Chordal Komatu-Loewner equation and Brownian motion with darning in multiply connected domains

Zhen-Qing Chen[*], **Masatoshi Fukushima**[†] and **Steffen Rohde**[‡]

(Appeared in *Trans. Amer. Math. Soc.* **368** (2016), 4065-4114.)

Abstract

Let $D = \mathbb{H} \setminus \cup_{k=1}^N C_k$ be a standard slit domain where $\mathbb{H}$ is the upper half plane and C_k, $1 \leq k \leq N$, are mutually disjoint horizontal line segments in $\mathbb{H}$. Given a Jordan arc $\gamma \subset D$ starting at $\partial\mathbb{H}$, let g_t be the unique conformal map from $D \setminus \gamma[0, t]$ onto a standard slit domain D_t satisfying the hydrodynamic normalization. We prove that g_t satisfies an ODE with the kernel on its righthand side being the complex Poisson kernel of the Brownian motion with darning (BMD) for D_t, generalizing the chordal Loewner equation for the simply connected domain $D = \mathbb{H}$. Such a generalization has been obtained by Y. Komatu in the case of circularly slit annuli and by R. O. Bauer and R. M. Friedrich in the present chordal case, but only in the sense of the left derivative in t. We establish the differentiability of g_t in t to make the equation a genuine ODE. To this end, we first derive the continuity of $g_t(z)$ in t with a certain uniformity in z from a probabilistic expression of $\Im g_t(z)$ in terms of the BMD for D, which is then combined with a Lipschitz continuity of the complex Poisson kernel under the perturbation of standard slit domains to get the desired differentiability.

AMS 2000 Mathematics Subject Classification: Primary 60H30; Secondary 30C20

Keywords and phrases: Komatu-Loewner equation, Brownian motion with darning, harmonic function with zero period, Green function, complex Poisson kernel, multiply connected domain

1 Introduction

A domain of the form $D = \mathbb{H} \setminus \bigcup_{k=1}^N C_k$ is called a *standard slit domain*, where $\mathbb{H}$ is the upper half-plane and $\{C_k\}$ are mutually disjoint horizontal line segments. We fix a standard slit domain D and consider a Jordan arc

$$\gamma : [0, t_\gamma] \to \overline{D}, \quad \gamma(0) \in \partial\mathbb{H}, \quad \gamma(0, t_\gamma] \subset D. \tag{1.1}$$

For each $t \in [0, t_\gamma]$, let g_t be the unique conformal map from $D \setminus \gamma(0, t]$ onto a standard slit domain $D_t = \mathbb{H} \setminus \bigcup_{k=1}^N C_{k,t}$ satisfying

$$g_t(z) = z + \frac{a_t}{z} + o(1/|z|), \qquad z \to \infty, \tag{1.2}$$

[*]Research partially supported by NSF Grant DMS-1206276 and NNSFC Grant 11128101.
[†]Research supported by Grant-in-Aid for Scientific Research of MEXT No. 22540125
[‡]Research partially supported by NSF Grant DMS-1068105

for some constant a_t. It can be shown (and we will give a self-contained proof in §8) that $t \mapsto a_t$ is a real-valued strictly increasing continuous function with $a_0 = 0$ so that the arc γ can be reparametrized in a way that $a_t = 2t$, $0 \le t \le t_\gamma$. Define

$$\xi(t) = g_t(\gamma(t)) \ (\in \partial\mathbb{H}), \quad 0 \le t \le t_\gamma. \tag{1.3}$$

In [3], R.O. Bauer and R.M. Friedrich have derived under the above reparametrization of γ a *chordal Komatu-Loewner equation*

$$\frac{\partial^- g_t(z)}{\partial t} = -2\pi\Psi_t(g_t(z), \xi(t)), \quad g_0(z) = z, \quad 0 < t \le t_\gamma, \tag{1.4}$$

where $\frac{\partial^- g_t(z)}{\partial t}$ denotes the left partial derivative in t. This is an extension of the Komatu-Loewner equation obtained first by Y. Komatu [12] for circularly slit annuli and later by Bauer-Friedrich [2] for circularly slit disks with an improved expression of the right hand side. The kernel $\Psi_t(z,\zeta)$, $z \in D_t$, $\zeta \in \partial\mathbb{H}$, appearing in (1.4) is an analytic function of $z \in D_t$ subjected to the normalization

$$\lim_{z \in D_t,\, z \to \infty} \Psi_t(z,\zeta) = 0. \tag{1.5}$$

However the differentiability of $g_t(z)$ in t has been established neither in the circularly slit cases ([12], [2]) nor in the chordal case ([3]). We have three goals in the present paper and a primary one of them is to prove in Theorem 9.9 this differentiability making (1.4) a genuine ordinary differential equation. While several results of this paper could be proved by methods of classical complex analysis, we will emphasize a more probabilistic approach.

The first goal is to present an alternative derivation of the Komatu-Loewner equation (1.4) by showing that the imaginary part $K_t^*(z,\zeta)$ of the kernel $\Psi_t(z,\zeta)$ is just the Poisson kernel of the *Brownian motion with darning* (*BMD* in abbreviation) for the standard slit domain D_t. In Chapter 7 of the book [5] by the first and second authors, the notion of BMD is introduced and its basic properties are studied. Roughly speaking, the BMD for a standard slit domain $D = \mathbb{H} \setminus \bigcup_{k=1}^N C_k$ is a diffusion process on a state space $D^* = D \cup \{c_1^*, \cdots, c_N^*\}$ obtained from the absorbing Brownian motion (ABM in abbreviation) on $\mathbb{H}$ by regarding (or "shorting") each "hole" C_k into one single point c_k^*. The BMD on D^* is symmetric with respect to the measure m^* that extends the Lebesgue measure m on D to D^* by setting $m^*(\{c_k^*\}) = 0$, $1 \le k \le N$. We are motivated by a paper [14] of G. Lawler in which it was claimed that $K^*(z,\zeta)$ is the Poisson kernel of the *excursion reflected Brownian motion* (*ERBM* in abbreviation) introduced there in a rather descriptive way using excursions. When the number of the slits is one, the ERBM was described more explicitly by S. Drenning [7] and, by that, the ERBM can be identified with our BMD as will be seen in §2. The paper [7] investigates a Komatu-Loewner type equation for standard slit domains with the goal to confirm Lawler's claim that K^* is the Poisson kernel of ERBM. The method in [7] is quite different from ours and particularly contains a comparison of a_t with its counterpart in the simply connected domain $\mathbb{H}$.

The BMD is known ([5]) to be the unique m^*-symmetric diffusion extension of the ABM on D to D^* that admits no killing on $\{c_k^*,\ 1 \le k \le N\}$. In the next two sections of this paper, we present a self-contained short exposition of BMD and its basic properties that will be used later in the paper. In §2, we give a simple and direct construction of BMD as well as its uniqueness proof by observing that the associated Dirichlet form on $L^2(D^*; m^*)$ is regular due to a nice property (2.1) of the two-dimensional Brownian motion. We show in Section 3 that the L^2-generator of BMD is characterized by the zero flux property at each c_k^*. In §3.3, we then derive from this zero flux condition the zero period property around c_k^* for any function that is BMD-harmonic in a neighborhood of c_k^*. This zero period property readily yields in §5 explicit expressions of the Green function and the Poisson kernel of BMD in terms of the periods of the harmonic basis.

It was proved in [3] that a_t is strictly increasing, based on a certain extremal property of g_t. In §6, we shall give a different proof, establish left-continuity, and by combining it with the above mentioned explicit expression of the BMD-Poisson kernel and a general formula of the period of a harmonic function presented in §4, we shall derive a chordal Komatu-Loewner equation formulated in terms of the left derivative in a_t.

The second goal of this paper is to present in §7 a probabilistic representation of the conformal map g_t and use it to derive in §8 the continuity of $g_t(z)$ in t with uniformity in z on each compact region including the boundary $\partial_p C_k$ of the set $\mathbb{H} \setminus C_k$ in the path distance topology, and thereby obtaining the continuity of a_t, D_t as well as $\xi(t)$. To this end, a construction of a BMD-harmonic function v^* on a general $(N+1)$-connected domain $D = \mathbb{H} \setminus \bigcup_{i=1}^N A_i$ with a compact $\mathbb{H}$-hull F removed will be carried out in Appendix 1 using a hitting probability of BMD by following the method of Lawler [14, §5] where ERBM was used in place of BMD. The analytic map f with imaginary part v^* will then be shown to be a conformal mapping from $D \setminus F$ onto a standard slit domain by invoking a degree theorem, which is presented in Appendix 2. Thus the imaginary part $\Im g_t(z)$ of $g_t(z)$ admits a probabilistic representation in terms of BMD on D^* and ABM on $\mathbb{H}$. The continuity of $g_t(z)$ in t then follows from the stochastic continuity of the first hitting time of the set $\gamma[0,t]$ by BMD and ABM (see Proposition 7.3 and Theorem 7.4).

The third goal of this paper is to show in §9 a Lipschitz continuity of the BMD complex Poisson kernel $\Psi(z,\zeta)$ under the perturbation of the standard slit domains, which combined with the continuity theorems obtained in §8 will enable us to derive the continuity of the right hand side of (1.4) in t and consequently the desired differentiability of $g_t(z)$ in t. The BMD complex Poisson kernel can be obtained from the classical Green function through operations of taking normal derivatives at $\partial\mathbb{H}$, taking periods around the slits and taking line integrals of normal derivatives along smooth curves. We shall thus utilize two perturbation formulae of Green's function whose proof will be given in Appendix 3 by following the interior variation method in Garabedian [10] of constructing a special parametrix of an elliptic partial differential operator for a transformed Green function and solving an associated Fredholm type integral equation. One of these formulae was considered in [2] in relation to slit motions but we shall make them considerably more detailed and precise to be usable for the present purpose.

Acknowledgement: The authors would like to thank R. Bauer for clarifying some details of some of the proofs in [2] and [3].

2 Brownian motion with darning

A subset A of $\mathbb{R}^2$ is said to be a *continuum* if A is a connected closed set containing at least two points. Suppose A is a continuum. Then, due to a Lebesgue's theorem, A is non-polar and each point $z \in A$ is a regular point of A with respect to the two-dimensional Brownian motion $Z = (Z_t, \mathbb{P}_x)$ (cf. [17]). In particular, for any $\alpha \geq 0$ and any bounded continuous function f on A, the function

$$w(z) := \mathbb{E}_z \left[e^{-\alpha \sigma_A} f(Z_{\sigma_A}) \right], \quad z \in \mathbb{R}^2$$

is a bounded continuous function on $\mathbb{R}^2$ with (cf. [17, Proposition 2.3.5]):

$$\lim_{z' \to z, z' \in \mathbb{R}^2 \setminus A} w(z') = f(z) \text{ for } z \in \partial(\mathbb{R}^2 \setminus A), \quad w(z) = f(z) \text{ for } z \in A. \tag{2.1}$$

Here σ_A denotes the first hitting time of A by the process Z; that is, $\sigma_A := \inf\{t > 0 : Z_t \in A\}$. We will use similar notation for other processes, such as Z^*, and its meaning should be clear from the context.

Consider a domain $E \subset \mathbb{R}^2$. We assume E is either $\mathbb{R}^2$ or otherwise $\mathbb{R}^2 \setminus E$ consists of continuum components. Let

$$D = E \setminus K, \quad K = \cup_{i\in\Lambda} A_i, \tag{2.2}$$

where $\{A_i, i \in \Lambda\}$ is a finite or a countably infinite family of mutually disjoint compact continuum contained in E that are locally finite on $\mathbb{R}^2$.

Let

$$D^* = D \cup K^*, \quad K^* = \{a_i^*, i \in \Lambda\} \tag{2.3}$$

and define a neighborhood U_i^* of each point a_i^* in D^* by $\{a_i^*\} \cup (U_i \setminus A_i)$ for some neighborhood U_i of A_i in E. That is, D^* is a topological space obtained from E by regarding each continuum A_i as one point a_i^*. Denote by m the Lebesgue measure in D that is extended to D^* by setting $m(K^*) = 0$.

Definition 2.1 *Brownian motion with darning* (BMD in abbreviation) $Z^* = (Z_t,^*, \zeta^*, \mathbb{P}_z^*)$ is an m-symmetric diffusion on D^* such that

(i) its subprocess killed upon leaving D has the same law as Brownian motion in D;

(ii) it admits no killings on K^*.

Recall that a Markov process Z^* is said to be m-symmetric if the transition semigroup of Z^* is self-adjoint in $L^2(D^*; m)$; and that Z^* admits no killings on K^* means $\mathbb{P}_x(Z^*_{\zeta^*-} = a^*, \zeta^* < \infty) = 0$ for m-a.e. $x \in D^*$. Here ζ^* denotes the lifetime of Z^*. In Definition 2.1, assuming (i), then it is easy to see that condition (ii) is equivalent to

$$\mathbb{P}_x(Z^*_{\zeta^*-} = a^*, \zeta^* < \infty) = 0 \qquad \text{for every } x \in D.$$

Observe that it follows from the m-symmetry of Z^* and the fact that $m(K^*) = 0$ that BMD Z^* spends zero Lebesgue amount of time (i.e. zero sojourn time) at K^*. We point out that D can be disconnected as $E \setminus A_i$ can be disconnected.

As a consequence of a more general result [5, Theorems 7.7.3 and 7.7.4], BMD on D^* exists and is unique in law. (In fact, BMD can be defined in any dimension as in [5, 4].) For the reader's convenience, following [4], we present below a direct proof of this fact through two theorems.

Remark 2.2 The BMD coincides with the excursion reflected Brownian motion (ERBM) described by Drenning [7] when K consists of just one single continuum A so that $\mathbb{C} \setminus A$ is connected. In the case that $E = \mathbb{C}$ and $A = \mathbb{D}$, the unit disk centered at the origin, a Feller transition semigroup $\{T_t, t \geq 0\}$ of the ERBM on $(\mathbb{C} \setminus \mathbb{D}) \cup \{a^*\}$ whose associated Hunt process extends the absorbing Brownian motion on $\mathbb{C} \setminus \mathbb{D}$ was explicitly given in [7, Proposition 3.1]. It is easy to see that T_t is m-symmetric. Therefore the ERBM must be equal to the BMD due to the stated uniqueness of the BMD. Based on Lawler's description of ERBM in the case of $E = \mathbb{C}$ and $A = \mathbb{D}$ using excursion laws, identification between BMD and ERBM is also done in [5, Remark 7.6.4]. The ERBM for a general open set E and a continuum $A \subset E$ so that $\mathbb{C} \setminus A$ is connected is defined in [7] by a conformal map from $\mathbb{C} \setminus \mathbb{D}$ with a due time change and a killing upon leaving E. Hence it can also be identified with the BMD on account of the conformal invariance of BMD formulated in [5, §7.8(1)]. However, even in this one single continuum case, BMD is more general as, for BMD, $\mathbb{C} \setminus A$ is allowed to have disconnected components, for example, when A is the unit circle in $\mathbb{C}$.

Consider next the case that $\Lambda = \{1, \cdots, N\}$ for $N \geq 2$ and each $\mathbb{C} \setminus A_i$ is connected. Then, by the locality ([4, Theorem 1.3.1]) and conformal invariance ([5, Theorem 7.8.1]) of BMD, our BMD Z^* on D^* has the properties described in [7, §3.4] and enjoys all the properties prescribed in [7, Definition 3.1] for ERBM. So Z^* may be identified with the ERBM if some uniqueness statement is available concerning [7, Definition 3.1]. □

Let $Z = (Z_t, \mathbb{P}_z)$ (resp. $Z^0 = (Z^0_t, \mathbb{P}^0_z)$) be an absorbing Brownian motion(ABM in abbreviation) on E (resp. on D). The Dirichlet form for Z on $L^2(E; dx)$ (resp. for Z^0 on $L^2(D; dx)$) is $(\mathbf{D}, W^{1,2}_0(E))$ (resp. $(\mathbf{D}, W^{1,2}_0(D))$). Here for an open set E, $W^{1,2}_0(E)$ is the $\mathbf{D}_1^{1/2}$-completion of the space $C^\infty_c(E)$ of smooth functions with compact support in E, where $\mathbf{D}$ denotes the Dirichlet integral on E and $\mathbf{D}_1(u, u) := \mathbf{D}(u, u) + \int_E u(x)^2 dx$.

Define for $i \in \Lambda$ and $z \in E$,

$$u^{(i)}(z) = \mathbb{E}_z \left[e^{-\sigma_K}; Z_{\sigma_K} \in A_i \right], \quad \varphi^{(i)}(z) = \mathbb{P}_z \left(Z_{\sigma_K} \in A_i \right). \tag{2.4}$$

We call the system $\{\varphi^{(i)}; i \in \Lambda\}$ the *harmonic basis* for the domain D. In complex analysis, $\varphi^{(i)}$ is called the *harmonic measure* of A_i. By (2.1), both $u^{(i)}$ and $\varphi^{(i)}$ are continuous functions on $\overline{E}$ that takes constant value 1 on A_i and vanish on ∂E and on A_j for $j \neq i$.

Now we define

$$\mathcal{F}^* = \mathbf{D}_1\text{-closure of linear span of } C^\infty_c(D) \text{ and } \{u^{(j)}|_D;\ i \in \Lambda\} \tag{2.5}$$

and for $u, v \in \mathcal{F}^*$,

$$\mathcal{E}^*(u, v) = \frac{1}{2} \int_D \nabla u(x) \cdot \nabla v(x) dx. \tag{2.6}$$

Observe that

$$\mathcal{F}^* = \left\{ u|_D :\ u \in W^{1,2}_0(E),\ u \text{ is constant } \mathbf{D}\text{-q.e. on each } A_j \right\}. \tag{2.7}$$

It is easy to check that $(\mathcal{E}^*, \mathcal{F}^*)$ is a Dirichlet form on $L^2(D; dx) = L^2(D^*; m)$.

Theorem 2.3 *The quadratic form $(\mathcal{E}^*, \mathcal{F}^*)$ defined by (2.5)-(2.6) is a regular Dirichlet form on $L^2(D^*; m)$. It is strongly local and each a^*_j has positive capacity. Consequently, there is an m-symmetric diffusion Z^* on D^* that starts from every point in D^* and admits no killings on D^*. The diffusion Z^* is BMD on D^* and every a^*_j is regular for itself (that is, $\mathbb{P}_{a^*}(\sigma_{\{a^*\}} = 0) = 1$).*

Proof. Let $\mathcal{C}$ be the linear span of $C^\infty_c(D)$ and $\{u^{(j)}; j \in \Lambda\}$. By defining $u(a^*_j)$ to be the value of u on A_j, we can view $\mathcal{C}$ as a subspace of $C(D^*) \cap \mathcal{F}^*$. Since $\mathcal{C}$ is an algebra that separates points in D^*, by Weierstrass theorem, $\mathcal{C}$ is uniformly dense in the space $C_\infty(D^*)$ of continuous functions in D^* vanishing at ∂D^*. Clearly $\mathcal{C}$ is $\mathcal{E}^*_1$-dense in $\mathcal{F}^*$. Hence $(\mathcal{E}^*, \mathcal{F}^*)$ is a regular Dirichlet form on $L^2(D^*; m)$. It is strongly local and its part Dirichlet form on D is $(\mathbf{D}, W^{1,2}_0(D))$. So there is an m-symmetric diffusion Z^* on D^* associated with $(\mathcal{E}^*, \mathcal{F}^*)$, whose part process in D is the killed Brownian motion in D. The diffusion Z^* is a BMD on D^*. Since Brownian motion X^E in E starting from $x \in D$ visits each A_j with positive probability, Z^* starting from $x \in D$ visits each a^*_j with positive probability. This implies that each a^*_j has positive capacity. Consequently, Z^* can be refined to start from every point in D^*. That each a^*_j is regular for itself follows from the general fact that for any nearly Borel measurable set A, $A \setminus A^r$ is semipolar and hence m-polar. □

Theorem 2.4 *BMD on D^* is unique in law.*

Proof. It suffices to show that if Z^* is a BMD on D^*, its associated quasi-regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(D^*; m)$ has to be $(\mathcal{E}^*, \mathcal{F}^*)$. First note that according to the definition of BMD, each a_j^* is non-polar for Z^* and that the part Dirichlet form $(\mathcal{E}, \mathcal{F}_D)$ of $(\mathcal{E}, \mathcal{F})$ in D is $(\mathbf{D}, W_0^{1,2}(D))$ (see [5, Theorem 3.3.8]). By the $\mathcal{E}_1$-orthogonal projection (see [5, Theorem 3.2.2]), for every $u \in \mathcal{F}$, $\mathbf{H}^1_{K^*}u(x) := \mathbb{E}_x\left[e^{-\sigma^*}u(Z^*_{\sigma^*})\right] \in \mathcal{F}$ and $u - \mathbf{H}^1_{K^*}u \in W_0^{1,2}(D)$. Here $K^* := \{a_i^*; \, i \in \Lambda\}$ and $\sigma^* := \inf\{t > 0 : Z_t^* \in K^*\}$. Now

$$\mathbf{H}^1_{K^*}u(x) = \sum_{j \in \Lambda} u(a_j^*)\, \mathbb{E}_x\left[e^{-\sigma^*}; Z^*_{\sigma^*} = a_j^*\right] \quad \text{for } x \in D.$$

By the continuity of Z^*, the definition of a_j^* and the fact that $X^{*,D}$ has the same distribution as the subprocess of X^E killed upon leaving D, we see that

$$\mathbb{E}_x\left[e^{-\sigma^*}; Z^*_{\sigma^*} = a_j^*\right] = \mathbb{E}_x\left[e^{-\sigma_E}; X^E_{\sigma_K} \in A_j\right] = u^{(j)}(x) \quad \text{for } x \in D.$$

It follows then $\mathbf{H}^1_{K^*}u = \sum_{j \in \Lambda} u(a_j^*)u^{(j)}(x)$. As each a_j^* is non-polar, for every finite subset Λ_1 of Λ having N elements,

$$\{(u(a_j^*), j \in \Lambda_1); u \in \mathcal{F}\} = \mathbb{R}^N$$

and so $\mathcal{F} = \mathcal{F}^*$. Note that $(\mathcal{E}, \mathcal{F})$ is strongly local so for every bounded $u \in \mathcal{F} = \mathcal{F}^*$,

$$\mathcal{E}(u,u) = \frac{1}{2}\mu^c_{\langle u \rangle}(D^*) = \frac{1}{2}\mu^c_{\langle u \rangle}(D) + \sum_{j \in \Lambda} \mu^c_{\langle u \rangle}(a_j^*) = \frac{1}{2}\mu^c_{\langle u \rangle}(D),$$

where in the last inequality, we used the fact that the energy measure $\mu^c_{\langle u \rangle}$ of u does not charge on level sets of u (cf. [5, Theorem 4.3.8]). As $(\mathcal{E}, \mathcal{F}_D) = (\mathbf{D}, W_0^{1,2}(D))$, we have by the strong local property of $\mu^c_{\langle u \rangle}$ (see [5, Proposition 4.3.1]) that $\mu^c_{\langle u \rangle}(dx) = |\nabla u(x)|^2 dx$ on D. Consequently $\mathcal{E}(u,u) = \frac{1}{2}\int_D |\nabla u(x)|^2 dx$ for every bounded and hence for any $u \in \mathcal{F}$. This completes the proof for $(\mathcal{E}, \mathcal{F}) = (\mathcal{E}^*, \mathcal{F}^*)$. □

It is easy to see from the definition of BMD and the conformal invariance of Brownian motion that BMD is invariant under conformal map up to a time change. See [5, Theorem 7.8.1].

3 Zero period property of BMD-harmonic functions

From now on, we assume that the index set Λ is a finite set $\{1, \ldots, N\}$ and so $K = \bigcup_{i=1}^N A_i$.

3.1 BMD harmonicity and its locality

Denote by $Z = (Z_t, \mathbb{P}_z)$, $Z^0 = (Z^0_t, \mathbb{P}^0_z)$ the ABM on E and $D = E \setminus K$, respectively, and by $Z^* = (Z_s, \mathbb{P}^*_z)$ the BMD on the set

$$D^* = D \cup K^*, \quad K^* = \{a_1^*, \cdots, a_N^*\} \tag{3.1}$$

obtained from Z by regarding each hole A_i as one point a_i^*.

A function u defined on a connected open subset O of D^* is said to be Z^*-*harmonic* or *BMD-harmonic* on O (with respect to Z^*) if u is continuous on O and, for any relatively compact open set O_1 with $\overline{O}_1 \subset O$,

$$\mathbb{E}^*_z\left[|u(Z^*_{\tau_{O_1}})|\right] < \infty \quad \text{and} \quad \mathbb{E}^*_z\left[u(Z^*_{\tau_{O_1}})\right] = u(z) \quad \text{for every } z \in O_1. \tag{3.2}$$

Here $\tau_{O_1} = \inf\{t \geq 0 : Z_t^* \notin O_1\}$ is the first exit time from O_1 by Z^*. The restriction to $O \cap D$ of any Z^*-harmonic function on O is harmonic there in the classical sense (with respect to Brownian motion) and takes constant boundary value $u(a_i^*)$ at ∂A_i whenever $a_i^* \in O$.

Theorem 3.1 *Suppose that D_1 and D_2 are two connected subsets of D^* and that $D_1 \cap D_2 \neq \emptyset$. If u is Z^*-harmonic in D_i for $i = 1, 2$, then u is Z^*-harmonic in $D_1 \cup D_2$.*

Proof. Let O be a relatively compact open subset of $D_1 \cup D_2$. Let $\{U_k^{(i)}; k \geq 1\}$ be an increasing sequence of relatively compact open subsets whose union is D_i and $\partial U_k^{(i)}$ is a smooth subset in D for $i = 1, 2$. Since $\{U_k^{(1)} \cup U_k^{(2)}; k \geq 1\}$ forms an open cover for $\overline{O}$, there is some $k_0 \geq 1$ so that $\overline{O} \subset U_{k_0}^{(1)} \cup U_{k_0}^{(2)}$. For notational simplicity, denote $U_{k_0}^{(i)}$ by U_i for $i = 1, 2$. Note that $O_i := O \cap U_i$ is a relatively compact open subset of D_i, $i = 1, 2$. We claim that for every $x \in O$, $u(x) = \mathbb{E}_x\left[u(Z_{\tau_O}^*)\right]$. In the following we show that the above holds for every $x \in O_1$. The case for $x \in O_2$ is analogous.

