}_{c,k}(\bar{x}')$. □

3 Topology of level set of weak KAM solutions

The parameter σ in Theorem 1.1 is introduced to indicate "volume" bounded by the graph of the weak KAM solution and the graph of a prescribed weak KAM solution. The introduction of this parameter relies on a good understanding of the topology of level set of weak KAM solutions. Given two globally elementary weak KAM solutions U and U', we denote the level set of $U - U'$ by

$$Z_{U,U'} = \{x \in \mathbb{R}^2 : U(x) - U'(x) = 0\},$$

and let

$$\Omega_{U,U'} = \{x \in \mathbb{R}^2 : U(x) > U'(x)\}.$$

3.1 Topology of the sets $Z_{U,U'}$, $\Omega_{U,U'}$ and $\Omega_{U',U}$

Theorem 3.1 *Let U and U' be two globally elementary backward (forward) weak KAM solutions of the Hamilton–Jacobi equation (1.1) on the universal covering space $\mathbb{R}^2$, corresponding to cohomology classes c and c' respectively. Then*

(1) If $\alpha(c) > \alpha(c')$ then $\Omega_{U',U}$ is connected and unbounded, the set $\Omega_{U,U'}$ may not be connected but contains only one unbounded connected component;
(2) If $\alpha(c) = \alpha(c') = E > \min\alpha$, with $c, c' \in \mathcal{E}_E$ and $c \neq c'$, then both $\Omega_{U,U'}$ and $\Omega_{U',U}$ are simply connected and unbounded.

In both cases, the level set $Z_{U,U'}$ has empty interior unless $\mathscr{L}_\beta^{-1}(c') = \mathscr{L}_\beta^{-1}(c)$ and the Mather set $\mathcal{M}(c) = \mathcal{M}(c')$ contains interior points.

Proof To prove the theorem, we introduce a lemma for the Hamiltonian satisfying the hypothesis:

(H1) There exists a constant C such that

$$|H(y, p) - H(x, q)| \le C(\|x - y\| + \|p - q\|), \quad \forall\, x, y \in \Omega, \quad p, q \in \mathbb{R}^n.$$

We note that this extra assumption (**H1**) is not needed in Theorem 3.1. Indeed, by the Tonelli condition, the weak KAM solutions are uniformly Lipschitz, so they do not depend on the behavior of $H(x, p)$ for $\|p\|$ large. Therefore, we can modify H such that $H(x, p) = A\|p\|$ for $\|p\| > K$ for some constants $K > 0$ and $A > 0$. In this case, the extra assumption **(H1)** is satisfied automatically.

The definition of viscosity solutions is provided in Definition A.1 in the "Appendix".

Lemma 3.1 *(a) Let Ω be an open subset of $\mathbb{R}^n$ and $f \in C(\Omega)$ satisfy $f(x) < 0$ for $x \in \Omega$. Let $u \in C(\bar{\Omega})$ be a viscosity subsolution of $H(x, Du) = f(x)$, and $v \in C(\bar{\Omega})$ be a viscosity* *supersolution of* $H(x, Dv) = 0$ *in* Ω *where* H *satisfies* (H1). *If* $\partial\Omega \neq \varnothing$, $u \leq v$ *on* $\partial\Omega$, *and both* u *and* v *are bounded in* Ω, *then* $u \leq v$ *on* Ω.

(*b*) *Let* $u \in C(\bar{\Omega})$ *be a viscosity subsolution of* $H(x, Du) = 0$ *and* $v \in C(\bar{\Omega})$ *be a viscosity supersolution of* $H(x, Dv) = 0$ *in* Ω. *If we assume further that*

(1) there is a function $\phi \in C^1(\Omega) \cap C(\bar{\Omega})$ *such that* $\phi \leq u$ *on* Ω *and*

$$\sup\{H(x, D\phi(x)) : x \in \omega, u \in \mathbb{R}\} < 0 \qquad \forall\, \omega \subset\subset \Omega;$$

(2) the function $p \to H(x, p)$ *is convex on* $\mathbb{R}^n$ *for each* $x \in \Omega$,

then $u \leq v$ *on* Ω *provided that* $u - v$ *is bounded in* Ω *and* $u \leq v$ *on* $\partial\Omega$ *with* $\partial\Omega \neq \varnothing$.

We postpone the proof of the lemma to the "Appendix", and return to the proof of the theorem. If U, U' are the elementary weak KAM solution on universal covering space, they are the viscosity solution of the Hamilton–Jacobi equation

$$H(x, \partial_x u) = E, \qquad x \in \mathbb{R}^2.$$

For each connected component Ω of $\Omega_{U',U}$ or $\Omega_{U,U'}$, $\exists$ a function g defined on $\partial\Omega$ such that $U_c|_{\partial\Omega} = U_{c'}|_{\partial\Omega} = g$. Thus, both U_c and $U_{c'}$ are the viscosity solution of the Dirichlet problem

$$\begin{cases} H(x, \partial_x u) = \alpha, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega \end{cases}$$

where α is valued as $\alpha(c)$ and $\alpha(c')$ respectively. Note, $\partial\Omega \neq \varnothing$ and Ω is not necessary bounded.

Since each elementary weak KAM solution admits a decomposition into a periodic function and a linear function, the difference of U and U' also admits such a decomposition: $U - U' = \Delta u + \langle c - c', x\rangle$. The level set $Z_{U,U'}$ is restricted a strip $\{|\langle c - c', x\rangle| \leq D\}$ for certain positive number $D > 0$, i.e. both $\Omega_{U',U}$ and $\Omega_{U,U'}$ contain a unbounded connected component if $c \neq c'$.

In the case that $\alpha(c) > \alpha(c')$, the set $\Omega_{U',U} = \{x : U'(x) > U(x)\}$ is connected. Otherwise it would contain a connected component $\mathcal{C}$ in the strip $\{|\langle c - c', x\rangle| \leq D\}$, since each of the two connected components of the complement of the strip lies in a connected component of $\Omega_{U',U}$ or $\Omega_{U,U'}$. On the connected component $\mathcal{C}$, we have that $U' - U$ is bounded, and $U = U'$ on $\partial\mathcal{C}$, thus by Lemma 3.1, we have $U'(x) \leq U(x)$ on $\mathcal{C}$. This is a contradiction to the definition of $\mathcal{C}$.

For the case that $\alpha(c) = \alpha(c') > \min\alpha$, if $\omega(c) \neq \omega(c')$, then $\exists\, 0 < \lambda < 1$ such that

$$\alpha(c^*) < \alpha(c), \qquad \text{where } c^* = \lambda c + (1 - \lambda)c'.$$

By the result in [2,9], we know that there exists a $C^{1,1}$ global sub-solution of the equation $H(x, \partial u + c^*) = \alpha(c^*)$, i.e. there exists $C^{1,1}$-function $\tilde{\phi}$: $\mathbb{T}^n \to \mathbb{R}$, such that $\phi(x) = \tilde{\phi}(x) + \langle c^*, x\rangle$ satisfies the condition

$$H(x, \partial\phi) - \alpha(c) < H(x, \partial\phi) - \alpha(c^*) \leq 0.$$

Since

$$U - \phi = \Delta u + (1 - \lambda)\langle c - c', x\rangle,$$

and

$$U' - \phi = \Delta u' - \lambda\langle c - c', x\rangle,$$

where both Δu and $\Delta u'$ are periodic, thus $U - \phi$ as well as $U' - \phi$ can be set positive in the strip $\{|\langle c - c', x\rangle| \le D\}$ by adding a suitable constant to ϕ. Since H is assumed convex in the action variable, by applying the second part of Lemma 3.1 we find that if there is a connected component $\mathcal{C}$ of $\Omega_{U,U'}$ contained entirely in the strip, then we have $U = U'$ on $\mathcal{C}$. This contradicts the fact that $\mathcal{C} \subset \Omega_{U,U'}$. Similarly there is no connected component of $\Omega_{U',U}$ lying entirely in the strip. This implies that both $\Omega_{U,U'}$ and $\Omega_{U',U}$ are simply connected and unbounded.

Finally, we show that the set $Z_{U,U'}$ contains no interior if the Mather sets $\mathcal{M}(c) \neq \mathcal{M}(c')$. Otherwise there exists a differentiable point $x_0 \in Z_{U,U'}$ of both U and U' and $DU(x_0) = DU'(x_0)$. The initial condition $(x_0, DU(x_0)) = (x_0, DU'(x_0))$ determines a backward (forward) semi-static curve γ whose α (ω)-limit set lies in both $\mathcal{M}(c)$ and $\mathcal{M}(c')$. This is impossible provided the Mather sets $\mathcal{M}(c)$ and $\mathcal{M}(c')$ are assumed to be different. The set $Z_{U^\pm_{c^l},U^\pm_{c^r}}$ does not contain interior points, if the Mather set does not contain interior points. To see this, we suppose $\omega = (0, \hat{\omega})$ up to a linear coordinate change. For any differentiable point x not in the Mather set, the initial values $(x, \partial U^\pm_{c^l}(x))$ and $(x, \partial U^\pm_{c^r}(x))$ produce different minimal orbits, one moves towards the left, the other moves towards the right. This completes the proof of the theorem. □

3.2 Normal direction of the set $Z_{U,U'}$

The set $Z_{U,U'}$ is described in Theorem 3.1. However, the description is not clear enough for later proof. For instance, even though $Z_{U,U'}$ contains no interior, it does not imply that $Z_{U,U'}$ is a curve. In this section, we will introduce a notion of *normal direction* of the set $Z_{U,U'}$ for later purpose.

For a convex function one define its sub-derivative.

Definition 3.1 The set of sub-derivative of a convex function ψ at x is defined as

$$D^-\psi(x) = \{y \in \mathbb{R}^n : \psi(x') - \psi(x) \ge \langle y, x' - x\rangle, \ \ \forall\, x' \in \mathbb{R}^n\}.$$

It is known that $D^-\psi(x)$ is a convex set.

Since each backward weak KAM solution u^- has a decomposition $u^- = \phi - \psi$ where ϕ is smooth and ψ is convex, we define the sup-derivative of the backward weak KAM solution u^- as

$$D^+u^-(x) = \{D\phi(x) - y : y \ \text{ is a sub-derivative of } \psi \ \text{ at } x\}. \tag{3.1}$$

Definition 3.2 (Calibrated curves).

