SHTUKAS AND THE TAYLOR EXPANSION OF L-FUNCTIONS

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Abstract. We define the Heegner–Drinfeld cycle on the moduli stack of Drinfeld Shtukas of rank two with r-modifications for an even integer r. We prove an identity between

(1) The r-th central derivative of the quadratic base change L-function associated to an everywhere unramified cuspidal automorphic representation π of PGL_2;

(2) The self-intersection number of the π-isotypic component of the Heegner–Drinfeld cycle.

This identity can be viewed as a function-field analog of the Waldspurger and Gross–Zagier formula for higher derivatives of L-functions.

Contents

1. Introduction

In this paper we prove a formula for the arbitrary order central derivative of a certain class of L-functions over a *function field* $F = k(X)$, for a curve X over a finite field k. The L-function under consideration is associated to a cuspidal automorphic representation of $\mathrm{PGL}_{2,F}$, or rather, its base change to a quadratic field extension of F. The r-th central derivative of our L-function is expressed in terms of the intersection number of the "Heegner–Drinfeld cycle" on a moduli stack denoted by Sht^r_G in the introduction, where $G = \mathrm{PGL}_2$. The moduli stack Sht^r_G is closely related to the moduli stack of Drinfeld Shtukas of rank two with r-modifications. One important feature of this stack is that it admits a natural fibration over the r-fold self-product X^r of the curve X over $\mathrm{Spec}\, k$

$$\mathrm{Sht}^r_G \longrightarrow X^r .$$

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The very existence of such moduli stacks presents a striking difference between a function field and a number field. In the number field case, the analogous spaces only exist (at least for the time being) when $r \leq 1$. When $r = 0$, the moduli stack Sht^0_G is the constant groupoid over k

$$\mathrm{Bun}_G(k) \simeq G(F)\backslash (G(\mathbb{A})/K), \tag{1.1}$$

where $\mathbb{A}$ is the ring of adèles, and K a maximal compact open subgroup of $G(\mathbb{A})$. The double coset in the RHS of (1.1) remains meaningful for a number field F (except that one cannot demand the archimedean component of K to be open). When $r = 1$ the analogous space in the case $F = \mathbb{Q}$ is the moduli stack of elliptic curves, which lives over $\mathrm{Spec}\,\mathbb{Z}$. From such perspectives, our formula can be viewed as a simultaneous generalization (for function fields) of the Waldspurger formula [24] (in the case of $r = 0$) and the Gross–Zagier formula [12] (in the case of $r = 1$).

Another noteworthy feature of our work is that we need not restrict ourselves to the leading coefficient in the Taylor expansion of the L-functions: our formula is about the r-th Taylor coefficient of the L-function regardless whether r is the central vanishing order or not. This leads us to speculate that, contrary to the usual belief, central derivatives of arbitrary order of motivic L-functions (for instance, those associated to elliptic curves) should bear some geometric meaning in the number field case. However, due to the lack of the analog of Sht^r_G for arbitrary r in the number field case, we could not formulate a precise conjecture.

Finally we note that, in the current paper, we restrict ourselves to everywhere unramified cuspidal automorphic representations. One consequence is that we only need to consider the even r case. Ramifications, particularly the odd r case, will be considered in subsequent work.

Now we give more details of our main theorems.

1.1. **Some notation.** Throughout the paper, let $k = \mathbb{F}_q$ be a finite field of characteristic p. Let X be a geometrically connected smooth proper curve over k. Let $\nu : X' \to X$ be a finite étale cover of degree 2 such that X' is also geometrically connected. Let $\sigma \in \mathrm{Gal}(X'/X)$ be the nontrivial involution. Let $F = k(X)$ and $F' = k(X')$ be their function fields. Let g and g' be the genera of X and X', then $g' = 2g - 1$.

We denote the set of closed points (places) of X by $|X|$. For $x \in |X|$, let $\mathcal{O}_x$ be the completed local ring of X at x and let F_x be its fraction field. Let $\mathbb{A} = \prod'_{x\in|X|} F_x$ be the ring of adèles, and $\mathbb{O} = \prod_{x\in|X|} \mathcal{O}_x$ the ring of integers inside $\mathbb{A}$. Similar notation applies to X'. Let

$$\eta_{F'/F} : F^\times\backslash\mathbb{A}^\times/\mathbb{O}^\times \longrightarrow \{\pm 1\}$$

be the character corresponding to the étale double cover X' via class field theory.

Let $G = \mathrm{PGL}_2$. Let $K = \prod_{x\in|X|} K_x$ where $K_x = G(\mathcal{O}_x)$. The (spherical) Hecke algebra $\mathscr{H}$ is the $\mathbb{Q}$-algebra of bi-K-invariant functions $C_c^\infty(G(\mathbb{A})/\!\!/K, \mathbb{Q})$ with the product given by convolution.

1.2. ***L*-functions.** Let $\mathcal{A} = C_c^\infty(G(F)\backslash G(\mathbb{A})/K, \mathbb{Q})$ be the space of everywhere unramified $\mathbb{Q}$-valued automorphic functions for G. Then $\mathcal{A}$ is an $\mathscr{H}$-module. By an everywhere unramified cuspidal automorphic representation π of $G(\mathbb{A}_F)$ we mean an $\mathscr{H}$-submodule $\mathcal{A}_\pi \subset \mathcal{A}$ that is irreducible over $\mathbb{Q}$.

For every such π, $\mathrm{End}_{\mathscr{H}}(\mathcal{A}_\pi)$ is a number field E_π, which we call the *coefficient field* of π. Then by the commutativity of $\mathscr{H}$, $\mathcal{A}_\pi$ is a one-dimensional E_π-vector space. If we extend scalars to $\mathbb{C}$, $\mathcal{A}_\pi$ splits into one-dimensional $\mathscr{H}_{\mathbb{C}}$-modules $\mathcal{A}_\pi \otimes_{E_\pi,\iota} \mathbb{C}$, one for each embedding $\iota : E_\pi \hookrightarrow \mathbb{C}$, and each $\mathcal{A}_\pi \otimes_{E_\pi,\iota} \mathbb{C} \subset \mathcal{A}_{\mathbb{C}}$ is the unramified vectors of an everywhere unramified cuspidal automorphic representation in the usual sense.

The standard (complete) L-function $L(\pi, s)$ is a polynomial of degree $4(g-1)$ in $q^{-s-1/2}$ with coefficients in the ring of integers $\mathcal{O}_{E_\pi}$. Let $\pi_{F'}$ be the base change to F', and let $L(\pi_{F'}, s)$ be the standard L-function of $\pi_{F'}$. This L-function is a product of two L-functions associated to cuspidal automorphic representations of G over F:

$$L(\pi_{F'}, s) = L(\pi, s)L(\pi \otimes \eta_{F'/F}, s).$$

Therefore $L(\pi_{F'}, s)$ is a polynomial of degree $8(g-1)$ in $q^{-s-1/2}$ with coefficients in E_π. It satisfies a functional equation

$$L(\pi_{F'}, s) = \epsilon(\pi_{F'}, s) L(\pi_{F'}, 1-s),$$

where the epsilon factor takes a simple form

$$\epsilon(\pi_{F'}, s) = q^{-8(g-1)(s-1/2)}.$$

Let $L(\pi, \mathrm{Ad}, s)$ be the adjoint L-function of π. Denote

$$\mathscr{L}(\pi_{F'}, s) = \epsilon(\pi_{F'}, s)^{-1/2} \frac{L(\pi_{F'}, s)}{L(\pi, \mathrm{Ad}, 1)}, \tag{1.2}$$

where the the square root is understood as

$$\epsilon(\pi_{F'}, s)^{-1/2} := q^{4(g-1)(s-1/2)}.$$

In particular, we have a functional equation:

$$\mathscr{L}(\pi_{F'}, s) = \mathscr{L}(\pi_{F'}, 1-s).$$

Consider the Taylor expansion at the central point $s = 1/2$:

$$\mathscr{L}(\pi_{F'}, s) = \sum_{r \geq 0} \mathscr{L}^{(r)}(\pi_{F'}, 1/2) \frac{(s-1/2)^r}{r!},$$

i.e.,

$$\mathscr{L}^{(r)}(\pi_{F'}, 1/2) = \frac{d^r}{ds^r}\Big|_{s=0} \left(\epsilon(\pi_{F'}, s)^{-1/2} \frac{L(\pi_{F'}, s)}{L(\pi, \mathrm{Ad}, 1)} \right).$$

If r is odd, by the functional equation we have

$$\mathscr{L}^{(r)}(\pi_{F'}, 1/2) = 0.$$

Since $L(\pi, \mathrm{Ad}, 1) \in E_\pi$, we have $\mathscr{L}(\pi_{F'}, s) \in E_\pi[q^{-s-1/2}, q^{s-1/2}]$. It follows that

$$\mathscr{L}^{(r)}(\pi_{F'}, 1/2) \in E_\pi \cdot (\log q)^r.$$

The main result of this paper is to relate each even degree Taylor coefficient to the self-intersection numbers of a certain algebraic cycle on the moduli stack of Shtukas. We give two formulations of our main results, one using certain subquotient of the rational Chow group, and the other using ℓ-adic cohomology.

1.3. **The Heegner–Drinfeld cycles.** From now on, we let r be an *even* integer. In §5.2, we will introduce moduli stack Sht_G^r of Drinfeld Shtukas with r-modifications for the group $G = \mathrm{PGL}_2$. The stack Sht_G^r is a Deligne–Mumford stack over X^r and the natural morphism

$$\pi_G : \mathrm{Sht}_G^r \longrightarrow X^r$$

is smooth of relative dimension r, and locally of finite type. Let $T = (\mathrm{Res}_{F'/F} \mathbb{G}_m)/\mathbb{G}_m$ be the non-split torus associated to the double cover X' of X.In §5.4, we will introduce the moduli stack Sht_T^μ of T-Shtukas, depending on the choice of an r-tuple of signs $\mu \in \{\pm\}^r$ satisfying certain balance conditions in §5.1.2. Then we have a similar map

$$\pi_T^\mu : \mathrm{Sht}_T^\mu \longrightarrow X'^r$$

which is a torsor under the finite Picard stack $\mathrm{Pic}_{X'}(k)/\mathrm{Pic}_X(k)$. In particular, Sht_T^μ is a proper smooth Deligne–Mumford stack over $\mathrm{Spec}\, k$.

There is a natural finite morphism of stacks over X^r

$$\mathrm{Sht}_T^\mu \longrightarrow \mathrm{Sht}_G^r .$$

It induces a finite morphism

$$\theta^\mu : \mathrm{Sht}_T^\mu \longrightarrow \mathrm{Sht}_G'^r := \mathrm{Sht}_G^r \times_{X^r} X'^r .$$

This defines a class in the Chow group

$$\theta_*^\mu[\mathrm{Sht}_T^\mu] \in \mathrm{Ch}_{c,r}(\mathrm{Sht}_G'^r)_{\mathbb{Q}}.$$

Here $\mathrm{Ch}_{c,r}(-)_{\mathbb{Q}}$ means the Chow group of proper cycles of dimension r, tensored over $\mathbb{Q}$. See §A.1 for details. In analogy to the classical Heegner cycles [12], we will call $\theta^{\mu}_{*}[\mathrm{Sht}^{\mu}_{T}]$ the *Heegner–Drinfeld cycle* in our setting.

1.4. **Main results: cycle-theoretic version.** The Hecke algebra $\mathscr{H}$ acts on the Chow group $\mathrm{Ch}_{c,r}(\mathrm{Sht}'^{r}_{G})_{\mathbb{Q}}$ as correspondences. Let $\widetilde{W}\subset \mathrm{Ch}_{c,r}(\mathrm{Sht}'^{r}_{G})_{\mathbb{Q}}$ be the sub $\mathscr{H}$-module generated by the Heegner–Drinfeld cycle $\theta^{\mu}_{*}[\mathrm{Sht}^{\mu}_{T}]$. There is a bilinear and symmetric intersection pairing [1]

$$\langle\cdot,\cdot\rangle_{\mathrm{Sht}'^{r}_{G}}:\widetilde{W}\times\widetilde{W}\longrightarrow\mathbb{Q}. \tag{1.3}$$

Let $\widetilde{W}_0$ be the kernel of the pairing, i.e.,

$$\widetilde{W}_0=\left\{z\in\widetilde{W}\,\middle|\,(z,z')=0,\ \text{for all } z'\in\widetilde{W}.\right\}$$

The pairing $\langle\cdot,\cdot\rangle_{\mathrm{Sht}'^{r}_{G}}$ then induces a *non-degenerate* pairing on the quotient $W:=\widetilde{W}/\widetilde{W}_0$

$$(\cdot,\cdot):W\times W\longrightarrow\mathbb{Q}\ . \tag{1.4}$$

The Hecke algebra $\mathscr{H}$ acts on W. For any ideal $\mathcal{I}\subset\mathscr{H}$, let

$$W[\mathcal{I}]=\left\{w\in W\,\middle|\,\mathcal{I}\cdot w=0\right\}.$$

Let π be an everywhere unramified cuspidal automorphic representation of G with coefficient field E_π, and let $\lambda_\pi:\mathscr{H}\to E_\pi$ be the associated character, whose kernel $\mathfrak{m}_\pi$ is a maximal ideal of $\mathscr{H}$. Let

$$W_\pi=W[\mathfrak{m}_\pi]\subset W$$

be the λ_π-eigenspace of W. This is an E_π-vector space. Let $\mathcal{I}_{\mathrm{Eis}}\subset\mathscr{H}$ be the Eisenstein ideal as defined in Definition 4.1 and define

$$W_{\mathrm{Eis}}=W[\mathcal{I}_{\mathrm{Eis}}].$$

Theorem 1.1. *We have an orthogonal decomposition of $\mathscr{H}$-modules*

$$W=W_{\mathrm{Eis}}\oplus\left(\bigoplus_{\pi}W_\pi\right), \tag{1.5}$$

where π runs over the finite set of everywhere unramified cuspidal automorphic representation of G, and W_π is an E_π-vector space of dimension at most one.

The proof will be given in §9.3.1. In fact one can also show that W_{Eis} is a free rank one module over $\mathbb{Q}[\mathrm{Pic}_X(k)]^{\iota_{\mathrm{Pic}}}$ (for notation see §4.1.2), but we shall omit the proof of this fact.

The $\mathbb{Q}$-bilinear pairing $(\cdot,\cdot)$ on W_π can be lifted to an E_π-bilinear symmetric pairing

$$(\cdot,\cdot)_\pi:W_\pi\times W_\pi\longrightarrow E_\pi \tag{1.6}$$

where for $w,w'\in W_\pi$, $(w,w')_\pi$ is the unique element in E_π such that $\mathrm{Tr}_{E_\pi/\mathbb{Q}}(e\cdot(w,w')_\pi)=(ew,w')$.

We now present the cycle-theoretic version of our main result.

Theorem 1.2. *Let π be an everywhere unramified cuspidal automorphic representation of G with coefficient field E_π. Let $[\mathrm{Sht}^{\mu}_{T}]_\pi\in W_\pi$ be the projection of the image of $\theta^{\mu}_{*}[\mathrm{Sht}^{\mu}_{T}]\in\widetilde{W}$ in W to the direct summand W_π under the decomposition* (1.5). *Then we have an equality in E_π*

$$\frac{1}{2(\log q)^r}\,|\omega_X|\,\mathscr{L}^{(r)}(\pi_{F'},1/2)=\Big([\mathrm{Sht}^{\mu}_{T}]_\pi,\quad[\mathrm{Sht}^{\mu}_{T}]_\pi\Big)_\pi,$$

where ω_X is the canonical divisor of X, and $|\omega_X|=q^{-\deg\omega_X}$.

The proof will be completed in §9.3.2.

[1] In this paper, the intersection pairing on the Chow groups will be denoted by $\langle\cdot,\cdot\rangle$, and other pairings (the ones on the quotient of the Chow groups, and the cup product pairing on cohomology) will be denoted by $(\cdot,\cdot)$.

Remark 1.3. Assume that $r = 0$. Then our formula is equivalent to the Waldspurger formula [24] for an everywhere unramified cuspidal automorphic representation π. More precisely, for any nonzero $\phi \in \pi^K$, the Waldspurger formula is the identity

$$\frac{1}{2}\,|\omega_X|\,\mathscr{L}(\pi_{F'}, 1/2) = \frac{\left|\int_{T(F)\backslash T(\mathbb{A})} \phi(t)\,dt\right|^2}{\langle \phi, \phi\rangle_{\mathrm{Pet}}},$$

where $\langle \phi, \phi\rangle_{\mathrm{Pet}}$ is the Petersson inner product (4.10), and the measure on $G(\mathbb{A})$ (resp. $T(\mathbb{A})$) is chosen such that $\mathrm{vol}(K) = 1$ (resp. $\mathrm{vol}(T(\mathbb{O})) = 1$).

Remark 1.4. Our E_π-valued intersection paring is similar to the Néron–Tate height pairing with coefficients in [25, §1.2.4].

1.5. **Main results: cohomological version.** Let ℓ be a prime number different from p. Consider the middle degree cohomology with compact support

$$V'_{\mathbb{Q}_\ell} = \mathrm{H}^{2r}_c((\mathrm{Sht}'^r_G) \otimes_k \overline{k}, \mathbb{Q}_\ell)(r).$$

In the main body of the paper, we simply denote this by V'. This vector space is endowed with the cup product

$$(\cdot,\cdot) : V'_{\mathbb{Q}_\ell} \times V'_{\mathbb{Q}_\ell} \longrightarrow \mathbb{Q}_\ell\ .$$

Then for any maximal ideal $\mathfrak{m} \subset \mathscr{H}_{\mathbb{Q}_\ell}$, we define the generalized eigenspace of $V'_{\mathbb{Q}_\ell}$ with respect to $\mathfrak{m}$ by

$$V'_{\mathbb{Q}_\ell,\mathfrak{m}} = \cup_{i>0} V'_{\mathbb{Q}_\ell}[\mathfrak{m}^i].$$

We also define the Eisenstein part of $V'_{\mathbb{Q}_\ell}$ by

$$V'_{\mathbb{Q}_\ell,\mathrm{Eis}} = \cup_{i>0} V'_{\mathbb{Q}_\ell}[\mathcal{I}^i_{\mathrm{Eis}}].$$

We remark that in the cycle-theoretic version (cf. §1.4), the generalized eigenspace coincides with the eigenspace because the space W is a cyclic module over the Hecke algebra.

Theorem 1.5 (see Theorem 7.16 for a more precise statement)**.** *We have an orthogonal decomposition of $\mathscr{H}_{\mathbb{Q}_\ell}$-modules*

$$V'_{\mathbb{Q}_\ell} = V'_{\mathbb{Q}_\ell,\mathrm{Eis}} \oplus \left(\bigoplus_{\mathfrak{m}} V'_{\mathbb{Q}_\ell,\mathfrak{m}}\right), \tag{1.7}$$

where $\mathfrak{m}$ runs over a finite set of maximal ideals of $\mathscr{H}_{\mathbb{Q}_\ell}$ whose residue fields $E_{\mathfrak{m}} := \mathscr{H}_{\mathbb{Q}_\ell}/\mathfrak{m}$ are finite extensions of $\mathbb{Q}_\ell$, and each $V'_{\mathbb{Q}_\ell,\mathfrak{m}}$ is an $\mathscr{H}_{\mathbb{Q}_\ell}$-module of finite dimension over $\mathbb{Q}_\ell$ supported at the maximal ideal $\mathfrak{m}$.

The action of $\mathscr{H}_{\mathbb{Q}_\ell}$ on $V'_{\mathbb{Q}_\ell,\mathfrak{m}}$ factors through the completion $\widehat{\mathscr{H}}_{\mathbb{Q}_\ell,\mathfrak{m}}$ with residue field $E_{\mathfrak{m}}$. Since $E_{\mathfrak{m}}$ is finite étale over $\mathbb{Q}_\ell$, and $\widehat{\mathscr{H}}_{\mathbb{Q}_\ell,\mathfrak{m}}$ is a complete local (hence henselian) $\mathbb{Q}_\ell$-algebra with residue field $E_{\mathfrak{m}}$, Hensel's lemma implies that there is a unique section $E_{\mathfrak{m}} \to \widehat{\mathscr{H}}_{\mathbb{Q}_\ell,\mathfrak{m}}$ (the minimal polynomial of every element $\overline{h} \in E_{\mathfrak{m}}$ over $\mathbb{Q}_\ell$ has a unique root $h \in \widehat{\mathscr{H}}_{\mathbb{Q}_\ell,\mathfrak{m}}$ whose reduction is $\overline{h}$). Hence each $V'_{\mathbb{Q}_\ell,\mathfrak{m}}$ is also an $E_{\mathfrak{m}}$-vector space in a canonical way. As in the case of W_π, using the $E_{\mathfrak{m}}$-action on $V'_{\mathbb{Q}_\ell,\mathfrak{m}}$, the $\mathbb{Q}_\ell$-bilinear pairing on $V'_{\mathbb{Q}_\ell,\mathfrak{m}}$ may be lifted to an $E_{\mathfrak{m}}$-bilinear symmetric pairing

$$(\cdot,\cdot)_{\mathfrak{m}} : V'_{\mathbb{Q}_\ell,\mathfrak{m}} \times V'_{\mathbb{Q}_\ell,\mathfrak{m}} \longrightarrow E_{\mathfrak{m}}\ .$$

Note that, unlike (1.5), in the decomposition (1.7) we can not be sure whether all $\mathfrak{m}$ are automorphic (i.e., the homomorphism $\mathscr{H} \to E_{\mathfrak{m}}$ is the character by which $\mathscr{H}$ acts on the unramified line of an irreducible automorphic representation). However, for an everywhere unramified cuspidal automorphic representation π of G with coefficient field E_π, we may extend $\lambda_\pi : \mathscr{H} \to E_\pi$ to $\mathbb{Q}_\ell$ to get

$$\lambda_\pi \otimes \mathbb{Q}_\ell : \mathscr{H}_{\mathbb{Q}_\ell} \longrightarrow E_\pi \otimes \mathbb{Q}_\ell \cong \prod\nolimits_{\lambda|\ell} E_{\pi,\lambda}$$

where λ runs over places of E_π above ℓ. Let $\mathfrak{m}_{\pi,\lambda}$ be the maximal ideal of $\mathscr{H}_{\mathbb{Q}_\ell}$ obtained as the kernel of the λ-component of the above map $\mathscr{H}_{\mathbb{Q}_\ell} \to E_{\pi,\lambda}$.

To alleviate notation, we denote $V'_{\mathbb{Q}_\ell,\mathfrak{m}_{\pi,\lambda}}$ simply by $V'_{\pi,\lambda}$, and denote the $E_{\pi,\lambda}$-bilinear pairing $(\cdot,\cdot)_{\mathfrak{m}_{\pi,\lambda}}$ on $V'_{\pi,\lambda}$ by

$$(\cdot,\cdot)_{\pi,\lambda} : V'_{\pi,\lambda} \times V'_{\pi,\lambda} \longrightarrow E_{\pi,\lambda}.$$

We now present the cohomological version of our main result.

Theorem 1.6. *Let π be an everywhere unramified cuspidal automorphic representation of G with coefficient field E_π. Let λ be a place of E_π above ℓ. Let $[\mathrm{Sht}^\mu_T]_{\pi,\lambda} \in V'_{\pi,\lambda}$ be the projection of the cycle class $\mathrm{cl}(\theta^\mu_*[\mathrm{Sht}^\mu_T]) \in V'_{\mathbb{Q}_\ell}$ to the direct summand $V'_{\pi,\lambda}$ under the decomposition* (1.7). *Then we have an equality in $E_{\pi,\lambda}$*

$$\frac{1}{2(\log q)^r} |\omega_X| \mathscr{L}^{(r)}(\pi_{F'}, 1/2) = \Big([\mathrm{Sht}^\mu_T]_{\pi,\lambda}, \quad [\mathrm{Sht}^\mu_T]_{\pi,\lambda}\Big)_{\pi,\lambda}.$$

In particular, the RHS also lies in E_π.

The proof will be completed in in §9.2.

1.6. **Two other results.** We have the following positivity result. This may be seen as an evidence of the Hodge standard conjecture (on the positivity of intersection pairing) for a subquotient of the Chow group of middle dimensional cycles on Sht'^r_G.

Theorem 1.7. *Let W_{cusp} be the orthogonal complement of W_{Eis} in W (cf.* (1.5)*). Then the restriction to W_{cusp} of the intersection pairing $(\cdot,\cdot)$ in* (1.4) *is positive definite.*

Proof. The assertion is equivalent to the positivity for the restriction to W_π of the intersection pairing, for all π in (1.5). Fix such a π. Then the coefficient field E_π is a totally real number field because the Hecke operators $\mathscr{H}$ act on the positive definite inner product space $\mathcal{A} \otimes_{\mathbb{Q}} \mathbb{R}$ (under the Petersson inner product) by self-adjoint operators. For an embedding $\iota : E_\pi \to \mathbb{R}$, we define

$$W_{\pi,\iota} := W_\pi \otimes_{E_\pi,\iota} \mathbb{R}.$$

Extending scalars from E_π to $\mathbb{R}$ via ι, the pairing (1.6) induces an $\mathbb{R}$-bilinear symmetric pairing

$$(\cdot,\cdot)_{\pi,\iota} : W_{\pi,\iota} \times W_{\pi,\iota} \longrightarrow \mathbb{R}.$$

It suffices to show that, for every embedding $\iota : E_\pi \to \mathbb{R}$, the pairing $(\cdot,\cdot)_{\pi,\iota}$ is positive definite. The $\mathbb{R}$-vector space $W_{\pi,\iota}$ is at most one-dimensional, with a generator given by $[\mathrm{Sht}^\mu_T]_{\pi,\iota} = [\mathrm{Sht}^\mu_T]_\pi \otimes 1$. The embedding ι gives an irreducible cuspidal automorphic representation π_ι with $\mathbb{R}$-coefficient. Then Theorem 1.2 implies that

$$\frac{1}{2(\log q)^r} |\omega_X| \mathscr{L}^{(r)}(\pi_{\iota,F'}, 1/2) = \Big([\mathrm{Sht}^\mu_T]_{\pi,\iota}, \quad [\mathrm{Sht}^\mu_T]_{\pi,\iota}\Big)_{\pi,\iota} \in \mathbb{R}.$$

It is easy to see that $L(\pi_\iota, \mathrm{Ad}, 1) > 0$. By Theorem B.2, we have

$$\mathscr{L}^{(r)}(\pi_{\iota,F'}, 1/2) \geq 0.$$

It follows that

$$\Big([\mathrm{Sht}^\mu_T]_{\pi,\iota}, \quad [\mathrm{Sht}^\mu_T]_{\pi,\iota}\Big)_{\pi,\iota} \geq 0.$$

This completes the proof. □

Another result is a "Kronecker limit formula" for function fields. Let $L(\eta, s)$ be the (complete) L-function associated to the Hecke character η.

Theorem 1.8. *When $r > 0$ is even, we have*

$$\langle \theta^\mu_*[\mathrm{Sht}^\mu_T], \quad \theta^\mu_*[\mathrm{Sht}^\mu_T]\rangle_{\mathrm{Sht}'^r_G} = \frac{2^{r+2}}{(\log q)^r} L^{(r)}(\eta, 0).$$

The proof will be given in §9.1.1. For the case $r = 0$, see Remark 9.3.

Remark 1.9. To obtain a similar formula for the odd order derivatives $L^{(r)}(\eta, 0)$, we need moduli spaces analogous to Sht^μ_T and Sht'^r_G for odd r. We will return this in future work.

1.7. **Outline of the proof of the main theorems.**

1.7.1. *Basic strategy.* The basic strategy is to compare two relative trace formulae. A relative trace formula (abbreviated as RTF) is an equality between a spectral expansion and an orbital integral expansion. We have two RTFs, an "analytic" one for the L-functions, and a "geometric" one for the intersection numbers, corresponding to the two sides of the desired equality in Theorem 1.6.

We may summarize the strategy of the proof into the following diagram

$$\begin{tikzcd} \text{Analytic:} & \sum_{u\in\mathbb{P}^1(F)-\{1\}} \mathbb{J}_r(u,f) \arrow[r, equal, "\S 2"] \arrow[d, dash, "\sim\text{Th } 8.1 \quad \Rightarrow"] & \mathbb{J}_r(f) \arrow[r, equal, "\S 4"] \arrow[d, dash, "\text{Th } 9.2 \ \Rightarrow"] & \sum_{\pi} \mathbb{J}_r(\pi,f) \arrow[d, dash, "\Rightarrow\text{Th } 1.6"] \\ \text{Geometric:} & \sum_{u\in\mathbb{P}^1(F)-\{1\}} \mathbb{I}_r(u,f) \arrow[r, equal, "\S 6"] & \mathbb{I}_r(f) \arrow[r, equal, "\S 7"] & \sum_{\mathfrak{m}} \mathbb{I}_r(\mathfrak{m},f) \end{tikzcd} \tag{1.8}$$

The vertical lines mean equalities after dividing the first row by $(\log q)^r$.

1.7.2. *The analytic side.* We start with the analytic RTF. To an $f \in \mathscr{H}_{\mathbb{Q}}$ (or more generally, $C_c^\infty(G(\mathbb{A}))$), one first associates an automorphic kernel function $\mathbb{K}_f$ on $G(\mathbb{A}) \times G(\mathbb{A})$ and then a regularized integral:

$$\mathbb{J}(f,s) = \int_{[A]\times[A]}^{\mathrm{reg}} \mathbb{K}_f(h_1,h_2)|h_1h_2|^s\eta(h_2)\, dh_1\, dh_2.$$

Here A is the diagonal torus of G, and $[A] = A(F)\backslash A(\mathbb{A})$. We refer to §2.2 for the definition of the weighted factors, and the regularization. Informally, we may view this integral as a weighted (naive) intersection number on the constant groupoid $\mathrm{Bun}_G(k)$ (the moduli stack of Shtukas with $r = 0$ modifications) between $\mathrm{Bun}_A(k)$ and its Hecke translation under f of $\mathrm{Bun}_A(k)$.

The resulting $\mathbb{J}(f,s)$ belongs to $\mathbb{Q}[q^{-s}, q^s]$. For an f in the Eisenstein ideal $\mathcal{I}_{\mathrm{Eis}}$ (cf. §4.1), the spectral decomposition of $\mathbb{J}(f,s)$ takes a simple form: it is the sum of

$$\mathbb{J}_\pi(f,s) = \frac{1}{2}\,|\omega_X|\,\mathscr{L}(\pi_{F'}, s+1/2)\,\lambda_\pi(f)$$

where π runs over all everywhere unramified cuspidal automorphic representations π of G with $\overline{\mathbb{Q}}_\ell$-coefficients (cf. Prop. 4.5). We define $\mathbb{J}_r(f)$ to be the r-th derivative

$$\mathbb{J}_r(f) := \left(\frac{d}{ds}\right)^r \Big|_{s=0} \mathbb{J}(f,s).$$

We point out that in the case of $r = 0$, the relative trace formula in question was first introduced by Jacquet [13], in his reproof of Waldspurger's formula. In the case of $r = 1$, a variant was first considered in [29] (for number fields).

1.7.3. *The geometric side.* Next we consider the geometric RTF. We consider the Heegner–Drinfeld cycle $\theta_*^\mu[\mathrm{Sht}_T^\mu]$ and its translation by the Hecke correspondence given by $f \in \mathscr{H}$, both being cycles on the ambient stack $\mathrm{Sht}_G'^r$. We define $\mathbb{I}_r(f)$ to be their intersection number

$$\mathbb{I}_r(f) := \langle \theta_*^\mu[\mathrm{Sht}_T^\mu], \quad f * \theta_*^\mu[\mathrm{Sht}_T^\mu]\rangle_{\mathrm{Sht}_G'^r} \quad \in \mathbb{Q}, \quad f \in \mathscr{H}_{\mathbb{Q}}.$$

To decompose this spectrally according to the Hecke action, we have two perspectives, one viewing the Heegner–Drinfeld cycle as an element in the Chow group modulo numerical equivalence, the other considering the cycle class of the Heegner–Drinfeld cycle in the ℓ-adic cohomology. In either case, when f is in a certain power of $\mathcal{I}_{\mathrm{Eis}}$, the spectral decomposition (§7, or Theorem 1.5) expresses $\mathbb{I}_r(f)$ as a sum of

$$\mathbb{I}_r(\pi,f) = \lambda_\pi(f)\Big([\mathrm{Sht}_T^\mu]_\pi, \quad [\mathrm{Sht}_T^\mu]_\pi\Big)$$

where π runs over all everywhere unramified cuspidal automorphic representations π of G with $\overline{\mathbb{Q}}_\ell$-coefficients. We remark that the method of the proof of the spectral decomposition in Theorem 1.5 can potentially be applied to moduli of Shtukas for more general groups G, which should lead to a better understanding of the cohomology of these moduli spaces.

We point out that here we use the same way as in [29] to set up the geometric RTF, although in [29] only the case of $r = 1$ was considered. In the case $r = 0$, Jacquet used an integration of kernel function to set up an RTF for the T-period integral, which is equivalent to our geometric

RTF because in this case Sht_T^μ and Sht_G^r become discrete stacks $\mathrm{Bun}_T(k)$ and $\mathrm{Bun}_G(k)$. Our geometric formulation treats all values of r uniformly.

1.7.4. *The key identity.* In view of the spectral decompositions of both $\mathbb{I}_r(f)$ and $\mathbb{J}_r(f)$, to prove the main Theorem 1.6 for all π simultaneously, it suffices to establish the following key identity (cf. Theorem 9.2)

$$\mathbb{I}_r(f) = (\log q)^{-r}\mathbb{J}_r(f) \in \mathbb{Q}, \quad \text{for all } f \in \mathscr{H}_{\mathbb{Q}}. \tag{1.9}$$

This key identity also allows us to deduce Theorem 1.1 on the spectral decomposition of the space W of cycles from the spectral decomposition of $\mathbb{J}_r$. Theorems 1.2 then follows easily from Theorem 1.6.

Since half of the paper is devoted to the proof of the key identity (1.9), we comment on its proof in more detail. The spectral decompositions allow us to reduce to proving (1.9) for sufficiently many functions $f \in \mathscr{H}_{\mathbb{Q}}$, indexed by effective divisors on X with large degree compared to the genus of X (cf. Theorem 8.1). Most of the algebro-geometric part of this paper is devoted to the proof of the key identity (1.9) for those Hecke functions.

In §3, we interpret the orbital integral expansion of $\mathbb{J}_r(f)$ (the northwestern sum in (1.8)) as a certain weighted counting of effective divisors on the curve X. The geometric ideas used in the part are close to those in the proof of various fundamental lemmas by Ngô [19] and by the first-named author [26], although the situation is much simpler in the current paper. In §6, we interpret the intersection number $\mathbb{I}_r(f)$ as the trace of a correspondence acting on the cohomology of a certain variety. This section involves new geometric ideas that did not appear in the treatment of the fundamental lemma type problems. This is also the most technical part of the paper, making use of the general machinery on intersection theory reviewed or improved in Appendix A.

After the preparations in §3 and §6, our situation can be summarized as follows. For an integer $d \geq 0$, we have fibrations

$$f_{\mathcal{N}} : \mathcal{N}_d = \coprod_{\underline{d}} \mathcal{N}_{\underline{d}} \longrightarrow \mathcal{A}_d, \quad f_{\mathcal{M}} : \mathcal{M}_d \longrightarrow \mathcal{A}_d,$$

where $\underline{d}$ runs over all quadruples $(d_{11}, d_{12}, d_{21}, d_{22}) \in \mathbb{Z}^4_{\geq 0}$ such that $d_{11} + d_{22} = d = d_{12} + d_{21}$. We need to show that the direct image complexes $\mathbf{R}f_{\mathcal{M},*}\mathbb{Q}_\ell$ and $\mathbf{R}f_{\mathcal{N},*}L_d$ are isomorphic to each other, where L_d is a local system of rank one coming from the double cover X'/X. When d is sufficiently large, we show that both complexes are shifted perverse sheaves, and are obtained by middle extension from a dense open subset of $\mathcal{A}_d$ over which both can be explicitly calculated (cf. Prop 8.2 and 8.5). The isomorphism between the two complexes over the entire base $\mathcal{A}_d$ then follows by the functoriality of the middle extension. The strategy used here is the *perverse continuation principle* coined by Ngô, which has already played a key role in all known geometric proofs of fundamental lemmas, see [19] and [26].

Remark 1.10. One feature of our proof of the key identity (1.9) is that it is entirely global, in the sense that we do *not* reduce to the comparison of local orbital integral identities, as opposed to what one usually does when comparing two trace formulae. Therefore our proof is different from Jacquet's in the case $r = 0$ in that his proof is essentially local (this is inevitable because he also considers the number field case).

Another remark is that our proof of (1.9) in fact gives a term-by-term identity of the orbital expansion of both $\mathbb{J}_r(f)$ and $\mathbb{I}_r(f)$, as indicated in the left column of (1.8), although this is not logically needed for our main results. However, such more refined identities (for more general G) will be needed in the proof of the arithmetic fundamental lemma for function fields, a project to be completed in near future [27].

1.8. **A guide for readers.** Since this paper uses a mixture of tools from automorphic representation theory, algebraic geometry and sheaf theory, we think it might help orient the readers by providing a brief summary of the contents and the background knowledge required for each section.

First we give the Leitfaden.

$$
\begin{tikzcd}
 & \boxed{\S2} \arrow[dl] \arrow[d] & & \boxed{\S5} \arrow[d] \arrow[dr] & \\
\boxed{\S4} \arrow[ddrr] & \boxed{\S3} \arrow[dr] & & \boxed{\S6} \arrow[dl] & \boxed{\S7} \arrow[ddll] \\
 & & \boxed{\S8} \arrow[d] & & \\
 & & \boxed{\S9} & &
\end{tikzcd}
$$

Section 2 sets up the relative trace formula following Jacquet's approach [13] to the Waldspurger formula. This section is purely representation-theoretic.

Section 3 gives a geometric interpretation of the orbital integrals involved in the relative trace formula introduced in §2. We express these orbital integrals as the trace of Frobenius on the cohomology of certain varieties, in the similar spirit of the proof of various fundamental lemmas ([19], [26]). This section involves both orbital integrals and some algebraic geometry but not yet perverse sheaves.

Section 4 relates the spectral side of the relative trace formula in §2 to automorphic L-functions. Again this section is purely representation-theoretic.

Section 5 introduces the geometric players in our main theorem: moduli stacks Sht_G^r of Drinfeld Shtukas, and Heegner–Drinfeld cycles on them. We give self-contained definitions of these moduli stacks, so no prior knowledge of Shtukas is assumed, although some experience with the moduli stack of bundles will help.

Section 6 is the technical heart of the paper, aiming to prove Theorem 6.5. The proof involves studying several auxiliary moduli stacks and uses heavily the intersection-theoretic tools reviewed and developed in Appendix A. The first-time readers may skip the proof and only read the statement of Theorem 6.5.

Section 7 gives a decomposition of the cohomology of Sht_G^r under the action of the Hecke algebra, generalizing the classical spectral decomposition for the space automorphic forms. The idea is to remove the analytic ingredients from the classical treatment of spectral decomposition, and to use solely commutative algebra (in particular, we crucially use the *Eisenstein ideal* introduced in §4). For first-time readers, we suggest read §7.1, then jump directly to Definition 7.12 and continue from there. What he/she will miss in doing this is the study of the geometry of Sht_G^r near infinity (horocycles), which requires some familiarity with the moduli stack of bundles, and the formalism of ℓ-adic sheaves.

Section 8 combines the geometric formula for orbital integrals established in §3 and the alternative formula for the intersection numbers established in §6 to prove the key identity (1.9) for most Hecke functions. The proofs in this section involve perverse sheaves.

Section 9 finishes the proofs of our main results. Assuming results from the previous sections, most argument in this section only involves commutative algebra.

Both appendices can be read independently of the rest of the paper.

Appendix A reviews the intersection theory on algebraic stacks following Kresch [15], with two key results that are used in §6 for the calculation of the intersection number of Heegner–Drinfeld cycles. The first result, called the *Octahedron Lemma* (Theorem A.10), is an elaborated version of the following simple principle: in calculating the intersection product of several cycles, one can combine terms and change the orders arbitrarily. The second result is a Lefschetz fixed point formula for certain correspondences, building on results of Varshavsky [22].

Appendix B proves a positivity result for central derivatives of automorphic L-functions, assuming the generalized Riemann hypothesis in the case of number fields. The main body of the paper only considers L-functions for function fields, for which the positivity result can be proved in an elementary way (see Remark B.4).

1.9. **Further notation.**

1.9.1. *Function field notation.* For $x \in |X|$, let ϖ_x be a uniformizer of $\mathcal{O}_x$, k_x be the residue field of x, $d_x = [k_x : k]$, and $q_x = \# k_x = q^{d_x}$.The valuation map is a homomorphism

$$\mathrm{val}\colon \mathbb{A}^\times \longrightarrow \mathbb{Z}$$

such that $\mathrm{val}(\varpi_x) = d_x$. The normalized absolute value on $\mathbb{A}^\times$ is defined as

$$\begin{aligned} |\cdot|\colon\ \mathbb{A}^\times &\longrightarrow \mathbb{Q}^\times_{>0} \subset \mathbb{R}^\times\ . \\ a &\longmapsto q^{-\mathrm{val}(a)} \end{aligned}$$

Denote the kernel of the absolute value by

$$\mathbb{A}^1 = \mathrm{Ker}(|\cdot|).$$

We have the global and local zeta function

$$\zeta_F(s) = \prod_{x \in |X|} \zeta_x(s), \quad \zeta_x(s) = \frac{1}{1 - q_x^{-s}}.$$

Denote by $\mathrm{Div}(X) \cong \mathbb{A}^\times / \mathbb{O}^\times$ the group of divisors on X.

1.9.2. *Group-theoretic notation.* Let $\mathbb{G}$ be an algebraic group over k. We will view it as an algebraic group over F by extension of scalar. We will abbreviate $[\mathbb{G}] = \mathbb{G}(F)\backslash\mathbb{G}(\mathbb{A})$. Unless otherwise stated, the Haar measure on the group $\mathbb{G}(\mathbb{A})$ will be chosen such that the natural maximal compact open subgroup $\mathbb{G}(\mathbb{O})$ has volume equal to one. For example, the measure on $\mathbb{A}^\times$, resp. $G(\mathbb{A})$ is such that $\mathrm{vol}\,(\mathbb{O}^\times) = 1$, resp. $\mathrm{vol}(K) = 1$.

1.9.3. *Algebro-geometric notation.* In the main body of the paper, all geometric objects are algebraic stacks over the finite field $k = \mathbb{F}_q$. For such a stack S, let $\mathrm{Fr}_S : S \to S$ be the absolute q-Frobenius endomorphism that raises functions to their q-th powers.

For an algebraic stack S over k, we write $\mathrm{H}^*(S \otimes_k \overline{k})$ (resp. $\mathrm{H}^*_c(S \otimes_k \overline{k})$) for the étale cohomology (resp. étale cohomology with compact support) of the base change $S \otimes_k \overline{k}$ with $\mathbb{Q}_\ell$-coefficients. The ℓ-adic homology $\mathrm{H}_*(S \otimes_k \overline{k})$ and Borel-Moore homology $\mathrm{H}^{\mathrm{BM}}_*(S \otimes_k \overline{k})$ are defined as the graded dual of $\mathrm{H}^*(S \otimes_k \overline{k})$ and $\mathrm{H}^*_c(S \otimes_k \overline{k})$ respectively. We use $D^b_c(S)$ to denote the derived category of $\mathbb{Q}_\ell$-complexes for the étale topology of S, as defined in [18]. We use $\mathbb{D}_S$ to denote the dualizing complex of S with $\mathbb{Q}_\ell$-coefficients.

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Part 1. **The analytic side**

2. The relative trace formula

In this section we set up the relative trace formula following Jacquet's approach [13] to the Waldspurger formula.

2.1. **Orbits.** In this subsection F is allowed to be an arbitrary field. Let F' be a semisimple quadratic F-algebra, i.e., it is either the split algebra $F \oplus F$ or a quadratic field extension of F. Denote by $\mathrm{Nm} : F' \to F$ the norm map.

Denote $G = \mathrm{PGL}_{2,F}$ and A the subgroup of diagonal matrices in G. We consider the action of $A \times A$ on G where $(h_1, h_2) \in A \times A$ acts by $(h_1, h_2)g = h_1^{-1} g h_2$. We define an $A \times A$-invariant morphism:

$$\begin{aligned} \mathrm{inv}\colon\ G &\longrightarrow \mathbb{P}^1_F - \{1\} \\ \gamma &\longmapsto \tfrac{bc}{ad} \end{aligned} \tag{2.1}$$

where $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2$ is a lifting of γ. We say that $\gamma \in G$ is $A \times A$-*regular semisimple* if

$$\mathrm{inv}(\gamma) \in \mathbb{P}^1_F - \{0, 1, \infty\},$$

or equivalently all a, b, c, d are invertible in terms of the lifting of γ. Let G_{rs} be the open subscheme of $A \times A$-regular semisimple locus. A section of the restriction of the morphism inv to G_{rs} is given by

$$\begin{aligned} \gamma\colon\ \mathbb{P}^1_F - \{0,1,\infty\} &\longrightarrow G \\ u &\longmapsto \gamma(u) = \begin{bmatrix} 1 & u \\ 1 & 1 \end{bmatrix}. \end{aligned} \tag{2.2}$$

Now we consider the induced map on the F-points $\mathrm{inv} : G(F) \to \mathbb{P}^1(F) - \{1\}$, and the action of $A(F) \times A(F)$ on $G(F)$. Denote by $\mathbb{O}_{\mathrm{rs}}(G) = A(F)\backslash G_{\mathrm{rs}}(F)/A(F)$ the set of orbits in $G_{\mathrm{rs}}(F)$ under the action of $A(F) \times A(F)$. They will be called the regular semisimple orbits. It is easy to see that the map $\mathrm{inv} : G_{\mathrm{rs}}(F) \to \mathbb{P}^1(F) - \{0, 1, \infty\}$ induces a bijection

$$\mathrm{inv} : \mathbb{O}_{\mathrm{rs}}(G) \longrightarrow \mathbb{P}^1(F) - \{0, 1, \infty\}.$$

A convenient set of representative of $\mathbb{O}_{\mathrm{rs}}(G)$ is given by

$$\mathbb{O}_{\mathrm{rs}}(G) \simeq \left\{ \gamma(u) = \begin{bmatrix} 1 & u \\ 1 & 1 \end{bmatrix} \,\middle|\, u \in \mathbb{P}^1(F) - \{0, 1, \infty\} \right\}.$$

There are six non-regular-semisimple orbits in $G(F)$, represented respectively by

$$\begin{aligned} 1 &= \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}, & n_+ &= \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}, & n_- &= \begin{bmatrix} 1 & \\ 1 & 1 \end{bmatrix}, \\ w &= \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}, & wn_+ &= \begin{bmatrix} & 1 \\ 1 & 1 \end{bmatrix}, & wn_- &= \begin{bmatrix} 1 & 1 \\ 1 & \end{bmatrix}, \end{aligned}$$

where the first three (the last three, resp.) have $\mathrm{inv} = 0$ (∞, resp.)

2.2. **Jacquet's RTF.** Now we return to the setting of the introduction. In particular, we have $\eta = \eta_{F'/F}$. In [13] Jacquet constructs an RTF to study the central value of L-functions of the same type as ours (mainly in the number field case). Here we modify his RTF to study higher derivatives.

For $f \in C_c^\infty(G(\mathbb{A}))$, we consider the automorphic kernel function

$$\mathbb{K}_f(g_1, g_2) = \sum_{\gamma \in G(F)} f(g_1^{-1} \gamma g_2), \quad g_1, g_2 \in G(\mathbb{A}). \tag{2.3}$$

We will define a distribution, given by a regularized integral

$$\mathbb{J}(f, s) = \int^{\mathrm{reg}}_{[A] \times [A]} \mathbb{K}_f(h_1, h_2) |h_1 h_2|^s \eta(h_2)\, dh_1\, dh_2.$$

Here we recall that $[A] = A(F)\backslash A(\mathbb{A})$, and for $h = \begin{bmatrix} a & \\ & d \end{bmatrix} \in A(\mathbb{A})$ we write for simplicity

$$\left|h\right| = \left|a/d\right|, \quad \eta(h) = \eta(a/d).$$

The integral is not always convergent but can be regularized in a way analogous to [13]. For an integer n, consider the "annulus"

$$\mathbb{A}^\times_n := \left\{ x \in \mathbb{A}^\times \,\middle|\, \mathrm{val}(x) = n \right\}.$$

This is a torsor under the group $\mathbb{A}^1 = \mathbb{A}^\times_0$. Let $A(\mathbb{A})_n$ be the subset of $A(\mathbb{A})$ defined by

$$A(\mathbb{A})_n = \left\{ \begin{bmatrix} a & \\ & d \end{bmatrix} \in A(\mathbb{A}) \,\middle|\, a/d \in \mathbb{A}^\times_n \right\}.$$

Then we define, for $(n_1, n_2) \in \mathbb{Z}^2$,

$$\mathbb{J}_{n_1,n_2}(f, s) = \int_{[A]_{n_1} \times [A]_{n_2}} \mathbb{K}_f(h_1, h_2)|h_1h_2|^s\eta(h_2)\, dh_1\, dh_2. \tag{2.4}$$

The integral (2.4) is clearly absolutely convergent and equal to a Laurent polynomial in q^s.

Proposition 2.1. *The integral $\mathbb{J}_{n_1,n_2}(f, s)$ vanishes when $|n_1| + |n_2|$ is sufficiently large.*

Granting this proposition, we then define

$$\mathbb{J}(f, s) := \sum_{(n_1,n_2)\in\mathbb{Z}^2} \mathbb{J}_{n_1,n_2}(f, s). \tag{2.5}$$

This is a Laurent polynomial in q^s.

The proof of Proposition 2.1 will occupy §2.3-2.5.

2.3. **A finiteness lemma.** For an $(A \times A)(F)$-orbit of γ, we define

$$\mathbb{K}_{f,\gamma}(h_1, h_2) = \sum_{\delta\in A(F)\gamma A(F)} f(h_1^{-1}\delta h_2), \quad h_1, h_2 \in A(\mathbb{A}). \tag{2.6}$$

Then we have

$$\mathbb{K}_f(h_1, h_2) = \sum_{\gamma\in A(F)\backslash G(F)/A(F)} \mathbb{K}_{f,\gamma}(h_1, h_2). \tag{2.7}$$

Lemma 2.2. *The sum in* (2.7) *has only finitely many non-zero terms.*

Proof. Denote by $G(F)_u$ the fiber of u under the (surjective) map (2.1)

$$\mathrm{inv} : G(F) \longrightarrow \mathbb{P}^1(F) - \{1\}.$$

We then have a decomposition of $G(F)$ as a disjoint union

$$G(F) = \coprod_{u\in\mathbb{P}^1(F)-\{1\}} G(F)_u.$$

There is exactly one (three, resp.) $(A\times A)(F)$-orbit in $G(F)_a$ when $u \in \mathbb{P}^1(F)-\{0, 1, \infty\}$ (when $u \in \{0, \infty\}$, resp.). It suffices to show that for all but finitely many $u \in \mathbb{P}^1(F) - \{0, 1, \infty\}$, the kernel function $\mathbb{K}_{f,\gamma(u)}(h_1, h_2)$ vanishes identically on $A(\mathbb{A}) \times A(\mathbb{A})$.