Let $\{\theta_t; t \geq 0\}$ be the shift operator for BMD Z^* on D^*. We use $\{\mathcal{F}_t; t \geq 0\}$ to denote the minimal augmented natural filtration generated by Z^*. Define a sequence of stopping times as follows. $T_1 := \tau_{O_1}$, $T_2 := \tau_{O_2}$, and for $k \geq 1$,

$$T_{2k+1} := T_{2k} + \tau_{O_1} \circ \theta_{T_{2k}} \quad \text{and} \quad T_{2k+2} := T_{2k+1} + \tau_{O_2} \circ \theta_{T_{2k+1}}.$$

Note that $\tau_O < \infty$ and $T_k \leq \tau_O$ for every $k \geq 1$.. Since u is Z^*-harmonic in both D_1 and D_2, we have for $x \in O_1$, $\mathbb{P}_x$-a.s.

$$u(X_{T_k}) = \mathbb{E}_{Z^*_{T_{k+1}}}\left[u(Z^*_{T_{k+1}})|\mathcal{F}_{T_k}\right] \qquad \text{for every } k \geq 1.$$

In other words, $\{u(Z^*_{T_k}); k \geq 1\}$ is an $\{\mathcal{F}_{T_k}\}_{k\geq 1}$-filtration under $\mathbb{P}_x$ for every $x \in O_1$. Let $T := \lim_{k\to\infty} T_k$. Since u is bounded and continuous on $\overline{O}$, we have

$$u(x) = \lim_{k\to\infty} \mathbb{E}_x\left[u(Z^*_{T_k})\right] = \mathbb{E}_x\left[u(Z^*_T)\right].$$

We next show that $T = \tau_O$. Clearly $T \leq \tau_O$ $\mathbb{P}_x$-a.s.. On $\{T < \tau_O\}$, $Z_T^*(\omega) \in O = O_1 \cup O_2$, say, $Z_T^*(\omega) \in O_2$. There is some large $k_0 = k_0(\omega)$ so that $Z^*_{T_k}(\omega) \in O_2$ for all $k \geq k_0$. This is impossible as for even $k \geq k_0$, $Z^*_{T_k} \notin O_2$. So we must have $T = \tau_O$ $\mathbb{P}_x$-a.s. and consequently, $u(x) = \mathbb{E}_x\left[u(Z^*_{\tau_O})\right]$ for every $x \in O_1$. This shows that u is Z^*-harmonic in O for every relatively compact subdomain O of $D_1 \cup D_2$ and so u is Z^*-harmonic in $D_1 \cup D_2$. □

Consider an open set $\widetilde{E}$ such that $\widetilde{E} \subset E$, $\mathbb{C} \setminus \widetilde{E}$ is a continuum and each compact continuum A_i is either contained in $\widetilde{E}$ or disjoint from $\widetilde{E}$. Let $\widetilde{K} = \widetilde{E} \cap K = \bigcup_{j=1}^{\ell} A_{i_j}$ for $1 \leq i_1 < \cdots < i_\ell$, $\ell \leq N$, and $\widetilde{D} = \widetilde{E} \setminus \widetilde{K}$. Denote by $\widetilde{Z}^*$ the BMD on the set

$$\widetilde{D}^* = \widetilde{D} \cup \widetilde{K}^*, \quad \widetilde{K}^* = \{a^*_{i_1}, \cdots, a^*_{i_\ell}\},$$

obtained from the ABM on $\widetilde{E}$ by rendering each hole A_{i_j} as a one point $a^*_{i_j}$ $1 \leq j \leq \ell$. Then $\widetilde{Z}^*$ is identical in law with the part process $Z^*_{\widetilde{D}^*}$ of Z^* on $\widetilde{D}^*$, namely, the subprocess of Z^* obtained by killing upon its exit time from $\widetilde{D}^*$. This is because both $\widetilde{Z}^*$ and $Z^*_{\widetilde{D}^*}$ are symmetric diffusion extension of ABM on $\widetilde{D}$ to $\widetilde{D}^*$ admitting no killing on $\widetilde{K}^*$, and the uniqueness result stated in §2 applies. Let O be an open connected subset of $\widetilde{D}^*(\subset D^*)$. A function u on O is Z^*-harmonic on O if and only if u is harmonic with respect to the part process $Z^*_{\widetilde{D}^*}$ of Z^* on $\widetilde{D}^*$. Therefore we have the following equivalence:

$$u \text{ is } Z^*\text{-harmonic on } O \iff u \text{ is } \widetilde{Z}^*\text{-harmonic on } O. \tag{3.3}$$

3.2 Zero flux condition for generator of BMD

Let $(\mathcal{A}^*, \mathcal{D}(\mathcal{A}^*))$ denote the L^2-infinitesimal generator of BMD Z^*, or equivalently, of the Dirichlet form $(\mathcal{E}^*, \mathcal{F}^*)$. That is, $u \in \mathcal{D}(\mathcal{A}^*)$ if and only if $u \in \mathcal{F}^*$ and there is some $f \in L^2(D; dx) = L^2(D^*; m)$ so that

$$\mathcal{E}^*(u,v) = -\int_D f(x)v(x)dx \qquad \text{for every } v \in \mathcal{F}^*. \tag{3.4}$$

We denote the above f as $\mathcal{A}^*u$. In view of (2.5), condition (3.4) is equivalent to

$$\frac{1}{2}\int_D \nabla u(x)\cdot\nabla v(x)dx = -\int_D f(x)v(x)dx \qquad \text{for every } v \in C_c^\infty(D) \tag{3.5}$$

and

$$\frac{1}{2}\int_D \nabla u(x)\cdot\nabla u_j(x)dx = -\int_D f(x)u_j(x)dx \qquad \text{for every } j = 1,\dots,N. \tag{3.6}$$

(3.5) says that Δu exists on D in the distributional sense and $f = \frac{1}{2}\Delta u \in L^2(D; dx)$. Let us define the flux $\mathcal{N}(u)(a_j^*)$ of u at a_j^* by

$$\mathcal{N}(u)(a_j^*) = \int_D \nabla u(x)\cdot\nabla u_j(x)dx + \int_D \Delta u(x)u_j(x)dx. \tag{3.7}$$

Then (3.6) is equivalent to

$$\mathcal{N}(u)(a_j^*) = 0 \qquad \text{for every } j = 1,\dots,N. \tag{3.8}$$

Hence we have established the following.

Theorem 3.2 *A function $u \in \mathcal{F}^*$ is in $\mathcal{D}(\mathcal{A}^*)$ if and only if the distributional Laplacian Δu of u exists as an L^2-integrable function on D and u has zero flux at every a_j^*. Moreover, for $u \in \mathcal{D}(\mathcal{A}^*)$, $\mathcal{A}^*u = \frac{1}{2}\Delta u$ on D.*

Note that when ∂K_j is smooth for $j = 1,\dots,N$, then by the Green-Gauss formula we have

$$\mathcal{N}(u)(a_j^*) = \int_{\partial K} \frac{\partial u(x)}{\partial \mathbf{n}} u^{(j)}(x)\sigma(dx),$$

where $\mathbf{n}$ is the unit outward normal vector field of D on ∂D and σ is the surface measure on ∂D. Since $u^{(j)}(x) = 1$ on K_j and $u^{(j)}(x) = 0$ on K_i with $i \neq j$,

$$\mathcal{N}(u)(a_j^*) = \int_{\partial K_j} \frac{\partial u(x)}{\partial \mathbf{n}}\sigma(dx). \tag{3.9}$$

For $\alpha \geq 0$, we use G_α^* to denote the α-order resolvent of Z^*; that is, $G_\alpha^* f(x) = \mathbb{E}_x\left[\int_0^\infty e^{-\alpha t} f(Z_t^*)dt\right]$ for $f \geq 0$ on D^*. Throughout this paper, we use the convention that ∂ is a cemetery point added to D^*, $Z_t^* = \partial$ for $t \geq \zeta^*$, and that every function f defined on D^* is extended to ∂ by setting $f(\partial) = 0$. When $\alpha = 0$, we will simply denote G_0^* by G^*.

Lemma 3.3 *If E is bounded, then for every $f \in L^\infty(D)(= L^\infty(D; m))$, $G^*f \in \mathcal{D}(\mathcal{A}^*)$ with $\mathcal{A}^*G^*f = -f$.*

Proof. Since $D = E \setminus K$ is bounded, $G^0(L^\infty(D)) \subset L^\infty(D) \subset L^2(D)$ for the 0-order resolvent G^0 of Z^0 in view of (4.6) below with $w_D = 0$. G^* has the same property because for $f \in L^\infty(D)$, G^*f is a linear combination of $G^0 f$ and $\varphi^{(i)}$ for $1 \leq i \leq N$. Hence the resolvent equation $G^*f = G_1^*f + G_1^*(G^*f)$ yields that $G^*f \in \mathcal{D}(\mathcal{A}^*)$ with $\mathcal{A}^*G^*f = -f$. □

3.3 Zero period property of BMD-harmonic functions

In this subsection, we assume that E is a planar domain whose complement is a continuum and $A_1, \cdots A_N$ are disjoint compact continua contained in E so that each $E \setminus A_j$ is connected. As before, $K = \cup_{j=1}^N A_j$. The domain $D := E \setminus K$ is called $(N+1)$-*connected*. In what follows, we mostly consider an $(N+1)$-connected domain with E being the upper half-plane $\mathbb{H}$. An $(N+1)$-connected domain $D = E \setminus K$ is called a *standard slit domain* if $E = \mathbb{H}$ and $A_i \subset \mathbb{H}$ is a line segment parallel to the x-axis for each $1 \leq i \leq N$.

Let γ be a C^1-smooth simple curve surrounding A_j, namely, $\gamma \subset D$, ins$\gamma \supset A_j$, $\overline{\text{ins}\gamma} \cap A_k = \emptyset$, $k \neq j$. Here insγ denotes the bounded component of $\mathbb{C} \setminus \gamma$ and is called the interior of γ. For a harmonic function u defined in a neighborhood of A_j, the value

$$\int_\gamma \frac{\partial u(\zeta)}{\partial \mathbf{n}_\zeta} ds(\zeta)$$

is independent of the choice of such curve γ with $\mathbf{n}$ denoting the normal vector pointing toward A_j and s the arc length of γ. This value is called the *period* of u around A_j.

Theorem 3.4 *Let O be a connected open subset of D^*. A Z^* -harmonic function in O has zero period round A_i for every i with $a_i^* \in O$.*

Proof. The assertion trivially holds if O does not contain any a_i^*. In view of Theorem 3.1 and the equivalence (3.3), without loss of generality, we may and do assume that E is bounded with smooth boundary ∂E, $D^* = O$ and that D^* contains exactly one a_1^* (that is, K consists of exactly one compact continuum A_1).

Suppose that v is Z^* harmonic in D^*. Clearly v is harmonic in D. Let U_1 and U_2 be relatively compact open subsets of E so that $A_1 \subset U_1 \subset \overline{U}_1 \subset U_2 \subset \overline{U}_2 \subset E$. Let $\psi \in C_c^\infty(\mathbb{R}^2)$ so that $\psi = 1$ on U_1 and $\psi = 0$ on U_2^c. Define $f(x) = -\frac{1}{2}\Delta(\psi v)(x)$ for $x \in D$. Note that $f \in L^\infty(D; dx)$ and $f = 0$ on $D \setminus (U_2 \setminus U_1)$. Hence $G^* f$ is Z^* harmonic in in $(U_1 \cap D) \cup \{a_1^*\}$ and so is $w := \psi v - G^* f$. On the other hand, Lemma 3.3 implies that w is harmonic and hence Z^*-harmonic in D. Thus by Theorem 3.1, w is Z^*-harmonic in D^*. Since both ψv and $G^* f$ vanish on $\partial E = \partial D^*$, $w = 0$ on ∂D^*. Thus by the maximum principle for the bounded Z^*-harmonic function w on D^* (note that a_1^* is an interior point of D^*), we have $w = 0$ on D^*, that is, $\psi v = G^* f$ on D^*.

Let $u_1(x) := \mathbb{E}_x \left[e^{-\sigma_{A_1}} \right]$, which is smooth, strictly smaller than 1 on D, and continuous on E with value 1 on A_1. For $\varepsilon \in (0,1)$, let η_ε be the boundary of the connected component of $\{x \in E : u_1(x) > 1 - \varepsilon\}$ that contains A_1. By Sard's theorem (see, e.g., [15]), there is a set $\mathcal{N}_0$ having zero Lebesgue measure so that for every $\varepsilon \in (0,1) \setminus \mathcal{N}_0$, η_ε is C^∞-smooth. Take a decreasing sequence $\{\varepsilon_n, n \geq 1\} \in (0,1) \setminus \mathcal{N}_0$ with $\lim_{n\to\infty} \varepsilon_n = 0$. Since $\{x \in E : u_1(x) > 1 - \varepsilon_n\}$ decreases to A_1, we may assume that each η_{ε_n} is contained inside U_1. As $\psi = 1$ on U_1, we can see by the Green-Gauss formula, Theorem 3.2 and Lemma 3.3 that the period of v at a_1^* equals

$$\begin{aligned} & \lim_{n\to\infty} \int_{\eta_{\varepsilon_n}} \frac{\partial(\psi v)(\xi)}{\partial \mathbf{n}_\xi} \sigma(d\xi) = \lim_{n\to\infty} \frac{1}{1-\varepsilon_n} \int_{\eta_{\varepsilon_n}} \frac{\partial G^* f(\xi)}{\partial \mathbf{n}_\xi} u_1(\xi) \sigma(d\xi) \\ = & \lim_{n\to\infty} \frac{1}{1-\varepsilon_n} \int_{D\setminus \text{ins}(\eta_n)} (\nabla u_1 \cdot \nabla G^* f + u_1 \Delta G^* f)\, dx \\ = & \int_D \nabla u_1(x) \cdot \nabla G^* f(x) dx + \int_D u_1(x) \Delta G^* f(x) dx = 2\mathcal{N}(G^* f)(a_1^*) = 0. \end{aligned}$$

Here $\mathbf{n}$ denotes the unit inward normal vector field on η_{ε_n} for the interior of η_{ε_n}. □

Remark 3.5 Theorem 3.2, Lemma 3.3 and Theorem 3.4 in fact hold in any dimension. Moreover, the statement in Theorem 3.4 is in fact a characterization for v to be Z^*-harmonic in O; see [4] for details.

By the zero period property of v, the value of

$$u(z) = -\int_\gamma \frac{\partial v(\xi)}{\partial \mathbf{n}_\xi} \sigma(d\xi) \tag{3.10}$$

is independent of the choice of the smooth C^2 simple curve γ that joins a fixed z_0 to z, and $f(z) := u(z) + iv(z)$ is an analytic function in D. Hence we obtain

Corollary 3.6 *If v is Z^*-harmonic on D^*, then $-v\big|_D$ admits a harmonic conjugate u on D uniquely up to an additive real constant so that $f(z) = u(z) + iv(z)$, $z \in D$, is an analytic function on D.*

4 Green function, harmonic functions and their periods

We first consider a planar domain D whose complement is non-empty and consists of continuum components. We recall the relationship between the 0-order resolvent density of the ABM Z^0 on D and the classical Green's function. Denote by $p_t^0(z,\zeta),\ z,\zeta \in D$, the transition density of the ABM Z^0 on D and $\tau_D := \inf\{t > 0 : Z_t \notin D\}$ the first exit time of the Brownian motion $Z = (Z_t, \mathbb{P}_z)$ from D. Since Z has transition density function $n(t, z-\zeta)$ with respect to the Lebesgue measure on $\mathbb{R}^2$, where $n(t,z) = (2\pi t)^{-1} \exp\left(-\frac{|z|^2}{2t}\right)$, it follows from the strong Markov property of Z that

$$p_t^0(z,\zeta) = n(t, z-\zeta) - \mathbb{E}_z[n(t-\tau_D, Z_{\tau_D} - \zeta); \tau_D < t]. \tag{4.1}$$

The *resolvent density* $G_\alpha^0(z,\zeta)$ and the 0-*order resolvent density* $G^0(z,\zeta)$ of Z^0 are defined respectively by

$$G_\alpha^0(z,\zeta) = \int_0^\infty e^{-\alpha t} p_t^0(z,\zeta)dt \quad \text{and} \quad G^0(z,\zeta) = \int_0^\infty p_t^0(z,\zeta)dt, \quad z,\ \zeta \in D.$$

The integral $\int_0^\infty n(t,z)dt$ is infinite but the re-centered integral

$$\begin{aligned} \int_0^\infty (n(t,z) - n(t,\mathbf{e}_1))\, dt &= \int_0^\infty \frac{1}{2\pi t}\left(e^{-|z|^2/2t} - e^{-1/2t}\right) dt \\ &= -\frac{1}{\pi} \log|z| \qquad \text{for } z \in \mathbb{C}, \end{aligned} \tag{4.2}$$

is finite where $\mathbf{e}_1 = (1,0) \in \mathbb{C}$. Since $n(t,z) \geq n(t,\mathbf{e}_1)$ when $|z| \leq 1$ and $n(t,z) < n(t,\mathbf{e}_1)$ when $|z| > 1$, in fact we have from (4.2) that

$$\int_0^\infty |n(t,z) - n(t,\mathbf{e}_1)| dt = \frac{1}{\pi} |\log|z|| \qquad \text{for } z \in \mathbb{C}. \tag{4.3}$$

Moreover, we know from [17, Theorem 3.4.2] and Harnack inequality that

$$\mathbb{E}_z\left[|\log|Z_{\tau_D} - \zeta||\right] < \infty \qquad \text{for every } z,\zeta \in D. \tag{4.4}$$

From (4.1), one has

$$
\begin{aligned}
p_t^0(z,\zeta) &= n(t,z-\zeta)-n(t,\mathbf{e}_1)-\mathbb{E}_z[n(t-\tau_D,Z_{\tau_D}-\zeta)-n(t-\tau_D,\mathbf{e}_1);\tau_D<t] \\
&\quad +n(t,\mathbf{e}_1)-\mathbb{E}_z[n(t-\tau_D,\mathbf{e}_1);\tau_D<t].
\end{aligned} \tag{4.5}
$$

Hence it follows from (4.3)-(4.4) that for distinct $z,\zeta\in D$,

$$
\begin{aligned}
&\int_0^\infty |n(t,\mathbf{e}_1)-\mathbb{E}_z[n(t-\tau_D,\mathbf{e}_1);\tau_D<t]|dt \\
&\le \; G^0(z,\zeta)+\frac{1}{\pi}|\log|z-\zeta||+\frac{1}{\pi}\mathbb{E}_z\left[|\log|Z_{\tau_D}-\zeta||\right]<\infty.
\end{aligned}
$$

Let $W_D(z)=\int_0^\infty \mathbb{E}_z\left[n(t,\mathbf{e}_1)-n(t-\tau_D,\mathbf{e}_1)\mathbf{1}_{\{t>\tau_D\}}\right]dt$. Then it follows from (4.5) that

$$
G^0(z,\zeta)=-\frac{1}{\pi}\log|z-\zeta|+\frac{1}{\pi}\mathbb{E}_z\log|Z_{\tau_D}-\zeta|+W_D(z), \quad z,\zeta\in D, \tag{4.6}
$$

By [17, Proposition 4.5.4], for each $\zeta\in D$, as a function of z, $G^0(z,\zeta)+\frac{1}{\pi}\log|z-\zeta|$ is harmonic in D and thus so is $W_D(z)$. Note that

$$
\begin{aligned}
W_D(z) &= \lim_{N\to\infty}\int_0^N \mathbb{E}_z\left[n(t,\mathbf{e}_1)-n(t-\tau_D,\mathbf{e}_1)\mathbf{1}_{\{t>\tau_D\}}\right]dt \\
&= \lim_{N\to\infty}\mathbb{E}_z\left[\int_0^N n(t,\mathbf{e}_1)dt-\int_0^{N\vee\tau_D-\tau_D} n(s,\mathbf{e}_1)ds\right] \\
&= \lim_{N\to\infty}\mathbb{E}_z\left[\int_{N\vee\tau_D-\tau_D}^N n(t,\mathbf{e}_1)dt\right]\ge 0.
\end{aligned}
$$

From the last display, we get the following monotonicity property on W:

$$
\text{if } D_1\supset D_2, \text{ then } W_{D_1}(z)\ge W_{D_2}(z) \text{ for every } z\in D_2.
$$

Denote by $\mathbb{H}$ the upper half plane $\{z=x+iy\in\mathbb{C}: y>0\}$ and by $G^{\mathbb{H},0}(z,\zeta)$ the 0-order resolvent density of the ABM $Z^{\mathbb{H},0}$ in $\mathbb{H}$. One deduces easily from (4.2) and the identity $p_t^{\mathbb{H},0}(z,\zeta)=n(t,z-\zeta)-n(t,z-\bar\zeta)$ that

$$
G^{\mathbb{H},0}(z,\zeta)=\frac{1}{\pi}\left(\log|z-\bar\zeta|-\log|z-\zeta|\right). \tag{4.7}
$$

By (4.6), $W_{\mathbb{H}}(z)=\frac{1}{\pi}\left(\log|z-\bar\zeta|-\mathbb{E}_z\log|Z_{\tau_{\mathbb{H}}}-\zeta|\right)$. Taking $\zeta=iy$ and letting $y\to+\infty$, one concludes from the above display that $W_{\mathbb{H}}\equiv 0$. Consequently, for any $D\subset\mathbb{H}$, we have by the monotonicity that $W_D=0$ on D and

$$
G^0(z,\zeta)=-\frac{1}{\pi}\log|z-\zeta|+\frac{1}{\pi}\mathbb{E}_z\log|Z_{\tau_D}-\zeta|, \quad z,\zeta\in D. \tag{4.8}
$$

In other words, the 0-order resolvent density of Z^0 coincides with the classical Green's function multiplied by the constant $\frac{1}{\pi}$ whenever the domain D is contained in a half space. For this reason, we call the 0-order resolvent density of the ABM Z^0 on D also its *Green function*. We point out that W_D can be non-trivial for some unbounded D. For example, when $D=\{z\in\mathbb{C}:|z|>r\}$ for $r>0$, $W_D(z)=\frac{1}{\pi}\log^+(|z|/r)$ (cf. [17, Proposition 4.9]). We take this opportunity to point out that some condition is needed for Proposition 2.36 of [13] to hold for a regular domain D in $\mathbb{R}^2$.

The bounded harmonic function in $\mathbb{H}\setminus\mathbb{D}$ with boundary values 1 on $\partial\mathbb{D}$ and 0 on $\mathbb{R}$ tends to zero at infinity. By the maximum principle, we easily obtain

Lemma 4.1 *Let $D = \mathbb{H} \setminus K$ be an $(N+1)$-connected domain.*
(i) *The function $\varphi^{(j)}(z)$, $z \in D$, defined by (2.4) with $E = \mathbb{H}$ is harmonic in D and satisfies*

$$\lim_{|z|\to\infty, z\in D} \varphi^{(j)}(z) = 0, \qquad j \geq 1. \tag{4.9}$$

(ii) *For any continuous function g on $\partial\mathbb{H}$ with compact support, the function*

$$\psi_g(z) := \mathbb{E}_z^0 \left[g(Z^0_{\zeta^0-}); Z^0_{\zeta^0-} \in \partial\mathbb{H} \right], \quad z \in D, \tag{4.10}$$

is a harmonic function in D and satisfies

$$\lim_{|z|\to\infty, z\in D} \psi_g(z) = 0. \tag{4.11}$$

Here ζ^0 denotes the lifetime of the ABM Z^0 on D.

Lemma 4.2 *Denote the positive x-axis and the positive y-axis by Γ_0 and Λ_0, respectively. Let $Z = (Z_t, \mathbb{P}_z)$ be the Brownian motion on $\mathbb{C}$. Then*

$$\mathbb{P}_z(\sigma_{\Lambda_0} < \sigma_{\Gamma_0}) = \frac{2}{\pi} \tan^{-1}(y/x) \quad \textit{for } z = x + iy \textit{ with } x > 0,\ y > 0. \tag{4.12}$$

Proof. The conformal map $\phi(z) = z^2$ of the first quadrant onto the upper half plane maps Λ_0 onto $(-\infty, 0)$ and Γ_0 onto $(0, \infty)$. Hence the Lemma follows from the conformal invariance of the absorbing Brownian motion and the formula

$$\omega(z, \mathbb{H}, [a,b]) = \frac{1}{\pi} \arg \frac{z-b}{z-a}$$

for the harmonic measure of an interval (a,b) in $\mathbb{H}$, see [11, I (1.1)]. □

Lemma 4.3 *For $a > 0$, consider the rectangle $R_a := \{z = x + iy : -a < x < a,\ 0 < y < a\}$ and put $\Sigma_a = \partial R_a \setminus \partial\mathbb{H}$. Let h be a harmonic function on $\mathbb{H} \setminus R_{\ell_0}$ for some $\ell_0 > 0$ such that*

$$\sup_{\zeta\in\mathbb{H}\setminus R_{\ell_0}} |h(\zeta)| := C < \infty, \quad h(x + i0+) = 0 \quad \text{for any } x \text{ with } |x| > \ell_0. \tag{4.13}$$

Then

$$\int_{\Sigma_\ell} \left| \frac{\partial h(\zeta)}{\partial \mathbf{n}_\zeta} \right| ds(\zeta) \leq 8C \quad \text{for any } \ell > 2\ell_0. \tag{4.14}$$

The estimate (4.14) follows readily from the Poisson formulae for the harmonic function h on the half-planes $\{z \in \mathbb{C} : y > \ell_1\}$ and $\{z \in \mathbb{C} : x > \ell_1\}$. The latter reads, for $x > \ell_1,\ y > 0$,

$$h(x+iy) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x - \ell_1}{(y-y')^2 + (x-\ell_1)^2} h(\ell_1 + iy') dy', \tag{4.15}$$

where h is extended from $\{z \in \mathbb{C} : x > \ell_0,\ y > 0\}$ to $\{z \in \mathbb{C} : x > \ell_0\}$ by the Schwarz reflection.

We will also need the following, see for instance [11, Theorem II.2.3] for the standard proof of (i). The second part easily follows by approximation of $\mathbb{H}$ with bounded domains, using Lemma 4.3.

Lemma 4.4 *Let $D = \mathbb{H} \setminus \bigcup_{i=1}^N A_i$ be an $(N+1)$-connected domain and f be a harmonic function on D.*

(i) *Take mutually disjoint C^1-smooth simple curves γ_i surrounding A_i, $1 \le i \le N$, and an analytic Jordan curve $\Gamma \subset \mathbb{H}$ containing $\bigcup_{i=1}^N \text{ins}\,\gamma_i$. Then, for any $z \in \text{ins}\,\Gamma \setminus \bigcup_{i=1}^N \text{ins}\,\gamma_i$,*

$$\begin{aligned} f(z) &= -\frac{1}{2}\sum_{k=1}^N \int_{\gamma_k} \frac{\partial G^0(z,\zeta)}{\partial \mathbf{n}_\zeta} f(\zeta) ds(\zeta) + \frac{1}{2}\sum_{k=1}^N \int_{\gamma_k} G^0(z,\zeta)\frac{\partial f(\zeta)}{\partial \mathbf{n}_\zeta} ds(\zeta) \\ &\quad -\frac{1}{2}\int_\Gamma \frac{\partial G^0(z,\zeta)}{\partial \mathbf{n}_\zeta} f(\zeta) ds(\zeta) + \frac{1}{2}\int_\Gamma G^0(z,\zeta)\frac{\partial f(\zeta)}{\partial \mathbf{n}_\zeta} ds(\zeta). \end{aligned} \tag{4.16}$$

(ii) *Suppose further that a harmonic function f on D takes a smooth boundary function with compact support on $\partial\mathbb{H}$ and satisfies*

$$\lim_{|\zeta|\to\infty,\ \zeta\in D} f(\zeta) = 0. \tag{4.17}$$

Then, for analytic smooth simple curves γ_i, $1 \le i \le N$, as in (i) *and for any $z \in \mathbb{H} \setminus \bigcup_{i=1}^N \gamma_i$,*

$$\begin{aligned} f(z) &= -\frac{1}{2}\sum_{k=1}^N \int_{\gamma_k} \frac{\partial G^0(z,\zeta)}{\partial \mathbf{n}_\zeta} f(\zeta) ds(\zeta) + \frac{1}{2}\sum_{k=1}^N \int_{\gamma_k} G^0(z,\zeta)\frac{\partial f(\zeta)}{\partial \mathbf{n}_\zeta} ds(\zeta) \\ &\quad -\frac{1}{2}\int_{\partial\mathbb{H}} \frac{\partial G^0(z,\zeta)}{\partial \mathbf{n}_\zeta} f(\zeta) ds(\zeta). \end{aligned} \tag{4.18}$$

Theorem 4.5 *Let $D = \mathbb{H} \setminus \bigcup_{i=1}^N A_i$ be an $(N+1)$-connected domain.*

(i) *For each $z \in D$, $1 \le i \le N$, $-2\varphi^{(i)}(z)$ equals the period of the harmonic function $G^0(z,\cdot)$ around A_i.*

(ii) *For any bounded continuous function g on $\partial\mathbb{H}$,*

$$\mathbb{E}_z^0\left[g(Z^0_{\zeta^0-}); Z^0_{\zeta^0-} \in \partial\mathbb{H}\right] = -\frac{1}{2}\int_{\partial\mathbb{H}} \frac{\partial G^0(z,\zeta)}{\partial \mathbf{n}_\zeta} g(\zeta) ds(\zeta),\ \ z \in D. \tag{4.19}$$

(iii) *Let f be a harmonic function on D taking a constant value f_i on each ∂A_i, taking a continuous boundary function with compact support on $\partial\mathbb{H}$ and satisfying*

$$\lim_{|z|\to\infty,\ z\in D} f(z) = 0. \tag{4.20}$$

It then holds for every $z \in D$ that

$$f(z) = \sum_{k=1}^N f_k \varphi^{(k)}(z) - \frac{1}{2}\int_{\partial\mathbb{H}} \frac{\partial G^0(z,\zeta)}{\partial \mathbf{n}_\zeta} f(\zeta) ds(\zeta). \tag{4.21}$$

It also holds that for each $1 \le i \le N$,

$$\text{the period of } f \text{ around } A_i = \sum_{k=1}^N f_k a_{ki} + \int_{\partial\mathbb{H}} \frac{\partial \varphi^{(i)}(\zeta)}{\partial \mathbf{n}_\zeta} f(\zeta) ds(\zeta), \tag{4.22}$$

where a_{ki} denotes the period of $\varphi^{(k)}$ around A_i.

Proof. (i) If the A_i are closed analytic curves, the claim follows from Lemma 4.4 with $f = \varphi^{(i)}$. In general, approximate A_i by the level sets $\{G^0(z,\zeta) = a\}$ which are closed analytic curves for almost all values of $a > 0$. Since the Green's function G^0_a of the domain $D_a = \{z : G^0(z,\zeta) > a\}$ satisfies $G^0_a = G^0 - a$, the claim follows from the weak convergence of harmonic measure of D_a to the harmonic measure of D, see [11, II Exercise 4].

(ii) can be proved similarly by applying Lemma 4.4 to the function f which coincides with g on $\partial\mathbb{H}$ and which is zero on $\partial D_a \setminus \partial\mathbb{H}$. Since Lemma 4.4 assumes smoothness of the boundary function, it should be applied to a mollification g_ϵ of g, rather than g itself. Passing to the limit $\epsilon \to 0$ poses no difficulties.

(iii) Let h be the difference of the functions on both sides of (4.21). Then h is a harmonic function on D taking value 0 on $\bigcup_{i=1}^N A_i$ and on $\partial\mathbb{H}$ as well. By condition (4.20) and Lemma 3.1, $\lim_{|z|\to\infty, z\in D} h(z) = 0$. Hence the maximum principle applies to h as in [16, p 9] and we can conclude that $h = 0$.