(1) A curve γ; $(-\infty, 0] \to M$ is called (u, L_c)-*calibrated* if

$$u(\gamma(t')) - u(\gamma(t)) = \int_t^{t'} L_c(\gamma(s), \dot{\gamma}(s))ds + (t' - t)\alpha(c)$$

holds for each $-\infty < t \le t' \le 0$.

(2) We say that (x, v) *determines* a (u_c^-, L_c)-calibrated curve γ: $(-\infty, 0] \to M$ if $\gamma(0) = x$ and $\dot{\gamma}(0) = v$.

(3) We say that $v \in Vu_c^-(x) \subset \mathbb{R}^n$, if (x, v) determines certain (u_c^-, L_c)-calibrated curve γ: $(-\infty, 0] \to M$.

Definition 3.3 Let u: $A \to \mathbb{R}$ be locally Lipschitz. A vector p is called a reachable gradient of u at $x \in A$ if a sequence $\{x_k\} \subset A\backslash\{x\}$ exists such that u is differentiable at x_k for each $k \in \mathbb{N}$ and

$$\lim_{k\to\infty} x_k = x, \qquad \lim_{k\to\infty} Du(x_k) = p.$$

The set of all reachable gradient of u at x is denoted by $D^*u(x)$.

If u is a semi-concave function then it is proved in Theorem 3.3.6 of [3] that

$$D^+u(x) = \text{co}D^*u(x). \tag{3.2}$$

Since we have $H(x, c+p) = \alpha(c)$, for all $p \in D^*u_c^-(x)$, the next lemma follows immediately from the strict convexity of H.

Lemma 3.2 *For each point x, the set of reachable gradients $D^*u_c^-(x)$ coincides with the set of extremal points of $D^+u_c^-(x)$.*

Theorem 3.2 *Let u_c^- be a backward weak KAM and γ_c: $(-\infty, 0] \to M$ be (u_c^-, L_c)-calibrated curve. Then, we have*

$$\frac{\partial L}{\partial \dot{x}}(x, Vu_c^-(x)) = D^*u_c^-(x), \quad x \in M.$$

In particular, this implies that y is an extremal point of $D^+u_c^-(x)$ only when there exists a calibrated curve γ such that $y = \frac{\partial L_c}{\partial \dot{x}}(\gamma_c(0), \dot{\gamma}_c(0))$.

Proof First, since u_c^- is differentiable along a calibrated curve, it is clear that

$$\frac{\partial L}{\partial \dot{x}}(x, Vu_c^-(x)) \subset D^*u_c^-(x).$$

Second, suppose u_c^- is differentiable at x_0, then there exists a backward calibrated curve $\gamma : (\infty, 0] \to M$ with $\gamma(0) = x_0$ and $Du_c^-(x_0) = \frac{\partial L_c}{\partial \dot{x}}(x_0, \dot{\gamma}(0))$. Next, by the upper-semi-continuity of Vu_c^-, we get

$$D^*u_c^-(x) \subset \frac{\partial L}{\partial \dot{x}}(x, Vu_c^-(x)).$$

The proof is now complete. □

Definition 3.4 Given two globally elementary weak KAM solutions U and U', the *derivative set* of $U - U'$ at the point x is defined by

$$D_{U,U',x} = \{y - y' : y \in D^+U(x), \quad y' \in D^+U'(x)\}. \tag{3.3}$$

Clearly, $D_{U,U',x}$ is closed and convex.

We define

$$\begin{aligned} S^{+,\delta}_{U,U',x} &= \left\{v \in \mathbb{R}^n : \langle y, v\rangle > \delta\|v\| \cdot \|y\|, \ \forall\ y \in D_{U,U',x}\right\}, \\ S^{-,\delta}_{U,U',x} &= \left\{v \in \mathbb{R}^n : \langle y, v\rangle < -\delta\|v\| \cdot \|y\|, \ \forall\ y \in D_{U,U',x}\right\}. \end{aligned} \tag{3.4}$$

If $0 \notin D_{U,U',x}$, neither of the two sets is empty provided $\delta > 0$ is suitably small.

For a C^1 function, it is well-known that its gradient is perpendicular to the level set. Here $D_{U,U',x}$ can be considered as the gradient of the level set $Z_{U,U'}$, and $S^{\pm,\delta}_{U,U',x}$ are cones containing $\pm D_{U,U',x}$. We naturally have the following.

Lemma 3.3 *Let U and U' be two globally elementary weak KAM solutions. For each x, there exists a suitably small $\epsilon > 0$ such that*

$$U(x_1) - U'(x_1) > 0, \qquad \forall\, x_1 - x \in S^{+,\delta}_{U,U',x} \cap (B_\epsilon(0) \setminus \{0\}),\ x \in Z_{U,U'}, \tag{3.5}$$

and

$$U(x_1) - U'(x_1) < 0, \qquad \forall\, x_1 - x \in S^{-,\delta}_{U,U',x} \cap (B_\epsilon(0) \setminus \{0\}),\ x \in Z_{U,U'}, \tag{3.6}$$

where $B_\epsilon(0)$ is a ball in $\mathbb{R}^n$, centered at 0 with radius ϵ.

Proof Note each backward weak KAM solution U has decomposition into a smooth function ϕ minus a convex function ψ, i.e. $U = \phi - \psi$. By definition, for $x_1 = x + te$ with $t > 0$ we have by Proposition 4.11.1 of [8]

$$\begin{aligned} U(x_1) - U(x) &= \langle y_e, x_1 - x\rangle + O(\|x_1 - x\|^2),\\ &\le \langle y, x_1 - x\rangle + O(\|x_1 - x\|^2), \qquad \forall\, y \in D^+U(x) \end{aligned} \tag{3.7}$$

where $y_e = \partial\phi(x) - y_{\psi,e}$. Similarly, we have

$$\begin{aligned} U'(x_1) - U'(x) &= \langle y'_e, x_1 - x\rangle + O(\|x_1 - x\|^2),\\ &\le \langle y', x_1 - x\rangle + O(\|x_1 - x\|^2), \qquad \forall\, y' \in D^+U'(x) \end{aligned} \tag{3.8}$$

where $y'_e = \partial\phi'(x) - y_{\psi',e}$. Therefore, if $x \in Z_{U,U'}$ and $x_1 - x \in S^{+,\delta}_{U,U',x}$, we subtract (3.8) from (3.7) and obtain

$$\begin{aligned} U(x_1) - U'(x_1) &\ge \langle y_e - y', x_1 - x\rangle - O(\|x_1 - x\|^2)\\ &\ge \delta\|x_1 - x\| \cdot \|y_e - y'\| - O(\|x_1 - x\|^2). \end{aligned}$$

This proves (3.5). The proof of (3.6) is similar. □

4 The order property of the derivative sets

We consider the level set of α-function $\alpha^{-1}(E)$ which is a closed and convex curve provided $E > \min\alpha$. By adding a closed 1-form to the Lagrangian, we assume that the α-function reaches its minimum at zero cohomology. In this case, $\alpha^{-1}(E)$ encircles the origin.

4.1 Crossing properties of minimal curves

For Hamiltonian systems with two degrees of freedom, the dynamics on energy level set resembles very much the dynamics of twist map.

Definition 4.1 We say a curve $\gamma : (-\infty, 0]$ is backward c-minimal or backward c-semi-static if

$$[A_c(\gamma)|_{[t,t']}] = \inf_{\xi}[A_c(\xi)|_{[\tau,\tau']}],$$

where the inf is taken among $\xi \in C^1$, $\xi(\tau) = \gamma(t)$ and $\xi(\tau') = \gamma(t')$. Similarly, we define forward c-minimal or forward c-semi-static curves defined on $[0, \infty)$.

Proposition 4.1 *Assume that γ and ξ are forward (backward) c-minimal and c'-minimal curves respectively with $c \neq c'$ and $\alpha(c) = \alpha(c') = E > \min \alpha$. Let $\bar{\gamma}$ and $\bar{\xi}$ denote their lifts to the universal covering space, if $\bar{\gamma}$ and $\bar{\xi}$ are not tangent at a point, then $\bar{\gamma}$ crosses $\bar{\xi}$ at most once.*

Proof We consider forward minimal case only, i.e. they are defined on $[0, \infty)$. The backward case is the same. If $\bar{\gamma}$ and $\bar{\xi}$ are tangent at a point, then they coincide for all the future time.

Let us assume that they cross twice, i.e. $\exists\, t_1 \neq t_2$ and $s_1 \neq s_2$ such that $\bar{\gamma}(t_i) = \bar{\xi}(s_i)$ for $i = 1, 2$. Clearly, $\theta_i = \dot{\gamma}(t_i) - \dot{\xi}(s_i) \neq 0$. Without loss of generality, we assume $t_1 \neq 0, s_1 \neq 0$. Otherwise, we extend the orbit backward to get an orbit defined on $[r, \infty)$ for some $r < 0$. By the lemma of Mather on page 186 of [15], we obtain that some $\epsilon > 0$ and $C > 0$ exist depending on the Lagrangian only such that

$$A(\gamma|_{[t_i-\epsilon,t_i+\epsilon]}) + A(\xi|_{[s_i-\epsilon,s_i+\epsilon]}) - A(a_i) - A(b_i) \geq C\epsilon\|\theta_i\|^2,$$

where $a_i : [-\epsilon, \epsilon] \to \mathbb{R}^2$ is a minimal curve of L joining $\bar{\xi}(s_1 - \epsilon)$ to $\bar{\gamma}(t_i + \epsilon)$, i.e. $a_i(-\epsilon) = \bar{\xi}(s_1 - \epsilon)$, $a_i(\epsilon) = \bar{\gamma}(t_i + \epsilon)$ and

$$A(a_i) = \int_{-\epsilon}^{\epsilon} L(a(t), \dot{a}(t))dt = \inf_{\substack{\zeta(-\epsilon)=a(-\epsilon)\\ \zeta(\epsilon)=a(\epsilon)}} \int_{-\epsilon}^{\epsilon} L(\zeta(t), \dot{\zeta}(t))dt,$$