Consider the map

$$\tau := \frac{\mathrm{inv}}{1 - \mathrm{inv}} : G(\mathbb{A}) \longrightarrow \mathbb{A}.$$

The map τ is continuous and takes constant values on $A(\mathbb{A}) \times A(\mathbb{A})$-orbits. For $\mathbb{K}_{f,\gamma(u)}(h_1, h_2)$ to be nonzero, the invariant $\tau(\gamma(u)) = \frac{u}{1-u}$ must be in the image of *supp*(f), the support of the function f. Since *supp*(f) is compact, so is its image under τ. On the other hand, the invariant $\tau(\gamma(u)) = \frac{u}{1-u}$ belongs to F. Since the intersection of a compact set *supp*(f) with a discrete set F in $\mathbb{A}$ must have finite cardinality, the kernel function $\mathbb{K}_{f,\gamma(u)}(h_1, h_2)$ is nonzero for only finitely many u. □

For $\gamma \in A(F)\backslash G(F)/A(F)$, we define

$$\mathbb{J}_{n_1,n_2}(\gamma, f, s) = \int_{[A]_{n_1} \times [A]_{n_2}} \mathbb{K}_{f,\gamma}(h_1, h_2)|h_1h_2|^s\eta(h_2)\, dh_1\, dh_2. \tag{2.8}$$

Then we have

$$\mathbb{J}_{n_1,n_2}(f, s) = \sum_{\gamma\in A(F)\backslash G(F)/A(F)} \mathbb{J}_{n_1,n_2}(\gamma, f, s).$$

By the previous lemma, the above sum has only finitely many nonzero terms. Therefore, to show Prop. 2.1, it suffices to show

Proposition 2.3. *For any $\gamma \in G(F)$, the integral $\mathbb{J}_{n_1,n_2}(\gamma, f, s)$ vanishes when $|n_1| + |n_2|$ is sufficiently large.*

Granting this proposition, we may define the (weighted) orbital integral

$$\mathbb{J}(\gamma, f, s) := \sum_{(n_1,n_2)\in\mathbb{Z}^2} \mathbb{J}_{n_1,n_2}(\gamma, f, s). \tag{2.9}$$

To show Prop. 2.3, we distinguish two cases according to whether γ is regular semisimple.

2.4. **Proof of Proposition 2.3: regular semisimple orbits.** For $u \in \mathbb{P}^1(F) - \{0, 1, \infty\}$, the fiber $G(F)_u = \mathrm{inv}^{-1}(u)$ is a single $A(F)\times A(F)$-orbit of $\gamma(u)$, and the stabilizer of $\gamma(u)$ is trivial. We may rewrite (2.8) as

$$\mathbb{J}_{n_1,n_2}(\gamma(u), f, s) = \int_{A(\mathbb{A})_{n_1}\times A(\mathbb{A})_{n_2}} f(h_1^{-1}\gamma(u)h_2)|h_1h_2|^s\eta(h_2)\,dh_1\,dh_2. \tag{2.10}$$

For the regular semisimple $\gamma = \gamma(u)$, the map

$$\begin{aligned} \iota_\gamma : (A\times A)(\mathbb{A}) &\longrightarrow G(\mathbb{A}) \\ (h_1, h_2) &\longmapsto h_1^{-1}\gamma h_2 \end{aligned}$$

is a closed embedding. It follows that the function $f\circ\iota_\gamma$ has compact support, hence belongs to $C_c^\infty((A\times A)(\mathbb{A}))$. Therefore, the integrand in (2.10) vanishes when $|n_1| + |n_2| \gg 0$ (depending on f and $\gamma(u)$).

2.5. **Proof of Proposition 2.3: non-regular-semisimple orbits.** Let $u \in \{0, \infty\}$. We only consider the case $u = 0$ since the other case is completely analogous. There are three orbit representatives $\{1, n_+, n_-\}$.

It is easy to see that for $\gamma = 1$, we have for all $(n_1, n_2) \in \mathbb{Z}^2$,

$$\mathbb{J}_{n_1,n_2}(\gamma, f, s) = 0,$$

because $\eta|_{\mathbb{A}^1}$ is a nontrivial character.

Now we consider the case $\gamma = n_+$; the remaining case $\gamma = n_-$ is similar. Define a function

$$\phi(x, y) = f\left(\begin{bmatrix} x & y \\ & 1 \end{bmatrix}\right), \quad (x, y) \in \mathbb{A}^\times \times \mathbb{A}. \tag{2.11}$$

Then we have $\phi \in C_c^\infty(\mathbb{A}^\times\times\mathbb{A})$. The integral $\mathbb{J}_{n_1,n_2}(n_+, f, s)$ is given by

$$\int_{\mathbb{A}^\times_{n_1}\times\mathbb{A}^\times_{n_2}} \phi\left(x^{-1}y, x^{-1}\right)\eta(y)|xy|^s\,d^\times x\,d^\times y, \tag{2.12}$$

where we use the multiplicative measure $d^\times x$ on $\mathbb{A}^\times$. We substitute y by xy, and then x by x^{-1}:

$$\int_{Z(n_1,n_2)} \phi\left(y, x\right)\eta(xy)|x|^{-2s}|y|^s\,d^\times x\,d^\times y,$$

where

$$Z(n_1, n_2) = \left\{(x, y) \mid x \in A(\mathbb{A})_{-n_1}, x^{-1}y \in A(\mathbb{A})_{n_2}\right\}.$$

Since $C_c^\infty(\mathbb{A}^\times\times\mathbb{A}) \simeq C_c^\infty(\mathbb{A}^\times)\otimes C_c^\infty(\mathbb{A})$, we may reduce to the case $\phi(x, y) = \phi_1(x)\phi_2(y)$ where $\phi_1 \in C_c^\infty(\mathbb{A}^\times), \phi_2 \in C_c^\infty(\mathbb{A})$. Moreover, by writing ϕ_1 as a finite linear combination, each supported on a single $\mathbb{A}^\times_n$, we may even assume that *supp*(ϕ_1) is contained in $\mathbb{A}^\times_n$, for some $n \in \mathbb{Z}$. The last integral is equal to

$$\left(\int_{\mathbb{A}^\times_n} \phi_1\left(y\right)\eta(y)|y|^s\,d^\times y\right)\left(\int_{\mathbb{A}^\times_{-n_1}\cap\mathbb{A}^\times_{-n_2+n}} \phi_2\left(x\right)\eta(x)|x|^{-2s}\,d^\times x\right).$$

Finally we recall that, from Tate's thesis, for any $\varphi \in C_c^\infty(\mathbb{A})$, the integral on an annulus

$$\int_{\mathbb{A}^\times_n} \varphi\left(x\right)\eta(x)|x|^{2s}\,d^\times x,$$

vanishes when $|n| \gg 0$. We briefly recall how this is proved. It is clear if $n \ll 0$. Now assume that $n \gg 0$. We rewrite the integral as

$$\int_{F^\times\backslash\mathbb{A}^\times_n} \sum_{\alpha\in F^\times} \varphi\left(\alpha x\right)\eta(x)|x|^{2s}\,d^\times x,$$

The Fourier transform of φ, denoted by $\widehat{\varphi}$, still lies in $C_c^\infty(\mathbb{A})$. By the Poisson summation formula, we have

$$\sum_{\alpha\in F^\times}\varphi(\alpha x) = -\varphi(0) + |x|^{-1}\widehat{\varphi}(0) + |x|^{-1}\sum_{\alpha\in F^\times}\widehat{\varphi}(\alpha/x), \tag{2.13}$$

By the boundedness of the support of $\widehat{\varphi}$, the sum over $F^\times$ on the RHS vanishes when $\mathrm{val}(x) = n \gg 0$. Finally we note that the the integral of the remaining two terms on the RHS of (2.13) vanishes because η is nontrivial on $F^\times\backslash\mathbb{A}^1$.

This completes the proof of Prop. 2.3, and Prop. 2.1.

2.6. **The distribution $\mathbb{J}$.** Now $\mathbb{J}(f,s)$ is a Laurent polynomial in q^s. Consider the r-th derivative

$$\mathbb{J}_r(f) := \left(\frac{d}{ds}\right)^r\Big|_{s=0}\mathbb{J}(f,s).$$

For $\gamma\in A(F)\backslash G(F)/A(F)$, we define

$$\mathbb{J}_r(\gamma,f) := \left(\frac{d}{ds}\right)^r\Big|_{s=0}\mathbb{J}(\gamma,f,s).$$

We then have an expansion (cf.(2.5))

$$\mathbb{J}(f,s) = \sum_{\gamma\in A(F)\backslash G(F)/A(F)}\mathbb{J}(\gamma,f,s),$$

and (cf. (2.9))

$$\mathbb{J}_r(f) = \sum_{\gamma\in A(F)\backslash G(F)/A(F)}\mathbb{J}_r(\gamma,f). \tag{2.14}$$

We define

$$\mathbb{J}(u,f,s) = \sum_{\gamma\in A(F)\backslash G(F)_u/A(F)}\mathbb{J}(\gamma,f,s),\quad u\in\mathbb{P}^1(F)-\{1\}. \tag{2.15}$$

and

$$\mathbb{J}_r(u,f) = \sum_{\gamma\in A(F)\backslash G(F)_u/A(F)}\mathbb{J}_r(\gamma,f),\quad u\in\mathbb{P}^1(F)-\{1\}. \tag{2.16}$$

Then we have a slightly coarser decomposition than (2.14)

$$\mathbb{J}_r(f) = \sum_{u\in\mathbb{P}^1(F)-\{1\}}\mathbb{J}_r(u,f). \tag{2.17}$$

2.7. **A special test function $f = \mathbf{1}_K$.**

Proposition 2.4. *For the test function*

$$f = \mathbf{1}_K,$$

we have

$$\mathbb{J}(u,\mathbf{1}_K,s) = \begin{cases} L(\eta,2s)+L(\eta,-2s) & \textit{if } u\in\{0,\infty\},\\ 1 & \textit{if } u\in k-\{0,1\},\\ 0 & \textit{otherwise}.\end{cases} \tag{2.18}$$

Proof. We first consider the case $u\in\mathbb{P}^1(F)-\{0,1,\infty\}$. In this case, we have

$$\begin{aligned}\mathbb{J}(u,\mathbf{1}_K,s) &= \int_{\mathbb{A}^\times\times\mathbb{A}^\times}\mathbf{1}_K\left(\begin{bmatrix}x^{-1}&0\\0&1\end{bmatrix}\begin{bmatrix}1&u\\1&1\end{bmatrix}\begin{bmatrix}y&0\\0&1\end{bmatrix}\right)|xy|^s\eta(y)\,d^\times x\,d^\times y\\ &= \sum_{x,y\in\mathbb{A}^\times/\mathbb{O}^\times}\mathbf{1}_K\left(\begin{bmatrix}x^{-1}y&x^{-1}u\\y&1\end{bmatrix}\right)|xy|^s\eta(y).\end{aligned}$$

The integrand is nonzero if and only if $g = \begin{bmatrix} x^{-1}y & x^{-1}u \\ y & 1 \end{bmatrix} \in K$. This is equivalent to the condition that $g_{ij}^2/\det(g) \in \mathbb{O}$, where $\{g_{ij}\}_{1\le i,j\le 2}$ are the entries of g. We have $\det(g) = x^{-1}y(1-u)$, therefore $g \in K$ is equivalent to

$$x^{-1}y(1-u)^{-1} \in \mathbb{O}, x^{-1}y^{-1}u^2(1-u)^{-1} \in \mathbb{O}, xy(1-u)^{-1} \in \mathbb{O}, \text{ and } xy^{-1}(1-u)^{-1} \in \mathbb{O}. \quad (2.19)$$

Multiplying the first and last condition we get $(1-u)^{-1} \in \mathbb{O}$. Therefore $1-u \in F^\times$ must be a constant function, i.e., $u \in k - \{0,1\}$. This shows that $\mathbb{J}(u, \mathbf{1}_K, s) = 0$ when $u \in F - k$.

When $u \in k - \{0,1\}$, the conditions (2.19) become

$$x^{-1}y \in \mathbb{O}, x^{-1}y^{-1} \in \mathbb{O}, xy \in \mathbb{O}, \text{ and } xy^{-1} \in \mathbb{O}.$$

These together imply that $x, y \in \mathbb{O}^\times$. Therefore the integrand is nonzero only when both x and y are in the unit coset of $\mathbb{A}^\times/\mathbb{O}^\times$, and the integrand is equal to 1 when this happens. This proves $\mathbb{J}(u, \mathbf{1}_K, s) = 1$ when $u \in k - \{0,1\}$.

Next we consider the case $u = 0$. For $f = \mathbf{1}_K$ and $\gamma = n_+$, we have in (2.11) $\phi = \phi_1 \otimes \phi_2$ where

$$\phi_1 = \mathbf{1}_{\mathbb{O}^\times}, \quad \phi_2 = \mathbf{1}_{\mathbb{O}}.$$

Therefore we have

$$\mathbb{J}(n_+, \mathbf{1}_K, s) = \int_{\mathbb{A}^\times} \phi_2(x)\eta(x)|x|^{-2s}\, d^\times x = L(\eta, -2s).$$

Similarly we have

$$\mathbb{J}(n_-, \mathbf{1}_K, s) = L(\eta, 2s).$$

This proves the equality (2.18) for $u = 0$. The case for $u = \infty$ is analogous. □

Corollary 2.5. *We have*

$$\mathbb{J}_r(\mathbf{1}_K) = \begin{cases} 4L(\eta,0) + q - 2 = 4\frac{\#\operatorname{Jac}_{X'}(k)}{\#\operatorname{Jac}_X(k)} + q - 2 & r = 0; \\ 2^{r+2}\left(\frac{d}{ds}\right)^r\Big|_{s=0} L(\eta, s) & r > 0 \text{ even}; \\ 0 & r > 0 \text{ odd}. \end{cases}$$

3. Geometric interpretation of orbital integrals

In this section, we will give a geometric interpretation the orbital integrals $\mathbb{J}(\gamma, f, s)$ (cf. (2.9)) as a certain weighted counting of effective divisors on the curve X, when f is in the unramified Hecke algebra.

3.1. A basis for the Hecke algebra.

Let $x \in |X|$. In the case $G = \mathrm{PGL}_2$, the local unramified Hecke algebra $\mathscr{H}_x$ is the polynomial algebra $\mathbb{Q}[h_x]$ where h_x is the characteristic function of the $G(\mathcal{O}_x)$-double coset of $\begin{bmatrix} \varpi_x & 0 \\ 0 & 1 \end{bmatrix}$, and $\varpi_x \in \mathcal{O}_x$ is a uniformizer. For each integer $n \ge 0$, consider the set $\mathrm{Mat}_2(\mathcal{O}_x)_{v_x(\det)=n}$ of matrices $A \in \mathrm{Mat}_2(\mathcal{O}_x)$ such that $v_x(\det(A)) = n$. Let $M_{x,n}$ be the image of $\mathrm{Mat}_2(\mathcal{O}_x)_{v_x(\det)=n}$ in $G(F_x)$. Then $M_{x,n}$ is a union of $G(\mathcal{O}_x)$-double cosets. We define h_{nx} to be the characteristic function

$$h_{nx} = \mathbf{1}_{M_{x,n}}. \quad (3.1)$$

Then $\{h_{nx}\}_{n\ge 0}$ is a $\mathbb{Q}$-basis for $\mathscr{H}_x$.

Now consider the global unramified Hecke algebra $\mathscr{H} = \otimes_{x\in|X|}\mathscr{H}_x$, which is a polynomial ring over $\mathbb{Q}$ with infinitely generators h_x. For each effective divisor $D = \sum_{x\in|X|} n_x \cdot x$, we can define an element $h_D \in \mathscr{H}$ using

$$h_D = \otimes_{x\in|X|} h_{n_x x} \quad (3.2)$$

where $h_{n_x x}$ is defined in (3.1). It is easy to see that the set $\{h_D | \text{D effective divisor on X}\}$ is a $\mathbb{Q}$-basis for $\mathscr{H}$.

The goal of the next few subsections is to give a geometric interpretation the orbital integral $\mathbb{J}(\gamma, h_D, s)$. We begin by defining certain moduli spaces.

3.2. Global moduli space for orbital integrals.

3.2.1. For $d \in \mathbb{Z}$, we consider the Picard stack Pic_X^d of lines bundles over X of degree d. Note that Pic_X^d is a $\mathbb{G}_m$-gerbe over its coarse moduli space. Let $\widehat{X}_d \to \mathrm{Pic}_X^d$ be the universal family of sections of line bundles, i.e., an S-point of $\widehat{X}_d$ is a pair $(\mathcal{L}, s)$, where $\mathcal{L}$ is a line bundle over $X \times S$ such that $\deg \mathcal{L}|_{X\times\{t\}} = d$ for all geometric points t of S, and $s \in \mathrm{H}^0(X \times S, \mathcal{L})$.

When $d < 0$, $\widehat{X}_d \cong \mathrm{Pic}_X^d$ since all global sections of all line bundles $\mathcal{L} \in \mathrm{Pic}_X^d$ vanish. When $d \geq 0$, let $X_d = X^d /\!\!/ S_d$ be the d-th symmetric power of X. Then there is an open embedding $X_d \hookrightarrow \widehat{X}_d$ as the open locus of nonzero sections, with complement isomorphic to Pic_X^d.

For $d_1, d_2 \in \mathbb{Z}$, we have a morphism

$$\widehat{\mathrm{add}}_{d_1,d_2} : \widehat{X}_{d_1} \times \widehat{X}_{d_2} \longrightarrow \widehat{X}_{d_1+d_2}$$

sending $((\mathcal{L}_1, s_1), (\mathcal{L}_2, s_2))$ to $(\mathcal{L}_1 \otimes \mathcal{L}_2, s_1 \otimes s_2)$. The restriction of $\widehat{\mathrm{add}}_{d_1,d_2}$ to the open subset $X_{d_1} \times X_{d_2}$ becomes the addition map for divisors $\mathrm{add}_{d_1,d_2} : X_{d_1} \times X_{d_2} \to X_{d_1+d_2}$.

3.2.2. *The moduli space $\mathcal{N}_{\underline{d}}$.* Let $d \geq 0$ be an integer. Let Σ_d be the set of quadruple of non-negative integers $\underline{d} = (d_{ij})_{i,j\in\{1,2\}}$ satisfying $d_{11} + d_{22} = d_{12} + d_{21} = d$.

For $\underline{d} \in \Sigma_d$, we consider the moduli functor $\widetilde{\mathcal{N}}_{\underline{d}}$ classifying $(\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}'_1, \mathcal{K}'_2, \varphi)$ where

- $\mathcal{K}_i, \mathcal{K}'_i \in \mathrm{Pic}_X$ with $\deg \mathcal{K}'_i - \deg \mathcal{K}_j = d_{ij}$.
- $\varphi : \mathcal{K}_1 \oplus \mathcal{K}_2 \to \mathcal{K}'_1 \oplus \mathcal{K}'_2$ is an $\mathcal{O}_X$-linear map. We express it as a matrix
$$\varphi = \begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix}$$
where $\varphi_{ij} : \mathcal{K}_j \to \mathcal{K}'_i$.
- If $d_{11} < d_{22}$, then $\varphi_{11} \neq 0$ otherwise $\varphi_{22} \neq 0$. If $d_{12} < d_{21}$ then $\varphi_{12} \neq 0$ otherwise $\varphi_{21} \neq 0$. Moreover, at most one of the four maps $\varphi_{ij}, i, j \in \{1, 2\}$ can be zero.

The Picard stack Pic_X acts on $\widetilde{\mathcal{N}}_{\underline{d}}$ by tensoring each $\mathcal{K}_i$ and $\mathcal{K}'_j$ with the same line bundle. Let $\mathcal{N}_{\underline{d}}$ be the quotient stack $\widetilde{\mathcal{N}}_{\underline{d}}/\mathrm{Pic}_X$, which will turn out to be representable by a scheme over k. We remark that the artificial-looking last condition in the definition of $\mathcal{N}_{\underline{d}}$ is to guarantee that $\mathcal{N}_{\underline{d}}$ is separated.

3.2.3. *The base $\mathcal{A}_d$.* Let $\mathcal{A}_d$ be the moduli stack of triples (Δ, a, b) where $\Delta \in \mathrm{Pic}_X^d$, a and b are sections of Δ with the open condition that a and b are not simultaneously zero. Then we have an isomorphism

$$\mathcal{A}_d \cong \widehat{X}_d \times_{\mathrm{Pic}_X^d} \widehat{X}_d - Z_d \tag{3.3}$$

where $Z_d \cong \mathrm{Pic}_X^d$ is the image of the diagonal zero sections $(0, 0) : \mathrm{Pic}_X^d \hookrightarrow \widehat{X}_d \times_{\mathrm{Pic}_X^d} \widehat{X}_d$.

We claim that $\mathcal{A}_d$ is a scheme. In fact, $\mathcal{A}$ is covered by two opens $V = \widehat{X}_d \times_{\mathrm{Pic}_X^d} X_d$ and $V' = X_d \times_{\mathrm{Pic}_X^d} \widehat{X}_d$. Both V and V' are schemes because the map $\widehat{X}_d \to \mathrm{Pic}_X^d$ is schematic.

We have a map

$$\delta : \mathcal{A}_d \longrightarrow \widehat{X}_d$$

given by $(\Delta, a, b) \mapsto (\Delta, a - b)$.

3.2.4. *The open part $\mathcal{A}_d^\heartsuit$.* Later we will consider the open subscheme $\mathcal{A}_d^\heartsuit \subset \mathcal{A}_d$ defined by the condition $a \neq b$, i.e., the preimage of X_d under the map $\delta : \mathcal{A}_d \to \widehat{X}_d$.

3.2.5. To a point $(\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}'_1, \mathcal{K}'_2, \varphi) \in \widetilde{\mathcal{N}}_{\underline{d}}$ we attach the following maps

- $a := \varphi_{11} \otimes \varphi_{22} : \mathcal{K}_1 \otimes \mathcal{K}_2 \to \mathcal{K}'_1 \otimes \mathcal{K}'_2$;
- $b := \varphi_{12} \otimes \varphi_{21} : \mathcal{K}_1 \otimes \mathcal{K}_2 \to \mathcal{K}'_2 \otimes \mathcal{K}'_1 \cong \mathcal{K}'_1 \otimes \mathcal{K}'_2$.

Both a and b can be viewed as sections of the line bundle $\Delta = \mathcal{K}'_1 \otimes \mathcal{K}'_2 \otimes \mathcal{K}_1^{-1} \otimes \mathcal{K}_2^{-1} \in \mathrm{Pic}_X^d$. Clearly this assignment $(\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}'_1, \mathcal{K}'_2, \varphi) \mapsto (\Delta, a, b)$ is invariant under the action of Pic_X on $\widetilde{\mathcal{N}}_{\underline{d}}$. Therefore we get a map

$$f_{\mathcal{N}_{\underline{d}}} : \mathcal{N}_{\underline{d}} \longrightarrow \mathcal{A}_d.$$

The composition $\delta \circ f_{\mathcal{N}_{\underline{d}}} : \mathcal{N}_{\underline{d}} \to \widehat{X}_d$ takes $(\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}'_1, \mathcal{K}'_2, \varphi)$ to $\det(\varphi)$ as a section of $\Delta = \mathcal{K}'_1 \otimes \mathcal{K}'_2 \otimes \mathcal{K}_1^{-1} \otimes \mathcal{K}_2^{-1}$.

3.2.6. *Geometry of $\mathcal{N}_{\underline{d}}$.* Fix $\underline{d} = (d_{ij}) \in \Sigma_d$. For $i, j \in \{1, 2\}$, we have a morphism $\jmath_{ij} : \mathcal{N}_{\underline{d}} \to \widehat{X}_{d_{ij}}$ sending $(\mathcal{K}_1, \cdots, \mathcal{K}'_2, \varphi)$ to the section φ_{ij} of the line bundle $\mathcal{L}_{ij} := \mathcal{K}'_i \otimes \mathcal{K}_j^{-1} \in \mathrm{Pic}_X^{d_{ij}}$. We have canonical isomorphisms $\mathcal{L}_{11} \otimes \mathcal{L}_{22} \cong \mathcal{L}_{12} \otimes \mathcal{L}_{21} \cong \Delta = \mathcal{K}'_1 \otimes \mathcal{K}'_2 \otimes \mathcal{K}_1^{-1} \otimes \mathcal{K}_2^{-1}$. Thus we get a morphism

$$\jmath_{\underline{d}} = (\jmath_{ij})_{i,j} : \mathcal{N}_{\underline{d}} \longrightarrow (\widehat{X}_{d_{11}} \times \widehat{X}_{d_{22}}) \times_{\mathrm{Pic}_X^d} (\widehat{X}_{d_{12}} \times \widehat{X}_{d_{21}}). \tag{3.4}$$

Here the fiber product on the right side is formed using the maps $\widehat{X}_{d_{11}} \times \widehat{X}_{d_{22}} \to \mathrm{Pic}_X^{d_{11}} \times \mathrm{Pic}_X^{d_{22}} \xrightarrow{\otimes} \mathrm{Pic}_X^d$ and $\widehat{X}_{d_{12}} \times \widehat{X}_{d_{21}} \to \mathrm{Pic}_X^{d_{12}} \times \mathrm{Pic}_X^{d_{21}} \xrightarrow{\otimes} \mathrm{Pic}_X^d$.

Proposition 3.1. *Let $\underline{d} \in \Sigma_d$.*

(1) The morphism $\jmath_{\underline{d}}$ is an open embedding, and $\mathcal{N}_{\underline{d}}$ is a geometrically connected scheme over k.

(2) If $d \geq 2g' - 1 = 4g - 3$, $\mathcal{N}_{\underline{d}}$ is smooth over k of dimension $2d - g + 1$.

(3) We have a commutative diagram

$$\begin{tikzcd} \mathcal{N}_{\underline{d}} \arrow[r, hook, "\jmath_{\underline{d}}"] \arrow[d, "f_{\mathcal{N}_{\underline{d}}}"] & (\widehat{X}_{d_{11}} \times \widehat{X}_{d_{22}}) \times_{\mathrm{Pic}_X^d} (\widehat{X}_{d_{12}} \times \widehat{X}_{d_{21}}) \arrow[d, "\widehat{\mathrm{add}}_{d_{11},d_{22}} \times \widehat{\mathrm{add}}_{d_{12},d_{21}}"] \\ \mathcal{A}_d \arrow[r, hook] & \widehat{X}_d \times_{\mathrm{Pic}_X^d} \widehat{X}_d \end{tikzcd} \tag{3.5}$$

Moreover, the map $f_{\mathcal{N}_{\underline{d}}}$ is proper.

Proof. (1) We abbreviate Pic_X^d by P^d. Let $Z_{\underline{d}} \subset (\widehat{X}_{d_{11}} \times \widehat{X}_{d_{22}}) \times_{\mathrm{Pic}_X^d} (\widehat{X}_{d_{12}} \times \widehat{X}_{d_{21}})$ be the closed substack consisting of $((\mathcal{L}_{ij}, s_{ij}) \in \widehat{X}_{d_{ij}})_{1 \leq i,j \leq 2}$ such that

- Either two of $\{s_{ij}\}_{1 \leq i,j \leq 2}$ are zero;
- Or $s_{11} = 0$ if $d_{11} < d_{22}$;
- Or $s_{22} = 0$ if $d_{11} \geq d_{22}$;
- Or $s_{12} = 0$ if $d_{12} < d_{21}$;
- Or $s_{21} = 0$ if $d_{12} \geq d_{21}$.

By the definition of $\mathcal{N}_{\underline{d}}$, we have a Cartesian diagram

$$\begin{tikzcd} \mathcal{N}_{\underline{d}} \arrow[r, "\jmath_{\underline{d}}"] \arrow[d, "\lambda"] & (\widehat{X}_{d_{11}} \times \widehat{X}_{d_{22}}) \times_{P^d} (\widehat{X}_{d_{12}} \times \widehat{X}_{d_{21}}) - Z_{\underline{d}} \arrow[d] \\ P^{d_{11}-d_{12}} \times P^{d_{11}} \times P^{d_{21}} \arrow[r, "\rho"] & (P^{d_{11}} \times P^{d_{22}}) \times_{P^d} (P^{d_{12}} \times P^{d_{21}}) \end{tikzcd}$$

Here λ sends $(\mathcal{K}_1, \cdots, \mathcal{K}'_2, \varphi)$ to $(\mathcal{X}_2 = \mathcal{K}_2 \otimes \mathcal{K}_1^{-1}, \mathcal{X}'_1 = \mathcal{K}'_1 \otimes \mathcal{K}_1^{-1}, \mathcal{X}'_2 = \mathcal{K}'_2 \otimes \mathcal{K}_1^{-1})$, and ρ sends $(\mathcal{X}_2, \mathcal{X}'_1, \mathcal{X}'_2)$ to $(\mathcal{X}'_1, \mathcal{X}'_2 \otimes \mathcal{X}_2^{-1}, \mathcal{X}'_1 \otimes \mathcal{X}_2^{-1}, \mathcal{X}'_2)$. Note that ρ is an isomorphism. Therefore $\jmath_{\underline{d}}$ is an isomorphism. Since the geometric fibers of λ are connected, and $P^{d_{11}-d_{12}} \times P^{d_{11}} \times P^{d_{21}}$ is geometrically connected, so is $\mathcal{N}_{\underline{d}}$.

The stack $\mathcal{N}_{\underline{d}}$ is covered by four open substacks $U_{ij}, i, j \in \{1, 2\}$ where U_{ij} is the locus where only φ_{ij} is allow to be zero. Each U_{ij} is a scheme over k. In fact, for example, U_{11} is an open substack of $(\widehat{X}_{d_{11}} \times X_{d_{22}}) \times_{P^d} (X_{d_{12}} \times X_{d_{21}})$, and the latter is a scheme since the morphism $\widehat{X}_{d_{11}} \to P^{d_{11}}$ is schematic.

(2) We first show that $\mathcal{N}_{\underline{d}}$ is smooth when $d \geq 2g' - 1 = 4g - 3$. For this we only need to show that U_{ij} is smooth (see the proof of part (1) for the definition of U_{ij}). By the definition of $\mathcal{N}_{\underline{d}}$, φ_{ij} is allowed to be zero only when $d_{ij} \geq d/2$, which implies that $d_{ij} \geq 2g - 1$. Therefore, we need U_{ij} to cover $\mathcal{N}_{\underline{d}}$ only when $d_{ij} \geq 2g - 1$; otherwise φ_{ij} is never zero and the rest of the $U_{i',j'}$ still cover $\mathcal{N}_{\underline{d}}$. Therefore, we only need to prove the smoothness of U_{ij} under the assumption that $d_{ij} \geq d/2$. Without loss of generality we argue for $i = j = 1$. Then $d_{11} \geq 2g - 1$ implies

that the Abel-Jacobi map $\mathrm{AJ}_{d_{11}} : \widehat{X}_{d_{11}} \to P^{d_{11}}$ is smooth of relative dimension $d_{11} - g + 1$. We have a Cartesian diagram

$$\begin{tikzcd} U_{11} \arrow[r] \arrow[d] & \widehat{X}_{d_{11}} \arrow[d, "\mathrm{AJ}_{d_{11}}"] \\ X_{d_{22}} \times X_{d_{12}} \times X_{d_{21}} \arrow[r] & P^{d_{11}} \end{tikzcd}$$

where the bottom horizontal map is given by $(\mathcal{L}_{22}, s_{22}, \mathcal{L}_{12}, s_{12}, \mathcal{L}_{21}, s_{21}) \mapsto \mathcal{L}_{12} \otimes \mathcal{L}_{21} \otimes \mathcal{L}_{22}^{-1}$. Therefore U_{11} is smooth over $X_{d_{22}} \times X_{d_{12}} \times X_{d_{21}}$ with relative dimension $d_{11} - g + 1$, and U_{11} is itself smooth over k of dimension $2d - g + 1$.

(3) The commutativity of the diagram (3.5) is clear from the definition of $\jmath_{\underline{d}}$. Finally we show that $f_{\mathcal{N}_{\underline{d}}} : \mathcal{N}_{\underline{d}} \to \mathcal{A}_d$ is proper. Note that $\mathcal{A}_d$ is covered by open subschemes $V = \widehat{X}_d \times_{P^d} X_d$ and $V' = X_d \times_{P^d} \widehat{X}_d$ whose preimages under $f_{\mathcal{N}_{\underline{d}}}$ are $U_{11} \cup U_{22}$ and $U_{12} \cup U_{21}$ respectively. Therefore it suffices to show that $f_V : U_{11} \cup U_{22} \to V$ and $f_{V'} : U_{12} \cup U_{21} \to V'$ are both proper.

We argue for the properness of f_V. There are two cases: either $d_{11} \geq d_{22}$ or $d_{11} < d_{22}$.

When $d_{11} \geq d_{22}$, by the last condition in the definition of $\mathcal{N}_{\underline{d}}$, φ_{22} is never zero, hence $U_{11} \cup U_{22} = U_{11}$. By part (2), the map f_V becomes

$$(\widehat{X}_{d_{11}} \times X_{d_{22}}) \times_{P^d} (X_{d_{12}} \times X_{d_{21}}) \longrightarrow \widehat{X}_d \times_{P^d} X_d.$$

Therefore it suffices to show that the restriction of the addition map

$$\alpha = \widehat{\mathrm{add}}_{d_{11},d_{22}}|_{\widehat{X}_{d_{11}} \times X_{d_{22}}} : \widehat{X}_{d_{11}} \times X_{d_{22}} \longrightarrow \widehat{X}_d$$

is proper. We may factor α as the composition of the closed embedding $\widehat{X}_{d_{11}} \times X_{d_{22}} \to \widehat{X}_d \times X_{d_{22}}$ sending $(\mathcal{L}_{11}, s_{11}, D_{22})$ to $(\mathcal{L}_{11}(D_{22}), s_{11}, D_{22})$ and the projection $\widehat{X}_d \times X_{d_{22}} \to \widehat{X}_d$, and the properness of α follows.

The case $d_{11} < d_{22}$ is argued in the same way. The properness of $f_{V'}$ is also proved in the similar way. This finishes the proof of the properness of $f_{\mathcal{N}_{\underline{d}}}$. □

3.3. **Relation with orbital integrals.** In this subsection we relate the derivative orbital integral $\mathbb{J}(\gamma, h_D, s)$ to the cohomology of fibers of $f_{\mathcal{N}_{\underline{d}}}$.

3.3.1. *The local system $L_{\underline{d}}$.* Recall that $\nu : X' \to X$ is a geometrically connected étale double cover with the nontrivial involution $\sigma \in \mathrm{Gal}(X'/X)$. Let $L = (\nu_* \mathbb{Q}_\ell)^{\sigma=-1}$. This is a rank one local system on X with $L^{\otimes 2} \cong \mathbb{Q}_\ell$. Since we have a canonical isomorphism $\mathrm{H}_1(X, \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{H}_1(\mathrm{Pic}^n_X, \mathbb{Z}/2\mathbb{Z})$, each Pic^n_X carries a rank one local system L_n corresponding to L. By abuse of notation, we also denote the pullback of L_n to $\widehat{X}_n$ by L_n. Note that the pullback of L_n to X_n via the Abel-Jacobi map $X_n \to \mathrm{Pic}^n_X$ is the descent of $L^{\boxtimes n}$ along the natural map $X^n \to X_n$.

Using the map $\jmath_{\underline{d}}$ (3.4), we define the following local system $L_{\underline{d}}$ on $\mathcal{N}_{\underline{d}}$:

$$L_{\underline{d}} := \jmath_{\underline{d}}^*(L_{d_{11}} \boxtimes \mathbb{Q}_\ell \boxtimes L_{d_{12}} \boxtimes \mathbb{Q}_\ell).$$

3.3.2. Fix $D \in X_d(k)$. Let $\mathcal{A}_D \subset \mathcal{A}_d$ be the fiber of $\mathcal{A}_d$ over D under the map $\delta : \mathcal{A}_d \to \widehat{X}_d$. Then $\mathcal{A}_D$ classifies triples $(\mathcal{O}_X(D), a, b)$ in $\mathcal{A}_d$ with the condition that $a - b$ is the tautological section $1 \in \Gamma(X, \mathcal{O}_X(D))$. Such a triple is determined uniquely by the section $a \in \Gamma(X, \mathcal{O}_X(D))$. Therefore we get canonical isomorphisms (viewing the RHS as an affine spaces over k)

$$\mathcal{A}_D \quad \cong \quad \Gamma(X, \mathcal{O}_X(D)). \tag{3.6}$$

On the level of k-points, we have an injective map

$$\begin{array}{rcl} \mathrm{inv}_D : \mathcal{A}_D(k) \cong \Gamma(X, \mathcal{O}_X(D)) & \hookrightarrow & \mathbb{P}^1(F) - \{1\} \\ (\mathcal{O}_X(D), a, a-1) \leftrightarrow a & \longmapsto & (a-1)/a = 1 - a^{-1}. \end{array}$$

Proposition 3.2. *Let $D \in X_d(k)$ and consider the test function h_D defined in* (3.2). *Let $u \in \mathbb{P}^1(F) - \{1\}$.*

(1) If u is not in the image of inv_D, *then $\mathbb{J}(\gamma, h_D, s) = 0$ for any $\gamma \in A(F)\backslash G(F)/A(F)$ with* $\mathrm{inv}(\gamma) = u$*;*

(2) *If* $u = \mathrm{inv}_D(a)$ *for some* $a \in \mathcal{A}_D(k) = \Gamma(X, \mathcal{O}_X(D))$*, and* $u \notin \{0, 1, \infty\}$ *(i.e.,* $a \notin \{0, 1\}$*), then*

$$\mathbb{J}(\gamma(u), h_D, s) = \sum_{\underline{d} \in \Sigma_d} q^{(2d_{12}-d)s} \operatorname{Tr}\Big(\mathrm{Frob}_a, \big(\mathbf{R}f_{\mathcal{N}_{\underline{d}},*} L_{\underline{d}}\big)_{\overline{a}}\Big).$$

(3) *Assume* $d \geq 2g' - 1 = 4g - 3$*. If* $u = 0$ *then it corresponds to* $a = 1 \in \mathcal{A}_D(k)$*; if* $u = \infty$ *then it corresponds to* $a = 0 \in \mathcal{A}_D(k)$*. In both cases we have*

$$\sum_{\mathrm{inv}(\gamma) = u} \mathbb{J}(\gamma, h_D, s) = \sum_{\underline{d} \in \Sigma_d} q^{(2d_{12}-d)s} \operatorname{Tr}\Big(\mathrm{Frob}_a, \big(\mathbf{R}f_{\mathcal{N}_{\underline{d}},*} L_{\underline{d}}\big)_{\overline{a}}\Big). \tag{3.7}$$

Here the sum on the LHS is over the three irregular double cosets $\gamma \in \{1, n_+, n_-\}$ *if* $u = 0$*, and over* $\gamma \in \{w, wn_+, wn_-\}$ *if* $u = \infty$*.*

Proof. We first make some general construction. Let $\widetilde{A} \subset \mathrm{GL}_2$ be the diagonal torus and let $\widetilde{\gamma} \in \mathrm{GL}_2(F) - (\widetilde{A}(F) \cup w\widetilde{A}(F))$ with image $\gamma \in G(F)$. Let $\alpha : \widetilde{A} \to \mathbb{G}_m$ be the simple root $\begin{bmatrix} a & \\ & d \end{bmatrix} \mapsto a/d$. Let $Z \cong \mathbb{G}_m \subset \widetilde{A}$ be the center of GL_2. We may rewrite $\mathbb{J}(\gamma, h_D, s)$ as an orbital integral on $\widetilde{A}(\mathbb{A})$-double cosets on $\mathrm{GL}_2(\mathbb{A})$ (cf. (2.10), (2.11), (2.12))

$$\mathbb{J}(\gamma, h_D, s) = \int_{\Delta(Z(\mathbb{A}))\backslash(\widetilde{A} \times \widetilde{A})(\mathbb{A})} \widetilde{h}_D(t'^{-1}\widetilde{\gamma}t)|\alpha(t)\alpha(t')|^s \eta(\alpha(t))\, dt\, dt'. \tag{3.8}$$

Here for $D = \sum_x n_x x$, $\widetilde{h}_D = \otimes_x \widetilde{h}_{n_x x}$ is an element in the global unramified Hecke algebra for GL_2, where $\widetilde{h}_{n_x x}$ is the characteristic function of $\mathrm{Mat}_2(\mathcal{O}_x)_{v_x(\det) = n_x}$, cf. §3.1.

Using the isomorphism $\widetilde{A}(\mathbb{A})/\prod_{x \in |X|} \widetilde{A}(\mathcal{O}_x) \cong (\mathbb{A}^\times/\mathbb{O}^\times)^2 \cong \mathrm{Div}(X)^2$ given by taking the divisors of the two diagonal entries, we may further write the RHS of (3.8) as a sum over divisors $E_1, E_2, E_1', E_2' \in \mathrm{Div}(X)$, up to simultaneous translation by $\mathrm{Div}(X)$. Suppose $t \in \widetilde{A}(\mathbb{A})$ gives the pair (E_1, E_2) and $t' \in \widetilde{A}(\mathbb{A})$ gives the pair (E_1', E_2'), then the integrand $\widetilde{h}_D(t'^{-1}\widetilde{\gamma}t)$ takes value 1 if and only if the rational map $\widetilde{\gamma} : \mathcal{O}_X^2 \dashrightarrow \mathcal{O}_X^2$ given by the matrix $\widetilde{\gamma}$ fits into a commutative diagram

$$\begin{tikzcd} \mathcal{O}_X^2 \arrow[rr, dashed, "\widetilde{\gamma}"] & & \mathcal{O}_X^2 \\ \mathcal{O}_X(-E_1) \oplus \mathcal{O}_X(-E_2) \arrow[u, hook] \arrow[rr, "\varphi_{\widetilde{\gamma}}"] & & \mathcal{O}_X(-E_1') \oplus \mathcal{O}_X(-E_2') \arrow[u, hook] \end{tikzcd} \tag{3.9}$$

Here the vertical maps are the natural inclusions, and $\varphi_{\widetilde{\gamma}}$ is an injective map of $\mathcal{O}_X$-modules such that $\det(\varphi_{\widetilde{\gamma}})$ has divisor D. The integrand $\widetilde{h}_D(t'^{-1}\widetilde{\gamma}t)$ is zero otherwise.

Let $\widetilde{\mathfrak{N}}_{D,\widetilde{\gamma}} \subset \mathrm{Div}(X)^4$ be the set of quadruples of divisors (E_1, E_2, E_1', E_2') such that $\widetilde{\gamma}$ fits into a diagram (3.9) and $\det(\varphi_{\widetilde{\gamma}})$ has divisor D. Let $\mathfrak{N}_{D,\widetilde{\gamma}} = \widetilde{\mathfrak{N}}_{D,\widetilde{\gamma}}/\mathrm{Div}(X)$ where $\mathrm{Div}(X)$ acts by simultaneous translation on the divisors E_1, E_2, E_1' and E_2'.

We have $|\alpha(t)\alpha(t')|^s = q^{-\deg(E_1 - E_2 + E_1' - E_2')s}$. Viewing η as a character on the idèle class group $F^\times\backslash\mathbb{A}_F^\times/\prod_{x \in |X|} \mathcal{O}_x^\times \cong \mathrm{Pic}_X(k)$, we have $\eta(\alpha(t)) = \eta(E_1)\eta(E_2) = \eta(E_1 - E_1')\eta(E_2 - E_1')$. Therefore

$$\mathbb{J}(\gamma, h_D, s) = \sum_{(E_1, E_2, E_1', E_2') \in \mathfrak{N}_{D,\widetilde{\gamma}}} q^{-\deg(E_1 - E_2 + E_1' - E_2')s} \eta(E_1 - E_1')\eta(E_2 - E_1'). \tag{3.10}$$

(1) Since $u = 0$ and ∞ are in the image of inv_D, we may assume that $u \notin \{0, 1, \infty\}$. For $\gamma \in G(F)$ with invariant u, any lifting $\widetilde{\gamma}$ of γ in $\mathrm{GL}_2(F)$ does not lie in $\widetilde{A}$ or $w\widetilde{A}$. Therefore the previous discussion applies to $\widetilde{\gamma}$. Suppose $\mathbb{J}(\gamma, h_D, s) \neq 0$, then $\mathfrak{N}_{D,\widetilde{\gamma}} \neq \varnothing$. Take a point $(E_1, E_2, E_1', E_2') \in \mathfrak{N}_{D,\widetilde{\gamma}}$, the map $\det(\varphi_{\widetilde{\gamma}})$ gives an isomorphism $\mathcal{O}_X(-E_1' - E_2') \cong \mathcal{O}_X(-E_1 - E_2 + D)$. Taking $a = \varphi_{\widetilde{\gamma},11}\varphi_{\widetilde{\gamma},22} : \mathcal{O}_X(-E_1 - E_2) \to \mathcal{O}_X(-E_1' - E_2')$, then a can be viewed as a section of $\mathcal{O}_X(D)$ satisfying $1 - a^{-1} = \mathrm{inv}(\gamma)$. Therefore $u = \mathrm{inv}(\gamma) = \mathrm{inv}_D(a)$ is in the image of inv_D.

(2) When $u \notin \{0, 1, \infty\}$, recall $\gamma(u)$ is the image of $\widetilde{\gamma}(u) = \begin{bmatrix} 1 & u \\ 1 & 1 \end{bmatrix}$. Let $\mathcal{N}_{\underline{d},a}$ be the fiber of $\mathcal{N}_{\underline{d}}$ over $a \in \mathcal{A}_D(k)$. Let $\mathcal{N}_a = \coprod_{\underline{d} \in \Sigma_d} \mathcal{N}_{\underline{d},a}$. We have a map

$$\begin{array}{rcl} \nu_u : \mathfrak{N}_{D,\widetilde{\gamma}(u)} & \longrightarrow & \mathcal{N}_a(k) \\ (E_1, E_2, E_1', E_2') & \longmapsto & (\mathcal{O}_X(-E_1), \mathcal{O}_X(-E_2), \mathcal{O}_X(-E_1'), \mathcal{O}_X(-E_2'), \varphi_{\widetilde{\gamma}(u)}). \end{array}$$

We show that this map is bijective by constructing an inverse. For $(\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_1', \mathcal{K}_2', \varphi_{ij}) \in \mathcal{N}_{\underline{d},a}(k)$, we may assume $\mathcal{K}_1 = \mathcal{O}_X$ (since we mod out by the action of Pic_X in the end). Let $S = |\mathrm{div}(a)| \cup |\mathrm{div}(a-1)| \cup |D|$ be a finite collection of places of X. Then each φ_{ij} is an isomorphism over $U = X - S$. In particular, we get isomorphisms $\varphi_{11} : \mathcal{O}_U \cong \mathcal{K}_1'|_U$, $\varphi_{21} : \mathcal{O}_U \cong \mathcal{K}_2'|_U$ and $\varphi_{22}^{-1}\varphi_{21} : \mathcal{O}_U \cong \mathcal{K}_2|_U$. Let E_1', E_2' and E_2 be the *negative* of the divisors of the isomorphisms φ_{11}, φ_{21} and $\varphi_{22}^{-1}\varphi_{21}$, viewed as rational maps between line bundles on X. Set $E_1 = 0$. Then we have $\mathcal{K}_i = \mathcal{O}_X(-E_i)$ and $\mathcal{K}_i' = \mathcal{O}_X(-E_i')$ for $i = 1, 2$. The map φ guarantees that the quadruple $(E_1 = 0, E_2, E_1', E_2') \in \mathfrak{N}_{D,\widetilde{\gamma}(u)}$. This gives a map $\mathcal{N}_a(k) \to \mathfrak{N}_{D,\widetilde{\gamma}(u)}$, which is easily seen to be inverse to ν_u.

By the Lefschetz trace formula, we have

$$\begin{aligned} &\sum_{\underline{d} \in \Sigma_d} q^{(2d_{12}-d)s} \operatorname{Tr}\left(\mathrm{Frob}_a, \left(\mathbf{R}f_{\mathcal{N}_{\underline{d}},*} L_{\underline{d}}\right)_{\overline{a}}\right) \\ = &\sum_{(\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_1', \mathcal{K}_2', \varphi) \in \mathcal{N}_a(k)} q^{(2d_{12}-d)s} \eta(D_{11}) \eta(D_{12}) \end{aligned}$$

where D_{ij} is the divisor of φ_{ij}. Moreover, under the isomorphism ν_u, the term $q^{-\deg(E_1 - E_2 + E_1' - E_2')s}$ corresponds to $q^{(2d_{12}-d)s}$ where $d_{12} = \deg(D_{12})$. Therefore Part (2) follows from the bijectivity of ν_u and (3.10).

(3) We treat the case $u = 0$ (i.e., $a = 1$), and the case $u = \infty$ is similar. Let $\mathfrak{N}'_{D,n_+}$ be the set of triples of *effective* divisors (D_{11}, D_{12}, D_{22}) such that $D_{11} + D_{22} = D$. Then we have a bijection

$$\begin{array}{rcl} \mathfrak{N}_{D,n_+} & \xrightarrow{\sim} & \mathfrak{N}'_{D,n_+} \\ (E_1, E_2, E_1', E_2') & \longmapsto & (E_1 - E_1', E_2 - E_1', E_2 - E_2'). \end{array}$$

Using this bijection, we may rewrite (3.10) as

$$\begin{aligned} \mathbb{J}(n_+, h_D, s) &= \sum_{(D_{11}, D_{12}, D_{22}) \in \mathfrak{N}'_{D,n_+}} q^{(2\deg(D_{12})-d)s} \eta(D_{11}) \eta(D_{12}) \\ &= q^{-ds} \sum_{D_{12} \geq 0} q^{2s\deg(D_{12})} \eta(D_{12}) \cdot \sum_{\substack{D_{11}+D_{22}=D \\ D_{11}, D_{22} \geq 0}} \eta(D_{11}) \\ &= q^{-ds} L(-2s, \eta) \sum_{0 \leq D_{11} \leq D} \eta(D_{11}) \end{aligned} \tag{3.11}$$

Similarly, let $\mathfrak{N}'_{D,n_-}$ be the set of triples of *effective* divisors (D_{11}, D_{21}, D_{22}) such that $D_{11} + D_{22} = D$. Then we have a bijection $\mathfrak{N}_{D,n_-} \leftrightarrow \mathfrak{N}'_{D,n_-}$ and an identity

$$\begin{aligned} \mathbb{J}(n_-, h_D, s) &= \sum_{(D_{11}, D_{21}, D_{22}) \in \mathfrak{N}'_{D,n_-}} q^{(d-2\deg(D_{21}))s} \eta(D_{21}) \eta(D_{22}) \\ &= q^{ds} L(2s, \eta) \sum_{0 \leq D_{22} \leq D} \eta(D_{22}). \end{aligned} \tag{3.12}$$

We now introduce a subset $\mathfrak{N}^{\heartsuit}_{D,n_+} \subset \mathfrak{N}'_{D,n_+}$ consisting of those (D_{11}, D_{12}, D_{22}) such that $\deg(D_{12}) < d/2$; similarly we introduce $\mathfrak{N}^{\heartsuit}_{D,n_-} \subset \mathfrak{N}'_{D,n_-}$ consisting of those (D_{11}, D_{21}, D_{22}) such that $\deg(D_{21}) \leq d/2$. Then the same argument as Part (2) gives a bijection

$$\nu_{n_\pm} : \mathfrak{N}^{\heartsuit}_{D,n_+} \coprod \mathfrak{N}^{\heartsuit}_{D,n_-} \xrightarrow{\sim} \mathcal{N}_a(k) := \coprod_{\underline{d} \in \Sigma_d} \mathcal{N}_{\underline{d},a}(k).$$

Here the degree constraints $\deg(D_{12}) < d/2$ or $\deg(D_{21}) \le d/2$ come from the last condition in the definition of $\mathcal{N}_{\underline{d}}$ in §3.2.2.