Finally (4.22) follows from (4.21) by taking the periods around A_i on both sides of (4.21) and by noting (i) and the symmetry of G^0. □

5 Green function and Poisson kernel of BMD

Since

$$\mathbb{P}^*_z(\zeta^* < \infty,\ Z^*_{\zeta^*-} \in \partial\mathbb{H}) \geq \mathbb{P}^0_z(\zeta^0 < \infty,\ Z^0_{\zeta^0-} \in \partial\mathbb{H}) > 0,\ z \in D,$$

Z^* is transient. Denote by G^* the 0-order resolvent of Z^*:

$$G^* f(z) = \mathbb{E}^*_z \left[\int_0^\infty f(Z_s)dt \right], \quad z \in D^*,\ f \in \mathcal{B}_+(D^*).$$

Lemma 5.1 *For any Borel measurable function $f \geq 0$ on D^*, $G^*f(z) = \int_D G^*(z,\zeta)f(\zeta)m(d\zeta)$, where*

$$G^*(z,\zeta) = G^0(z,\zeta) + 2\Phi(z)\mathcal{A}^{-1}\Phi(\zeta)^{tr} \quad z \in D^*,\ \zeta \in D. \tag{5.1}$$

Here $\Phi(z) = (\varphi^{(1)}(z), \cdots, \varphi^{(N)}(z))$, $\mathcal{A}$ is an $N \times N$-matrix whose (i,j)-component a_{ij} is the period of $\varphi^{(j)}$ around A_i, $1 \leq i,j \leq N$, and the superscript "tr" denotes vector transpose.

Proof. Let $f \in C_c(D)$. Then $G^*f \in \mathcal{F}^*$ is Z^*-harmonic in $D^* \setminus \mathrm{supp}[f]$ and

$$G^*f(z) = G^0 f(z) + \sum_{i=1}^N \varphi^{(i)}(z)\lambda_i, \quad \text{where } \lambda_i = G^*f(a^*_i).$$

By Theorem 3.4, we have for a smooth simple curve γ_j surrounding A_j

$$\frac{1}{2}\sum_{i=1}^N a_{ij}\lambda_i = -\frac{1}{2}\int_{\gamma_j} \frac{\partial G^0 f(\zeta)}{\partial \mathbf{n}_\zeta} ds(\zeta), \quad 1 \leq i,j \leq N.$$

Since the right hand side in the above equals $(\varphi^{(j)}, f)$ in view of the symmetry of G^0, and Theorem 3.5(i), it suffices to solve the above linear equation in $\lambda_i,\ 1 \leq i \leq N$. □

We call $G^*(z,\zeta),\ z\in D^*,\zeta\in D$, of Lemma 5.1 the *Green function of* Z^*. One can deduce the symmetry of the matrix $\check{\mathcal{A}}$ from the symmetry of the Green functions of Z^0 and Z^*. We now define the *Poisson kernel of* Z^* by

$$K^*(z,\zeta) = -\frac{1}{2}\frac{\partial}{\partial \mathbf{n}_\zeta} G^*(z,\zeta),\ z\in D^*,\zeta\in\partial\mathbb{H}.$$

Note that

$$K^*(z,\zeta) = -\frac{1}{2}\frac{\partial}{\partial \mathbf{n}_\zeta} G^0(z,\zeta) - \Phi(z)\mathcal{A}^{-1}\frac{\partial}{\partial \mathbf{n}_\zeta}\Phi(\zeta)^{tr},\ z\in D^*,\ \zeta\in\partial\mathbb{H}. \tag{5.2}$$

Thus for each $\zeta\in\partial\mathbb{H}$, $z\mapsto K^*(z,\zeta)$ is a Z^*-harmonic function of z on D^*.

Lemma 5.2 (i) *For any closed interval* $J\subset\partial\mathbb{H}$, $K^*(z,\zeta)$ *can be uniquely extended to a jointly continuous function on* $(\overline{\mathbb{H}}\setminus J)\times J$.

(ii) *For any* $g\in bC(\partial\mathbb{H})$, *the integral*

$$H^*g(z) = \int_{\partial\mathbb{H}} K^*(z,\zeta)g(\zeta)ds(\zeta),\quad z\in D^*,$$

gives a well defined bounded Z^**-harmonic function on* D^* *and*

$$\lim_{z\to\zeta, z\in D} H^*g(z) = g(\zeta),\quad \zeta\in\partial\mathbb{H}. \tag{5.3}$$

(iii) *It holds that*

$$\lim_{z\to\infty} K^*(z,\zeta) = 0 \quad \text{uniformly in } \ \zeta \text{ on any compact interval of } \partial\mathbb{H}. \tag{5.4}$$

(iv) *It holds for any* $g\in bC(\partial\mathbb{H})$ *that*

$$\mathbb{E}_z^*\left[g(Z^*_{\zeta^*-})\right] = H^*g(z),\quad x\in D^* \tag{5.5}$$

where ζ^* *denotes the lifetime of* Z^*.

Proof. (i) Note that every point in C_j (resp. $\partial\mathbb{H}$) is a regular point for C_j (resp. $\partial\mathbb{H}$). Thus $G^0(\cdot,\zeta)$ extends to be a continuous function on $\overline{\mathbb{H}}\setminus\{\zeta\}$ by taking values 0 on $\partial\mathbb{H}\cup K$, and $\varphi^{(k)}$ extends to be a continuous function on $\overline{\mathbb{H}}$ by taking value δ_{kj} on A_j and 0 on $\partial\mathbb{H}$. Since $G^0(z,\zeta)$ and $\varphi^{(k)}(\zeta),\ 1\le k\le N$, are, as functions of ζ, harmonic in D and vanish on $\partial\mathbb{H}$, they admit harmonic extensions across $\partial\mathbb{H}$ by the Schwarz reflection. Now by the integral representation of harmonic functions in balls in terms of Poisson kernels, the right hand side of (5.2) is jointly continuous on the desired set.
(ii) Since $\varphi^{(k)}$ vanishes on $\partial\mathbb{H}$, $\frac{\partial\varphi^{(j)}(x+iy)}{\partial y}\big|_{y=0}\ge 0$ for every $x\in\mathbb{R}$. For every non-negative function g on $\partial\mathbb{H}$, we have by (4.19) and (5.2)

$$H^*g(z) = \mathbb{E}_z^0\left[g(Z^0_{\zeta^0-})\right] + \frac{1}{2}\sum_{i,j=1}^N \varphi^{(i)}(z)(\mathcal{A}^{-1})_{ij}\int_{-\infty}^{\infty}\frac{\partial\varphi^{(j)}(x+iy)}{\partial y}\big|_{y=0} g(x)dx.$$

So it suffices to show that

$$\int_{-\infty}^{\infty}\frac{\partial\varphi^{(k)}(x+iy)}{\partial y}\big|_{y=0}dx \le 2\nu^{(k)}(A_k). \tag{5.6}$$

To verify (5.6), consider the ABM $Z^{\mathbb{H},0} = (Z_t^{\mathbb{H},0}, \mathbb{P}_z^{\mathbb{H},0})$ on the upper half-plane $\mathbb{H}$ and let $\psi^{(k)}(z) = \mathbb{P}_z^{\mathbb{H},0}(\sigma_{A_k} < \infty),\ z \in \mathbb{H},\ 1 \le k \le N$. The function $\psi^{(k)}$ is a 0-order equilibrium potential of the compact set A_k. So by Corollary 3.4.3 and the 0-order version of Lemma 2.3.10 of [5], there is a finite positive measure $\nu^{(k)}$ concentrated on A_k so that

$$\psi^{(k)}(z) = \int_{A_k} G^{\mathbb{H},0}(z, z') d\nu^{(k)}(z'), \quad z \in \mathbb{H}.$$

Here $G^{\mathbb{H},0}(z, z')$ is the Green function of $Z^{\mathbb{H},0}$ given by (4.7).

The measure $\nu^{(k)}$ is called the 0-order equilibrium measure of A_k relative to $Z^{\mathbb{H},0}$, and $\nu^{(k)}(A_k)$ is the (0-order) capacity of A_k in $\mathbb{H}$ (cf. [5]). Since $\varphi^{(k)} \le \psi^{(k)}$, we have

$$\frac{\partial \varphi^{(k)}(x+iy)}{\partial y}\Big|_{y=0} \le \frac{2}{\pi} \int_{A_k} \frac{y'}{(x-x')^2 + (y')^2} \nu^{(k)}(dz'),$$

which yields (5.6).

(iii) In view of the domination $G^0(z, \zeta) \le G^{\mathbb{H},0}(z, \zeta)$ and (4.7), we have for $z = x + iy \in D,\ \zeta = \xi + i0 \in \partial\mathbb{H}$,

$$-\frac{1}{2} \frac{\partial}{\partial \mathbf{n}_\zeta} G^0(z, \zeta) \le \frac{1}{\pi} \frac{y}{(x-\xi)^2 + y^2}. \tag{5.7}$$

Hence the first term of the expression (5.2) of $K^*(z, \zeta)$ converges to 0 as $z \to \infty$ uniformly in ζ on any compact interval of $\partial\mathbb{H}$, and so does its second term on account of (4.9) and the continuity of $\frac{\partial \varphi^{(j)}(\zeta)}{\partial \mathbf{n}_\zeta}$ in $\zeta \in \partial\mathbb{H}$ for every $1 \le j \le N$. Thus we get (5.4).

(iv) It is enough to prove (5.5) for a continuous function g on $\partial\mathbb{H}$ with compact support. Both sides of (5.5) are Z^*-harmonic functions with the same boundary value $g(\zeta),\ \zeta \in \partial\mathbb{H}$. As $z \to \infty$, the right hand side vanishes by (5.4) and so does the left hand side because it is a linear combination of $\psi_g(z)$ of (4.10) and $\varphi^{(i)}(z),\ 1 \le i \le N$, and (4.9) and (4.11) apply. Hence (5.5) holds by the maximum principle. □

6 Complex Poisson kernel of BMD and chordal Komatu-Loewner equation

For $z \in \mathbb{C}$ and $r > 0$, we use either $B_r(z)$ or $B(z, r)$ to denote the open ball with radius r centered at z.

6.1 Complex Poisson kernel of BMD on a slit domain

Let $D = \mathbb{H} \setminus \{C_1, \cdots, C_N\}$ be a standard slit domain and $K^*(z, \zeta),\ z \in D^*,\ \zeta \in \partial\mathbb{H}$, be the Poisson kernel (5.2) of the BMD on D^*. As $K^*(z, \zeta)$ is BMD-harmonic in $z \in D$ for each fixed $\zeta \in \partial\mathbb{H}$, by Corollary 3.6 there is an analytic function $\Psi(z, \zeta)$, unique up to an additive real constant, having $K^*(z, \zeta)$ as its imaginary part.

Lemma 6.1 (i) *The limit* $\lim_{z\to\infty} \Psi(z, \zeta)$ *exists and is real-valued. Furthermore* $\limsup_{z\to\infty} \sup_{\zeta \in J} |\Psi(z, \zeta)| < \infty$ *for any compact interval* $J \subset \partial\mathbb{H}$.
(ii) $\Psi(z, \zeta)$ *is determined uniquely by the normalization condition*

$$\lim_{z\to\infty} \Psi(z, \zeta) = 0. \tag{6.1}$$

$\Psi(z,\zeta)$ is then jointly continuous in (z,ζ) on $\big(D\cup\big(\cup_{i=1}^N(C_i^+\cup C_i^-)\big)\cup(\partial\mathbb{H}\setminus J)\big)\times J$. Here J is any compact subinterval of $\partial\mathbb{H}$ and C_i^+ (resp. C_i^-) denotes the upper (resp. lower) side of the (closed) slit C_i equipped with the topology induced from the path distance in D.

Proof. (i) By (5.7), we have

$$-\frac{1}{2}\frac{\partial}{\partial y}\frac{\partial}{\partial \mathbf{n}_\zeta}G^0(z,\zeta)\big|_{y=0}\le\frac{1}{\pi}\frac{1}{(x-\xi)^2},\tag{6.2}$$

which combined with the expression (5.2) and (5.6) implies that for any interval $I\subset\partial\mathbb{H}$ containing ζ,

$$\int_{\partial\mathbb{H}\setminus I}\left|\frac{\partial K^*(x+iy,\zeta)}{\partial y}\right|_{y=0}dx \text{ is finite and continuous in } \zeta\in I.\tag{6.3}$$

Since $z\mapsto\Psi(z,\zeta)$ is an analytic function on D whose imaginary part vanishes on $\partial\mathbb{H}\setminus\zeta$ by (5.2) and (5.7), it can be extended to an analytic function on $E=\mathbb{C}\setminus(\cup_{k=1}^N C_k)\cup(\cup_{k=1}^N\pi(C_k))\setminus\{\zeta\}$ by the reflection principle, where π denotes the reflection with respect to $\partial\mathbb{H}$. Denote the extended analytic function $z\mapsto\Psi(z,\zeta)$ as $u(z)+iv(z)$ with $v(z)=K^*(z,\zeta)$. The real part u can then be evaluated by

$$u(z)-u(z_0)=\int_C v_y dx-v_x dy\left(=-\int_C\frac{\partial v(z')}{\partial\mathbf{n}_{z'}}ds(z')\right)\tag{6.4}$$

where C is any smooth simple curve connecting z_0 with z in D. We fix an arbitrary compact interval $J\subset\partial\mathbb{H}$ and take $\zeta\in J,\ z_0\in\partial\mathbb{H}\setminus J$. Choose $\ell_0>0$ with $J\subset(-\ell_0,\ell_0),\ \cup_{i=1}^N C_i\subset R_{\ell_0}$.

On account of (5.4), we see that $v(z)=K^*(z,\zeta)$ is bounded on $\mathbb{H}\setminus R_{\ell_0}$ uniformly in $\zeta\in J$. Therefore by Lemma 4.3,

$$\sup_{\zeta\in J,\ell>2\ell_0}\int_{\Sigma_\ell}\left|\frac{\partial K^*(z,\zeta)}{\partial\mathbf{n}_z}\right|ds(z)<\infty.$$

For $z\in\mathbb{H}\setminus R_{2\ell_0}$, let $\ell\ge 2\ell_0$ be such that $z\in\Sigma_l$ and C be a smooth simple curve connecting z_0 to z in D. By the zero period property of v, the integral of $\dfrac{\partial v(z')}{\partial\mathbf{n}_{z'}}ds(z')$ along a closed Jordan curve consisting of C, a part of Σ_ℓ and a part of x-axis vanishes. Hence we can deduce from Lemma 4.3 and (6.3) that $u(z)$ is bounded near ∞ uniformly in $\zeta\in J$. This combined with (5.4) yields the second assertion of (i).

In particular, the analytic function $\Psi(\frac{1}{z})$ is uniformly bounded near the origin $\mathbf{0}$, which is therefore a removable singularity of this analytic function near $\mathbf{0}$, yielding the first assertion of (i).

(ii) Write $\Psi(z,\zeta)=u(z,\zeta)+iK^*(z,\zeta)$. By Lemma 5.2(i), $K^*(z,\zeta)$ is continuous in (z,ζ) on $[\mathbb{C}\setminus\bigcup_{k=1}^N(C_k\cup\pi C_k)\setminus J]\times J$, and so are $\frac{\partial}{\partial x}K^*(z,\zeta)$ and $\frac{\partial}{\partial y}K(z,\zeta)$ because $K^*(z,\zeta)$ is harmonic in z. Fix some $z_0=x_0+i0\in\partial\mathbb{H}\setminus J$ located to the right of J. As $\lim_{x\to\infty}u(x+i0,\zeta)=0$ by (6.1), we see from (6.4) that $u(z,\zeta),\ z\in D\cup(\partial\mathbb{H}\setminus J),\ \zeta\in J$, is determined by

$$u(z,\zeta)=\int_C-\frac{\partial}{\partial y}K^*(x+iy,\zeta)dx+\frac{\partial}{\partial x}K^*(x+iy,\zeta)dy+\int_{x_0}^\infty\frac{\partial}{\partial y}K^*(x+i0,\zeta)dx,$$

for any smooth simple curve C connecting z_0 to z in $D\cup(\partial\mathbb{H}\setminus J)$. Consequently, for any $z_1,\ z_2\in D\cup(\partial\mathbb{H}\setminus J)$,

$$u(z_2,\zeta)-u(z_1,\zeta)=\int_C\frac{\partial}{\partial y}K^*(x+iy,\zeta)dx-\frac{\partial}{\partial x}K^*(x+iy,\zeta)dy\tag{6.5}$$

for a smooth simple curve C joining z_1 to z_2 through $D \cup (\partial\mathbb{H} \setminus J)$. The joint continuity of $u(x, \zeta)$ on $(D \cup \partial\mathbb{H} \setminus J) \times J$ follows from these two formulae and (6.3).

Since, as a function of z, $K^*(z, \zeta)$ takes a constant boundary value on each slit C_i, it can be extended to be harmonic function and hence a smooth function across C_j^+ and C_j^- except at the end points for every $j = 1, \dots, N$. It follows from the above that $u(z, \zeta)$ can be extended continuously to C_j^+ and C_j^- except at the end points for every $j = 1, \dots, N$. We next show that $u(z, \zeta)$ can also be extended to the end points of C_j. Let z_1 be the left endpoint of C_j. Take $\varepsilon > 0$ small so that it is less than one half of the length of C_j and that $B(z_1, \varepsilon) \setminus C_j \subset D$. Then $\psi(z) = (z - z_1)^{1/2}$ maps $B(z_1, \varepsilon) \setminus C_j$ conformally onto $B(0, \sqrt{\varepsilon}) \cap \mathbb{H}$. Clearly $f(z, \zeta) := K^*(z^2 + z_1, \zeta)$ is a harmonic function in $z \in B(0, \sqrt{\varepsilon}) \cap \mathbb{H}$ that is continuous up to $B(0, \sqrt{\varepsilon}) \cap \partial\mathbb{H}$ and takes a constant value there. Hence by Schwarz reflection principle, $f(z, \zeta)$ can be extended to a harmonic function in $z \in B(0, \sqrt{\varepsilon})$ and so is smooth there. Moreover, by the integral representation of harmonic functions in disks using Poisson kernels, $\nabla_z f(z, \zeta)$ is in fact jointly continuous in $(z, \zeta) \in B(0, \sqrt{\varepsilon}) \times \partial\mathbb{H}$. This combined with (6.5) shows that $z \mapsto u(z, \zeta)$ can be extended continuously to z_1 and the resulting function is jointly continuous in (z, ζ). The case for the right endpoint of C_j can be dealt with analogously. This completes the proof of the lemma. □

We call $\Psi(z, \zeta)$, $z \in D$, $\zeta \in \partial\mathbb{H}$, subjected to the normalization condition (6.1) the *complex Poisson kernel* of BMD for the standard slit domain D.

6.2 The map $g_{t,s}$ and an expression of $a_t - a_s$

We are in a position to consider a fixed standard slit domain $D = \mathbb{H} \setminus \{C_1, \cdots, C_N\}$ together with a Jordan arc

$$\gamma : [0, t_\gamma] \to \overline{D}, \quad \gamma(0) \in \partial\mathbb{H}, \quad \gamma(0, t_\gamma] \subset D. \tag{6.6}$$

For each $t \in [0, t_\gamma]$, let g_t be a conformal map from $D \setminus \gamma(0, t]$ onto a standard slit domain D_t satisfying the condition

$$g_t(z) = z + \frac{a_t}{z} + o(\frac{1}{|z|}), \qquad z \to \infty. \tag{6.7}$$

a_t is then a real-valued function of t.

The unique existence of such a map g_t can be seen as follows. Let

$$G = \mathbb{C} \setminus \bigcup_{k=1}^{N} (C_k \cup \pi(C_k)) \setminus (\gamma[0, t] \cup \pi(\gamma[0, t])),$$

where π is the mirror reflection map with respect to $\partial\mathbb{H}$. If g_t satisfies the above mentioned properties, then it sends $\partial\mathbb{H} \setminus \{\gamma(0)\}$ onto $\partial\mathbb{H}$ minus a compact interval homeomorphically so that g_t can be extended by the Schwarz reflection to a conformal map $\widetilde{g}_t$ from G onto $\mathbb{C} \setminus \bigcup_{i=1}^{2N+1} \ell_i$ for some mutually disjoint horizontal line segments $\{\ell_i\}$ satisfying the condition (6.7) for some complex a_t. As is well known (e.g. [18, Theorem IX.23]), $\widetilde{g}_t$ with the stated properties is unique and so is its restriction g_t to $D \setminus \gamma(0, t]$. The existence of a map $f = \widetilde{g}_t$ on G with the stated properties is also well known (e.g. [18, Theorem IX 22]). Since the map $\widehat{f}$ defined by $\widehat{f}(z) = \overline{f(\bar{z})}$, $z \in G$, has the same properties as f, we have $f = \widehat{f}$ and in particular f sends real to real. So $f\big|_{D \setminus \gamma[0,t]}$ gives the desired map g_t.

For $0 < s < t < t_\gamma$, define

$$g_{t,s} = g_s \circ g_t^{-1}, \tag{6.8}$$

which is a conformal map from D_t onto $D_s \setminus g_s(\gamma[s,t])$. We can then deduce from (6.7) that

$$g_{t,s}(z) = z + \frac{a_s - a_t}{z} + o(\frac{1}{|z|}), \qquad z \to \infty. \tag{6.9}$$

Define

$$\xi(t) = g_t(\gamma(t))(= \lim_{z \to \gamma(t), z \in D \setminus \gamma[0,t]} g_t(z)). \tag{6.10}$$

By the well-known boundary correspondence for g_t (cf. [6]), $\xi(t) \in \partial\mathbb{H}$. Furthermore, by that for $g_{t,s}$, there exist unique points $\beta_0(t,s) \in \partial\mathbb{H}$ and $\beta_1(t,s) \in \partial\mathbb{H}$ such that

$$\beta_0(t,s) < \xi(t) < \beta_1(t,s), \quad g_{t,s}(\beta_0(t,s)) = g_{t,s}(\beta_1(t,s)) = \xi(s), \tag{6.11}$$

and it holds that

$$\Im g_{t,s}(x + i0+) \begin{cases} = & 0 \qquad x \in \partial\mathbb{H} \setminus (\beta_0(t,s), \beta_1(t,s)), \\ > & 0 \qquad x \in (\beta_0(t,s), \beta_1(t,s)). \end{cases} \tag{6.12}$$

We let

$$D_t = \mathbb{H} \setminus \bigcup_{k=1}^{N} C_{t,k}, \quad \ell_{t,s} = [\beta_0(t,s), \beta_1(t,s)](\subset \partial\mathbb{H}).$$

Because of (6.12), $g_{t,s}$ extends by the Schwarz reflection to an analytic function on

$$\mathbb{C} \setminus \Gamma_t, \quad \text{where} \quad \Gamma_t = \bigcup_{k=1}^{N} (C_{t,k} \cup \pi(C_{t,k})) \cup \ell_{t,s}. \tag{6.13}$$

In what follows, we may and will assume that $0 \notin \ell_{t,s}$.

Let $\widetilde{\Gamma}_t = \bigcup_{k=1}^{N} (\widetilde{C}_{t,k} \cup \widetilde{\pi(C_{t,k})}) \cup \widetilde{\ell}_{t,s}$ be the image of Γ_t under the inversion $w = 1/z$. Then $g_{t,s}(1/w)$ is analytic on $\mathbb{C} \setminus \widetilde{\Gamma}_t \setminus \{\mathbf{0}\}$, and so is $f(w) = \left(g_{t,s}(1/w) - \frac{1}{w}\right)/w$. Since (6.9) implies

$$\lim_{w \to 0} f(w) = \lim_{z \to \infty} z\{g_{t,s}(z) - z\} = a_s - a_t,$$

$\mathbf{0}$ is a removable singularity of f and f extends to an analytic function on $\mathbb{C} \setminus \widetilde{\Gamma}_t$ with

$$\mathbf{0} \in \mathbb{C} \setminus \widetilde{\Gamma}_t, \quad f(\mathbf{0}) = a_s - a_t. \tag{6.14}$$

We note that $\lim_{\zeta \to \infty} f(\zeta) = \lim_{z \to 0} z\{g_{t,s}(z) - z\} = 0$ and so $\lim_{R \to \infty} \int_{|\zeta|=R} \frac{f(\zeta)}{\zeta} d\zeta = 0$. Therefore we have from (6.14) and Cauchy's integral formula

$$a_s - a_t = \frac{1}{2\pi i} \int_{\bigcup_{k=1}^{2N+1} \widetilde{\gamma}_k} \frac{f(\zeta)}{\zeta} d\zeta, \tag{6.15}$$

where $\widetilde{\gamma}_1, \cdots, \widetilde{\gamma}_N$ (resp. $\widetilde{\gamma}_{N+1}, \cdots, \widetilde{\gamma}_{2N}$) are analytic contours surrounding $\widetilde{C}_{t,1}, \cdots, \widetilde{C}_{t,N}$ (resp. $\widetilde{\pi(C_{t,1})}, \cdots, \widetilde{\pi(C_{t,N})}$) and $\widetilde{\gamma}_{2N+1}$ is an analytic contour surrounding $\widetilde{\ell_{t,s}}$. They are disjoint and of clockwise orientation, and $\mathbf{0}$ is located outside all of them.

Lemma 6.2 *It holds for $0 \le s < t \le t_\gamma$ that*

$$a_t - a_s = \frac{1}{\pi} \int_{\beta_0(t,s)}^{\beta_1(t,s)} \Im g_{t,s}(x + i0+) dx. \tag{6.16}$$

Proof. From (6.15) and a change of variables of the line integral,

$$\begin{aligned} a_s - a_t &= \frac{1}{2\pi i}\int_{\bigcup_{k=1}^{2N+1}\widetilde{\gamma}_k} \frac{g_{t,s}(\frac{1}{\zeta}) - \frac{1}{\zeta}}{\zeta^2} d\zeta \\ &= -\frac{1}{2\pi i}\int_{\bigcup_{k=1}^{2N+1}\gamma_k} (g_{t,s}(\eta) - \eta)\, d\eta, \end{aligned}$$

where γ_k is the image of $\widetilde{\gamma}_k$ under $\eta = \frac{1}{\zeta}$ for $1 \le k \le 2N+1$. Hence $\gamma_1, \cdots, \gamma_N$ (resp. $\gamma_{N+1}, \cdots, \gamma_{2N}$) are analytic contours surrounding $C_{t,1}, \cdots, C_{t,N}$ (resp. $\pi(C_{t,1}), \cdots, \pi(C_{t,N})$) and γ_{2N+1} is an analytic contour surrounding $\ell_{t,s}$, all oriented clockwise.

Since $\Im g_{t,s}(\eta)$ is constant on $C_{t,k}$, $g_{t,s}(\eta)$ admits analytic extensions across $C_{t,k}$ from both sides. Hence the integral $\int_{\gamma_k}(g_{t,s}(\eta)-\eta)d\eta$ equals $\int_{C_{t,k}}(g_{t,s}(\eta)-\eta)d\eta$ for $1 \le k \le N$ and $\int_{\pi(C_{t,k})}(g_{t,s}(\eta) - \eta)d\eta$ for $N+1 \le k \le 2N$, the integral on $C_{t,k}$ (and on $\pi(C_{t,k})$) being understood to be the sum of the integral along its upper side and lower side with clockwise orientation. Taking as γ_{2N+1} a rectangle with width 2ε surrounding $\ell_{t,s}$, we then have

$$\begin{aligned} a_s - a_t &= -\frac{1}{2\pi i}\sum_{k=1}^{N}\int_{C_k\cup\pi(C_k)} (g_{t,s}(\eta) - \eta)\, d\eta \\ &\quad - \lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_{\beta_0(t,s)}^{\beta_1(t,s)} \Big((g_{t,s}(x+i\varepsilon) - x - i\varepsilon) - (\overline{g_{t,s}(x+i\varepsilon)} - x + i\varepsilon) \Big)\, dx. \end{aligned}$$

Since $a_s - a_t$ is real, we conclude that $a_s - a_t$ equals

$$-\frac{1}{2\pi}\sum_{k=1}^{N}\int_{C_k\cup\pi(C_k)} \Im\,(g_{t,s}(\eta)-\eta)\, d\eta - \frac{1}{\pi}\int_{\beta_0(t,s)}^{\beta_1(t,s)} \Im g_{t,s}(x+i0+)dx.$$

But $\Im(g_{t,s}(\eta) - \eta)$ takes the same constant value from both sides of each C_k and of each $\pi(C_k)$ so that the sum in the above vanishes. This establishes (6.16). □

Corollary 6.3 *a_t is a strictly increasing left-continuous function in $t > 0$ with $a_0 = 0$.*

Proof. a_t is strictly increasing in view of (6.12) and (6.16). Denote by γ_s^+, γ_s^- the points of 'both sides' of the arc γ corresponding to γ_s. As $s \uparrow t$, $\gamma_s^- \to \gamma_t$ (resp. $\gamma_s^+ \to \gamma_t$) so that $\beta_0(t,s) = g_t(\gamma_s^-) \uparrow g_t(\gamma_t) = \xi_t$ (resp. $\beta_1(t,s) = g_t(\gamma_s^+) \downarrow g_t(\gamma_t) = \xi_t$). a_t is thus left continuous. Since $g_0(z) = z,\ z \in D$, on account of the uniqueness of g_0 and (6.7), we have $a_0 = 0$. □

6.3 Chordal Komatu-Loewner differential equation

Keeping the setting of the preceding subsection, we now derive the Komatu-Loewner differential equation (1.3), but with its left hand side being replaced by the left derivative $\frac{\partial^- g_t(z)}{\partial a_t}$ with respect to the strictly increasing left continuous function a_t studied in the preceding subsection.