$b_i : [-\epsilon, \epsilon] \to \mathbb{R}^2$ is a minimal curve of L joining $\bar{\gamma}(s_1 - \epsilon)$ to $\bar{\xi}(t_i + \epsilon)$ and $A(b_i)$ is defined in the same way as for $A(a_i)$. Let $\Delta s = s_2 - s_1$, $\Delta t = t_2 - t_1$, we define two curves

$$\gamma'(t) = \begin{cases} \gamma(t), & t \in [0, t_1 - \epsilon], \\ b_1(t - t_1), & t - t_1 \in [-\epsilon, \epsilon], \\ \xi(t - t_1), & t - t_1 \in [s_1 + \epsilon, s_2 - \epsilon], \\ a_2(t - t_1 + \Delta s), & t - t_1 + \Delta s \in [-\epsilon, \epsilon], \\ \gamma(t - t_1 + \Delta s), & t - t_1 + \Delta s \in [t_2 + \epsilon, \infty), \end{cases}$$

and

$$\xi'(t) = \begin{cases} \xi(t), & t \in [0, s_1 - \epsilon], \\ a_1(t - s_1), & t - s_1 \in [-\epsilon, \epsilon], \\ \gamma(t - s_1), & t - s_1 \in [t_1 + \epsilon, t_2 - \epsilon], \\ b_2(t - s_1 + \Delta t), & t - s_1 + \Delta t \in [-\epsilon, \epsilon], \\ \xi(t - s_1 + \Delta t), & t - s_1 + \Delta t \in [s_2 + \epsilon, \infty). \end{cases}$$

By the construction, we have $\gamma(t_1 - \epsilon) = \gamma'(t_1 - \epsilon)$, $\gamma(t_2 + \epsilon) = \gamma'(t_1 + \Delta s + \epsilon)$, $\xi(s_1 - \epsilon) = \xi'(s_1 - \epsilon)$, $\xi(s_2 + \epsilon) = \xi'(s_1 + \Delta t + \epsilon)$ and

$$\begin{aligned} &[A_c(\gamma|_{[t_1-\epsilon,t_2+\epsilon]})] + [A_{c'}(\xi|_{[s_1-\epsilon,s_2+\epsilon]})] - [A_c(\gamma'|_{[t_1-\epsilon,t_1+\Delta s+\epsilon]})] \\ &\qquad - [A_{c'}(\xi'|_{[s_1-\epsilon,s_1+\Delta t+\epsilon]})] \\ &\quad = A(\gamma|_{[t_1-\epsilon,t_1+\epsilon]}) + A(\xi|_{[s_1-\epsilon,s_1+\epsilon]}) - A(a_1) - A(b_1) \\ &\qquad + A(\gamma|_{[t_2-\epsilon,t_2+\epsilon]}) + A(\xi|_{[s_2-\epsilon,s_2+\epsilon]}) - A(a_2) - A(b_2) \\ &\quad \geq C\epsilon(\|\theta_1\|^2 + \|\theta_2\|^2). \end{aligned}$$

Note that the c-dependences are all canceled. This contradicts that fact that γ and ξ are c- and c'-minimal curves respectively. The absurdity verifies the fact that $\bar{\gamma}$ crosses $\bar{\xi}$ at most once. □

Corollary 4.1 *Assume that both γ and ξ are forward (backward) c-minimal curves, $d\gamma$ and $d\xi$ share the same ω (α)-limit set. Let $\bar{\gamma}$ and $\bar{\xi}$ denote their lift to the universal covering space, then they do not cross anywhere.*

This corollary follows from the same argument as the Aubry's asymptotic crossing lemma (see Lemma 3.9 of [1]).

Lemma 4.2 *Assume the autonomous Lagrangian is defined on $T\mathbb{T}^2$. For each $c \in \alpha^{-1}(E)$ with $E > \min\alpha$, the rotation vectors of all orbits in $\tilde{\mathcal{M}}(c)$ have the same direction, i.e. if $d\gamma_1, d\gamma_2 \in \tilde{\mathcal{M}}(c)$, then*

$$\langle \omega(\gamma_1), \omega(\gamma_2)\rangle = \|\omega(\gamma_2)\| \cdot \|\omega(\gamma_2)\| > 0.$$

Proof Since the system is defined on $T\mathbb{T}^2$, the minimal measure is uniquely ergodic if the rotation vector is irrational. Therefore, for each orbit $d\gamma$ lying in the Mather set with irrational rotation vector, its rotation vector $\omega(\gamma)$ is well defined by ergodic theorem. Next we consider only the case of rational rotation vector, in which case orbits in the Mather sets are periodic.

We claim that for each orbit $d\gamma\colon \mathbb{T}^2 \to TM$ in $\tilde{\mathcal{M}}(c)$, we have $\langle c - c^*, [\gamma]\rangle \neq 0$, where c^* is the minimum point of the α-function, which is defaulted to be zero here. The convexity of the α function implies that

$$\alpha(0) \geq \alpha(c) - \langle \omega, c\rangle, \quad \forall\omega \in \partial\alpha(c).$$

By assumption, we have $\alpha(c) > \alpha(0)$ so we get $\langle \omega, c\rangle > 0$. In the rational case, the rotation vector $\omega(\gamma)$ is positively proportional to $[\gamma]$. The inequality $\langle \omega, c\rangle > 0$ implies that all the rotation vectors in $\partial\alpha(c)$ are positively proportional to each other. □

4.2 Order property

Each energy level has a natural fiberation over $\mathbb{T}^2$. The fiber over a point x is denoted by

$$\mathbf{Y}_{x,E} = \{y : (x, y) \in H^{-1}(E)\}.$$

If $E > \min\alpha$, $\mathbf{Y}_{x,E}$ is a smooth, convex and closed curve for each $x \in M$. Next, let

$$\mathbf{V}_{x,E} = \{v = \partial_y H(x, y) : y \in \mathbf{Y}_{x,E}\},$$

which is also a smooth, convex and closed curve in $\mathbb{R}^2$ encircling the origin. The two curves $\mathbf{Y}_{x,E}$ and $\mathbf{V}_{x,E}$ are related by Legendre transform.

Definition 4.2 (Circle order). Given three vectors $v_1, v_2, v_3 \in \mathbb{R}^2$, we say that they are in clock-wise order, denoted by $v_1 \prec v_2 \prec v_3$ if the points $\frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|}$ are in clock-wise order on the unit circle.

Via the one-to-one correspondence $v \to y = \partial_{\dot{x}} L(x, v)$, v_1, v_2, v_3 in $\mathbf{V}_{x,E}$ uniquely determine three points y_1, y_2, y_3 in $\mathbf{Y}_{x,E}$. As L is strictly positive definite in the speed, we have $y_1 - y_x \prec y_2 - y_x \prec y_3 - y_x$ where y_x is the minimal point of $H(x, y)$ with fixed x.

Proposition 4.3 (Order property). *Consider three cohomology classes $c_1, c_2, c_3 \in \mathcal{E}_E$ with $E > \min\alpha$ satisfying $c_1 \prec c_2 \prec c_3$.*

(1) *Suppose $\gamma_i : (-\infty, 0] \to \mathbb{T}^2$ is a backward c_i-semi-static curve with $\gamma_i(0) = x$, $i = 1, 2, 3$, then one has*

$$\dot{\gamma}_1(0) \prec \dot{\gamma}_2(0) \prec \dot{\gamma}_3(0).$$

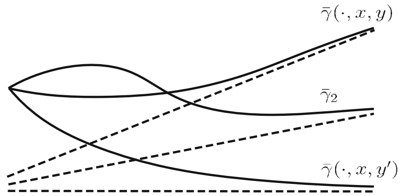

Fig. 2 The order of rotation vectors implies the order of minimal curves

Denote by $y_i = \partial_{\dot{x}} L(x, \dot{\gamma}_i^-(0))$*, then one has that*

$$y_1 - y_x \prec y_2 - y_x \prec y_3 - y_x.$$

(2) *Denote by* U_i^- *the globally elementary weak KAM solutions associated to the cohomology classes* c_i, $i = 1, 2, 3$. *Then the order relation*

$$y_1 - y_x \prec y_2 - y_x \prec y_3 - y_x$$

holds for each $y_i \in D^+U_i^-(x)$, $i = 1, 2, 3$. *So the following order property is well-defined*

$$D^+U_1^-(x) - y_x \prec D^+U_2^-(x) - y_x \prec D^+U_3^-(x) - y_x.$$

Moreover, the order relation is independent of the base point x.

Proof For fixed x, the Legendre transform $\partial_{\dot{x}} L(x, \cdot) : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $L(x, v) = \max_y \langle y, v\rangle - H(x, y)$ where v is sent to y attaining the max. The sublevel set $\{H(x, y) \le E\}$ for each fixed x is a convex set. It turns out that the Legendre transform of the set $\{H(x, y) \le E\}$ is also a convex set and it sends the boundary to boundary(Theorem 26.5 of [19]). Moreover, the Legendre transform is a diffeomorphism whose Jacobian det $\partial^2_{\dot{x}\dot{x}} L > 0$, so the diffeomorphism $\partial_{\dot{x}} L(x, \cdot)$ preserves orientation on the boundary. By the positive-definiteness of $\partial^2_{\dot{x}\dot{x}} L$, it is enough for us to prove $\dot{\gamma}_1(0) \prec \dot{\gamma}_2(0) \prec \dot{\gamma}_3(0)$. Suppose the circle order is violated and without loss of generality suppose we have $\dot{\gamma}_1(0) \prec \dot{\gamma}_3(0) \prec \dot{\gamma}_2(0)$.

Let $\bar{\gamma}_i(\cdot)$ denote the lift of the curves $\gamma_i(\cdot)$, $i = 1, 2, 3$, to the universal covering space $\mathbb{R}^2$ respectively such that $\bar{\gamma}_1(0) = \bar{\gamma}_2(0) = \bar{\gamma}_3(0)$. The order relation $c_1 \prec c_2 \prec c_3$ induces the order relation $\omega_1 \prec \omega_2 \prec \omega_3$ for rotation vectors. Indeed, restricted to each energy level, we can reduce the Hamiltonian system into a twist map, for which the order relation of cohomology classes agrees with that of the rotation numbers.