Using the Lefschetz trace formula, we get

$$\begin{aligned}
&\sum_{\underline{d}\in\Sigma_d} q^{(2d_{12}-d)s}\operatorname{Tr}\left(\mathrm{Frob}_a, \mathrm{H}_c^*(\mathcal{N}_{\underline{d},a}\otimes_k \overline{k}, L_{\underline{d}})\right)\\
=& \sum_{(D_{11},D_{12},D_{22})\in\mathfrak{N}^{\heartsuit}_{D,n_+}} q^{(2\deg(D_{12})-d)s}\eta(D_{11})\eta(D_{12})\\
&+\sum_{(D_{11},D_{21},D_{22})\in\mathfrak{N}^{\heartsuit}_{D,n_-}} q^{(d-2\deg(D_{21}))s}\eta(D_{21})\eta(D_{22})\\
=& \; q^{-ds}\sum_{D_{12}\ge 0,\deg(D_{12})<d/2} q^{2\deg(D_{12})s}\eta(D_{12})\sum_{0\le D_{11}\le D}\eta(D_{11}) \qquad (3.13)\\
&+q^{ds}\sum_{D_{21}\ge 0,\deg(D_{21})\le d/2} q^{-2\deg(D_{21})s}\eta(D_{21})\sum_{0\le D_{22}\le D}\eta(D_{22}). \qquad (3.14)
\end{aligned}$$

The only difference between the term in (3.13) and the RHS of (3.11) is that we have restricted the range of the summation to effective divisors D_{12} satisfying $\deg(D_{12}) < d/2$. However, since η is a nontrivial idèle class character, the Dirichlet L-function $L(s,\eta) = \sum_{E\ge 0} q^{-\deg(E)s}\eta(E)$ is a polynomial in q^{-s} of degree $2g-2 < d/2$. Therefore(3.13) is the same as (3.11). Similarly, (3.14) is the same as (3.12). We conclude that

$$\sum_{\underline{d}\in\Sigma_d} q^{(2d_{12}-d)s}\operatorname{Tr}\left(\mathrm{Frob}_a, \mathrm{H}_c^*(\mathcal{N}_{\underline{d},a}\otimes_k \overline{k}, L_{\underline{d}})\right) = \mathbb{J}(n_+,h_D,s)+\mathbb{J}(n_-,h_D,s). \qquad (3.15)$$

Finally, observe that

$$\mathbb{J}(1,h_D,s)=0 \qquad (3.16)$$

because η restricts nontrivially to the centralizer of $\gamma = 1$. Putting together (3.15) and the vanishing (3.16), we get (3.7). □

Corollary 3.3. *For $D \in X_d(k)$ and $u \in \mathbb{P}^1(F) - \{1\}$, we have*

$$\mathbb{J}_r(u,h_D) = \begin{cases} (\log q)^r \sum_{\underline{d}\in\Sigma_d}(2d_{12}-d)^r \operatorname{Tr}\left(\mathrm{Frob}_a, \left(\mathbf{R}f_{\mathcal{N}_{\underline{d}},*}L_{\underline{d}}\right)_{\overline{a}}\right) & \text{if } u = \mathrm{inv}_D(a), a\in\mathcal{A}_D(k);\\ 0 & \text{otherwise.}\end{cases}$$

4. Analytic spectral decomposition

In this section we express the spectral side of the relative trace formula in §2 in terms of automorphic L-functions.

4.1. The Eisenstein ideal.

Consider the Hecke algebra $\mathscr{H} = \otimes_{x\in|X|}\mathscr{H}_x$. We also consider the Hecke algebra $\mathscr{H}_A$ for the diagonal torus $A = \mathbb{G}_m$ of G. Then $\mathscr{H}_A = \otimes_{x\in|X|}\mathscr{H}_{A,x}$ with $\mathscr{H}_{A,x} = \mathbb{Q}[F_x^\times/\mathcal{O}_x^\times] = \mathbb{Q}[t_x, t_x^{-1}]$, and t_x stands for the characteristic function of $\varpi_x^{-1}\mathcal{O}_x^\times$, where ϖ_x is a uniformizer of F_x.

Recall we have a basis $\{h_D\}$ for $\mathscr{H}$ indexed by effective divisors D on X. For fixed $x\in|X|$, $h_x\in\mathscr{H}_x$ and $\mathscr{H}_x\cong\mathbb{Q}[h_x]$ is a polynomial algebra with generator h_x.

4.1.1. *The Satake transform.* To avoid introducing $\sqrt{q}$, we normalize the Satake transform in the following way

$$\begin{aligned}\mathrm{Sat}_x : \mathscr{H}_x &\longrightarrow \mathscr{H}_{A,x}\\ h_x &\longmapsto t_x + q_x t_x^{-1}\end{aligned}$$

where $q_x = \#k_x$. Consider the involution ι_x on $\mathscr{H}_{A,x}$ sending t_x to $q_x t_x^{-1}$. Then Sat_x identifies $\mathscr{H}_x$ with the subring of ι_x-invariants of $\mathscr{H}_{A,x}$. This normalization of the Satake transform is designed to make it compatible with constant term operators, see Lemma 7.8. Let

$$\mathrm{Sat} : \mathscr{H} \longrightarrow \mathscr{H}_A$$

be the tensor product of all Sat_x.

4.1.2. We have natural homomorphisms between abelian groups:

$$\begin{tikzcd} \mathbb{A}^\times/\mathbb{O}^\times \arrow[r, "\simeq"] \arrow[d] & \mathrm{Div}(X) \arrow[d] \\ F^\times\backslash\mathbb{A}^\times/\mathbb{O}^\times \arrow[r, "\simeq"] & \mathrm{Pic}_X(k). \end{tikzcd}$$

In particular, the top row above gives a canonical isomorphism $\mathscr{H}_A = \mathbb{Q}[\mathbb{A}^\times/\mathbb{O}^\times] \cong \mathbb{Q}[\mathrm{Div}(X)]$, the group algebra of $\mathrm{Div}(X)$.

Define an involution ι_{Pic} on $\mathbb{Q}[\mathrm{Pic}_X(k)]$ by

$$\iota_{\mathrm{Pic}}(\mathbf{1}_{\mathcal{L}}) = q^{\deg \mathcal{L}}\mathbf{1}_{\mathcal{L}^{-1}}.$$

Here $\mathbf{1}_{\mathcal{L}} \in \mathbb{Q}[\mathrm{Pic}_X(k)]$ is the characteristic function of the point $\mathcal{L} \in \mathrm{Pic}_X(k)$. Since the action of $\otimes_x \iota_x$ on $\mathscr{H}_A \cong \mathbb{Q}[\mathrm{Div}(X)]$ is compatible with the involution ι_{Pic} on $\mathbb{Q}[\mathrm{Pic}_X(k)]$ under the projection $\mathbb{Q}[\mathrm{Div}(X)] \to \mathbb{Q}[\mathrm{Pic}_X(k)]$, we see that the image of the composition

$$\mathscr{H} \xrightarrow{\mathrm{Sat}} \mathscr{H}_A \cong \mathbb{Q}[\mathrm{Div}(X)] \twoheadrightarrow \mathbb{Q}[\mathrm{Pic}_X(k)]$$

lies in the ι_{Pic}-invariants. Therefore the above composition gives a ring homomorphism

$$a_{\mathrm{Eis}} : \mathscr{H} \longrightarrow \mathbb{Q}[\mathrm{Pic}_X(k)]^{\iota_{\mathrm{Pic}}} =: \mathscr{H}_{\mathrm{Eis}}. \tag{4.1}$$

Definition 4.1. We define the *Eisenstein ideal* $\mathcal{I}_{\mathrm{Eis}} \subset \mathscr{H}$ to be the kernel of the ring homomorphism a_{Eis} in (4.1).

The ideal $\mathcal{I}_{\mathrm{Eis}}$ is the analog of the Eisenstein ideal of Mazur in the function field setting. Taking the spectra we get a morphism of affine schemes

$$\mathrm{Spec}(a_{\mathrm{Eis}}) : Z_{\mathrm{Eis}} := \mathrm{Spec}\,\mathbb{Q}[\mathrm{Pic}_X(k)]^{\iota_{\mathrm{Pic}}} \longrightarrow \mathrm{Spec}\,\mathscr{H}.$$

Lemma 4.2. *(1) For any $x \in |X|$, under the ring homomorphism a_{Eis}, $\mathbb{Q}[\mathrm{Pic}_X(k)]^{\iota_{\mathrm{Pic}}}$ is finitely generated as an $\mathscr{H}_x$-module.*

(2) The map a_{Eis} is surjective, hence $\mathrm{Spec}(a_{\mathrm{Eis}})$ is a closed embedding. [2]

Proof. (1) We have an exact sequence $0 \to \mathrm{Jac}_X(k) \to \mathrm{Pic}_X(k) \to \mathbb{Z} \to 0$ with $\mathrm{Jac}_X(k)$ finite. Let $x \in |X|$, then the map $\mathbb{Z} \to \mathrm{Pic}_X(k)$ sending $n \mapsto \mathcal{O}_X(nx)$ has finite cokernel since $\mathrm{Jac}_X(k)$ is finite. Therefore $\mathbb{Q}[\mathrm{Pic}_X(k)]$ is finitely generated as a $\mathscr{H}_{A,x} \cong \mathbb{Q}[t_x, t_x^{-1}]$-module. On the other hand, via Sat_x, $\mathscr{H}_{A,x}$ is a finitely generated $\mathscr{H}_x$-module (in fact a free module of rank two over $\mathscr{H}_x$). Therefore $\mathbb{Q}[\mathrm{Pic}_X(k)]$ is a finitely generated module over the noetherian ring $\mathscr{H}_x$, hence so is its submodule $\mathbb{Q}[\mathrm{Pic}_X(k)]^{\iota_{\mathrm{Pic}}}$.

(2) For proving surjectivity we may base change the situation to $\overline{\mathbb{Q}}_\ell$. Let $\mathfrak{Z}_{\mathrm{Eis}} = \mathrm{Spec}\,\overline{\mathbb{Q}}_\ell[\mathrm{Pic}_X(k)]^{\iota_{\mathrm{Pic}}}$, and we still use $\mathrm{Spec}(a_{\mathrm{Eis}})$ to denote $\mathfrak{Z}_{\mathrm{Eis}} \to \mathrm{Spec}\,\mathscr{H}_{\overline{\mathbb{Q}}_\ell}$. We first check that $\mathrm{Spec}(a_{\mathrm{Eis}})$ is injective on $\overline{\mathbb{Q}}_\ell$-points. Identifying $\mathrm{Pic}_X(k)$ with the abelianized Weil group $W(X)^{\mathrm{ab}}$ via class field theory, the set $\mathfrak{Z}_{\mathrm{Eis}}(\overline{\mathbb{Q}}_\ell)$ are in natural bijection with Galois characters $\chi : W(X) \to \overline{\mathbb{Q}}_\ell^\times$ up to the equivalence relation $\chi \sim \chi^{-1}(-1)$ (where (-1) means Tate twist). Suppose χ_1 and χ_2 are two such characters that pullback to the same homomorphism $\mathscr{H} \to \overline{\mathbb{Q}}_\ell[\mathrm{Pic}_X(k)] \xrightarrow{\chi_i} \overline{\mathbb{Q}}_\ell$, then $\chi_1(a_{\mathrm{Eis}}(h_x)) = \chi_1(\mathrm{Frob}_x) + q_x\chi_1(\mathrm{Frob}_x^{-1}) = \chi_2(\mathrm{Frob}_x) + q_x\chi_2(\mathrm{Frob}_x^{-1}) = \chi_2(a_{\mathrm{Eis}}(h_x))$ for all x. Consider the two-dimensional representation $\rho_i = \chi_i \oplus \chi_i^{-1}(-1)$ of $W(X)$. Then $\mathrm{Tr}(\rho_1(\mathrm{Frob}_x)) = \mathrm{Tr}(\rho_2(\mathrm{Frob}_x))$ for all x. By Chebotarev density, this implies that ρ_1 and ρ_2 are isomorphic to each other (since they are already semisimple). Therefore either $\chi_1 = \chi_2$ or $\chi_1 = \chi_2^{-1}(-1)$. In any case χ_1 and χ_2 define the same $\overline{\mathbb{Q}}_\ell$-point of $\mathfrak{Z}_{\mathrm{Eis}}$. We are done.

Next, we show that $\mathrm{Spec}(a_{\mathrm{Eis}})$ is injective on tangent spaces at $\overline{\mathbb{Q}}_\ell$-points. Let $\widetilde{\mathfrak{Z}}_{\mathrm{Eis}} = \mathrm{Spec}\,\overline{\mathbb{Q}}_\ell[\mathrm{Pic}_X(k)]$. Then $\widetilde{\mathfrak{Z}}_{\mathrm{Eis}}$ is a disjoint union of components indexed by characters $\chi_0 : \mathrm{Jac}_X(k) \to \overline{\mathbb{Q}}_\ell^\times$, and each component is a torsor under $\mathbb{G}_m$. The scheme $\mathfrak{Z}_{\mathrm{Eis}}$ is the quotient $\widetilde{\mathfrak{Z}}_{\mathrm{Eis}} /\!\!/ \langle\iota_{\mathrm{Pic}}\rangle$. For a character $\chi : \mathrm{Pic}_X(k) \to \overline{\mathbb{Q}}_\ell^\times$ with restriction χ_0 to $\mathrm{Jac}_X(k)$, we may identify its component $\widetilde{\mathfrak{Z}}_{\chi_0}$ with $\mathbb{G}_m$ in such a way that $s \in \mathbb{G}_m$ corresponds to the character

[2] This result is not used in an essential way in the rest of paper.

$\chi \cdot s^{\deg} : \mathrm{Pic}_X(k) \to \overline{\mathbb{Q}}_\ell^\times, \mathcal{L} \mapsto \chi(\mathcal{L}) s^{\deg \mathcal{L}}$. The map $\mathrm{Spec}(a_{\mathrm{Eis}})$ pulled back to $\widetilde{\mathfrak{Z}}_{\chi_0}$ then gives a morphism

$$b : \mathbb{G}_m \cong \widetilde{\mathfrak{Z}}_{\chi_0} \longrightarrow \mathfrak{Z}_{\mathrm{Eis}} \longrightarrow \mathrm{Spec}\, \mathscr{H}_{\overline{\mathbb{Q}}_\ell} \cong \mathbb{A}^{|X|}$$

given by the formula

$$\mathbb{G}_m \ni s \longmapsto \left(\chi(t_x) s^{d_x} + q_x \chi(t_x^{-1}) s^{-d_x}\right)_{x \in |X|} \tag{4.2}$$

where $d_x = [k_x : k]$. The derivative $\frac{db}{ds}$ at $s = 1$ is then the vector $(d_x(\chi(t_x) - q_x \chi(t_x^{-1})))_{x \in |X|}$. This is identically zero only when $\chi(t_x) = \pm q_x^{1/2}$ for all x, hence if and only if $\chi^2 = q^{\deg} = \overline{\mathbb{Q}}_\ell(-1)$. Therefore when $\chi^2 \neq \overline{\mathbb{Q}}_\ell(-1)$, we have proved that the tangent map of b at $s = 1$ is nonzero, hence a fortiori the tangent map of $\mathrm{Spec}(a_{\mathrm{Eis}})$ at the image of χ is nonzero. If $\chi^2 = \overline{\mathbb{Q}}_\ell(-1)$, χ is a fixed point under ι_{Pic}. The component $\widetilde{\mathfrak{Z}}_{\chi_0}$ is then stable under ι_{Pic} which acts by $s \mapsto s^{-1}$, and its image $\mathfrak{Z}_{\chi_0} \subset \mathfrak{Z}_{\mathrm{Eis}}$ is a component isomorphic to $\mathbb{A}^1$ with affine coordinate $z = s + s^{-1}$. Therefore we may factor b into two steps

$$b : \widetilde{\mathfrak{Z}}_{\chi_0} \cong \mathbb{G}_m \xrightarrow{s \longmapsto z = s + s^{-1}} \mathbb{A}^1 \cong \mathfrak{Z}_{\chi_0} \xrightarrow{c} \mathrm{Spec}\, \mathscr{H}_{\overline{\mathbb{Q}}_\ell} \cong \mathbb{A}^{|X|}$$

where c is the restriction of $\mathrm{Spec}(a_{\mathrm{Eis}})$ to Z_{χ_0}. By chain rule we have $\frac{dc}{dz}\frac{dz}{ds} = \frac{db}{ds}$. Using this we see that the derivative $\frac{dc}{dz}$ at $z = s + s^{-1}$ is the vector

$$\left(d_x \chi(t_x) \frac{s^{d_x} - s^{-d_x}}{s - s^{-1}} \right)_{x \in |X|}$$

(using that $\chi(t_x) = q_x \chi(t_x^{-1})$). Evaluating at $s = 1$ we get the vector $(\chi(t_x) d_x^2)_{x \in |X|}$, which is nonzero. We have checked that the tangent map of $\mathrm{Spec}(a_{\mathrm{Eis}})$ is also injective at the image of those points $\chi \in \widetilde{\mathfrak{Z}}_{\mathrm{Eis}}(\overline{\mathbb{Q}}_\ell)$ such that $\chi^2 = \mathbb{Q}_\ell(-1)$. Therefore the tangent map of $\mathrm{Spec}(a_{\mathrm{Eis}})$ is injective at all $\overline{\mathbb{Q}}_\ell$-points. Combining the two injectivity results we conclude that $\mathrm{Spec}(a_{\mathrm{Eis}})$ is a closed immersion and hence a_{Eis} is surjective. □

4.2. **Spectral decomposition of the kernel function.** Recall that we have defined the automorphic kernel function by (2.3). For a cuspidal automorphic representation π (in the usual sense, i.e., an irreducible sub-representation of the $\mathbb{C}$-values automorphic functions), we define the π-component of the kernel function as (cf. [13, §7.1(1)])

$$\mathbb{K}_{f,\pi}(x, y) = \sum_{\phi} \pi(f)\phi(x)\overline{\phi(y)}, \tag{4.3}$$

where the sum runs over an orthonormal basis $\{\phi\}$ of π. The cuspidal kernel function is defined as

$$\mathbb{K}_{f,\mathrm{cusp}} = \sum_{\pi} \mathbb{K}_{f,\pi}, \tag{4.4}$$

where the sum runs over all cuspidal automorphic representations π of G. Note that this is a finite sum.

Similarly, we define the special (residual) kernel function (cf. [13, §7.4])

$$\mathbb{K}_{f,\mathrm{sp}}(x, y) := \sum_{\chi} \pi(f)\chi(x)\overline{\chi(y)},$$

where the sum runs over all one-dimensional automorphic representations $\pi = \chi$, indeed solely characters of order two:

$$\chi \colon G(\mathbb{A}) \longrightarrow F^\times \backslash \mathbb{A}^\times / (\mathbb{A}^\times)^2 \longrightarrow \{\pm 1\}.$$

Theorem 4.3. *Let $f \in \mathcal{I}_{\mathrm{Eis}}$ be in the Eisenstein ideal $\mathcal{I}_{\mathrm{Eis}} \subset \mathscr{H}$. Then we have*

$$\mathbb{K}_f = \mathbb{K}_{f,\mathrm{cusp}} + \mathbb{K}_{f,\mathrm{sp}}.$$

Proof. To show this, we need to recall the Eisenstein series (cf. [13, §8.4]). We fix an $\alpha \in \mathbb{A}^\times$ with valuation one and we then have a direct product

$$\mathbb{A}^\times = \mathbb{A}^1 \times \alpha^{\mathbb{Z}}.$$

For a character $\chi : F^\times\backslash\mathbb{A}^1 \to \mathbb{C}^\times$, we extend it as a character of $F^\times\backslash\mathbb{A}^\times$, by demanding $\chi(\alpha) = 1$. Moreover, we define a character for any $u \in \mathbb{C}$

$$\begin{array}{rrcl} \chi_u : & \mathbb{A}^\times & \longrightarrow & \mathbb{C}^\times \\ & a & \longmapsto & \chi(a)|a|^u \end{array} .$$

We also define

$$\begin{array}{rrcl} \delta_B : & B(\mathbb{A}) & \longrightarrow & \mathbb{A}^\times \\ & \begin{bmatrix} a & b \\ & d \end{bmatrix} & \longmapsto & a/d \end{array}, \qquad \text{and} \qquad \begin{array}{rrcl} \chi : & B(\mathbb{A}) & \longrightarrow & \mathbb{C}^\times \\ & b & \longmapsto & \chi(a/d) \end{array} .$$

For $u \in \mathbb{C}$, the (induced) representation $\rho_{\chi,u}$ of $G(\mathbb{A}) = \mathrm{PGL}_2(\mathbb{A})$ is defined to be the right translation on the space $V_{\chi,u}$ of smooth functions

$$\phi : G(\mathbb{A}) \longrightarrow \mathbb{C}$$

such that

$$\phi(bg) = \chi(b)\,|\delta_B(b)|^{u+\frac{1}{2}}\phi(g), \quad b \in B(\mathbb{A}), \quad g \in G(\mathbb{A}).$$

Note that we have $\rho_{\chi,u} = \rho_{\chi,u+\frac{2\pi i}{\log q}}$. By restriction to K, the space $V_{\chi,u}$ is canonically identified with the space of smooth functions

$$V_\chi : = \left\{ \phi : K \longrightarrow \mathbb{C}, \text{ smooth} \;\middle|\; \phi(bk) = \chi(b)\,\phi(k), \quad b \in K \cap B(\mathbb{A}) \right\}.$$

This space is endowed with a natural inner product

$$(\phi, \phi') = \int_K \phi(k)\overline{\phi'(k)}dk. \tag{4.5}$$

Let $\phi \in V_\chi$, we denote by $\phi(g, u, \chi)$ the corresponding function in $V_{\chi,u}$, i.e.,

$$\phi(g, u, \chi) = \chi(b)\left|\delta_B(b)\right|^{u+\frac{1}{2}}\phi(k)$$

if we write $g = bk$ where $b \in B(\mathbb{A}), k \in K$.

For $\phi \in V_\chi$, the Eisenstein series is defined as (the analytic continuation of)

$$E(g, \phi, u, \chi) = \sum_{\gamma \in B(F)\backslash G(F)} \phi(\gamma g, u, \chi).$$

Let $\{\phi_i\}_i$ be an orthonormal basis of the Hermitian space V_χ. We define

$$\mathbb{K}_{f,\mathrm{Eis},\chi}(x, y) := \frac{\log q}{2\pi i} \sum_{i,j} \int_0^{\frac{2\pi i}{\log q}} (\rho_{\chi,u}(f)\phi_i, \phi_j)\, E(x, \phi_i, u, \chi)\overline{E(y, \phi_j, u, \chi)}\, du, \tag{4.6}$$

where the inner product is given by (4.5) via the identification $V_{\chi,u} \simeq V_\chi$. We set (cf., [13, §8.4])

$$\mathbb{K}_{f,\mathrm{Eis}} := \sum_\chi \mathbb{K}_{f,\mathrm{Eis},\chi} \tag{4.7}$$

where the sum runs over all characters χ of $F^\times\backslash\mathbb{A}^1$. Since our test function f is in the spherical Hecke algebra $\mathscr{H}$, for $\mathbb{K}_{f,\mathrm{Eis},\chi}$ to be nonzero, the character χ is necessarily unramified everywhere. Therefore the sum over χ is in fact finite.

By [13, §7.1(4)], we have a spectral decomposition of the automorphic kernel function $\mathbb{K}_f$ defined by (2.3)

$$\mathbb{K}_f = \mathbb{K}_{f,\mathrm{cusp}} + \mathbb{K}_{f,\mathrm{sp}} + \mathbb{K}_{f,\mathrm{Eis}}. \tag{4.8}$$

Therefore it remains to show that $\mathbb{K}_{f,\mathrm{Eis}}$ vanishes if f lies in the Eisenstein ideal $\mathcal{I}_{\mathrm{Eis}}$.

We may assume that χ is unramified. Then we have

$$\mathbb{K}_{f,\mathrm{Eis},\chi}(x, y) = \frac{\log q}{2\pi i} \int_0^{\frac{2\pi i}{\log q}} (\rho_{\chi,u}(f)\phi, \phi)\, E(x, \phi, u, \chi)\overline{E(y, \phi, u, \chi)}\, du, \tag{4.9}$$

where $\phi = \mathbf{1}_K \in V_\chi$ (we are taking the Haar measure on $G(\mathbb{A})$ such that $\mathrm{vol}(K) = 1$).

Recall that the Satake transform Sat has the property that, for all unramified characters χ, and all $u \in \mathbb{C}$, we have

$$\operatorname{tr} \rho_{\chi,u}(f) = \chi_{u+1/2}(\operatorname{Sat}(f)),$$

where we extend $\chi_{u+1/2}$ to a homomorphism $\mathscr{H}_{A,\mathbb{C}} \simeq \mathbb{C}[\operatorname{Div}(X)] \to \mathbb{C}$. Since $\chi_u : A(\mathbb{A})/(A(\mathbb{A}) \cap K) \simeq \operatorname{Div}(X) \to \mathbb{C}^\times$ factors through $\operatorname{Pic}_X(k)$, we have

$$\operatorname{tr} \rho_{\chi,u}(f) = \chi_{u+1/2}(a_{\mathrm{Eis}}(f)),$$

Then we may rewrite (4.9) as

$$\mathbb{K}_{f,\mathrm{Eis},\chi}(x,y) = \frac{\log q}{2\pi i} \int_0^{\frac{2\pi i}{\log q}} \chi_{u+1/2}(a_{\mathrm{Eis}}(f))\, E(x,\phi,u,\chi)\overline{E(y,\phi,u,\chi)}\, du.$$

In particular, if f lies in the Eisenstein ideal, then $a_{\mathrm{Eis}}(f) = 0$, and hence the integrand vanishes. This completes the proof. □

4.3. **The cuspidal kernel.** Let π be a cuspidal automorphic representation of $G(\mathbb{A})$, endowed with the natural Hermitian form given by the Petersson inner product:

$$\langle \phi, \phi' \rangle_{\mathrm{Pet}} := \int_{[G]} \phi(g)\overline{\phi'(g)} dg, \quad \phi, \phi' \in \pi. \tag{4.10}$$

We abbreviate the notation to $\langle \phi, \phi' \rangle$. For a character $\chi : F^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times$, the $(A,\chi)-$period integral for $\phi \in \pi$ is defined as

$$\mathscr{P}_\chi(\phi, s) := \int_{[A]} \phi(h)\chi(h)\left|h\right|^s dh. \tag{4.11}$$

We simply write $\mathscr{P}(\phi, s)$ if $\chi = \mathbf{1}$ is trivial. This is absolutely convergent for all $s \in \mathbb{C}$.

The spherical character (relative to $(A \times A, 1 \times \eta)$) associated to π is a distribution on $G(\mathbb{A})$ defined by

$$\mathbb{J}_\pi(f, s) = \sum_\phi \frac{\mathscr{P}(\pi(f)\phi, s)\mathscr{P}_\eta(\overline{\phi}, s)}{\langle \phi, \phi \rangle}, \quad f \in C_c^\infty(G(\mathbb{A})), \tag{4.12}$$

where the sum runs over an orthogonal basis $\{\phi\}$ of π. This is a finite sum, and the result is independent of the choice of the basis.

Lemma 4.4. *Let f be a function in the Eisenstein ideal $\mathcal{I}_{\mathrm{Eis}} \subset \mathscr{H}$. Then we have*

$$\mathbb{J}(f,s) = \sum_\pi \mathbb{J}_\pi(f,s),$$

where the sum runs over all (everywhere unramified) cuspidal automorphic representations π of $G(\mathbb{A})$.

Proof. For $* = \mathrm{cusp}, \mathrm{sp}$ or π, we define $\mathbb{J}_*(f,s)$ by replacing $\mathbb{K}_f$ by $\mathbb{K}_{f,*}$ in both (2.4) and (2.5). To make sense of this, we need to show the analogous statements to Proposition 2.1. When $* = \mathrm{sp}$, we note that, for any character $\chi : \mathbb{A}^\times \to \mathbb{C}^\times$, one of χ and $\chi\eta$ must be nontrivial on $\mathbb{A}^1$. It follows that for any $(n_1, n_2) \in \mathbb{Z}^2$ we have

$$\int_{[A]_{n_1} \times [A]_{n_2}} \chi(h_1)\chi^{-1}(h_2)|h_1 h_2|^s \eta(h_2)\, dh_1\, dh_2 = 0.$$

Consequently we have

$$\mathbb{J}_{\mathrm{sp}}(f,s) = 0.$$

When $* = \pi$, we need to show that, for any $\phi \in \pi$, the following integral vanishes if $|n| \gg 0$

$$\int_{[A]_n} \phi(h)\chi(h)\left|h\right|^s dh.$$

But this follows from the fact that ϕ is cuspidal, particularly $\phi(h) = 0$ if $h \in [A]_n$ and $|n| \gg 0$. This also shows that this definition of $\mathbb{J}_\pi(f,s)$ coincides with (4.12). The case $* = \mathrm{cusp}$ follows from the case for $* = \pi$ and the finite sum decomposition (4.4). We then have

$$\mathbb{J}_{\mathrm{cusp}}(f,s) = \sum_\pi \mathbb{J}_\pi(f,s).$$

The proof is complete, noting that, by Theorem 4.3, we have

$$\mathbb{J}(f,s)=\mathbb{J}_{\rm cusp}(f,s)+\mathbb{J}_{\rm sp}(f,s).$$

□

Proposition 4.5. *Let π be a cuspidal automorphic representation of $G(\mathbb{A})$, unramified everywhere. Let $\lambda_\pi:\mathscr{H}\to\mathbb{C}$ be the homomorphism associated to π. Then we have*

$$\mathbb{J}_\pi(f,s)=\frac{1}{2}\,|\omega_X|\,\mathscr{L}(\pi_{F'},s+1/2)\lambda_\pi(f).$$

Proof. Write $\pi=\otimes_{x\in|X|}\pi_x$ and let ϕ be a nonzero vector in the one-dimensional space π^K. Since $f\in\mathscr{H}$ is bi-K-invariant, the sum in (4.12) is reduced to one term

$$\mathbb{J}_\pi(f,s)=\frac{\mathscr{P}(\phi,s)\mathscr{P}_\eta(\overline{\phi},s)}{\langle\phi,\phi\rangle_{\rm Pet}}\lambda_\pi(f)\,{\rm vol}(K),\tag{4.13}$$

where we may choose any measure on $G(\mathbb{A})$, and then define the Petersson inner product using the same measure. We will choose the Tamagawa measure on $G(\mathbb{A})$ in this proof. To decompose the Tamagawa measure into local measures, we fix a nontrivial additive character associated to a nonzero meromorphic differential form c on X:

$$\psi:\ \mathbb{A}\longrightarrow\mathbb{C}^\times\ .$$

We note that the character ψ is defined by $\psi(a)=\psi_{\mathbb{F}_p}\left(\sum_{x\in|X|}{\rm Tr}_{k_x/\mathbb{F}_p}\left({\rm Res}_x(ca)\right)\right)$ where $\psi_{\mathbb{F}_p}$ is a fixed nontrivial character $\mathbb{F}_p\to\mathbb{C}^\times$.

We decompose $\psi=\prod_{x\in|X|}\psi_x$ where ψ_x is a character of F_x. This gives us a self-dual measure $dt=dt_{\psi_x}$ on F_x, a measure $d^\times t=\zeta_x(1)\frac{dt}{|t|}$ on $F_x^\times$, and the product measure on $\mathbb{A}^\times$. We then choose the Haar measure $dg_x=\zeta_x(1)|\det(g_x)|^{-2}\prod_{1\le i,j\le 2}dg_{ij}$ on ${\rm GL}_2(F_x)$ where $g_x=(g_{ij})\in{\rm GL}_2(F_x)$. The measure on $G(F_x)$ is then the quotient measure, and the Tamagawa measure on $G(\mathbb{A})$ decomposes $dg=\prod_{x\in|X|}dg_x$. Note that under such a choice of measures, we have

$${\rm vol}(\mathbb{O}^\times)={\rm vol}(\mathbb{O})=|\omega_X|^{1/2},\tag{4.14}$$

$${\rm vol}(K)=\zeta_F(2)^{-1}\,{\rm vol}(\mathbb{O})^3=\zeta_F(2)^{-1}|\omega_X|^{3/2}.\tag{4.15}$$

To compute the period integrals, we use the Whittaker models with respect to the character ψ. Denote the Whittaker model of π_x by W_{ψ_x}. Write the ψ-Whittaker coefficient W_ϕ as a product $\otimes_{x\in|X|}W_x$, where $W_x\in W_{\psi_x}$.

Let $L(\pi_x\times\widetilde{\pi}_x,s)$, resp. $L(\pi\times\widetilde{\pi},s)$ denote the local, resp. global Rankin–Selberg L-functions. By [30, Prop. 3.1] there are invariant inner products $\theta_x^\natural$ on the Whittaker models W_{ψ_x}

$$\theta_x^\natural(W_x,W_x'):=\frac{1}{L(\pi_x\times\widetilde{\pi}_x,1)}\int_{F_x^\times}W_x\left(\begin{bmatrix}a&\\&1\end{bmatrix}\right)\overline{W'}_x\left(\begin{bmatrix}a&\\&1\end{bmatrix}\right)d^\times a,$$

such that

$$\langle\phi,\phi\rangle_{\rm Pet}=2\frac{{\rm Res}_{s=1}L(\pi\times\widetilde{\pi},s)}{{\rm vol}(F^\times\backslash\mathbb{A}^1)}\prod_{x\in|X|}\theta_x^\natural(W_x,W_x).$$

Note that

$${\rm Res}_{s=1}L(\pi\times\widetilde{\pi},s)=L(\pi,{\rm Ad},1)\ {\rm Res}_{s=1}\zeta_F(s)=L(\pi,{\rm Ad},1)\ {\rm vol}(F^\times\backslash\mathbb{A}^1).$$

Hence we have

$$\langle\phi,\phi\rangle_{\rm Pet}=2L(\pi,{\rm Ad},1)\prod_{x\in|X|}\theta_x^\natural(W_x,W_x).$$

Moreover, when ψ_x is unramified, we have $\theta_x^\natural(W_x,W_x)={\rm vol}(K_x)=\zeta_x(2)^{-1}$ (cf. *loc. cit.*).

In [30, Prop. 3.3] there are linear functionals $\lambda_x^\natural$ on the Whittaker models W_{ψ_x}

$$\lambda_x^\natural(W_x,\chi_x,s):=\frac{1}{L(\pi_x\otimes\chi_x,s+1/2)}\int_{F_x^\times}W_x\left(\begin{bmatrix}a&\\&1\end{bmatrix}\right)\chi_x(a)|a|^s\,d^\times a$$

such that

$$\mathscr{P}_\chi(\phi, s) = L(\pi \otimes \chi, s+1/2) \prod_{x\in|X|} \lambda_x^\natural(W_x, \chi_x, s).$$

While in *loc. cit.* we only treated the case $s=0$, the same argument goes through. Moreover, when ψ_x and χ_x are unramified, we have $\lambda_x^\natural = 1$.

We now have

$$\frac{\mathscr{P}(\phi, s)\mathscr{P}_\eta(\overline{\phi}, s)}{\langle \phi, \phi\rangle_{\mathrm{Pet}}} = |\omega_X|^{-1} \frac{L(\pi_{F'}, s+1/2)}{2L(\pi, \mathrm{Ad}, 1)} \prod_{x\in|X|} \xi_{x,\psi_x}(W_x, \eta_x, s), \tag{4.16}$$

where the constant $|\omega_X|^{-1}$ is caused by the choice of measures (cf. (4.14)), and the local term at a place x is

$$\xi_{x,\psi_x}(W_x, \eta_x, s) := \frac{\lambda_x^\natural(W_x, \mathbf{1}_x, s)\lambda_x^\natural(\overline{W}_x, \eta_x, s)}{\theta_x^\natural(W_x, W_x)}. \tag{4.17}$$

Note that the local term ξ_{x,ψ_x} is now independent of the choice of the nonzero vector W_x in the one-dimensional space $W_{\psi_x}^{K_x}$. We thus simply write it as

$$\xi_{x,\psi_x}(\eta_x, s) := \xi_{x,\psi_x}(W_x, \eta_x, s).$$

When ψ_x is unramified, we have

$$\xi_{x,\psi_x}(\eta_x, s) = \zeta_x(2).$$

We want to know how ξ_{x,ψ_x} depends on ψ_x. Let $c_x \in F_x^\times$, and denote by ψ_{x,c_x} the twist $\psi_{x,c_x}(t) = \psi_x(c_x t)$.

Lemma 4.6. *For any unramified character χ_x of $F_x^\times$, we have*

$$\xi_{x,\psi_{x,c_x}}(\chi_x, s) = \chi^{-1}(c_x)|c_x|^{-2s+1/2}\xi_{x,\psi_x}(\chi_x, s).$$

Proof. The self-dual measure on F_x changes according to the following rule

$$dt_{\psi_{x,c_x}} = |c_x|^{1/2}\, dt_{\psi_x}.$$

Then the multiplicative measure on $F_x^\times$ changes by the same multiple. Now we compare ξ_{x,ψ_x} and $\xi_{x,\psi_{x,c_x}}$ using the same measure on $F_x^\times$ to define the integrals.

There is a natural isomorphism between the Whittaker models $W_{\psi_x} \simeq W_{\psi_{x,c_x}}$, preserving the natural inner product $\theta_x^\natural$. We write $\lambda_{\psi_x}^\natural$ to indicate the dependence on ψ_x. Then we have for any character $\chi_x : F_x^\times \to \mathbb{C}^\times$:

$$\lambda_{\psi_{x,c_x}}^\natural(W_x, \chi_x, s) = \chi^{-1}(c_x)|c_x|^{-s}\lambda_{\psi_x}^\natural(W_x, \chi_x, s).$$

This completes the proof of Lemma 4.6. □

Let ψ_x have conductor $c_x^{-1}\mathcal{O}_x$. Then the idèle class of $(c_x)_{x\in|X|}$ in $\mathrm{Pic}_X(k)$ is the class of $\mathrm{div}(c)$ and hence the class of ω_X. Hence we have

$$\prod_{x\in|X|} |c_x| = |\omega_X| = q^{-\deg \omega_X} = q^{-(2g-2)}.$$

This shows that the product in (4.16) is equal to

$$\begin{aligned}\prod_{x\in|X|} \xi_{x,\psi_x}(\eta_x, s) &= \eta(\omega_X) \prod_{x\in|X|} \zeta_x(2)|c_x|^{-2s+1/2} \\ &= \eta(\omega_X)|\omega_X|^{1/2}\, \zeta_F(2)\, q^{4(g-1)s}.\end{aligned}$$

We claim that

$$\eta(\omega_X) = 1.$$

In fact, this follows from

$$\eta(\omega_X) = \prod_{x\in|X|} \epsilon(\eta_x, 1/2, \psi_x) = \epsilon(\eta, 1/2) = 1,$$

where $\epsilon(\eta, s)$ is in the functional equation (of the complete L-function) $L(\eta, s) = \epsilon(\eta, s)L(\eta, 1-s)$.

We thus have

$$\frac{\mathscr{P}(\phi,s)\mathscr{P}_\eta(\overline{\phi},s)}{\langle\phi,\phi\rangle_{\mathrm{Pet}}}=\frac{1}{2}\,|\omega_X|^{-1/2}\,\zeta_F(2)\,\mathscr{L}(\pi_{F'},s+1/2).$$

Together with (4.13) and (4.15), the proof of Proposition 4.5 is complete. □

4.4. **Change of coefficients.** Let E be algebraic closed field containing $\mathbb{Q}$. We consider the space of E-valued automorphic functions $\mathcal{A}_E=C_c^\infty(G(F)\backslash G(\mathbb{A})/K,E)$, and its subspace $\mathcal{A}_{E,0}$ of cuspidal automorphic functions. For an irreducible $\mathscr{H}_E$-module π in $\mathcal{A}_{E,0}$, let $\lambda_\pi:\mathscr{H}\to E$ be the associated homomorphism. The L-function $\mathscr{L}(\pi_{F'},s+1/2)$ is a well-defined element in $E[q^{-s},q^s]$. Recall that $f\in\mathscr{H}$, the distribution $\mathbb{J}(f,s)$ defines an element in $\mathbb{Q}[q^{-s},q^s]$ (cf. §2).

Theorem 4.7. *Let f be a function in the Eisenstein ideal $\mathcal{I}_{\mathrm{Eis}}\subset\mathscr{H}$. Then we have an equality in $E[q^{-s},q^s]$:*

$$\mathbb{J}(f,s)=\frac{1}{2}\,|\omega_X|\,\sum_\pi\mathscr{L}(\pi_{F'},s+1/2)\,\lambda_\pi(f),$$

where the sum runs over all irreducible $\mathscr{H}_E$-module π in the E-vector space $\mathcal{A}_{E,0}$.

Proof. It suffices to show this when $E=\overline{\mathbb{Q}}$, and we fix an embedding $\overline{\mathbb{Q}}\hookrightarrow\mathbb{C}$. For $f\in\mathcal{I}_{\mathrm{Eis}}$, then Theorem 4.3 on the kernel functions remains valid if we understand the sum in (4.4) over π as $\mathscr{H}_E$-submodule. In fact, to prove Theorem 4.3, we are allowed to extend $E=\overline{\mathbb{Q}}$ to $\mathbb{C}$.

Since a cuspidal ϕ has compact support, the integral $\mathscr{P}_\chi(\phi,s)$ defined by (4.11) for $\chi\in\{1,\eta\}$ reduces to a finite sum. In particular, it defines an element in $E[q^{-s},q^s]$. Therefore the equalities in Lemma 4.4 and Proposition 4.5 hold, when both sides are viewed as elements in $E[q^{-s},q^s]$, and λ_π as an $E=\overline{\mathbb{Q}}$-valued homomorphism. This completes the proof. □

Part 2. The geometric side

5. Moduli spaces of Shtukas

The notion of rank n Shtukas (or F-sheaves) with one upper and one lower modifications was introduced by Drinfeld [6]. It was generalized to an arbitrary reductive group G and arbitrary number and type of modifications by Varshavsky [21]. In this section, we will review the definition of rank n Shtukas, and then specialize to the case of $G=\mathrm{PGL}_2$ and the case of T a nonsplit torus. Then we define Heegner–Drinfeld cycles to set up notation for the geometric side of the main theorem.

5.1. The moduli of rank n Shtukas.

5.1.1. We fix the following data.

- $r\geq 0$ is an integer;
- $\mu=(\mu_1,...,\mu_r)$ is an ordered sequence of dominant coweights for GL_n, where each μ_i is either equal to $\mu_+=(1,0,...,0)$ or equal to $\mu_-=(0,...,0,-1)$.

To such a tuple μ we assign an r-tuple of signs

$$\mathrm{sgn}(\mu)=(\mathrm{sgn}(\mu_1),\cdots,\mathrm{sgn}(\mu_r))\in\{\pm1\}^r$$

where $\mathrm{sgn}(\mu_\pm)=\pm1$.

5.1.2. *Parity condition.* At certain places we will impose the following conditions on the data (r,μ) above:

- r is even;
- Exactly half of μ_i are μ_+, and the other half are μ_-. Equivalently $\sum_{i=1}^r\mathrm{sgn}(\mu_i)=0$.

5.1.3. *The Hecke stack.* We denote by Bun_n the moduli stack of rank n vector bundles on X. By definition, for any k-scheme S, $\mathrm{Bun}_n(S)$ is the groupoid of vector bundles over $X \times S$ of rank n. It is well-known that Bun_n is a smooth algebraic stack over k of dimension $n^2(g-1)$.

Definition 5.1. Let μ be as in §5.1.1. The *Hecke stack* Hk_n^μ is the stack whose S-points $\mathrm{Hk}_n^\mu(S)$ is the groupoid of the following data:

(1) A sequence of vector bundles $(\mathcal{E}_0, \mathcal{E}_1, \cdots, \mathcal{E}_r)$ of rank n on $X \times S$;

(2) Morphisms $x_i : S \to X$ for $i = 1, \cdots, r$, with graphs $\Gamma_{x_i} \subset X \times S$;

(3) Isomorphisms of vector bundles

$$f_i : \mathcal{E}_{i-1}|_{X\times S-\Gamma_{x_i}} \xrightarrow{\sim} \mathcal{E}_i|_{X\times S-\Gamma_{x_i}}, \quad i = 1, 2, ..., r,$$

such that

- If $\mu_i = \mu_+$, then f_i extends to an injective map $\mathcal{E}_{i-1} \to \mathcal{E}_i$ whose cokernel is an invertible sheaf on the graph Γ_{x_i};
- If $\mu_i = \mu_-$, then f_i^{-1} extends to an injective map $\mathcal{E}_i \to \mathcal{E}_{i-1}$ whose cokernel is an invertible sheaf on the graph Γ_{x_i}.

For each $i = 0, \cdots, r$, we have a map

$$p_i : \mathrm{Hk}_n^\mu \longrightarrow \mathrm{Bun}_n$$

sending $(\mathcal{E}_0, \cdots, \mathcal{E}_r, x_1, \cdots, x_r, f_1, \cdots, f_r)$ to $\mathcal{E}_i$. We also have a map

$$p_X : \mathrm{Hk}_n^\mu \longrightarrow X^r$$

recording the points $(x_1, ..., x_r) \in X^r$.

Remark 5.2. The morphism $(p_0, p_X) : \mathrm{Hk}_n^\mu \to \mathrm{Bun}_n \times X^r$ is representable, proper and smooth of relative dimension $r(n-1)$. Its fibers are iterated $\mathbb{P}^{n-1}$-bundles. In particular, Hk_n^μ is a smooth algebraic stack over k because Bun_n is.

5.1.4. *The moduli stack of rank n Shtukas.*

Definition 5.3. Let μ satisfy the conditions in §5.1.2. The *moduli stack* Sht_n^μ *of* GL_n*-Shtukas of type* μ is the fiber product

$$\begin{tikzcd} \mathrm{Sht}_n^\mu \arrow[r] \arrow[d] & \mathrm{Hk}_n^\mu \arrow[d, "{(p_0,p_r)}"'] \\ \mathrm{Bun}_n \arrow[r, "{(\mathrm{id},\mathrm{Fr})}"] & \mathrm{Bun}_n \times \mathrm{Bun}_n \end{tikzcd} \tag{5.1}$$

By definition, we have a morphism

$$\pi_n^\mu : \mathrm{Sht}_n^\mu \longrightarrow \mathrm{Hk}_n^\mu \xrightarrow{p_X} X^r.$$

5.1.5. Let S be a scheme over k. For a vector bundle $\mathcal{E}$ on $X \times S$, we denote

$${}^\tau\mathcal{E} := (\mathrm{id}_X \times \mathrm{Fr}_S)^*\mathcal{E}.$$

An object in the groupoid $\mathrm{Sht}_n^\mu(S)$ is called a *Shtuka of type* μ *over* S. Concretely, a Shtuka of type μ over S is the following data:

(1) $(\mathcal{E}_0, \mathcal{E}_1, \cdots, \mathcal{E}_r; x_1, \cdots, x_r; f_1, \cdots, f_r)$ as in Definition 5.1;

(2) An isomorphism $\iota : \mathcal{E}_r \simeq {}^\tau\mathcal{E}_0$.

The basic geometric properties of Sht_n^μ are summarized in the following theorem.

Theorem 5.4 (Drinfeld [6] for $r = 2$; Varshavsky [22, Prop 2.16, Thm 2.20] in general)**.**

(1) The stack Sht_n^μ *is a Deligne–Mumford stack locally of finite type.*

(2) The morphism $\pi_n^\mu : \mathrm{Sht}_n^\mu \to X^r$ *is separated and smooth of relative dimension* $r(n-1)$*.*

We briefly comment on the proof of the separatedness of π_n^μ. Pick a place $x \in |X|$, and consider the restriction of π_n^μ to $(X-\{x\})^r$. By [22, Prop 2.16(a)], $\mathrm{Sht}_n^\mu|_{(X-\{x\})^r}$ is an increasing union of open substacks $\mathfrak{X}_1 \subset \mathfrak{X}_2 \subset \cdots$ where each $\mathfrak{X}_i \cong [V_i/G_i]$ is the quotient of a quasi-projective scheme V_i over k by a finite discrete group G_i. These V_i are obtained as moduli of Shtukas with level structures at x and then truncated using stability conditions. Therefore each map $\mathfrak{X}_i \to (X-\{x\})^r$ is separated, hence so is $\pi_n^\mu|_{(X-\{x\})^r}$. Since X^r is covered by open subschemes of the form $(X-\{x\})^r$, the map π_n^μ is separated.

5.1.6. The Picard stack Pic_X of line bundles on X acts on Bun_n and on Hk_n^μ by tensoring on the vector bundles.

Similarly, the groupoid $\mathrm{Pic}_X(k)$ of line bundles over X acts on Sht_n^μ. For a line bundle $\mathcal{L}$ over X and $(\mathcal{E}_i; x_i; f_i; \iota) \in \mathrm{Sht}_n^\mu(S)$, we define $\mathcal{L}\cdot(\mathcal{E}_i; x_i; f_i; \iota)$ to be $(\mathcal{E}_i \otimes_{\mathcal{O}_X} \mathcal{L}; x_i; f_i \otimes \mathrm{id}_{\mathcal{L}}; \iota')$ where ι' is the isomorphism

$$\mathcal{E}_r \otimes_{\mathcal{O}_X} \mathcal{L} \xrightarrow{\iota\otimes \mathrm{id}_{\mathcal{L}}} ((\mathrm{id}_X \times \mathrm{Fr}_S)^*\mathcal{E}_0) \otimes_{\mathcal{O}_X} \mathcal{L} \cong (\mathrm{id}_X \times \mathrm{Fr}_S)^*(\mathcal{E}_0 \otimes_{\mathcal{O}_X} \mathcal{L}) = {}^\tau(\mathcal{E}_0 \otimes_{\mathcal{O}_X} \mathcal{L}).$$

5.2. **Moduli of Shtukas for $G = \mathrm{PGL}_2$.** Now we move on to G-Shtukas where $G = \mathrm{PGL}_2$. Let Bun_G be the moduli stack of G-torsors over X, then $\mathrm{Bun}_G = \mathrm{Bun}_2/\mathrm{Pic}_X$.