Let $Z^{t,*}$ be the BMD on the standard slit domain D_t and $\Psi_t(z,\zeta),\ z \in D_t,\ \zeta \in \partial\mathbb{H}$, be the complex Poisson kernel of $Z^{t,*}$, namely, an analytic function on D_t with imaginary part being the Poisson kernel $K_t^*(z,\zeta)$ of $Z^{t,*}$ subject to the normalization (6.1).

Theorem 6.4 *The map g_t satisfies the following chordal Komatu-Loewner differential equation: for $t \in (0, t_\gamma]$ and $z \in D \setminus \gamma[0,t]$, $g_t(z)$ is left differentiable with respect to a_t and*

$$\frac{\partial^- g_t(z)}{\partial a_t} = -\pi \Psi_t(g_t(z), \xi(t)), \quad g_0(z) = z. \tag{6.17}$$

Proof. We proceed along the same line as in the proof of Theorem 3.1 in Bauer and Friedrich [3], but with several modifications stated in §1.

We consider the analytic function

$$F(z) = g_{t,s}(z) - z, \qquad z \in D_t,$$

which satisfies

$$F(z) = \frac{a_s - a_t}{z} + o(1/|z|). \tag{6.18}$$

Then, $f(z) := \Im F(z)$ is harmonic on D_t and takes a constant, say f_i on each slit C_i. As $\lim_{z\to\infty} f(z) = 0$ by (6.18), the formula (4.21) holds: for $z \in D_t$,

$$f(z) = \sum_{k=1}^N f_k \varphi_t^{(k)}(z) - \frac{1}{2}\int_{\partial\mathbb{H}} \frac{\partial G_t^0(z,\zeta)}{\partial \mathbf{n}_\zeta} f(\zeta) ds(\zeta), \tag{6.19}$$

where $\{\varphi_t^{(i)}\}$ is the harmonic basis and $G_t^0(z,\zeta)$ is the Green function of the ABM on the domain $D_t = \mathbb{H} \setminus \bigcup_{k=1}^N C_{t,k}$.

Since f is the imaginary part of the analytic function F, its period around $C_{t,i}$ vanishes and we have from (4.22)

$$0 = -\sum_{k=1}^N f_k a_{t,ki} - \int_{\partial\mathbb{H}} \frac{\partial \varphi_t^{(i)}(\zeta)}{\partial \mathbf{n}_\zeta} f(\zeta) ds(\zeta), \tag{6.20}$$

for every $1 \le i \le N$, where $a_{t,ki}$ denotes the period of $\varphi_t^{(i)}$ around the slit $C_{t,k}$.

Denote by $\mathcal{A}_t$ the symmetric matrix with (i,j)-component $a_{t,ij}$ and by $q_{t,ij}$ the (i,j)-component of $\mathcal{A}_t^{-1}$. Multiply both sides of (6.20) by $\sum_{j=1}^N \varphi_t^{(j)}(z) q_{t,ji}$ and add up in i, and finally add the resulting identity to (6.19). We are then left with

$$f(z) = \int_{\partial\mathbb{H}} K_t^*(z,\zeta) f(\zeta) ds(\zeta),$$

in view of the expression (5.2) of the Poisson kernel $K_s^*(z,\zeta)$ of $Z^{t,*}$. Since f vanishes on $\partial\mathbb{H} \setminus [\beta_0(t,s), \beta_1(t,s)]$, we have

$$f(z) = \int_{\beta_0(t,s)}^{\beta_1(t,s)} K_t^*(z,x) f(x) dx,$$

and accordingly,

$$g_{t,s}(z) - z = \int_{\beta_0(t,s)}^{\beta_1(t,s)} \Psi_t(z,x) \Im g_{t,s}(x) dx + c, \tag{6.21}$$

for some real constant c. By taking the normalization (6.8) and Lemma 6.1 into account, we let $z \to \infty$ in (6.21) to get $c = 0$. We then substitute $z = g_t(w)$ to obtain

$$g_s(z) - g_t(z) = \int_{\beta_0(t,s)}^{\beta_1(t,s)} \Psi_t(g_t(z), x) \Im g_{t,s}(x) dx. \tag{6.22}$$

By Lemma 6.1 (ii), $\Psi_t(z,\zeta) = u_t(z,\zeta) + iK_t^*(z,\zeta)$ is continuous in $\zeta \in \partial\mathbb{H}$ for each $z \in D$, and consequently (6.16), (6.22) and the mean value theorem of integration imply that, for some $x', x'' \in (\beta_0(t,s), \beta_1(t,s))$,

$$\frac{g_s(z) - g_t(z)}{a_s - a_t} = -\pi u_t(g_t(z), x') - i\pi K_t^*(g_t(z), x''). \tag{6.23}$$

If we let $s \uparrow t$, then both $\beta_0(t,s)$ and $\beta_1(t,s)$ converge to $\xi(t)$ as was observed in the proof of Corollary 6.3, and we arrive at the desired equation (6.17). □

7 A probabilistic representation of $\Im g_t$

For the conformal map g_t studied in the preceding two subsections, we shall give in this section a probabilistic representation of $\Im g_t$ in terms of the absorbing Brownian motion (ABM in abbreviation) on $\mathbb{H}$ and the BMD on $(\mathbb{H} \setminus \bigcup_{i=1}^N C_i) \cup \{c_1^*, \cdots, c_N^*\}$. This representation will readily yield basic properties of the family $\{\Im g_t\}$ that will be utilized in the next section.

More generally we start with an $(N+1)$-connected domain (see §3.3)

$$D = \mathbb{H} \setminus K \quad \text{with} \quad K = \bigcup_{i=1}^N A_i. \tag{7.1}$$

We call a set $F \subset \mathbb{H}$ a *compact $\mathbb{H}$-hull* if $\overline{F}$ is compact, $F = \overline{F} \cap \mathbb{H}$ and $\mathbb{H} \setminus F$ is simply connected. We consider a compact $\mathbb{H}$-hull F satisfying $F \subset D$. Denote by $Z^{\mathbb{H}} = (Z_t^{\mathbb{H}}, \mathbb{P}_z^{\mathbb{H}})$ the ABM on $\mathbb{H}$ and by $Z^{\mathbb{H},*} = (Z_t^{\mathbb{H},*}, \mathbb{P}_z^{\mathbb{H},*})$ the BMD on $D^* = D \cup K^*$ with $K^* = \{a_1^*, \cdots, a_N^*\}$ obtained from $Z^{\mathbb{H}}$ by regarding each compact continuum A_i as one point a_i^*.

For $r > 0$, let $\Gamma_r = \{z = x + iy : y = r\}$ and

$$v^*(z) := \lim_{r\to\infty} r \cdot \mathbb{P}_z^{\mathbb{H},*}(\sigma_{\Gamma_r} < \sigma_F), \quad z \in D^* \setminus F. \tag{7.2}$$

Theorem 7.1 (i) *The function v^* on $D^* \setminus F$ is well defined and is Z^*-harmonic on $D^* \setminus F$. Furthermore*

$$v^*(z) = v(z) + \sum_{j=1}^N \mathbb{P}_z^{\mathbb{H}}\left(\sigma_K < \sigma_F,\, Z_{\sigma_K}^{\mathbb{H}} \in A_j\right) v^*(a_j^*),\ z \in D \setminus F, \tag{7.3}$$

where

$$v(z) = \Im z - \mathbb{E}_z^{\mathbb{H}}\left[\Im Z_{\sigma_{F\cup K}}^{\mathbb{H}};\ \sigma_{F\cup K} < \infty\right]\ (\geq 0), \tag{7.4}$$

$$v^*(a_i^*) = \sum_{j=1}^N \frac{M_{ij}}{1 - R_j^*} \int_{\eta_j} v(z)\nu_j(dz), \quad 1 \leq i \leq N. \tag{7.5}$$

Here $\eta_1, \cdots, \eta_N$ are mutually disjoint smooth Jordan curve surrounding $A_1, \cdots, A_N$, respectively,

$$\nu_i(dz) = \mathbb{P}_{a_i^*}^{\mathbb{H},*}\left(Z_{\sigma_{\eta_i}}^{\mathbb{H},*} \in dz\right), \quad 1 \leq i \leq N, \tag{7.6}$$

$$R_i^* = \int_{\eta_i} \mathbb{P}_z^{\mathbb{H}}\left(\sigma_K < \sigma_F,\ Z_{\sigma_K}^{\mathbb{H}} \in A_i\right) \nu_i(dz), \quad 1 \leq i \leq N, \tag{7.7}$$

and M_{ij} *is the* (i,j)*-entry of the matrix* $M=\sum_{n=0}^{\infty}(Q^*)^n$ *for a matrix* Q^* *with entries*

$$q^*_{ij}=\begin{cases}\mathbb{P}^{\mathbb{H},*}_{a^*_i}\big(\sigma_{K^*}<\sigma_F,\ Z^{\mathbb{H},*}_{\sigma_{K^*}}=a^*_j\big)/(1-R^*_i) & \text{if } i\neq j,\\ 0 & \text{if } i=j,\end{cases}\qquad 1\le i,j\le N. \tag{7.8}$$

(ii) $v^*\big|_D$ *admits a unique harmonic conjugate* u^* *such that* $f(z)=u^*(z)+iv^*(z),\ z\in D$, *is analytic on* D *and*

$$f(z)=z+\frac{a}{z}+o(\frac{1}{z}),\qquad z\to\infty \tag{7.9}$$

for some positive constant a.

A proof of this theorem will be given in Appendix 1 by a series of lemmas. We remark that the way of constructing v^* in the above theorem is due to G. Lawler [13] where ERBM on the N-connected domain was utilized in place of the current BMD.

Theorem 7.2 *The analytic function* f *of Theorem* 7.1 *is a conformal mapping from* $\mathbb{H}\setminus(F\cup K)$ *onto* $\mathbb{H}\setminus\bigcup_{i=1}^N\widetilde{C}_i$ *where* $\widetilde{C}_i$, $1\le i\le N$, *are mutually disjoint horizontal line segments in* $\mathbb{H}$.

Proof. Since $v^*=0$ on $\partial(\mathbb{H}\setminus\overline{F})$, by Schwarz reflection we can extend f to an analytic map from

$$D_1=\overline{\mathbb{C}}\setminus(\overline{F}\cup K\cup\pi(\overline{F}\cup K)) \tag{7.10}$$

into $\overline{\mathbb{C}}$, which satisfies

$$f(z)=z+\frac{a}{z}+o(\frac{1}{|z|}),\quad z\to\infty. \tag{7.11}$$

Let $D_2=\overline{\mathbb{C}}\setminus f(\partial D_1)$, where $f(\partial D_1)$ is defined in the sense of (11.1). Since $\Im f=v^*(a^*_j)$ on A_j and $\Im f=0$ on $\overline{F}$,

$$f(\partial D_1)\subset\left\{z=x+iy\in\mathbb{C}:y=\pm v^*(a^*_j)\text{ for }j=1,\cdots,N\text{ or }y=0\right\}. \tag{7.12}$$

This implies that the complement of D_2 has empty interior and so

$$f(\partial D_1)=\partial D_2. \tag{7.13}$$

We next show that ∞ is an interior point of D_2. Since f has multiplicity 1 near infinity, this will then imply by Theorem 11.2 that f is a conformal mapping from D_1 to D_2. To this end, it suffices to show that f maps F and each A_j (understood in the sense the limit points (11.1)) into a bounded subset in $\mathbb{C}$.

Let ϕ be the conformal map of $\mathbb{H}\setminus F$ onto $\mathbb{H}$ that satisfies the hydrodynamic normalization (7.11) at infinity (but with a possibly different constant a), and set $g=f\circ\phi^{-1}$. Clearly, g is well-defined and analytic in $\mathbb{H}$. Let O be an open neighborhood of the interval $I=\phi(F)$ (understood in the sense of limit points) that is disjoint from $\phi(K)$. The imaginary part of $g(z)$ tends to 0 as $y=\Im z\to 0$ in $O\cap\mathbb{H}$. Thus g extends analytically across the real line to be an analytic function in the disk O. In particular, g is bounded in every compact subset O. This in particular implies that $f(F)$ is bounded.

Similarly, for each $j=1,\dots,N$, since A_j is a compact continuum, there is a conformal map ϕ_j of the complement of the interval $[0,1]$ in $\mathbb{C}$ onto the complement of A_j. Let O_j be a relatively compact open neighborhood of A_j that is disjoint from the other boundary components. Then $\widehat{O}_j=\phi_j^{-1}(O_j)$ is an open neighborhood of $[0,1]$, and $g_j=f\circ\phi_j$ is analytic in $\widehat{O}_j$. Since $\phi_j(z)\to A_j$

as $z \to [0,1]$, the imaginary part $\Im g_j(z) = v^*(\phi_j(z) \to v^*(a_j^*)$ as $z \to [0,1]$. By the argument similar to that in the previous paragraph, g_j is bounded on every compact subset of O_i and so $f = g_j \circ \phi_j^{-1}$ is bounded near A_j.

In view of the above, we conclude from (7.11) that the pre-image of $\infty \in D_2$ under f is ∞ with multiplicity 1. Theorem 11.2 together with (7.12) implies that f is conformal from D_1 onto D_2. As f is a topological homeomorphism between D_1 and D_2, D_2 has to be $(2N+1)$-connected as well. Thus D_2 is a slit domain with $2N+1$ disjoint horizontal line segments symmetric relative to $\partial\mathbb{H}$. None of them degenerates to a single point. In fact, if one of them reduces to a point p, then, due to (7.11), p is not a pole nor an essential singularity of the analytic function f^{-1} so that p is removable, contradicting the assumption that the boundary components of D_1 are continua. □

Let us return to the setting in the last two sections: $D = \mathbb{H} \setminus K$ is a standard slit domain with $K = \bigcup_{i=1}^N C_i$ and $\gamma = \{\gamma(t),\ 0 \le t \le t_\gamma\}$ is a Jordan arc satisfying condition (6.6). For each fixed $t \in [0, t_\gamma]$, let g_t be the unique conformal map from $D \setminus \gamma(0,t]$ onto a standard slit domain satisfying property (6.7) at infinity. g_0 is the identity map. $Z^{\mathbb{H},*} = (Z_t^{\mathbb{H},*},\ \mathbb{P}_z^{\mathbb{H},*})$ will denote the BMD on $D = D \cup K^*$ with $K^* = \{c_1,^*, \cdots, c_N^*\}$ obtained from the ABM $Z^{\mathbb{H}}$ by regarding each slit C_i as a point c_i^*.

For each $t \in [0, t_\gamma]$, we let $F_t = \gamma[0,t]$. The functions v^*, v and the quantities R_i^*, q_{ij}^* specified by Theorem 7.1 but for C_i, c_i^*, $1 \le i \le N$, and F_t in place of A_i, a_i^*, $1 \le i \le N$, and F will be designated by v_t^*, v_t and $R_i^*(t)$, $q_{ij}^*(t)$, respectively.

By virtue of Theorem 7.2, it then holds that

$$v_t^*(z) = \Im g_t(z), \qquad z \in D \setminus F_t. \tag{7.14}$$

Proposition 7.3 *The function $v_t^*(z)$ can be continuously extended to $K \cup \partial\mathbb{H} \setminus \{\gamma(0)\}$ as a jointly continuous function in $[0,s] \times \left(\overline{\mathbb{H}} \setminus \gamma[0,s]\right)$ for each $s \in (0, t_\gamma]$. Moreover,*

$$0 < \inf_{t \in [0,t_\gamma],\ 1 \le k \le N} v_t^*(c_k^*) \le \sup_{t \in [0,t_\gamma],\ 1 \le k \le N} v_t^*(c_k^*) < \infty, \tag{7.15}$$

and for every $\varepsilon > 0$,

$$\lim_{t \downarrow 0} v_t^*(z) = v_0^*(z) = \Im z, \quad z \in \overline{\mathbb{H}} \setminus \gamma[0,\varepsilon]. \tag{7.16}$$

Proof. Since any one-point set is polar with respect to the planar Brownian motion, we have by the continuity of Brownian motion and the curve γ that for every $z \in \mathbb{H}$,

$$\mathbb{P}_z^{\mathbb{H}} \left(\lim_{s \to t} \sigma_{F_s} = \sigma_{F_t} \right) = 1, \quad \mathbb{P}_z^{\mathbb{H}} \left(\lim_{s \to t} \sigma_{F_s \cup K} = \sigma_{F_t \cup K} \right) = 1. \tag{7.17}$$

Similarly it holds that

$$\mathbb{P}_z^{\mathbb{H},*} \left(\lim_{s \to t} \sigma_{F_s} = \sigma_{F_t} \right) = 1 \quad \text{for every } z \in D^*. \tag{7.18}$$

(7.17) implies that $h(z,t) = \mathbb{E}_z^{\mathbb{H}}[\Im Z^{\mathbb{H}}_{\sigma_{F_t \cup K}},\ \sigma_{F_t \cup K} < \infty]$ is continuous in $t \in [0,s]$ for each $z \in \mathbb{H} \setminus \gamma[0,s]$. Since $h(z,t)$ is harmonic in $z \in \mathbb{H} \setminus \gamma[0,s]$ and taking a constant value on each slit C_i, we get the joint continuity of $v_t(z)$ in $(t,z) \in [0,s] \times \left(\overline{\mathbb{H}} \setminus \gamma[0,s]\right)$.

On the other hand, in view of (7.18), $R_i^*(t)$ is a decreasing continuous function in t. For each $1 \le i \le N$,

$$\sum_{j: j \ne i} q_{ij}^*(t) = \frac{\mathbb{P}_{c_i^*}^{\mathbb{H},*}\left(\sigma_{K^*} < \sigma_{F_t},\ Z^{\mathbb{H},*}_{\sigma_{K^*}} \ne c_i^*\right)}{1 - R_i^*(t)}, \tag{7.19}$$

which by (7.18) is continuous in t and so

$$\lambda := \max_{1\le i\le N,\,0\le t\le t_\gamma} \sum_{j:j\ne i} q^*_{ij}(t) < 1.$$

It follows that $M_{ij}(t) \le 1+1/(1-\lambda) \le 2/(1-\lambda)$ for every $t\in[0,t_\gamma]$ and that $M_{ij}(t)$ is continuous in t. One concludes from (7.5) and the joint continuity of $v_t(z)$ that $v^*_t(c^*_i)$ is a continuous function in t for every $1\le i\le N$. This and (7.3) yield the joint continuity of $v^*_t(z)$ in $(t,z)\in[0,s]\times\big(\overline{\mathbb{H}}\setminus\gamma[0,s]\big)$, as well as (7.15) and (7.16). □

In the remainder of this paper, for a Borel set $A\subset\overline{\mathbb{H}}$, we use $\partial_p A$ to denote the boundary of A with respect to the topology induced by the path distance in $\mathbb{H}\setminus A$. For instance, when $A(\subset\mathbb{H})$ is a horizontal line segment, $\partial_p A$ consists of the upper part A^+ and the lower part A^- of the line segment A.

Theorem 7.4 *For each $t>0$, the conformal map $g_t(z)$ extends continuously to $\partial_p K$. Moreover, $\{g_s(z); s\in[0,t]\}$ are locally equi-continuous and locally uniformly bounded in $z\in D\cup\partial_p K\cup\partial_p(\mathbb{H}\setminus\gamma[0,t])$.*

Proof. Let $\widehat{F}_t=\gamma[0,t]\cup\pi(\gamma[0,t])$, where, as before, π denotes the mirror reflection with respect to $\partial\mathbb{H}$. By the Schwarz reflection principle, for each $t>0$, we can extend $g_t(z)$ to be an analytic function in $\mathbb{C}\setminus\big(K\cup\pi(K)\cup\widehat{F}_t\big)$. Let $u^*_t(z)=\Re g_t(z)$ and $v^*_t(z)=\Im g_t(z)$. We note that, owing to Proposition 7.3, $v^*_s(z)$ is continuously extendable in z to $\mathbb{C}\setminus\widehat{F}_t$ to be jointly continuous in $(s,z)\in[0,t]\times(\mathbb{C}\setminus\widehat{F}_t)$.

Since for each $s\ge0$, $v^*_s(z)$ is harmonic in $z\in\mathbb{C}\setminus\big(K\cup\pi(K)\cup\widehat{F}_t\big)$, it follows from the integral representation for harmonic functions in disks in terms of Poisson kernels that $\nabla_z v^*_s(z)$ is jointly continuous in $(s,z)\in[0,t]\times\big(\mathbb{C}\setminus\big(K\cup\pi(K)\cup\widehat{F}_t\big)\big)$. Hence by Cauchy-Riemann equation, so is $\nabla_z u^*_s(z)$. This implies that $\{u^*_s(z); s\in[0,t]\}$ are locally equi-continuous in $z\in\mathbb{C}\setminus\big(K\cup\pi(K)\cup\widehat{F}_t\big)$. On the other hand, it follows from (6.23) that for each fixed $z\in D$, $t\mapsto u^*_t(z)$ is bounded in $t\in(0,t_\gamma]$. These together with Proposition 7.3 yield that $\{g_s(z); s\in[0,t]\}$ are locally equi-continuous and locally uniformly bounded in $z\in D\cup\partial\mathbb{H}\setminus\gamma[0,t]$.

We next show that $\{g_s(z); s\in[0,t]\}$ are in fact locally equi-continuous and locally uniformly bounded in $z\in D\cup\partial_p K\cup\partial_P(\mathbb{H}\setminus\gamma[0,t])$. Since $\Im g_s(z)$ takes constant value on each slit ∂C_j, by Schwarz reflection principle, $g_s(z)=u^*_s(z)+iv^*_s(z)$ can be extended to be an analytic function across C^+_j (resp. C^-_j) except at the two endpoints of C_j. As the harmonic function $v^*_s(z)$ is jointly continuous in (s,z), the same argument as above applies and we conclude that $\{g_s(z); s\in[0,t]\}$ are locally equi-continuous and locally uniformly bounded in a neighborhood of every point in $C^+_j\cup C^-_j$ with the two endpoints removed.

For the left endpoint z_1 of the slit C_j, take $\varepsilon>0$ small so that it is less than one half of the length of C_j and that $B(z_1,\varepsilon)\setminus C_j\subset D$. Then $\psi(z)=(z-z_1)^{1/2}$ maps $B(z_1,\varepsilon)\setminus C_j$ conformally onto $B(0,\sqrt{\varepsilon})\cap\mathbb{H}$. Consequently, $g_s\circ\psi^{-1}(z)=g_s(z^2+z_1)$ is an analytic function in $z\in B(0,\sqrt{\varepsilon})\cap\mathbb{H}$ that is continuous up to $B(0,\sqrt{\varepsilon})\cap\partial\mathbb{H}\setminus\{0\}$ and $v^*_s(z^2+z_1)=\Im g_s(z^2+z_1)$ takes constant value there. By Schwarz reflection, $g_s(z^2+z_1)$ extends to $B(0,\sqrt{\varepsilon})\setminus\{0\}$. Thus 0 is an isolated singularity of $g_s(z^2+z_1)$. Since $\Im g_s(z^2+z_1)$ is bounded near the origin, it has to be a removable singularity. It follows that $g_s(z^2+z_1)$ can be extended to be analytic in $z\in B(0,\sqrt{\varepsilon})$ and $\Im g_s(z^2+z_1)=v^*_s(z^2+z_1)$ is jointly continuous in (s,z). Thus the same argument as above is applicable again in concluding that $\{g_s(z); s\in[0,t]\}$ are equi-continuous and uniformly bounded in $B(z_1,\varepsilon/2)\cap(D\cup\partial_p K)$. The case for the right endpoint of C_j can be dealt with analogously.

The same argument with trivial modification establishes the analogous assertion with $\gamma[0,t]$ in place of C_j. By an obvious covering argument, we arrive at the desired conclusion of the theorem. $\square$

We will show in Theorem 8.3 below that for every $t \in (0, t_\gamma]$, $g_s(z)$ is in fact jointly continuous in $(s,z) \in [0,t] \times (D \cup \partial_p K \cup \partial\mathbb{H} \setminus \gamma[0,t])$.

8 Continuity of g_t, a_t, D_t and $\xi(t)$

Throughout this section, we maintain the setting of §6.2 and §6.3. Let $g_t(z)$ be the conformal map from $D \setminus \gamma(0,t]$ onto a standard slit domain D_t satisfying the hydrodynamic normalization condition (6.7) at infinity. The goal of this section is to show, using Proposition 7.3 and Theorem 7.4, the continuity of $g_t(z)$ in t with certain uniformity in z, and thereby derive the continuity of the function a_t in (6.7), the standard slit domain D_t and the position $\xi(t) \in \partial\mathbb{H}$ defined in (6.10).

Recall the conformal map $g_{t,s} = g_s \circ g_t^{-1}$ defined in §6.2 for $0 \le s < t < t_\gamma$, which sends D_t onto $D_s \setminus g_s(\gamma[s,t])$. Denote by $C_{t,i}^+$ (resp. $C_{t,i}^-$) the upper (resp. lower) side of the slit $C_{t,i}$, and $\partial_p K_t := \cup_{j=1}^N (C_{t,i}^+ \cup C_{t,i}^-)$. In the proof of the next proposition, we will use the following identity that follows from (6.21) with $c = 0$, Lemma 6.1 and Theorem 7.4:

$$g_{t,s}(z) - z = \int_{\ell_{t,s}} \Psi_t(z,x) \Im g_{t,s}(x) dx \quad \text{for } s < t \text{ and } z \in D_t \cup \partial_p K_t \cup \partial\mathbb{H}. \tag{8.1}$$

Here $\ell_{t,s}$ denotes the interval $(\beta_0(t,s), \beta_1(t,s)) (\subset \partial\mathbb{H})$.

Theorem 8.1 *For a fixed $t \in (0, t_\gamma]$, $\lim_{s \uparrow t} g_{t,s}(z) = z$ uniformly in z on each compact subset of $D_t \cup \partial_p K_t \cup (\partial\mathbb{H} \setminus \{\xi(t)\})$.*

Proof. We let $M_\gamma = \sup_{t \in [0,t_\gamma]} \Im\gamma(t)$. We have by (7.3), (7.4) and (7.15),

$$\begin{aligned} \sup_{0 \le s < t} \sup_{x \in \ell_{t,s}} \Im g_{t,s}(x) &= \sup_{0 \le s < t} \sup_{s \le s' \le t} \Im g_s(\gamma(s')) = \sup_{0 \le s' \le t} \sup_{0 \le s \le s'} v_s^*(\gamma(s')) \\ &\le M_\gamma + \sum_{j=1}^N \sup_{0 \le s \le t_\gamma} v_s^*(c_j^*) =: M_1 < \infty. \end{aligned} \tag{8.2}$$

For any compact subset L of $D_t \cup \partial_p K_t \cup (\partial\mathbb{H} \setminus \{\xi(t)\})$, choose $\varepsilon > 0$ and $\delta > 0$ such that $L \cap B_\varepsilon(\xi(t)) = \emptyset$ and $\ell_{t,t-\delta} \subset B_\varepsilon(\xi(t))$. We then see from Lemma 6.1(ii) that $M_2 = \sup\{|\Psi_t(z,\zeta)| : z \in L,\ \zeta \in \ell_{t,t-\delta}\}$ is finite. Hence (8.1) implies that, for any $s \in (t-\delta, t)$, $\sup_{z \in L} |g_{t,s}(z) - z| \le M_1 M_2 |\ell_{t,s}| < 2M_1 M_2\, \varepsilon$. $\square$

The inverse $g_{t,s}^{-1} = g_t \circ g_s^{-1}$ of $g_{t,s}$ is a conformal map from $D_s \setminus g_s(\gamma[s,t])$ onto the standard slit domain D_t satisfying

$$g_{t,s}^{-1}(z) = z + \frac{a_t - a_s}{z} + o(\frac{1}{|z|}), \quad \text{as } |z| \to \infty, \quad 0 \le s < t \le t_\gamma. \tag{8.3}$$

By the Schwarz reflection, $g_{t,s}^{-1}$, $0 \le s < t \le t_\gamma$, extends to a conformal map from $\mathbb{C} \setminus \Lambda_s$ onto $\mathbb{C} \setminus \Gamma_t$ still satisfying (8.3), where

$$\Lambda_s = \bigcup_{k=1}^N (C_{s,k} \cup \pi(C_{s,k})) \cup g_s(\gamma[s,t]) \cup \pi(g_s(\gamma[s,t])),\ \Gamma_t = \bigcup_{k=1}^N (C_{t,k} \cup \pi(C_{t,k})) \cup \ell_{t,s}.$$

Theorem 8.2 *For a fixed* $s \in [0, t_\gamma)$, $\lim_{t \downarrow s} g_{t,s}^{-1}(z) = z$ *uniformly in* z *on each compact subset of* $D_s \cup \partial_p K_s \cup (\partial \mathbb{H} \setminus \{\xi(s)\})$.