In this case, $\bar{\gamma}_2$ either crosses $\bar{\gamma}_1(\cdot)$ or crosses $\bar{\gamma}_3(\cdot)$ at another point, because of the order $c_1 \prec c_2 \prec c_3$, see Fig. 2. (It may happen that two of the three vectors $\omega_1, \omega_2, \omega_3$ coincide, say $\omega_1 = \omega_2$, in which case we have that $\bar{\gamma}_1$ and $\bar{\gamma}_2$ approaches two neighboring lifts of periodic orbits in the Mather set $\tilde{\mathcal{M}}(c_1) = \tilde{\mathcal{M}}(c_2)$. The crossing will also occur if the order $\dot{\gamma}_1(0) \prec \dot{\gamma}_2(0) \prec \dot{\gamma}_3(0)$ is violated.) But this violates the property that all these three curves are backward semi-static by Proposition 4.1. This contradiction verifies our claim.

***

Next, we consider part (2). Given $x \in \mathbb{T}^2$, there might be two backward c-semi-static curve originating from this point, denoted by γ_c and γ_c' with $v = \dot{\gamma}_c(0) \neq v' = \dot{\gamma}_c'(0)$. In this case, the elementary weak KAM solution U_c^- is not differentiable at this point, there are at least two points $y_c, y_c' \in \mathbf{Y}^-_{x,E}$ such that $y_c = \partial_{\dot{x}} L(v, x)$ and $y_c' = \partial_{\dot{x}} L(v', x)$, and $y_c, y_c' \in \partial(D^+U_c^-(x))$.

Lemma 4.4 *Let* $c, c' \in \mathcal{E}_E$ *and* $c \neq c'$. *Then we have*

$$\mathrm{Int}(D^+U_c^-(x)) \cap D^+U_{c'}^-(x) = \emptyset \quad \text{and} \quad D^+U_c^-(x) \cap \mathrm{Int}(D^+U_{c'}^-(x)) = \emptyset.$$

Proof of Lemma 4.4 Suppose without loss of generality that $D^+U_c^-(x) \cap \mathrm{Int}(D^+U_{c'}^-(x)) \neq \emptyset$. Denote by $y_{c'}^l$ and $y_{c'}^r$ the extremal points of $D^+U_{c'}^-(x)$ and by y_c is the extremal point of $D^+U_c^-(x)$ such that

$$y_{c'}^r - y_x \prec y_c - y_x \prec y_{c'}^l - y_x.$$

The three points y_c, $y_{c'}^r$, $y_{c'}^l$ are distinct and are all reachable gradients by Lemma 3.2. Denote by γ_c, $\gamma_{c'}^r$ and $\gamma_{c'}^l$ the calibrated curves terminated at x determined by the three points respectively. We get the order relation after Legendre transform

$$\dot\gamma_{c'}^r(0) \prec \dot\gamma_c(0) \prec \dot\gamma_{c'}^l(0).$$

We also know that $\dot\gamma_{c'}^r(0)$, $\dot\gamma_c(0)$, $\dot\gamma_{c'}^l(0)$ are distinct since $\partial_{\dot x}L(x,\cdot)$ is a diffeomorphism.

We next pick c^* satisfying $c^* \prec c \prec c'$ and a c^*-calibrated curve γ_{c^*} with $\gamma_{c^*}(0) = x$. By part (1) of the lemma, we get $\dot\gamma_{c^*}(0) \prec \dot\gamma_c(0) \prec \dot\gamma_{c'}^l(0)$ and $\dot\gamma_{c^*}(0) \prec \dot\gamma_c(0) \prec \dot\gamma_{c'}^r(0)$. By the same argument as in the proof of part (1), we see that γ_c would intersect $\gamma_{c'}^l$ or $\gamma_{c'}^r$ at a second point, which contradicts Proposition 4.1. □

Part (3) follows directly from the above lemma. □

For each $y \in \mathbf{Y}_{x,E}$, there is a unique smooth curve $\gamma(t,x,y)\colon \mathbb{R} \to M$ such that $\gamma(0,x,y) = x$, $\partial_{\dot x}L(x,\dot\gamma(0,x,y)) = y$. Indeed, $(\gamma(t),\dot\gamma(t))$ is a trajectory of the Lagrange flow. However, it is not necessary that each of these curve is semi-static.

Definition 4.3 Let

$$\mathbf{Y}_{x,E}^- = \{y \in \mathbf{Y}_{x,E} : \gamma(\cdot,x,y)|_{t\in\mathbb{R}^-} = \gamma_c^-(\cdot,x) \text{ for some } c \in \alpha^{-1}(E)\},$$

namely, each point $y \in \mathbf{Y}_{x,E}^-$ determines a backward semi-static curve $\gamma(\cdot,x,y)$ for certain cohomology class c such that $\gamma(0,x,y) = x$ and $\partial_{\dot x}L(\gamma(0,x,y),\dot\gamma(0,x,y)) = y$. It approaches to certain Aubry set in the following sense

$$\alpha(d\gamma(\cdot,x,y)) \subseteq \tilde{\mathcal{A}}(c).$$

Since the configuration space is two-dimensional, each orbit in $\alpha(d\gamma(\cdot,x,y)) \cap \tilde{\mathcal{A}}(c)$ has the same rotation vector, denoted by $\omega(x,y)$. Because of upper semi-continuity of backward semi-static curves on cohomology classes, the set $\mathbf{Y}_{x,E}^-$ is closed in $\mathbf{Y}_{x,E}$.

By Lemma 3.2, we get that

$$\mathbf{Y}_{x,E}^- = \cup_{c\in\mathcal{E}_E} D^*U_c^-(x).$$

Since the complementary set $\mathbf{Y}_{x,E}\backslash\mathbf{Y}_{x,E}^-$ is composed of open intervals in $\mathbf{Y}_{x,E}$, an equivalence relation $\sim$ in $\mathbf{Y}_{x,E}^-$ is introduced such that $y \sim y'$ if y and y' are the two boundary points of an open interval in $\mathbf{Y}_{x,E}\backslash\mathbf{Y}_{x,E}^-$.

Proposition 4.5 *If $E > \min\alpha$, $y \sim y'$ in $\mathbf{Y}_{x,E}^-$, then $\omega(x,y)$ and $\omega(x,y')$ have the same direction, i.e.*

$$\langle\omega(x,y),\omega(x,y')\rangle = \|\omega(x,y)\| \cdot \|\omega(x,y')\|.$$

If c and c' are the cohomlogy classes such that $\gamma(\cdot,x,y)$, $\gamma(\cdot,x,y')$ are the c-, c'-semi static respectively, then they stay in the same flat of the α-function.

Proof Let us assume the contrary, *i.e*. the direction of $\omega(x, y)$ is different from that of $\omega(x, y')$. Under such assumption, the curves $\gamma(\cdot, x, y)|_{t\in\mathbb{R}^-}$ and $\gamma(\cdot, x, y')|_{t\in\mathbb{R}^-}$ can not be semi-static for the same cohomology class, which is guaranteed by Lemma 4.2.

Denoted by c and c' the cohomology classes such that $\gamma(\cdot, x, y)|_{t\in\mathbb{R}^-}$ is c-semi-static and $\gamma(\cdot, x, y')|_{t\in\mathbb{R}^-}$ is c'-semi-static. We claim that c and c' are not contained in one flat of the α-function under the assumption that the direction of $\omega(x, y)$ is different from that of $\omega(x, y')$. Indeed, by Proposition 6 of [18], $\mathcal{A}(c_1) = \mathcal{A}(c_2)$ whenever both c_1 and c_2 are in the relative interior of the flat, $\mathcal{A}(c_1) \supset \mathcal{A}(c_2)$ if c_1 is on the boundary of the flat while c_2 is in the relative interior. Suppose c and c' are contained in the same flat. Choose c^* from the relative interior of the flat where c, c' are, then we have $\mathcal{A}(c) \supseteq \mathcal{A}(c^*)$ and $\mathcal{A}(c') \supseteq \mathcal{A}(c^*)$. But this places us in a dilemma: if the direction of the rotation vector for $\tilde{\mathcal{A}}(c^*)$ is the same as that for $\tilde{\mathcal{A}}(c)$, then it is different from that for $\tilde{\mathcal{A}}(c')$, and vise versa.

Next, we have the following lemma since the level set $\alpha^{-1}(E),\ E > \min\alpha$, is a C^1 curve.

Lemma 4.6 *If an autonomous Lagrangian is defined on $T\mathbb{T}^2$, then, any flat of the α-function is disjoint from any other flat.*

By this lemma, we can choose $c_1, c_2 \in \alpha^{-1}(E)$ with

$$c_1 \prec c \prec c_2 \prec c' \prec c_1,$$

such that no flat contains any two of the four cohomology classes.

By choosing suitable c_1 and c_2, we assume that any two of these rotation vectors are not colinear. Notice there exists a backward c_i-semi static curve γ_i^- $(i = 1, 2)$ such that $\gamma_i^-(0) = x$. Then by Proposition 4.3, we have

$$y_1 - y_x \prec y - y_x \prec y_2 - y_x \prec y' - y_x \prec y_1 - y_x.$$

However, this contradicts the assumption that $y \sim y'$. The contradiction implies that $\omega(x, y)$ and $\omega(x, y')$ are in the same direction. Consequently, c and c' are in one flat of the α-function. □

5 The modulus of continuity of weak KAM solutions

In this section, we give the proof of Theorem 1.1.

Consider cohomology classes $c_1, c_2, c_3 \in \mathcal{E}_E$ with $c_1 \prec c_2 \prec c_3$, and the corresponding elementary weak KAM solutions U_1^-, U_2^- and U_3^- respectively. By Proposition 4.3 (3), one has

$$D^+U_i^-(x) - y_x \prec D^+U_j^-(x) - y_x \prec D^+U_k^-(x) - y_x \tag{5.1}$$

and the relation is independent of x.

By choosing suitable constants we assume that $U_i^-(x_0) = 0$ for each i and for any gien point x_0. Denoted by

$$Z_{i,j}^- = \{x \in \mathbb{R}^2 : U_j^-(x) - U_i^-(x) = 0\}.$$

Since these rotation directions are all different, any two of the cohomology classes are not in the same flat. In virtue of Theorem 3.1, each set $Z_{i,j}^-$ is a connected set without interior. Indeed, because both functions admit a decomposition of periodic and linear functions $U_i^-(x) = u_i^-(x) + \langle c_i, x\rangle$ and $U_j^-(x) = u_j^-(x) + \langle c_j, x\rangle$, the level set $Z_{i,j}^-$ is a curve without self-intersection, remaining in a strip $\{x \in \mathbb{R}^2 : |\langle c_i - c_j, x\rangle| < d_{i,j}\}$ for some $d_{i,j} > 0$ and extending to infinity in both directions.