For each μ as in §5.1.1, we define

$$\mathrm{Hk}_G^\mu := \mathrm{Hk}_2^\mu/\mathrm{Pic}_X\,.$$

For μ satisfying §5.1.2, we define

$$\mathrm{Sht}_G^\mu := \mathrm{Sht}_2^\mu/\mathrm{Pic}_X(k).$$

The actions of Pic_X and $\mathrm{Pic}_X(k)$ are those introduced in §5.1.6. The maps $p_i : \mathrm{Hk}_2^\mu \to \mathrm{Bun}_2$ are Pic_X-equivariant, and induce maps

$$p_i : \mathrm{Hk}_G^\mu \longrightarrow \mathrm{Bun}_G, \quad 0 \le i \le r. \tag{5.2}$$

Lemma 5.5. *For different choices μ and μ' as in §5.1.1, there are canonical isomorphisms $\mathrm{Hk}_2^\mu \cong \mathrm{Hk}_2^{\mu'}$ and $\mathrm{Hk}_G^\mu \cong \mathrm{Hk}_G^{\mu'}$. Moreover, these isomorphisms respect the maps p_i in* (5.2)*.*

Proof. For $\mu_+^r := (\mu_+, \cdots, \mu_+)$, we denote the corresponding Hecke stack by Hk_2^r. The S-points of Hk_2^r classify a sequence of rank two vector bundles on $X \times S$ together with embeddings

$$\mathcal{E}_0 \xrightarrow{f_1} \mathcal{E}_1 \xrightarrow{f_2} \cdots \xrightarrow{f_r} \mathcal{E}_r$$

such that the cokernel of f_i is an invertible sheaf supported on the graph of a morphism $x_i : S \to X$.

We construct a morphism

$$\phi_\mu : \mathrm{Hk}_2^\mu \longrightarrow \mathrm{Hk}_2^r.$$

Consider a point $(\mathcal{E}_i; x_i; f_i) \in \mathrm{Hk}_2^\mu(S)$. For $i = 1, \cdots, r$, we define a divisor on $X \times S$

$$D_i := \sum_{1\le j \le i, \mu_j = \mu_-} \Gamma_{x_j}.$$

Then we define

$$\mathcal{E}_i' = \mathcal{E}_i(D_i).$$

If $\mu_i = \mu_+$, then $D_{i-1} = D_i$, the map f_i induces an embedding $f_i' : \mathcal{E}_{i-1}' = \mathcal{E}_{i-1}(D_{i-1}) \to \mathcal{E}_i(D_{i-1}) = \mathcal{E}_i'$. If $\mu_i = \mu_-$, then $D_i = D_{i-1} + \Gamma_{x_i}$, and the map $f_i : \mathcal{E}_i \to \mathcal{E}_{i-1}$ induces an embedding $\mathcal{E}_{i-1} \to \mathcal{E}_i(\Gamma_{x_i})$, and hence an embedding $f_i' : \mathcal{E}_{i-1}' = \mathcal{E}_{i-1}(D_{i-1}) \to \mathcal{E}_i(D_{i-1} + \Gamma_{x_i}) = \mathcal{E}_i'$. The map ϕ_μ sends $(\mathcal{E}_i; x_i; f_i)$ to $(\mathcal{E}_i'; x_i; f_i')$.

We also have a morphism

$$\begin{array}{rcl} \psi_\mu : \mathrm{Hk}_2^r & \longrightarrow & \mathrm{Hk}_2^\mu \\ (\mathcal{E}_i'; x_i; f_i') & \longmapsto & (\mathcal{E}_i'(-D_i); x_i; f_i). \end{array}$$

It is easy to check that ϕ_μ and ψ_μ are inverse to each other. This way we get a canonical isomorphism $\mathrm{Hk}_2^\mu \cong \mathrm{Hk}_2^r$, which is clearly Pic_X-equivariant. Therefore all Hk_G^μ are also canonically isomorphic to each other. In the construction of ϕ_μ, the vector bundles $\mathcal{E}_i$ only change by tensoring with line bundles, therefore the image of $\mathcal{E}_i$ in Bun_G remain unchanged. This shows that the canonical isomorphisms between the Hk_G^μ respect the maps p_i in (5.2). $\square$

Lemma 5.6. *There is a canonical Cartesian diagram*

$$\begin{tikzcd} \mathrm{Sht}_G^\mu \arrow[r] \arrow[d] & \mathrm{Hk}_G^\mu \arrow[d, "(p_0,p_r)"] \\ \mathrm{Bun}_G \arrow[r, "(\mathrm{id},\mathrm{Fr})"] & \mathrm{Bun}_G \times \mathrm{Bun}_G \end{tikzcd} \tag{5.3}$$

In particular, for different choices of μ satisfying the conditions in §5.1.2, the stacks Sht_G^μ are canonically isomorphic to each other.

Proof. This follows from the Cartesian diagram (5.1) divided termwisely by the Cartesian diagram

$$\begin{tikzcd} \mathrm{Pic}_X(k) \arrow[r] \arrow[d] & \mathrm{Pic}_X \arrow[d, "\Delta"'] \\ \mathrm{Pic}_X \arrow[r, "(\mathrm{id},\mathrm{Fr})"] & \mathrm{Pic}_X \times \mathrm{Pic}_X \end{tikzcd}$$

□

By the above lemmas, we may unambiguously use the notation

$$\mathrm{Sht}_G^r; \quad \mathrm{Hk}_G^r \tag{5.4}$$

for Sht_G^μ and Hk_G^μ with any choice of μ. If r is fixed from the context, we may also drop r from the notation and write simply Sht_G. The morphism $\pi_2^\mu : \mathrm{Sht}_2^\mu \to X^r$ is invariant under the action of $\mathrm{Pic}_X(k)$ and induces a morphism

$$\pi_G : \mathrm{Sht}_G^r \longrightarrow X^r.$$

Theorem 5.4 has the following immediate consequence.

Corollary 5.7. *(1) The stack Sht_G^r is a Deligne–Mumford stack locally of finite type.*

(2) The morphism $\pi_G : \mathrm{Sht}_G^r \to X^r$ is separated and smooth of relative dimension r.

5.3. **Hecke correspondences.** We define the rational Chow group of proper cycles $\mathrm{Ch}_{c,i}(\mathrm{Sht}_G^r)_\mathbb{Q}$ as in §A.1. As in §A.1.6, we also have a $\mathbb{Q}$-algebra ${}_c\mathrm{Ch}_{2r}(\mathrm{Sht}_G^r \times \mathrm{Sht}_G^r)_\mathbb{Q}$ that acts on $\mathrm{Ch}_{c,i}(\mathrm{Sht}_G^r)_\mathbb{Q}$. The goal of this subsection is to define a ring homomorphism from the unramified Hecke algebra $\mathscr{H} = C_c(K\backslash G(\mathbb{A})/K, \mathbb{Q})$ to ${}_c\mathrm{Ch}_{2r}(\mathrm{Sht}_G^r \times \mathrm{Sht}_G^r)_\mathbb{Q}$.

5.3.1. *The stack* $\mathrm{Sht}_G^r(h_D)$. Recall from §3.1 that we have a basis h_D of $\mathscr{H}$ indexed by effective divisors D on X. For each effective divisor $D = \sum_{x\in|X|} n_x x$ we shall define a self-correspondence $\mathrm{Sht}_G^r(h_D)$ of Sht_G^r over X^r:

$$\begin{tikzcd} & \mathrm{Sht}_G^r(h_D) \arrow[dl, "\overleftarrow{p}"'] \arrow[dr, "\overrightarrow{p}"] & \\ \mathrm{Sht}_G^r \arrow[dr] & & \mathrm{Sht}_G^r \arrow[dl] \\ & X^r & \end{tikzcd}$$

For this, we first fix a μ as in §5.1.2. We introduce a self-correspondence $\mathrm{Sht}_2^\mu(h_D)$ of Sht_2^μ whose S-points is the groupoid classifying the data

(1) Two objects $(\mathcal{E}_i; x_i; f_i; \iota)$ and $(\mathcal{E}'_i; x_i; f'_i; \iota')$ of $\mathrm{Sht}_2^\mu(S)$ with the same collection of points $x_1, \cdots, x_r$ in $X(S)$;

(2) For each $i = 0, \cdots, r$, an embedding of coherent sheaves $\phi_i : \mathcal{E}_i \hookrightarrow \mathcal{E}'_i$ such that $\det(\phi_i) : \det \mathcal{E}_i \hookrightarrow \det \mathcal{E}'_i$ has divisor $D \times S \subset X \times S$.

(3) The following diagram is commutative

$$\begin{tikzcd} \mathcal{E}_0 \arrow[r, dashed, "f_1"] \arrow[d, "\phi_0"] & \mathcal{E}_1 \arrow[r, dashed, "f_2"] \arrow[d, "\phi_1"] & \cdots \arrow[r, dashed, "f_r"] & \mathcal{E}_r \arrow[r, "\iota"] \arrow[d, "\phi_r"] & {}^\tau\mathcal{E}_0 \arrow[d, "{}^\tau\phi_0"] \\ \mathcal{E}'_0 \arrow[r, dashed, "f'_1"] & \mathcal{E}'_1 \arrow[r, dashed, "f'_2"] & \cdots \arrow[r, dashed, "f'_r"] & \mathcal{E}'_r \arrow[r, "\iota'"] & {}^\tau\mathcal{E}'_0 \end{tikzcd} \tag{5.5}$$

There is a natural action of $\mathrm{Pic}_X(k)$ on $\mathrm{Sht}^\mu_2(h_D)$ by tensoring on each $\mathcal{E}_i$ and $\mathcal{E}'_i$. We define

$$\mathrm{Sht}^r_G(h_D) := \mathrm{Sht}^\mu_2(h_D)/\mathrm{Pic}_X(k).$$

Using Lemma 5.5, it is easy to check that $\mathrm{Sht}^r_G(h_D)$ is canonically independent of the choice of μ. The two maps $\overleftarrow{p}, \overrightarrow{p} : \mathrm{Sht}^r_G(h_D) \to \mathrm{Sht}^r_G$ send the data above to the image of $(\mathcal{E}_i; x_i; f_i; \iota)$ and $(\mathcal{E}'_i; x_i; f'_i; \iota')$ in Sht^r_G respectively.

Lemma 5.8. *The maps $\overleftarrow{p}, \overrightarrow{p}$ as well as $(\overleftarrow{p}, \overrightarrow{p}) : \mathrm{Sht}^r_G(h_D) \to \mathrm{Sht}^r_G \times \mathrm{Sht}^r_G$ are representable and proper.*

Proof. Once the bottom row of the diagram (5.5) is fixed, the choices of the vertical maps ϕ_i for $i = 1, \cdots, r$ form a closed subscheme of the product of Quot schemes $\prod_{i=1}^r \mathrm{Quot}^d(\mathcal{E}'_i)$, where $d = \deg D$, which is proper. Therefore $\overrightarrow{p}$ is representable and proper. Same argument applied to the dual of the diagram (5.5) proves that $\overleftarrow{p}$ is proper.

The representability of $(\overleftarrow{p}, \overrightarrow{p})$ is obvious from the definition, since its fibers are closed subschemes of $\prod_{i=1}^r \mathrm{Hom}(\mathcal{E}_i, \mathcal{E}'_i)$. Since Sht^r_G is separated by Corollary 5.7 and $\overleftarrow{p}$ is proper, $(\overleftarrow{p}, \overrightarrow{p})$ is also proper. □

Lemma 5.9. *The geometric fibers of the map $\mathrm{Sht}^r_G(h_D) \to X^r$ have dimension r.*

The proof of this lemma will be postponed to §6.4.4, because the argument will involve some auxiliary moduli spaces that we will introduce in §6.3.

Granting Lemma 5.9, we have $\dim \mathrm{Sht}^r_G(h_D) = 2r$. By Lemma 5.8, it makes sense to push forward the fundamental cycle of $\mathrm{Sht}^r_G(h_D)$ along the proper map $(\overleftarrow{p}, \overrightarrow{p})$. Therefore $(\overleftarrow{p}, \overrightarrow{p})_*[\mathrm{Sht}^r_G(h_D)]$ defines an element in ${}_c\mathrm{Ch}_{2r}(\mathrm{Sht}^r_G \times \mathrm{Sht}^r_G)_\mathbb{Q}$ (because $\overleftarrow{p}$ is also proper). We define the $\mathbb{Q}$-linear map

$$\begin{aligned} H : \mathscr{H} &\longrightarrow {}_c\mathrm{Ch}_{2r}(\mathrm{Sht}^r_G \times \mathrm{Sht}^r_G)_\mathbb{Q} && (5.6)\\ h_D &\longmapsto (\overleftarrow{p} \times \overrightarrow{p})_*[\mathrm{Sht}^r_G(h_D)], \quad \text{for all effective divisors } D && (5.7) \end{aligned}$$

Proposition 5.10. *The linear map H in (5.6) is a ring homomorphism.*

Proof. Let D, D' be two effective divisors, and we would like to show the equality

$$H(h_D h_{D'}) = H(h_D) * H(h_{D'}) \in {}_c\mathrm{Ch}_{2r}(\mathrm{Sht}^r_G \times \mathrm{Sht}^r_G)_\mathbb{Q}. \tag{5.8}$$

Let $U = X - |D| - |D'|$. Since $h_D h_{D'}$ is a linear combination of h_E for effective divisors $E \le D + D'$ such that $D + D' - E$ has even coefficients, the cycle $H(h_D h_{D'})$ is supported on $\cup_{E \le D+D', D+D'-E \text{ even}} \mathrm{Sht}^r_G(h_E) = \mathrm{Sht}^r_G(h_{D+D'})$. The cycle $H(h_D) * H(h_{D'})$ is supported on the image of the projection

$$\mathrm{pr}_{13} : \mathrm{Sht}^r_G(h_D) \times_{\overrightarrow{p}, \mathrm{Sht}^r_G, \overleftarrow{p}} \mathrm{Sht}^r_G(h_{D'}) \longrightarrow \mathrm{Sht}^r_G \times \mathrm{Sht}^r_G$$

which is easily seen to be contained in $\mathrm{Sht}^r_G(h_{D+D'})$. We see that both sides of (5.8) are supported on $Z := \mathrm{Sht}^r_G(h_{D+D'})$.

By Lemma 5.9 applied to $Z = \mathrm{Sht}^r_G(h_{D+D'})$, the dimension of $Z - Z|_{U^r}$ is strictly less than $2r$. Therefore, the restriction map induces an isomorphism

$$\mathrm{Ch}_{2r}(Z)_\mathbb{Q} \xrightarrow{\sim} \mathrm{Ch}_{2r}(Z|_{U^r})_\mathbb{Q} \tag{5.9}$$

Restricting the definition of H to U^r, we get a linear map $H_U : \mathscr{H} \to {}_c\mathrm{Ch}_{2r}(\mathrm{Sht}^r_G|_{U^r} \times \mathrm{Sht}^r_G|_{U^r})_\mathbb{Q}$. For any effective divisor E supported on $|D| \cup |D'|$, the two projections $\overleftarrow{p}, \overrightarrow{p} : \mathrm{Sht}^r_G(h_E)|_{U^r} \to \mathrm{Sht}^r_G|_{U^r}$ are finite étale. The equality

$$H_U(h_D h_{D'}) = H_U(h_D) * H_U(h_{D'}) \in \mathrm{Ch}_{2r}(Z|_{U^r})_\mathbb{Q} \tag{5.10}$$

is well-known. By (5.9), this implies the equality (5.8) where both sides are interpreted as elements in $\mathrm{Ch}_{2r}(Z)_\mathbb{Q}$, and a fortiori as elements in ${}_c\mathrm{Ch}_{2r}(\mathrm{Sht}^r_G \times \mathrm{Sht}^r_G)_\mathbb{Q}$. □

Remark 5.11. Let $g = (g_x) \in G(\mathbb{A})$, and let $f = \mathbf{1}_{KgK} \in \mathscr{H}$ be the characteristic function of the double coset KgK in $G(\mathbb{A})$. Traditionally, one defines a self-correspondence $\Gamma(g)$ of $\mathrm{Sht}^r_G|_{(X-S)^r}$ over $(X-S)^r$, where S is the finite set of places where $g_x \notin K_x$ (see [16, Construction 2.20]). The two projections $\overleftarrow{p}, \overrightarrow{p} : \Gamma(g) \to \mathrm{Sht}^r_G|_{(X-S)^r}$ are finite étale. The disadvantage of this definition is that we need to remove the bad points S which depend on f, so one is forced to work only with the generic fiber of Sht^r_G over X^r if one wants to consider the actions of all Hecke functions. Our definition of $H(f)$ for any $f \in \mathscr{H}$ gives a correspondence for the whole Sht^r_G. It is easy to check that for $f = \mathbf{1}_{KgK}$, our cycle $H(f)|_{(X-S)^r}$, which is a linear combination of the cycles $\mathrm{Sht}^r_G(h_D)|_{(X-S)^r}$ for divisors D supported on S, is the same cycle as $\Gamma(g)$. Therefore our definition of the Hecke algebra action extends the traditional one.

5.3.2. *A variant.* Later we will consider the stack $\mathrm{Sht}'^r_G := \mathrm{Sht}^r_G \times_{X^r} X'^r$ defined using the double cover $X' \to X$. Let $\mathrm{Sht}'^r_G(h_D) = \mathrm{Sht}^r_G(h_D) \times_{X^r} X'^r$. Then we have natural maps

$$\overleftarrow{p}', \overrightarrow{p}' : \mathrm{Sht}'^r_G(h_D) \longrightarrow \mathrm{Sht}'^r_G.$$

The analogs of Lemma 5.8 and 5.9 for $\mathrm{Sht}'^r_G(h_D)$ follow from the original statements. The map $h_D \mapsto (\overleftarrow{p}' \times \overrightarrow{p}')_*[\mathrm{Sht}'^r_G(h_D)] \in {}_c\mathrm{Ch}_{2r}(\mathrm{Sht}'^r_G \times \mathrm{Sht}'^r_G)_{\mathbb{Q}}$ then gives a ring homomorphism H':

$$H' : \mathscr{H} \longrightarrow {}_c\mathrm{Ch}_{2r}(\mathrm{Sht}'^r_G \times \mathrm{Sht}'^r_G)_{\mathbb{Q}}.$$

5.3.3. *Notation.* By §A.1.6, the $\mathbb{Q}$-algebra ${}_c\mathrm{Ch}_{2r}(\mathrm{Sht}'^r_G \times \mathrm{Sht}'^r_G)_{\mathbb{Q}}$ acts on $\mathrm{Ch}_{c,*}(\mathrm{Sht}'^r_G)_{\mathbb{Q}}$. Hence the Hecke algebra $\mathscr{H}$ also acts on $\mathrm{Ch}_{c,*}(\mathrm{Sht}'^r_G)_{\mathbb{Q}}$ via the homomorphism H'. For $f \in \mathscr{H}$, we denote its action on $\mathrm{Ch}_{c,*}(\mathrm{Sht}'^r_G)_{\mathbb{Q}}$ by

$$f * (-) : \mathrm{Ch}_{c,*}(\mathrm{Sht}'^r_G)_{\mathbb{Q}} \longrightarrow \mathrm{Ch}_{c,*}(\mathrm{Sht}'^r_G)_{\mathbb{Q}}.$$

Recall the Chow group $\mathrm{Ch}_{c,*}(\mathrm{Sht}^r_G)_{\mathbb{Q}}$ (or $\mathrm{Ch}_{c,*}(\mathrm{Sht}'^r_G)_{\mathbb{Q}}$) is equipped with an intersection pairing between complementary degrees, see §A.1.4.

Lemma 5.12. *The action of any $f \in \mathscr{H}$ on $\mathrm{Ch}_{c,*}(\mathrm{Sht}^r_G)_{\mathbb{Q}}$ or $\mathrm{Ch}_{c,*}(\mathrm{Sht}'^r_G)_{\mathbb{Q}}$ is self-adjoint with respect to the intersection pairing.*

Proof. It suffices to prove self-adjointness for h_D for all effective divisors D. We give the argument for Sht^r_G and the case of Sht'^r_G can be proved in the same way. For $\zeta_1 \in \mathrm{Ch}_{c,i}(\mathrm{Sht}^r_G)_{\mathbb{Q}}$ and $\zeta_2 \in \mathrm{Ch}_{c,2r-i}(\mathrm{Sht}^r_G)_{\mathbb{Q}}$, the intersection pairing $(h_D * \zeta_1, \zeta_2)_{\mathrm{Sht}^r_G}$ is the same as the following intersection number in $\mathrm{Sht}^r_G \times \mathrm{Sht}^r_G$

$$(\zeta_1 \times \zeta_2, (\overleftarrow{p}, \overrightarrow{p})_*[\mathrm{Sht}^r_G(h_D)])_{\mathrm{Sht}^r_G \times \mathrm{Sht}^r_G}.$$

We will construct an involution τ on $\mathrm{Sht}^r_G(h_D)$ such that the following diagram is commutative

$$\begin{tikzcd} \mathrm{Sht}^r_G(h_D) \arrow[r, "\tau"] \arrow[d, "(\overleftarrow{p}, \overrightarrow{p})"] & \mathrm{Sht}^r_G(h_D) \arrow[d, "(\overleftarrow{p}, \overrightarrow{p})"] \\ \mathrm{Sht}^r_G \times \mathrm{Sht}^r_G \arrow[r, "\sigma_{12}"] & \mathrm{Sht}^r_G \times \mathrm{Sht}^r_G \end{tikzcd} \tag{5.11}$$

Here σ_{12} in the bottom row means flipping two factors. Once we have such a diagram, we can apply τ to $\mathrm{Sht}^r_G(h_D)$ and σ_{12} to $\mathrm{Sht}^r_G \times \mathrm{Sht}^r_G$ and get

$$(\zeta_1 \times \zeta_2, (\overleftarrow{p}, \overrightarrow{p})_*[\mathrm{Sht}^r_G(h_D)])_{\mathrm{Sht}^r_G \times \mathrm{Sht}^r_G} = (\zeta_2 \times \zeta_1, (\overleftarrow{p}, \overrightarrow{p})_*[\mathrm{Sht}^r_G(h_D)])_{\mathrm{Sht}^r_G \times \mathrm{Sht}^r_G}$$

which is the same as the self-adjointness for $h * (-)$:

$$(h_D * \zeta_1, \zeta_2)_{\mathrm{Sht}^r_G} = (h_D * \zeta_2, \zeta_1)_{\mathrm{Sht}^r_G} = (\zeta_1, h_D * \zeta_2)_{\mathrm{Sht}^r_G}.$$

We pick any μ as in §5.1.2 and identify Sht^r_G with $\mathrm{Sht}^\mu_G = \mathrm{Sht}^\mu_2 / \mathrm{Pic}_X(k)$. We use $-\mu$ to denote the negated tuple if we think of $\mu \in \{\pm 1\}^r$ using the sgn map. We consider the composition

$$\delta : \mathrm{Sht}^\mu_G \xrightarrow{\delta'} \mathrm{Sht}^{-\mu}_G \cong \mathrm{Sht}^\mu_G$$

where $\delta'(\mathcal{E}_i; x_i; f_i; \iota) = (\mathcal{E}_i^\vee; x_i; (f_i^\vee)^{-1}; (\iota^\vee)^{-1})$ and the second map is the canonical isomorphism $\mathrm{Sht}^{-\mu}_G \cong \mathrm{Sht}^\mu_G$ given by Lemma 5.6.

Similarly we define τ as the composition

$$\tau : \mathrm{Sht}^\mu_G(h_D) \xrightarrow{\tau'} \mathrm{Sht}^{-\mu}_G(h_D) \cong \mathrm{Sht}^\mu_G(h_D) \tag{5.12}$$

where τ' sends the diagram (5.5) to the diagram

$$\begin{tikzcd}
\mathcal{E}_0'^{\vee} \arrow[r, dashed, "f_1'^{\vee-1}"] \arrow[d, "\phi_0^{\vee}"] & \mathcal{E}_1'^{\vee} \arrow[r, dashed, "f_2'^{\vee-1}"] \arrow[d, "\phi_1^{\vee}"] & \cdots \arrow[r, dashed, "f_r'^{\vee-1}"] & \mathcal{E}_r'^{\vee} \arrow[r, "\iota'^{\vee-1}"] \arrow[d, "\phi_r^{\vee}"] & {}^{\tau}\mathcal{E}_0'^{\vee} \arrow[d, "{}^{\tau}\phi_0^{\vee}"] \\
\mathcal{E}_0^{\vee} \arrow[r, dashed, "f_1^{\vee-1}"] & \mathcal{E}_1^{\vee} \arrow[r, dashed, "f_2^{\vee-1}"] & \cdots \arrow[r, dashed, "f_r^{\vee-1}"] & \mathcal{E}_r^{\vee} \arrow[r, "\iota^{\vee-1}"] & {}^{\tau}\mathcal{E}_0^{\vee}
\end{tikzcd} \tag{5.13}$$

and the second map in (5.12) is the canonical isomorphism $\mathrm{Sht}_G^{-\mu}(h_D) \cong \mathrm{Sht}_G^{\mu}(h_D)$ given by the analog of Lemma 5.6. It is clear from the definition that if we replace the bottom arrow of (5.11) with $\sigma_{12} \circ (\delta \times \delta)$ (i.e., the map $(a, b) \mapsto (\delta(b), \delta(a))$), the diagram is commutative.

We claim that δ is the identity map for Sht_G^{μ}. In fact, δ turns $(\mathcal{E}_i; x_i; f_i; \iota) \in \mathrm{Sht}_G^{\mu}$ into $(\mathcal{E}_i^{\vee}(D_i); x_i; (f_i^{\vee})^{-1}; (\iota^{\vee})^{-1})$, where $D_i = \sum_{1\le j\le i} \mathrm{sgn}(\mu_j)\Gamma_{x_j}$. Note that we have a canonical isomorphism $\mathcal{E}_i^{\vee} \cong \mathcal{E}_i \otimes (\det \mathcal{E}_i)^{-1}$, and isomorphisms $\det \mathcal{E}_i \cong (\det \mathcal{E}_0)(D_i)$ induced by the f_i. Therefore we get a canonical isomorphism $\mathcal{E}_i^{\vee}(D_i) \cong \mathcal{E}_i \otimes (\det \mathcal{E}_i)^{-1} \otimes \mathcal{O}(D_i) \cong \mathcal{E}_i \otimes (\det \mathcal{E}_0)^{-1}$ compatibly with the maps $(f_i^{\vee})^{-1}$ and f_i, and also compatible with $(\iota^{\vee})^{-1}$ and ι. Therefore $\delta(\mathcal{E}_i; x_i; f_i; \iota)$ is canonically isomorphic to $(\mathcal{E}_i; x_i; f_i; \iota)$ up to tensoring with $\det(\mathcal{E}_0)$. This shows that δ is the identity map of Sht_G^{μ}.

Since $\delta = \mathrm{id}$, the diagram (5.11) is also commutative. This finishes the proof. $\square$

5.4. **Moduli of Shtukas for the torus T.**

5.4.1. Recall that $\nu : X' \to X$ is an étale double covering with X' also geometrically connected. Let $\sigma \in \mathrm{Gal}(X'/X)$ be the non-trivial involution.

Let $\widetilde{T}$ be the two-dimensional torus over X defined as

$$\widetilde{T} := \mathrm{Res}_{X'/X}\, \mathbb{G}_m.$$

We have a natural homomorphism $\mathbb{G}_m \to \widetilde{T}$. We define a one-dimensional torus over X

$$T := \widetilde{T}/\mathbb{G}_m = (\mathrm{Res}_{X'/X}\, \mathbb{G}_m)/\mathbb{G}_m.$$

Let Bun_T be the moduli stack of T-torsors over X. Then we have a canonical isomorphism of stacks

$$\mathrm{Bun}_T \cong \mathrm{Pic}_{X'} / \mathrm{Pic}_X .$$

In particular, Bun_T is a Deligne–Mumford stack whose coarse moduli space is a group scheme with two components, and its neutral component is an abelian variety over k.

5.4.2. Specializing Definition 5.1 to the case $n = 1$ and replacing the curve X with its double cover X', we get the Hecke stack $\mathrm{Hk}^{\mu}_{1,X'}$. This makes sense for any tuple μ as in §5.1.1.

Now assume that μ satisfies the conditions in §5.1.2. We may view each μ_i as a coweight for $\mathrm{GL}_1 = \mathbb{G}_m$ in an obvious way: μ_+ means 1 and μ_- means -1. Specializing Definition 5.3 to the case $n = 1$ and replacing X with X', we get the moduli stack $\mathrm{Sht}^{\mu}_{1,X'}$ of rank one Shtukas over X' of type μ. We define

$$\mathrm{Sht}^{\mu}_{\widetilde{T}} := \mathrm{Sht}^{\mu}_{1,X'}.$$

We have a Cartesian diagram

$$\begin{tikzcd}
\mathrm{Sht}^{\mu}_{\widetilde{T}} \arrow[r] \arrow[d] & \mathrm{Hk}^{\mu}_{1,X'} \arrow[d, "{(p_0,p_r)}"] \\
\mathrm{Pic}_{X'} \arrow[r, "{(\mathrm{id},\mathrm{Fr})}"] & \mathrm{Pic}_{X'} \times \mathrm{Pic}_{X'}
\end{tikzcd}$$

We also have a morphism

$$\pi^{\mu}_{\widetilde{T}} : \mathrm{Sht}^{\mu}_{\widetilde{T}} \longrightarrow \mathrm{Hk}^{\mu}_{1,X'} \xrightarrow{p_{X'}} X'^r.$$

5.4.3. Fix μ as in §5.1.2. Concretely, for any k-scheme S, $\mathrm{Sht}^{\mu}_{\widetilde{T}}(S)$ classifies the following data

(1) A line bundle $\mathcal{L}$ over $X' \times S$;

(2) Morphisms $x'_i : S \to X'$ for $i = 1, \cdots, r$, with graphs $\Gamma_{x'_i} \subset X' \times S$;

(3) An isomorphism

$$\iota : \mathcal{L}\left(\sum_{i=1}^{r} \mathrm{sgn}(\mu_i)\Gamma_{x'_i}\right) \xrightarrow{\sim} {}^{\tau}\mathcal{L} := (\mathrm{id} \times \mathrm{Fr}_S)^{*}\mathcal{L}.$$

Here the signs $\mathrm{sgn}(\mu_{\pm}) = \pm 1$ are defined in §5.1.1.

This description of points appears to be simpler than its counterpart in §5.1.5: the other line bundles $\mathcal{L}_i$ are canonically determined by $\mathcal{L}_0$ and x'_i using the formula

$$\mathcal{L}_i = \mathcal{L}_0\left(\sum_{1 \le j \le i} \mathrm{sgn}(\mu_j)\Gamma_{x'_j}\right). \tag{5.14}$$

5.4.4. The Picard stack $\mathrm{Pic}_{X'}$, and hence Pic_X, acts on $\mathrm{Hk}^{\mu}_{1,X'}$. We consider the quotient

$$\mathrm{Hk}^{\mu}_{T} := \mathrm{Hk}^{\mu}_{1,X'} / \mathrm{Pic}_X. \tag{5.15}$$

In fact we have a canonical isomorphism $\mathrm{Hk}^{\mu}_{T} \cong \mathrm{Bun}_T \times X'^r$ sending $(\mathcal{L}_i; x'_i; f_i)$ to $(\mathcal{L}_0; x'_i)$. In particular, Hk^{μ}_{T} is a smooth and proper Deligne–Mumford stack of pure dimension $r+g-1$ over k.

5.4.5. The groupoid $\mathrm{Pic}_{X'}(k)$ acts on $\mathrm{Sht}^{\mu}_{\widetilde{T}}$ by tensoring on the line bundle $\mathcal{L}$. We consider the restriction of this action to $\mathrm{Pic}_X(k)$ via the pullback map $\nu^* : \mathrm{Pic}_X(k) \to \mathrm{Pic}_{X'}(k)$. We define

$$\mathrm{Sht}^{\mu}_{T} := \mathrm{Sht}^{\mu}_{\widetilde{T}} / \mathrm{Pic}_X(k).$$

The analog of Lemma 5.6 gives a Cartesian diagram

$$\begin{tikzcd} \mathrm{Sht}^{\mu}_{T} \arrow[r] \arrow[d] & \mathrm{Hk}^{\mu}_{T} \arrow[d, "(p_0,p_r)"] \\ \mathrm{Bun}_T \arrow[r, "(\mathrm{id},\mathrm{Fr})"] & \mathrm{Bun}_T \times \mathrm{Bun}_T \end{tikzcd} \tag{5.16}$$

Since the morphism $\pi^{\mu}_{\widetilde{T}}$ is invariant under $\mathrm{Pic}_X(k)$, we get a morphism

$$\pi^{\mu}_{T} : \mathrm{Sht}^{\mu}_{T} \longrightarrow X'^r.$$

Lemma 5.13. *The morphism π^{μ}_{T} is a torsor under the finite Picard groupoid $\mathrm{Pic}_{X'}(k)/\mathrm{Pic}_X(k)$. In particular, π^{μ}_{T} is finite étale, and the stack Sht^{μ}_{T} is a smooth proper Deligne–Mumford stack over k of pure dimension r.*

Proof. This description given in §5.4.3 gives a Cartesian diagram

$$\begin{tikzcd} \mathrm{Sht}^{\mu}_{\widetilde{T}} \arrow[r] \arrow[d, "\pi^{\mu}_{\widetilde{T}}"] & \mathrm{Pic}_{X'} \arrow[d, "\mathrm{id}-\mathrm{Fr}"] \\ X'^r \arrow[r, "\phi"] & \mathrm{Pic}^0_{X'} \end{tikzcd} \tag{5.17}$$

where $\phi(x'_1, \cdots, x'_r) = \mathcal{O}_{X'}(\sum_{i=1}^{r} \mathrm{sgn}(\mu_i)x'_i)$. Dividing the top row of the diagram (5.17) by $\mathrm{Pic}_X(k)$ we get a Cartesian diagram

$$\begin{tikzcd} \mathrm{Sht}^{\mu}_{T} \arrow[r] \arrow[d, "\pi^{\mu}_{T}"] & \mathrm{Pic}_{X'} /(\mathrm{Pic}_X(k)) \arrow[d, "\mathrm{id}-\mathrm{Fr}"] \\ X'^r \arrow[r, "\phi"] & \mathrm{Pic}^0_{X'} \end{tikzcd}$$

Since the right vertical map $\mathrm{id}-\mathrm{Fr} : \mathrm{Pic}_{X'} /(\mathrm{Pic}_X(k)) \to \mathrm{Pic}^0_{X'}$ is a torsor under $\mathrm{Pic}_{X'}(k)/\mathrm{Pic}_X(k)$, so is π^{μ}_{T}. $\square$

5.4.6. *Changing* μ. For a different choice μ' as in 5.1.1, we have a canonical isomorphism

$$\mathrm{Hk}^{\mu}_{\widetilde{T}} \xrightarrow{\sim} \mathrm{Hk}^{\mu'}_{\widetilde{T}} \tag{5.18}$$

sending $(\mathcal{L}_i; x'_i; f_i)$ to $(\mathcal{K}_i; y'_i; g_i)$ where

$$y'_i = \begin{cases} x'_i & \text{if } \mu_i = \mu'_i \\ \sigma(x'_i) & \text{if } \mu_i \neq \mu'_i \end{cases} \tag{5.19}$$

and

$$\mathcal{K}_i = \mathcal{L}_0 \left(\sum_{1 \le j \le i} \mathrm{sgn}(\mu'_j) \Gamma_{y'_j} \right). \tag{5.20}$$

The rational maps $g_i : \mathcal{K}_{i-1} \dashrightarrow \mathcal{K}_i$ is the one corresponding to the identity map on $\mathcal{L}_0$ via the description (5.20). Note that we have

$$\mathcal{K}_i = \mathcal{L}_i \otimes_{\mathcal{O}_{X \times S}} \mathcal{O}_{X \times S} \left(\sum_{1 \le j \le i} \frac{\mathrm{sgn}(\mu'_j) - \mathrm{sgn}(\mu_j)}{2} \Gamma_{x_j} \right)$$

where $x_i : S \to X$ is the image of x'_i. Therefore $\mathcal{K}_i$ has the same image as $\mathcal{L}_i$ in Bun_T. The isomorphism (5.18) induces an isomorphism

$$\mathrm{Hk}^{\mu}_{T} \xrightarrow{\sim} \mathrm{Hk}^{\mu'}_{T}. \tag{5.21}$$

From the construction and the above discussion, this isomorphism preserves the maps p_i to Bun_T but *does not preserve* the projections to X'^r (it only preserves the further projection to X^r).

Since the isomorphism (5.21) preserves the maps p_0 and p_r, the diagram (5.16) implies a canonical isomorphism

$$\iota_{\mu,\mu'} : \mathrm{Sht}^{\mu}_{T} \xrightarrow{\sim} \mathrm{Sht}^{\mu'}_{T}. \tag{5.22}$$

Just as the map (5.21), $\iota_{\mu,\mu'}$ does not respect the maps π^{μ}_T and $\pi^{\mu'}_T$ from Sht^{μ}_T and $\mathrm{Sht}^{\mu'}_T$ to X'^r: it only respects their further projections to X^r.

5.5. **The Heegner–Drinfeld cycles.**

5.5.1. We have a morphism

$$\begin{array}{rcl} \Pi : \mathrm{Bun}_T & \longrightarrow & \mathrm{Bun}_G \\ (\mathcal{L} \mod \mathrm{Pic}_X) & \longmapsto & (\nu_* \mathcal{L} \mod \mathrm{Pic}_X) \end{array}$$

5.5.2. For any μ as in §5.1.2 we define a morphism

$$\widetilde{\theta}^{\mu} : \mathrm{Sht}^{\mu}_{\widetilde{T}} \longrightarrow \mathrm{Sht}^{\mu}_{2}$$

as follows. Let $(\mathcal{L}; x'_i; \iota) \in \mathrm{Sht}^{\mu}_{\widetilde{T}}(S)$ as in the description in §5.4.3. Let $\mathcal{L}_0 = \mathcal{L}$ and we may define the line bundles $\mathcal{L}_i$ using (5.14). Then there are natural maps $g_i : \mathcal{L}_{i-1} \hookrightarrow \mathcal{L}_i$ if $\mu_i = \mu_+$ or $g_i : \mathcal{L}_i \hookrightarrow \mathcal{L}_{i-1}$ if $\mu_i = \mu_-$. Let $\nu_S = \nu \times \mathrm{id}_S : X' \times S \to X \times S$ the base change of ν. We define

$$\mathcal{E}_i = \nu_{S*} \mathcal{L}_i$$

with the maps $f_i : \mathcal{E}_{i-1} \to \mathcal{E}_i$ or $\mathcal{E}_i \to \mathcal{E}_{i-1}$ induced from g_i. The isomorphism ι then induces an isomorphism

$$\jmath : \mathcal{E}_r = \nu_{S*} \mathcal{L}_r \xrightarrow{\nu_{S*}\iota} \nu_{S*} (\mathrm{id}_{X'} \times \mathrm{Fr}_S)^* \mathcal{L}_0 \cong (\mathrm{id}_X \times \mathrm{Fr}_S)^* \nu_{S*} \mathcal{L}_0 = {}^{\tau}\mathcal{E}_0.$$

Let $x_i = \nu \circ x'_i$. The morphism $\widetilde{\theta}^{\mu}$ then sends $(\mathcal{L}; x'_i; \iota)$ to $(\mathcal{E}_i; x_i; f_i; \jmath)$. Clearly $\widetilde{\theta}^{\mu}$ is equivariant with respect to the $\mathrm{Pic}_X(k)$-actions. Passing to the quotients, we get a morphism

$$\overline{\theta}^{\mu} : \mathrm{Sht}^{\mu}_{T} \longrightarrow \mathrm{Sht}^{\mu}_{G}.$$

For a different μ', the canonical isomorphism $\iota_{\mu,\mu'}$ in (5.22) intertwines the maps $\overline{\theta}^{\mu}$ and $\overline{\theta}^{\mu'}$, i.e., we have a commutative diagram

$$\begin{tikzcd} \mathrm{Sht}_T^{\mu} \arrow[r, "\overline{\theta}^{\mu}"] \arrow[d, "\iota_{\mu,\mu'}"] & \mathrm{Sht}_G^{\mu} \arrow[d] \\ \mathrm{Sht}_T^{\mu'} \arrow[r, "\overline{\theta}^{\mu'}"] & \mathrm{Sht}_G^{\mu'} \end{tikzcd}$$

where the right vertical map is the canonical isomorphism in Lemma 5.6. By our identification of Sht_G^{μ} for different μ (cf. (5.4)), we get a morphism, still denoted by $\overline{\theta}^{\mu}$,

$$\overline{\theta}^{\mu} : \mathrm{Sht}_T^{\mu} \longrightarrow \mathrm{Sht}_G^{r}.$$

5.5.3. By construction we have a commutative diagram

$$\begin{tikzcd} \mathrm{Sht}_T^{\mu} \arrow[r, "\overline{\theta}^{\mu}"] \arrow[d, "\pi_T^{\mu}"] & \mathrm{Sht}_G^{r} \arrow[d, "\pi_G"] \\ X'^r \arrow[r, "\nu^r"] & X^r \end{tikzcd}$$

Recall that

$$\mathrm{Sht}'^r_G := \mathrm{Sht}_G^r \times_{X^r} X'^r.$$

Then the map $\overline{\theta}^{\mu}$ factors through a morphism

$$\theta^{\mu} : \mathrm{Sht}_T^{\mu} \longrightarrow \mathrm{Sht}'^r_G$$

over X'^r. Since Sht_T^{μ} is proper of dimension r, $\theta^{\mu}_*[\mathrm{Sht}_T^{\mu}]$ is a proper cycle class in Sht'^r_G of dimension r.

Definition 5.14. The *Heegner–Drinfeld cycle* of type μ is the direct image of $[\mathrm{Sht}_T^{\mu}]$ under θ^{μ}:

$$\theta^{\mu}_*[\mathrm{Sht}_T^{\mu}] \in \mathrm{Ch}_{c,r}(\mathrm{Sht}'^r_G)_{\mathbb{Q}}.$$

Recall from Proposition 5.10 and §5.3.3 that we have an action of $\mathscr{H}$ on $\mathrm{Ch}_{c,r}(\mathrm{Sht}'^r_G)_{\mathbb{Q}}$. Since

$$\dim \mathrm{Sht}_T^{\mu} = r = \frac{1}{2}\dim \mathrm{Sht}'^r_G,$$

both $\theta^{\mu}_*[\mathrm{Sht}_T^{\mu}]$ and $f * \theta^{\mu}_*[\mathrm{Sht}_T^{\mu}]$ for any function $f \in \mathscr{H}$ are *proper* cycle classes in Sht_G^r of complementary dimension, and they define elements in $\mathrm{Ch}_{c,r}(\mathrm{Sht}'^r_G)_{\mathbb{Q}}$. The following definition then makes sense.

Definition 5.15. Let $f \in \mathscr{H}$ be an unramified Hecke function. We define the following intersection number

$$\mathbb{I}_r(f) := \langle \theta^{\mu}_*[\mathrm{Sht}_T^{\mu}], \quad f * \theta^{\mu}_*[\mathrm{Sht}_T^{\mu}]\rangle_{\mathrm{Sht}'^r_G} \in \mathbb{Q}.$$

5.5.4. *Changing* μ. For different μ and μ' as in §5.1.2, the Heegner–Drinfeld cycles $\theta^{\mu}_*[\mathrm{Sht}_T^{\mu}]$ and $\theta^{\mu'}_*[\mathrm{Sht}_T^{\mu'}]$ are different. Therefore, a priorily the intersection number $\mathbb{I}_r(f)$ depends on μ. However we have

Lemma 5.16. *The intersection number $\mathbb{I}_r(f)$ for any $f \in \mathscr{H}$ is independent of the choice of μ.*

Proof. Let Z^{μ} denote the cycle $\theta^{\mu}_*[\mathrm{Sht}_T^{\mu}]$. Using the isomorphism $\iota_{\mu,\mu'}$ in (5.22), we see that Z^{μ} and $Z^{\mu'}$ are transformed to each other under the involution $\sigma(\mu,\mu') : \mathrm{Sht}'^r_G = \mathrm{Sht}_G^r \times_{X^r} X'^r \to \mathrm{Sht}_G^r \times_{X^r} X'^r = \mathrm{Sht}'^r_G$ which is the identity on Sht_G^r and on X'^r sends $(x'_1, \cdots, x'_r)$ to $(y'_1, \cdots, y'_r)$ using the formula (5.19). Since $\sigma(\mu,\mu')$ is the identity on Sht_G^r, it commutes with the Hecke action on $\mathrm{Ch}_{c,r}(\mathrm{Sht}'^r_G)_{\mathbb{Q}}$. Therefore we have

$$\begin{aligned} & \langle Z^{\mu}, f * Z^{\mu}\rangle_{\mathrm{Sht}'^r_G} = \langle \sigma(\mu,\mu')_* Z^{\mu}, \sigma(\mu,\mu')_*(f * Z^{\mu})\rangle_{\mathrm{Sht}'^r_G} \\ = \quad & \langle \sigma(\mu,\mu')_* Z^{\mu}, f * (\sigma(\mu,\mu')_* Z^{\mu})\rangle_{\mathrm{Sht}'^r_G} = \langle Z^{\mu'}, f * Z^{\mu'}\rangle_{\mathrm{Sht}'^r_G}. \end{aligned}$$

$\square$

6. Alternative calculation of intersection numbers

The goal of this section is to turn the intersection number $\mathbb{I}_r(h_D)$ into the trace of an operator acting on the cohomology of a certain variety. This will be accomplished in Theorem 6.5. To state the theorem, we need to introduce certain moduli spaces similar to $\mathcal{N}_{\underline{d}}$ defined in §3.2.2.

6.1. Geometry of $\mathcal{M}_d$.

6.1.1. Recall $\nu : X' \to X$ is a geometrically connected étale double cover. We will use the notation $\widehat{X}'_d$ and X'_d as in §3.2.1. We have the norm map $\widehat{\nu}_d : \widehat{X}'_d \to \widehat{X}_d$ sending $(\mathcal{L}, \alpha \in \Gamma(X', \mathcal{L}))$ to $(\mathrm{Nm}(\mathcal{L}), \mathrm{Nm}(\alpha) \in \Gamma(X, \mathrm{Nm}(\mathcal{L})))$.

Let $d \geq 0$ be an integer. Let $\widetilde{\mathcal{M}}_d$ be the moduli functor whose S-points is the groupoid of $(\mathcal{L}, \mathcal{L}', \alpha, \beta)$ where

- $\mathcal{L}, \mathcal{L}' \in \mathrm{Pic}(X' \times S)$ such that $\deg(\mathcal{L}'_s) - \deg(\mathcal{L}_s) = d$ for all geometric points $s \in S$;
- $\alpha : \mathcal{L} \to \mathcal{L}'$ is an $\mathcal{O}_{X'}$-linear map;
- $\beta : \mathcal{L} \to \sigma^*\mathcal{L}'$ is an $\mathcal{O}_{X'}$-linear map;
- For each geometric point $s \in S$, the restrictions $\alpha|_{X' \times s}$ and $\beta|_{X' \times s}$ are not both zero.

There is a natural action of Pic_X on $\widetilde{\mathcal{M}}_d$ by tensoring: $\mathcal{K} \in \mathrm{Pic}_X$ sends $(\mathcal{L}, \mathcal{L}', \alpha, \beta)$ to $(\mathcal{L} \otimes \nu^*\mathcal{K}, \mathcal{L}' \otimes \nu^*\mathcal{K}, \alpha \otimes \mathrm{id}_{\mathcal{K}}, \beta \otimes \mathrm{id}_{\mathcal{K}})$. We define

$$\mathcal{M}_d := \widetilde{\mathcal{M}}_d / \mathrm{Pic}_X .$$

6.1.2. To $(\mathcal{L}, \mathcal{L}', \alpha, \beta) \in \widetilde{\mathcal{M}}_d$, we may attach

- $a := \mathrm{Nm}(\alpha) : \mathrm{Nm}(\mathcal{L}) \to \mathrm{Nm}(\mathcal{L}')$;
- $b := \mathrm{Nm}(\beta) : \mathrm{Nm}(\mathcal{L}) \to \mathrm{Nm}(\sigma^*\mathcal{L}') = \mathrm{Nm}(\mathcal{L}')$.

Both a and b are sections of the same line bundle $\Delta = \mathrm{Nm}(\mathcal{L}') \otimes \mathrm{Nm}(\mathcal{L})^{-1} \in \mathrm{Pic}^d_X$, and they are not simultaneously zero. The assignment $(\mathcal{L}, \mathcal{L}', \alpha, \beta) \mapsto (\Delta, a, b)$ is invariant under the the action of Pic_X on $\widetilde{\mathcal{M}}_d$, and it induces a morphism

$$f_{\mathcal{M}} : \mathcal{M}_d \longrightarrow \mathcal{A}_d.$$

Here $\mathcal{A}_d$ is defined in §3.2.3.

6.1.3. Given $(\mathcal{L}, \mathcal{L}', \alpha, \beta) \in \widetilde{\mathcal{M}}_d$, there is a canonical way to attach an $\mathcal{O}_X$-linear map $\psi : \nu_*\mathcal{L} \to \nu_*\mathcal{L}'$ and vice versa. In fact, by adjunction, a map $\psi : \nu_*\mathcal{L} \to \nu_*\mathcal{L}'$ is the same as a map $\nu^*\nu_*\mathcal{L} \to \mathcal{L}'$. Since $\nu^*\nu_*\mathcal{L} \cong \mathcal{L} \oplus \sigma^*\mathcal{L}$ canonically, the datum of ψ is the same as a map of $\mathcal{O}_{X'}$-modules $\mathcal{L} \oplus \sigma^*\mathcal{L} \to \mathcal{L}'$, and we name the two components of this map by α and $\sigma^*\beta$. Note that the determinant of the map ψ is given by

$$\det(\psi) = \mathrm{Nm}(\alpha) - \mathrm{Nm}(\beta) = a - b : \mathrm{Nm}(\mathcal{L}) = \det \nu_*\mathcal{L} \longrightarrow \det \nu_*\mathcal{L}' = \mathrm{Nm}(\mathcal{L}'). \tag{6.1}$$

The composition $\delta \circ f_{\mathcal{M}} : \mathcal{M}_d \to \mathcal{A}_d \to \widehat{X}_d$ takes $(\mathcal{L}, \mathcal{L}', \alpha, \beta)$ to the pair $(\Delta = \mathrm{Nm}(\mathcal{L}') \otimes \mathrm{Nm}(\mathcal{L})^{-1}, \det(\psi))$.

6.1.4. We give another description of $\mathcal{M}_d$. We have a map $\iota_\alpha : \mathcal{M}_d \to \widehat{X}'_d$ sending $(\mathcal{L}, \mathcal{L}', \alpha, \beta)$ to the line bundle $\mathcal{L}' \otimes \mathcal{L}^{-1}$ and its section given by α. Similarly we have a map $\iota_\beta : \mathcal{M}_d \to \widehat{X}'_d$ sending $(\mathcal{L}, \mathcal{L}', \alpha, \beta)$ to the line bundle $\sigma^*\mathcal{L}' \otimes \mathcal{L}^{-1}$ and its section given by β. Note that the line bundles underlying $\iota_\alpha(\mathcal{L}, \mathcal{L}', \alpha, \beta)$ and $\iota_\beta(\mathcal{L}, \mathcal{L}', \alpha, \beta)$ have the same norm $\Delta = \mathrm{Nm}(\mathcal{L}') \otimes \mathrm{Nm}(\mathcal{L})^{-1} \in \mathrm{Pic}^d_X$. Since α and β are not both zero, we get a map

$$\iota = (\iota_\alpha, \iota_\beta) : \mathcal{M}_d \longrightarrow \widehat{X}'_d \times_{\mathrm{Pic}^d_X} \widehat{X}'_d - Z'_d$$

where the fiber product on the RHS is taken with respect to the map $\widehat{X}'_d \to \mathrm{Pic}^d_{X'} \xrightarrow{\mathrm{Nm}} \mathrm{Pic}^d_X$, and $Z'_d := \mathrm{Pic}^d_{X'} \times_{\mathrm{Pic}^d_X} \mathrm{Pic}^d_{X'}$ is embedded into $\widehat{X}'_d \times_{\mathrm{Pic}^d_X} \widehat{X}'_d$ by viewing $\mathrm{Pic}^d_{X'}$ as the zero section of $\widehat{X}'_d$ in both factors.