Proof. Without loss of generality, we may assume that $s = 0$ and so $g_{t,s}^{-1} = g_t$. For any $0 < \varepsilon < R$ with $\overline{B(\gamma(0), \varepsilon)} \cap K = \emptyset$, $K \subset B(\gamma(0), R)$, let $A_{\varepsilon,R} = \big((\overline{\mathbb{H}} \setminus K) \cap (B(\gamma(0), R) \setminus B(\gamma(0), \varepsilon))\big) \cup \partial_p K$ and take $\delta > 0$ such that $\gamma[0, \delta] \subset B(\gamma(0), \varepsilon)$. In view of Theorem 7.4, the family $\{g_t(z) = u_t^*(z) + iv_t^*(z); 0 \le t \le \delta\}$ is uniformly bounded and equi-continuous in $z \in A_{\varepsilon,R}$. Therefore, every sequence $t_n \downarrow 0$ admits a subsequence still denoted as t_n such that $\widetilde{g}_{t_n}(z) := g_{t_n}(z) - z$ converges uniformly on $A_{\varepsilon,R}$, as $n \to \infty$, to a function f, which is analytic on $D \cap (B(\gamma(0), R) \setminus \overline{B(\gamma(0), \varepsilon)})$. It follows from (7.16) that $\Im f(z) = 0$ and so f is a constant.

To show this constant is zero, extend g_t by Schwarz reflection as before. Since g_t is a conformal map from $\overline{\mathbb{C}} \setminus \overline{B(\gamma(0), R)}$ to the outside of $g_t(\partial \overline{B(\gamma(0), R)})$, it easily follows that $g_t(z) - z$ also converges uniformly (with respect to the spherical metric) on $\overline{\mathbb{C}} \setminus \overline{B(\gamma(0), R)}$, to an analytic function that extends f. Near ∞ the Laurent series of g_t begins with $g_t(z) = z + O(1/z)$ so that $f(z) \to 0$ as $z \to \infty$. □

Theorems 8.1-8.2 together with Theorem 7.4 immediately yield the following.

Theorem 8.3 *For every* $0 < s \le t_\gamma$, $g_t(z)$ *is jointly continuous in* $(t, z) \in [0, s] \times ((D \cup \partial_p K \cup \partial \mathbb{H}) \setminus \gamma[0, s])$.

Theorem 8.2 in particular implies that, for a fixed $s \in [0, t_\gamma]$,

$$\lim_{t \downarrow s} g_{t,s}^{-1}(z) = z \text{ uniformly on each compact subset of } \mathbb{C} \setminus \Lambda_s^0, \tag{8.4}$$

for $\Lambda_s^0 = \bigcup_{k=1}^N (C_{s,k} \cup \pi(C_{s,k})) \cup \{\xi(s)\}$. This immediately leads us to the right continuity of a_t in the following manner.

We may assume that $\xi(s) \neq \mathbf{0}$. Let $\widetilde{\Lambda}_s^0 = \bigcup_{k=1}^N (\widetilde{C_{s,k}} \cup \widetilde{\pi(C_{s,k})}) \cup \widetilde{\{\xi(s)\}}$ be the image of Λ_s^0 under the inversion $w = \frac{1}{z}$. Then $h_{t,s}(w) = \frac{1}{w}\left(g_{t,s}^{-1}(\frac{1}{w}) - \frac{1}{w}\right)$ is analytic on $\mathbb{C} \setminus \widetilde{\Lambda}_s^0 \setminus \{\mathbf{0}\}$. Just as (6.15), we can then get from (8.3) the integral formula

$$a_t - a_s = \frac{1}{2\pi i} \int_{\bigcup_{k=1}^{2N+1} \widetilde{\gamma}_k} \frac{h_{t,s}(\zeta)}{\zeta} d\zeta, \quad s < t, \tag{8.5}$$

where $\widetilde{\gamma}_1, \cdots, \widetilde{\gamma}_N$ (resp. $\widetilde{\gamma}_{N+1}, \cdots, \widetilde{\gamma}_{2N}$) are analytic contours surrounding $\widetilde{C_{s,1}}, \cdots, \widetilde{C_{s,N}}$ (resp. $\widetilde{\pi(C_{s,1})}, \cdots, \widetilde{\pi(C_{s,N})}$) and $\widetilde{\gamma_{2N+1}}$ is an analytic contour containing $\widetilde{\{\xi(s)\}}$ inside such that $\mathbf{0} \notin \cup_{k=1}^{2N+1} \widetilde{\gamma}_k$.

By virtue of (8.4), $h_{t,s}(\zeta)$ converges to 0 as $t \downarrow s$ uniformly in ζ on each $\widetilde{\gamma}_k$, $1 \le k \le 2N+1$, and consequently we get $\lim_{t \downarrow s} a_t = a_s$ from (8.5). This together with Corollary 6.3 yields

Theorem 8.4 a_t *is a strictly increasing continuous function in* $t \in [0, t_\gamma]$ *with* $a_0 = 0$.

We next denote by $\mathcal{D}$ the collection of 'labeled (ordered)' standard slit domains. For instance, $\mathbb{H} \setminus \{C_1, C_2, C_3, \cdots, C_N\}$ and $\mathbb{H} \setminus \{C_2, C_1, C_3, \cdots, C_N\}$ are considered as different elements of $\mathcal{D}$ in general although they correspond to the same subset $\mathbb{H} \setminus \bigcup_{i=1}^N C_i$ of $\mathbb{H}$. For $D, \widetilde{D} \in \mathcal{D}$, define their distance $d(D, \widetilde{D})$ by

$$d(D, \widetilde{D}) = \max_{1 \le i \le N} (|z_i - \widetilde{z}_i| + |z_i' - \widetilde{z}_i'|), \tag{8.6}$$

where, for $D = \mathbb{H} \setminus \{C_1, C_2, \cdots, C_N\}$, z_i (resp. z_i') denotes the left (resp. right) end point of C_i, $1 \le i \le N$. $\widetilde{z}_i$, $\widetilde{z}_i'$, $1 \le i \le N$, are the corresponding points to $\widetilde{D}$. $\{D_t : 0 \le t \le t_\gamma\}$ is a one parameter subfamily of $\mathcal{D}$.

The following theorem follows immediate from Theorems 8.1-8.2.

Theorem 8.5 *$\{D_t : 0 \le t \le t_\gamma\}$ is continuous in t in the sense that, for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for any s with $|t - s| < \delta$, $d(D_t, D_s) < \varepsilon$.*

Theorem 8.6 *$\xi(t)$ is continuous in $t \in [0, t_\gamma]$. Moreover*

$$\lim_{t \downarrow s} \beta_0(t, s) = \xi(s), \quad \lim_{t \downarrow s} \beta_1(t, s) = \xi(s). \tag{8.7}$$

Just as in Duren [8, p85], the left continuity of $\xi(t)$ follows from Theorem 8.1, while its right continuity as well as (8.7) follow from Theorem 8.2. So the proof of Theorem 8.6 is omitted.

9 Lipschitz continuity of Ψ and differentiability of g_t

The goal of this section is to establish the following theorem, which combined with theorems obtained in the preceding section will enable us to derive the Komatu-Loewner differential equation (1.4) for the map $g_t(z)$ but with the left derivative $\frac{\partial^-}{\partial t}$ being replaced by the derivative $\frac{\partial}{\partial t}$.

Recall the distance d defined by (8.6) on the space $\mathcal{D}$ of all 'labeled' standard slit domains. For each $D \in \mathcal{D}$, the associated BMD-complex Poisson kernel $\Psi(z, \zeta)$, $(z, \zeta) \in D \times \partial\mathbb{H}$, is well defined as in §6.1. Recall also that, for $D = \mathbb{H} \setminus K$, $K = \bigcup_{i=1}^N C_i$, C_i^+ (resp. C_i^-) denotes the upper (resp. lower) side of the slit C_i, and $\partial_p K$ denotes the set $\bigcup_{i=1}^N (C_i^+ \cup C_i^-)$ with topology induced from the path distance on $\mathbb{H} \setminus K$. By Lemma 6.1, $\Psi(z, \zeta)$ can be extended to be a continuous function on $(D \cup \partial_p K \cup (\partial\mathbb{H} \setminus J)) \times J$ for any compact interval $J \subset \partial\mathbb{H}$.

Theorem 9.1 *The correspondence $\mathcal{D} \mapsto \Psi(z, \zeta)$ is Lipschitz continuous in the following sense: Let U_j, V_j, $1 \le j \le N$, be any relatively compact open subsets of $\mathbb{H}$ with*

$$\overline{U}_j \subset V_j \subset \overline{V}_j \subset \mathbb{H} \quad \text{and} \quad \overline{V}_j \cap \overline{V}_k = \emptyset \text{ for } j \ne k. \tag{9.1}$$

We fix any $a > 0$ and $b > 0$ so that the subcollection $\mathcal{D}_0$ of $\mathcal{D}$ defined by

$$\mathcal{D}_0 = \left\{ \mathbb{H} \setminus \cup_{j=1}^N C_j \in \mathcal{D} : C_j \subset U_j,\ |z_j - z_j'| > a,\ \mathrm{dist}(C_j, \partial U_j) > b,\ 1 \le j \le N \right\} \tag{9.2}$$

is non-empty. There exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and for any $D \in \mathcal{D}_0$ and $\widetilde{D} \in \mathcal{D}$ with $d(D, \widetilde{D}) < \varepsilon$, there exists a diffeomorphism $\widetilde{f}_\varepsilon$ from $\mathbb{H}$ onto $\mathbb{H}$ satisfying

(i) *$\widetilde{f}_\varepsilon$ is sending D onto $\widetilde{D}$, linear on $\bigcup_{j=1}^N U_j$ and the identity map on $\mathbb{H} \setminus \bigcup_{j=1}^N \overline{V}_j$;*

(ii) *for some positive constant L_1 independent of $\varepsilon \in (0, \varepsilon_0)$, $D \in \mathcal{D}_0$ and $\widetilde{D} \in \mathcal{D}$,*

$$|z - \widetilde{f}_\varepsilon(z)| \le L_1 \varepsilon, \quad z \in \mathbb{H}; \tag{9.3}$$

(iii) *for any compact subset Q of $\overline{\mathbb{H}}$ containing $\bigcup_{j=1}^N U_j$ and for any compact subset J of $\partial\mathbb{H}$,*

$$|\Psi(z, \zeta) - \widetilde{\Psi}(\widetilde{f}_\varepsilon(z), \zeta)| \le L_{Q,J} \cdot \varepsilon, \quad z \in (Q \setminus K) \cup \partial_p K,\ \zeta \in J, \tag{9.4}$$

where $\widetilde{\Psi}$ denotes the BMD-complex Poisson kernel for $\widetilde{D}$ and $L_{Q,J}$ is a positive constant independent of $\varepsilon \in (0, \varepsilon_0)$, $D \in \mathcal{D}_0$ and $\widetilde{D} \in \mathcal{D}$.

The proof of Theorem 9.1 will be carried out through Lemmas 9.2, 9.5 and 9.7.

Let U_j and V_j be as in the statement of Theorem 9.1. For any $\varepsilon > 0$ and any $D \in \mathcal{D}_0$, take any $\widetilde{D} \in \mathcal{D}$ with $d(D, \widetilde{D}) < \varepsilon$. The quantities associated with $\widetilde{D}$ will be designated with $\widetilde{\ }$. For each $1 \leq j \leq N$, let $\delta_j \in \mathbb{R},\ b_j \in \mathbb{C}$ be constants that are uniquely determined by

$$\begin{cases} \widetilde{z}_j - z_j = \delta_j z_j + b_j, \\ \widetilde{z}'_j - z'_j = \delta_j z'_j + b_j, \end{cases}$$

where $z_j = x_{j1} + ix_{j2}$ (resp. $z'_j = x'_{j1} + ix'_{j2}$) is the left (resp. right) end point of the slit C_j. Since $\delta_j = \frac{(\widetilde{x}_{j1}-x_{j1})+(x'_{j1}-\widetilde{x}'_{j1})}{x_{j1}-x'_{j1}}$ and $|b_j| \leq |\widetilde{z}_j - z_j| + |\delta_j||z_j|$, we have

$$\frac{|\delta_j|}{\varepsilon} \leq \frac{2}{a}, \quad \frac{|b_j|}{\varepsilon} \leq 1 + \frac{2M}{a}, \qquad \text{where } M = \sup_{z \in \cup_{j=1}^N U_j} |z|. \tag{9.5}$$

Define a linear map

$$F_{j,\varepsilon}(z) = \frac{1}{\varepsilon}(\delta_j z + b_j), \quad 1 \leq j \leq N, \tag{9.6}$$

whose coefficients are bounded uniformly in $\varepsilon > 0$, $D \in \mathcal{D}_0$ and $\widetilde{D} \in \mathcal{D}$ by (9.5). Choose a smooth function $q(x_1, x_2),\ z = x_1 + ix_2 \in \mathbb{H}$, taking value in $[0, 1]$ such that

$$q(x_1, x_2) = \begin{cases} 1 & \text{if } x_1 + ix_2 \in U_j, \quad 1 \leq j \leq N, \\ 0 & \text{if } x_1 + ix_2 \in \mathbb{H} \setminus \bigcup_{j=1}^N \overline{V}_j, \end{cases}$$

and define a map $\widetilde{f}_\varepsilon$ by

$$\begin{cases} \widetilde{f}_\varepsilon(z) = z + \varepsilon F_\varepsilon(x_1, x_2), \quad \text{where} \\ F_\varepsilon(x_1, x_2) = q(x_1, x_2) \sum_{j=1}^N \mathbf{1}_{V_j}(z) F_{j,\varepsilon}(z), \quad z = x_1 + ix_2. \end{cases} \tag{9.7}$$

Lemma 9.2 *There exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$ and for any $D \in \mathcal{D}_0$ and $\widetilde{D} \in \mathcal{D}$ with $d(D, \widetilde{D}) < \varepsilon$, the map $\widetilde{f}_\varepsilon$ defined by* (9.7) *is a diffeomorphism from $\mathbb{H}$ onto $\mathbb{H}$ satisfying the properties* (i) *and* (ii) *of Theorem* 9.1.

Proof. In view of (9.5), (9.6) and (9.7), $F_\varepsilon(x_1, x_2)$ and its derivatives are bounded on $\mathbb{H}$ uniformly in $\varepsilon > 0$, $D \in \mathcal{D}_0$ and $\widetilde{D} \in \mathcal{D}$. So we can find $\varepsilon_1 > 0$ independent of $D \in \mathcal{D}_0$ such that (9.7) defines a continuous one-to-one map $\widetilde{f}_\varepsilon$ from $\mathbb{H}$ into $\mathbb{H}$ for any $\varepsilon \in (0, \varepsilon_1)$. $\widetilde{f}_\varepsilon$ is linear on each U_j, sends C_j onto $\widetilde{C}_j$ for $1 \leq j \leq N$, and an identity map on $\mathbb{H} \setminus \bigcup_{j=1}^N \overline{V}_j$. Further, it satisfies (9.3) for a constant L_1 independent of $\varepsilon > 0$, $D \in \mathcal{D}_0$ and $\widetilde{D} \in \mathcal{D}$. In what follows, we write $\widetilde{f}_\varepsilon(z) = \widetilde{z},\ z = x_1 + ix_2,\ \widetilde{z} = \widetilde{x}_1 + i\widetilde{x}_2$. We then have

$$\frac{\partial(\widetilde{x}_1, \widetilde{x}_2)}{\partial(x_1, x_2)} = \begin{cases} 1 + \varepsilon L(x_1, x_2) & \text{if } x_1 + ix_2 \in \bigcup_{j=1}^N \overline{V}_j, \\ 1 & \text{if } x_1 + ix_2 \in \mathbb{H} \setminus \bigcup_{j=1}^N \overline{V}_j, \end{cases}$$

where $L(x_1, x_2)$ is a uniformly bounded function on $\bigcup_{j=1}^N \overline{V}_j$ in $\varepsilon > 0$, $D \in \mathcal{D}_0$ and $\widetilde{D} \in \mathcal{D}$. Hence $\frac{\partial(\widetilde{x}_1, \widetilde{x}_2)}{\partial(x_1, x_2)} > 0$ for $x_1 + ix_2 \in \mathbb{H}$, and $\widetilde{f}_\varepsilon$ is an open map for any $\varepsilon \in (0, \varepsilon_2)$ for some ε_2 independent of $D \in \mathcal{D}_0$. We let $\varepsilon_0 = \varepsilon_1 \wedge \varepsilon_2$. For $\varepsilon \in (0, \varepsilon_0)$, $\widetilde{U} = \widetilde{f}_\varepsilon(\mathbb{H})$ is a connected open subset of $\mathbb{H}$. On the other hand, note that f_ε is an identity map on the relative closure of $\mathbb{H} \setminus \bigcup_{j=1}^N \overline{V}_j$ in $\mathbb{H}$. Since $\bigcup_{j=1}^N \overline{V}_j$ is a compact subset of $\mathbb{H}$, its image under f_ε is also compact. It follows that

$$\widetilde{U} = f(\mathbb{H}) = \left(\mathbb{H} \cap \overline{\mathbb{H} \setminus \cup_{j=1}^N \overline{V}_j}\right) \cup f(\cup_{j=1}^N \overline{V}_j)$$

is relatively closed in $\mathbb{H}$. Hence $\widetilde{U} = \mathbb{H}$. □

We remark that F_ε in (9.7) depends on $\varepsilon > 0$, D and $\widetilde{D}$, while Garabedian [10, §15.1] treated a case for a fixed map from D independent of $\varepsilon > 0$.

We denote by $G(z, w)$ the Green function of the domain D defined by (4.8) but with the superscript ^{0} dropped. The Green function of $\widetilde{D}$ is denoted by $\widetilde{G}(\widetilde{z}, \widetilde{w})$ and we define

$$g(z, w, \varepsilon) = \widetilde{G}(\widetilde{f}_\varepsilon(z), \widetilde{f}_\varepsilon(w)), \quad z, w \in D. \tag{9.8}$$

We introduce a second order self-adjoint elliptic differential operator A_ε by

$$\begin{cases} (A_\varepsilon u)(x_1, x_2) = \displaystyle\sum_{k,\ell=1}^2 \frac{\partial}{\partial x_k}\left(A_{k\ell}^{(\varepsilon)} \frac{\partial u}{\partial x_\ell}\right), \quad \textit{where} \\ A_{k\ell}^{(\varepsilon)} = \dfrac{1}{2} \dfrac{\partial(\widetilde{x}_1, \widetilde{x}_2)}{\partial(x_1, x_2)} \displaystyle\sum_{j=1}^2 \frac{\partial x_k}{\partial \widetilde{x}_j} \frac{\partial x_\ell}{\partial \widetilde{x}_j}, \quad 1 \le k, \ell \le 2. \end{cases} \tag{9.9}$$

Proposition 9.3 (i) *$g(z, w, \varepsilon)$ is a fundamental solution of A_ε in the sense that*

$$A_\varepsilon(g_\varepsilon f)(z) = -f(z), \quad z \in D \tag{9.10}$$

for any $f \in C_c(D)$, where $(g_\varepsilon f)(z) = \int_D g(z, w, \varepsilon) f(w) dw_1 dw_2$.

(ii) *$A_\varepsilon = \frac{1}{2}\Delta + \varepsilon B^{(\varepsilon)}$, where*

$$B^{(\varepsilon)} = \sum_{k,\ell=1}^2 b_{k\ell}^{(\varepsilon)} \frac{\partial^2}{\partial x_k \partial x_\ell} + \sum_{k,\ell=1}^2 \frac{\partial b_{k\ell}^{(\varepsilon)}}{\partial x_k} \frac{\partial}{\partial x_\ell}. \tag{9.11}$$

Here $b_{k\ell}^{(\varepsilon)}$, $1 \le k, \ell \le 2$, are smooth functions on $\mathbb{H}$ with $b_{k\ell}^{(\varepsilon)} = b_{\ell k}^{(\varepsilon)}$ vanishing on $(\mathbb{H} \setminus \bigcup_{i=1}^2 \overline{V}_i) \cup (\bigcup_{i=1}^N U_i)$ that together with their derivatives are uniformly bounded on $\mathbb{H}$ in $\varepsilon \in (0, \varepsilon_0)$, $D \in \mathcal{D}_0$ and $\widetilde{D} \in \mathcal{D}$.

(iii) *Put $F = \bigcup_{i=1}^N (\overline{V}_i \setminus U_i)$. Then for any $\zeta \in \overline{\mathbb{H}} \setminus F$ and $w \in \overline{\mathbb{H}}$,*

$$g(\zeta, w, \varepsilon) - G(\zeta, w) = \varepsilon \int_F B_z^{(\varepsilon)} G(z, \zeta) g(z, w, \varepsilon) dx_1 dx_2, \quad z = x_1 + ix_2, \ \varepsilon \in (0, \varepsilon_0). \tag{9.12}$$

(iv) *There exists $\widetilde{\varepsilon}_0 \in (0, \varepsilon_0]$ independent of $D \in \mathcal{D}_0$ such that for any $\zeta \in \overline{\mathbb{H}} \setminus F$ and $w \in \overline{\mathbb{H}}$*

$$g(\zeta, w, \varepsilon) - G(\zeta, w) = \varepsilon \int_F B_z^{(\varepsilon)} G(z, \zeta)(G(z, w) + \varepsilon \eta^{(\varepsilon)}(z, w)) dx_1 dx_2, \quad \varepsilon \in (0, \widetilde{\varepsilon}_0), \tag{9.13}$$

where $\eta^{(\varepsilon)}$ is a continuous function on $\overline{\mathbb{H}} \times \overline{\mathbb{H}}$ that is uniformly bounded in $\varepsilon \in (0, \widetilde{\varepsilon}_0)$, $D \in \mathcal{D}_0$ and $\widetilde{D} \in \mathcal{D}$.

(v) *For each compact* $J \subset \partial\mathbb{H}$, *the function* $\frac{\partial}{\partial \mathbf{n}_\zeta} B_z^{(\varepsilon)} G(z,\zeta)$ *is bounded in* $(z,\zeta) \in F \times J$ *that is uniform in* $\varepsilon \in (0,\varepsilon_0)$, $D \in \mathcal{D}_0$ *and* $\widetilde{D} \in \mathcal{D}$.

(vi) *For each* $1 \le i \le N$, $B_z^{(\varepsilon)} \varphi^{(i)}(z)$ *is bounded in* z *on* F *that is uniform in* $\varepsilon \in (0,\varepsilon_0)$, $D \in \mathcal{D}_0$ *and* $\widetilde{D} \in \mathcal{D}$. *Here* $\varphi^{(i)}(z) = \mathbb{P}_z^{\mathbb{H}}(\sigma_K < \infty,\ Z_{\sigma_K}^{\mathbb{H}} \in C_i)$, $z \in D$, *and* $K = \cup_{j=1}^N C_j$.

(vii) *For each compact set* $J \subset \partial\mathbb{H}$ *and for* $k = 1,2$,

$$\sup_{x_1 \in \mathbb{R}, \zeta \in J} \int_0^\infty \mathbf{1}_F(z) \left| \frac{\partial^2}{\partial x_k \partial \zeta_2} G(\zeta, z) \right| dx_2, \quad \sup_{x_2 > 0, \zeta \in J} \int_{-\infty}^\infty \mathbf{1}_F(z) \left| \frac{\partial^2}{\partial x_k \partial \zeta_2} G(\zeta, z) \right| dx_1, \qquad (9.14)$$

are bounded in $D \in \mathcal{D}_0$.

(viii) *Fix* $1 \le j \le N$. *It holds for* $k = 1,2$ *that*

$$\sup_{x_1 \in \mathbb{R}} \int_0^\infty \mathbf{1}_F(z) \left| \frac{\partial}{\partial x_k} \varphi^{(j)}(z) \right| dx_2, \quad \sup_{x_2 > 0} \int_{-\infty}^\infty \mathbf{1}_F(z) \left| \frac{\partial}{\partial x_k} \varphi^{(j)}(z) \right| dx_1, \qquad (9.15)$$

are bounded in $D \in \mathcal{D}_0$.

In what follows, the constant $\widetilde{\varepsilon}_0$ in the statement of (iv) above will simply be denoted as ε_0. A proof of this proposition will be given in Appendix 3 through a series of lemmas. We now proceed to prove Theorem 9.1(iii) with $\widetilde{f}_\varepsilon$ defined by (9.7), using the perturbation formulae (9.12) and (9.13).

For a smooth function u on $D \cup \partial\mathbb{H}$ and a smooth simple curve $\gamma \subset D \cup \partial\mathbb{H}$, define

$$I(u;\gamma) = \int_\gamma \frac{\partial u(w)}{\partial \mathbf{n}_w} ds(w). \qquad (9.16)$$

The corresponding quantity for $\widetilde{D}$ is denoted by $\widetilde{I}(\widetilde{u};\widetilde{\gamma})$.

Lemma 9.4 *For a smooth simple curve* γ *in* $D \cup \partial\mathbb{H}$ *of finite length and a smooth function* $\widetilde{u}$ *on* $\widetilde{D} \cup \partial\mathbb{H}$, *define* $\widetilde{\gamma} = \widetilde{f}_\varepsilon(\gamma)$. *Then*

$$\widetilde{I}(\widetilde{u};\ \widetilde{\gamma}) = 2 \int_\gamma \sum_{k,\ell=1}^2 A_{k\ell}^{(\varepsilon)}(w) \frac{\partial \widetilde{u}(\widetilde{f}_\varepsilon(w))}{\partial w_k} \frac{\partial w_\ell}{\partial \mathbf{n}} ds(w). \qquad (9.17)$$

Proof. (9.17) can be established by a direct computation via a change of variable. Here we provide a different proof using Green's first identity. Extend γ to a smooth simple closed curve in $D \cup \partial\mathbb{H}$ and denote its enclosed interior by U. Define $\widetilde{U} = \widetilde{f}_\varepsilon(U)$. By Green's first formula for the self-adjoint operator A_ε,

$$\sum_{i,j=1}^2 \int_U A_{ij}^{(\varepsilon)} \frac{\partial \widetilde{v}}{\partial x_i} \frac{\partial \widetilde{u}}{\partial x_j} dx_1 dx_2 + \int_U \widetilde{v} A_\varepsilon \widetilde{u}\ dx_1 dx_2 = \int_{\partial U} \widetilde{v} \sum_{i,j=1}^2 A_{ij}^{(\varepsilon)} \frac{\partial \widetilde{u}}{\partial x_i} \frac{\partial x_j}{\partial \mathbf{n}} ds.$$

In view of the proof of Lemma 12.1 below, the left hand side equals

$$\frac{1}{2} \int_{\widetilde{U}} \sum_{j=1}^2 \frac{\partial \widetilde{v}}{\partial \widetilde{x}_j} \frac{\partial \widetilde{u}}{\partial \widetilde{x}_j} d\widetilde{x}_1 d\widetilde{x}_2 + \frac{1}{2} \int_{\widetilde{U}} \widetilde{v} \widetilde{\Delta} \widetilde{u} d\widetilde{x}_1 d\widetilde{x}_2 = \frac{1}{2} \int_{\partial \widetilde{U}} \widetilde{v} \frac{\partial \widetilde{u}}{\partial \widetilde{\mathbf{n}}} d\widetilde{s}.$$

Therefore the right hand sides of the above two identities coincide for any bounded smooth functions $\widetilde{v}$. A measure theoretical argument shows that they remain the same for any bounded Borel function $\widetilde{v}$ on $\partial\widetilde{U}$. Taking $\widetilde{v}$ to be the indicator function of $\widetilde{\gamma}$ establishes (9.17). □

For a function $\phi(z,\zeta)$ on $((\overline{\mathbb{H}}\setminus K)\cup\partial_p K)\times\partial\mathbb{H}$ and a function $\psi(z,\zeta,\varepsilon)$ on $((\overline{\mathbb{H}}\setminus K)\cup\partial_p K)\times\partial\mathbb{H}\times(0,\varepsilon_0)$, we write

$$\phi(z,\zeta)\sim\psi(z,\zeta,\varepsilon),\quad z\in(\overline{\mathbb{H}}\setminus K)\cup\partial_p K,\quad \zeta\in\partial\mathbb{H},$$

if, for any compact set $Q\subset\overline{\mathbb{H}}$ containing $\bigcup_{j=1}^N U_j$ and any compact set $J\subset\partial\mathbb{H}$, there exists a positive constant $L_{Q,J}$ independent of $\varepsilon\in(0,\varepsilon_0)$, $D\in\mathcal{D}_0$ and $\widetilde{D}=\widetilde{f}_\varepsilon(D)\in\mathcal{D}$ such that

$$|\phi(z,\zeta)-\psi(z,\zeta,\varepsilon)|\le L_{Q,J}\cdot\varepsilon,\quad z\in(Q\setminus K)\cup\partial_p K,\ \zeta\in J,\ \varepsilon\in(0,\varepsilon_0).$$

Using this notation, the third assertion of Theorem 9.1 can be simply expressed as $\Psi(z,\zeta)\sim\widetilde{\Psi}(\widetilde{f}_\varepsilon(z),\zeta)$, $z\in(\mathbb{H}\setminus K)\cup\partial_p K$, $\zeta\in\partial\mathbb{H}$.