Lemma 5.1 *Let the cohomology classes c_i, c_j, c_k satisfy that $\alpha(c_i) = \alpha(c_j) = \alpha(c_k) = E > \min\alpha$, then $Z^-_{i,j}$ intersects $Z^-_{i,k}$ only at $x = x_0$.*

We postpone the proof of the lemma and complete the proof of the main theorem by assuming the lemma. We next show that the order property (5.1) induces an order property on the sets $Z^-_{i,j}$.

Definition 5.1 Given level sets $Z^-_{j_\ell,i_\ell}$, $\ell = 1, 2, \ldots$, we say they are in the order

$$Z^-_{j_1,i_1} \prec Z^-_{j_2,i_2} \prec Z^-_{j_3,i_3} \prec \cdots,$$

if they intersect each other only at x_0, and at x_0 the "gradient" of $U^-_{j_\ell} - U^-_{i_\ell}$ are in the order

$$D_{U^-_{j_1},U^-_{i_1},x_0} \prec D_{U^-_{j_2},U^-_{i_2},x_0} \prec D_{U^-_{j_3},U^-_{i_3},x_0} \prec \cdots.$$

Given two points y_ℓ, $y_k \in \mathbf{Y}^-_{x_0,E}$, we choose other two points y_0, $y_1 \in \mathbf{Y}^-_{x_0,E}$ such that they are in the order $y_0 \prec y_\ell \prec y_k \prec y_1$. We assume that all these four points determine backward semi-static curves which approach the Aubry sets with different rotation direction. One has the order (see the figure below):

$$y_\ell - y_0 \prec y_k - y_0 \prec y_k - y_\ell \prec y_1 - y_\ell \prec y_1 - y_k.$$

By applying Lemma 5.1 to the order $y_\ell - y_0 \prec y_k - y_0 \prec y_k - y_\ell$ as well as to the order $y_k - y_\ell \prec y_1 - y_\ell \prec y_1 - y_k$, one sees that the following order property holds by (5.1), Definition 5.1 and Definition 3.4

$$Z^-_{\ell,0} \prec Z^-_{k,0} \prec Z^-_{k,\ell} \prec Z^-_{1,\ell} \prec Z^-_{1,k}.$$

Geometrically, the set $Z^-_{k,\ell}$ lies in the sector-shaped region bounded by the sets $Z^-_{\ell,0}$ and $Z^-_{1,k}$ containing the sets $Z^-_{k,0}$ and $Z^-_{1,\ell}$, see the figure followed.

By the same argument, for any two points $y_i, y_j \in \mathbf{Y}^-_{x_0,E}$ between y_ℓ and y_k in the following sense

$$y_0 \prec y_\ell \prec y_i \prec y_j \prec y_k \prec y_1,$$

we have

$$Z^-_{\ell,0} \prec Z^-_{i,0} \prec Z^-_{j,i} \prec Z^-_{1,j} \prec Z^-_{1,k}.$$

Therefore, $Z^-_{j,i}$ lies in the sector-shaped region bounded by the lines $Z^-_{\ell,0}$ and $Z^-_{1,k}$, containing the curve $Z^-_{k,\ell}$.

The straight line $\{\langle c_0 - c_\ell, x - x_0\rangle = 0\}$ is not parallel to the line $\{\langle c_k - c_1, x - x_0\rangle = 0\}$. The curves $Z^-_{\ell,0}$ and $Z^-_{1,k}$ lie in the strips $\{|\langle c_0 - c_\ell, x - x_0\rangle| < d_{\ell,0}\}$ and $\{|\langle c_k - c_1, x - x_0\rangle| < d_{l,k}\}$ respectively.

Let us normalize $\{U^-_c\}$ by the condition $U^-_c(0) = 0,\ c \in \mathcal{E}_E$. Given any bounded domain Ω, choosing $m \in \mathbb{Z}^2$ properly and making the translation $x \mapsto x + m$, we can guarantee that Ω falls into the sector-shaped bounded by the curves $Z^-_{\ell,0}$ and $Z^-_{1,k}$, but not containing the curve $Z^-_{k,\ell}$. It implies that $Z^-_{j,i} \cap \Omega = \varnothing$ if the relation (5) holds. Without loss of generality, we assume Ω is a disk. Otherwise, we replace Ω by a disk containing Ω.

Let U^-_0, U^-_ℓ, U^-_k, U^-_1 be the global elementary weak KAM solution for the cohomology classes c_0, c_ℓ, c_k, c_1 respectively. It follows from the above argument that the global elementary weak KAM solutions for all cohomology classes $\{c : c_0 \prec c_\ell \prec c \prec c_k \prec c_1\}$ are totally ordered on the set Ω (Fig. 3). For any two functions $U^-_c + \langle c, m\rangle$ and $U^-_{c'} + \langle c', m\rangle$ for different classes c and c' in this set we have either $U^-_c(x) - U^-_{c'}(x) + \langle c - c', m\rangle > 0$ for or alternatively, $U_c^-(x) - U_{c'}^-(x) + \langle c - c', m\rangle < 0$ for all $x \in \Omega$. Therefore, these elementary weak KAM solutions can be parameterized by the "volume" in the following way. Fix any $c = c(0)$, we introduce the parametrization σ such that

$$\sigma = \int_\Omega (U^-_{c(\sigma)} - U^-_{c(0)})dx + \int_\Omega \langle c(\sigma) - c(0), m\rangle\, dx,$$

where the second term is caused by the m-translation. The order property guarantees that the map $\sigma \to c$ is well-defined.

Suppose the cohomology classes of the elementary weak KAM solutions under consideration is bounded by M. Denote by L the Lipschitz constant for all the weak KAM solutions u_c^-. It is known that L depends only on the Tonelli Hamiltonian H. So we get that

$$\max_{x\in\Omega,c} |U_c^- - U_{c'}^- + \langle c - c', m\rangle| = \max_{x\in\Omega,c} |u_c^- - u_{c'}^- + \langle c - c', x + m\rangle| \le (M + L)(\|m\| + 1).$$

We fix a number $d \ge L + M$ such that

$$\min_{a\in\Omega} \mathrm{Area}(\Omega \cap B_r(a)) \ge \frac{1}{4}\mathrm{Area}(B_r(a)),$$

where $B_r(a)$ is a disk centered at $a \in \Omega$ with radius $r \le (M + L)(\|m\| + 1)/d$.

We next show that in the parameter σ, the weak KAM solution is $\frac{1}{3}$-Hölder continuous in C^0-topology. Indeed, given $U^-_{c(\sigma)} + \langle c(\sigma), m\rangle > U^-_{c(\sigma')} + \langle c(\sigma'), m\rangle$, the region

$$D_{\sigma,\sigma'} = \{(x, z) \in \Omega \times \mathbb{R} : U^-_{c(\sigma)}(x) + \langle c(\sigma), m\rangle \le z \le U^-_{c(\sigma')}(x) + \langle c(\sigma'), m\rangle\}$$

contains at least two cones (Fig. 4). The height of each is

$$h := \frac{1}{2}\|U^-_{c(\sigma)} - U^-_{c(\sigma')} + \langle c(\sigma) - c(\sigma'), m\rangle\|_{C^0(\Omega)},$$

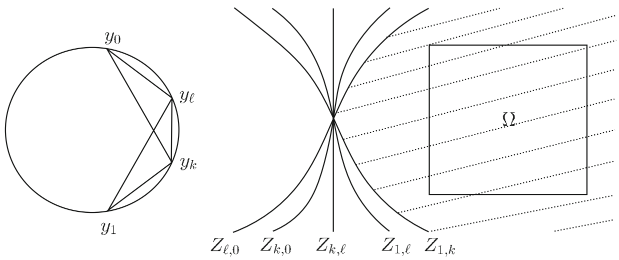

Fig. 3 The graphs of the weak KAM solutions are ordered above Ω

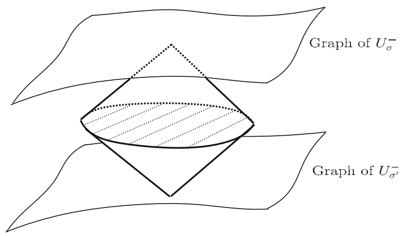

Fig. 4 Inserting cones between the graphs of two weak KAM solutions

and the radius of its bottom disc is $\frac{h}{d}$, so the volume is $\frac{\pi h^3}{3d^2}$. By the choice of d, we have

$$|\sigma-\sigma'|=\left|\int_\Omega (U^-_{c(\sigma)}-U^-_{c(\sigma')})\,dx+\int_\Omega\langle c(\sigma)-c(\sigma'),m\rangle\,dx\right|\ge\frac{1}{4}\frac{\pi h^3}{3d^2},$$

from which one obtains the $\frac{1}{3}$-Hölder regularity immediately

$$\|U^-_{c(\sigma)}-U^-_{c(\sigma')}\|_{C^0(\Omega)}\le 2h+\|c(\sigma)-c(\sigma')\|\cdot\|m\|\le C(\Omega,H)\Big(|\sigma-\sigma'|^{1/3}+\|c(\sigma)-c(\sigma')\|\Big).$$

As $\mathbf{Y}^-_{x_0,E}$ can be covered by finitely many such arcs, and each global elementary weak KAM solition admits a decomposition of periodic and linear functions, we complete the proof of the main theorem. □