Proposition 6.1. *(1) The morphism ι is an isomorphism of functors, and $\mathcal{M}_d$ is a proper Deligne–Mumford stack over k.* [3]

[3]The properness of $\mathcal{M}_d$ will not be used elsewhere in this paper.

(2) For $d \geq 2g'-1$, $\mathcal{M}_d$ is a smooth Deligne–Mumford stack over k of pure dimension $2d-g+1$.

(3) The morphism $\widehat{\nu}_d : \widehat{X}'_d \to \widehat{X}_d$ is proper.

(4) We have a Cartesian diagram

$$\begin{tikzcd} \mathcal{M}_d \arrow[r, hook, "\iota"] \arrow[d, "f_{\mathcal{M}}"] & \widehat{X}'_d \times_{\mathrm{Pic}^d_X} \widehat{X}'_d \arrow[d, "\widehat{\nu}_d \times \widehat{\nu}_d"] \\ \mathcal{A}_d \arrow[r, hook] & \widehat{X}_d \times_{\mathrm{Pic}^d_X} \widehat{X}_d \end{tikzcd} \tag{6.2}$$

Moreover, the map $f_{\mathcal{M}}$ is proper.

Proof. (1) Let $(\mathrm{Pic}_{X'} \times \mathrm{Pic}_{X'})_d$ be the disjoint union of $\mathrm{Pic}^i_{X'} \times \mathrm{Pic}^{i+d}_{X'}$ over all $i \in \mathbb{Z}$. Consider the morphism $\theta : (\mathrm{Pic}_{X'} \times \mathrm{Pic}_{X'})_d / \mathrm{Pic}_X \to \mathrm{Pic}^d_{X'} \times_{\mathrm{Pic}^d_X} \mathrm{Pic}^d_{X'}$ (the fiber product is taken with respect to the norm map) that sends $(\mathcal{L}, \mathcal{L}')$ to $(\mathcal{L}' \otimes \mathcal{L}^{-1}, \sigma^*\mathcal{L}' \otimes \mathcal{L}^{-1}, \tau)$, where τ is the tautological isomorphism between $\mathrm{Nm}(\mathcal{L}' \otimes \mathcal{L}^{-1}) \cong \mathrm{Nm}(\mathcal{L}') \otimes \mathrm{Nm}(\mathcal{L})^{-1}$ and $\mathrm{Nm}(\sigma^*\mathcal{L}' \otimes \mathcal{L}^{-1}) \cong \mathrm{Nm}(\mathcal{L}') \otimes \mathrm{Nm}(\mathcal{L})^{-1}$. By definition, we have a Cartesian diagram

$$\begin{tikzcd} \mathcal{M}_d \arrow[r, "\iota"] \arrow[d, "\omega"] & \widehat{X}'_d \times_{\mathrm{Pic}^d_X} \widehat{X}'_d - Z'_d \arrow[d] \\ (\mathrm{Pic}_{X'} \times \mathrm{Pic}_{X'})_d / \mathrm{Pic}_X \arrow[r, "\theta"] & \mathrm{Pic}^d_{X'} \times_{\mathrm{Pic}^d_X} \mathrm{Pic}^d_{X'} \end{tikzcd} \tag{6.3}$$

where the map ω sends $(\mathcal{L}, \mathcal{L}', \alpha, \beta)$ to $(\mathcal{L}, \mathcal{L}')$. Therefore it suffices to check that θ is an isomorphism. For this we will construct an inverse to θ.

From the exact sequence of étale sheaves

$$1 \longrightarrow \mathcal{O}^\times_X \xrightarrow{\nu^*} \nu_*\mathcal{O}^\times_{X'} \xrightarrow{\mathrm{id}-\sigma} \nu_*\mathcal{O}^\times_{X'} \xrightarrow{\mathrm{Nm}} \mathcal{O}^\times_X \longrightarrow 1$$

we get an exact sequence of Picard stacks

$$1 \longrightarrow \mathrm{Pic}_{X'} / \mathrm{Pic}_X \xrightarrow{\mathrm{id}-\sigma} \mathrm{Pic}^0_{X'} \xrightarrow{\mathrm{Nm}} \mathrm{Pic}^0_X \longrightarrow 1.$$

Given $(\mathcal{K}_1, \mathcal{K}_2, \tau) \in \mathrm{Pic}^d_{X'} \times_{\mathrm{Pic}^d_X} \mathrm{Pic}^d_{X'}$ (where $\tau : \mathrm{Nm}(\mathcal{K}_1) \cong \mathrm{Nm}(\mathcal{K}_2)$), there is a unique object $\mathcal{L}' \in \mathrm{Pic}_{X'} / \mathrm{Pic}_X$ such that $\mathcal{L}' \otimes \sigma^*\mathcal{L}'^{-1} \cong \mathcal{K}_1 \otimes \mathcal{K}_2^{-1}$ compatible with the trivializations of the norms to X of both sides. We then define $\psi(\mathcal{K}_1, \mathcal{K}_2, \tau) = (\mathcal{L}' \otimes \mathcal{K}_1^{-1}, \mathcal{L}')$, which is a well-defined object in $(\mathrm{Pic}_{X'} \times \mathrm{Pic}_{X'})_d / \mathrm{Pic}_X$. It is easy to check that ψ is an inverse to θ. This proves that θ is an isomorphism, and so is ι.

We show that $\mathcal{M}_d$ is a proper Deligne–Mumford stack over k. By extending k we may assume that X' contains a k-point, and we fix a point $y \in X'(k)$. We consider the moduli stack $\widehat{\mathcal{M}}_d$ classifying $(\mathcal{K}_1, \gamma_1, \mathcal{K}, \rho, \alpha, \beta)$ where $\mathcal{K}_1 \in \mathrm{Pic}^d_{X'}$, γ_1 is a trivialization of the stalk $\mathcal{K}_{1,y}$, $\mathcal{K} \in \mathrm{Pic}^0_{X'}$, ρ is an isomorphism $\mathrm{Nm}(\mathcal{K}) \cong \mathcal{O}_X$, α is a section of $\mathcal{K}_1$ and β is a section of $\mathcal{K}_1 \otimes \mathcal{K}$ such that α and β are not both zero. There is a canonical map $p : \widehat{\mathcal{M}}_d \to \widehat{X}'_d \times_{\mathrm{Pic}^d_X} \widehat{X}'_d - Z'_d$ sending $(\mathcal{K}_1, \gamma_1, \mathcal{K}, \rho, \alpha, \beta)$ to $(\mathcal{K}_1, \mathcal{K}_2 := \mathcal{K}_1 \otimes \mathcal{K}, \tau, \alpha, \beta)$ (the isomorphism $\tau : \mathrm{Nm}(\mathcal{K}_1) \cong \mathrm{Nm}(\mathcal{K}_2)$ is induced from the trivialization ρ). Clearly p is the quotient map for the $\mathbb{G}_m$-action on $\widehat{\mathcal{M}}_d$ that scales γ_1. There is another $\mathbb{G}_m$-action on $\widehat{\mathcal{M}}_d$ that scales α and β simultaneously. Using automorphisms of $\mathcal{K}_1$, we have a canonical identification of the two $\mathbb{G}_m$-actions on $\widehat{\mathcal{M}}_d$; however, to distinguish them, we call the first torus $\mathbb{G}_m(y)$ and the second $\mathbb{G}_m(\alpha, \beta)$. By the above discussion, $\iota^{-1} \circ p$ gives an isomorphism $\widehat{\mathcal{M}}_d / \mathbb{G}_m(y) \cong \mathcal{M}_d$, hence also an isomorphism $\widehat{\mathcal{M}}_d / \mathbb{G}_m(\alpha, \beta) \cong \mathcal{M}_d$.

Let $\mathrm{Prym}_{X'/X} := \ker(\mathrm{Nm} : \mathrm{Pic}^0_{X'} \to \mathrm{Pic}^0_X)$ which classifies a line bundle $\mathcal{K}$ on X' together with a trivialization of $\mathrm{Nm}(\mathcal{L})$. This is a Deligne–Mumford stack isomorphic to the usual Prym variety divided by the trivial action of μ_2. Let $J^d_{X'}$ be the degree d-component of the Picard *scheme* of X', which classifies a line bundle $\mathcal{K}_1$ on X' of degree d together with a trivialization of the stalk $\mathcal{K}_{1,y}$. We have a natural map $h : \widehat{\mathcal{M}}_d \to J^d_{X'} \times \mathrm{Prym}_{X'/X}$ sending $(\mathcal{K}_1, \gamma_1, \mathcal{K}, \rho, \alpha, \beta)$

to $(\mathcal{K}_1,\gamma_1)\in J^d_{X'}$ and $(\mathcal{K},\rho)\in \mathrm{Prym}_{X'/X}$. The map h is invariant under the $\mathbb{G}_m(\alpha,\beta)$-action, hence induces a map

$$\overline{h}:\widehat{\mathcal{M}}_d/\mathbb{G}_m(\alpha,\beta)\cong\mathcal{M}_d\longrightarrow J^d_{X'}\times\mathrm{Prym}_{X'/X} \tag{6.4}$$

The fiber of $\overline{h}$ over a point $((\mathcal{K}_1,\gamma_1),(\mathcal{K},\rho))\in J^d_{X'}\times\mathrm{Prym}_{X'/X}$ is the projective space $\mathbb{P}(\Gamma(X',\mathcal{K}_1)\oplus\Gamma(X',\mathcal{K}_1\otimes\mathcal{K}))$. In particular, the map $\overline{h}$ is proper and schematic. Since $J^d_{X'}\times\mathrm{Prym}_{X'/X}$ is a proper Deligne–Mumford stack over k, so is $\mathcal{M}_d$.

(2) Since $\mathcal{M}_d$ is covered by open substacks $X'_d\times_{\mathrm{Pic}^d_X}\widehat{X}'_d$ and $\widehat{X}'_d\times_{\mathrm{Pic}^d_X}X'_d$, it suffices to show that both of them are smooth over k. For $d\geq 2g'-1$ the Abel-Jacobi map $\mathrm{AJ}_d:X'_d\to\mathrm{Pic}^d_{X'}$ is smooth of relative dimension $d-g'+1$, hence X'_d is smooth over Pic^d_X of relative dimension $d-g+1$. Therefore both $X'_d\times_{\mathrm{Pic}^d_X}\widehat{X}'_d$ and $\widehat{X}'_d\times_{\mathrm{Pic}^d_X}X'_d$ are smooth over X'_d of relative dimension $d-g+1$. We conclude that $\mathcal{M}_d$ is a smooth Deligne–Mumford stack of dimension $2d-g+1$ over k.

(3) We introduce a compactification $\overline{X}'_d$ of $\widehat{X}'_d$ as follows. Consider the product $\widehat{X}'_d\times\mathbb{A}^1$ with the natural $\mathbb{G}_m$-action scaling both the section of the line bundle and the scalar in $\mathbb{A}^1$. Let $z_0:\mathrm{Pic}^d_{X'}\hookrightarrow\widehat{X}'_d\times\mathbb{A}^1$ sending $\mathcal{L}$ to $(\mathcal{L},0,0)$. Let $\overline{X}'_d:=(\widehat{X}'_d\times\mathbb{A}^1-z_0(\mathrm{Pic}^d_{X'}))/\mathbb{G}_m$. Then the fiber of $\overline{X}'_d$ over $\mathcal{L}\in\mathrm{Pic}^d_{X'}$ is the projective space $\mathbb{P}(\Gamma(X',\mathcal{L}\oplus\mathcal{O}_{X'}))$. In particular, $\overline{X}'_d$ is proper and schematic over $\mathrm{Pic}^d_{X'}$. The stack $\overline{X}'_d$ contains $\widehat{X}'_d$ as an open substack where the $\mathbb{A}^1$-coordinate is invertible, whose complement is isomorphic to the projective space bundle $X'_d/\mathbb{G}_m$ over $\mathrm{Pic}^d_{X'}$. Similarly we a compactification $\overline{X}_d$ of $\widehat{X}_d$.

Consider the quadratic map $\widehat{X}'_d\times\mathbb{A}^1\to\widehat{X}_d\times\mathbb{A}^1$ sending $(\mathcal{L},s,\lambda)\mapsto(\mathrm{Nm}(\mathcal{L}),\mathrm{Nm}(s),\lambda^2)$. This quadratic map passes to the projectivizations because $(\mathrm{Nm}(s),\lambda^2)=(0,0)$ implies $(s,\lambda)=(0,0)$ on the level of field-valued points. The resulting map $\overline{\nu}_d:\overline{X}'_d\to\overline{X}_d$ extends $\widehat{\nu}_d$. We may factorize $\overline{\nu}_d$ as the composition

$$\overline{\nu}_d:\overline{X}'_d\longrightarrow\overline{X}_d\times_{\mathrm{Pic}^d_X}\mathrm{Pic}^d_{X'}\longrightarrow\overline{X}_d$$

Here the first map is proper because both the source and the target are proper over $\mathrm{Pic}^d_{X'}$; the second map is proper by the properness of the norm map $\mathrm{Nm}:\mathrm{Pic}^d_{X'}\to\mathrm{Pic}^d_X$. We conclude that $\overline{\nu}_d$ is proper. Since $\widehat{\nu}_d$ is the restriction of $\overline{\nu}_d$ to $\widehat{X}_d\hookrightarrow\overline{X}_d$, it is also proper.

(4) The commutativity of the diagram (6.2) is clear from the construction of ι. Note that Z'_d is the preimage of Z_d under $\widehat{\nu}_d\times\widehat{\nu}_d$, and $\mathcal{M}_d$ and $\mathcal{A}_d$ are complements of Z'_d and Z_d respectively. Therefore (6.2) is also Cartesian. Now the properness of $f_{\mathcal{M}}$ follows from the properness of $\widehat{\nu}_d$ proved in part (3) together with the Cartesian diagram (6.2).

□

6.2. A formula for $\mathbb{I}_r(h_D)$.

6.2.1. *The correspondence* $\mathrm{Hk}^{\mu}_{\mathcal{M},d}$. Fix any tuple $\mu=(\mu_1,\cdots,\mu_r)$ as in §5.1.1. We define $\widetilde{\mathrm{Hk}}^{\mu}_{\mathcal{M},d}$ to be the moduli functor whose S-points classify the following data

(1) For $i=1,\cdots,r$, a map $x'_i:S\to X'$ with graph $\Gamma_{x'_i}$.

(2) For each $i=0,1,\cdots,r$, an S-point $(\mathcal{L}_i,\mathcal{L}'_i,\alpha_i,\beta_i)$ of $\widetilde{\mathcal{M}}_d$:

$$\alpha_i:\mathcal{L}_i\longrightarrow\mathcal{L}'_i,\quad \beta_i:\mathcal{L}_i\longrightarrow\sigma^*\mathcal{L}'_i.$$

In particular, $\deg\mathcal{L}'_i-\deg\mathcal{L}_i=d$ and α_i and β_i are not both zero.

(3) A commutative diagram of $\mathcal{O}_{X'}$-linear maps between line bundles on X'

$$\begin{tikzcd}
\mathcal{L}_0 \arrow[r, dashed, "f_1"] \arrow[d, "\alpha_0"] & \mathcal{L}_1 \arrow[r, dashed, "f_2"] \arrow[d, "\alpha_1"] & \cdots \arrow[r, dashed, "f_r"] & \mathcal{L}_r \arrow[d, "\alpha_r"] \\
\mathcal{L}'_0 \arrow[r, dashed, "f'_1"] & \mathcal{L}'_1 \arrow[r, dashed, "f'_2"] & \cdots \arrow[r, dashed, "f'_r"] & \mathcal{L}'_r
\end{tikzcd} \tag{6.5}$$

where the top and bottom rows are S-points of Hk_T^μ over the same point $(x'_1,\cdots,x'_r)\in X'^r(S)$, such that the following diagram is also commutative

$$\begin{tikzcd}
\mathcal{L}_0 \arrow[r, dashed, "f_1"] \arrow[d, "\beta_0"] & \mathcal{L}_1 \arrow[r, dashed, "f_2"] \arrow[d, "\beta_1"] & \cdots \arrow[r, dashed, "f_r"] & \mathcal{L}_r \arrow[d, "\beta_r"] \\
\sigma^*\mathcal{L}'_0 \arrow[r, dashed, "\sigma^* f'_1"] & \sigma^*\mathcal{L}'_1 \arrow[r, dashed, "\sigma^* f'_2"] & \cdots \arrow[r, dashed, "\sigma^* f'_r"] & \sigma^*\mathcal{L}'_r
\end{tikzcd} \tag{6.6}$$

There is an action of Pic_X on $\widetilde{\mathrm{Hk}}^\mu_{\mathcal{M},d}$ by tensoring on the line bundles $\mathcal{L}_i$ and $\mathcal{L}'_i$. We define

$$\mathrm{Hk}^\mu_{\mathcal{M},d} := \widetilde{\mathrm{Hk}}^\mu_{\mathcal{M},d}/\operatorname{Pic}_X .$$

The same argument as §5.4.6 (applying the isomorphism (5.21) to both rows of (6.6)) shows that for different choices of μ, the stacks $\mathrm{Hk}^\mu_{\mathcal{M},d}$ are canonically isomorphic to each other. However, as in the case for Hk^μ_T, the morphism $\mathrm{Hk}^\mu_{\mathcal{M},d}\to X'^r$ does depend on μ.

6.2.2. Let $\gamma_i:\mathrm{Hk}^\mu_{\mathcal{M},d}\to\mathcal{M}_d$ be the projections given by taking the diagram (6.5) to its i-th column. It is clear that this map is schematic, therefore $\mathrm{Hk}^\mu_{\mathcal{M},d}$ itself is a scheme.

In the diagram (6.5), the line bundles $\Delta_i=\mathrm{Nm}(\mathcal{L}'_i)\otimes\mathrm{Nm}(\mathcal{L}_i)^{-1}$ are all canonically isomorphic to each other for $i=0,\cdots,r$. Also the sections $a_i=\mathrm{Nm}(\alpha_i)$ (resp. $b_i=\mathrm{Nm}(\beta_i)$) of Δ_i can be identified with each other for all i under the isomorphisms between the Δ_i's. Therefore, composing γ_i with the map $f_{\mathcal{M}}:\mathcal{M}_d\to\mathcal{A}_d$ all give the same map. We may view $\mathrm{Hk}^\mu_{\mathcal{M},d}$ as a self-correspondence of $\mathcal{M}_d$ over $\mathcal{A}_d$ via the maps (γ_0,γ_r).

There is a stronger statement. Let us define $\widetilde{\mathcal{A}}_d\subset\widehat{X}'_d\times_{\mathrm{Pic}^d_X}\widehat{X}_d$ to be preimage of $\mathcal{A}_d$ under $\mathrm{Nm}\times\mathrm{id}:\widehat{X}'_d\times_{\mathrm{Pic}^d_X}\widehat{X}_d\to\widehat{X}_d\times_{\mathrm{Pic}^d_X}\widehat{X}_d$. Then $\widetilde{\mathcal{A}}_d$ classifies triples $(\mathcal{K},\alpha,b)$ where $\mathcal{K}\in\mathrm{Pic}_{X'}$, α is a section of $\mathcal{K}$ and b is a section of $\mathrm{Nm}(\mathcal{K})$ such that α and b are not simultaneously zero. Then $f_{\mathcal{M}}$ factors through the map

$$\widetilde{f}_{\mathcal{M}}:\mathcal{M}_d\longrightarrow\widetilde{\mathcal{A}}_d$$

sending $(\mathcal{L},\mathcal{L}',\alpha,\beta)$ to $(\mathcal{L}'\otimes\mathcal{L}^{-1},\alpha,\mathrm{Nm}(\beta))$.

Consider a point of $\mathrm{Hk}^\mu_{\mathcal{M},d}$ giving among others the diagram (6.5). Since the maps f_i and f'_i are simple modifications at the same point x'_i, the line bundles $\mathcal{L}'_i\otimes\mathcal{L}_i^{-1}$ are all isomorphic to each other for all $i=0,1,\cdots,r$. Under these isomorphisms, their sections given by α_i correspond to each other. Therefore the maps $\widetilde{f}_{\mathcal{M}}\circ\gamma_i:\mathrm{Hk}^\mu_{\mathcal{M},d}\to\widetilde{\mathcal{A}}_d$ are the same for all i.

6.2.3. The particular case $r=1$ and $\mu=(\mu_+)$ gives a moduli space $\mathcal{H}:=\mathrm{Hk}^1_{\mathcal{M},d}$ classifying commutative diagrams up to simultaneous tensoring by Pic_X:

$$\begin{tikzcd}
\mathcal{L}_0 \arrow[r, "f"] \arrow[d, "\alpha_0"] & \mathcal{L}_1 \arrow[d, "\alpha_1"] \\
\mathcal{L}'_0 \arrow[r, "f'"] & \mathcal{L}'_1
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{L}_0 \arrow[r, "f"] \arrow[d, "\beta_0"] & \mathcal{L}_1 \arrow[d, "\beta_1"] \\
\sigma^*\mathcal{L}'_0 \arrow[r, "\sigma^* f'"] & \sigma^*\mathcal{L}'_1
\end{tikzcd} \tag{6.7}$$

such that the cokernel of f and f' are invertible sheaves supported at the same point $x'\in X'$, and the data $(\mathcal{L}_0,\mathcal{L}'_0,\alpha_0,\beta_0)$ and $(\mathcal{L}_1,\mathcal{L}'_1,\alpha_1,\beta_1)$ are objects of $\mathcal{M}_d$.

We have two maps $(\gamma_0,\gamma_1):\mathcal{H}\to\mathcal{M}_d$, and we view $\mathcal{H}$ as a self-correspondence of $\mathcal{M}_d$ over $\mathcal{A}_d$. We also have a map $p:\mathcal{H}\to X'$ recording the point x' (support of $\mathcal{L}_1/\mathcal{L}_0$ and $\mathcal{L}'_1/\mathcal{L}'_0$).

The following lemma follows directly from the definition of $\mathrm{Hk}^\mu_{\mathcal{M},d}$.

Lemma 6.2. *As a self-correspondence of $\mathcal{M}_d$, $\mathrm{Hk}^\mu_{\mathcal{M},d}$ is canonically isomorphic to the r-fold composition of $\mathcal{H}$*

$$\mathrm{Hk}^\mu_{\mathcal{M},d}\cong\mathcal{H}\times_{\gamma_1,\mathcal{M}_d,\gamma_0}\times\mathcal{H}\times_{\gamma_1,\mathcal{M}_d,\gamma_0}\times\cdots\times_{\gamma_1,\mathcal{M}_d,\gamma_0}\mathcal{H}.$$

6.2.4. Let $\mathcal{A}_d^{\diamondsuit} \subset \mathcal{A}_d$ be the open subset consisting of (Δ, a, b) where $b \neq 0$, i.e., $\mathcal{A}_d^{\diamondsuit} = \widehat{X}_d \times_{\mathrm{Pic}_X^d} X_d$ under the isomorphism (3.3). Let $\mathcal{M}_d^{\diamondsuit}$, $\mathrm{Hk}^{\mu}_{\mathcal{M}^{\diamondsuit},d}$ and $\mathcal{H}^{\diamondsuit}$ be the preimages of $\mathcal{A}_d^{\diamondsuit}$ in $\mathcal{M}_d$, $\mathrm{Hk}^{\mu}_{\mathcal{M},d}$ and $\mathcal{H}$.

Lemma 6.3. *Let $I'_d \subset X'_d \times X'$ be the incidence scheme, i.e., $I'_d \to X'_d$ is the universal family of degree d effective divisors on X'. There is a natural map $\mathcal{H}^{\diamondsuit} \to I'_d$ such that the diagram*

$$\begin{tikzcd} & \mathcal{H}^{\diamondsuit} \arrow[r] \arrow[d, "\gamma_1"] \arrow[rr, bend left, "p"] & I'_d \arrow[r, "p_{I'}"'] \arrow[d, "q"] & X' \\ \mathcal{M}_d^{\diamondsuit} \arrow[r, equal] & \widehat{X}'_d \times_{\mathrm{Pic}_X^d} X'_d \arrow[r, "\mathrm{pr}_2"] & X'_d & \end{tikzcd} \tag{6.8}$$

is commutative and the square is Cartesian. Here the $q : I'_d \to X'_d$ sends $(D, y) \in X'_d \times X'$ to $D - y + \sigma(y)$, and $p_{I'} : I'_d \to X'$ sends (D, y) to y.

Proof. A point in $\mathcal{H}^{\diamondsuit}$ is a diagram as in (6.7) with β_i nonzero (hence injections). Such a diagram is uniquely determined by $(\mathcal{L}_0, \mathcal{L}'_0, \alpha_0, \beta_0) \in \mathcal{M}_d^{\diamondsuit}$ and $y = \mathrm{div}(f) \in X'$ for then $\mathcal{L}_1 = \mathcal{L}_0(y)$, $\mathcal{L}'_1 = \mathcal{L}'_0(y)$ are determined, and f, f' are the obvious inclusions and α_1 the unique map making the first diagram in (6.7) commutative; the commutativity of the second diagram uniquely determines β_1, but there is a condition on y to make it possible:

$$\mathrm{div}(\beta_0) + \sigma(y) = \mathrm{div}(\beta_1) + y \in X'_{d+1}.$$

Since σ acts on X' without fixed points, y must appear in $\mathrm{div}(\beta_0)$. The assignment $\mathcal{H}^{\diamondsuit} \ni (y, \mathcal{L}_0, \cdots, \beta_0, \mathcal{L}_1, \cdots, \beta_1) \mapsto (\mathrm{div}(\beta_0), y)$ then gives a point in I'_d. The above argument shows that the square in (6.8) is Cartesian and the triangle therein is commutative. □

Lemma 6.4. *We have*

(1) The map $\gamma_0 : \mathrm{Hk}^{\mu}_{\mathcal{M}^{\diamondsuit},d} \to \mathcal{M}_d^{\diamondsuit}$ is finite and surjective. In particular, $\dim \mathrm{Hk}^{\mu}_{\mathcal{M}^{\diamondsuit},d} = \dim \mathcal{M}_d^{\diamondsuit} = 2d - g + 1$.

(2) The dimension of the image of $\mathrm{Hk}^{\mu}_{\mathcal{M},d} - \mathrm{Hk}^{\mu}_{\mathcal{M}^{\diamondsuit},d}$ in $\mathcal{M}_d \times \mathcal{M}_d$ is at most $d + 2g - 2$.

Proof. (1) In the case $r = 1$, this follows from the Cartesian square in (6.8), because the map $q : I'_d \to X'_d$ is finite. For general r, the statement follows by induction from Lemma 6.2.

(2) The closed subscheme $Y = \mathrm{Hk}^{\mu}_{\mathcal{M},d} - \mathrm{Hk}^{\mu}_{\mathcal{M}^{\diamondsuit},d}$ classifies diagrams (6.5) only because all the β_i are zero. Its image $Z \subset \mathcal{M}_d \times \mathcal{M}_d$ under (γ_0, γ_r) consists of pairs of points $(\mathcal{L}_0, \mathcal{L}'_0, \alpha_0, 0)$ and $(\mathcal{L}_r, \mathcal{L}'_r, \alpha_r, 0)$ in $\mathcal{M}_d$ such that there exists a diagram of the form (6.5) connecting them. In particular, the divisors of α_0 and α_r are the same. Therefore such a point in Z is completely determined by two points $\mathcal{L}_0, \mathcal{L}_r \in \mathrm{Bun}_T$ and a divisor $D \in X'_d$ (as the divisor of α_0 and α_r). We see that $\dim Z \leq 2 \dim \mathrm{Bun}_T + \dim X'_d = d + 2g - 2$. □

6.2.5. Recall $\mathcal{H} = \mathrm{Hk}^{1}_{\mathcal{M},d}$ is a self-correspondence of $\mathcal{M}_d$ over $\mathcal{A}_d$ (see the discussion in §6.2.2). Let

$$[\mathcal{H}^{\diamondsuit}] \in \mathrm{Ch}_{2d-g+1}(\mathcal{H})_{\mathbb{Q}}$$

denote the class of the closure of $\mathcal{H}^{\diamondsuit}$. The image of $[\mathcal{H}^{\diamondsuit}]$ in the Borel-Moore homology group $\mathrm{H}^{\mathrm{BM}}_{2(2d-g+1)}(\mathcal{H} \otimes_k \overline{k})(-2d + g - 1)$ defines a cohomological self-correspondence of the constant sheaf $\mathbb{Q}_\ell$ on $\mathcal{M}_d$. According the discussion in §A.4.1, it induces an endomorphism

$$f_{\mathcal{M},!}[\mathcal{H}^{\diamondsuit}] : \mathbf{R}f_{\mathcal{M},!}\mathbb{Q}_\ell \longrightarrow \mathbf{R}f_{\mathcal{M},!}\mathbb{Q}_\ell$$

For a point $a \in \mathcal{A}_d(k)$, we denote the action of $f_{\mathcal{M},!}[\mathcal{H}^{\diamondsuit}]$ on the geometric stalk $(\mathbf{R}f_{\mathcal{M},!}\mathbb{Q}_\ell)_{\overline{a}} = \mathrm{H}^*_c(f_{\mathcal{M}}^{-1}(\overline{a}) \otimes_k \overline{k})$ by $(f_{\mathcal{M},!}[\mathcal{H}^{\diamondsuit}])_a$.

Recall from §3.3.2 that $\mathcal{A}_D = \delta^{-1}(D) \subset \mathcal{A}_d^{\heartsuit}$ is the fiber of D under $\delta : \mathcal{A}_d \to \widehat{X}_d$. The main result of this section is the following.

Theorem 6.5. *Suppose D is an effective divisor on X of degree $d \geq \max\{2g'-1, 2g\}$. Then we have*

$$\mathbb{I}_r(h_D) = \sum_{a\in\mathcal{A}_D(k)} \mathrm{Tr}\left((f_{\mathcal{M},!}[\mathcal{H}^{\diamondsuit}])_a^r \circ \mathrm{Frob}_a, (\mathbf{R}f_{\mathcal{M},!}\mathbb{Q}_\ell)_{\overline{a}}\right). \tag{6.9}$$

6.2.6. *Orbital decomposition of* $\mathbb{I}_r(h_D)$. According Theorem 6.5, we may write

$$\mathbb{I}_r(h_D) = \sum_{u\in\mathbb{P}^1(F)-\{1\}} \mathbb{I}_r(u, h_D) \tag{6.10}$$

where

$$\mathbb{I}_r(u,h_D) = \begin{cases} \mathrm{Tr}\left((f_{\mathcal{M},!}[\mathcal{H}^{\diamondsuit}])_a^r \circ \mathrm{Frob}_a, (\mathbf{R}f_{\mathcal{M},!}\mathbb{Q}_\ell)_{\overline{a}}\right) & \text{if } u = \mathrm{inv}_D(a) \text{ for some } a\in\mathcal{A}_D(k); \\ 0 & \text{otherwise.}\end{cases} \tag{6.11}$$

The rest of the section is devoted to the proof of this theorem. In the rest of this subsection we assume $d \geq \max\{2g'-1, 2g\}$.

6.2.7. We apply the discussion in Appendix §A.4.4 to $M = \mathcal{M}_d \xrightarrow{f_{\mathcal{M}}} S = \mathcal{A}_d$ and the self-correspondence $C = \mathrm{Hk}^{\mu}_{\mathcal{M},d}$ of $\mathcal{M}_d$. We define $\mathrm{Sht}^{\mu}_{\mathcal{M},d}$ by the Cartesian diagram

$$\begin{tikzcd} \mathrm{Sht}^{\mu}_{\mathcal{M},d} \arrow[r] \arrow[d] & \mathrm{Hk}^{\mu}_{\mathcal{M},d} \arrow[d, "(\gamma_0,\gamma_r)"] \\ \mathcal{M}_d \arrow[r, "(\mathrm{id},\mathrm{Fr}_{\mathcal{M}_d})"] & \mathcal{M}_d\times\mathcal{M}_d \end{tikzcd} \tag{6.12}$$

This fits into the situation of §A.4.4 because $f_{\mathcal{M}}\circ\gamma_0 = f_{\mathcal{M}}\circ\gamma_r$ by the discussion in §6.2.2, hence $\mathrm{Hk}^{\mu}_{\mathcal{M},d}$ is a self-correspondence of $\mathcal{M}_d$ over $\mathcal{A}_d$ while $(\mathrm{id},\mathrm{Fr}_{\mathcal{M}_d})$ covers the map $(\mathrm{id},\mathrm{Fr}_{\mathcal{A}_d}) : \mathcal{A}_d \to \mathcal{A}_d\times\mathcal{A}_d$. In particular we have a decomposition

$$\mathrm{Sht}^{\mu}_{\mathcal{M},d} = \coprod_{a\in\mathcal{A}_d(k)} \mathrm{Sht}^{\mu}_{\mathcal{M},d}(a). \tag{6.13}$$

For $D\in X_d(k)$, we let

$$\mathrm{Sht}^{\mu}_{\mathcal{M},D} := \coprod_{a\in\mathcal{A}_D(k)} \mathrm{Sht}^{\mu}_{\mathcal{M},d}(a) \subset \mathrm{Sht}^{\mu}_{\mathcal{M},d}. \tag{6.14}$$

Using the decompositions (6.13) and (6.14), we get a decomposition

$$\mathrm{Ch}_0(\mathrm{Sht}^{\mu}_{\mathcal{M},d})_{\mathbb{Q}} = \left(\bigoplus_{D\in X_d(k)} \mathrm{Ch}_0(\mathrm{Sht}^{\mu}_{\mathcal{M},D})_{\mathbb{Q}}\right) \oplus \left(\bigoplus_{a\in\mathcal{A}_d(k)-\mathcal{A}_d^{\heartsuit}(k)} \mathrm{Ch}_0(\mathrm{Sht}^{\mu}_{\mathcal{M},d}(a))_{\mathbb{Q}}\right). \tag{6.15}$$

Let $\zeta\in\mathrm{Ch}_{2d-g+1}(\mathrm{Hk}^{\mu}_{\mathcal{M},d})_{\mathbb{Q}}$. Since $\mathcal{M}_d$ is a smooth Deligne–Mumford stack by Proposition 6.1(2), $(\mathrm{id},\mathrm{Fr}_{\mathcal{M}_d})$ is a regular local immersion, the refined Gysin map (which is the same as intersecting with the Frobenius graph $\Gamma(\mathrm{Fr}_{\mathcal{M}_d})$ of $\mathcal{M}_d$) is defined

$$(\mathrm{id},\mathrm{Fr}_{\mathcal{M}_d})^! : \mathrm{Ch}_{2d-g+1}(\mathrm{Hk}^{\mu}_{\mathcal{M},d})_{\mathbb{Q}} \longrightarrow \mathrm{Ch}_0(\mathrm{Sht}^{\mu}_{\mathcal{M},d})_{\mathbb{Q}}$$

Under the decomposition (6.15), we denote the component of $(\mathrm{id},\mathrm{Fr}_{\mathcal{M}_d})^!\zeta$ in the direct summand $\mathrm{Ch}_0(\mathrm{Sht}^{\mu}_{\mathcal{M},D})_{\mathbb{Q}}$ by

$$\left((\mathrm{id},\mathrm{Fr}_{\mathcal{M}_d})^!\zeta\right)_D \in \mathrm{Ch}_0(\mathrm{Sht}^{\mu}_{\mathcal{M},D})_{\mathbb{Q}}.$$

Composing with the degree map (which exists because $\mathrm{Sht}^{\mu}_{\mathcal{M},D}$ is proper over k, see the discussion after (A.27)), we define

$$\langle\zeta,\Gamma(\mathrm{Fr}_{\mathcal{M}_d})\rangle_D := \deg\left((\mathrm{id},\mathrm{Fr}_{\mathcal{M}_d})^!\zeta\right)_D \in \mathbb{Q}.$$

As the first step towards the proof of Theorem 6.5, we have the following result.

Theorem 6.6. *Suppose D is an effective divisor on X of degree $d\geq\max\{2g'-1,2g\}$, then there exists a class $\zeta\in\mathrm{Ch}_{2d-g+1}(\mathrm{Hk}^{\mu}_{\mathcal{M},d})_{\mathbb{Q}}$ whose restriction to $\mathrm{Hk}^{\mu}_{\mathcal{M},d}|_{\mathcal{A}_d^{\heartsuit}\cap\mathcal{A}_d^{\diamondsuit}}$ is the fundamental cycle, such that*

$$\mathbb{I}_r(h_D) = \langle\zeta,\Gamma(\mathrm{Fr}_{\mathcal{M}_d})\rangle_D. \tag{6.16}$$

This theorem will be proved in §6.3.6, after introducing some auxiliary moduli stacks in the next subsection.

6.2.8. *Proof of Theorem 6.5.* Granting Theorem 6.6, we now prove Theorem 6.5. Let $\zeta \in \mathrm{Ch}_{2d-g+1}(\mathrm{Hk}^{\mu}_{\mathcal{M},d})_{\mathbb{Q}}$ be the class as in Theorem 6.6. By (6.14), we have a decomposition

$$\mathrm{Ch}_0(\mathrm{Sht}^{\mu}_{\mathcal{M},D})_{\mathbb{Q}} = \bigoplus_{a\in\mathcal{A}_D(k)} \mathrm{Ch}_0(\mathrm{Sht}^{\mu}_{\mathcal{M},d}(a))_{\mathbb{Q}}. \tag{6.17}$$

We write

$$\langle\zeta,\Gamma(\mathrm{Fr}_{\mathcal{M}_d})\rangle_D = \sum_{a\in\mathcal{A}_D(k)} \langle\zeta,\Gamma(\mathrm{Fr}_{\mathcal{M}_d})\rangle_a$$

under the decomposition (6.17), where $\langle\zeta,\Gamma(\mathrm{Fr}_{\mathcal{M}_d})\rangle_a$ is the degree of $\big((\mathrm{id},\mathrm{Fr}_{\mathcal{M}_d})^!\zeta\big)_a \in \mathrm{Ch}_0(\mathrm{Sht}^{\mu}_{\mathcal{M},d}(a))_{\mathbb{Q}}$. Combining this with Theorem 6.6 we get

$$\mathbb{I}_r(h_D) = \sum_{a\in\mathcal{A}_D(k)} \langle\zeta,\Gamma(\mathrm{Fr}_{\mathcal{M}_d})\rangle_a. \tag{6.18}$$

On the other hand, by Proposition A.12, we have for any $a\in\mathcal{A}_D(k)$

$$\langle\zeta,\Gamma(\mathrm{Fr}_{\mathcal{M}_d})\rangle_a = \mathrm{Tr}((f_{\mathcal{M},!}\mathrm{cl}(\zeta))_a\circ\mathrm{Frob}_a, (\mathbf{R}f_{\mathcal{M},!}\mathbb{Q}_\ell)_{\overline{a}}). \tag{6.19}$$

Here we are viewing the cycle class $\mathrm{cl}(\zeta)\in \mathrm{H}^{\mathrm{BM}}_{2(2d-g+1)}(\mathrm{Hk}^{\mu}_{\mathcal{M},d})(-2d+g-1)$ as a cohomological self-correspondence of the constant sheaf $\mathbb{Q}_\ell$ on $\mathcal{M}_d$, which induces an endomorphism

$$f_{\mathcal{M},!}\mathrm{cl}(\zeta) : \mathbf{R}f_{\mathcal{M},!}\mathbb{Q}_\ell \longrightarrow \mathbf{R}f_{\mathcal{M},!}\mathbb{Q}_\ell, \tag{6.20}$$

and $(f_{\mathcal{M},!}\mathrm{cl}(\zeta))_a$ is the induced endomorphism on the geometric stalk $(\mathbf{R}f_{\mathcal{M},!}\mathbb{Q}_\ell)_{\overline{a}}$. Since we only care about the action of $f_{\mathcal{M},!}\mathrm{cl}(\zeta)$ on stalks in $\mathcal{A}^{\heartsuit}_d$, only the restriction $\zeta^{\heartsuit} := \zeta|_{\mathcal{A}^{\heartsuit}_d} \in Z_{2d-g+1}(\mathrm{Hk}^{\mu}_{\mathcal{M},d}|_{\mathcal{A}^{\heartsuit}_d})_{\mathbb{Q}}$ matters. Combining (6.19) with (6.18), we see that in order to prove (6.9), it suffices to show that $f_{\mathcal{M},!}\mathrm{cl}(\zeta^{\heartsuit})$ and $(f_{\mathcal{M},!}[\mathcal{H}^{\diamondsuit}])^r$ give the same endomorphism of the complex $\mathbf{R}f_{\mathcal{M},!}\mathbb{Q}_\ell|_{\mathcal{A}^{\heartsuit}_d}$. This is the following lemma, which is applicable because $d\geq 3g-2$ is implied by $d\geq 2g'-1=4g-3$ (since $g\geq 1$).

Lemma 6.7. *Suppose $d\geq 3g-2$, and $\zeta^{\heartsuit}\in Z_{2d-g+1}(\mathrm{Hk}^{\mu}_{\mathcal{M},d}|_{\mathcal{A}^{\heartsuit}_d})_{\mathbb{Q}}$. Suppose the restriction of $\zeta^{\heartsuit}$ to $\mathrm{Hk}^{\mu}_{\mathcal{M},d}|_{\mathcal{A}^{\heartsuit}_d\cap\mathcal{A}^{\diamondsuit}_d}$ is the fundamental cycle, then the endomorphism $f_{\mathcal{M},!}\mathrm{cl}(\zeta^{\heartsuit})$ of $\mathbf{R}f_{\mathcal{M},!}\mathbb{Q}_\ell|_{\mathcal{A}^{\heartsuit}_d}$ is equal to the r-th power of the endomorphism $f_{\mathcal{M},!}[\mathcal{H}^{\diamondsuit}]$.*

Proof. Let $[\mathcal{H}^{\diamondsuit}]^r$ denotes the r-th self-convolution of $[\mathcal{H}^{\diamondsuit}]$, which is a cycle on the r-th self composition of $\mathcal{H}$, hence on $\mathrm{Hk}^{\mu}_{\mathcal{M},d}$ by Lemma (6.2). We have two cycle $\zeta^{\heartsuit}$ and (the restriction of) $[\mathcal{H}^{\diamondsuit}]^r$ in $Z_{2d-g+1}(\mathrm{Hk}^{\mu}_{\mathcal{M},d}|_{\mathcal{A}^{\heartsuit}_d})_{\mathbb{Q}}$. We temporarily denote $\mathcal{M}_d|_{\mathcal{A}^{\heartsuit}_d}$ by $\mathcal{M}^{\heartsuit}_d$ (although the same notation will be defined in an a priorily different way in §6.3). We need to show that they are in the same cycle class when projected to $\mathcal{M}^{\heartsuit}_d\times\mathcal{M}^{\heartsuit}_d$ under $(\gamma_0,\gamma_r):\mathrm{Hk}^{\mu}_{\mathcal{M},d}|_{\mathcal{A}^{\heartsuit}_d}\to\mathcal{M}^{\heartsuit}_d\times\mathcal{M}^{\heartsuit}_d$.

By assumption, when restricted to $\mathrm{Hk}^{\mu}_{\mathcal{M},d}|_{\mathcal{A}^{\heartsuit}_d\cap\mathcal{A}^{\diamondsuit}_d}$, both $\zeta^{\heartsuit}$ and $[\mathcal{H}^{\diamondsuit}]^r$ are the fundamental cycle. Therefore the difference $(\gamma_0,\gamma_r)_*(\zeta^{\heartsuit}-[\mathcal{H}^{\diamondsuit}]^r)\in Z_{2d-g+1}(\mathcal{M}^{\heartsuit}_d\times\mathcal{M}^{\heartsuit}_d)_{\mathbb{Q}}$ is supported on the image of $\mathrm{Hk}^{\mu}_{\mathcal{M},d}|_{\mathcal{A}^{\heartsuit}_d-\mathcal{A}^{\diamondsuit}_d}$ in $\mathcal{M}^{\heartsuit}_d\times\mathcal{M}^{\heartsuit}_d$, which is contained in the image of $\mathrm{Hk}^{\mu}_{\mathcal{M},d}-\mathrm{Hk}^{\mu}_{\mathcal{M}^{\diamondsuit},d}$ in $\mathcal{M}_d\times\mathcal{M}_d$. By Lemma 6.4(2), the latter has dimension $\leq d+2g-2$. Since $d>3g-3$, we have $d+2g-2<2d-g+1$, therefore $(\gamma_0,\gamma_r)_*(\zeta^{\heartsuit}-[\mathcal{H}^{\diamondsuit}]^r)=0\in Z_{2d-g+1}(\mathcal{M}^{\heartsuit}_d\times\mathcal{M}^{\heartsuit}_d)_{\mathbb{Q}}$, and the lemma follows. □

6.3. **Auxiliary moduli stacks.** The goal of this subsection is to prove Theorem 6.6. Below we fix an integer $d\geq\max\{2g'-1,2g\}$. In this subsection, we will introduce moduli stacks $\mathrm{Hk}'^{r}_{G,d}$

and H_d that will fit into the following commutative diagram

$$\begin{tikzcd}
\mathrm{Hk}^\mu_T \arrow[d, "{(\gamma_0,\gamma_r)}"'] & \times\, \mathrm{Hk}^\mu_T \arrow[d, "{(\gamma_0,\gamma_r)}"] \arrow[r, "\Pi^\mu\times\Pi^\mu"] & \mathrm{Hk}'^r_G \arrow[d, "{(\gamma'_0,\gamma'_r)}"'] & \times\, \mathrm{Hk}'^r_G \arrow[d, "{(\gamma'_0,\gamma'_r)}"] & \mathrm{Hk}'^r_{G,d} \arrow[l, "{(\overleftarrow{p}',\overrightarrow{p}')}"'] \arrow[d, "{(\gamma'_0,\gamma'_r)}"] \\
(\mathrm{Bun}_T)^2 & \times\, (\mathrm{Bun}_T)^2 \arrow[r, "\Pi\times\Pi\times\Pi\times\Pi"] & (\mathrm{Bun}_G)^2 & \times\, (\mathrm{Bun}_G)^2 & H_d \times H_d \arrow[l, "\overleftrightarrow{p}_{13}\times\overleftrightarrow{p}_{24}"'] \\
\mathrm{Bun}_T \arrow[u, "{(\mathrm{id},\mathrm{Fr})}"] & \times\, \mathrm{Bun}_T \arrow[u, "{(\mathrm{id},\mathrm{Fr})}"'] \arrow[r, "\Pi\times\Pi"] & \mathrm{Bun}_G \arrow[u, "{(\mathrm{id},\mathrm{Fr})}"] & \times\, \mathrm{Bun}_G \arrow[u, "{(\mathrm{id},\mathrm{Fr})}"'] & H_d \arrow[l, "{\overleftrightarrow{p}=(\overleftarrow{p},\overrightarrow{p})}"'] \arrow[u, "{(\mathrm{id},\mathrm{Fr})}"']
\end{tikzcd} \tag{6.21}$$

The maps in this diagram will be introduced later. The fiber products of the three columns are

$$\mathrm{Sht}^\mu_T \times \mathrm{Sht}^\mu_T \xrightarrow{\theta^\mu\times\theta^\mu} \mathrm{Sht}'^r_G \times \mathrm{Sht}'^r_G \xleftarrow{(\overleftarrow{p}',\overrightarrow{p}')} \mathrm{Sht}'^r_{G,d} \tag{6.22}$$

where $\mathrm{Sht}'^r_{G,d}$ is defined as the fiber product of the third column.

The fiber products of the three rows will be denoted

$$\begin{tikzcd}
\mathrm{Hk}^\mu_{\mathcal{M}^\heartsuit,d} \arrow[d, "{(\gamma_0,\gamma_r)}"] \\
\mathcal{M}^\heartsuit_d \times \mathcal{M}^\heartsuit_d \\
\mathcal{M}^\heartsuit_d \arrow[u, "{(\mathrm{id},\mathrm{Fr})}"']
\end{tikzcd} \tag{6.23}$$

These stacks will turn out to be the restrictions of $\mathcal{M}_d$ and $\mathrm{Hk}^\mu_{\mathcal{M},d}$ to $\mathcal{A}^\heartsuit_d$, as we will see in Lemma 6.8(2) and Lemma 6.9.

6.3.1. In §A.3 we discuss an abstract situation as in the above diagrams, which can be pictured using a subdivided octahedron. By Lemma A.9, the fiber products of the two diagrams (6.22) and (6.23) are canonically isomorphic. We denote this stack by

$$\mathrm{Sht}^\mu_{\mathcal{M}^\heartsuit,d}.$$

Below we will introduce H_d and $\mathrm{Hk}'^r_{G,d}$.