We also use the obvious analogous notations $u(z)\sim v(z,\varepsilon),\ z\in(\overline{\mathbb{H}}\setminus K)\cup\partial_p K$, for functions on $(\overline{\mathbb{H}}\setminus K)\cup\partial_p K$, and on $(\overline{\mathbb{H}}\setminus K)\cup\partial_p K\times(0,\varepsilon_0)$, $f(\zeta)\sim g(\zeta,\varepsilon),\ \zeta\in\partial\mathbb{H}$, for functions on $\partial\mathbb{H}$ and on $\partial\mathbb{H}\times(0,\varepsilon_0)$, and further $\alpha\sim\beta(\varepsilon)$ for a constant and a function of $\varepsilon\in(0,\varepsilon_0)$.

Recall from (5.2) the explicit formula for the BMD-Poisson kernel $K^*(z,\zeta)$ for D: for $z\in D,\ \zeta\in\partial\mathbb{H}$

$$K^*(z,\zeta)=-\frac{1}{2}\frac{\partial}{\partial\mathbf{n}_\zeta}G(z,\zeta)-\Phi(z)\mathcal{A}^{-1}\frac{\partial}{\partial\mathbf{n}_\zeta}\Phi(\zeta)^{\rm tr},\tag{9.18}$$

where $\Phi(z)$ is the vector with entries $\varphi^{(i)}(z),\ 1\le i\le N$, $\mathcal{A}$ is an $N\times N$-matrix whose (i,j)-component a_{ij} is the period of $\varphi^{(j)}$ around the slit $C_i,\ 1\le i,j\le N$.

Lemma 9.5 *It holds that*

$$K^*(z,\zeta)\sim\widetilde{K}^*(\widetilde{f}_\varepsilon(z),\zeta),\quad z\in(\overline{\mathbb{H}}\setminus K)\cup\partial_p K,\quad \zeta\in\partial\mathbb{H}.\tag{9.19}$$

Proof. By taking the partial derivative in ζ_2 at $\zeta\in\partial\mathbb{H}$ on both sides of (9.13) and using the symmetry of G, Proposition 9.3(v) as well as the property $G\le G^{\mathbb{H}}$, we have

$$\frac{\partial}{\partial\mathbf{n}_\zeta}G(w,\zeta)\sim\frac{\partial}{\partial\mathbf{n}_\zeta}\widetilde{G}(\widetilde{f}_\varepsilon(w),\zeta),\quad w\in(\overline{\mathbb{H}}\setminus K)\cup\partial_p K,\quad \zeta\in\partial\mathbb{H}.\tag{9.20}$$

Let $\gamma_i\subset U_i$ be a smooth Jordan curve surrounding C_i for each $1\le i\le N$. By Theorem 4.5 (i), $\varphi^{(i)}(w)=-\frac{1}{2}I(G(\cdot,w);\gamma_i)$. On the other hand, since $A^{(\varepsilon)}=\frac{1}{2}\Delta$ on U by Proposition 9.3, (9.17) applied to $\widetilde{u}(\cdot)=\widetilde{G}(\cdot,\widetilde{f}_\varepsilon(w))$ reads

$$\widetilde{\varphi}^{(i)}(\widetilde{f}_\varepsilon(w))=-\frac{1}{2}\widetilde{I}(\widetilde{u};\widetilde{\gamma}_i)=-\frac{1}{2}I(g(\cdot,w,\varepsilon);\gamma_i),\tag{9.21}$$

where $\widetilde{u}(\cdot)=\widetilde{G}(\cdot,\widetilde{f}_\varepsilon(w))$. By taking the period around C_i with respect to ζ on both sides of (9.13), we have by Proposition 9.3(vi)

$$\varphi^{(i)}(w)\sim\widetilde{\varphi}^{(i)}(\widetilde{f}_\varepsilon(w)),\quad w\in(\overline{\mathbb{H}}\setminus K)\cup\partial_p K,\quad 1\le i\le N.\tag{9.22}$$

In the rest of the proof, we will use (9.12) instead of (9.13). In both sides of (9.12), we take the period around C_i with respect to w and use (9.21) to obtain

$$\varphi^{(i)}(\zeta)-\widetilde{\varphi}^{(i)}(\widetilde{f}_\varepsilon(\zeta))=\varepsilon\int_F B_z^{(\varepsilon)}G(z,\zeta)\widetilde{\varphi}^{(i)}(\widetilde{f}_\varepsilon(z))dx_1dx_2.\tag{9.23}$$

Keeping in mind that $0 \leq \widetilde{\varphi}^{(i)}(\widetilde{f}_\varepsilon(z)) \leq 1$, we perform two operations in (9.23) with respect to ζ. Firstly we take the partial derivative in ζ_2 at $\zeta \in \partial\mathbb{H}$ and use Proposition 9.3 (v) to obtain

$$\frac{\partial}{\partial \mathbf{n}_\zeta}\varphi^{(i)}(\zeta) \sim \frac{\partial}{\partial \mathbf{n}_\zeta}\widetilde{\varphi}^{(i)}(\widetilde{f}_\varepsilon(\zeta)), \quad \zeta \in \partial\mathbb{H}, \quad 1 \leq i \leq N. \tag{9.24}$$

Secondly we take the period for the curve $\gamma_j \subset U_j$ surrounding C_j and use (9.17) as well as Proposition 9.3 (vi) again to have

$$a_{ij} - \widetilde{a}_{ij} = \varepsilon\eta_{ij}(\varepsilon), \quad 1 \leq i, j \leq N, \tag{9.25}$$

for $\eta_{ij}(\varepsilon)$ bounded uniformly in $(0, \varepsilon_0)$, $D \in \mathcal{D}_0$ and $\widetilde{f}_\varepsilon(D)$. This implies that $a_{ij} = a_{ij}(D)$ is, as a function of $D \in \mathcal{D}$, uniformly Lipschitz continuous on $\mathcal{D}_0$ and consequently

$$b_{ij} \sim \widetilde{b}_{ij}, \quad 1 \leq i, j \leq N, \tag{9.26}$$

for (i,j)-component b_{ij} (resp. $\widetilde{b}_{ij}$) of $\mathcal{A}^{-1}$ (resp. $\widetilde{\mathcal{A}}^{-1}$). (9.19) follows from (9.18), (9.20), (9.22), (9.24) and (9.26). □

We write $\Psi(z,\zeta) = u(z,\zeta) + iK^*(z,\zeta),\ z \in D,\ \zeta \in \partial\mathbb{H}$. Fix a compact interval $J \subset \partial\mathbb{H}$, take a point $z_0 = x_0 + i0 \in \partial\mathbb{H}$ on the right of J and consider the half line $\Gamma = \{z \in \partial\mathbb{H} : x_0 \leq \Re z\} \subset \partial\mathbb{H}$. For each $z \in D$, let $\gamma(z)$ be any smooth simple curve in D joining z to z_0. As we saw in the proof of Theorem 6.1(ii), $u(z,\zeta),\ u \in D,\ \zeta \in J$, can be evaluated as

$$u(z,\zeta) = I(K^*(\cdot,\zeta);\gamma(z)) + I(K^*(\cdot,\zeta);\Gamma), \tag{9.27}$$

independent of the choice of the curve $\gamma(z)$.

We let $V = \cup_{j=1}^N V_j$, $U = \cup_{j=1}^N U_j$ so that $F = \overline{V} \setminus U$. We make special choices of x_0 and $\gamma(z)$ so that the straight line through z_0 is parallel to the y-axis does not intersect with $\overline{V}$, while $\gamma(z)$ consists of at most two line segments γ_1 (starting at z_0), γ_2 (ending at z) parallel to the y-axis and one line segment γ_3 parallel to the x-axis through the slit domain D. Define $\widetilde{\gamma}(\widetilde{f}_\varepsilon(z)) = \widetilde{f}_\varepsilon(\gamma(z))$.

Lemma 9.6 *There exists a positive $\varepsilon_0' \leq \varepsilon_0$ independent of $D \in \mathcal{D}_0$ such that, for any positive $\varepsilon < \varepsilon_0'$ and for any relatively compact open set Q with $\overline{V} \subset Q \subset \overline{\mathbb{H}}$,, the inequality*

$$|\widetilde{I}(\widetilde{G}(\cdot,\widetilde{f}_\varepsilon(z));\widetilde{\gamma}(\widetilde{f}_\varepsilon(w))| \ \leq C_1 + C_2\log_+\frac{1}{|x_1 - w_1|} + C_3\log_+\frac{1}{|x_2 - w_2|} \tag{9.28}$$

holds for any $z = x_1 + ix_2 \in F$, $w = w_1 + iw_2 \in \overline{Q}$ with $x_i \neq w_i$ for $i = 1, 2$, where $C_1,\ C_2,\ C_3$ are positive constants independent of $\varepsilon \in (0, \varepsilon_0')$, $D \in \mathcal{D}_0$ and $\widetilde{f}_\varepsilon(D)$. Here $\log_+ a := 0 \vee \log a$ for $a > 0$.

Proof. Let $Q = (A_1, B_1) \times (0, B_2)$. By (9.17) and the definition (9.8) of $g(\cdot,\cdot,\varepsilon)$, we have

$$\widetilde{I}(\widetilde{G}(\cdot,\widetilde{f}_\varepsilon(z)),\widetilde{\gamma}(\widetilde{f}_\varepsilon(w)) = 2\int_{\gamma(w)}\sum_{k,\ell=1}^{2} A_{k\ell}^{(\varepsilon)}\frac{\partial g(z,w',\varepsilon)}{\partial w_k'}\frac{\partial w_\ell'}{\partial \mathbf{n}}ds(w'), \tag{9.29}$$

which equals $\widetilde{I}_1 + \widetilde{I}_2 + \widetilde{I}_3$, where

$$\begin{cases} \widetilde{I}_j = 2\sum_{k=1}^2 \int_{\gamma_j} A_{k1}^{(\varepsilon)}(w')g_{w_k'}(z,w',\varepsilon)dw_2', \quad j = 1, 2, \\ \widetilde{I}_3 = 2\sum_{k=1}^2 \int_{\gamma_3} A_{k2}^{(\varepsilon)}(w')g_{w_k'}(z,w',\varepsilon)dw_1'. \end{cases}$$

We show that $|\widetilde{I}_2|$ admits a bound as (9.28). The bound for $|\widetilde{I}_1|$ is the same but simpler, and that for $|\widetilde{I}_3|$ is quite analogous. Let $\beta > 0$ be a uniform bound of $|A^{(\varepsilon)}_{k\ell}(w)|,\ w \in \mathbb{H}, 1 \le k, \ell \le 2$. Then

$$|\widetilde{I}_2| \le 2\beta(\widetilde{J}_1 + \widetilde{J}_2),\ \text{for}\ \widetilde{J}_1 = \int_{\gamma_2} \left|\frac{\partial}{\partial w_1'} g(z, w', \varepsilon)\right| dw_2',\ \widetilde{J}_2 = \int_{\gamma_2} \mathbf{1}_F(w') \left|\frac{\partial}{\partial w_2'} g(z, w', \varepsilon)\right| dw_2'.$$

Notice that the integrand for $\widetilde{J}_2$ vanishes for $w' \in \mathbb{H} \setminus F$ because $A^{(\varepsilon)}_{21} = 0$ there.

In view of (4.8),

$$g(w', z, \varepsilon) = \frac{1}{2\pi} \log |\widetilde{f}_\varepsilon(w') - \widetilde{f}_\varepsilon(z)|^2 - \frac{1}{2\pi} \int_{\partial \widetilde{D}} \widetilde{\Pi}(\widetilde{f}_\varepsilon(z), d\widetilde{z}) \log |\widetilde{z} - \widetilde{f}_\varepsilon(w')|^2, \tag{9.30}$$

where $\widetilde{\Pi}(\widetilde{\zeta}, \cdot)$ is the exit distribution by the Brownian motion starting at $\widetilde{\zeta}$ from $\widetilde{D}$. Denote by $F_1(x_1, x_2)$ and $F_2(x_1, x_2)$ the first and second components of $F_\varepsilon(x_1, x_2)$ defined by (9.7). They are Lipschitz continuous on $\mathbb{H}$ and we take $M > 0$ larger than their Lipschitz constants. We also take $M' > 0$ to be a uniform bound of $|\frac{\partial}{\partial x_k} F_i(x_1, x_2)|,\ 1 \le i, k \le 2$. It is then easy to see that, for any positive ε smaller than $\varepsilon_1 = (8M)^{-1} \wedge (M')^{-1}$ and for $z, w' \in \mathbb{H}$,

$$\frac{1}{2\pi} \left|\frac{\partial}{\partial w_k'} \log |\widetilde{f}_\varepsilon(w') - \widetilde{f}_\varepsilon(z)|^2\right| \le \frac{5}{\pi} \frac{|w_1' - x_1| + |w_2' - x_2|}{(w_1' - x_1)^2 + (w_2' - x_2)^2},\ k = 1, 2.$$

We take $\varepsilon_0' = \varepsilon_0 \wedge \varepsilon_1$. For $\varepsilon \in (0, \varepsilon_0')$, the contributions of the first term of the right hand side of (9.30) to $\widetilde{J}_k,\ k = 1, 2$, are therefore less than (for $w \in Q,\ z \in F,\ w_1 \ne x_1$)

$$5 + \frac{5}{2\pi} \left|\log\left[((w_1 - x_1)^2 + (B_2 - x_2)^2)((w_1 - x_1)^2 + x_2^2)\right]\right| + \frac{10}{\pi} \log_+ \frac{1}{|w_1 - x_1|}.$$

On the other hand, we have for $\varepsilon \in (0, \varepsilon_0')$

$$\frac{1}{2\pi} \left|\frac{\partial}{\partial w_1'} \log |\widetilde{z} - \widetilde{f}_\varepsilon(w')|^2\right| \le k_1(\widetilde{z}, w') \quad \text{for } \widetilde{z} \in \partial \widetilde{D},\ w' \in Q, \quad \text{where}$$

$$k_1(\widetilde{z}, w') = \frac{1}{\pi} \frac{2|\widetilde{x}_1 - w_1' - \varepsilon F_1(w')| + \mathbf{1}_F(w')|\widetilde{x}_2 - w_2' - \varepsilon F_2(w')|}{(\widetilde{x}_1 - w_1' - \varepsilon F_1(w'))^2 + (\widetilde{x}_2 - w_2' - \varepsilon F_2(w'))^2}.$$

For the constant $b > 0$ of (9.2), we define

$$b_0 = b \wedge \left(\min_{1 \le i \le N} \mathrm{dist}(\partial V_i, \partial \mathbb{H})\right) \tag{9.31}$$

and set $U_0 = U \cup \{z \in Q; 0 < y < b_0\}$. Denote $\varepsilon_0' \wedge \frac{a}{4} \wedge \frac{b}{2L_1} \wedge \frac{b_0}{4\sqrt{2}M''}$ by ε_0' again for a uniform bound M'' of $|F_1|$ and $|F_2|$ on $\mathbb{H}$. Then, for $\widetilde{z} \in \partial \widetilde{D}$ and $w' \in Q \setminus U_0$, $\mathrm{dist}(\widetilde{z}, w') \ge b_0/2$ and the denominator of $k_1(\widetilde{z}, w')$ is greater than $b_0^2/16$ for $\varepsilon \in (0, \varepsilon_0')$ so that $k_1(\widetilde{z}, w') \mathbf{1}_{Q \setminus U_0}(w') \le M_1$ for any $\widetilde{z} \in \partial \widetilde{D}$ and $\varepsilon \in (0, \varepsilon_0')$ with constant M_1 depending only on ℓ_0, M'' and Q. When $w' \in U$, $\varepsilon F_i(w') = \delta_i w_i' + b_i$, $i = 1, 2$. By (9.5), $1 + \delta_i > \frac{1}{2}$ for $i = 1, 2$. Hence $\int_0^{B_2} k_1(\widetilde{z}, w') dw_2' \le 3 + M_1 B_2$ for $\widetilde{z} \in \partial \widetilde{D}$. The integration with respect to $\widetilde{\Pi}$ in $\widetilde{z}$ yields the same bound for the contribution to $\widetilde{J}_1$ of the second term of the right hand side of (9.30).

Similarly, we see that $\frac{1}{2\pi} \left|\frac{\partial}{\partial w_2'} \log |\widetilde{z} - \widetilde{f}_\varepsilon(w')|^2\right|$ is dominated by $k_2(\widetilde{z}, w')$ for $\widetilde{z} \in \partial \widetilde{D},\ w' \in Q$, where $k_2(\widetilde{z}, w')$ is obtained from $k_1(\widetilde{z}, w')$ by interchanging the subscripts 1 and 2. Since $F \subset Q \setminus U_0$, it follows that $k_2(\widetilde{z}, w') \mathbf{1}_F(w')$ admits a bound M_2 like M_1 as above in $\widetilde{z} \in \partial \widetilde{D}$ and so the contribution to $\widetilde{J}_2$ of the second term of the right hand side of (9.30) has a bound $M_2 B_2$. □

The constant ε_0' in Lemma 9.6 will be designated by ε_0 again.

Lemma 9.7 *It holds that*

$$u(z,\zeta) \sim \widetilde{u}(\widetilde{f}_\varepsilon(z),\zeta), \quad z \in (\overline{\mathbb{H}} \setminus K) \cup \partial_p K, \quad \zeta \in \partial\mathbb{H}. \tag{9.32}$$

Proof. **(a)**. Let L be a linear operator sending a smooth function $f(w)$ on D to

$$(Lf)(w) = 2\int_{\gamma(w)} \sum_{k,\ell=1}^{2} A_{kl}^{(\varepsilon)}(w') \frac{\partial f(w')}{\partial w'_k} \frac{\partial w'_\ell}{\partial \mathbf{n}} ds(w'), \quad w \in D,$$

for the curve $\gamma(w)$ specified in the paragraph above Lemma 9.6. On both sides of (9.12), we take the normal derivative at $\zeta \in J$ and apply the linear operator L in w. By virtue of (9.17) and (9.29), we have for $w \in D$, $\zeta \in J$

$$\begin{aligned} &\widetilde{I}(\widetilde{G}_{\zeta_2}(\zeta,\cdot);\widetilde{\gamma}(\widetilde{f}_\varepsilon(w))) - I(G_{\zeta_2}(\zeta,\cdot);\gamma(w)) \\ = \;& \varepsilon \int_F B_z^{(\varepsilon)} G(z,\zeta)_{\zeta_2} \widetilde{I}(\widetilde{G}(\widetilde{f}_\varepsilon(z),\cdot);\widetilde{\gamma}(\widetilde{f}_\varepsilon(w)))dx_1 dx_2 \\ &+ 2\varepsilon \sum_{k=1}^{2} \int_{\gamma_1\cup\gamma_2} \mathbf{1}_F(w') b_{k1}^{(\varepsilon)}(w') G_{\zeta_2 w'_k}(\zeta,w')dw'_2 + 2\varepsilon \sum_{k=1}^{2} \int_{\gamma_3} \mathbf{1}_F(w') b_{k2}^{(\varepsilon)}(w') G_{\zeta_2 w'_k}(\zeta,w')dw'_1. \end{aligned}$$

This combined with Lemma 9.6 and Proposition 9.3 (ii), (v), (vii), yields

$$I(G_{\zeta_2}(\zeta,\cdot);\gamma(w)) \sim \widetilde{I}(\widetilde{G}_{\zeta_2}(\zeta,\cdot);\widetilde{\gamma}(\widetilde{f}_\varepsilon(w))), \quad w \in (\overline{\mathbb{H}} \setminus K) \cup \partial_p K,\ \zeta \in \partial\mathbb{H}. \tag{9.33}$$

(b). As $g(z,w,\varepsilon) = \widetilde{G}(\widetilde{f}_\varepsilon(z),w) \le G^{\mathbb{H}}(\widetilde{f}_\varepsilon(z),w)$ for $z \in \mathbb{H}$ and $w \in \mathbb{H} \setminus \bigcup_{j=1}^N \overline{V}_j$,

$$I(g(z,\cdot,\varepsilon);\Gamma) \le \int_{-\infty}^{\infty} G^{\mathbb{H}}_{w_2}(\widetilde{f}_\varepsilon(z),w)\big|_{w_2=0} dw_1 = 2.$$

So we can use (9.12) and Proposition 9.3 (v) again to get

$$I(G_{\zeta_2}(\zeta,\cdot);\Gamma)) \sim I(g_{\zeta_2}(\zeta,\cdot,\varepsilon);\Gamma), \quad \zeta \in \partial\mathbb{H}.$$

Since $\widetilde{f}_\varepsilon(z) = z$ near $\partial\mathbb{H}$, we have $\widetilde{I}(\widetilde{G}_{\zeta_2}(\zeta,\cdot);\Gamma) = I(g_{\zeta_2}(\zeta,\cdot,\varepsilon);\Gamma)$ for $\zeta \in \partial\mathbb{H}$ and

$$I(G_{\zeta_2}(\zeta,\cdot);\Gamma) \sim \widetilde{I}(\widetilde{G}_{\zeta_2}(\zeta,\cdot);\Gamma), \quad \zeta \in \partial\mathbb{H}. \tag{9.34}$$

(c). In **(a)**, we replace the operation of taking the partial derivative with respect to ζ_2 at $\zeta \in \partial\mathbb{H}$ by the operation of taking a period around C_i in $\zeta \in \gamma_i \subset U_i$ and use (9.21). We then utilize Proposition 9.3 (ii), (vi), (viii) as well as Lemma 9.6. We can also make the same replacement in **(b)** and use Proposition 9.3 (vi). We thus arrive at

$$I(\varphi^{(i)};\gamma(w)) \sim \widetilde{I}(\widetilde{\varphi}^{(i)};\widetilde{\gamma}(\widetilde{f}_\varepsilon(w)),\ w \in D \cup \partial_p K \cup \partial\mathbb{H} \quad \text{and} \quad I(\varphi^{(i)};\Gamma) \sim \widetilde{I}(\widetilde{\varphi}^{(i)};\Gamma). \tag{9.35}$$

(9.32) follows from (9.27), (9.33), (9.34), (9.35) combined with (9.24) and (9.26). □

Lemmas 9.5 and 9.7 yield Theorem 9.1(iii).

Finally we return to the setting of §6.2 and §6.3. That is, $D = \mathbb{H} \setminus K$ is a fixed standard slit domain with $K = \bigcup_{i=1}^N C_i$, γ is a Jordan arc satisfying (6.6), g_t, $t \in [0,t_\gamma]$, is a conformal map from $D \setminus \gamma[0,t]$ onto a standard slit domain $D_t = \mathbb{H} \setminus K_t$ with $K_t = \bigcup_{i=1}^N C_{t,i}$ satisfying (6.8), and $\Psi_t(z,\zeta)$ is the complex Poisson kernel of BMD on D_t. $\Psi_t(z,\zeta)$ is continuous in $z \in D_t \cup \partial_p K_t \cup (\partial\mathbb{H} \setminus \{\zeta\})$ for each $(t,\zeta) \in [0,t_\gamma] \times \partial\mathbb{H}$.

By combining Theorem 9.1 with Lemma 6.1 and Theorem 8.5, we are led to

Theorem 9.8 *$\Psi_t(z,\zeta)$ is jointly continuous in (t,z,ζ).*

Proof. Fix $t^* \in [0,t_\gamma]$ and a compact interval $J \subset \partial\mathbb{H}$. We shall apply Theorem 9.1 to the single fixed $D_{t^*} = \mathbb{H} \setminus \cup_{j=1}^N C_{t^*,j} \in \mathcal{D}_0$ by choosing $U_j,\ V_j$ as $C_{t^*,j} \subset U_j \Subset V_j \Subset \mathbb{H}$, and a and b less than the minimum of the width of $C_{t^*,j}$ and the minimum of $\mathrm{dist}(C_{t^*,j}, \partial U_j)$, respectively. Take any $\varepsilon \in (0,\varepsilon_0)$ and any relatively compact open subset G_1 of $\overline{\mathbb{H}}$ such that, $K_{t^*} \subset G_1$ and, if we define $G_2 = U_\varepsilon(G_1) \cap \overline{\mathbb{H}},\ G_3 = U_\varepsilon(G_2) \cap \overline{\mathbb{H}}$, then $\overline{G}_3 \cap J = \emptyset$. Here $U_\varepsilon(G_i)$ denotes the ε-neighborhood of $G_i,\ i=1,2$. We write $G_{i,t} = G_i \setminus K_t,\ t \in [0,t_\gamma],\ 1 \le i \le 3$. By virtue of Lemma 6.1, there exists $\delta > 0$ such that

$$|\Psi_{t^*}(z_1^*,\zeta_1) - \Psi_{t^*}(z_2^*,\zeta_2)| < \varepsilon \tag{9.36}$$

for any $z_1^*,\ z_2^* \in G_{3,t^*} \cup \partial_p K_{t^*}$ with $|z_1^* - z_2^*| < \delta$ and any $\zeta_1,\ \zeta_2 \in J$ with $|\zeta_1 - \zeta_2| < \delta$.

Let $\varepsilon' = \dfrac{\varepsilon \wedge \delta}{1 + 2L_1} (< \varepsilon \wedge \delta)$. By virtue of Theorem 8.5, there exists $\delta_0 > 0$ such that

$$t \in (t^* - \delta_0, t^* + \delta_0) \cap [0,t_\gamma] \implies d(D_t,\ D_{t^*}) < \varepsilon', \tag{9.37}$$

which particularly implies that $K_t \subset G_2$ whenever $|t - t^*| < \delta_0$.

Now take any $t_1,\ t_2 \in (t^* - \delta_0, t^* + \delta_0) \cap [0,t_\gamma]$, any $z_1 \in G_{2,t_1} \cup \partial_p K_{t_1},\ z_2 \in G_{2,t_2} \cup \partial_p K_{t_2}$ with $|z_1 - z_2| < \varepsilon'$ and any $\zeta_1, \zeta_2 \in J$ with $|\zeta_1 - \zeta_2| < \delta$. Denote by $f^i_{\varepsilon'}$ the diffeomorphism sending D_{t^*} onto D_{t_i} that appears in Theorem 9.1, $i=1,2$. There exist unique $z_i^* \in D_{t^*}$ such that $f^i_{\varepsilon'}(z_i^*) = z_i$, $i=1,2$. By (9.3), $|z_i - z_i^*| \le L_1\varepsilon' < \varepsilon$ so that $z_i^* \in G_{3,t^*} \cup \partial_p K_{t^*},\ i=1,2$. We further have $|z_1^* - z_2^*| \le |z_1 - z_1^*| + |z_2 - z_2^*| + |z_1 - z_2| < 2L_1\varepsilon' + \varepsilon' = \varepsilon \wedge \delta$. Therefore we obtain from (9.4) and (9.36) the following desired estimate

$$\begin{aligned}
|\Psi_{t_1}(z_1,\zeta_1) - \Psi_{t_2}(z_2,\zeta_2)| &\le |\Psi_{t_1}(z_1,\zeta_1) - \Psi_{t^*}(z_1^*,\zeta_1)| + |\Psi_{t_2}(z_2,\zeta_2) - \Psi_{t^*}(z_2^*,\zeta_2)| \\
&\quad + |\Psi_{t^*}(z_1^*,\zeta_1) - \Psi_{t^*}(z_2^*,\zeta_2)| \\
&< (2L^* + 1)\varepsilon,
\end{aligned}$$

where $L^* = L_{\overline{G_3},J}$.

□

Theorem 9.9 *The curve γ can be reparametrized in a way that $a_t = 2t,\ t \in [0,t_\gamma]$. Under this parametrization, $g_t(z)$ is differentiable in $t \in [0,t_\gamma]$ for each $z \in (D \cup \partial_p K) \setminus \gamma[0,t_\gamma]$ and satisfies the Komatu-Loewner differential equation*

$$\frac{\partial g_t(z)}{\partial t} = -2\pi \Psi_t(g_t(z), \xi(t)), \quad g_0(z) = z, \quad 0 < t \le t_\gamma. \tag{9.38}$$

Proof. The stated reparametrization of γ is possible by Theorem 8.4. Under this parametrization, the equation (6.17) in Theorem 6.4 is converted into (1.4). We notice that, in view of the proof of Theorem 6.4, equation (6.17) remains valid for any $z \in (D \cup \partial_p K) \setminus \gamma[0,t_\gamma]$. For a fixed $z \in (D \cup \partial_p K) \setminus \gamma[0,t_\gamma]$, $\Psi_t(g_t(z), \xi(t))$ is continuous in $t \in [0,t_\gamma]$ by virtue of Theorem 8.3, Theorem 8.6 and Theorem 9.8. Therefore (1.4) means that the left derivative of $g_t(z)$ in t is continuous on $(0,t_\gamma]$ possessing a limit at $t=0$, and accordingly $g_t(z)$ is differentiable in t on $(0,t_\gamma)$ and right differentiable at 0 as well (see e.g. [14, Lemma 4.3]). Equation (9.38) is established. □

10 Appendix 1: Proof of Theorem 7.1

We prove Theorem 7.1 through a series of lemmas. We consider a strip $\mathbb{H}_r = \{z = x+iy : 0 < y < r\}$ and take a large $r > 0$ so that $\mathbb{H}_r \supset F \cup K$. We put $D_r = \mathbb{H}_r \setminus K$. Then $D_r \setminus F = \mathbb{H}_r \setminus (F \cup K)$. Recall that $\Gamma_r := \{z = x + iy : \ y = r\}$.