Proof of Lemma 5.1 By Theorem 3.2, the vertex of $D^+U_i^-(x)$ must be on $\mathbf{Y}^-_{x,E}$. Thus, by Proposition 4.3(3) one deduces from the assumption $\omega_i\neq\omega_j\neq\omega_k\neq\omega_i$ that the derivative sets (see Definition 3.4) $D_{U_i^-,U_j^-,x}$, $D_{U_i^-,U_k^-,x}$ and $D_{U_j^-,U_k^-,x}$ are disjoint from each other for each $x\in\mathbb{T}^2$. If $c_i\prec c_j\prec c_k$, then $y_j-y_i\prec y_k-y_i\prec y_k-y_j$, namely

$$D_{U_i^-,U_j^-,x}\prec D_{U_i^-,U_k^-,x}\prec D_{U_j^-,U_k^-,x}. \tag{5.2}$$

We claim that each intersection point of $Z^-_{i,j}$ with $Z^-_{i,k}$ is isolated. To see it, let us note that $0\notin \mathrm{co}D_{U_i^-,U_j^-,x}$ holds for any $x\in\mathbb{T}^2$ and $i\neq j$. If $Z^-_{i,j}$ intersects $Z^-_{i,k}$ at some point x, then $U_i^-(x)=U_j^-(x)=U_k^-(x)$. Let $x'\in Z^-_{i,j}$ be a point close to x, we obtain from Lemma 3.3 that $x'-x$ is almost orthogonal to certain vector $y\in D_{U_i^-,U_j^-,x}$, i.e. the inequality $|\langle x'-x,y\rangle|\le\delta\|x'-x\|\|y\|$ holds for sufficiently small $\delta>0$ and some $y\in D_{U_i^-,U_j^-,x}$. Therefore, we obtain from (5.2) that

$$Z^-_{i,j}\cap B_\epsilon(x)\subset B_\epsilon(x)\backslash(S^{+,\delta}_{U_i^-,U_j^-,x}\cup S^{-,\delta}_{U_i^-,U_j^-,x}),$$
$$Z^-_{i,k}\cap B_\epsilon(x)\subset S^{+,\delta}_{U_i^-,U_j^-,x}\cup S^{-,\delta}_{U_i^-,U_j^-,x}$$

provided $\delta>0$ is sufficiently small. It implies that x is the only point in $B_\epsilon(x)$ where $Z^-_{i,j}$ intersects $Z^-_{i,k}$ and the intersection is topologically transversal.

Therefore, given an intersection point x of $Z^-_{i,j}$ with $Z^-_{i,k}$, it makes sense to find another intersection point x' next to x when there are more than one intersection point. In this case, $Z^-_{j,k}$ also passes through the points x and x'. Note that the intersection of these sets is always topologically transversal. If the derivative sets of these curves at x are in clock-wise order

$$D_{U_i^-,U_j^-,x}\prec D_{U_i^-,U_k^-,x}\prec D_{U_j^-,U_k^-,x},$$

then at x' they would be in anti clock-wise order (see Fig. 5)

$$D_{U_i^-,U_j^-,x'}\succ D_{U_i^-,U_k^-,x'}\succ D_{U_j^-,U_k^-,x'}.$$

However, the order property (5.1) is independent of x by Proposition 4.3(3). The contradiction verifies the fact: they intersect each other only at $x=x_0$. □

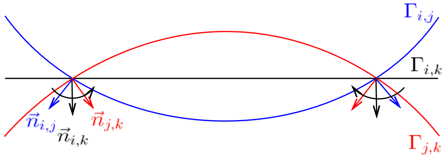

Fig. 5 The order property is violated if the level curves intersect twice

6 Extension of weak KAM solution by normal hyperbolicity

In this section, we consider a special case where the main theorem admits a higher dimensional generalization. This is the case when the Hamiltonian system admits a normally hyperbolic invariant manifold diffeomorphic to $T^*\mathbb{T}^2$. This structure exists widely in nearly integrable Hamiltonian systems.

If such modulus continuity is established on certain normally hyperbolic invariant manifold (NHIM), we can extend it to the stable and unstable fibers. We first recall the definition of a NHIM. The theory of NHIM can be found in [10].

Definition 6.1 Let $f : M \to M$ be a C^r-diffeomorphism on a smooth manifold M with $r > 1$. Let $N \subset M$ be a submanifold (probably with boundary) invariant under f, $f(N) = N$. We say that N is a *normally hyperbolic invariant manifold* if there exist a constant $C \geq 1$, rates $0 < \lambda < \mu^{-1} < 1$ and an invariant splitting for every $x \in N$

$$T_x M = T_x N \oplus E^s_x \oplus E^u_x$$

in such a way that

$$\begin{aligned} v \in T_x N \quad &\Leftrightarrow \quad |Df^k(x)v| \leq C\mu^k|v|, \quad k \in \mathbb{Z}, \\ v \in E^s_x \quad &\Leftrightarrow \quad |Df^k(x)v| \leq C\lambda^k|v|, \quad k \geq 0, \\ v \in E^u_x \quad &\Leftrightarrow \quad |Df^k(x)v| \leq C\lambda^{|k|}|v|, \quad k \leq 0. \end{aligned}$$

We denote $\ell = \min\{\frac{|\ln\lambda|}{|\ln\mu|}, r\}$ where r is the regularity of the time-1 map of the Hamiltonian flow.

Theorem 6.1 *Let $\mathbb{T}^k \times \mathbb{R}^k (\subset \mathbb{T}^n \times \mathbb{R}^n)$, $k < n$, be a normally hyperbolic invariant manifold for the Hamiltonian flow with $\ell \geq 2$ and let $u^{\pm}_{c(\sigma)}$ be elementary weak KAMs defined on $\mathbb{T}^n$ for $c(\cdot) : \Sigma \to H^1(\mathbb{T}^k, \mathbb{R})$ continuous and one-to-one, where Σ is a compact subset of $\mathbb{R}^k$. If $\bar{u}^{\pm}_{c(\sigma)} := \bar{u}^{\pm}_{c(\sigma)}|_{\mathbb{T}^k}$ is ν-Hölder continuous in σ, then the weak KAM solutions $u^{\pm}_{c(\sigma)}$ satisfy the following estimate*

$$\|u^{\pm}_{c(\sigma)} - u^{\pm}_{c(\sigma')}\|_{C^0(\mathbb{T}^n)} \leq C(\|\sigma - \sigma'\|^{\nu} + \|c(\sigma) - c(\sigma')\|).$$

for some constant C.

Proof In this setting of normal hyperbolicity, the unstable (stable respectively) manifold W^u (W^s respectively) of an Aubry set $\mathcal{A}(c) \subset \mathbb{T}^k \times \mathbb{R}^k$ is given by the graph of the $(x, \partial u^-_c(x))$ ($(x, \partial u^+_c(x))$ respectively) where $u^{\pm}_c$ are the elementary weak KAM solutions determined by the Aubry set $\mathcal{A}(c)$. By the NHIM theorem, we know that for any point $\bar{z} \in \mathbb{T}^k \times \mathbb{R}^k$, in its neighborhood $W^u_{\bar{z}}$ is the image of $E^u_{\bar{z}}$ under the exponential map which is near identity if the neighborhood is small enough. Moreover, the unstable manifold $W^u_{\bar{z}}$ is C^{ℓ} in $\bar{z}$. So we get that in a small neighborhood U of $\mathbb{T}^k$, the weak KAM solutions $u^-_{c(\sigma)}$'s are ν-Hölder in σ as $\bar{u}^-_{c(\sigma)}$'s are.

Next, we show how to extend the estimate from a neighborhood of $\mathbb{T}^k$ to the whole $\mathbb{T}^n$ as stated in the theorem. The argument can be found in [5] Lemma 6.4. For each $x \in \mathbb{T}^n \setminus U$ there exists $x_c \in U$ and $k_c \in \mathbb{R}$ such that

$$u_c^-(x) = u_c^-(x_c) + h_c((x_c, k_c), (x, 0)),$$

where the notation $h_c((x_c, k_c), (x, 0))$ was given in Sect. 1.1. Here the number $|k_c|$ are uniformly bounded for all $x \in \mathbb{T}^n \setminus U$ and all $c \in c(\Sigma)$, and the upper bound depends only on the size of U. We denote by K the upper bound (Fig. 8).

Next, let $\gamma_c : [k_c, 0] \to \mathbb{T}^n$ the curve with $\gamma_c(k_c) = x_c$ and $\gamma_c(0) = x$ and attaining the quantity

$$h_c((x_c, k_c), (x, 0)) = \int_{k_c}^0 L(d\gamma_c(t)) - \langle c, \dot{\gamma}_c(t)\rangle + \alpha(c)\, dt.$$

It is clear that we have

$$u_{c'}^-(x) \le \int_{k_c}^0 L(d\gamma_c(t)) - \langle c', \dot{\gamma}_c(t)\rangle + \alpha(c')\, dt + u_{c'}^-(x_c).$$

This gives

$$\begin{aligned}
&\left| h_c((x_c, k_c), (x, 0)) - \int_{k_c}^0 L(d\gamma_c(t)) - \langle c', \dot{\gamma}_c(t)\rangle + \alpha(c')\, dt \right| \\
&\le \left| \int_{k_c}^0 \langle c - c', \dot{\gamma}_c(t)\rangle + \alpha(c) - \alpha(c')\, dt \right| \\
&\le |\tilde{\gamma}_c(0) - \tilde{\gamma}_c(k_c)\| \cdot |c' - c\| + |k_c| \cdot |\alpha(c) - \alpha(c')|
\end{aligned}$$

where $\tilde{\gamma}_c$ denotes the lift of γ_c to $\mathbb{R}^n$. By the convexity of α and the compactness of the set Σ, we have that $|\alpha(c) - \alpha(c')| \le C\|c - c'\|$ for some constant C. We also have the uniform bound $|\tilde{\gamma}_c(0) - \tilde{\gamma}_c(k_c)| \le CK$ for all $c \in c(\Sigma)$ due to the uniform bound on $|k_c|$. Now we get

$$\begin{aligned}
|u_{c'}^-(x) - u_c^-(x)| &\le \left| h_c((x_c, k_c), (x, 0)) - \int_{k_c}^0 L(d\gamma_c(t)) - \langle c', \dot{\gamma}_c(t)\rangle + \alpha(c')\, dt \right| \\
&\quad + |u_c^-(x_c) - u_{c'}^-(x_c)| \le (CK + C)\|c - c'\| + C\|\sigma - \sigma'\|^{\nu}.
\end{aligned}$$

This completes the proof. □

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Appendix A: Proof of Lemma 3.1

In this appendix, we prove Lemma 3.1. We first introduce the definition of viscosity solutions.

Definition A.1 (viscosity solution). Let $U(\subset M)$ be an open set and $H : T^*M \to \mathbb{R}$ continuous.