6.3.2. We define $\widetilde{H}_d$ to be the moduli stack whose S-points is the groupoid of maps

$$\phi : \mathcal{E} \hookrightarrow \mathcal{E}'$$

where $\mathcal{E}, \mathcal{E}'$ are vector bundles over $X \times S$ of rank two, ϕ is an injective map of $\mathcal{O}_{X\times S}$-modules (so its cokernel has support finite over S) and $\mathrm{pr}_{S*}\operatorname{coker}(\phi)$ is a locally free $\mathcal{O}_S$-module of rank d (where $\mathrm{pr}_S : X \times S \to S$ is the projection). We have an action of Pic_X on $\widetilde{H}_d$ by tensoring, and we form the quotient

$$H_d := \widetilde{H}_d / \mathrm{Pic}_X$$

Taking the map ϕ to its source and target gives two maps $\overleftarrow{p}, \overrightarrow{p} : H_d \to \mathrm{Bun}_G$. The map $\overleftrightarrow{p}_{13} \times \overleftrightarrow{p}_{24}$ that appears in (6.21) is the map

$$\begin{aligned}
\overleftrightarrow{p}_{13} \times \overleftrightarrow{p}_{24} : H_d \times H_d &\longrightarrow \mathrm{Bun}_G \times \mathrm{Bun}_G \times \mathrm{Bun}_G \times \mathrm{Bun}_G \\
(h, h') &\longmapsto (\overleftarrow{p}(h), \overleftarrow{p}(h'), \overrightarrow{p}(h), \overrightarrow{p}(h')).
\end{aligned}$$

On the other hand we have the morphism $\Pi : \mathrm{Bun}_T \to \mathrm{Bun}_G$ sending $\mathcal{L}$ to $\nu_*\mathcal{L}$, see §5.5.1. We form the following Cartesian diagram, and take it as the definition of $\mathcal{M}^\heartsuit_d$

$$\begin{tikzcd}
\mathcal{M}^\heartsuit_d \arrow[r] \arrow[d] & H_d \arrow[d, "{(\overleftarrow{p},\overrightarrow{p})}"] \\
\mathrm{Bun}_T \times \mathrm{Bun}_T \arrow[r, "\Pi\times\Pi"] & \mathrm{Bun}_G \times \mathrm{Bun}_G
\end{tikzcd} \tag{6.24}$$

Lemma 6.8. *(1) The morphisms* $\overleftarrow{p}, \overrightarrow{p} : H_d \to \mathrm{Bun}_G$ *are representable and smooth of pure relative dimension* $2d$*. In particular,* H_d *is a smooth algebraic stack over* k *of pure dimension* $2d + 3g - 3$*.*

(2) There is a canonical open embedding $\mathcal{M}_d^{\heartsuit} \hookrightarrow \mathcal{M}_d$ *whose image is* $f_{\mathcal{M}}^{-1}(\mathcal{A}_d^{\heartsuit})$ *(for the definition of* $\mathcal{A}_d^{\heartsuit}$*, see §3.2.4). In particular,* $\mathcal{M}_d^{\heartsuit}$ *is a smooth Deligne–Mumford stack over* k *of pure dimension* $2d - g + 1$*.*

Proof. (1) Let R be a local artinian k-algebra, and let $\phi : \mathcal{E} \hookrightarrow \mathcal{E}'$ be an R-point of H_d, where $\mathcal{E}$ and $\mathcal{E}'$ are rank two vector bundles over X_R. The tangent complex of H_d at ϕ is $\mathrm{H}^*(X_R, \mathcal{K})$ where $\mathcal{K}$ is the two-term complex

$$(\mathrm{End}(\mathcal{E}) \oplus \mathrm{End}(\mathcal{E}'))/\mathcal{O}_{X_R} \cdot (\mathrm{id}_{\mathcal{E}}, \mathrm{id}_{\mathcal{E}'}) \xrightarrow{\delta} \mathrm{Hom}(\mathcal{E}, \mathcal{E}')$$

where δ sends a pair $(a, b) \in \mathrm{End}(\mathcal{E}) \oplus \mathrm{End}(\mathcal{E}')$ to $\phi \circ a - b \circ \phi$. Here $\mathcal{K}$ is placed in degrees -1 and 0. The tangent complex of Bun_G at $\mathcal{E} \in \mathrm{Bun}_G(R)$ is given by

$$\mathrm{H}^*(X_R, \mathrm{End}(\mathcal{E})/\mathcal{O}_{X_R} \cdot \mathrm{id}_{\mathcal{E}})[1].$$

The tangent map of $\overleftarrow{p}$ is $\mathrm{H}^*(X_R, \mathcal{K}) \to \mathrm{H}^*(X_R, \mathrm{End}(\mathcal{E})/\mathcal{O} \cdot \mathrm{id})[1]$ induced from the projection $\mathcal{K} \to \mathrm{End}(\mathcal{E})/\mathcal{O} \cdot \mathrm{id}_{\mathcal{E}}[1]$. Therefore the relative tangent complex of $\overleftarrow{p}$ is $\mathrm{H}^*(X_R, \mathcal{V})$ where $\mathcal{V}$ is the two-term complex

$$\mathrm{End}(\mathcal{E}') \xrightarrow{b \longmapsto -b \circ \phi} \mathrm{Hom}(\mathcal{E}, \mathcal{E}') \tag{6.25}$$

in degrees -1 and 0. Since ϕ is generically an isomorphism, the map (6.25) is generically an isomorphism, and $\mathcal{V}$ is quasi-isomorphic to the torsion sheaf $\mathcal{H}^0\mathcal{V}$, which is the cokernel of the map (6.25). Therefore $\mathrm{H}^*(X_R, \mathcal{V})$ is concentrated in degree zero, and $\overleftarrow{p}$ is smooth. The relative dimension of $\overleftarrow{p}$ at a $\overline{k}$-point ϕ as above (for $R = \overline{k}$) is equal to the Euler characteristic of $\mathrm{H}^*(X_{\overline{k}}, \mathcal{V})$, or the length of the torsion sheaf $\mathcal{H}^0\mathcal{V}$, which is $2d$ (using that $\deg \mathrm{End}(\mathcal{E}') = 0$ and $\deg \mathrm{Hom}(\mathcal{E}, \mathcal{E}') = 2d$). Similar argument works for $\overrightarrow{p}$.

(2) By the diagram (6.24), $\mathcal{M}_d^{\heartsuit}$ classifies $(\mathcal{L}, \mathcal{L}', \psi)$ up to the action of Pic_X, where $\mathcal{L}$ and $\mathcal{L}'$ are as in the definition of $\widetilde{\mathcal{M}}_d$, and ψ is an injective $\mathcal{O}_X$-linear map $\nu_*\mathcal{L} \to \nu_*\mathcal{L}'$.

The discussion in §6.1.3 turns a point $(\mathcal{L}, \mathcal{L}', \psi : \nu_*\mathcal{L} \to \nu_*\mathcal{L}') \in \mathcal{M}_d^{\heartsuit}$ into a point $(\mathcal{L}, \mathcal{L}', \alpha : \mathcal{L} \to \mathcal{L}', \beta : \mathcal{L} \to \sigma^*\mathcal{L}') \in \mathcal{M}_d$. The condition that ψ be injective is precisely the condition that $\det(\psi) \neq 0$, which is equivalent to saying that $f_{\mathcal{M}}(\mathcal{L}, \mathcal{L}', \psi) \in \mathcal{A}_d^{\heartsuit}$, according to (6.1).

Proposition 6.1(2) shows that $\mathcal{M}_d$ is a smooth Deligne–Mumford stack over k of pure dimension $2d - g + 1$, hence the same is true for its open substack $\mathcal{M}_d^{\heartsuit}$. $\square$

6.3.3. Recall the Hecke stacks Hk_G^r and Hk_T^μ defined in (5.4) and (5.15). Let $\mathrm{Hk}_{2,d}^\mu$ be the moduli stack of commutative diagrams

$$\begin{array}{ccccccc} \mathcal{E}_0 & \dashrightarrow & \mathcal{E}_1 & \dashrightarrow & \cdots & \dashrightarrow & \mathcal{E}_r \\ \downarrow{\scriptstyle \phi_0} & & \downarrow{\scriptstyle \phi_1} & & & & \downarrow{\scriptstyle \phi_r} \\ \mathcal{E}'_0 & \dashrightarrow & \mathcal{E}'_1 & \dashrightarrow & \cdots & \dashrightarrow & \mathcal{E}'_r \end{array} \tag{6.26}$$

where both rows are points in Hk_2^μ with the same image in X^r, and the vertical maps ϕ_j are points in H_d (i.e., injective maps with colength d). Let

$$\mathrm{Hk}_{G,d}^r = \mathrm{Hk}_{2,d}^\mu / \mathrm{Pic}_X$$

where Pic_X simultaneously acts on all $\mathcal{E}_i$ and $\mathcal{E}'_i$ by tensor product. The same argument of Lemma 5.5 shows that $\mathrm{Hk}_{G,d}^r$ is independent of μ.

There are natural maps $\mathrm{Hk}_G^r \to X^r$ and $\mathrm{Hk}_{G,d}^r \to X^r$. We define

$$\mathrm{Hk}_G'^r = \mathrm{Hk}_G^r \times_{X^r} X'^r; \quad \mathrm{Hk}_{G,d}'^r := \mathrm{Hk}_{G,d}^r \times_{X^r} X'^r.$$

The map $\mathrm{Hk}_T^\mu \to \mathrm{Hk}_G^r$ given by $\mathcal{E}_i = \nu_*\mathcal{L}_i$ induces a map

$$\Pi^\mu : \mathrm{Hk}_T^\mu \longrightarrow \mathrm{Hk}_G'^r.$$

We have two maps

$$\overleftarrow{p}, \overrightarrow{p} : \mathrm{Hk}_{G,d}^r \longrightarrow \mathrm{Hk}_G^r$$

sending the diagram (6.26) to its top and bottom row. We denote their base change to X'^r by

$$\overleftarrow{\rho}', \overrightarrow{\rho}' : \mathrm{Hk}'^r_{G,d} \longrightarrow \mathrm{Hk}'^r_G$$

We define $\mathrm{Hk}^\mu_{\mathcal{M}^\heartsuit,d}$ by the following Cartesian diagram

$$\begin{tikzcd} \mathrm{Hk}^\mu_{\mathcal{M}^\heartsuit,d} \arrow[r] \arrow[d] & \mathrm{Hk}'^r_{G,d} \arrow[d, "(\overleftarrow{\rho}', \overrightarrow{\rho}')"] \\ \mathrm{Hk}^\mu_T \times \mathrm{Hk}^\mu_T \arrow[r, "\Pi^\mu \times \Pi^\mu"] & \mathrm{Hk}'^r_G \times \mathrm{Hk}'^r_G \end{tikzcd} \tag{6.27}$$

The same argument of Lemma 6.8(2) shows the following result. Recall that the stack $\mathrm{Hk}^\mu_{\mathcal{M},d}$ is defined in §6.2.1.

Lemma 6.9. *There is a canonical isomorphism between* $\mathrm{Hk}^\mu_{\mathcal{M}^\heartsuit,d}$ *and the preimage of* $\mathcal{A}^\heartsuit_d$ *under the natural map* $f_\mathcal{M} \circ \gamma_0 : \mathrm{Hk}^\mu_{\mathcal{M},d} \to \mathcal{A}_d$.

6.3.4. We have a map

$$s : \mathrm{Hk}^r_{G,d} \longrightarrow X_d \times X^r$$

which sends a diagram (6.26) to $(D; x_1, \cdots, x_r)$ where D is the divisor of $\det(\phi_i)$ for all i. Let $(X_d \times X^r)^\circ \subset X_d \times X^r$ be the open subscheme consisting of those $(D; x_1, \cdots, x_r)$ where x_i is disjoint from the support of D for all i. Let

$$\mathrm{Hk}^{r,\circ}_{G,d} = s^{-1}((X_d \times X^r)^\circ).$$

be an open substack of $\mathrm{Hk}^r_{G,d}$. Let $\mathrm{Hk}'^{r,\circ}_{G,d} \subset \mathrm{Hk}'^r_{G,d}$ and $\mathrm{Hk}^{\mu,\circ}_{\mathcal{M}^\heartsuit,d} \subset \mathrm{Hk}^\mu_{\mathcal{M}^\heartsuit,d}$ be the preimages of $\mathrm{Hk}^{r,\circ}_{G,d}$.

Lemma 6.10. *(1) The stacks* $\mathrm{Hk}^{r,\circ}_{G,d}$ *and* $\mathrm{Hk}'^{r,\circ}_{G,d}$ *are smooth of pure dimension* $2d+2r+3g-3$.

(2) The dimensions of all geometric fibers of s *are* $d+r+3g-3$. *In particular,* $\dim \mathrm{Hk}^r_{G,d} = \dim \mathrm{Hk}'^r_{G,d} = 2d+2r+3g-3$.

(3) Recall that $\mathrm{Hk}^\mu_{\mathcal{M}^\diamondsuit,d}$ *is the restriction* $\mathrm{Hk}^\mu_{\mathcal{M},d}|_{\mathcal{A}^\diamondsuit_d}$, *where* $\mathcal{A}^\diamondsuit_d \subset \mathcal{A}_d$ *is defined in §6.2.4. Suppose* $d \geq \max\{2g'-1, 2g\}$. *Let* $\mathrm{Hk}^{\mu,\circ}_{\mathcal{M}^\diamondsuit,d}$ *be the intersection of* $\mathrm{Hk}^\mu_{\mathcal{M}^\diamondsuit,d}$ *with* $\mathrm{Hk}^{\mu,\circ}_{\mathcal{M}^\heartsuit,d}$ *inside* $\mathrm{Hk}^\mu_{\mathcal{M},d}$. *Then* $\dim(\mathrm{Hk}^\mu_{\mathcal{M}^\diamondsuit,d} - \mathrm{Hk}^{\mu,\circ}_{\mathcal{M}^\diamondsuit,d}) < 2d-g+1 = \dim \mathrm{Hk}^\mu_{\mathcal{M}^\diamondsuit,d}$.

The proof of this lemma will be postponed to §6.4.1-§6.4.3.

Lemma 6.11. *Suppose* $d \geq \max\{2g'-1, 2g\}$.

(1) The diagram (6.27) *satisfies the conditions in §A.2.8. In particular, the refined Gysin map*

$$(\Pi^\mu \times \Pi^\mu)^! : \mathrm{Ch}_*(\mathrm{Hk}'^r_{G,d})_\mathbb{Q} \longrightarrow \mathrm{Ch}_{*-2(2g-2+r)}(\mathrm{Hk}^\mu_{\mathcal{M}^\heartsuit,d})_\mathbb{Q}$$

is defined.

(2) Let

$$\zeta^\heartsuit = (\Pi^\mu \times \Pi^\mu)^![\mathrm{Hk}'^r_{G,d}] \in \mathrm{Ch}_{2d-g+1}(\mathrm{Hk}^\mu_{\mathcal{M}^\heartsuit,d})_\mathbb{Q}. \tag{6.28}$$

Then the restriction of $\zeta^\heartsuit$ *to* $\mathrm{Hk}^\mu_{\mathcal{M}^\heartsuit,d}|_{\mathcal{A}^\diamondsuit_d \cap \mathcal{A}^\heartsuit_d}$ *is the fundamental cycle.*

Proof. (1) We first check that $\mathrm{Hk}^\mu_{\mathcal{M}^\heartsuit,d}$ admits a finite flat presentation. The map $\gamma_0 : \mathrm{Hk}^\mu_{\mathcal{M}^\heartsuit,d} \to \mathcal{M}^\heartsuit_d$ is schematic, so it suffices to check that $\mathcal{M}^\heartsuit_d$ or $\mathcal{M}_d$ admits a finite flat presentation. In the proof of Proposition 6.1(1) we constructed a proper and schematic map $\overline{h} : \mathcal{M}_d \to J^d_{X'} \times \mathrm{Prym}_{X'/X}$, see (6.4). Since $J^d_{X'}$ is a scheme and $\mathrm{Prym}_{X'/X}$ is the quotient of the usual Prym variety by the trivial action of μ_2, $J^d_{X'} \times \mathrm{Prym}_{X'/X}$ admits a finite flat presentation, hence so do $\mathcal{M}_d$ and $\mathrm{Hk}^\mu_{\mathcal{M}^\heartsuit,d}$.

Next we verify the condition (2) of §A.2.8. Extending k if necessary, we may choose a point $y \in X(k)$ that is split into $y', y'' \in X'(k)$. Let $\mathrm{Bun}_G(y)$ be the moduli stack of G-torsors over X with a Borel reduction at y. Let $\mathrm{Hk}'^r_G(y) = \mathrm{Hk}'^r_G \times_{\mathrm{Bun}_G} \mathrm{Bun}_G(y)$ where the map $\mathrm{Hk}'^r_G \to \mathrm{Bun}_G$ sends $(\mathcal{E}_i; x_i; f_i)$ to $\mathcal{E}_0$. We may lift the morphism Π^μ to a morphism

$$\Pi^\mu(y) : \mathrm{Hk}^\mu_T \longrightarrow \mathrm{Hk}'^r_G(y)$$

where the Borel reduction of $\mathcal{E}_0 = \nu_*\mathcal{L}_0$ at y (i.e., a line in the stalk $\mathcal{E}_{0,y}$) is given by the stalk of $\mathcal{L}_0$ at y'. The projection $p : \mathrm{Hk}'^r_G(y) \to \mathrm{Hk}'^r_G$ is smooth, and $\Pi^\mu = p \circ \Pi^\mu(y)$. So to check the condition (2) of §A.2.8, it suffices to show that $\Pi^\mu(y)$ is a regular local immersion.

We will show by tangential calculations that $\mathrm{Hk}'^r_G(y)$ is a Deligne–Mumford stack in a neighborhood of the image of $\Pi^\mu(y)$, and the tangent map of $\Pi^\mu(y)$ is injective. For this it suffices to make tangential calculations at geometric points of Hk^μ_T and its image in $\mathrm{Hk}'^r_G(y)$. We identify Hk^μ_T with $\mathrm{Bun}_T \times X'^r$ as in §5.4.4. Fix a geometric point $(\mathcal{L}; \underline{x'}) \in \mathrm{Pic}_{X'}(K) \times X'(K)^r$. For notational simplicity, we base change the situation from k to K without changing notation. So X means $X \otimes_k K$, etc.

The relative tangent space of $\mathrm{Hk}^\mu_T \to X'^r$ at $(\mathcal{L}; \underline{x'})$ is $\mathrm{H}^1(X, \mathcal{O}_{X'}/\mathcal{O}_X)$. The relative tangent complex of $\mathrm{Hk}'^r_G(y) \to X'^r$ at $\Pi^\mu(y)(\mathcal{L}; \underline{x'}) = (\nu_*\mathcal{L} \to \nu_*\mathcal{L}(x'_1) \to \cdots ; \mathcal{L}_{y'})$ is $\mathrm{H}^*(X, \mathrm{Ad}^{\underline{x'},y}(\nu_*\mathcal{L}))[1]$, where $\mathrm{Ad}^{\underline{x'},y}(\nu_*\mathcal{L}) = \underline{\mathrm{End}}^{\underline{x'},y}(\nu_*\mathcal{L})/\mathcal{O}_X \cdot \mathrm{id}$, and $\underline{\mathrm{End}}^{\underline{x'},y}(\nu_*\mathcal{L})$ is the endomorphism sheaf of the chain of vector bundles $\nu_*\mathcal{L} \to \nu_*\mathcal{L}(x'_1) \to \cdots$ preserving the line $\mathcal{L}_{y'}$ of the stalk $(\nu_*\mathcal{L})_y$. Note that

$$\begin{aligned}\underline{\mathrm{End}}^{\underline{x'},y}(\nu_*\mathcal{L}) &\subset \underline{\mathrm{End}}^{y}(\nu_*\mathcal{L}) = \nu_*\underline{\mathrm{Hom}}(\mathcal{L} \oplus (\sigma^*\mathcal{L})(y''), \mathcal{L}) \\ &= \nu_*\mathcal{O}_{X'} \oplus \nu_*(\mathcal{L} \otimes \sigma^*\mathcal{L}^{-1}(-y''))\end{aligned} \tag{6.29}$$

We also have a natural inclusion

$$\gamma : \nu_*\mathcal{O}_{X'} \hookrightarrow \underline{\mathrm{End}}^{\underline{x'},y}(\nu_*\mathcal{L})$$

identifying the LHS as those endomorphisms of $\nu_*\mathcal{L}$ that are $\mathcal{O}_{X'}$-linear. Now $\gamma(\nu_*\mathcal{O}_{X'})$ maps isomorphically to $\nu_*\mathcal{O}_{X'}$ on the RHS of (6.29). Combining these we get a canonical decomposition $\underline{\mathrm{End}}^{\underline{x'},y}(\nu_*\mathcal{L}) = \nu_*\mathcal{O}_{X'} \oplus \mathcal{K}$ for some line bundle $\mathcal{K}$ on X with $\deg(\mathcal{K}) < 0$. Consequently, we have a canonical decomposition

$$\mathrm{Ad}^{\underline{x'},y}(\nu_*\mathcal{L}) = \mathcal{O}_{X'}/\mathcal{O}_X \oplus \mathcal{K}. \tag{6.30}$$

In particular $\mathrm{H}^0(X, \mathrm{Ad}^{\underline{x'},y}(\nu_*\mathcal{L})) = \mathrm{H}^0(X, \mathcal{O}_{X'}/\mathcal{O}_X) = 0$. This shows that $\mathrm{Hk}'^r_G(y)$ is a Deligne–Mumford stack in a neighborhood of $\Pi^\mu(y)(\mathcal{L}; \underline{x'})$.

The tangent map of $\Pi^\mu(y)$ is the map $\mathrm{H}^1(X, \mathcal{O}_{X'}/\mathcal{O}_X) \to \mathrm{H}^1(X, \mathrm{Ad}^{\underline{x'}}(\nu_*\mathcal{L}))$ induced by γ, hence it corresponds to the inclusion of the first factor in the decomposition (6.30). In particular, the tangent map of $\Pi^\mu(y)$ is injective. This finishes the verification of all conditions in §A.2.8 for the diagram (6.27).

(2) Let $\mathrm{Hk}^{\mu,\circ}_{\mathcal{M}^\heartsuit,d}$ be the preimage of $\mathrm{Hk}'^{r,\circ}_{G,d}$. By Lemma 6.10(1), $\mathrm{Hk}'^{r,\circ}_{G,d}$ is smooth of dimension $2d + 2r + 3g - 3$. On the other hand, by Lemma 6.4, $\mathrm{Hk}^\mu_{\mathcal{M}^\diamondsuit,d}$ has dimension $2d - g + 1$. Combining these facts, we see that $\mathrm{Hk}^{\mu,\circ}_{\mathcal{M}^\heartsuit,d} \cap \mathrm{Hk}^\mu_{\mathcal{M}^\diamondsuit,d}$ has the expected dimension in the Cartesian diagram (6.27). This implies that $\zeta^\heartsuit|_{\mathrm{Hk}^{\mu,\circ}_{\mathcal{M}^\heartsuit,d} \cap \mathrm{Hk}^\mu_{\mathcal{M}^\diamondsuit,d}}$ is the fundamental cycle. By Lemma 6.10(3), $\mathrm{Hk}^\mu_{\mathcal{M}^\diamondsuit,d} - \mathrm{Hk}^{\mu,\circ}_{\mathcal{M}^\heartsuit,d}$ has lower dimension than $\mathrm{Hk}^\mu_{\mathcal{M}^\diamondsuit,d}$, therefore $\zeta^\heartsuit|_{\mathrm{Hk}^\mu_{\mathcal{M}^\diamondsuit,d}}$ must be the fundamental cycle. $\square$

6.3.5. There are $r + 1$ maps γ_i $(0 \le i \le r)$ from the diagram (6.27) to (6.24): it sends the diagram (6.26) to its i-th column, etc. In particular, we have maps $\gamma_i : \mathrm{Hk}^r_{G,d} \to H_d$ and $\gamma'_i : \mathrm{Hk}'^r_{G,d} \to H_d$. The maps γ'_0 and γ'_r appear in the diagram (6.21).

We define the stack $\mathrm{Sht}^r_{G,d}$ by the following Cartesian diagram

$$\begin{tikzcd} \mathrm{Sht}^r_{G,d} \arrow[r] \arrow[d] & \mathrm{Hk}^r_{G,d} \arrow[d, "(\gamma_0,\gamma_r)"] \\ H_d \arrow[r, "(\mathrm{id},\mathrm{Fr})"] & H_d \times H_d \end{tikzcd} \tag{6.31}$$

Similarly we define $\mathrm{Sht}'^r_{G,d}$ as the fiber product of the third column of (6.21):

$$\begin{tikzcd} \mathrm{Sht}'^r_{G,d} \arrow[r] \arrow[d] & \mathrm{Hk}'^r_{G,d} \arrow[d, "(\gamma'_0,\gamma'_r)"] \\ H_d \arrow[r, "(\mathrm{id},\mathrm{Fr})"] & H_d \times H_d \end{tikzcd} \tag{6.32}$$

We have $\mathrm{Sht}'^r_{G,d} \cong \mathrm{Sht}^r_{G,d} \times_{X^r} X'^r$.

Lemma 6.12. *There are canonical isomorphisms of stacks*

$$\mathrm{Sht}^r_{G,d} \cong \coprod_{D \in X_d(k)} \mathrm{Sht}^r_G(h_D);$$
$$\mathrm{Sht}'^r_{G,d} \cong \coprod_{D \in X_d(k)} \mathrm{Sht}'^r_G(h_D).$$

For the definitions of $\mathrm{Sht}^r_G(h_D)$ *and* $\mathrm{Sht}'^r_G(h_D)$*, see §5.3.1 and §5.3.2.*

Proof. From the definitions, (γ_0, γ_r) factors through the map $\mathrm{Hk}^r_{G,d} \to H_d \times_{X_d} H_d$. On the other hand, $(\mathrm{id}, \mathrm{Fr}) : H_d \to H_d \times H_d$ covers the similar map $(\mathrm{id}, \mathrm{Fr}) : X_d \to X_d \times X_d$. By the discussion in §A.4.5, we have a decomposition

$$\mathrm{Sht}^r_{G,d} = \coprod_{D \in X_d(k)} \mathrm{Sht}^r_{G,D}$$

Let H_D and $\mathrm{Hk}^r_{G,D}$ be the fibers of H_d and $\mathrm{Hk}^r_{G,d}$ over D. Then the D-component $\mathrm{Sht}^r_{G,D}$ of $\mathrm{Sht}^r_{G,d}$ fits into a Cartesian diagram

$$\begin{tikzcd} \mathrm{Sht}^r_{G,D} \arrow[r] \arrow[d] & \mathrm{Hk}^r_{G,D} \arrow[d, "(\gamma_0,\gamma_r)"] \\ H_D \arrow[r, "(\mathrm{id},\mathrm{Fr})"] & H_D \times H_D \end{tikzcd} \tag{6.33}$$

Comparing this with the definition in §5.3.1, we see that $\mathrm{Sht}^r_{G,D} \cong \mathrm{Sht}^r_G(h_D)$. The statement for $\mathrm{Sht}'^r_{G,d}$ follows from the statement for $\mathrm{Sht}^r_{G,d}$ by base change to X'^r. □

Corollary 6.13. *Let* $D \in X_d(k)$ *(i.e., an effective divisor on* X *of degree* d*). Recall the stack* $\mathrm{Sht}^\mu_{\mathcal{M},d}$ *defined in* (6.12) *and* $\mathrm{Sht}^\mu_{\mathcal{M}^\heartsuit,d}$ *defined in §6.3.1. Then* $\mathrm{Sht}^\mu_{\mathcal{M}^\heartsuit,d}$ *is canonically isomorphic to the restriction of* $\mathrm{Sht}^\mu_{\mathcal{M},d}$ *to* $\mathcal{A}^\heartsuit_d(k) \subset \mathcal{A}_d(k)$.

Moreover, there is a canonical decomposition

$$\mathrm{Sht}^\mu_{\mathcal{M}^\heartsuit,d} = \coprod_{D \in X_d(k)} \mathrm{Sht}^\mu_{\mathcal{M},D},$$

where $\mathrm{Sht}^\mu_{\mathcal{M},D}$ *is defined in* (6.14)*. In particular, we have a Cartesian diagram*

$$\begin{tikzcd} \mathrm{Sht}^\mu_{\mathcal{M},D} \arrow[r] \arrow[d] & \mathrm{Sht}'^r_G(h_D) \arrow[d, "(\overleftarrow{p}', \overrightarrow{p}')"] \\ \mathrm{Sht}^\mu_T \times \mathrm{Sht}^\mu_T \arrow[r, "\theta^\mu \times \theta^\mu"] & \mathrm{Sht}'^r_G \times \mathrm{Sht}'^r_G \end{tikzcd} \tag{6.34}$$

Proof. Note that $\mathrm{Sht}^\mu_{\mathcal{M}^\heartsuit,d}$ is defined as a fiber product in two ways: one as the fiber product of (6.22) and the other as the fiber product of (6.23). Using the first point of view and the decomposition of $\mathrm{Sht}'^r_{G,d}$ given by Lemma 6.12, we get a decomposition of $\mathrm{Sht}^\mu_{\mathcal{M}^\heartsuit,d} = \coprod_{D \in X_d(k)} \mathrm{Sht}^\mu_{\mathcal{M}^\heartsuit,D}$, where $\mathrm{Sht}^\mu_{\mathcal{M}^\heartsuit,D}$ is by definition the stack to put in the northwest corner of (6.34) to make the diagram Cartesian.

On the other hand, using the second point of view of $\mathrm{Sht}^\mu_{\mathcal{M}^\heartsuit,d}$ as the fiber product of (6.23), and using the fact that $\mathrm{Hk}^\mu_{\mathcal{M}^\heartsuit,d}$ is the restriction of $\mathrm{Hk}^\mu_{\mathcal{M},d}$ over $\mathcal{A}^\heartsuit_d$ by Lemma 6.9, we see that $\mathrm{Sht}^\mu_{\mathcal{M}^\heartsuit,d}$ is the restriction of $\mathrm{Sht}^\mu_{\mathcal{M},d}$ over $\mathcal{A}^\heartsuit_d$ by comparing (6.23) and (6.12). By (6.13) and (6.14), and the fact that $\mathcal{A}^{\heartsuit}_d(k) = \coprod_{D\in X_d(k)} \mathcal{A}_D(k)$, we get a decomposition $\mathrm{Sht}^{\mu}_{\mathcal{M}^{\heartsuit},d} = \coprod_{D\in X_d(k)} \mathrm{Sht}^{\mu}_{\mathcal{M},D}$. Therefore, both $\mathrm{Sht}^{\mu}_{\mathcal{M}^{\heartsuit},D}$ and $\mathrm{Sht}^{\mu}_{\mathcal{M},D}$ are the fiber of the map $\mathrm{Sht}^{\mu}_{\mathcal{M},d} \to \mathcal{A}_d \to \widehat{X}_d$ over D, and they are canonically isomorphic. Hence we may replace the northwest corner of (6.34) by $\mathrm{Sht}^{\mu}_{\mathcal{M}^{\heartsuit},D}$, and the new diagram is Cartesian by definition. □

Lemma 6.14. *(1) The diagram* (6.32) *satisfies the conditions in* §*A.2.10. In particular, the refined Gysin map*

$$(\mathrm{id}, \mathrm{Fr}_{H_d})^! : \mathrm{Ch}_*(\mathrm{Hk}'^r_{G,d})_{\mathbb{Q}} \longrightarrow \mathrm{Ch}_{*-\dim H_d}(\mathrm{Sht}'^r_{G,d})_{\mathbb{Q}}$$

is defined.

(2) We have

$$[\mathrm{Sht}'^r_{G,d}] = (\mathrm{id}, \mathrm{Fr}_{H_d})^! [\mathrm{Hk}'^r_{G,d}] \in \mathrm{Ch}_{2r}(\mathrm{Sht}'^r_{G,d}).$$

Proof. (1) Since $\overleftarrow{p} : \mathrm{Sht}^r_G(h_D) \to \mathrm{Sht}^r_G$ is representable by Lemma 5.8, $\mathrm{Sht}^r_G(h_D)$ is also a Deligne–Mumford stack. Since $\mathrm{Sht}^r_{G,d}$ is the disjoint union of $\mathrm{Sht}^r_G(h_D)$ by Lemma 6.12, $\mathrm{Sht}^r_{G,d}$ is Deligne–Mumford, hence so is $\mathrm{Sht}'^r_{G,d}$. The map $\gamma'_0 : \mathrm{Hk}'^r_{G,d} \to H_d$ is representable because its fibers are closed subschemes of iterated Quot schemes (fixing $\mathcal{E}_0 \hookrightarrow \mathcal{E}'_0$, building $\mathcal{E}_i$ and $\mathcal{E}'_i$ step by step and imposing commutativity of the maps). Therefore (γ'_0, γ'_r) is also representable. This verifies the condition (1) in §A.2.10.

Since H_d is smooth by Lemma 6.8, the normal cone stack of the map $(\mathrm{id}, \mathrm{Fr}_{H_d}) : H_d \to H_d \times H_d$ is the vector bundle stack $\mathrm{Fr}^* TH_d$, the Frobenius pullback of the tangent bundle stack of H_d. Therefore $(\mathrm{id}, \mathrm{Fr}_{H_d})$ satisfies condition (2) in §A.2.10. It also satisfies condition (3) of §A.2.10 by the discussion in Remark A.7.

Finally the dimension condition (4) in §A.2.10 for $\mathrm{Hk}'^r_{G,d}$ and $\mathrm{Sht}'^r_{G,d} = \coprod_D \mathrm{Sht}'^r_G(h_D)$ follow from Lemma 6.10(2) and Lemma 5.9. We have verified all conditions in §A.2.10.

(2) Take the open substack $\mathrm{Hk}'^{r,\circ}_{G,d} \subset \mathrm{Hk}'^r_{G,d}$ as in Lemma 6.10. Then $\mathrm{Hk}'^{r,\circ}_{G,d}$ is smooth of pure dimension $2d + 2r + 3g - 3$. According to Lemma 6.12, the corresponding open part $\mathrm{Sht}'^{r,\circ}_{G,d}$ is the disjoint union of $\mathrm{Sht}'^{r,\circ}_G(h_D)$, where

$$\mathrm{Sht}'^{r,\circ}_G(h_D) = \mathrm{Sht}'^r_G(h_D)|_{(X' - \nu^{-1}(D))^r}.$$

It is easy to see that both projections $\mathrm{Sht}'^{r,\circ}_G(h_D) \to \mathrm{Sht}'^r_G$ are étale, hence $\mathrm{Sht}'^{r,\circ}_G(h_D)$ is smooth of dimension $2r = \dim \mathrm{Hk}'^{r,\circ}_{G,d} - \mathrm{codim}(\mathrm{id}, \mathrm{Fr}_{H_d})$, the expected dimension. This implies that if we replace $\mathrm{Hk}'^r_{G,d}$ with $\mathrm{Hk}'^{r,\circ}_{G,d}$, and replace $\mathrm{Sht}'^r_{G,d}$ with $\mathrm{Sht}'^{r,\circ}_{G,d}$ in the diagram (6.32), it becomes a complete intersection diagram. Therefore $(\mathrm{id}, \mathrm{Fr}_{H_d})^! [\mathrm{Hk}'^r_{G,d}]$ is the fundamental cycle when restricted to $\mathrm{Sht}'^{r,\circ}_{G,d}$. Since $\mathrm{Sht}'^r_{G,d} - \mathrm{Sht}'^{r,\circ}_{G,d}$ has lower dimension than $2r$ by Lemma 5.9, we see that $(\mathrm{id}, \mathrm{Fr}_{H_d})^! [\mathrm{Hk}'^r_{G,d}]$ must be equal to the fundamental cycle over the whole $\mathrm{Sht}'^r_{G,d}$. □

6.3.6. *Proof of Theorem 6.6.* Consider the diagram (6.34). Since Sht^{μ}_T is a proper Deligne–Mumford stack over k and the map $(\overleftarrow{p}', \overrightarrow{p}')$ is proper and representable, $\mathrm{Sht}^{\mu}_{\mathcal{M},D}$ is also a proper Deligne–Mumford stack over k. A simple manipulation using the functoriality of Gysin maps gives

$$\mathbb{I}_r(h_D) = \langle \theta^{\mu}_* [\mathrm{Sht}^{\mu}_T], h_D * \theta^{\mu}_* [\mathrm{Sht}^{\mu}_T] \rangle_{\mathrm{Sht}^r_G} = \deg \left((\theta^{\mu} \times \theta^{\mu})^! [\mathrm{Sht}'^r_G(h_D)] \right).$$

Here $(\theta^{\mu} \times \theta^{\mu})^! : \mathrm{Ch}_{2r}(\mathrm{Sht}'^r_G(h_D))_{\mathbb{Q}} \to \mathrm{Ch}_0(\mathrm{Sht}^{\mu}_{\mathcal{M},D})_{\mathbb{Q}}$ is the refined Gysin map attached to the map $\theta^{\mu} \times \theta^{\mu}$. By Corollary 6.13, $(\theta^{\mu} \times \theta^{\mu})^! [\mathrm{Sht}'^r_G(h_D)]$ is the D-component of the 0-cycle

$$(\theta^{\mu} \times \theta^{\mu})^! [\mathrm{Sht}'^r_{G,d}] \in \mathrm{Ch}_0(\mathrm{Sht}^{\mu}_{\mathcal{M}^{\heartsuit},d})_{\mathbb{Q}} = \bigoplus_{D\in X_d(k)} \mathrm{Ch}_0(\mathrm{Sht}^{\mu}_{\mathcal{M},D})_{\mathbb{Q}}.$$

Therefore, to prove (6.16) simultaneously for all D of degree d, it suffices to find a cycle class $\zeta^{\heartsuit} \in \mathrm{Ch}_{2d-g+1}(\mathrm{Hk}^{\mu}_{\mathcal{M}^{\heartsuit},d})_{\mathbb{Q}}$ whose restriction to $\mathrm{Hk}^{\mu}_{\mathcal{M}^{\heartsuit},d} \cap \mathrm{Hk}^{\mu}_{\mathcal{M}^{\diamondsuit},d} = \mathrm{Hk}^{\mu}_{\mathcal{M},d}|_{\mathcal{A}^{\heartsuit}_d \cap \mathcal{A}^{\diamondsuit}_d}$ is the fundamental class, and that

$$(\theta^{\mu} \times \theta^{\mu})^! [\mathrm{Sht}'^r_{G,d}] = (\mathrm{id}, \mathrm{Fr}_{\mathcal{M}^{\heartsuit}_d})^! \zeta^{\heartsuit} \in \mathrm{Ch}_0(\mathrm{Sht}^{\mu}_{\mathcal{M}^{\heartsuit},d})_{\mathbb{Q}}. \tag{6.35}$$

The statement of Theorem 6.6 asks for a cycle ζ on $\mathrm{Hk}^{\mu}_{\mathcal{M},d}$, but we may extend the above $\zeta^{\heartsuit}$ arbitrarily to a $(2d - g + 1)$-cycle in $\mathrm{Hk}^{\mu}_{\mathcal{M},d}$.

To prove (6.35), we would like to apply Theorem A.10 to the situation of (6.21). We check the assumptions:

(1) The smoothness of Bun_T and Bun_G is well-known. The smoothness of Hk'^r_G and Hk^μ_T follow from Remark 5.2 and §5.4.4. Finally, by Lemma 6.8, H_d is smooth of pure dimension $2d+3g-3$. This checks the smoothness of all members in (6.21) except $B = \mathrm{Hk}'^r_{G,d}$.

(2) By Corollary 5.7, Sht^r_G and hence Sht'^r_G is smooth of pure dimension $2r$; by Lemma 5.13, Sht^μ_T is smooth of pure dimension r. By Lemma 6.8, $\mathcal{M}^\heartsuit_d$ is smooth of pure dimension $2d-g+1$. All of them have the dimension expected from the Cartesian diagrams defining them.

(3) The diagram (6.32) satisfies the conditions in §A.2.10 by Lemma 6.14. The diagram (6.27) satisfies the conditions in §A.2.8 by Lemma 6.11.

(4) We check that the Cartesian diagram formed by (6.23), or rather (6.12), satisfies the conditions in §A.2.8. The map $\mathrm{Sht}^\mu_{\mathcal{M},d} \to \mathcal{M}_d$ is representable because $\mathrm{Hk}^\mu_{\mathcal{M},d} \to \mathcal{M}_d \times \mathcal{M}_d$ is. In the proof of Lemma 6.11(1) we have proved that $\mathcal{M}_d$ admits a finite flat presentation, hence so does $\mathrm{Sht}^\mu_{\mathcal{M},d}$. This verifies the first condition in §A.2.8. Since $\mathcal{M}_d$ is a smooth Deligne–Mumford stack by Lemma 6.1(2), $(\mathrm{id}, \mathrm{Fr}_{\mathcal{M}_d}) : \mathcal{M}_d \to \mathcal{M}_d \times \mathcal{M}_d$ is a regular local immersion, which verifies condition (2) of §A.2.8.

Finally we consider the Cartesian diagram formed by (6.22) (or equivalently, the disjoint union of the diagrams (6.34) for all $D \in X_d(k)$). We have already showed above that $\mathrm{Sht}^\mu_{\mathcal{M},d}$ admits a finite flat presentation. All members in these diagrams are Deligne–Mumford stacks, and Sht^μ_T and Sht'^r_G are smooth Deligne–Mumford stacks by Lemma 5.13 and Corollary 5.7. Hence the map $\theta^\mu \times \theta^\mu$ satisfies the conditions (2) of §A.2.8 by Remark A.4.

Now we can apply Theorem A.10 to the situation (6.21). Let $\zeta^\heartsuit = (\Pi^\mu \times \Pi^\mu)^![\mathrm{Hk}'^r_{G,d}] \in \mathrm{Ch}_{2d-g+1}(\mathrm{Hk}^\mu_{\mathcal{M}^\heartsuit,d})_\mathbb{Q}$ as defined in (6.28). Then the restriction of $\zeta^\heartsuit$ to $\mathrm{Hk}^\mu_{\mathcal{M},d}|_{\mathcal{A}^\diamondsuit_d \cap \mathcal{A}^\heartsuit_d}$ is the fundamental cycle by Lemma 6.11(2). Finally,

$$\begin{aligned}
&(\mathrm{id}, \mathrm{Fr}_{\mathcal{M}^\heartsuit_d})^!\zeta^\heartsuit = (\mathrm{id}, \mathrm{Fr}_{\mathcal{M}^\heartsuit_d})^!(\Pi^\mu \times \Pi^\mu)^![\mathrm{Hk}'^r_{G,d}]\\
= \quad & (\theta^\mu \times \theta^\mu)^!(\mathrm{id}, \mathrm{Fr}_{H_d})^![\mathrm{Hk}'^r_{G,d}] \quad \text{(Theorem A.10)}\\
= \quad & (\theta^\mu \times \theta^\mu)^![\mathrm{Sht}'^r_{G,d}] \quad \text{(Lemma 6.14(2))}
\end{aligned}$$

which is (6.35). This finishes the proof of (6.16).

6.4. **Some dimension calculation.** In this subsection, we give the proofs of several lemmas we stated previously concerning the dimensions of certain moduli stacks.

6.4.1. *Proof of Lemma 6.10*(1). In the diagram (6.26), when the divisors of the ϕ_i are disjoint from the divisors of the horizontal maps, namely the x_i's, the diagram is uniquely determined by its left column $\phi_0 : \mathcal{E}_0 \to \mathcal{E}'_0$ and top row. Therefore we have

$$\mathrm{Hk}^{r,\circ}_{G,d} = (H_d \times_{\mathrm{Bun}_G} \mathrm{Hk}^r_G)|_{(X_d \times X^r)^\circ}.$$

Since H_d is smooth of pure dimension $2d+3g-3$ by Lemma 6.8, and the map $p_0 : \mathrm{Hk}^r_G \to \mathrm{Bun}_G$ is smooth of relative dimension $2r$, we see that $H_d \times_{\mathrm{Bun}_G} \mathrm{Hk}^r_G$ is smooth of pure dimension $2d+2r+3g-3$.

6.4.2. *Proof of Lemma 6.10*(2). Over $(X_d \times X^r)^\circ$, we have $\dim \mathrm{Hk}^{r,\circ}_{G,d} = 2d+2r+3g-3$, therefore the generic fiber of s has dimension $d+r+3g-3$. By the semicontinuity of fiber dimensions, it suffices to show that the geometric fibers of s have dimension $\le d+r+3g-3$. We will actually show that the geometric fibers of the map $(s, p_0) : \mathrm{Hk}^r_{G,d} \to X_d \times X^r \times \mathrm{Bun}_G$ sending the diagram (6.26) to $(D; x_i; \mathcal{E}'_r)$ have dimension $\le d+r$.

We present $\mathrm{Hk}^r_{G,d}$ as the quotient of $\mathrm{Hk}^\mu_{2,d}/\operatorname{Pic}_X$ with $\mu = \mu^r_+$. Therefore a point in $\mathrm{Hk}^r_{G,d}$ is a diagram of the form (6.26) with all arrows f_i, f'_i pointing to the right.

Let $(D; \underline{x} = (x_i)) \in X_d \times X^r$ and $\mathcal{E}'_r \in \mathrm{Bun}_G$ be geometric points. For notational simplicity we base change the whole situation to the field of definition of this point without changing notation.

Let $H_{D,\underline{x},\mathcal{E}'_r}$ be the fiber of (s,p_0) over $(D;x_i;\mathcal{E}'_r)$. We consider the scheme $H'=H'_{D,\underline{x},\mathcal{E}'_r}$ classifying commutative diagrams

$$\begin{tikzcd} \mathcal{E}_0 \arrow[r, "f_1"] \arrow[d, "\phi_0"] & \mathcal{E}_1 \arrow[r, "f_2"] & \cdots \arrow[r, "f_r"] & \mathcal{E}_r \arrow[d, "\phi_r"] \\ \mathcal{E}'_0 \arrow[r, "f'_1"] & \mathcal{E}'_1 \arrow[r, "f'_2"] & \cdots \arrow[r, "f'_r"] & \mathcal{E}'_r \end{tikzcd} \tag{6.36}$$

where $\mathrm{div}(\det\phi_0)=D=\mathrm{div}(\det\phi_r)$ and $\mathrm{div}(\det f_i)=x_i=\mathrm{div}(\det f'_i)$. The only difference between H' and $H_{D,\underline{x},\mathcal{E}'_r}$ is that we do not require the maps ϕ_i for $1\le i\le r-1$ to exist (they are unique if exist). There is a natural embedding $H_{D,\underline{x},\mathcal{E}'_r}\hookrightarrow H'$, and it suffices to show that $\dim(H')\le d+r$. We isolate this part of the argument into a separate Lemma below, because it will be used in another proof. This finishes the proof of Lemma 6.10(2).

Lemma 6.15. *Consider the scheme $H'=H'_{D,\underline{x},\mathcal{E}'_r}$ introduced in the proof of Lemma 6.10(2). We have* $\dim H'=d+r$.

Proof. We only give the argument for the essential case where all x_i are equal to the same point x and $D=dx$. The general case can be reduced to this case by factorizing H' into a product indexed by points that appear in $|D|\cup\{x_1,\cdots,x_r\}$. Let $\mathrm{Gr}_{1^r,d}$ be the iterated version of the affine Schubert variety classifying chains of lattices $\Lambda_0\subset\Lambda_1\subset\Lambda_2\subset\cdots\subset\Lambda_r\subset\Lambda'_r=\mathcal{O}_x^2$ in F_x^2 where all inclusions have colength 1 except for the last one, which has colength d. Similarly let $\mathrm{Gr}_{d,1^r}$ be the iterated affine Schubert variety classifying chains of lattices $\Lambda_0\subset\Lambda'_0\subset\Lambda'_1\subset\cdots\subset\Lambda'_r=\mathcal{O}_x^2$ in F_x^2 where the first inclusion has colength d and all other inclusions have colength 1. Let $\mathrm{Gr}_{d+r}\subset\mathrm{Gr}_{G,x}$ be the affine Schubert variety classifying $\mathcal{O}_x$-lattices $\Lambda\subset\mathcal{O}_x^2$ with colength $d+r$. We have natural maps $\pi:\mathrm{Gr}_{1^r,d}\to\mathrm{Gr}_{d+r}$ and $\pi':\mathrm{Gr}_{d,1^r}\to\mathrm{Gr}_{d+r}$ sending the lattice chains to Λ_0. By the definition of H', after choosing a trivialization of $\mathcal{E}'_r$ in the formal neighborhood of x, we have an isomorphism

$$H'\cong\mathrm{Gr}_{1^r,d}\times_{\mathrm{Gr}_{d+r}}\mathrm{Gr}_{d,1^r}. \tag{6.37}$$

Since π and π' are surjective, therefore $\dim H'\ge\dim\mathrm{Gr}_{d+r}=d+r$.

Now we show $\dim H'\le d+r$. Since the natural projections $\mathrm{Gr}_{1^{d+r}}\to\mathrm{Gr}_{1^r,d}$ and $\mathrm{Gr}_{1^{d+r}}\to\mathrm{Gr}_{d,1^r}$ are surjective, it suffices to show that $\dim(\mathrm{Gr}_{1^{d+r}}\times_{\mathrm{Gr}_{d+r}}\mathrm{Gr}_{1^{d+r}})\le d+r$. In other words, letting $m=d+r$, we have to show that $\pi_m:\mathrm{Gr}_{1^m}\to\mathrm{Gr}_m$ is a semismall map. This is a very special case of the semismallness of convolution maps in the geometric Satake equivalence, and we shall give a direct argument. The scheme Gr_m is stratified into Y_m^i $(0\le i\le[m/2])$ where Y_m^i classifies those $\Lambda\subset\mathcal{O}_x^2$ such that $\mathcal{O}_x^2/\Lambda\cong\mathcal{O}_x/\varpi_x^i\oplus\mathcal{O}_x/\varpi_x^{m-i}$. We may identify Y_m^i with the open subscheme $Y_{m-2i}^0\subset\mathrm{Gr}_{m-2i}$ by sending $\Lambda\in Y_m^i$ to $\varpi_x^{-i}\Lambda\subset\mathcal{O}_x^2$, hence $\dim Y_m^i=m-2i$ and $\mathrm{codim}_{\mathrm{Gr}_m}Y_m^i=i$. We need to show that for $\Lambda\in Y_i$, $\dim\pi_m^{-1}(\Lambda)\le i$. We do this by induction on m. By definition, $\pi_m^{-1}(\Lambda)$ classifies chains $\Lambda=\Lambda_0\subset\Lambda_1\subset\cdots\subset\Lambda_m=\mathcal{O}_x^2$ each step of which has colength one. For $i=0$ such a chain is unique. For $i>0$, the choices of Λ_1 are parametrized by $\mathbb{P}^1$, and the map $\rho:\pi_m^{-1}(\Lambda)\to\mathbb{P}^1$ recording Λ_1 has fibers $\pi_{m-1}^{-1}(\Lambda_1)$. Either $\mathcal{O}_x^2/\Lambda_1\cong\mathcal{O}_x/\varpi_x^{i-1}\oplus\mathcal{O}_x/\varpi_x^{m-i}$, in which case $\dim\rho^{-1}(\Lambda_1)=\dim\pi_{m-1}^{-1}(\Lambda_1)\le i-1$ by inductive hypothesis, or $\mathcal{O}_x^2/\Lambda_1\cong\mathcal{O}_x/\varpi_x^{i}\oplus\mathcal{O}_x/\varpi_x^{m-i-1}$ (which happens for exactly one Λ_1), in which case $\dim\rho^{-1}(\Lambda_1)=\dim\pi_{m-1}^{-1}(\Lambda_1)\le i$. These imply that $\dim\pi_m^{-1}(\Lambda)\le i$. The lemma is proved.

□

6.4.3. *Proof of Lemma 6.10*(3). We denote $\mathrm{Hk}^\mu_{\mathcal{M}^\diamond,d}-\mathrm{Hk}^{\mu,\circ}_{\mathcal{M}^\diamond,d}$ by $\partial\mathrm{Hk}^\mu_{\mathcal{M}^\diamond,d}$. By Lemma 6.2 and Lemma 6.3, $\mathrm{Hk}^\mu_{\mathcal{M}^\diamond,d}\cong\widehat{X}'_d\times_{\mathrm{Pic}^d_X}B_{r,d}$, where $B_{r,d}$ classifies $(r+1)$-triples of divisors $(D_0,D_1,\cdots,D_r)$ of degree d on X', such that for each $1\le i\le r$, D_i is obtained from D_{i-1} by changing some point $x'_i\in D_{i-1}$ to $\sigma(x'_i)$. In particular, all D_i have the same image $D_b:=\pi(D_i)\in X_d$. We denote a point in $\mathrm{Hk}^\mu_{\mathcal{M}^\diamond,d}$ by $z=(\mathcal{L},\alpha,D_0,\cdots,D_r)\in\widehat{X}'_d\times_{\mathrm{Pic}^d_X}B_{r,d}$, where $(\mathcal{L},\alpha)\in\widehat{X}'_d$ denotes a line bundle $\mathcal{L}$ on X' and a section α of it, together with an isomorphism $\mathrm{Nm}(\mathcal{L})\cong\mathcal{O}_X(D_b)$. Therefore both $\mathrm{Nm}(\alpha)$ and 1 give sections of $\mathcal{O}_X(D_b)$. The image of z under $\mathrm{Hk}^\mu_{\mathcal{M}^\diamond,d}\to\mathcal{A}_d\xrightarrow{\delta}\widehat{X}_d$ is the pair $(\mathcal{O}_X(D_b),\mathrm{Nm}(\alpha)-1)$. Therefore $z\in\partial\mathrm{Hk}^\mu_{\mathcal{M}^\diamond,d}$ if and only if $\operatorname{div}(\operatorname{Nm}(\alpha)-1)$ contains $\pi(x'_i)$ for some $1 \le i \le r$ ($\operatorname{Nm}(\alpha)=1$ is allowed). Since $x'_i \in D_{i-1}$, we have $\pi(x'_i) \in \pi(D_{i-1}) = D_b$, therefore $\pi(x'_i)$ also appears in the divisor of $\operatorname{Nm}(\alpha)$. So we have two cases: either $\alpha = 0$ or $\operatorname{div}(\operatorname{Nm}(\alpha))$ shares a common point with D_b.