Lemma 10.1 *Let*

$$v_{D_r\setminus F}(z) = \mathbb{P}_z^{\mathbb{H}} \left(\sigma_{\Gamma_r} < \sigma_{F\cup K}\right), \quad z \in D_r \setminus F. \tag{10.1}$$

Then the limit

$$v(z) = \lim_{r\to\infty} r\, v_{D_r\setminus F}(z), \quad z \in D, \tag{10.2}$$

exists and admits the expression (7.4).

Proof. Obviously

$$v_{\mathbb{H}_r}(z) = \mathbb{P}_z^{\mathbb{H}}(Z_{\sigma_{\Gamma_r}} < \infty) = \frac{y}{r}, \quad z = x + iy \in \mathbb{H}_r.$$

By the strong Markov property of $Z^{\mathbb{H}}$,

$$v_{\mathbb{H}_r}(z) = v_{D_r\setminus F}(z) + \mathbb{E}_z^{\mathbb{H}} \left[v_{\mathbb{H}_r}(Z^{\mathbb{H}}_{\sigma_{F\cup K}}); \sigma_{F\cup K} < \sigma_{\Gamma_r}\right]$$

Consequently the limit (10.2) exists and satisfies (7.4). □

Let $E = \mathbb{H} \setminus F$. We consider the BMD $Z^* = (Z_t^*, \mathbb{P}_z^*)$ on $E^* = (E \setminus K) \cup \{a_1^*, \cdots, a_N^*\}$ obtained from the ABM on E by rendering each compact continuum $A_j \subset E$ into a one point a_j^*, $1 \le j \le N$. As we have observed in §3.1, Z^* is nothing but the part process on E^* of the BMD $Z^{\mathbb{H},*}$ on $D^* = (\mathbb{H} \setminus K) \cup \{a_1, \cdots a_N^*\}$. In particular, ν_i and q_{ij}^* defined by (7.6) and (7.8) can be rewritten in terms of Z^* as

$$\nu_i(dz) = \mathbb{P}^*_{a_i^*} \left(Z^*_{\sigma_{\eta_i}} \in dz\right), \quad 1 \le i \le N, \tag{10.3}$$

$$q_{ij}^* = \begin{cases} \mathbb{P}^*_{a_i^*} \left(\sigma_{K^*} < \infty,\ Z^*_{\sigma_{K^*}} = a_j^*\right)/(1 - R_i^*) & \text{if } i \ne j, \\ 0 & \text{if } i = j, \end{cases} \quad 1 \le i, j \le N, \tag{10.4}$$

where R_i^*, $1 \le i \le N$, are defined in the same way as (7.7). Then $0 < R_i^* < 1$ for $1 \le i \le N$. Since

$$\sum_{j:j\ne i} q_{ij}^* = \frac{\int_{\eta_i} \mathbb{P}_z^*(\sigma_{K^*} < \infty)\nu_i(dz) - R_i^*}{1 - R_i^*} < 1, \quad 1 \le i \le N, \tag{10.5}$$

$I - Q^*$ admits the inverse M with positive entries

$$M_{ij} = \sum_{n=0}^{\infty} q_{ij}^{*n} \quad \text{where} \quad q_{ij}^{*n} = (Q^*)^n_{ij},\ 1 \le i, j \le N. \tag{10.6}$$

Lemma 10.2 *The limit $v^*(z)$ in* (7.2) *exists for $z \in D^* \setminus F$ and satisfies* (7.3) *with* (7.4) *and* (7.5).

Proof. Denote by ζ^* be the lifetime of the BMD Z^* on E^* and let

$$v_r^*(z) := \mathbb{P}_z^* \left(\sigma_{\Gamma_r} < \zeta^*\right), \quad z \in (D_r \setminus F) \cup K^*. \tag{10.7}$$

Then $v_r^*(z) = \mathbb{P}_z^{\mathbb{H},*}(\sigma_{\Gamma_r} < \sigma_F)$ and, for $z \in (D_r \setminus F) \cup K^*$,

$$v_r^*(z) = v_{D_r\setminus F}(z) + \sum_{j=1}^N \mathbb{P}_z^* \left(\sigma_{K^*} < \sigma_{\Gamma_r},\ Z^*_{\sigma_{K^*}} = a_j^*\right) v_r^*(a_j^*). \tag{10.8}$$

Since $v_r^*(a_i^*) = \int_{\eta_i} v_r^*(z)\,\nu_i(dz)$, and

$$\int_{\eta_i} \mathbb{P}_z^* \left(\sigma_{K^*} < \sigma_{\Gamma_r},\ Z^*_{\sigma_{K^*}} = a_j^*\right) \nu_i(dz) = \Pi_{ij}^{r*}, \quad j \neq i,$$

where $\Pi_{ij}^{r*} = \mathbb{P}^*_{a_i^*}\left(\sigma_{K^*} < \sigma_{\Gamma_r},\ Z^{r,*}_{\sigma_{K^*}} = a_j^*\right)$, we get by integrating the both hand sides of (10.8) with respect to $\nu_i(dz)$ that

$$v_r^*(a_i^*) = \int_{\eta_i} v_{D_r\setminus F}(z)\,\nu_i(dz) + \sum_{j\neq i} \Pi_{ij}^{r*} v_r^*(a_j^*) + R_i^{r*} v_r^*(a_i^*). \tag{10.9}$$

where R_i^{r*} is defined by (7.7) with $F \cup \Gamma_r$ in place of F. Define for $i \neq j$, $q_{ij}^{r*} := \Pi_{ij}^{r*}/(1 - R_i^{r*})$, and

$$f_r^*(a_i^*) := \frac{1}{1 - R_i^{r*}} \int_{\eta_i} v_{D_r\setminus F}(z)\,\nu_i(dz), \quad 1 \leq i \leq N.$$

Clearly $q_{ij}^{r*} > 0$ for $i \neq j$,

$$\sum_{j:j\neq i} q_{ij}^{r*} = \frac{\int_{\eta_i} \mathbb{P}_z^*(\sigma_{K^*} < \sigma_{\Gamma_r})\nu_i(dz) - R_i^{r*}}{1 - R_i^{r,*}} < 1, \quad 1 \leq i \leq N, \tag{10.10}$$

and the equation (10.9) can be rewritten as

$$v_r^*(a_i^*) - \sum_{j:j\neq i} q_{ij}^{r*} v_r^*(a_j^*) = f_r^*(a_i^*), \quad 1 \leq i \leq N. \tag{10.11}$$

In view of (10.10), the equation (10.11) admits a unique solution

$$v_r^*(a_i^*) = \sum_{j=1}^N M_{ij}^r\, f_r^*(a_j^*), \quad 1 \leq i \leq N. \tag{10.12}$$

Here $M_{ij}^r,\ 1 \leq i, j \leq N$, are the entries of the inverse of $I - Q^{r*}$ for the matrix Q^{r*} with off-diagonal elements $q_{ij}^{r*},\ i \neq j$ and zero diagonal elements.

Observe that, as $r \uparrow \infty$, R_i^{r*} for $1 \leq i \leq N$ and Π_{ij}^{r*} for $1 \leq i, j \leq N$ increase to R_i^* and Π_{ij}^*, respectively. Accordingly q_{ij}^{r*} and M_{ij}^{r*} increase to q_{ij}^* and M_{ij}^*, respectively. Furthermore by Lemma 10.1,

$$\lim_{r\to\infty} r f_r^*(a_i^*) = \frac{1}{1 - R_i^*} \int_{\eta_i} v(z)\,\nu_i(dz), \quad 1 \leq i \leq N.$$

Therefore we deduce from (10.12) that the limit $v^*(a_i^*) := \lim_{r\to\infty} r v_r^*(a_i^*)$ exists and satisfies (7.5). Moreover it follows from Lemma 10.1 and (10.8) that the limit $v^*(z) = \lim_{r\to\infty} r v_r^*(z)$ exists for every $z \in D$ and satisfies (7.3). □

The next lemma concerns the behaviors of $v^*(z)$ when $|z|$ gets large.

Lemma 10.3
$$v^*(x+iy) = y + o(1), \quad y \to \infty, \quad \text{uniformly in } x \in \mathbb{R}. \tag{10.13}$$
$$\lim_{y\to\infty} v_x^*(x+iy) = 0 \quad \text{uniformly in } x \in \mathbb{R}. \tag{10.14}$$
$$\lim_{x\to\pm\infty} |v^*(x+iy) - y| = 0, \quad \text{uniformly in } y > 0. \tag{10.15}$$
$$\lim_{x\to\pm\infty} |v_y^*(x+iy) - 1| = 0, \quad \text{uniformly in } y > 0. \tag{10.16}$$

Proof. (10.13) follows immediately from (7.3) and (7.4).

Take $r_0 > 0$ such that $\mathbb{H}_{r_0} \supset F \cup K$. Let $p(\xi, \eta) = \frac{1}{\pi}\frac{\eta}{\xi^2+\eta^2}$, $\eta > 0$, be the Poisson kernel of the upper half plane for Brownian motion. In view of (7.3) and (7.4), $v^*(z) - y$ is a bounded harmonic function in $\{z = x+iy : y > r_0\}$ and so $v^*(x+iy) = y + \int_{-\infty}^{\infty} p(x-\xi, y-r_0)(v^*(\xi+ir_0)-r_0)d\xi$, $y > r_0$. We then get (10.14) from
$$|p_x(x-\xi, y-r_0)| \le \frac{2}{(y-r_0)} p(x-\xi, y-r_0) \quad \text{for} \quad y > r_0.$$

Choose $\ell > 0$ such that $F \cup K \subset \{z = x+iy \in \mathbb{H} : |x| < \ell\}$. Let
$$\Lambda_\ell = \{z = x+iy \in \mathbb{H} : x = \ell\}.$$
Note that it follows from (7.3) and (7.4) that $h(z) := v^*(z) - y$ is a bounded harmonic function on $\{z = x+iy \in \mathbb{H} : |x| > \ell\}$vanishing continuously on the x-axis and admitting the expression
$$h(z) = \mathbb{E}_z\left[h(Z_{\sigma_{\Lambda_\ell}}); \sigma_{\Lambda_\ell} < \sigma_{\partial\mathbb{H}}\right], \quad |x| > \ell.$$
Combining this with (4.12), we get (10.15).

(10.16) follows from another expression of h given by (4.15) for ℓ in place of ℓ_1. □

The function v^* on D^* constructed in Lemma 7.3 is obviously Z^*-harmonic. In view of Corollary 3.6, $-v^*\big|_D$ admits a harmonic conjugate u^* on D uniquely up to an additive constant so that $f(z) = u^*(z) + iv^*(z)$, $z \in D$, is an analytic function.

Further u^* admits the expression as (6.4) in terms of v^* up to a real additive constant independently of a choice of a rectifiable curve C connecting z_0 with z. In particular, choosing $C = C_1 + C_2$ for a straight line segment C_1 (resp. C_2) connecting $z_0 = x_0 + iy_0$ (resp. $x + iy_0$) with $z = x + iy_0$ (resp. $z = x + iy$), we get
$$u^*(x+iy) - u^*(x_0+iy_0) = \int_{x_0}^{x} v_y^*(\xi + iy_0)d\xi - \int_{y_0}^{y} v_x^*(x+i\eta)d\eta, \quad y > 0. \tag{10.17}$$

The next lemma concerns behaviors of $f(z)$ as $|z|$ gets large.

Lemma 10.4 *It holds for each $x \in \mathbb{R}$,*
$$\lim_{y\to\infty} \frac{f(x+iy)}{x+iy} = 1. \tag{10.18}$$
Furthermore
$$\limsup_{z\to\infty, z\in D} \left|\frac{f(z)}{z}\right| \le 1. \tag{10.19}$$

Proof. The real part of $f(x+iy)/(x+iy)$ equals $[xu^*(x+iy)+yv^*(x+iy)]/(x^2+y^2)$. As $y \to \infty$, the first term of this sum goes to zero in view of (10.14) and (10.17), while the second term goes to 1 in view of (10.13). The imaginary part equals $[-yu^*(x+iy)+xv^*(x+iy)]/(x^2+y^2)$, which goes to zero as $y \to \infty$ by the analogous observation. By (10.13) there exists, for any $\varepsilon > 0$, $L > 0$ such that for any $y > L$ and any $x \in \mathbb{R}$, $\frac{v^*(x+iy)}{\sqrt{x^2+y^2}} \le \frac{v^*(x+iy)}{y} < 1+\varepsilon$, so that $\limsup_{z\to\infty, z\in D} \dfrac{v^*(z)}{|z|} \le 1+\varepsilon$.

On the other hand, (10.14) and (10.16) imply that, for any $\varepsilon > 0$, there exist $y_0 > 0$, $\ell > 0$ such that

$$|v_x^*(x+i\eta)| < \varepsilon \quad \text{for any } x \in \mathbb{R} \text{ and for any } \eta > y_0,$$

$$|v_y^*(\xi+iy_0)| < 1+\varepsilon \quad \text{for any } \xi \text{ with } |\xi| > \ell.$$

Let $M = \sup_{|\xi|\le\ell} |v_y^*(\xi+iy_0)|$. We get from (10.17) with $x_0 = 0$

$$\frac{|u^*(z)|}{|z|} \le \frac{|u^*(z_0)|}{|z|} + \frac{M\ell + |x|(1+\varepsilon)}{|z|} + \frac{|y-y_0|}{|z|}\varepsilon,$$

and so $\limsup_{y\to\infty} \sup_{x\in\mathbb{R}} \dfrac{|u^*(z)|}{|z|} \le 1+2\varepsilon$. Similarly we get $\limsup_{x\to\pm\infty} \sup_{y>0} \dfrac{|u^*(z)|}{|z|} \le 1$. □

Lemma 10.5 *u^* can be chosen uniquely in such a way that $f = u^* + iv^*$ satisfies* (7.9).

Proof. Since $v^* = 0$ on $\partial(\mathbb{H} \setminus \overline{F})$, by reflection principle, f extends to be an analytic function on $\mathbb{C} \setminus ((K \cup \overline{F}) \cup \pi(K \cup \overline{F}))$, which will still be denoted as f. Here $\pi(K \cup \overline{F})$ is the mirror reflection of $K \cup \overline{F}$ with respect to the x-axis.

Let $g(z) := zf(1/z)$. Then g is analytic on $\mathbb{C} \setminus \{\mathbf{0}\}$ and bounded near $\mathbf{0}$ by (10.19). So $\mathbf{0}$ is a removable singularity of $g(z)$ and $\lim_{z\to\mathbf{0}} g(z) = a_0$ exists. By (10.18) with $x = 0$, $\lim_{y\downarrow 0} g(-iy) = 1$ so that $a_0 = 1$ and g can be expanded near zero as $g(z) = 1 + a_1 z + a_2 z^2 + \cdots$. Therefore

$$f(z) = z + a_1 + \frac{a_2}{z} + \cdots \quad \text{near} \quad \infty. \tag{10.20}$$

Since $\Im f = 0$ on $\partial(\mathbb{H} \setminus \overline{F})$, $a_1, a_2, \cdots$, are real. f satisfies (7.9) if and only if $\lim_{z\to\infty}(f(z) - z) = 0$. The functions f and u^* are uniquely determined by v^* under this condition. □

The proof of Theorem 7.1 is now complete.

11 Appendix 2: Proper maps and the degree

Recall that a continuous map $f : X \to Y$ between topological spaces is called *proper* if pre-images of compact sets are compact. Intuitively, if X and Y are subsets of larger spaces, this means that the boundary of X maps into the boundary of Y (though it not required that f extends continuously to the boundary). In rather general situations (for instance smooth orientation preserving maps between manifolds with boundary, see e.g. [15]), such maps have the property that every $y \in Y$ has the same number of pre-images $x \in X$ (counted according to multiplicity). In this section, we formulate and prove a simple version of this principle, in the setting of analytic functions that is suitable for our purpose. This is certainly standard, but lacking a reference we provide the details for the sake of completeness. We allow ∞ to be in the domain and in the range of f and adopt the usual definition that a function f (defined in a neighborhood of ∞ with $f(\infty) = \infty$) is *analytic* if $1/f(1/z)$ is analytic in a neighborhood of 0.

Lemma 11.1 *Let D_1, D_2 be connected open subsets of the Riemann sphere $\overline{\mathbb{C}}$ and f analytic in D_1. If f is a proper map between D_1 and D_2 (that is, if $f(D_1) \subset D_2$ and if $f^{-1}(K)$ is a compact subset of D_1 whenever K is a compact subset of D_2), then there is a finite number d such that every $w \in D_2$ has precisely d pre-images in D_1, counting multiplicity.*

Proof. Fix $w_0, w_1 \in D_2$. In order to show that they have the same number of pre-images under f, we first assume that neither w_0 nor w_1 are critical values (that is, we assume $f' \neq 0$ for every pre-image of w_0 and w_1). Then there is a simple curve $\gamma \subset D_2$ joining w_0 and w_1 that is disjoint from the set of critical values. For every pre-image z_0 of w_0, the branch g_0 of the inverse function f^{-1} with $g_0(w_0) = z_0$, defined in a neighborhood of w_0, can be analytically continued along γ and yields a branch g_1 of f^{-1} near w_1.

This is where both the assumption on the critical values, and the assumption on properness are used: During the process of analytic continuation, the curve $f^{-1}(\gamma(t))$ cannot escape from D_1 by properness, and one can always analytically continue further because one does not meet critical values. Formally, one considers the subset $S = \{s \in [0,1] : g_0$ can be analytically continued along $\gamma[0,s]\}$ of the interval $[0,1]$ and shows that it is both open and closed.

Conversely, continuing g_1 along the reversed curve γ^{-1} leads us back to g_0. Thus we have a bijection between the sets $f^{-1}(w_0)$ and $f^{-1}(w_1)$. If a pre-image z_0 (or z_1) of w_0 (or w_1) is a critical point (so that the local degree of f at z_0 is more than 1), simply replace w_0 (or w_1) by nearby points w_0' or w_1' and use the fact that the number of pre-images of w_0' near z_0 equals the local degree of f at z_0. □

Now we will show that the assumption $f(D_1) \subset D_2$ can be removed if the degree is one and if the complement of D_2 has empty interior. For a function f and an open set $D \subset \overline{\mathbb{C}}$, we define by

$$f(\partial D) = \bigcap_{K \Subset D} \overline{f(D \setminus K)} \tag{11.1}$$

the set of limit points of f as z approaches ∂D (the intersection is over all compact subsets of D). It is easy to see that any proper map f from D_1 onto D_2 satisfies $f(\partial D_1) = \partial D_2$. The next theorem goes in the opposite direction. Notice that we do not assume a priori that $f(D_1) \subset D_2$.

Theorem 11.2 *Let D_1, D_2 be connected open subsets of the Riemann sphere $\overline{\mathbb{C}}$ and f analytic in D_1. Assume that the complement of D_2 has empty interior, that*

$$f(\partial D_1) = \partial D_2, \tag{11.2}$$

and that there is one point $w_0 \in D_2$ that has precisely one pre-image z_0 under f (counting multiplicity). Then f is a conformal map from D_1 onto D_2.

Proof. Let $D \subset D_1$ be a connected component of $f^{-1}(D_2)$. Then $f : D \to D_2$ is proper.

To see this, suppose K is a compact subset of D_2 and $z_n \in f^{-1}(K) \cap D$. We need to show that every subsequential limit of z_n is in D. Assume to the contrary that z_n converges to a point $z \in \partial D$. Then if $z \in D_1$, it follows that $f(z) \in D_2$ and thus a neighborhood of z belongs to D, a contradiction. If $z \in \partial D_1$, then every limit point of $\{f(z_n)\}$ belongs to $f(\partial D_1) = \partial D_2$, which contradicts the assumption that K is a compact subset of D_2.

Now Lemma 11.1 implies that every $w \in D_2$ has the same number of pre-images in D. This number is trivially at least one, and it is at most one by assumption on w_0. Thus the degree of f on D equals one, and there is only one such component D. In particular, f is a conformal bijection between D and D_2. It remains to show that $D = D_1$. If not, there is a point $z_1 \in D_1 \cap \partial D$, and it follows that $w_1 = f(z_1) \in \partial D_2$. By the assumption $f(\partial D_1) = \partial D_2$, there is a sequence $z_n \in D_1$ converging to a boundary point such that $f(z_n) \to w_1$. Since f is an open mapping and since the complement of D_2 has empty interior, there are $z_n' \in D_1$ so that $f(z_n') \to w_1$ and that $f(z_n') \in D_2$, in particular $z_n' \in D$. Since the equation $f(z) = f(z_n')$ has another solution near z_1, we obtain a contradiction to the injectivity of f on D. □

Notice that the last claim $D = D_1$ is not true if the degree is more than one, as for instance the map $f(z) = z^2$ with $D_1 = D_2 = \mathbb{C} \setminus [0, 1]$ shows: Here $D = \mathbb{C} \setminus [-1, 1]$.

12 Appendix 3: Proof of Proposition 9.3

The assertions (i) and (ii) of Proposition 9.3 will be proved by the next lemma.

Lemma 12.1 *$g(z, w, \varepsilon)$ is a fundamental solution of A_ε in the sense of* (9.10). *The coefficients $A^{(\varepsilon)}_{k\ell}$ of A_ε admit expressions*

$$A^{(\varepsilon)}_{k\ell} = \frac{1}{2}\delta_{k\ell} + \varepsilon b^{(\varepsilon)}_{k\ell}, \quad 1 \le k, \ell \le 2, \tag{12.1}$$

where $b^{(\varepsilon)}_{k\ell}$, $1 \le k, \ell \le 2$, are smooth functions on $\mathbb{H}$ with $b^{(\varepsilon)}_{k\ell} = b^{(\varepsilon)}_{\ell k}$ vanishing on $(\mathbb{H} \setminus \bigcup_{i=1}^N \overline{V}_i) \cup (\bigcup_{i=1}^N U_i)$ which together with their derivatives are uniformly bounded in $\varepsilon \in (0, \varepsilon_0)$, $D \in \mathcal{D}_0$ and $\widetilde{D} = \widetilde{f}_\varepsilon(D) \in \mathcal{D}$.

Proof. Denote the Jacobian determinant $\frac{\partial(\widetilde{x}_1, \widetilde{x}_2)}{\partial(x_1, x_2)}$ of the map $\widetilde{f}_\varepsilon$ by $J = J(x_1, x_2)$. Let $\widetilde{\mathcal{E}}$ and $\widetilde{A}$ be the Dirichlet form and the L^2-generator on $L^2(\widetilde{D})$ of the ABM on $\widetilde{D}$, respectively. Then $\widetilde{u} \in \mathcal{D}(\widetilde{A})$ and $\widetilde{A}\widetilde{u} = \widetilde{f} \in L^2(\widetilde{D})$ if and only if $\widetilde{u} \in H^1_0(\widetilde{D})$ and $\widetilde{\mathcal{E}}(\widetilde{u}, \widetilde{v}) = -\frac{1}{2}\int_{\widetilde{D}} \widetilde{f}\widetilde{v} d\widetilde{x}_1 d\widetilde{x}_2$. It follows from

$$\begin{aligned} \widetilde{\mathcal{E}}(\widetilde{u}, \widetilde{v}) &= \frac{1}{2}\int_{\widetilde{D}} \sum_{j=1}^2 \frac{\partial \widetilde{v}}{\partial \widetilde{x}_j}\frac{\partial \widetilde{u}}{\partial \widetilde{x}_j} d\widetilde{x}_1 d\widetilde{x}_2 \\ &= \frac{1}{2}\int_D \sum_{j=1}^2 \sum_{k,\ell=1}^2 \frac{\partial \widetilde{v}}{\partial x_k}\frac{\partial x_k}{\partial \widetilde{x}_j}\frac{\partial \widetilde{u}}{\partial x_\ell}\frac{\partial x_\ell}{\partial \widetilde{x}_j} J(x_1, x_2) dx_1 dx_2 \\ &= -\int_D \widetilde{v}(x_1, x_2) A_\varepsilon \widetilde{u}(x_1, x_2) dx_1 dx_2 \end{aligned} \tag{12.2}$$

and $\int_{\widetilde{D}} \widetilde{f}\widetilde{v} d\widetilde{x}_1 d\widetilde{x}_2 = \int_D \widetilde{f}(x_1, x_2)\widetilde{v}(x_1, x_2) J(x_1, x_2) dx_1 dx_2$ for every $\widetilde{v} \in C^1_c(\widetilde{D})$ that

$$(\widetilde{A}\widetilde{u})(x_1, x_2) = J^{-1} A_\varepsilon \widetilde{u}(x_1, x_2). \tag{12.3}$$

On the other hand, if we define $\widetilde{G}\widetilde{f}(\widetilde{z}) = \int_{\widetilde{D}} \widetilde{G}(\widetilde{z}, \widetilde{w})\widetilde{f}(\widetilde{w}) d\widetilde{w}$, then

$$\widetilde{G}\widetilde{f}(\widetilde{z}) = \int_D g(z, w, \varepsilon)(\widetilde{f}J)(w) dw_1 dw_2 =: g_\varepsilon(\widetilde{f}J)(z).$$

Since $\widetilde{A}(\widetilde{G}\widetilde{f})(\widetilde{z}) = -\widetilde{f}(\widetilde{z})$, we have by (12.3) that $J^{-1} A_\varepsilon \cdot g_\varepsilon(\widetilde{f}J)(z) = -\widetilde{f}(z)$. This establishes (9.10) by taking $f = \widetilde{f}J$.

The stated expression and properties of coefficients of A_ε follow from (9.7), (9.9) and the uniform boundedness of the coefficients of the linear map (9.6). In particular, $b_{k\ell}^{(\varepsilon)}$, $1 \le k, \ell \le N$, vanish on $\mathbb{H} \setminus \bigcup_{i=1}^N \overline{V}_i$ because $\widetilde{f}_\varepsilon$ is an identity map there, and on each U_i as well because it is an analytic (actually linear) map there. □

To derive the perturbation formulae (9.12) and (9.13), we first construct an appropriate parametrix for the elliptic differential operator A_ε by following an interior variation method presented in section 15.1 of Garabedian's book [10].

Denote by $a^{(\varepsilon)} = (a_{k\ell}^{(\varepsilon)})_{1\le k,\ell\le 2}$ the inverse matrix of $A^{(\varepsilon)} = (A_{k\ell}^{(\varepsilon)})_{1\le k,\ell\le 2}$. Since $\det A^{(\varepsilon)} = \frac{1}{4}$, we have

$$a_{11}^{(\varepsilon)} = 2 + 4\varepsilon b_{22}^{(\varepsilon)}, \quad a_{22}^{(\varepsilon)} = 2 + 4\varepsilon b_{11}^{(\varepsilon)}, \quad a_{12}^{(\varepsilon)} = a_{21}^{(\varepsilon)} = -4\varepsilon b_{12}^{(\varepsilon)}. \tag{12.4}$$

Define

$$\Gamma(z,\zeta) = \frac{1}{2}\sum_{i,j=1}^{2} a_{ij}^{(\varepsilon)}(\zeta)(x_i - \zeta_i)(x_l - \zeta_j), \quad z = x_1 + ix_2,\ \zeta = \zeta_1 + i\zeta_2. \tag{12.5}$$

$-\frac{1}{2\pi}\log\Gamma(z,\zeta)$ has the same singularity as the fundamental solution $g(z,\zeta,\varepsilon)$ of the elliptic differential operator A_ε (cf. [10, (5.80)]).

Recall the constant b_0 defined in (9.31). We fix an arbitrary $\ell_0 \in (0, b_0]$ and consider a smooth non-positive real function $\alpha(t)$, $t \in \mathbb{R}$, with

$$\alpha(0) = -1/(2\pi), \quad \alpha(t) = 0 \quad \text{if} \quad t \notin (-\ell_0^2, \ell_0^2). \tag{12.6}$$

Let

$$\begin{cases} P_\varepsilon(z,\zeta) = \alpha(|z-\zeta|^2)\log\Gamma(z,\zeta), \\ P_0(z,\zeta) = \alpha(|z-\zeta|^2)\log|z-\zeta|^2, \quad z,\zeta \in \overline{\mathbb{H}},\ z \ne \zeta. \end{cases} \tag{12.7}$$

For a function $u(z,\zeta,\varepsilon)$, $z,\zeta \in D \cup \partial\mathbb{H}$, $z \ne \zeta$, $\varepsilon \in (0,\varepsilon_0)$, we write $u(z,\zeta,\varepsilon) = O(\varepsilon/r)$, $r = |z-\zeta|$ if

$$|u(z,\zeta,\varepsilon)| \le \varepsilon\frac{M_1}{|z-\zeta|} + \varepsilon M_2, \quad z,\ \zeta \in D \cup \partial\mathbb{H}, \tag{12.8}$$

for positive constants M_1, M_2 independent of $\varepsilon \in (0,\varepsilon_0)$, $D \in \mathcal{D}_0$ and $\widetilde{D} = \widetilde{f}_\varepsilon(D) \in \mathcal{D}$.

In what follows, the set $\bigcup_{i=1}^N (\overline{V}_i \setminus U_i)$ will be denoted by F.