We say a function $u : U \to \mathbb{R}$ to be a *viscosity subsolution* of the Hamilton–Jacobi equation $H(q, d_q u(q)) = a$, if for any C^1 function ψ, and any $q_0 \in U$ such that $u - \psi$ has a max at q_0, then we have $H(q_0, d_q\psi(q_0)) \leq a$;

We say a function $u : U \to \mathbb{R}$ to be a *viscosity supersolution* of the Hamilton–Jacobi equation $H(q, d_q u(q)) = a$, if for any C^1 function ψ, and any $q_0 \in U$ such that $u - \psi$ has a min at q_0, then we have $H(q_0, d_q\psi(q_0)) \geq a$;

We say a function $u : U \to \mathbb{R}$ is a *viscosity solution*, if it is both a viscosity subsolution and a viscosity supersolution on U.

To prove Lemma 3.1, we use the same method used to prove Lemma 1 and Theorem 1 in [11] and Theorem 2.1 in [6].

We choose a smooth function β such that $0 \leq \beta \leq 1$, $\beta(0) = 1$ and $\beta(x) = 0$ whenever $\|x\| \geq 1$. Let $\beta_\epsilon(x) = \beta(x/\epsilon)$.

To prove the first part of the lemma, we assume the contrary, i.e. $\exists\, x_0 \in \Omega$ such that $u(x_0) - v(x_0) > 0$. Let $M = \max\{\|u\|_{C^0}, \|v\|_{C^0}\}$ and introduce Φ: $\Omega \times \Omega \to \mathbb{R}$ be given by

$$\Phi(x, x') = u(x) - v(x') + 3M\beta_\epsilon(x - x').$$

As $u - v \leq 0$ on $\partial\Omega$ and $\Phi \leq 2M$ if $\|x - x'\| \geq \epsilon$, one deduces that

$$\Phi(x_0, x_0) = u(x_0) - v(x_0) + 3M\beta_\epsilon(0) > 3M,$$

and

$$\Phi_{\partial(\Omega\times\Omega)} < \Phi(x_0, x_0) \qquad \text{if } \epsilon > 0 \text{ is sufficiently small.}$$

Choose $\delta > 0$ very small and then (x_1, x_1') so that

$$\Phi(x_1, x_1') > \sup_{\Omega\times\Omega} \Phi(x, x') - \delta.$$

We choose a smooth function $\zeta: \Omega\times\Omega \to \mathbb{R}$ such that $0 \leq \zeta \leq 1$, $\zeta(x_1, x_1') = 1$, $\zeta(x, x') = 0$ whenever $\|x - x_1\|^2 + \|x' - x_1'\| > 1$ and $|D\zeta| \leq 2$ in $\Omega \times \Omega$. Finally, we set

$$\Psi(x, x') = \Phi(x, x') + 2\delta\zeta(x, x').$$

Clearly, Ψ has a global maximum point $(\bar{x}, \bar{x}') \in \Omega \times \Omega$. Indeed,

$$\Psi(x_1, x_1') = \Phi(x_1, x_1') + 2\delta > \sup_{\Omega\times\Omega} \Phi + \delta$$

whereas

$$\limsup_{\|x\|+\|x'\|\to\infty} \Psi(x, x') \leq \sup_{\Omega\times\Omega} \Phi$$

and

$$\Psi(x, x')|_{\partial(\Omega\times\Omega)} < \sup_{\Omega\times\Omega} \Phi + \delta \qquad \text{if } \delta \text{ and } \epsilon \text{ are sufficiently small.}$$

Obviously, we have

$$\|\bar{x} - \bar{x}'\| < \epsilon.$$

Now $\bar{x}$ is a maximum point of $x \mapsto u(x) - (v(\bar{x}') - 3M\beta_\epsilon(x - \bar{x}') - 2\delta\zeta(x, \bar{x}'))$ and thus, by assumption

$$H(\bar{x}, -3M\partial\beta_\epsilon(\bar{x} - \bar{x}') - 2\delta\partial_x\zeta(x, \bar{x}')) \leq f(\bar{x}). \tag{A.1}$$

Similarly, $\bar{x}'$ is a minimum point of $x' \mapsto v(x') - (u(\bar{x}) + 3M\beta_\epsilon(x - \bar{x}') + 2\delta\zeta(x, \bar{x}'))$ and so

$$H(\bar{x}', -3M\partial\beta_\epsilon(\bar{x} - \bar{x}') + 2\delta\partial_{x'}\zeta(x, \bar{x}')) \geq 0. \tag{A.2}$$

Since $f(\bar{x}) < 0$ and δ can be chosen arbitrarily small, (A.1) contradicts to (A.2). This contradiction implies that $u \leq v$ on Ω. This completes the proof of the first part.

For the second part of the statement, we set

$$u_\theta(x) = \theta u(x) + (1-\theta)\phi(x) \qquad \text{for } x \in \bar{\Omega},$$

where $\theta \in (0, 1)$. By the assumptions, we are able to choose function $f \in C(\bar{\Omega})$ such that $H(x, \partial\phi(x)) \leq f(x) < 0$. Thus, we see that $u_\theta \leq u$ on Ω and $u_\theta \in C(\bar{\Omega})$ since $\phi \leq u$. Note that $p \to H(\cdot, p)$ is convex, a formal calculation reveals that

$$H(x, \partial u_\theta) \leq \theta H(x, \partial u) + (1-\theta)H(x, \partial\phi) \leq (1-\theta)f.$$

Applying the first part of the lemma we get $u_\theta \leq v$. Noting that θ can be arbitrarily close to 1, we complete the proof for the second part.

Appendix B: Further structure of the weak KAM solutions in the rational case

In Sect. 2, we have considered the globally weak KAM solutions when the rotation vector is irrational, and in the rational case assuming $c = \mathscr{L}_\beta(\omega)$ when $\mathscr{L}_\beta(\omega)$ is a point and c is an endpoint of $\mathscr{L}_\beta(\omega)$ when it is an interval. In this appendix, we consider a cohomology class c in the interior of $I_\omega = \mathscr{L}_\beta(\omega)$. The result in this appendix will not be used in the proof of the main theorem. We include it since it gives us some new information of the weak KAM solutions.

We fix a periodic curve $\gamma_{\omega,\lambda}$, denote by $u^\pm_{c,\lambda}$ the corresponding elementary weak KAM solution. Without loss of generality, we assume the rotation vector has the form $\omega = (0, \hat{\omega})$.

In the covering space $k\mathbb{T}^2$ with $k = (k_1, k_2)$ and $k_1 > 1$, denote by $\gamma^*_{\omega,\lambda}$ the curve in the lift of $\gamma_{\omega,\lambda}$ which passes through the interval $\{x \in \mathbb{R}^2 : x_1 \in [0, 1), x_2 = 0\}$. Any other curve in the lift is obtained by certain Deck transformation. Denoted by $u^\pm_{c,\lambda,k}$ the elementary weak KAM solution determined by $\gamma^*_{\omega,\lambda}$. Treating it as the k-periodic function defined on $\mathbb{R}^2$, one has

Proposition B.1 *Assume $c \in \mathrm{int} I_\omega$ where $I_\omega = \mathscr{L}_\beta(\omega)$ is a line segment for $\omega = (0, \hat{\omega})$ for some $\hat{\omega} \in \mathbb{R}$. For any bounded domain $\Omega \subset \mathbb{R}^2$, there is a positive number $\mathsf{k}(\Omega)$ such that for any $k, k' \in \mathbb{Z}^2$ with $k_1, k'_1 \geq \mathsf{k}(\Omega)$ and $k_2, k'_2 \geq 1$, one has*

$$u^\pm_{c,\lambda,k}|_\Omega = u^\pm_{c,\lambda,k'}|_\Omega.$$

Proof Clearly, $u^\pm_{c,\lambda,k}$ is always 1-periodic in x_1. To study how this function is valued, let us consider the quantity $h_c(g)$ for $g \in H_1(\mathbb{T}^2, \mathbb{Z})$ defined as follows

$$h_c(g) = \min_{x \in \mathbb{T}^2} \liminf_{T \to \infty} \inf_{\substack{\xi(0)=\xi(T)=x \\ [\xi]=g}} [A_c(\xi)].$$

Since the system is autonomous, the limit infimum is indeed a limit. Clearly, $h_c(g) = 0$ if $g = [\gamma_{\omega,\lambda}]$. Notice that $H_1(\mathbb{T}^2, \gamma_{\omega,\lambda}, \mathbb{Z})$ is generated by $\{\mathsf{e}, -\mathsf{e}\}$ where $\mathsf{e} = (1, 0)$ and $\gamma_{\omega,\lambda}$ is codimension 1, one has $h_c(\pm m\mathsf{e}) = mh_c(\pm\mathsf{e})$ for $m \in \mathbb{N}$. If $\cup_\lambda \gamma_{\omega,\lambda}$ does not make up a 2-torus, one has $h_c(\mathsf{e}) + h_c(-\mathsf{e}) > 0$, both $h_c(\mathsf{e}) > 0$ and $h_c(-\mathsf{e}) > 0$ if $c \in \mathrm{int} I_\omega$.

For the rotation vector $\omega(c) = (0, \hat{\omega})$ and $c \in \mathrm{int} I_\omega$, we consider the elementary weak KAM determined by $\gamma^*_{\omega,\lambda}$ in the covering space $k\mathbb{T}^2$, denoted by $u^{\pm}_{c,\lambda,k}$. For $k_1 \geq 4$, the shifts of $\gamma^*_{\omega,\lambda}$,

$$\gamma^{*,-}_{\omega,\lambda} = \gamma^*_{\omega,\lambda} - (1,0) \quad \text{and} \quad \gamma^{*,+}_{\omega,\lambda} = \gamma^*_{\omega,\lambda} + (1,0)$$

cut the torus into two annuli. Let us study how $u^{\pm}_{c,\lambda,k}$ is valued when it is restricted on the annulus which contains the curve $\gamma^*_{\omega,\lambda}$, denoted by $\mathbb{A} = \mathbb{A}^- \cup \mathbb{A}^+$, where $\mathbb{A}^-$ is bounded by the curves $\gamma^{*,-}_{\omega,\lambda}$ and $\gamma^*_{\omega,\lambda}$, $\mathbb{A}^+$ is bounded by the curves $\gamma^*_{\omega,\lambda}$ and $\gamma^{*,+}_{\omega,\lambda}$. Let $\bar{x} \in \gamma^*_{\omega,\lambda}$ and $\bar{x}' \in \mathbb{A}^+ \cup \mathbb{A}^-$, one obtains from the definition that

$$u^-_{c,\lambda,k}(\bar{x}') - u^-_{c,\lambda,k}(\bar{x}) = \lim_{T\to\infty} \inf_{\substack{\xi_T(0)=\bar{x} \\ \xi_T(T)=\bar{x}'}} [A_c(\xi_T)].$$

Notice $c \in \mathrm{int} I_\omega$. For sufficiently large k_1 and T, the minimal curve ξ_T falls into the annulus $\mathbb{A}^+ \cup \mathbb{A}^-$, it does not cross the annulus $k\mathbb{T}^2 \backslash (\mathbb{A}^+ \cup \mathbb{A}^-)$. Otherwise the quantity $[A_c(\xi_T)]$ shall approach infinity as $k_1 \to \infty$, guaranteed by $h_c(\mathsf{e}) > 0$ and $h_c(-\mathsf{e}) > 0$. For the same reason, the minimal curve for $u^+_{c,\lambda,k}$ also stays in $\mathbb{A}^+ \cup \mathbb{A}^-$. Therefore, the restriction of $u^{\pm}_{c,\lambda,k}$ on $\mathbb{A}^+ \cup \mathbb{A}^-$ is independent of k_1 provided it is suitably large.