In the former case, z is contained in $\operatorname{Pic}^d_{X'} \times_{\operatorname{Pic}^d_X} B_{r,d}$ which has dimension $g-1+d < 2d-g+1$ since $d \ge 2g$.

In the latter case, the image of z in $\mathcal{A}_d$ lies in the subscheme $\mathcal{C}_d \subset X_d \times_{\operatorname{Pic}^d_X} X_d$ consisting of triples $(D_1, D_2, \gamma : \mathcal{O}(D_1) \cong \mathcal{O}(D_2))$ such that the divisors D_1 and D_2 have a common point. There is a surjection $X \times (X_{d-1} \times_{\operatorname{Pic}^{d-1}_X} X_{d-1}) \to \mathcal{C}_d$ which implies that $\dim \mathcal{C}_d \le 1 + 2(d-1) - g + 1 = 2d - g$. Here we are using the fact that $d - 1 \ge 2g - 1$ to compute the dimension of $X_{d-1} \times_{\operatorname{Pic}^{d-1}_X} X_{d-1}$. The conclusion is that in the latter case, z lies in the preimage of $\mathcal{C}_d$ in $\operatorname{Hk}^\mu_{\mathcal{M}^\diamond,d}$, which has dimension equal to $\dim \mathcal{C}_d$ (because $\operatorname{Hk}^\mu_{\mathcal{M},d} \to \mathcal{A}_d$ is finite when restricted to $\mathcal{C}_d \subset X_d \times_{\operatorname{Pic}^d_X} X_d$), which is $\le 2d - g < 2d - g + 1$.

Combining the two cases we conclude that $\dim \partial \operatorname{Hk}^\mu_{\mathcal{M}^\diamond,d} < 2d - g + 1 = \dim \operatorname{Hk}^\mu_{\mathcal{M}^\diamond,d}$.

6.4.4. *Proof of Lemma 5.9.* Let $\underline{x} = (x_1, \cdots, x_r) \in X^r$ be a geometric point. Let $\operatorname{Sht}^r_G(h_D)_{\underline{x}}$ be the fiber of $\operatorname{Sht}^r_G(h_D)$ over $\underline{x}$. When $\underline{x}$ is disjoint from $|D|$, $\overleftarrow{p} : \operatorname{Sht}^r_G(h_D)_{\underline{x}} \to \operatorname{Sht}^r_{G,\underline{x}}$ is étale, hence in this case $\dim \operatorname{Sht}^r_G(h_D)_{\underline{x}} = r$. By semicontinuity of fiber dimensions, it remains to show that $\dim \operatorname{Sht}^r_G(h_D)_{\underline{x}} \le r$ for all geometric points $\underline{x}$ over closed points of X^r. To simplify notation we assume $x_i \in X(k)$. The general case can be argued similarly.

We use the same notation as in §6.4.2. In particular, we will use $\operatorname{Hk}^r_{G,d}$, and think of it as $\operatorname{Hk}^\mu_{2,d}/\operatorname{Pic}_X$ with $\mu = \mu^r_+$. Let H_D be the fiber over D of $H_d \to X_d$ sending $(\phi : \mathcal{E} \hookrightarrow \mathcal{E}')$ to the divisor of $\det(\phi)$. Let $\operatorname{Hk}^r_{D,\underline{x}}$ be the fiber of $s : \operatorname{Hk}^r_{G,d} \to X_d \times X^r$ over $(D; \underline{x})$.

Taking the fiber of the diagram (6.33) over $\underline{x}$ we get a Cartesian diagram

$$\begin{tikzcd} \operatorname{Sht}^r_G(h_D)_{\underline{x}} \arrow[r] \arrow[d] & \operatorname{Hk}^r_{D,\underline{x}} \arrow[d, "(p_0,p_r)"] \\ H_D \arrow[r, "(\mathrm{id},\mathrm{Fr})"] & H_D \times H_D \end{tikzcd} \tag{6.38}$$

For each divisor $D' \le D$ such that $D - D'$ has even coefficients, we have a closed embedding $H_{D'} \hookrightarrow H_D$ sending $(\phi : \mathcal{E} \to \mathcal{E}') \in H_{D'}$ to $\mathcal{E} \xrightarrow{\phi} \mathcal{E}' \hookrightarrow \mathcal{E}'(\frac{1}{2}(D - D'))$. Let $H_{D,\le D'}$ be the image of this embedding. Also let $H_{D,D'} = H_{D,\le D'} - \cup_{D''<D'} H_{D,\le D''}$. Then $\{H_{D,D'}\}$ give a stratification of H_D indexed by divisors $D' \le D$ such that $D - D'$ is even. We may restrict the diagram (6.38) to $H_{D,D'} \times H_{D,D'} \hookrightarrow H_D \times H_D$ and get a Cartesian diagram

$$\begin{tikzcd} \operatorname{Sht}^r_G(h_D)_{D',\underline{x}} \arrow[r] \arrow[d] & \operatorname{Hk}^r_{D,D',\underline{x}} \arrow[d, "(p_0,p_r)"] \\ H_{D,D'} \arrow[r, "(\mathrm{id},\mathrm{Fr})"] & H_{D,D'} \times H_{D,D'} \end{tikzcd} \tag{6.39}$$

We will show that $\dim \operatorname{Sht}^r_G(h_D)_{D',\underline{x}} \le r$ for each $D' \le D$ and $D - D'$ even.

The embedding $H_{D'} \hookrightarrow H_D$ above restricts to an isomorphism $H_{D',D'} \cong H_{D,D'}$. Similarly we have an isomorphism $\operatorname{Hk}^r_{D',D',\underline{x}} \cong \operatorname{Hk}^r_{D,D',\underline{x}}$ sending a diagram of the form (6.26) to the diagram of the same shape with each $\mathcal{E}'_i$ changed to $\mathcal{E}'(\frac{1}{2}(D - D'))$. Therefore we have $\operatorname{Sht}_G(h_{D'})_{D',\underline{x}} \cong \operatorname{Sht}_G(h_D)_{D',\underline{x}}$, and it suffices to show that the open stratum $\operatorname{Sht}_G(h_D)_{D,\underline{x}}$ has dimension at most r. This way we reduce to treating the case $D' = D$.

Let $\widetilde{D} = D + \underline{x} = D + x_1 + \cdots + x_r \in X_{d+r}$ be the effective divisor of degree $d + r$. Let $\operatorname{Bun}_{G,\widetilde{D}}$ be the moduli stack of G bundles with a trivialization over $\widetilde{D}$. A point of $\operatorname{Bun}_{G,\widetilde{D}}$ is a pair $(\mathcal{E}', \tau : \mathcal{E}'_{\widetilde{D}} \cong \mathcal{O}^2_{\widetilde{D}})$ (where $\mathcal{E}'$ is a vector bundle of rank two over X) up to the action of $\operatorname{Pic}_X(\widetilde{D})$ (line bundles with a trivialization over $\widetilde{D}$). There is a map $h : \operatorname{Bun}_{G,\widetilde{D}} \to H_{D,D}$ sending $(\mathcal{E}', \tau)$ to $(\phi : \mathcal{E} \hookrightarrow \mathcal{E}')$ where $\mathcal{E}$ is the preimage of the first copy of $\mathcal{O}_D$ under the surjective map $\mathcal{E}' \twoheadrightarrow \mathcal{E}'_D \xrightarrow{\tau} \mathcal{O}^2_{\widetilde{D}} \twoheadrightarrow \mathcal{O}^2_D$. Let $B_D \subset \operatorname{Res}^{\mathcal{O}_D}_k G = \operatorname{PGL}_2(\mathcal{O}_D)$ be the subgroup stabilizing the first copy of $\mathcal{O}^2_D$, and let $\widetilde{B}_D \subset \operatorname{Res}^{\mathcal{O}_{\widetilde{D}}}_k G = \operatorname{PGL}_2(\mathcal{O}_{\widetilde{D}})$ be the preimage of B_D. Then h is a $\widetilde{B}_D$-torsor. In particular, $H_{D,D}$ is smooth, and the map h is also smooth. Since smooth maps have sections étale locally, we may choose an étale surjective map $\omega : Y \to H_{D,D}$ and a map $s : Y \to \mathrm{Bun}_{G,\widetilde{D}}$ such that $hs = \omega$.

Let $W = \mathrm{Hk}^r_{D,D,\underline{x}} \times_{H_{D,D}} Y$ (using the projection $\gamma_r : \mathrm{Hk}^r_{D,D,\underline{x}} \to H_{D,D}$). We claim that the projection $W \to Y$ is in fact a trivial fibration. In fact, let T be the moduli space of diagrams of the form (6.26) with $\mathcal{E}'_r = \mathcal{O}^2_X$ and $\mathcal{E}_r = \mathcal{O}_X(-D) \oplus \mathcal{O}_X$ and ϕ_r is the obvious embedding $\mathcal{E}_r \hookrightarrow \mathcal{E}'_r$. In such a diagram all $\mathcal{E}_i$ and $\mathcal{E}'_i$ contain $\mathcal{E}'_r(-\widetilde{D})$, therefore it contains the same amount of information as the diagram formed by the torsion sheaves $\mathcal{E}_i/\mathcal{E}'_r(-\widetilde{D})$ and $\mathcal{E}'_i/\mathcal{E}'_r(-\widetilde{D})$. For a point $y \in Y$ with image $(\phi_r : \mathcal{E}_r \hookrightarrow \mathcal{E}'_r) \in H_{D,D}$, $s(y) \in \mathrm{Bun}_{G,\widetilde{D}}$ gives a trivialization of $\mathcal{E}'_r|_{\widetilde{D}}$. Therefore, completing ϕ_r into a diagram of the form (6.26) is the same as completing the standard point $(\mathcal{E}_r = \mathcal{O}_X(-D) \oplus \mathcal{O}_X \hookrightarrow \mathcal{O}^2_X) \in H_{D,D}$ into such a diagram. This shows that $W \cong Y \times T$. We have a diagram

$$\begin{tikzcd}
\mathcal{U} \arrow[r] \arrow[d, "u"] & W \arrow[r, "\sim"] \arrow[d, "w"] & Y \times T \arrow[r, "\omega\times\mathrm{id}"] & H_{D,D} \times T \\
\mathrm{Sht}^r_G(h_D)_{D,\underline{x}} \arrow[r] \arrow[d] & \mathrm{Hk}^r_{D,D,\underline{x}} \arrow[d, "(\gamma_0,\gamma_r)"] & & \\
H_{D,D} \arrow[r, "(\mathrm{id},\mathrm{Fr})"] & H_{D,D} \times H_{D,D} & &
\end{tikzcd}$$

where $\mathcal{U}$ is defined so that the top square is Cartesian. The outer Cartesian diagram fits into the situation of [16, Lemme 2.13], and we have used the same notation as in *loc.cit*, except that we take $Z = H_{D,D}$. Applying *loc. cit.*, we conclude that the map $\mathcal{U} \to T$ is étale. Since $w : W \to \mathrm{Hk}^r_{D,D,\underline{x}}$ is étale surjective, so is $u : \mathcal{U} \to \mathrm{Sht}^r_G(h_D)_{D,\underline{x}}$. Therefore $\mathrm{Sht}^r_G(h_D)_{D,\underline{x}}$ is étale locally isomorphic to T, and in particular they have the same dimension.

It remains to show that $\dim T \le r$. Recall the moduli space $H' = H'_{D,\underline{x},\mathcal{E}'_r}$ introduced in the proof of Lemma 6.10(2) classifying diagrams of the form (6.36). Here we fix $\mathcal{E}'_r = \mathcal{O}^2_X$. Let T' be subscheme of H' consisting of diagrams of the form (6.36) where $(\phi_r : \mathcal{E}_r \hookrightarrow \mathcal{E}'_r)$ is fixed to be $(\mathcal{E}_r = \mathcal{O}_X(-D) \oplus \mathcal{O}_X \hookrightarrow \mathcal{O}^2_X)$. Then we have a natural embedding $T \hookrightarrow T'$, and it suffices to show that $\dim T' \le r$. Again we treat only the case where D and $\underline{x}$ are both supported at a single point $x \in X$. The general case easily reduces to this by factorizing T' into a product indexed by points in $|D| \cup \{x_1, \cdots, x_r\}$.

Let $\mathrm{Gr}_d \subset \mathrm{Gr}_{G,x}$ be the affine Schubert variety classifying lattices $\Lambda \subset \mathcal{O}^2_x$ of colength d. Let $\mathrm{Gr}^\heartsuit_d \subset \mathrm{Gr}_d$ be the open Schubert stratum consisting of lattices $\Lambda \subset \mathcal{O}^2_x$ such that $\mathcal{O}^2_x/\Lambda \cong \mathcal{O}_x/\varpi^d_x$ (ϖ_x is a uniformizer at x). We have a natural projection $\rho : H' \to \mathrm{Gr}_d$ sending the diagram (6.36) to $\Lambda := \mathcal{E}_r|_{\mathrm{Spec}\,\mathcal{O}_x} \hookrightarrow \mathcal{E}'_r|_{\mathrm{Spec}\,\mathcal{O}_x} = \mathcal{O}^2_x$. Then T' is the fiber of ρ at the point $\Lambda = \varpi^d_x\mathcal{O}_x \oplus \mathcal{O}_x$. Let $H^\heartsuit = \rho^{-1}(\mathrm{Gr}^\heartsuit_d)$. There is a natural action of the positive loop group L^+_xG on both H' and Gr_d making ρ equivariant under these actions. Since the action of L^+_xG on $\mathrm{Gr}^\heartsuit_d$ is transitive, all fibers of ρ over points of $\mathrm{Gr}^\heartsuit_d$ have the same dimension, i.e.,

$$\dim T' = \dim H^\heartsuit - \dim \mathrm{Gr}^\heartsuit_d = \dim H^\heartsuit - d. \tag{6.40}$$

By Lemma 6.15, $\dim H' = d + r$. Therefore $\dim H^\heartsuit = d + r$ and $\dim T' \le r$ by (6.40). We are done.

7. Cohomological spectral decomposition

In this section, we give a decomposition of the cohomology of Sht^r_G under the action of the Hecke algebra $\mathscr{H}$, generalizing the classical spectral decomposition for the space of automorphic forms. The main result is Theorem 7.14 which shows that $\mathrm{H}^{2r}_c(\mathrm{Sht}^r_{G,\overline{k}}, \mathbb{Q}_\ell)$ is an orthogonal direct sum of an Eisenstein part and finitely many (generalized) Hecke eigenspaces. We then use a variant of such a decomposition for Sht'^r_G to make a decomposition for the Heegner-Drinfeld cycle.

7.1. **Cohomology of the moduli stack of Shtukas.**

7.1.1. *Truncation of* Bun_G *by index of instability.* For a rank two vector bundle $\mathcal{E}$ over X, we define its *index of instability* to be

$$\mathrm{inst}(\mathcal{E}) := \max\{2\deg\mathcal{L} - \deg\mathcal{E}\},$$

where $\mathcal{L}$ runs over line subbundle of $\mathcal{E}$. When $\mathrm{inst}(\mathcal{E}) > 0$, $\mathcal{E}$ is called unstable, in which case there is a unique line subbundle $\mathcal{L} \subset \mathcal{E}$ such that $\deg\mathcal{L} > \frac{1}{2}\deg\mathcal{E}$. We call this line subbundle the *maximal line subbundle* of $\mathcal{E}$. Note that there is a constant $c(g)$ depending only on the genus g of X such that $\mathrm{inst}(\mathcal{E}) \geq c(g)$ for all rank two vector bundles $\mathcal{E}$ on X.

The function $\mathrm{inst} : \mathrm{Bun}_2 \to \mathbb{Z}$ is upper semi-continuous, and descends to a function $\mathrm{inst} : \mathrm{Bun}_G \to \mathbb{Z}$. For an integer a, $\mathrm{inst}^{-1}((-\infty, a]) =: \mathrm{Bun}_G^{\leq a}$ is an open substack of Bun_G of finite type over k.

7.1.2. *Truncation of* Sht_G^r *by index of instability.* For Sht_G^r we define a similar stratification by the index of instability of the various $\mathcal{E}_i$. We choose μ as in §5.1.2 and present Sht_G^r as $\mathrm{Sht}_2^\mu / \mathrm{Pic}_X(k)$.

Consider the set $\mathcal{D}$ of functions $d : \mathbb{Z}/r\mathbb{Z} \to \mathbb{Z}$ such that $d(i) - d(i-1) = \pm 1$ for all i. There is a partial order on $\mathcal{D}$ by pointwise comparison.

For any $d \in \mathcal{D}$, let $\mathrm{Sht}_2^{\mu,\leq d}$ be the open substack of Sht_2^μ consisting of those $(\mathcal{E}_i; x_i; f_i)$ such that $\mathrm{inst}(\mathcal{E}_i) \leq d(i)$. Then each $\mathrm{Sht}_2^{\mu,\leq d}$ is preserved by the $\mathrm{Pic}_X(k)$-action, and we define $\mathrm{Sht}_G^{\mu,\leq d} := \mathrm{Sht}_2^{\mu,\leq d}/\mathrm{Pic}_X(k)$, an open substack of Sht_G^r of finite type. If we change μ to μ', the canonical isomorphism $\mathrm{Sht}_G^\mu \cong \mathrm{Sht}_G^{\mu'}$ in Lemma 5.6 preserves the G-torsors $\mathcal{E}_i$, therefore the open substacks $\mathrm{Sht}_G^{\mu,\leq d}$ and $\mathrm{Sht}_G^{\mu',\leq d}$ correspond to each other under the isomorphism. This shows that $\mathrm{Sht}_G^{\mu,\leq d}$ is canonically independent of the choice of μ, and we will simply denote it by $\mathrm{Sht}_G^{\leq d}$.

In the sequel, the superscript on Sht_G will be reserved for the truncation parameters $d \in \mathcal{D}$, and we will omit r from the superscripts. In the rest of the section, Sht_G means Sht_G^r.

Define $\mathrm{Sht}_G^d := \mathrm{Sht}_G^{\leq d} - \cup_{d'<d}\mathrm{Sht}_G^{\leq d'}$. This is a locally closed substack of Sht_G of finite type classifying Shtukas $(\mathcal{E}_i; x_i; f_i)$ with $\mathrm{inst}(\mathcal{E}_i) = d(i)$ for all i. A priori we could define Sht_G^d for any function $d : \mathbb{Z}/r\mathbb{Z} \to \mathbb{Z}$; however, only for those $d \in \mathcal{D}$ is Sht_G^d nonempty, because for $(\mathcal{E}_i; x_i; f_i) \in \mathrm{Sht}_2^\mu$, $\mathrm{inst}(\mathcal{E}_i) = \mathrm{inst}(\mathcal{E}_{i-1}) \pm 1$. The locally closed substacks $\{\mathrm{Sht}_G^d\}_{d\in\mathcal{D}}$ give a stratification of Sht_G.

7.1.3. *Cohomology of* Sht_G. Let $\pi_G^{\leq d} : \mathrm{Sht}_G^{\leq d} \to X^r$ be the restriction of π_G, and similarly define $\pi_G^{\leq d}$ and π_G^d. For $d \leq d' \in \mathcal{D}$ we have a map induced by the open inclusion $\mathrm{Sht}_G^{\leq d} \hookrightarrow \mathrm{Sht}_G^{\leq d'}$:

$$\iota_{d,d'} : \mathbf{R}\pi_{G,!}^{\leq d}\mathbb{Q}_\ell \longrightarrow \mathbf{R}\pi_{G,!}^{\leq d'}\mathbb{Q}_\ell$$

The total cohomology $\mathrm{H}_c^*(\mathrm{Sht}_G \otimes_k \overline{k})$ is defined as the inductive limit

$$\mathrm{H}_c^*(\mathrm{Sht}_G \otimes_k \overline{k}) := \varinjlim_{d\in\mathcal{D}} \mathrm{H}_c^*(\mathrm{Sht}_G^{\leq d} \otimes_k \overline{k}) = \varinjlim_{d\in\mathcal{D}} \mathrm{H}^*(X^r \otimes_k \overline{k}, \mathbf{R}\pi_{G,!}^{\leq d}\mathbb{Q}_\ell).$$

7.1.4. *The action of Hecke algebra on the cohomology of* Sht_G. For each effective divisor D of X, we have defined in §5.3.1 a self-correspondence $\mathrm{Sht}_G(h_D)$ of Sht_G over X^r.

For any $d \in \mathcal{D}$, let ${}^{\leq d}\mathrm{Sht}_G(h_D) \subset \mathrm{Sht}_G(h_D)$ be the preimage of $\mathrm{Sht}_G^{\leq d}$ under $\overleftarrow{p}$. For a point $(\mathcal{E}_i \hookrightarrow \mathcal{E}'_i)$ of ${}^{\leq d}\mathrm{Sht}_G(h_D)$, we have $\mathrm{inst}(\mathcal{E}_i) \leq d(i)$, hence $\mathrm{inst}(\mathcal{E}'_i) \leq d(i) + \deg D$. Therefore the image of ${}^{\leq d}\mathrm{Sht}_G(h_D)$ under $\overrightarrow{p}$ lies in $\mathrm{Sht}_G^{\leq d+\deg D}$. For any $d' \geq d + \deg D$, we may view ${}^{\leq d}\mathrm{Sht}_G(h_D)$ as a correspondence between $\mathrm{Sht}_G^{\leq d}$ and $\mathrm{Sht}_G^{\leq d'}$ over X^r. By Lemma 5.9, $\dim \mathrm{Sht}_G(h_D) = \dim \mathrm{Sht}_G = 2r$, the fundamental cycle of ${}^{\leq d}\mathrm{Sht}_G(h_D)$ gives a cohomological correspondence between the constant sheaf on $\mathrm{Sht}_G^{\leq d}$ and the constant sheaf on $\mathrm{Sht}_G^{\leq d'}$ (see §A.4.1), and induces a map

$$C(h_D)_{d,d'} : \mathbf{R}\pi_{G,!}^{\leq d}\mathbb{Q}_\ell \longrightarrow \mathbf{R}\pi_{G,!}^{\leq d'}\mathbb{Q}_\ell. \tag{7.1}$$

Here we are using the fact that ${}^{\leq d}\overleftarrow{p} : {}^{\leq d}\mathrm{Sht}_G(h_D) \to \mathrm{Sht}_G^{\leq d}$ is proper (which is necessary for the construction (A.25)), which follows from the properness of $\overleftarrow{p} : \mathrm{Sht}_G(h_D) \to \mathrm{Sht}_G$ by Lemma 5.8.

For any $e \geq d$ and $e' \geq e + \deg D$ and $e' \geq d'$, we have a commutative diagram

$$\begin{tikzcd}
\mathbf{R}\pi^{\leq d}_{G,!}\mathbb{Q}_\ell \arrow[r, "C(h_D)_{d,d'}"] \arrow[d, "\iota_{d,e}"] & \mathbf{R}\pi^{\leq d'}_{G,!}\mathbb{Q}_\ell \arrow[d, "\iota_{d',e'}"] \\
\mathbf{R}\pi^{\leq e}_{G,!}\mathbb{Q}_\ell \arrow[r, "C(h_D)_{e,e'}"] & \mathbf{R}\pi^{\leq e'}_{G,!}\mathbb{Q}_\ell
\end{tikzcd}$$

which follows from the definition of cohomological correspondences. Taking $\mathrm{H}^*(X^r \otimes_k \overline{k}, -)$ and taking inductive limit over d and e, we get an endomorphism of $\mathrm{H}^*_c(\mathrm{Sht}_G \otimes_k \overline{k})$

$$\begin{aligned} C(h_D) \quad : \quad & \mathrm{H}^*_c(\mathrm{Sht}_G \otimes_k \overline{k}) = \varinjlim_{d\in\mathcal{D}} \mathrm{H}^*(X^r \otimes_k \overline{k}, \mathbf{R}\pi^{\leq d}_{G,!}\mathbb{Q}_\ell) \\ & \xrightarrow{\varinjlim C(h_D)_{d,d'}} \varinjlim_{d'\in\mathcal{D}} \mathrm{H}^*(X^r \otimes_k \overline{k}, \mathbf{R}\pi^{\leq d'}_{G,!}\mathbb{Q}_\ell) = \mathrm{H}^*_c(\mathrm{Sht}_G \otimes_k \overline{k}). \end{aligned}$$

The following result is a cohomological analog of Proposition 5.10.

Proposition 7.1. *The assignment $h_D \mapsto C(h_D)$ gives a ring homomorphism for each $i \in \mathbb{Z}$*

$$C : \mathscr{H} \longrightarrow \mathrm{End}(\mathrm{H}^i_c(\mathrm{Sht}_G \otimes_k \overline{k})).$$

Proof. The argument is similar to that of Proposition 5.10, for this reason we only give a sketch here. For two effective divisors D and D', we need to check that the action of $C(h_D h_{D'})$ is the same as the composition $C(h_D) \circ C(h_{D'})$.

Let $d, d^\dagger$ and $d' \in \mathcal{D}$ satisfy $d^\dagger \geq d + \deg D'$ and $d' \geq d^\dagger + \deg D$, then the map

$$C(h_D)_{d^\dagger,d'} \circ C(h_{D'})_{d,d^\dagger} : \mathbf{R}\pi^{\leq d}_{G,!}\mathbb{Q}_\ell \longrightarrow \mathbf{R}\pi^{\leq d^\dagger}_{G,!}\mathbb{Q}_\ell \longrightarrow \mathbf{R}\pi^{\leq d'}_{G,!}\mathbb{Q}_\ell$$

is induced from a cohomological correspondence ζ between the constant sheaves on $\mathrm{Sht}^{\leq d}_G$ and on $\mathrm{Sht}^{\leq d'}_G$ supported on ${}^{\leq d}\mathrm{Sht}_G(h_D) * {}^{\leq d^\dagger}\mathrm{Sht}_G(h_{D'}) := {}^{\leq d}\mathrm{Sht}_G(h_D) \times_{\overrightarrow{p},\mathrm{Sht}_G,\overleftarrow{p}} {}^{\leq d^\dagger}\mathrm{Sht}_G(h_{D'})$; i.e., $\zeta \in \mathrm{H}^{\mathrm{BM}}_{4r}({}^{\leq d}\mathrm{Sht}_G(h_D) * {}^{\leq d^\dagger}\mathrm{Sht}_G(h_{D'}) \otimes_k \overline{k})$.

On the other hand, the Hecke function $h_D h_{D'}$ is a linear combination of h_E where $E \leq D+D'$ and $D + D' - E$ is even. Since $d \in \mathcal{D}$ and $d' \geq d + \deg D + \deg D'$, the map

$$C(h_D h_{D'})_{d,d'} : \mathbf{R}\pi^{\leq d}_{G,!}\mathbb{Q}_\ell \longrightarrow \mathbf{R}\pi^{\leq d'}_{G,!}\mathbb{Q}_\ell$$

is induced from a cohomological correspondence ξ between the constant sheaves on $\mathrm{Sht}^{\leq d}_G$ and on $\mathrm{Sht}^{\leq d'}_G$ supported on the union of ${}^{\leq d}\mathrm{Sht}_G(h_E)$ for $E \leq D + D'$ and $D + D' - E$ even, i.e., supported on ${}^{\leq d}\mathrm{Sht}_G(h_{D+D'})$. In other words, $\xi \in \mathrm{H}^{\mathrm{BM}}_{4r}({}^{\leq d}\mathrm{Sht}_G(h_{D+D'}) \otimes_k \overline{k})$.

There is a proper map of correspondences $\theta : {}^{\leq d}\mathrm{Sht}_G(h_D) * {}^{\leq d^\dagger}\mathrm{Sht}_G(h_{D'}) \to {}^{\leq d}\mathrm{Sht}_G(h_{D+D'})$, and the action of $C(h_D)_{d^\dagger,d'} \circ C(h_{D'})_{d,d^\dagger}$ is also induced from the class $\theta_*\zeta \in \mathrm{H}^{\mathrm{BM}}_{4r}({}^{\leq d}\mathrm{Sht}_G(h_{D+D'}) \otimes_k \overline{k})$, viewed as a cohomological correspondence supported on ${}^{\leq d}\mathrm{Sht}_G(h_{D+D'})$. Let $U = X - |D| - |D'|$. It is easy to check that $\xi|_{U^r} = \theta_*\zeta|_{U^r}$ using that, over U^r, the correspondences $\mathrm{Sht}(h_D), \mathrm{Sht}(h_{D'})$ and $\mathrm{Sht}(h_{D+D'})$ are finite étale over Sht_G. By Lemma 5.9, ${}^{\leq d}\mathrm{Sht}_G(h_{D+D'}) - {}^{\leq d}\mathrm{Sht}_G(h_{D+D'})|_{U^r}$ has dimension $< 2r$, therefore $\xi = \theta_*\zeta$ holds as elements in $\mathrm{H}^{\mathrm{BM}}_{4r}({}^{\leq d}\mathrm{Sht}_G(h_{D+D'}) \otimes_k \overline{k})$, and hence $C(h_D h_{D'})_{d,d'} = C(h_D)_{d^\dagger,d'} \circ C(h_{D'})_{d,d^\dagger}$. Applying $\mathrm{H}^*(X^r \otimes_k \overline{k}, -)$ and taking inductive limit over d and d', we see that $C(h_D h_{D'}) = C(h_D) \circ C(h_{D'})$ as endomorphisms of $\mathrm{H}^*_c(\mathrm{Sht}_G \otimes_k \overline{k})$. □

7.1.5. *Notation.* For $\alpha \in \mathrm{H}^*_c(\mathrm{Sht}_G \otimes_k \overline{k})$ and $f \in \mathscr{H}$, we denote the action of $C(f)$ on α simply by $f * \alpha \in \mathrm{H}^*_c(\mathrm{Sht}_G \otimes_k \overline{k})$.

7.1.6. Cup product gives a symmetric bilinear pairing on $\mathrm{H}^*_c(\mathrm{Sht}_G \otimes_k \overline{k})$

$$(-,-) : \mathrm{H}^i_c(\mathrm{Sht}_G \otimes_k \overline{k}) \times \mathrm{H}^{4r-i}_c(\mathrm{Sht}_G \otimes_k \overline{k}) \longrightarrow \mathrm{H}^{4r}_c(\mathrm{Sht}_G \otimes_k \overline{k}) \cong \mathbb{Q}_\ell(-2r).$$

We have a cohomological analog of Lemma 5.12.

Lemma 7.2. *The action of any $f \in \mathscr{H}$ on $\mathrm{H}^*_c(\mathrm{Sht}_G \otimes_k \overline{k})$ is self-adjoint with respect to the cup product pairing.*

Proof. Since $\{h_D\}$ span $\mathscr{H}$, it suffices to show that the action of h_D is self-adjoint. From the construction of the endomorphism $C(h_D)$ of $\mathrm{H}^i_c(\mathrm{Sht}_G \otimes_k \overline{k})$, we see that for $\alpha \in \mathrm{H}^i_c(\mathrm{Sht}_G \otimes_k \overline{k})$ and $\beta \in \mathrm{H}^{4r-i}_c(\mathrm{Sht}_G \otimes_k \overline{k})$, the pairing $(h_D * \alpha, \beta)$ is the same as the pairing $([\mathrm{Sht}_G(h_D)], \overleftarrow{p}^*\alpha \cup \overrightarrow{p}^*\beta)$ (i.e, the pairing of $\overleftarrow{p}^*\alpha \cup \overrightarrow{p}^*\beta \in \mathrm{H}^{4r}_c(\mathrm{Sht}_G(h_D) \otimes_k \overline{k})$ with the fundamental class of $\mathrm{Sht}_G(h_D)$). Similarly, $(\alpha, h_D*\beta)$ is the pairing $([\mathrm{Sht}_G(h_D)], \overleftarrow{p}^*\beta \cup \overrightarrow{p}^*\alpha)$. Applying the involution τ on $\mathrm{Sht}_G(h_D)$ constructed in the proof of Lemma 5.12 that switches the two projections $\overleftarrow{p}$ and $\overrightarrow{p}$, we get

$$\Big([\mathrm{Sht}_G(h_D)], \overleftarrow{p}^*\alpha \cup \overrightarrow{p}^*\beta\Big) = \Big([\mathrm{Sht}_G(h_D)], \overleftarrow{p}^*\beta \cup \overrightarrow{p}^*\alpha\Big)$$

which is equivalent to the self-adjointness of h_D: $(h_D * \alpha, \beta) = (\alpha, h_D * \beta)$. □

7.1.7. The cycle class map gives a $\mathbb{Q}$-linear map (see §A.1.5)

$$\mathrm{cl} : \mathrm{Ch}_{c,i}(\mathrm{Sht}_G)_{\mathbb{Q}} \longrightarrow \mathrm{H}^{4r-2i}_c(\mathrm{Sht}_G \otimes_k \overline{k})(2r-i).$$

Lemma 7.3. *The map* cl *is* $\mathscr{H}$*-equivariant for any* i*.*

Proof. Since $\{h_D\}$ span $\mathscr{H}$, it suffices to show that cl intertwines the actions of h_D on $\mathrm{Ch}_{c,i}(\mathrm{Sht}_G)$ and on $\mathrm{H}^{4r-2i}_c(\mathrm{Sht}_G \otimes_k \overline{k})(2r-i)$. Let $\zeta \in \mathrm{Ch}_{c,i}(\mathrm{Sht}_G)$. By the definition of the h_D-action on $\mathrm{Ch}_{c,i}(\mathrm{Sht}_G)$, $h_D * \zeta \in \mathrm{Ch}_{c,i}(\mathrm{Sht}_G)$ is $\mathrm{pr}_{2*}((\mathrm{pr}_1^*\zeta) \cdot_{\mathrm{Sht}_G \times \mathrm{Sht}_G} (\overleftarrow{p}, \overrightarrow{p})_*[\mathrm{Sht}_G(h_D)])$. Taking its cycle class we get that $\mathrm{cl}(h_D * \zeta) \in \mathrm{H}^{4r-2i}_c(\mathrm{Sht}_G \otimes_k \overline{k})(2r-i)$ can be identified with the class

$$\overrightarrow{p}_*(\overleftarrow{p}^*\mathrm{cl}(\zeta) \cap [\mathrm{Sht}_G(h_D)]) \in \mathrm{H}_{2i}(\mathrm{Sht}_G \otimes_k \overline{k})(-i)$$

under the Poincaré duality isomorphism $\mathrm{H}^{4r-2i}_c(\mathrm{Sht}_G \otimes_k \overline{k}) \cong \mathrm{H}_{2i}(\mathrm{Sht}_G \otimes_k \overline{k})(-2r)$.

On the other hand, by (A.25), the action of h_D on $\mathrm{H}^{4r-2i}_c(\mathrm{Sht}_G \otimes_k \overline{k})$ is the composition

$$\mathrm{H}^{4r-2i}_c(\mathrm{Sht}_G \otimes_k \overline{k}) \xrightarrow{\overleftarrow{p}^*} \mathrm{H}^{4r-2i}_c(\mathrm{Sht}_G(h_D) \otimes_k \overline{k}) \xrightarrow{\cap[\mathrm{Sht}_G(h_D)]}$$
$$\mathrm{H}_{2i}(\mathrm{Sht}_G(h_D) \otimes_k \overline{k})(-2r) \xrightarrow{\overrightarrow{p}_*} \mathrm{H}_{2i}(\mathrm{Sht}_G \otimes_k \overline{k})(-2r) \cong \mathrm{H}^{4r-2i}_c(\mathrm{Sht}_G \otimes_k \overline{k}).$$

Therefore we have $\mathrm{cl}(h_D * \zeta) = h_D * \mathrm{cl}(\zeta)$. □

7.1.8. We are most interested in the middle dimensional cohomology

$$V_{\mathbb{Q}_\ell} := \mathrm{H}^{2r}_c(\mathrm{Sht}_G \otimes_k \overline{k}, \mathbb{Q}_\ell)(r).$$

This is a $\mathbb{Q}_\ell$-vector space with an action of $\mathscr{H}$. In the sequel, we simply write V for $V_{\mathbb{Q}_\ell}$.

For the purpose of proving our main theorems, it is the cohomology of Sht'_G rather than Sht_G that matters. However, for most of this section, we will study V. The main result in this section (Theorem 7.14) provides a decomposition of V into a direct sum of two $\mathscr{H}$-modules, an infinite-dimensional one called the Eisenstein part and a finite-dimensional complement. The same result holds when Sht_G is replaced by Sht'_G with the same proof. This will mention this in the final subsection §7.5 and use it to decompose the Heegner-Drinfeld cycle.

7.2. **Study of horocycles.** Let $B \subset G$ be a Borel subgroup with quotient torus $H \cong \mathbb{G}_m$. We think of H as the universal Cartan of G, which is to be distinguished with the subgroup A of G. We shall define horocycles in Sht_G corresponding to B-Shtukas.

7.2.1. Bun_B. Let $\widetilde{B} \subset \mathrm{GL}_2$ be the preimage of B. Then $\mathrm{Bun}_{\widetilde{B}}$ classifies pairs $(\mathcal{L} \hookrightarrow \mathcal{E})$ where $\mathcal{E}$ is a rank two vector bundle over X and $\mathcal{L}$ is a line subbundle of it. We have $\mathrm{Bun}_B = \mathrm{Bun}_{\widetilde{B}}/\mathrm{Pic}_X$ where Pic_X acts by simultaneous tensoring on $\mathcal{E}$ and on $\mathcal{L}$. We have a decomposition

$$\mathrm{Bun}_B = \coprod_{n \in \mathbb{Z}} \mathrm{Bun}^n_B$$

where $\mathrm{Bun}^n_B = \mathrm{Bun}^n_{\widetilde{B}}/\mathrm{Pic}_X$, and $\mathrm{Bun}^n_{\widetilde{B}}$ is the open and closed substack of $\mathrm{Bun}_{\widetilde{B}}$ classifying those $(\mathcal{L} \hookrightarrow \mathcal{E})$ such that $2\deg\mathcal{L} - \deg\mathcal{E} = n$.

7.2.2. *Hecke stack for* $\widetilde{B}$. Fix $d \in \mathcal{D}$. Choose any μ as in §5.1.2. Consider the moduli stack $\mathrm{Hk}^{\mu,d}_{\widetilde{B}}$ whose S-points classify the data $(\mathcal{L}_i \hookrightarrow \mathcal{E}_i; x_i; f_i)$ where

(1) A point $(\mathcal{E}_i; x_i; f_i) \in \mathrm{Hk}^{\mu}_2(S)$.

(2) For each $i = 0, \cdots, r$, $(\mathcal{L}_i \hookrightarrow \mathcal{E}_i) \in \mathrm{Bun}^{d(i)}_{\widetilde{B}}$ such that the isomorphism $f_i : \mathcal{E}_{i-1}|_{X\times S-\Gamma_{x_i}} \cong \mathcal{E}_i|_{X\times S-\Gamma_{x_i}}$ restricts to an isomorphism $\alpha'_i : \mathcal{L}_{i-1}|_{X\times S-\Gamma_{x_i}} \cong \mathcal{L}_i|_{X\times S-\Gamma_{x_i}}$.

We have $(r+1)$ maps $p_i : \mathrm{Hk}^{\mu,d}_{\widetilde{B}} \to \mathrm{Bun}^{d(i)}_{\widetilde{B}}$ by sending the above data to $(\mathcal{L}_i \hookrightarrow \mathcal{E}_i)$, $i = 0, 1, \cdots, r$. We define $\mathrm{Sht}^{\mu,d}_{\widetilde{B}}$ by the Cartesian diagram

$$\begin{tikzcd}
\mathrm{Sht}^{\mu,d}_{\widetilde{B}} \arrow[r] \arrow[d] & \mathrm{Hk}^{\mu,d}_{\widetilde{B}} \arrow[d, "{(p_0,p_r)}"] \\
\mathrm{Bun}^{d(0)}_{\widetilde{B}} \arrow[r, "{(\mathrm{id},\mathrm{Fr})}"] & \mathrm{Bun}^{d(0)}_{\widetilde{B}} \times \mathrm{Bun}^{d(0)}_{\widetilde{B}}
\end{tikzcd} \tag{7.2}$$

In other words, $\mathrm{Sht}^{\mu,d}_{\widetilde{B}}$ classifies $(\mathcal{L}_i \hookrightarrow \mathcal{E}_i; x_i; f_i; \iota)$, where $(\mathcal{L}_i \hookrightarrow \mathcal{E}_i; x_i; f_i)$ is a point in $\mathrm{Hk}^{\mu,d}_{\widetilde{B}}$ and ι is an isomorphism $\mathcal{E}_r \cong {}^\tau\mathcal{E}_0$ sending $\mathcal{L}_r$ isomorphically to ${}^\tau\mathcal{L}_0$.

We may summarize the data classified by $\mathrm{Sht}^{\mu,d}_{\widetilde{B}}$ as a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{L}_0 \arrow[r] \arrow[d, dashed, "\alpha'_1"] & \mathcal{E}_0 \arrow[r] \arrow[d, dashed, "f_1"] & \mathcal{M}_0 \arrow[r] \arrow[d, dashed, "\alpha''_1"] & 0 \\
& \cdots \arrow[d, dashed, "\alpha'_r"] & \cdots \arrow[d, dashed, "f_r"] & \cdots \arrow[d, dashed, "\alpha''_r"] & \\
0 \arrow[r] & \mathcal{L}_r \arrow[r] \arrow[d, "\iota'"', "\wr"] & \mathcal{E}_r \arrow[r] \arrow[d, "\iota"', "\wr"] & \mathcal{M}_r \arrow[r] \arrow[d, "\iota''"', "\wr"] & 0 \\
0 \arrow[r] & {}^\tau\mathcal{L}_0 \arrow[r] & {}^\tau\mathcal{E}_0 \arrow[r] & {}^\tau\mathcal{M}_0 \arrow[r] & 0
\end{tikzcd} \tag{7.3}$$

Here we denote the quotient line bundle $\mathcal{E}_i/\mathcal{L}_i$ by $\mathcal{M}_i$.

7.2.3. *B-Shtukas.* There is an action of $\mathrm{Pic}_X(k)$ on $\mathrm{Sht}^{\mu,d}_{\widetilde{B}}$ by tensoring each member in (7.3) by a line bundle defined over k. We define

$$\mathrm{Sht}^d_B := \mathrm{Sht}^{\mu,d}_{\widetilde{B}} / \mathrm{Pic}_X(k).$$

Equivalently we may first define $\mathrm{Hk}^{\mu,d}_B := \mathrm{Hk}^{\mu,d}_{\widetilde{B}} / \mathrm{Pic}_X$ and define Sht^d_B by a diagram similar to (7.2), using Hk^d_B and $\mathrm{Bun}^{d(0)}_B$ instead of $\mathrm{Hk}^{\mu,d}_{\widetilde{B}}$ and $\mathrm{Bun}^{d(0)}_{\widetilde{B}}$. The same argument as Lemma 5.5 shows that $\mathrm{Hk}^{\mu,d}_B$ is canonically independent of the choice of μ and these isomorphisms preserve the maps p_i, hence Sht^d_B is also independent of the choice of μ.

7.2.4. *Indexing by degrees.* In the definition of Shtukas in §5.1.4, we may decompose Sht^{μ}_n according to the degrees of $\mathcal{E}_i$. More precisely, for $d \in \mathcal{D}$, we let $\mu(d) \in \{\pm 1\}^r$ be defined as

$$\mu_i(d) = d(i) - d(i-1). \tag{7.4}$$

Let $\mathrm{Sht}^d_n \subset \mathrm{Sht}^{\mu(d)}_n$ be the open and closed substack classifying rank n Shtukas $(\mathcal{E}_i; \cdots)$ with $\deg \mathcal{E}_i = d(i)$.

Consider the action of $\mathbb{Z}$ on $\mathcal{D}$ by adding a constant integer to a function $d \in \mathcal{D}$. The assignment $d \mapsto \mu(d)$ descends to a function $\mathcal{D}/\mathbb{Z} \to \{\pm 1\}^r$. For a $\mathbb{Z}$-orbit $\delta \in \mathcal{D}/\mathbb{Z}$, we write $\mu(d)$ as $\mu(\delta)$ for any $d \in \delta$. Then for any $\delta \in \mathcal{D}/\mathbb{Z}$, we have a decomposition

$$\mathrm{Sht}^{\mu(\delta)}_n = \coprod_{d\in\delta} \mathrm{Sht}^d_n \tag{7.5}$$

In particular, after identifying H with $\mathbb{G}_m$, we define Sht^d_H to be Sht^d_1 for any $d \in \mathcal{D}$.

7.2.5. *The horocycle correspondence.* From the definition of Sht_B^d, we have a forgetful map

$$p_d : \mathrm{Sht}_B^d \longrightarrow \mathrm{Sht}_G$$

sending the data in (7.3) to the middle column.

On the other hand, mapping the diagram (7.3) to $(\mathcal{L}_i \otimes \mathcal{M}_i^{-1}; x_i; \alpha_i' \otimes \alpha_i''; \iota' \otimes \iota'')$ we get a morphism

$$q_d : \mathrm{Sht}_B^d \longrightarrow \mathrm{Sht}_H^d.$$

Via the maps p_d and q_d, we may view Sht_B^d as a correspondence between Sht_G and Sht_H^d over X^r:

$$\begin{tikzcd} & \mathrm{Sht}_B^d \arrow[dl, "p_d"'] \arrow[dd, "\pi_B^d"] \arrow[dr, "q_d"] & \\ \mathrm{Sht}_G \arrow[dr, "\pi_G"'] & & \mathrm{Sht}_H^d \arrow[dl, "\pi_H^d"] \\ & X^r & \end{tikzcd} \tag{7.6}$$

Lemma 7.4. *Let $\mathcal{D}^+ \subset \mathcal{D}$ be the subset consisting of functions d such that $d(i) > 0$ for all i. Suppose $d \in \mathcal{D}^+$. Then the map $p_d : \mathrm{Sht}_B^d \to \mathrm{Sht}_G$ has image Sht_G^d and induces an isomorphism $\mathrm{Sht}_B^d \cong \mathrm{Sht}_G^d$.*

Proof. We first show that $p_d(\mathrm{Sht}_B^d) \subset \mathrm{Sht}_G^d$. If $(\mathcal{L}_i \hookrightarrow \mathcal{E}_i; x_i; f_i; \iota) \in \mathrm{Sht}_B^d$ (up to tensoring with a line bundle), then $\deg \mathcal{L}_i \geq \frac{1}{2}(\deg \mathcal{E}_i + d(i)) > \frac{1}{2}\deg \mathcal{E}_i$, hence $\mathcal{L}_i$ is the maximal line subbundle of $\mathcal{E}_i$. Therefore $\mathrm{inst}(\mathcal{E}_i) = d(i)$ and $(\mathcal{E}_i; x_i; f_i) \in \mathrm{Sht}_G^d$.

Conversely, we will define a map $\mathrm{Sht}_G^d \to \mathrm{Sht}_B^d$. Let $(\mathcal{E}_i; x_i; f_i; \iota) \in \mathrm{Sht}_G^d(S)$, then the maximal line bundle $\mathcal{L}_i \hookrightarrow \mathcal{E}_i$ is well-defined since each $\mathcal{E}_i$ is unstable.

We claim that for each geometric point $s \in S$, the generic fibers of $\mathcal{L}_i|_{X \times \{s\}}$ map isomorphically to each other under the rational maps f_i between the $\mathcal{E}_i$'s. For this we may assume $S = \mathrm{Spec}(K)$ for some field K and we base change the situation to K without changing notation. Let $\mathcal{L}_{i+1}' \subset \mathcal{E}_{i+1}$ be the line bundle obtained by saturating $\mathcal{L}_i$ under the rational map $f_{i+1} : \mathcal{E}_i \dashrightarrow \mathcal{E}_{i+1}$. Then $d'(i+1) := 2\mathcal{L}_{i+1}' - \deg \mathcal{E}_{i+1} = d(i) \pm 1$. If $d'(i+1) > 0$, then $\mathcal{L}_{i+1}'$ is also the maximal line subbundle of $\mathcal{E}_{i+1}$, hence $\mathcal{L}_{i+1}' = \mathcal{L}_{i+1}$. If $d'(i+1) \leq 0$, then we must have $d(i) = 1$ and $d'(i+1) = 0$. Since $d \in \mathcal{D}^+$, we must have $d(i+1) = 2$. In this case the map $\mathcal{L}_{i+1}' \oplus \mathcal{L}_{i+1} \to \mathcal{E}_{i+1}$ cannot be injective because the source has degree at least $\frac{1}{2}(\deg \mathcal{E}_{i+1} + d'(i+1)) + \frac{1}{2}(\deg \mathcal{E}_{i+1} + d(i+1)) = \deg \mathcal{E}_{i+1} + 1 > \deg \mathcal{E}_{i+1}$. Therefore $\mathcal{L}_{i+1}'$ and $\mathcal{L}_{i+1}$ have the same generic fiber, which is impossible since they are both line subbundles of $\mathcal{E}_{i+1}$ but have different degrees. This proves the claim.

Moreover, the isomorphism $\iota : \mathcal{E}_r \cong {}^\tau\mathcal{E}_0$ must send $\mathcal{L}_r$ isomorphically onto ${}^\tau\mathcal{L}_0$ by the uniqueness of the maximal line subbundle. This together with the claim above implies that $(\mathcal{L}_i; x_i; f_i|_{\mathcal{L}_i}; \iota|_{\mathcal{L}_r})$ is a rank one sub-Shtuka of $(\mathcal{E}_i; x_i; f_i; \iota)$, and therefore $(\mathcal{L}_i \hookrightarrow \mathcal{E}_i; x_i; f_i; \iota)$ gives a point in Sht_B^d. This way we have defined a map $\mathrm{Sht}_G^d \to \mathrm{Sht}_B^d$. It is easy to check that this map is inverse to $p_d : \mathrm{Sht}_B^d \to \mathrm{Sht}_G^d$. $\square$

Lemma 7.5. *Let $d \in \mathcal{D}$ be such that $d(i) > 2g-2$ for all i. Then the morphism $q_d : \mathrm{Sht}_B^d \to \mathrm{Sht}_H^d$ is smooth of relative dimension $r/2$, and its geometric fibers are isomorphic to $[\mathbb{G}_a^{r/2}/Z]$ for some finite étale group scheme Z acting on $\mathbb{G}_a^{r/2}$ via a homomorphism $Z \to \mathbb{G}_a^{r/2}$.*

Proof. We pick μ as in §5.1.2 to realize Sht_G as the quotient $\mathrm{Sht}_2^\mu/\mathrm{Pic}_X(k)$, and Sht_B^d as the quotient $\mathrm{Sht}_{\widetilde{B}}^{\mu,d}/\mathrm{Pic}_X(k)$.

In the definition of Shtukas in §5.1.4, we may allow some coordinates μ_i of the modification type μ to be 0, which means that the corresponding f_i is an isomorphism. Therefore we may define Sht_n^μ for more general $\mu \in \{0, \pm 1\}^r$ such that $\sum \mu_i = 0$.