Lemma 12.2 *$G(z,\zeta) - P_0(z,\zeta) + P_\varepsilon(z,\zeta)$ is a parametrix of the operator $A_\varepsilon = A_{\varepsilon,z}$ in a specific sense that*

$$A_{\varepsilon,z}(G(z,\zeta) - P_0(z,\zeta) + P_\varepsilon(z,\zeta)) = O(\varepsilon/r), \quad r = |z-\zeta|. \tag{12.9}$$

Proof. It suffices to show that

$$A_\varepsilon P_\varepsilon(z,\zeta) - \frac{1}{2}\Delta P_0 = O(\varepsilon/r), \quad r = |z-\zeta|, \tag{12.10}$$

$$(A_\varepsilon - \frac{1}{2}\Delta)(G - P_0) = O(\varepsilon/r), \quad r = |z-\zeta|. \tag{12.11}$$

Note that they imply (12.9) because $\Delta_z G(z,\zeta) = 0$ for $z \ne \zeta$. Designating $A_{ij}^{(\varepsilon)}$ by A_{ij}, the left hand side of (12.10) can be rewritten as $I_\varepsilon + II_\varepsilon + III_\varepsilon + IV_\varepsilon + V_\varepsilon$ with

$$\begin{cases} I_\varepsilon = \alpha(|z-\zeta|^2)\sum_{ij=1}^2 A_{ij}(z)(\log\Gamma(z,\zeta))_{x_ix_j} \\ II_\varepsilon = \sum_{i,j=1}^2 A_{ij,x_i}(z)[\alpha(|z-\zeta|^2)\log\Gamma(z,\zeta)]_{x_j} \\ III_\varepsilon = \sum_{i=1}^2 \alpha(|z-\zeta|^2)_{x_ix_i}[A_{ii}(z)\log\Gamma(z,\zeta) - \frac{1}{2}\log|z-\zeta|^2] \\ IV_\varepsilon = 4\alpha'(|z-\zeta|^2)[\sum_{i,j=1}^2 A_{ij}(z)(x_i-\zeta_i)(\log\Gamma(z,\zeta))_{x_j} - 1] \\ V_\varepsilon = \sum_{i\ne j} A_{ij}(z)\alpha(|z-\zeta|^2)_{x_ix_j}\log\Gamma(z,\zeta), \end{cases}$$

where the function $\Gamma(z,\zeta)$ is defined by (12.5). The sum $\sum_{ij=1}^2 A_{ij}(z)(\log\Gamma(z,\zeta))_{x_ix_j}$ equals

$$\frac{\sum_{i,j=1}^2 A_{ij}(z)a_{ij}(\zeta)\Gamma(z,\zeta)-\sum_{i,j=1}^2 A_{ij}(z)\Gamma_{x_i}(z,\zeta)\Gamma_{x_j}(z,\zeta)}{\Gamma(z,\zeta)^2}. \tag{12.12}$$

Since

$$A_{ij}(z)=A_{ij}(\zeta)+\varepsilon\sum_{k=1}^2 C_{i,j,k}(z,\zeta)(x_k-\zeta_k), \tag{12.13}$$

for $C_{i,j,k}$ involving only derivatives of b_{ij}, and

$$\Gamma(z,\zeta)=|z-\zeta|^2(1+\varepsilon\eta_1(z,\zeta)) \tag{12.14}$$

for η_1 uniformly bounded in $\varepsilon>0$, $D\in\mathcal{D}_0$ and $\widetilde{f}_\varepsilon(D)$ on account of (12.4), we see that (12.12) can be written as

$$\varepsilon\frac{\eta_2(z,\zeta)|z-\zeta|\Gamma(z,\zeta)-\eta_3(z,\zeta)|z-\zeta|^3}{\Gamma(z,\zeta)^2}=\varepsilon\eta_4(z,\zeta)\frac{1}{|z-\zeta|}$$

for $\eta_k,\ 2\le k\le 4$, uniformly bounded in $\varepsilon>0,\ D\in\mathcal{D}_0$ and $\widetilde{f}_\varepsilon(D)$ yielding that $I_\varepsilon=O(\varepsilon/r)$.

All other terms II_ε, III_ε, IV_ε and V_ε can also be verified to satisfy (12.8) on account of (12.13), (12.14), Lemma 12.1 and (12.4), yielding (12.10).

By Lemma 12.1, $A_\varepsilon-\frac{1}{2}\Delta$ is equal to $\varepsilon B^{(\varepsilon)}$ for the differential operator $B^{(\varepsilon)}$ defined by (9.11). Using (4.7) and (12.24) below, the left hand side of (12.11) can be written as $\varepsilon B_z^{(\varepsilon)}u(z,\zeta)-\varepsilon B_z^{(\varepsilon)}S(z,\zeta)$ for

$$u(z,\zeta)=\frac{1}{2\pi}\log|z-\bar{\zeta}|^2-\left(\alpha(|z-\zeta|^2)+\frac{1}{2\pi}\right)\log|z-\zeta|^2,$$

and for the function $S(z,\zeta)$ defined by (12.24). By taking (12.6) into account, we can readily verify that $\epsilon B_z^{(\varepsilon)}u(z,\zeta)=O(\varepsilon/r)$. $S(\zeta,z)$ depends on D. Since the coefficients of $B^{(\varepsilon)}$ are supported by F however,

$$|B_z^{(\varepsilon)}S(\zeta,z)|\le M\mathbb{P}_\zeta^{\mathbb{H}}(\sigma_K<\infty)\le M\mathbb{P}_\zeta^{\mathbb{H}}(\sigma_{\cup_{j=1}^N\overline{V}_j}<\infty),\quad \zeta\in\mathbb{H},$$

where $M=\sup\{|B_z^{(\varepsilon)}G^{\mathbb{H}}(w,z)|: z\in F,\ w\in K,\ \varepsilon\in(0,\varepsilon_0),\ D\in\mathcal{D}_0,\ \widetilde{D}\in\mathcal{D}\}$ which is finite by (9.2) and Lemma 12.1. Hence $\varepsilon B_z^{(\varepsilon)}S(\zeta,z)=O(\varepsilon/r)$. □

The ℓ_0-neighborhood of F will be denoted by W_{ℓ_0}.

Lemma 12.3 *For any $w,\zeta\in D,\ w\neq\zeta$, it holds that*

$$\begin{aligned} g(\zeta,w,\varepsilon) &= G(w,\zeta)-P_0(w,\zeta)+P_\varepsilon(w,\zeta)\\ &\quad+\int_{W_{\ell_0}} g(z,w,\varepsilon)A_{\varepsilon,z}[G-P_0+P_\varepsilon](z,\zeta)dx_1dx_2. \end{aligned} \tag{12.15}$$

Proof. According to [10, (15.14)], the self-adjoint elliptic differential operator A_ε admits Green's second formula

$$\int_E(vA_\varepsilon u-uA_\varepsilon v)dx_idx_2=\int_{\partial E}\Lambda_\varepsilon[u,v]ds, \tag{12.16}$$

where E is a bounded domain in $\mathbb{H}$ with smooth boundary ∂E and

$$\Lambda_\varepsilon[u,v]=\sum_{k,\ell=1}^2 A_{k\ell}^{(\varepsilon)}\left(v\frac{\partial u}{\partial x_k}\frac{\partial x_\ell}{\partial\mathbf{n}}-u\frac{\partial v}{\partial x_k}\frac{\partial x_\ell}{\partial\mathbf{n}}\right). \tag{12.17}$$

Here $\mathbf{n}$ is the unit outward normal at ∂E.

We fix $w, \zeta \in D,\ w \neq \zeta$. We then take a large $\ell > 0$ such that the rectangle $R_\ell = \{x_1 + ix_2 \in \mathbb{H} : |x_1| < \ell,\ 0 < x_2 < \ell\}$ contains the points $w,\ \zeta$ and the set W_{ℓ_0} as well. For each $1 \le i \le N$, we choose a smooth Jordan curve γ_i surrounding C_i in such a way that $\gamma_i \subset U_i \setminus \overline{W}_{\ell_0}$ and that its enclosure does not contain w nor ζ. We apply the identity (12.16) to

$$\begin{cases} E = R_\ell \setminus (\bigcup_{i=1}^N \mathrm{ins}\gamma_i) \setminus \overline{B_\delta(w)} \setminus \overline{B_\delta(\zeta)}, \\ u(z) = G(z,\zeta) - P_0(z,\zeta) + P_\varepsilon(z,\zeta), \\ v(z) = g(z,w,\varepsilon). \end{cases}$$

for a sufficiently small $\delta > 0$.

By Lemma 12.1, $A_\varepsilon v = 0$ on E. We can further observe the implication

$$z \in E \setminus W_{\ell_0} \implies u(z) = G(z,\zeta), \quad A_\varepsilon u(z) = 0. \tag{12.18}$$

Indeed, if $\zeta \in F,\ z \in \mathbb{H}\setminus W_{\ell_0}$, then $|z-\zeta| > \ell_0$ so that $\alpha(|z-\zeta|^2) = 0$ and $u(z) = G(z,\zeta)$. If $\zeta \in \mathbb{H}\setminus F$, then by Lemma 12.1 and (12.4) $a_{ij}^{(\varepsilon)}(\zeta) = 2\delta_{ij}$ so that $P_\varepsilon(z,\zeta) = P_0(z,\zeta)$ and $u(z) = G(z,\zeta)$. Since $A_{k\ell}^{(\varepsilon)} = \frac{1}{2}\delta_{k\ell}$ on $\mathbb{H} \setminus F$ in view of Lemma 12.1, we obtain from (12.16) and (12.17) the identity

$$\begin{aligned} \int_{W_{\ell_0}} v(z) A_\varepsilon u(z) dx_1 dx_2 \;&=\; \frac{1}{2}\int_{\Sigma_\ell}\left(v\frac{\partial u}{\partial \mathbf{n}} - \frac{\partial v}{\partial \mathbf{n}}u\right)ds + \frac{1}{2}\sum_{i=1}^N \int_{\gamma_i}\left(v\frac{\partial u}{\partial \mathbf{n}} - \frac{\partial v}{\partial \mathbf{n}}u\right)ds \\ &\quad + \int_{\partial B_\delta(w)} \Lambda_\varepsilon[u,v]ds + \int_{\partial B_\delta(\zeta)} \Lambda_\varepsilon[u,v]ds, \end{aligned} \tag{12.19}$$

where $\Sigma_\ell = \partial R_\ell \setminus \partial\mathbb{H}$.

Along Σ_ℓ, $u(z) = G(z,\zeta)$, $v(z) = g(z,w,\varepsilon)$, and both are harmonic and converge to 0 when $|z| \to \infty$ as they are dominated by $G^{\mathbb{H}}$ of (4.7). Hence we can use Lemma 4.3 to conclude that the first integral on the right hand side of (12.19) tends to 0 as $\ell \to \infty$. Along γ_i, we again have $u(z) = G(z,\zeta)$, $v(z) = g(z,w,\varepsilon)$, and both are harmonic and converge to 0 as $z \to C_i$, $1 \le i \le N$. Therefore, in the same way as in the proof of Theorem 4.5, the second term on the right hand side of (12.19) can be seen to converge to 0 as we take γ_i to be the level curve of u shrinking to C_i.

Recall the function $\Gamma(z,w)$ defined in (12.5). As for the third term, we can replace $v(z) = g(z,w,\varepsilon)$ by $\widehat{g}(z,w) = -\frac{1}{2\pi}\log\Gamma(z,w)$ of the same singularity at w to conclude that it tends to $-u(w) = -G(w,\zeta) + P_0(w,\zeta) - P_\varepsilon(w,\zeta)$ as $\delta \downarrow 0$. As for the forth term, replacing $u(z) = G(z,\zeta) - P_0(z,\zeta) + P_\varepsilon(z,\zeta)$ by $\widehat{g}(z,\zeta)$ of the same singularity at ζ, we see that it tends to $v(\zeta) = g(\zeta,w,\varepsilon)$ as $\delta \downarrow 0$. □

If we write

$$\begin{cases} \widehat{K}_\varepsilon(\zeta,z) = A_{\varepsilon,z}[G - P_0 + P_\varepsilon](z,\zeta) \\ G_\varepsilon(\zeta,w) = G(w,\zeta) - P_0(w,\zeta) + P_\varepsilon(w,\zeta), \end{cases}$$

then the identity (12.15) is converted into a Fredholm type integral equation:

$$g(\zeta,w,\varepsilon) = G_\varepsilon(\zeta,w) + \int_{W_{\ell_0}} \widehat{K}_\varepsilon(\zeta,z) g(z,w,\varepsilon) dx_1 dx_2. \tag{12.20}$$

In view of (12.4) and (12.7), we have

$$|P_\varepsilon(w,\zeta) - P_0(w,\zeta)| \le -2\varepsilon\alpha(|w-\zeta|^2)\left(|b_{11}^{(\varepsilon)}(\zeta)| + |b_{22}^{(\varepsilon)}(\zeta)| + |b_{12}^{(\varepsilon)}(\zeta)|\right)$$

so that

$$P_\varepsilon(w,\zeta) - P_0(w,\zeta) = \varepsilon\eta_1^{(\varepsilon)}(w,\zeta), \quad w,\ \zeta \in \overline{\mathbb{H}}, \tag{12.21}$$

where $\eta_1^{(\varepsilon)}(w,\zeta)$ a continuous function on $\overline{\mathbb{H}} \times \overline{\mathbb{H}}$ bounded uniformly in $\varepsilon \in (0,\varepsilon_0)$, $D \in \mathcal{D}_0$ and $\widetilde{f}_\varepsilon(D)$.

For a function $u(\zeta,w)$, $\zeta, w \in D \cup \partial\mathbb{H}$, we let $\|u\|_\infty = \sup_{\zeta,w\in D\cup\partial\mathbb{H}} |u(\zeta,w)|$. If we write $(\widehat{K}_\varepsilon G_\varepsilon)(\zeta,w) = \int_{W_{\ell_0}} \widehat{K}_\varepsilon(\zeta,z)G_\varepsilon(z,w)dx_1dx_2$, then $|(\widehat{K}_\varepsilon G_\varepsilon)(\zeta,w)| \le \int_{W_{\ell_0}} |\widehat{K}_\varepsilon(\zeta,z)|(G^{\mathbb{H}}(z,w) + |P_\varepsilon(z,w) - P_0(z,w)|)dx_1dx_2$, and we have from Lemma 12.2 and (12.21)

$$\|\widehat{K}_\varepsilon G_\varepsilon\|_\infty \le \varepsilon\, C_1 \quad \text{for } C_1 \text{ independent of } \varepsilon \in (0,\varepsilon_0),\ D \in \mathcal{D}_0 \text{ and } \widetilde{f}_\varepsilon(D).$$

From Lemma 12.2, we also have, for a constant $C_2 > 0$ independent of $\varepsilon \in (0,\varepsilon_0)$, $D \in \mathcal{D}_0$ and $\widetilde{f}_\varepsilon(D)$, $\int_{W_{\ell_0}} |\widehat{K}_\varepsilon(\zeta,z)|dx_1dx_2 \le \varepsilon C_2$ for any $\zeta \in D \cup \partial\mathbb{H}$. Hence $\|\widehat{K}_\varepsilon^{(2)} G_\varepsilon\|_\infty \le \varepsilon^2 C_1C_2$ for $\widehat{K}_\varepsilon^{(2)}G_\varepsilon(\zeta,w) = \widehat{K}_\varepsilon(\widehat{K}_\varepsilon G_\varepsilon)(\zeta,w)$. Similarly, we have $\|\widehat{K}_\varepsilon^{(n)}G_\varepsilon\|_\infty \le \varepsilon^n C_1 C_2^{n-1}$ for every $n \ge 1$.

Denote $\varepsilon_0 \wedge (1/(2C_2))$ by $\widehat{\varepsilon}_0$. For $0 < \varepsilon < \widehat{\varepsilon}_0$, $\sum_{n=1}^\infty \|\widehat{K}_\varepsilon^{(n)}G_\varepsilon\|_\infty \le \sum_{n=1}^\infty \varepsilon^n C_1 C_2^{n-1} < 2\varepsilon C_1$ and so the convergence

$$\sum_{n=1}^\infty \widehat{K}_\varepsilon^{(n)} G_\varepsilon(\zeta,w) = \varepsilon\eta_2^{(\varepsilon)}(\zeta,w) \tag{12.22}$$

is uniform on $D\cup\partial\mathbb{H} \times D \cup \partial\mathbb{H}$, where $\varepsilon\eta_2^{(\varepsilon)}(\zeta,w)$ is a continuous function there that is uniformly bounded in $\varepsilon \in (0,\widehat{\varepsilon}_0)$, $D \in \mathcal{D}_0$ and $\widetilde{f}_\varepsilon(D)$. Moreover, using the bound $g(z,w,\varepsilon) \le G^{\mathbb{H}}(\widetilde{f}_\varepsilon(z), \widetilde{f}_\varepsilon(w))$, the second term of one can check that the right hand side of (12.20) is bounded in $\zeta, w \in D$ for any $\varepsilon \in (0,\widetilde{\varepsilon}_0)$ for some $\widetilde{\varepsilon}_0 \in (0,\widehat{\varepsilon}_0]$.

Solving the equation (12.20) for $g(\zeta,w,\varepsilon)$ and setting $\eta^{(\varepsilon)}(\zeta,w) = \eta_1^{(\varepsilon)}(w,\zeta) + \eta_2^{(\varepsilon)}(\zeta,w)$, we get the following from (12.21) and (12.22).

Lemma 12.4 *For any $\zeta,\ w \in D,\ \zeta \neq w$,*

$$g(\zeta,w,\varepsilon) = G(\zeta,w) + \varepsilon\eta^{(\varepsilon)}(\zeta,w), \quad \varepsilon \in (0,\widetilde{\varepsilon}_0), \tag{12.23}$$

where $\eta^{(\varepsilon)}(\zeta,w)$ is a continuous function on $D\cup\partial\mathbb{H}\times D\cup\partial\mathbb{H}$ bounded there uniformly in $\varepsilon \in (0,\widetilde{\varepsilon}_0)$, $D \in \mathcal{D}_0$ and $\widetilde{D} = \widetilde{f}_\varepsilon(D) \in \mathcal{D}$.

Let us take $\zeta \in \mathbb{H} \setminus F$ in (12.15). By noting that $P_\varepsilon(z,\zeta) = P_0(z,\zeta)$ and letting $\ell_0 \downarrow 0$, we arrive at (9.12) as $A_{\varepsilon,z}G(z,\zeta) = \varepsilon B_z^{(\varepsilon)}G(z,\zeta)$, $z \in F$. By substituting (12.23) into (9.12), we obtain (9.13). The proof for assertions (iii) and (iv) of Proposition 9.3 is now complete.

We finally prove the assertions (v)-(viii) of Proposition 9.3.

(v): G can be expressed by the Green function $G^{\mathbb{H}}$ of the ABM $Z^{\mathbb{H}}$ as

$$G(\zeta,z) = G^{\mathbb{H}}(\zeta,z) - S(\zeta,z) \quad \text{where} \quad S(\zeta,z) = \mathbb{E}_\zeta^{\mathbb{H}}\left[G^{\mathbb{H}}(Z_{\sigma_K}^{\mathbb{H}}, z); \sigma_K < \infty\right]. \tag{12.24}$$

By the expression (4.7) of $G^{\mathbb{H}}$ and Proposition 9.3(ii), $B_z^{(\varepsilon)}G^{\mathbb{H}}(z,\zeta)_{\zeta_2}$ is uniformly bounded on $F \times J$ in $\varepsilon \in (0,\varepsilon_0)$, $D \in \mathcal{D}_0$ and $\widetilde{f}_\varepsilon(D)$. Furthermore, for any $\varepsilon \in (0,\varepsilon_0)$, $z \in F$ and $\zeta \in \mathbb{H} \setminus \bigcup_{j=1}^N V_j$,

$$M = \sup_{\varepsilon\in(0,\varepsilon_0), w\in K, \xi\in F, D\in\mathcal{D}_0, \widetilde{D}\in\mathcal{D}} |B_z^{(\varepsilon)}G^{\mathbb{H}}(w,\xi)| < \infty \quad \text{and} \quad |B_z^{(\varepsilon)}S(z,\zeta)| \le M\, \mathbb{P}_\zeta^{\mathbb{H}}(\sigma_{\overline{U}} < \infty),$$

where $U = \bigcup_{j=1}^N U_j$. As was observed in the proof of Lemma 5.2(ii), $\mathbb{P}_\zeta^{\mathbb{H}}(\sigma_{\overline{U}} < \infty) = \int_{\overline{U}} G^{\mathbb{H}}(\zeta, w)\mu(dw)$ for a finite measure μ concentrated on $\overline{U}$. Since both $B_z^{(\varepsilon)} S(z, \zeta)$ and $\mathbb{P}_\zeta^{\mathbb{H}}(\sigma_{\overline{U}} < \infty)$ vanish when $\zeta \in \partial\mathbb{H}$, we deduce from above that

$$\left| B_z^{(\varepsilon)} \frac{\partial}{\partial \zeta_2} S(z, \zeta) \right| \leq M \sup_{w \in \overline{U},\, \zeta \in J} \frac{\partial}{\partial \zeta_2} G^{\mathbb{H}}(\zeta, w) \cdot \mu(\overline{U}), \quad \zeta \in J.$$

(vi): Let $G^{\mathbb{H},i}(z, \zeta)$ be the Green function of the ABM on $D^i = D \cup C_i (= \mathbb{H} \setminus (K \setminus C_i))$. Then

$$G^{\mathbb{H},i}(z, \zeta) = G^{\mathbb{H}}(z, \zeta) - \mathbb{E}_\zeta^{\mathbb{H}} \left[G^{\mathbb{H}}(Z_{\sigma_{K \setminus C_i}}^{\mathbb{H}}, z);\ \sigma_{K \setminus C_i} < \infty \right].$$

According to [5, Corollary 3.4.3], $\varphi^{(i)}$ is the 0-order equilibrium potential of the compact set C_i with respect to the transient extended Sobolev space $(H_{0,e}^1(D^i), \frac{1}{2}\mathbf{D})$ and it admits an expression

$$\varphi^{(i)}(z) = G^{\mathbb{H},i}\nu_i(z) = \int_{C_i} G^{\mathbb{H},i}(z, \zeta)\nu_i(d\zeta)$$

for some finite positive measure ν_i concentrated on C_i in view of the 0-order version of [5, Lemma 2.3.10] and [9, Exercise 4.2.2]. Hence

$$\varphi^{(i)}(z) = G^{\mathbb{H}}\nu_i(z) - \mathbb{E}_{\nu_i}^{\mathbb{H}} \left[G^{\mathbb{H}}(Z_{\sigma_{K \setminus C_i}}^{\mathbb{H}}, z);\ \sigma_{K \setminus C_i} < \infty \right]. \tag{12.25}$$

Consequently, we have for the same constant M as in the proof of **(v)**,

$$|B_z^{(\varepsilon)} \varphi^{(i)}(z)| \leq 2M\ \nu_i(C_i), \quad \varepsilon \in (0, \varepsilon_0),\ z \in F. \tag{12.26}$$

For an open set $G \subset \mathbb{H}$, denote by $\mathrm{Cap}(B; G)$ the 0-order capacity of $B \subset G$ relative to $(H_{0,e}^1(G), \frac{1}{2}\mathbf{D})$. It increases as B increases or G decreases. Moreover, we have $\mathrm{Cap}(C_i, D^i) = \nu_i(C_i)$ (cf. [9]). Hence (12.26) leads us to a uniform bound

$$|B_z^{(\varepsilon)} \varphi^{(i)}(z)| \leq 2M\ \mathrm{Cap}(\overline{U}_i; \mathbb{H} \setminus (\cup_{k \neq i} \overline{V}_k)), \quad \varepsilon \in (0, \varepsilon_0),\ z \in F. \tag{12.27}$$

(vii): Choose A_1, B_1, B_2 and ℓ_0 as in the proof of Lemma 9.6. F is a subset of $(A_1, B_1) \times (\ell_0, B_2)$. By (4.7), we see that the first integral in (9.14) for $k = 1, 2$ evaluated for $G^{\mathbb{H}}(\zeta, z)$ in place of $G(\zeta, z)$ is bounded by $\dfrac{2}{\pi} \dfrac{|x_1 - \zeta_1|}{(x_1 - \zeta_1)^2 + \ell_0^2}$ and $\dfrac{2}{\pi \ell_0}$, respectively. The second integral evaluated for $G^{\mathbb{H}}(\zeta, z)$ is bounded by $\dfrac{4}{\pi \ell_0}$ and $\dfrac{2}{\ell_0}$, respectively.

On the other hand, we have for $\zeta \in \mathbb{H}$ and $x_1 \in \mathbb{R}$ or $x_2 > 0$,

$$\int_0^\infty \left| \frac{\partial}{\partial x_1} G^{\mathbb{H}}(\zeta, x_1 + ix_2) \right| dx_2 \leq 1, \qquad \int_{-\infty}^\infty \left| \frac{\partial}{\partial x_2} G^{\mathbb{H}}(\zeta, x_1 + ix_2) \right| dx_1 \leq 2. \tag{12.28}$$

Since $|z - \zeta| > b$ for $(z, \zeta) \in F \times K$, we have for any $\zeta \in K$,

$$\int_{-\infty}^\infty \mathbf{1}_F(z) \left| \frac{\partial}{\partial x_1} G^{\mathbb{H}}(\zeta, z) \right| dx_1 \leq \frac{2}{\pi b^2}(B_1 - A_1)^2,\ \int_0^\infty \mathbf{1}_F(z) \left| \frac{\partial}{\partial x_2} G^{\mathbb{H}}(\zeta, z) \right| dx_2 \leq \frac{2}{\pi b^2} B_2^2. \tag{12.29}$$

Denote by c the maximum of the constants in (12.29) and 2. Then each of the four integrals in (9.14) evaluated for the function S in (12.24) admits a bound $c\,\frac{\partial}{\partial \zeta_2}\mathbb{P}^{\mathbb{H}}_\zeta(\sigma_{\overline{U}} < \infty)$, which is in turn dominated by

$$c \sup_{w\in\overline{U},\,\zeta\in J} \frac{\partial}{\partial \zeta_2} G^{\mathbb{H}}(\zeta, w)\ \mu(\overline{U}).$$

as in the proof of **(v)** above.

(viii): By virtue of (12.25), (12.28) and (12.29), we see that each of the four integrals in (9.15) is dominated by

$$2c\,\nu_j(C_j) \le 2c\,\mathrm{Cap}\left(\overline{U}_j;\, \mathbb{H}\setminus(\cup_{k\neq j}\overline{V}_k)\right).$$

□

References

[1] L. V. Ahlfors, *Complex Analysis*, McGraw-Hill, 1979

[2] R. O. Bauer and R. M. Friedrich, On radial stochastic Loewner evolution in multiply connected domains, *J. Funct. Anal.* **237** (2006), 565-588

[3] R. O. Bauer and R. M. Friedrich, On chordal and bilateral SLE in multiply connected domains, *Math. Z.* **258** (2008), 241-265

[4] Z.-Q. Chen, Browniam Motion with Darning. Lecture notes for talks given at RIMS, Kyoto University. 2012.

[5] Z.-Q. Chen and M. Fukushima, *Symmetric Markov Processes, Time Change and Boundary Theory.* Princeton University Press, 2012.

[6] J. B. Conway, *Functions of One Complex Variable* II, Springer, 1995

[7] S. Drenning, *Excursion Reflected Brownian Motions and Loewner Equations in multiply connected domains*, arXiv:1112.4123

[8] P. L. Duren, *Univalent Functions*, Springer, 1983

[9] M. Fukushima, Y. Oshima and M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*, De Gruyter, 1994, Second Extended Edition, 2011

[10] P. R. Garabedian, *Partial Differential Equations*, AMS Chelsia, 2007, republication of 1964 edition

[11] J. B. Garnett and D. E. Marshall, *Harmonic Measure*, Cambridge University Press, 2005

[12] Y. Komatu, On conformal slit mapping of multiply-connected domains, *Proc. Japan Acad.* **26**(1950), 26-31

[13] G. F. Lawler, Conformally invariant processes in the plane, Mathematical Surveys and Monographs, AMS, 2005

[14] G. F. Lawler, The Laplacian-b random walk and the Schramm-Loewner evolution. *Illinois J. Math.* **50** (2006), 701-746 (Special volume in memory of Joseph Doob)

[15] J. Milnor, *Topology from the Differentiable Viewpoint*, Princeton University Press, 1965

[16] Z. Nehari, *Conformal Mapping*, Dover, 1975

[17] S. C. Port and C. J. Stone, *Brownian Motion and Classical Potential Theory*, Academic Press, 1978

[18] M. Tsuji, *Potential Theory in Modern Function Theory*, Marzen, Tokyo, 1959

Zhen-Qing Chen

Department of Mathematics, University of Washington, Seattle, WA 98195, USA
E-mail: `zqchen@uw.edu`

Masatoshi Fukushima:

Branch of Mathematical Science, Osaka University, Toyonaka, Osaka 560-0043, Japan.
Email: `fuku2@mx5.canvas.ne.jp`

Steffen Rohde

Department of Mathematics, University of Washington, Seattle, WA 98195, USA
E-mail: `rohde@math.washington.edu`