In the covering space, each curve in the lift of $\gamma_{\omega,\lambda}$ takes the form of $\gamma^*_{\omega,\lambda} + (j, 0)$ with $j = 1, \ldots, k_1$, mod k_1. Let $\mathbb{A}_m$ denote the annulus bounded by the curves $\gamma^*_{\omega,\lambda} + (m, 0)$ and $\gamma^*_{\omega,\lambda} + (m + 1, 0)$, we claim that, for a positive integer m, there exists $k^*_1 = k^*_1(m)$ such that for any $k_1 \geq k^*_1$,

$$\begin{aligned} &u^{\pm}_{c,\lambda,k}(\bar{x}) = u^{\pm}_{c,\lambda,k}(\bar{x} - (m, 0)) + mh_c(\mp\mathsf{e}), && \bar{x} \in \mathbb{A}_m; \\ &u^{\pm}_{c,\lambda,k}(\bar{x}) = u^{\pm}_{c,\lambda,k}(\bar{x} + (m-1, 0)) + (m-1)h_c(\pm\mathsf{e}), && \bar{x} \in \mathbb{A}_{-m}. \end{aligned} \tag{B.1}$$

To verify, let $\xi\colon [-T, 0] \to k\mathbb{T}^2$ be the minimal curve connecting $\bar{x}_0 \in \gamma^*_{\omega,\lambda}$ to $\bar{x}$ such that $\xi(-T) = \bar{x}_0$ and $\xi(0) = \bar{x}$. Because $h_c(\mp\mathsf{e}) > 0$, the curve ξ will cross the annulus $\mathbb{A}_j$ with $j = 1, \ldots, m-1$ if $m > 0$ is suitably smaller than k_1 as $A_c(\xi) > A_c(\zeta)$ holds for any curve $\zeta : [-T, 0] \to k\mathbb{T}^2$ which connects $\bar{x}_0$ to $\bar{x}$ but does not cross the annulus $\mathbb{A}_j$ with $j = 1, \ldots, m-1$. Denoted by t_j the time when $\xi(t_j) = \gamma^*_{\omega,\lambda}(t_j) + (j, 0)$, then one has $|\dot{\xi}(t_j) - \dot{\gamma}^*_{\omega,\lambda}(t_j)| \to 0$ as $T \to \infty$. Therefore, there exists a point $\bar{x}_m \in \mathbb{A}_m$ on the curve ξ such that $|\bar{x}_m - (m, 0) - \bar{x}_0| \to 0$ when $T \to \infty$. Note $t_m \in [-T, 0]$, $T + t_m \to \infty$ and $-t_m \to \infty$ as $T \to \infty$, one obtains from the definition that

$$h^{\infty}_c(\bar{x}_0, \bar{x}) \leftarrow h^T_c(\bar{x}_0, \bar{x}) = h^{T+t_m}_c(\bar{x}_0, \bar{x}_m) + h^{-t_m}_c(\bar{x}_m, \bar{x}),$$

namely, $u^-_{c,\lambda,k}(\bar{x}) = u^-_{c,\lambda,k}(\bar{x} - (m, 0)) + mh_c(\mathsf{e})$. Other formulae can be proved in the similar way. Therefore, for any domain $\Omega \subset \mathbb{R}^2$, some positive number $\mathsf{k}(\Omega)$ exists such that $u^{\pm}_{c,\lambda,k}|_\Omega = u^{\pm}_{c,\lambda,k'}|_\Omega$ provided $k_1, k'_1 \geq \mathsf{k}(\Omega)$. □

B.1 An example

To illustrate what the lemma means, let us consider the weak KAM solution of

$$H(x, \partial_x u + c) = \frac{1}{2}(\partial_x u + c)^2 - (1 - \cos x)$$

for $c = 0$. The figures (Fig. 6, Fig. 7, Fig. 8) below are the graphs of the weak KAM solutions when they are lifted to the universal covering space. They are equal for $x \in [-\pi, \pi]$. The

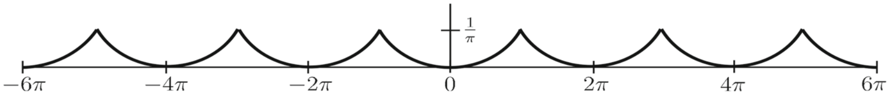

Fig. 6 The graph of u_0^-, by treating $\{x \in \mathbb{R} : x \mod 2\pi\}$ as the configuration space

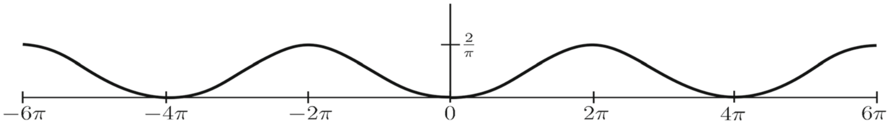

Fig. 7 The graph of u_0^- with respect to the Aubry class $\{x = 0\}$, by treating $\{x \in \mathbb{R} : x \mod 4\pi\}$ as the configuration space

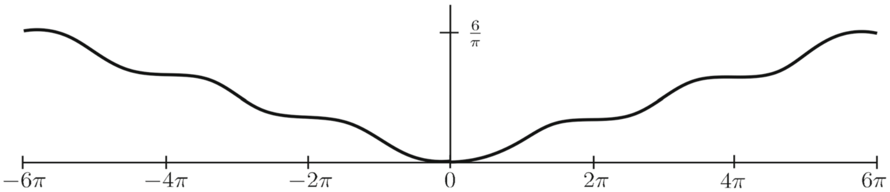

Fig. 8 The graph of u_0^- with respect to the Aubry class $\{x = 0\}$, by treating $\{x \in \mathbb{R} : x \mod 12\pi\}$ as the configuration space

functions for $\{x \in \mathbb{R} : x \bmod 4\pi\}$ and for $\{x \in \mathbb{R} : x \bmod 12\pi\}$ are equal when they are restricted on $[-2\pi, 2\pi]$.

References

1. Bangert, V.: Mather sets for twist maps and geodesics on tori. Dynamics Reported, pp. 1–56. Vieweg+ Teubner, Weisbaden (1988)
2. Bernard, P.: Existence of $C^{1,1}$ critical sub-solutions of the Hamilton–Jacobi equation on compact manifolds. Ann. Sci. École Norm. Sup. **40**, 445–452 (2007)
3. Cannarsa, P., Sinestrari, C.: Semiconcave functions, Hamilton–Jacobi equations and optimal control. Progress in Nonlinear Differential Equations and Their Appications, vol. 58. Birkhäuser, Basel (2004)
4. Cheng, C.-Q., Yan, J.: Existence of diffusion orbits in a priori unstable Hamiltonian systems. J. Differ. Geom. **67**, 457–517 (2004)
5. Cheng, C.-Q., Yan, J.: Arnold diffusion in Hamiltonian systems: a priori unstable case. J. Differ. Geom. **82**, 229–277 (2009)
6. Crandall, M., Evans, L.C., Loins, P.-L.: Some properties of viscosity solutions of Hamilton–Jacobi equations. Trans. Am. Math. Soc. **282**, 487–502 (1984)
7. Fathi, A.: Théorème KAM faible et théorie de Mather sue les systèmes. C. R. Acad. Sci. Paris Sér. I Math. **324**, 1043–1046 (1997)
8. Fathi, A.: Weak KAM theorem in Lagrangian dynamics, preliminary version 10, (2008)
9. Fathi, A., Siconolfi, A.: Existence of C^1 critical subsolutions of the Hamilton–Jacobi equation. Invent. Math. **155**, 363–388 (2004)
10. Hirsch, M., Pugh, C., Shub, M.: Invariant manifolds. Lecture Notes in Mathematics, vol. 583. Springer, Berlin (1977)
11. Ishii, H.: A simple, direct proof of uniqueness for solutions of the Hamilton–Jacobi equations. Proc. Am. Math. Soc. **100**, 247–251 (1987)
12. Lions, P.-L., Papanicolaou, G., Varadhan, S.R.S: Homogenization of Hamilton–Jacobi equations. (1986) (unpublished preprint)

13. Mather, J.: Modulus of continuity for Peierls's barrier. Periodic solutions of Hamiltonian systems and related topics (Il Ciocco, 1986), 177-202, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 209, Reidel, Dordrecht, (1987)
14. Mather, J.: Destruction of invariant circles. Ergod. Theory Dyn. Syst. **8**, 199–214 (1988)
15. Mather, J.: Action minimizing invariant measures for positive definite Lagrangian systems. Math. Z. **207**(2), 169–207 (1991)
16. Mather, J.: Variational construction of connecting orbits. Ann. Inst. Fourier (Grenoble) **43**, 1349–1386 (1993)
17. Mañé, R.: Generic properties and problems of minimizing measures of Lagrangian systems. Nonlinearity **9**, 273–310 (1996)
18. Massart, D.: On Aubry sets and Mather's action functional. Israel J. Math. **134**, 157–171 (2003)
19. Rockafellar, T.: Convex analysis. Princeton University Press, Princeton (1970)
20. Zhou, Min: Hölder regularity of weak KAM solutions in a priori unstable systems. Math. Res. Lett. **18**(1), 75–92 (2011)