We define the sequence $\mu'(d) = (\mu_1'(d), \cdots, \mu_r'(d)) \in \{0, \pm 1\}^r$ by

$$\mu_i'(d) := \frac{1}{2}(\mathrm{sgn}(\mu_i) + d(i) - d(i-1))$$

We also define $\mu''(d) = (\mu_1''(d), \cdots, \mu_r''(d)) \in \{0, \pm 1\}^r$ by

$$\mu_i''(d) := \frac{1}{2}(\mathrm{sgn}(\mu_i) - d(i) + d(i-1)) = \mathrm{sgn}(\mu_i) - \mu_i'(d)$$

We write $\mu'(d)$ and $\mu''(d)$ simply as μ' and μ''. Mapping the diagram (7.3) to the rank one Shtuka $(\mathcal{L}_i; x_i; \alpha_i'; \iota')$ defines a map $\mathrm{Sht}_{\widetilde{B}}^{\mu,d} \to \mathrm{Sht}_1^{\mu'}$; similarly, sending the diagram (7.3) to the rank one Shtuka $(\mathcal{M}_i; x_i; \alpha_i''; \iota'')$ defines a map $\mathrm{Sht}_{\widetilde{B}}^{\mu,d} \to \mathrm{Sht}_1^{\mu''}$. Combining the two maps we get

$$\widetilde{q}_d : \mathrm{Sht}_{\widetilde{B}}^{\mu,d} \longrightarrow \mathrm{Sht}_1^{\mu'} \times_{X^r} \mathrm{Sht}_1^{\mu''}.$$

Fix a pair $\mathcal{L}_\bullet := (\mathcal{L}_i; x_i; \alpha_i'; \iota') \in \mathrm{Sht}_1^{\mu'}(S)$ and $\mathcal{M}_\bullet := (\mathcal{M}_i; x_i; \alpha_i''; \iota'') \in \mathrm{Sht}_1^{\mu''}(S)$. Then the fiber of q_d over $(\mathcal{L}_i \otimes \mathcal{M}_i^{-1}; x_i; \cdots) \in \mathrm{Sht}_H^d(S)$ is isomorphic to the fiber of $\widetilde{q}_d$ over $(\mathcal{L}_\bullet, \mathcal{M}_\bullet)$, the latter being the moduli stack $E_{\mathrm{Sht}}(\mathcal{M}_\bullet, \mathcal{L}_\bullet)$ (over S) of extensions of $\mathcal{M}_\bullet$ by $\mathcal{L}_\bullet$ as Shtukas.

Since $\deg(\mathcal{L}_i) - \deg(\mathcal{M}_i) = d(i) > 2g-2$, we have $\mathrm{Ext}^1(\mathcal{M}_i, \mathcal{L}_i) = 0$. For each i, let $E(\mathcal{M}_i, \mathcal{L}_i)$ be the stack classifying extensions of $\mathcal{M}_i$ by $\mathcal{L}_i$. Then $E(\mathcal{M}_i, \mathcal{L}_i)$ is canonically isomorphic to the classifying space of the additive group $H_i := \underline{\mathrm{Hom}}(\mathcal{M}_i, \mathcal{L}_i)$ over S. For each $i = 1, \cdots, r$, we have another moduli stack C_i classifying commutative diagrams of extensions

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{L}_{i-1} \arrow[r] \arrow[d, dashed, "\alpha_i'"] & \mathcal{E}_{i-1} \arrow[r] \arrow[d, dashed, "f_i"] & \mathcal{M}_{i-1} \arrow[r] \arrow[d, dashed, "\alpha_i''"] & 0 \\
0 \arrow[r] & \mathcal{L}_i \arrow[r] & \mathcal{E}_i \arrow[r] & \mathcal{M}_i \arrow[r] & 0
\end{tikzcd}$$

Here the left and right columns are fixed. We have four cases:

(1) When $(\mu_i', \mu_i'') = (1, 0)$, then $\alpha_i' : \mathcal{L}_{i-1} \hookrightarrow \mathcal{L}_i$ with colength one and α_i'' is an isomorphism. In this case, the bottom row is the pushout of the top row along α_i', hence determined by the top row. Therefore $C_i = E(\mathcal{M}_{i-1}, \mathcal{L}_{i-1})$ in this case.

(2) When $(\mu_i', \mu_i'') = (-1, 0)$, then $\alpha_i'^{-1} : \mathcal{L}_i \hookrightarrow \mathcal{L}_{i-1}$ with colength one and α_i'' is an isomorphism. In this case, the top row is the pushout of the bottom row along $\alpha_i'^{-1}$, hence determined by the bottom row. Therefore $C_i = E(\mathcal{M}_i, \mathcal{L}_i)$ in this case.

(3) When $(\mu_i', \mu_i'') = (0, 1)$, then α_i' is an isomorphism and $\alpha_i'' : \mathcal{M}_{i-1} \hookrightarrow \mathcal{M}_i$ with colength one. In this case, the top row is the pullback of the bottom row along α_i'', hence determined by the bottom row. Therefore $C_i = E(\mathcal{M}_i, \mathcal{L}_i)$ in this case.

(4) When $(\mu_i', \mu_i'') = (0, -1)$, then α_i' is an isomorphism and $\alpha_i''^{-1} : \mathcal{M}_i \hookrightarrow \mathcal{M}_{i-1}$ with colength one. In this case, the bottom row is the pullback of the top row along $\alpha_i''^{-1}$, hence determined by the top row. Therefore $C_i = E(\mathcal{M}_{i-1}, \mathcal{L}_{i-1})$ in this case.

From the combinatorics of μ' and μ'' we see that the cases (1)(4) and (2)(3) each appear $r/2$ times. In all cases, we view C_i as a correspondence

$$E(\mathcal{M}_{i-1}, \mathcal{L}_{i-1}) \leftarrow C_i \rightarrow E(\mathcal{M}_i, \mathcal{L}_i)$$

then C_i is the graph of a natural map $E(\mathcal{M}_{i-1}, \mathcal{L}_{i-1}) \to E(\mathcal{M}_i, \mathcal{L}_i)$ in cases (1) and (4) and the graph of a natural map $E(\mathcal{M}_i, \mathcal{L}_i) \to E(\mathcal{M}_{i-1}, \mathcal{L}_{i-1})$ in cases (2) and (3). We see that C_i is canonically the classifying space of an additive group scheme Ω_i over S, which is either H_{i-1} in cases (1) and (4) or H_i in cases (2) and (3).

Consider the composition of these correspondences

$$C(\mathcal{M}_\bullet, \mathcal{L}_\bullet) := C_1 \times_{E(\mathcal{M}_1, \mathcal{L}_1)} C_2 \times_{E(\mathcal{M}_2, \mathcal{L}_2)} \cdots \times_{E(\mathcal{M}_{r-1}, \mathcal{L}_{r-1})} C_r.$$

This is viewed as a correspondence

$$E(\mathcal{M}_0, \mathcal{L}_0) \leftarrow C(\mathcal{M}_\bullet, \mathcal{L}_\bullet) \rightarrow E(\mathcal{M}_r, \mathcal{L}_r) \cong E({}^\tau\mathcal{M}_0, {}^\tau\mathcal{L}_0).$$

To compute $C(\mathcal{M}_\bullet, \mathcal{L}_\bullet)$ more explicitly, we consider the following situation. Let $\mathcal{G}$ be a group scheme over S with two subgroup schemes $\mathcal{G}_1$ and $\mathcal{G}_2$. Then we have a canonical isomorphism of stacks over S

$$\mathbb{B}(\mathcal{G}_1) \times_{\mathbb{B}(\mathcal{G})} \mathbb{B}(\mathcal{G}_2) \cong \mathcal{G}_1 \backslash \mathcal{G} / \mathcal{G}_2.$$

Using this fact repeatedly, and using that $E(\mathcal{M}_i, \mathcal{L}_i) = \mathbb{B}(H_i)$ and $C_i = \mathbb{B}(\Omega_i)$, we see that

$$C(\mathcal{M}_\bullet, \mathcal{L}_\bullet) \cong \Omega_1 \backslash H_1 \overset{\Omega_2}{\times} H_2 \overset{\Omega_3}{\times} \cdots \overset{\Omega_{r-1}}{\times} H_{r-1}/\Omega_r. \tag{7.7}$$

where $H_{i-1} \overset{\Omega_i}{\times} H_i$ means dividing by the diagonal action of Ω_i on both H_{i-1} and H_i by translations. Let

$$A(\mathcal{M}_\bullet, \mathcal{L}_\bullet) := H_0 \overset{\Omega_1}{\times} H_1 \overset{\Omega_2}{\times} \cdots \overset{\Omega_{r-1}}{\times} H_{r-1} \overset{\Omega_r}{\times} H_r$$

Since Ω_i is always the smaller of H_{i-1} and H_i, $A(\mathcal{M}_\bullet, \mathcal{L}_\bullet)$ is an additive group scheme over S. Then we have

$$C(\mathcal{M}_\bullet, \mathcal{L}_\bullet) \cong H_0 \backslash A(\mathcal{M}_\bullet, \mathcal{L}_\bullet)/H_r. \tag{7.8}$$

Note that $H_r \cong {}^\tau H_0$ is the pullback of H_0 via Fr_S. We have a relative Frobenius map over S

$$\mathrm{Fr}_{/S} : E(\mathcal{M}_0, \mathcal{L}_0) = \mathbb{B}(H_0) \xrightarrow{\mathrm{Fr}_{H_0/S}} \mathbb{B}(H_r) = E(\mathcal{M}_r, \mathcal{L}_r).$$

By the moduli meaning of $E_{\mathrm{Sht}}(\mathcal{M}_\bullet, \mathcal{L}_\bullet)$, we have a Cartesian diagram of stacks

$$\begin{tikzcd}
E_{\mathrm{Sht}}(\mathcal{M}_\bullet, \mathcal{L}_\bullet) \arrow[r] \arrow[d] & C(\mathcal{M}_\bullet, \mathcal{L}_\bullet) \arrow[d] \\
E(\mathcal{M}_0, \mathcal{L}_0) \arrow[r, "(\mathrm{id}, \mathrm{Fr}_{/S})"] & E(\mathcal{M}_0, \mathcal{L}_0) \times E(\mathcal{M}_r, \mathcal{L}_r)
\end{tikzcd}$$

Using the isomorphism (7.8), the above diagram becomes

$$\begin{tikzcd}
E_{\mathrm{Sht}}(\mathcal{M}_\bullet, \mathcal{L}_\bullet) \arrow[r] \arrow[d] & H_0 \backslash A(\mathcal{M}_\bullet, \mathcal{L}_\bullet)/H_r \arrow[d] \\
\mathbb{B}(H_0) \arrow[r, "(\mathrm{id}, \mathrm{Fr}_{H_0/S})"] & \mathbb{B}(H) \times \mathbb{B}(H_r)
\end{tikzcd} \tag{7.9}$$

This implies that

$$E_{\mathrm{Sht}}(\mathcal{M}_\bullet, \mathcal{L}_\bullet) \cong [A(\mathcal{M}_\bullet, \mathcal{L}_\bullet)/_{(\mathrm{id}, \mathrm{Fr}_{H_0/S})} H_0] \tag{7.10}$$

where H_0 acts on $A(\mathcal{M}_\bullet, \mathcal{L}_\bullet)$ via the embedding $(\mathrm{id}, \mathrm{Fr}_{H_0/S}) : H_0 \to H_0 \times H_r$ and the natural action of $H_0 \times H_r$ on $A(\mathcal{M}_\bullet, \mathcal{L}_\bullet)$. Since A is an additive group scheme over S, hence smooth over S, the isomorphism (7.10) shows that $E_{\mathrm{Sht}}(\mathcal{M}_\bullet, \mathcal{L}_\bullet)$ is smooth over S.

To compute the dimension of $A(\mathcal{M}_\bullet, \mathcal{L}_\bullet)$, we compare $\dim \Omega_i$ with $\dim H_i$. We have $\dim H_i - \dim \Omega_i = 1$ in cases (1) and (4) and $\dim H_i - \dim \Omega_i = 0$ in cases (2) and (3). Since (1)(4) and (2)(3) each appear $r/2$ times, we have

$$\dim A(\mathcal{M}_\bullet, \mathcal{L}_\bullet) = \dim H_0 + \sum_{i=1}^{r} (\dim H_i - \dim \Omega_i) = \dim H_0 + r/2.$$

This implies $E_{\mathrm{Sht}}(\mathcal{M}_\bullet, \mathcal{L}_\bullet)$ is smooth of dimension $r/2$.

When S is a geometric point $\mathrm{Spec}(K)$, H_0 and H_r can be viewed as subspaces of the K-vector space $A := A(\mathcal{M}_\bullet, \mathcal{L}_\bullet)$, and $\phi = \mathrm{Fr}_{H_0/K} : H_0 \to H_r$ is a morphism of group schemes over K. Choose a K-subspace $L \subset A$ complement to H_0, then $L \cong \mathbb{G}_a^{r/2}$ as a group scheme over K. Consider the homomorphism

$$\alpha : H_0 \times L \longrightarrow A$$

given by $(x, y) \mapsto x + y + \phi(x)$. By computing the tangent map of α at the origin, we see that α is étale, therefore $Z = \ker(\alpha)$ is a finite étale group scheme over K. We conclude that in this case the fiber of q_d over $S = \mathrm{Spec}(K)$ is

$$E_{\mathrm{Sht}}(\mathcal{M}_\bullet, \mathcal{L}_\bullet) \cong [A/_{(\mathrm{id}, \phi)} H_0] \cong [L/Z] \cong [\mathbb{G}_a^{r/2}/Z].$$

□

Corollary 7.6. *Suppose $d \in \mathcal{D}$ satisfies $d(i) > 2g - 2$ for all i, then the cone of the map $\mathbf{R}\pi_{G,!}^{<d} \mathbb{Q}_\ell \to \mathbf{R}\pi_{G,!}^{\leq d} \mathbb{Q}_\ell$ is isomorphic to $\pi_{H,!}^{d} \mathbb{Q}_\ell[-r](-r/2)$, which is a local system concentrated in degree r.*

Proof. The cone of $\mathbf{R}\pi^{<d}_{G,!}\mathbb{Q}_\ell \to \mathbf{R}\pi^{\le d}_{G,!}\mathbb{Q}_\ell$ is isomorphic to $\mathbf{R}\pi^d_{G,!}\mathbb{Q}_\ell$, where $\pi^d_G : \mathrm{Sht}^d_G \to X^r$. By Lemma 7.4, for $d \in \mathcal{D}^+$, we have $\mathbf{R}\pi^d_{G,!}\mathbb{Q}_\ell \cong \mathbf{R}\pi^d_{B,!}\mathbb{Q}_\ell$. By Lemma 7.5, q_d is smooth of relative dimension $r/2$, the relative fundamental cycles gives $\mathbf{R}q_{d,!}\mathbb{Q}_\ell \to \mathbf{R}^r q_{d,!}\mathbb{Q}_\ell[-r] \to \mathbb{Q}_\ell[-r](-r/2)$, which is an isomorphism by checking the stalks (using the description of the geometric fibers of q_d given in Lemma 7.5). Therefore $\mathbf{R}\pi^d_{B,!}\mathbb{Q}_\ell \cong \mathbf{R}\pi^d_{H,!}\mathbb{Q}_\ell[-r](-r/2)$. Finally, $\pi^d_H : \mathrm{Sht}^d_H \to X^r$ is a $\mathrm{Pic}^0_X(k)$-torsor by an argument similar to Lemma 5.13. Therefore $\mathbf{R}\pi^d_{H,!}\mathbb{Q}_\ell$ is a local system on X^r, and $\mathbf{R}\pi^d_{G,!}\mathbb{Q}_\ell \cong \pi^d_{H,!}\mathbb{Q}_\ell[-r](-r/2)$ is a local system shifted to degree r. □

7.3. **Horocycles in the generic fiber.** Fix a geometric generic point $\overline{\eta}$ of X^r. For a stack $\mathfrak{X}$ over X^r, we denote its fiber over $\overline{\eta}$ by $\mathfrak{X}_{\overline{\eta}}$. Next we study the cycles in $\mathrm{Sht}_{G,\overline{\eta}}$ given by images of $\mathrm{Sht}^d_{B,\overline{\eta}}$.

Lemma 7.7 (Drinfeld [6, Prop 4.2] for $r = 2$; Varshavsky [21, Prop 5.7] in general)**.** *For each $d \in \mathcal{D}$, the map $p_{d,\overline{\eta}} : \mathrm{Sht}^d_{B,\overline{\eta}} \to \mathrm{Sht}_{G,\overline{\eta}}$ is finite and unramified.*

7.3.1. *The cohomological constant term.* Taking the geometric generic fiber of the diagram (7.6), we view $\mathrm{Sht}^d_{B,\overline{\eta}}$ as a correspondence between $\mathrm{Sht}_{G,\overline{\eta}}$ and $\mathrm{Sht}^d_{H,\overline{\eta}}$. The fundamental cycle of $\mathrm{Sht}^d_{B,\overline{\eta}}$ (of dimension $r/2$) gives a cohomological correspondence between the constant sheaf on $\mathrm{Sht}_{G,\overline{\eta}}$ and the shifted constant sheaf $\mathbb{Q}_\ell[-r](-r/2)$ on $\mathrm{Sht}^d_{H,\overline{\eta}}$. Therefore $[\mathrm{Sht}^d_{B,\overline{\eta}}]$ induces a map

$$\gamma_d : \mathrm{H}^r_c(\mathrm{Sht}_{G,\overline{\eta}})(r/2) \xrightarrow{p^*_{d,\overline{\eta}}} \mathrm{H}^r_c(\mathrm{Sht}^d_{B,\overline{\eta}})(r/2) \xrightarrow{[\mathrm{Sht}^d_{B,\overline{\eta}}]} \mathrm{H}_0(\mathrm{Sht}^d_{B,\overline{\eta}}) \xrightarrow{q_{d,\overline{\eta},!}} \mathrm{H}_0(\mathrm{Sht}^d_{H,\overline{\eta}}). \tag{7.11}$$

Here we are implicitly using Lemma 7.7 to conclude that $p_{d,\overline{\eta}}$ is proper, hence $p^*_{d,\overline{\eta}}$ induces a map between compactly supported cohomology groups.

Taking the product of γ_d for all d in a fixed $\mathbb{Z}$-orbit $\delta \in \mathcal{D}/\mathbb{Z}$, using the decomposition (7.5), we get a map

$$\gamma_\delta : \mathrm{H}^r_c(\mathrm{Sht}_{G,\overline{\eta}})(r/2) \longrightarrow \prod_{d\in\delta} \mathrm{H}_0(\mathrm{Sht}^d_{H,\overline{\eta}}) \cong \mathrm{H}^0(\mathrm{Sht}^{\mu(\delta)}_{1,\overline{\eta}}). \tag{7.12}$$

When $r = 0$, (7.12) is exactly the constant term map for automorphic forms. Therefore we may call γ_δ the *cohomological constant term map.*

The RHS of (7.12) carries an action of the Hecke algebra $\mathscr{H}_H = \otimes_{x\in|X|}\mathbb{Q}[t_x, t_x^{-1}]$. In fact, $\mathrm{Sht}^{\mu(\delta)}_{1,\overline{\eta}}$ is a $\mathrm{Pic}_X(k)$-torsor over $\mathrm{Spec}\, k(\overline{\eta})$. The action of $\mathscr{H}_H$ on $\mathrm{Sht}^{\mu(\delta)}_{H,\overline{\eta}}$ is via the natural map $\mathscr{H}_H \cong \mathbb{Q}[\mathrm{Div}(X)] \to \mathbb{Q}[\mathrm{Pic}_X(k)]$.

Lemma 7.8. *The map γ_δ in (7.12) intertwines the $\mathscr{H}$-action on the LHS and the $\mathscr{H}_H$-action on the RHS via the Satake transform* $\mathrm{Sat} : \mathscr{H} \hookrightarrow \mathscr{H}_H$*.*

Proof. Since $\mathscr{H}$ is generated by $\{h_x\}_{x\in|X|}$ as a $\mathbb{Q}$-algebra, it suffices to show that for any $x \in |X|$, the following diagram is commutative

$$\begin{tikzcd} \mathrm{H}^r_c(\mathrm{Sht}_{G,\overline{\eta}}) \arrow[r, "\gamma_\delta"] \arrow[d, "C(h_x)"] & \prod_{d\in\delta} \mathrm{H}_0(\mathrm{Sht}^d_{H,\overline{\eta}}) \arrow[d, "t_x+q_xt_x^{-1}"] \\ \mathrm{H}^r_c(\mathrm{Sht}_{G,\overline{\eta}}) \arrow[r, "\gamma_\delta"] & \prod_{d\in\delta} \mathrm{H}_0(\mathrm{Sht}^d_{H,\overline{\eta}}) \end{tikzcd} \tag{7.13}$$

Let $U = X - \{x\}$. For a stack $\mathfrak{X}$ over X^r, we use $\mathfrak{X}_{U^r}$ to denote its restriction to U^r. Similar notation applies to morphisms over X^r.

Recall that $\mathrm{Sht}_{G,U^r}(S)$ classifies $(\mathcal{E}_i; x_i; f_i; \iota)$ such that x_i are disjoint from x. Hence the composition $\iota \circ f_r \cdots f_1 : \mathcal{E}_0 \dashrightarrow {}^\tau\mathcal{E}_0$ is an isomorphism near x. In particular, the fiber $\mathcal{E}_{0,x} = \mathcal{E}_0|_{S\times\{x\}}$ carries a Frobenius structure $\mathcal{E}_{0,x} \cong {}^\tau\mathcal{E}_{0,x}$, hence $\mathcal{E}_{0,x}$ descends to a two-dimensional vector space over $\mathrm{Spec}\, k_x$ (k_x is the residue field of X at x) up to tensoring with a line. In other words, there is a morphism $\omega_x : \mathrm{Sht}_{G,U^r} \to \mathbb{B}(G(k_x))$ sending $(x_i; \mathcal{E}_i; f_i; \iota)$ to the descent of $\mathcal{E}_{0,x}$ to $\mathrm{Spec}\, k_x$. In the following we shall understand that $\mathcal{E}_{0,x}$ is a 2-dimensional vector space over k_x, up to tensoring with a line over k_x.

The correspondence $\mathrm{Sht}_G(h_x)_{U^r}$ classifies diagrams of the form (5.5) where the vertical maps have divisor x. Therefore, if the first row in (5.5) is fixed, the bottom row is determined by $\mathcal{E}'_0$, which in turn is determined by the line $e_x = \ker(\mathcal{E}_{0,x} \to \mathcal{E}'_{0,x})$ over k_x. Recall that $\overleftarrow{p}$ and

$\overrightarrow{p} : \mathrm{Sht}_G(h_x) \to \mathrm{Sht}_G$ are the projections sending (5.5) to the top and bottom row respectively. Then we have a Cartesian diagram

$$\begin{tikzcd} \mathrm{Sht}_G(h_x)_{U^r} \arrow[r] \arrow[d, "\overleftarrow{p}_{U^r}"] & \mathbb{B}(B(k_x)) \arrow[d] \\ \mathrm{Sht}_{G,U^r} \arrow[r, "\omega_x"] & \mathbb{B}(G(k_x)) \end{tikzcd}$$

where $B \subset G$ is a Borel subgroup. We have a similar Cartesian diagram where $\overleftarrow{p}_{U^r}$ is replaced with $\overrightarrow{p}_{U^r}$. In particular, $\overleftarrow{p}_{U^r}$ and $\overrightarrow{p}_{U^r}$ are finite étale of degree $q_x + 1$.

Let $\mathrm{Sht}^d_B(h_x)$ be the base change of $\overleftarrow{p}$ along $\mathrm{Sht}^d_B \to \mathrm{Sht}_G$. Let $\overleftarrow{p}_B : \mathrm{Sht}^d_B(h_x)_{U^r} \to \mathrm{Sht}^d_{B,U^r}$ be the base-changed map restricted to U^r. A point $(\mathcal{L}_i \hookrightarrow \mathcal{E}_i; x_i; f_i; \iota) \in \mathrm{Sht}^d_B$ gives another line $\ell_x := \mathcal{L}_{0,x} \subset \mathcal{E}_{0,x}$. Therefore, for a point $(\mathcal{L}_i \hookrightarrow \mathcal{E}_i \to \mathcal{E}'_i; \cdots) \in \mathrm{Sht}^d_B(h_x)|_{U^r}$, we get two lines ℓ_x and e_x inside $\mathcal{E}_{0,x}$. In other words we have a morphism

$$\omega : \mathrm{Sht}^d_B(h_x)_{U^r} \longrightarrow \mathbb{B}(B(k_x)) \times_{\mathbb{B}(G(k_x))} \mathbb{B}(B(k_x)) = B(k_x)\backslash G(k_x)/B(k_x)$$

This allows us to decompose $\mathrm{Sht}^d_B(h_x)_{U^r}$ into the disjoint union of two parts

$$\mathrm{Sht}^d_B(h_x)_{U^r} = C_1 \coprod C_2$$

where C_1 is the preimage of the unit coset $B(k_x)\backslash B(k_x)/B(k_x)$ and C_2 is the preimage of the complement.

For a point $(\mathcal{L}_i \hookrightarrow \mathcal{E}_i \hookrightarrow \mathcal{E}'_i; \cdots) \in C_1$, $\mathcal{E}'_i$ is determined by $e_x = \ell_x = \mathcal{L}_{0,x}$. Therefore the map $\overleftarrow{p_{B,1}} := \overleftarrow{p}_B|_{C_1} : C_1 \to \mathrm{Sht}^d_{B,U^r}$ is an isomorphism. In this case, $\mathcal{E}'_i$ is obtained via the pushout of $\mathcal{L}_i \to \mathcal{E}_i$ along $\mathcal{L}_i \hookrightarrow \mathcal{L}_i(x)$. This way we get an exact sequence $0 \to \mathcal{L}_i(x) \to \mathcal{E}_i(x) \to \mathcal{M}_i \to 0$ where $\mathcal{M}_i = \mathcal{E}_i/\mathcal{L}_i$. We define a map $\overrightarrow{p_{B,1}} : C_1 \to \mathrm{Sht}^{d+d_x}_{B,U^r}$ sending $(\mathcal{L}_i \hookrightarrow \mathcal{E}_i \hookrightarrow \mathcal{E}'_i; \cdots) \in C_1$ to $(\mathcal{L}_i(x) \hookrightarrow \mathcal{E}'_i; \cdots)$. Since $\overleftarrow{p_{B,1}}$ is an isomorphism, C_1 viewed as a correspondence between Sht^d_{B,U^r} and $\mathrm{Sht}^{d+d_x}_{B,U^r}$ can be identified with the graph of the map $\varphi_x := \overrightarrow{p_{B,1}} \circ \overleftarrow{p_{B,1}}^{-1} : \mathrm{Sht}^d_{B,U^r} \to \mathrm{Sht}^{d+d_x}_{B,U^r}$. Note that φ_x is a finite étale map of degree q_x. We have a commutative diagram

$$\begin{tikzcd} \mathrm{Sht}^d_{H,U^r} & \Gamma(t_x) \arrow[l, "\mathrm{id}"'] \arrow[r, "t_x"] & \mathrm{Sht}^{d+d_x}_{H,U^r} \\ \mathrm{Sht}^d_{B,U^r} \arrow[u, "q_d"] \arrow[d, "p_d"] & C_1 = \Gamma(\varphi_x) \arrow[l, "\overleftarrow{p_{B,1}}"', "\sim"] \arrow[r, "\overrightarrow{p_{B,1}}"] \arrow[u] \arrow[d] & \mathrm{Sht}^{d+d_x}_{B,U^r} \arrow[u, "q_{d+d_x}"] \arrow[d, "p_{d+d_x}"] \\ \mathrm{Sht}_{G,U^r} & \mathrm{Sht}_G(h_x)_{U^r} \arrow[l, "\overleftarrow{p}"'] \arrow[r, "\overrightarrow{p}"] & \mathrm{Sht}_{G,U^r} \end{tikzcd}$$

Here $\Gamma(t_x)$ is the graph of the isomorphism $\mathrm{Sht}^d_{H,U^r} \to \mathrm{Sht}^{d+d_x}_{H,U^r}$ given by tensoring the line bundles with $\mathcal{O}(x)$. Therefore the action of $[C_1]$ on the compactly supported cohomology of the generic fiber of Sht^d_B fits into a commutative diagram

$$\begin{tikzcd} \mathrm{H}^r_c(\mathrm{Sht}^d_{B,\overline{\eta}})(r/2) \arrow[r, "{[\mathrm{Sht}^d_{B,\overline{\eta}}]}"] \arrow[d, "{[C_1]}"] & \mathrm{H}_0(\mathrm{Sht}^d_{B,\overline{\eta}}) \arrow[r] \arrow[d, "\varphi_{x,*}"] & \mathrm{H}_0(\mathrm{Sht}^d_{H,\overline{\eta}}) \arrow[d, "t_x"] \\ \mathrm{H}^r_c(\mathrm{Sht}^{d+d_x}_{B,\overline{\eta}})(r/2) \arrow[r, "{[\mathrm{Sht}^{d+d_x}_{B,\overline{\eta}}]}"] & \mathrm{H}_0(\mathrm{Sht}^{d+d_x}_{B,\overline{\eta}}) \arrow[r] & \mathrm{H}_0(\mathrm{Sht}^{d+d_x}_{H,\overline{\eta}}) \end{tikzcd} \tag{7.14}$$

Similarly for C_2, we define a morphism $\overrightarrow{p_{B,2}} : C_2 \to \mathrm{Sht}^{d-d_x}_{B,U^r}$ sending $(\mathcal{L}_i \hookrightarrow \mathcal{E}_i \hookrightarrow \mathcal{E}'_i; \cdots) \in C_2$ to $(\mathcal{L}_i \hookrightarrow \mathcal{E}'_i; \cdots)$. Then $\overrightarrow{p_{B,2}}$ is an isomorphism while $\overleftarrow{p}_{B,2} = \overleftarrow{p}_B|_{C_2}$ is finite étale of degree q_x. Therefore C_2 viewed as a correspondence between Sht^d_{B,U^r} and $\mathrm{Sht}^{d-d_x}_{B,U^r}$ can be identified with the *transpose* of the graph of the map $\varphi_x : \mathrm{Sht}^{d-d_x}_{B,U^r} \to \mathrm{Sht}^d_{B,U^r}$ defined previously. We also have a commutative diagram

$$\begin{tikzcd}
\mathrm{Sht}^d_{H,U^r} & \Gamma(t_x^{-1}) \arrow[l, "\mathrm{id}"'] \arrow[r, "t_x^{-1}"] & \mathrm{Sht}^{d-d_x}_{H,U^r} \\
\mathrm{Sht}^d_{B,U^r} \arrow[u, "q_d"] \arrow[d, "p_d"] & C_2 = {}^t\Gamma(\varphi_x) \arrow[l, "\overleftarrow{p_{B,2}}"'] \arrow[r, "\overrightarrow{p_{B,2}}", "\sim"'] \arrow[u] \arrow[d] & \mathrm{Sht}^{d-d_x}_{B,U^r} \arrow[u, "q_{d-d_x}"'] \arrow[d, "p_{d-d_x}"'] \\
\mathrm{Sht}_{G,U^r} & \mathrm{Sht}_G(h_x)_{U^r} \arrow[l, "\overleftarrow{p}"'] \arrow[r, "\overrightarrow{p}"] & \mathrm{Sht}_{G,U^r}
\end{tikzcd}$$

The action of $[C_2]$ on the compactly supported cohomology of the generic fibers of Sht^d_B fits into a commutative diagram

$$\begin{tikzcd}
\mathrm{H}^r_c(\mathrm{Sht}^d_{B,\overline{\eta}})(r/2) \arrow[r, "{[\mathrm{Sht}^d_{B,\overline{\eta}}]}"] \arrow[d, "{[C_2]=\varphi_x^*}"] & \mathrm{H}_0(\mathrm{Sht}^d_{B,\overline{\eta}}) \arrow[r] & \mathrm{H}_0(\mathrm{Sht}^d_{H,\overline{\eta}}) \\
\mathrm{H}^r_c(\mathrm{Sht}^{d-d_x}_{B,\overline{\eta}})(r/2) \arrow[r, "{[\mathrm{Sht}^{d-d_x}_{B,\overline{\eta}}]}"] & \mathrm{H}_0(\mathrm{Sht}^{d-d_x}_{B,\overline{\eta}}) \arrow[r] \arrow[u, "q_x^{-1}\varphi_{x,*}"] & \mathrm{H}_0(\mathrm{Sht}^{d-d_x}_{H,\overline{\eta}}) \arrow[u, "q_x^{-1}t_x"]
\end{tikzcd} \tag{7.15}$$

The appearance of q_x in the above diagram is because the degree of φ_x is q_x. Combining (7.14) and (7.15) we get a commutative diagram

$$\begin{tikzcd}
\prod_{d\in\delta} \mathrm{H}^r_c(\mathrm{Sht}^d_{B,\overline{\eta}})(r/2) \arrow[r] \arrow[d, "{[C_1]+[C_2]}"] & \prod_{d\in\delta} \mathrm{H}_0(\mathrm{Sht}^d_{H,\overline{\eta}}) \arrow[d, "t_x+q_xt_x^{-1}"] \\
\prod_{d\in\delta} \mathrm{H}^r_c(\mathrm{Sht}^d_{B,\overline{\eta}})(r/2) \arrow[r] & \prod_{d\in\delta} \mathrm{H}_0(\mathrm{Sht}^d_{H,\overline{\eta}})
\end{tikzcd} \tag{7.16}$$

Finally, let $\overrightarrow{p_B} : \mathrm{Sht}^d_B(h_x)_{U^r} \to \mathrm{Sht}^d_{B,U^r}$ be $\overrightarrow{p_{B,1}}$ on C_1 and $\overrightarrow{p_{B,2}}$ on C_2. Consider the commutative diagram

$$\begin{tikzcd}
\coprod_{d\in\delta} \mathrm{Sht}^d_B(h_x)_{U^r} \arrow[r, "\overrightarrow{p_B}"] \arrow[d, "(p_d)_{d\in\delta}"] & \coprod_{d\in\delta} \mathrm{Sht}^d_{B,U^r} \arrow[d, "(p_d)_{d\in\delta}"] \\
\mathrm{Sht}_G(h_x)_{U^r} \arrow[r, "\overrightarrow{p}"] & \mathrm{Sht}_{G,U^r}
\end{tikzcd}$$

Since $\overrightarrow{p_B}$ and $\overrightarrow{p}$ are both finite étale of degree q_x+1, by examining geometric fibers we conclude that the above diagram is Cartesian. The similar diagram with $\overrightarrow{p_B}$ and $\overrightarrow{p}$ replaced with $\overleftarrow{p_B}$ and $\overleftarrow{p}$ is Cartesian by definition. From these facts we get a commutative diagram

$$\begin{tikzcd}
\mathrm{H}^r_c(\mathrm{Sht}_{G,\overline{\eta}}) \arrow[r, "(p_d^*)_{d\in\delta}"] \arrow[d, "C(h_x)"] & \prod_{d\in\delta} \mathrm{H}^r_c(\mathrm{Sht}^d_{B,\overline{\eta}}) \arrow[d, "{[C_1]+[C_2]}"] \\
\mathrm{H}^r_c(\mathrm{Sht}_{G,\overline{\eta}}) \arrow[r, "(p_d^*)_{d\in\delta}"] & \prod_{d\in\delta} \mathrm{H}^r_c(\mathrm{Sht}^d_{B,\overline{\eta}})
\end{tikzcd}$$

Combining this with (7.16) we obtain (7.13), as desired. □

7.4. **Finiteness.** For fixed $d \in \mathcal{D}$, the Leray spectral sequence associated with the map $\pi^{\le d}_G$ gives an increasing filtration $L_{\le i}\mathrm{H}^{2r}_c(\mathrm{Sht}^{\le d}_G \otimes_k \overline{k})$ on $\mathrm{H}^{2r}_c(\mathrm{Sht}^{\le d}_G \otimes_k \overline{k})$, with $L_{\le i}\mathrm{H}^{2r}_c(\mathrm{Sht}^{\le d}_G \otimes_k \overline{k})$ being the image of $\mathrm{H}^{2r}(X^r \otimes_k \overline{k}, \tau_{\le i}\mathbf{R}\pi^{\le d}_{G,!}\mathbb{Q}_\ell) \to \mathrm{H}^{2r}(X^r \otimes_k \overline{k}, \mathbf{R}\pi^{\le d}_{G,!}\mathbb{Q}_\ell) \cong \mathrm{H}^{2r}_c(\mathrm{Sht}^{\le d}_G \otimes_k \overline{k})$. Here $\tau_{\le i}$ means the truncation in the usual t-structure of $D^b_c(X^r, \mathbb{Q}_\ell)$. Let $L_{\le i}V$ be the inductive limit $\varinjlim_{d\in\mathcal{D}} L_{\le i}\mathrm{H}^{2r}_c(\mathrm{Sht}^{\le d}_G \otimes_k \overline{k})(r)$, which is a subspace of V. This way we get a filtration on V

$$0 \subset L_{\le 0}V \subset L_{\le 1}V \subset \cdots \subset L_{\le 2r}V = V.$$

Lemma 7.9. *Each $L_{\le i}V$ is stable under the action of $\mathscr{H}$.*

Proof. The map $C(h_D)_{d,d'}$ in (7.1) induces $\tau_{\le i}C(h_D)_{d,d'}:\tau_{\le i}\mathbf{R}\pi^{\le d}_{G,!}\mathbb{Q}_\ell\to\tau_{\le i}\mathbf{R}\pi^{\le d'}_{G,!}\mathbb{Q}_\ell$. By the construction of $C(h_D)$ we have a commutative diagram

$$\begin{tikzcd}
\varinjlim_d \mathrm{H}^{2r}(X^r\otimes_k\overline{k},\tau_{\le i}\mathbf{R}\pi^{\le d}_{G,!}\mathbb{Q}_\ell) \arrow[r, "\varinjlim \tau_{\le i}C(h_D)_{d,d'}"] \arrow[d] & \varinjlim_{d'} \mathrm{H}^{2r}(X^r\otimes_k\overline{k},\tau_{\le i}\mathbf{R}\pi^{\le d'}_{G,!}\mathbb{Q}_\ell) \arrow[d] \\
\mathrm{H}^{2r}_c(\mathrm{Sht}_G\otimes_k\overline{k}) \arrow[r, "C(h_D)"] & \mathrm{H}^{2r}_c(\mathrm{Sht}_G\otimes_k\overline{k})
\end{tikzcd}$$

The image of the vertical maps are both $L_{\le i}V$ up to a Tate twist, therefore $L_{\le i}V$ is stable under $C(h_D)$. When D runs over all effective divisors on X, $C(h_D)$ span $\mathscr{H}$, hence $L_{\le i}V$ is stable under $\mathscr{H}$. □

Lemma 7.10. *For $i\ne r$, $\mathrm{Gr}^L_iV:=L_{\le i}V/L_{\le i-1}V$ is finite-dimensional over $\mathbb{Q}_\ell$.*

Proof. We say $d\in\mathcal{D}$ is large if $d(i)>2g-2$ for all i. In the following argument it is convenient to choose a total order on $\mathcal{D}$ that extends its partial order. Under the total order, $\mathrm{Sht}^{<d}_G=\coprod_{d'<d}\mathrm{Sht}^{d'}_G$ and $\mathrm{Sht}^{\le d}_G=\coprod_{d'\le d}\mathrm{Sht}^{d}_G$ are different from their original meanings, and we will use the new notion during the proof.

By Corollary 7.6, the inductive system $\tau_{\le r-1}\mathbf{R}\pi^{\le d}_{G,!}\mathbb{Q}_\ell$ stabilizes for d large. Hence so does $L_{\le r-1}\mathrm{H}^{2r}_c(\mathrm{Sht}^{\le d}_G\otimes_k\overline{k})$. Therefore $L_{\le r-1}V$ is finite dimensional.

It remains to show that $V/L_{\le r}V$ is finite dimensional. Again by Corollary 7.6, for d large, the map $\mathbf{R}^{r+1}\pi^{<d}_{G,!}\mathbb{Q}_\ell\to\mathbf{R}^{r+1}\pi^{\le d}_{G,!}\mathbb{Q}_\ell$ is surjective because the next term in the long exact sequence is $\mathbf{R}^{r+1}\pi^{d}_{G,!}\mathbb{Q}_\ell=0$. This implies that the inductive system $\mathbf{R}^{r+1}\pi^{\le d}_{G,!}\mathbb{Q}_\ell$ is eventually stable because any chain of surjections $\mathcal{F}_1\twoheadrightarrow\mathcal{F}_2\twoheadrightarrow\cdots$ of constructible sheaves on X^r has to stabilize (i.e., constructible $\mathbb{Q}_\ell$-sheaves satisfy the ascending chain condition). Also by Corollary 7.6, the inductive system $\tau_{>r+1}\mathbf{R}\pi^{\le d}_{G,!}\mathbb{Q}_\ell$ is stable. Combined with the stability of $\mathbf{R}^{r+1}\pi^{\le d}_{G,!}\mathbb{Q}_\ell$, we see that the system $\tau_{>r}\mathbf{R}\pi^{\le d}_{G,!}\mathbb{Q}_\ell$ is stable. In other words, there exists a large $d_0\in\mathcal{D}$ such that for any $d\ge d_0$, the natural map $\tau_{>r}\mathbf{R}\pi^{<d}_{G,!}\mathbb{Q}_\ell\to\tau_{>r}\mathbf{R}\pi^{\le d}_{G,!}\mathbb{Q}_\ell$ is an isomorphism.

We abbreviate $\mathrm{H}^{2r}_c(\mathrm{Sht}^{<d}_G\otimes_k\overline{k})$ by $H_{<d}$ and $\mathrm{H}^{2r}_c(\mathrm{Sht}^{\le d}_G\otimes_k\overline{k})$ by $H_{\le d}$. For $d\ge d_0$, the distinguished triangle of functors $\tau_{\le r}\to\mathrm{id}\to\tau_{>r}\to$ applied to $\mathbf{R}\pi^{<d}_{G,!}\mathbb{Q}_\ell$ and $\mathbf{R}\pi^{\le d}_{G,!}\mathbb{Q}_\ell$ gives a morphism of exact sequences

$$\begin{tikzcd}
L_{\le r}H_{<d} \arrow[r] \arrow[d] & H_{<d} \arrow[r] \arrow[d] & \mathrm{H}^{2r}(X^r\otimes_k\overline{k},\tau_{>r}\mathbf{R}\pi^{<d}_{G,!}\mathbb{Q}_\ell) \arrow[r] \arrow[d, "\wr"] & \cdots \\
L_{\le r}H_{\le d} \arrow[r] & H_{\le d} \arrow[r] & \mathrm{H}^{2r}(X^r\otimes_k\overline{k},\tau_{>r}\mathbf{R}\pi^{\le d}_{G,!}\mathbb{Q}_\ell) \arrow[r] & \cdots
\end{tikzcd}$$

Therefore the inductive system $H_{\le d}/L_{\le r}H_{\le d}$ is a subsystem of $\mathrm{H}^{2r}(X^r\otimes_k\overline{k},\tau_{>r}\mathbf{R}\pi^{\le d}_{G,!}\mathbb{Q}_\ell)$ which is stable with finite-dimensional inductive limit. Hence the inductive system $H_{\le d}/L_{\le r}H_{\le d}$ is itself stable with finite-dimensional inductive limit. Taking inductive limit on d, using that $V=\varinjlim_d H_{\le d}(r)$ and $L_{\le i}V=\varinjlim_d L_{\le i}H_{\le d}(r)$, we see that $V/L_{\le r}V\cong\varinjlim_d H_{\le d}(r)/L_{\le r}H_{\le d}(r)$ is finite dimensional. □

Lemma 7.11. *The space $\mathcal{I}_{\mathrm{Eis}}\cdot(L_{\le r}V)$ is finite-dimensional over $\mathbb{Q}_\ell$.*

Proof. Let $U\subset\mathrm{Sht}_G$ be the union of those $\mathrm{Sht}^{\le d}_G$ for $d\in\mathcal{D}$ such that $\min_{i\in\mathbb{Z}/r\mathbb{Z}}\{d(i)\}\le 2g-2$. Since $\mathrm{inst}(\mathcal{E})$ has an absolute lower bound, there are only finitely many such d with $\mathrm{Sht}^d_G\ne\varnothing$, hence U is an open substack of finite type. Let $\pi^U_G:U\to X^r$ be the restriction of π_G. For $f\in\mathscr{H}$, and any $d\in\mathcal{D}$, its action defines a map $C(f)_{d,d'}:\mathbf{R}^i\pi^{\le d}_{G,!}\mathbb{Q}_\ell\to\mathbf{R}^i\pi^{\le d'}_{G,!}\mathbb{Q}_\ell$ for sufficiently large d'. We may assume $d'>2g-2$, which means $d'(j)>2g-2$ for all j. We shall show that when $f\in\mathcal{I}_{\mathrm{Eis}}$ and $i\le r$, the image of $C(f)_{d,d'}$ is contained in the image of the map $\iota_{U,d'}:\mathbf{R}^i\pi^U_{G,!}\mathbb{Q}_\ell\to\mathbf{R}^i\pi^{\le d'}_{G,!}\mathbb{Q}_\ell$ induced by the inclusion $U\subset\mathrm{Sht}^{\le d'}_G$, which implies the proposition. By Corollary 7.6, either $\iota_{U,d'}$ is an isomorphism (if $i<r$) or when $i=r$, the

cokernel of ι is a local system on X^r. Therefore, it suffices to show the same statement for the generic stalk of the relevant complexes.

Let $\overline{\eta}$ be a geometric generic point of X^r and we use a subscript $\overline{\eta}$ to denote the fibers over $\overline{\eta}$, as in §7.3. Let $\iota_U : \mathrm{H}^r_c(U_{\overline{\eta}}) \to \mathrm{H}^r_c(\mathrm{Sht}_{G,\overline{\eta}})$ be the map induced by the inclusion of U. It suffices to show that for $f \in \mathcal{I}_{\mathrm{Eis}}$, the composition $\mathrm{H}^r_c(\mathrm{Sht}_{G,\overline{\eta}}) \xrightarrow{f*} \mathrm{H}^r_c(\mathrm{Sht}_{G,\overline{\eta}}) \twoheadrightarrow \mathrm{H}^r_c(\mathrm{Sht}_{G,\overline{\eta}})/\iota_U(\mathrm{H}^r_c(U_{\overline{\eta}}))$ is zero.

Recall from (7.11) the cohomological constant term map $\gamma_d : \mathrm{H}^r_c(\mathrm{Sht}_{G,\overline{\eta}}) \to \mathrm{H}_0(\mathrm{Sht}^d_{H,\overline{\eta}})$. By the definition of γ_d, for $d > 2g-2$, γ_d factors through the quotient $\mathrm{H}^r_c(\mathrm{Sht}_{G,\overline{\eta}})/\iota_U(\mathrm{H}^r_c(U_{\overline{\eta}}))$, and induces a map

$$\gamma_+ := \prod_{d>2g-2} \gamma_d : \mathrm{H}^r_c(\mathrm{Sht}_{G,\overline{\eta}})/\iota_U(\mathrm{H}^r_c(U_{\overline{\eta}})) \longrightarrow \prod_{d>2g-2} \mathrm{H}_0(\mathrm{Sht}^d_{H,\overline{\eta}}).$$

Both sides of the above map admit filtrations indexed by the poset $\{d \in \mathcal{D}; d > 2g-2\}$: on the LHS this is given by the image of $\mathrm{H}^r_c(\mathrm{Sht}^{\le d}_{G,\overline{\eta}})$ and on the RHS this is given by $\prod_{2g-2<d'\le d} \mathrm{H}_0(\mathrm{Sht}^{d'}_{H,\overline{\eta}})$. The map γ_+ respects these filtrations and by Corollary 7.6, the associated graded map of γ_+ under these filtrations is injective. Therefore γ_+ is injective.

By Lemma 7.8, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{H}^r_c(\mathrm{Sht}_{G,\overline{\eta}})(r/2) \arrow[r, "f*"] \arrow[d, "\prod \gamma_d"] & \mathrm{H}^r_c(\mathrm{Sht}_{G,\overline{\eta}})(r/2) \arrow[r] \arrow[d, "\prod \gamma_d"] & \mathrm{H}^r_c(\mathrm{Sht}_{G,\overline{\eta}})(r/2)/\iota_U(\mathrm{H}^r_c(U_{\overline{\eta}}))(r/2) \arrow[d, "\gamma_+"] \\
\prod_{d\in\mathcal{D}} \mathrm{H}_0(\mathrm{Sht}^d_{H,\overline{\eta}}) \arrow[r, "\mathrm{Sat}(f)*"] & \prod_{d\in\mathcal{D}} \mathrm{H}_0(\mathrm{Sht}^d_{H,\overline{\eta}}) \arrow[r] & \prod_{d>2g-2} \mathrm{H}_0(\mathrm{Sht}^d_{H,\overline{\eta}})
\end{tikzcd}$$

Since the action of $\mathscr{H}_H$ on $\prod_{d\in\mathcal{D}} \mathrm{H}_0(\mathrm{Sht}^d_{H,\overline{\eta}})$ factors through $\mathbb{Q}_\ell[\mathrm{Pic}_X(k)]$, $\mathrm{Sat}(f)$ acts by zero in the bottom arrow above. Since γ_+ is injective, the composition of the top row is also zero, as desired. □

Definition 7.12. We define the $\mathbb{Q}_\ell$-algebra $\overline{\mathscr{H}_\ell}$ to be the image of the map

$$\mathscr{H} \otimes \mathbb{Q}_\ell \longrightarrow \mathrm{End}_{\mathbb{Q}_\ell}(V) \times \mathbb{Q}_\ell[\mathrm{Pic}_X(k)]^{\iota_{\mathrm{Pic}}},$$

the product of the action map on V and $a_{\mathrm{Eis}} \otimes \mathbb{Q}_\ell$.

Lemma 7.13. *(1) For any $x \in |X|$, V is a finitely generated $\mathscr{H}_x \otimes \mathbb{Q}_\ell$-module.*

(2) The $\mathbb{Q}_\ell$-algebra $\overline{\mathscr{H}_\ell}$ is finitely generated over $\mathbb{Q}_\ell$ and is a ring with Krull dimension one.

Proof. (1) Let $\mathcal{D}_{\le n} \subset \mathcal{D}$ be the subset of those d such that $\min_i\{d(i)\} \le 2g-2+nd_x$. Let $\mathcal{D}_n = \mathcal{D}_{\le n} - \mathcal{D}_{\le n-1}$. For each $n \ge 0$, let $U_n = \cup_{d\in\mathcal{D}_{\le n}} \mathrm{Sht}^{\le d}_G$, then $U_0 \subset U_1 \subset \cdots$ are finite type open substacks of Sht_G which exhaust Sht_G. Let $\pi_n : U_n \to X^r$ be the restriction of π_G, and let $K_n = \mathbf{R}\pi_{n,!}\mathbb{Q}_\ell$. The inclusion $U_n \hookrightarrow U_{n+1}$ induces maps $\iota_n : K_n \to K_{n+1}$. Let C_{n+1} be the cone of ι_n. Then by Corollary 7.6, when $n \ge 0$, C_{n+1} is a successive extension of shifted local systems $\pi^d_{H,!}\mathbb{Q}_\ell[-r](-r/2)$ for those $d \in \mathcal{D}_{n+1}$. In particular, for $n \ge 0$, C_{n+1} is a shifted local system in degree r and pure of weight 0 as a complex.

By construction, the action of $h_x \in \mathscr{H}_x$ on $\mathrm{H}^*_c(\mathrm{Sht}_G \otimes_k \overline{k})$ is induced from the correspondence $\mathrm{Sht}_G(h_x)$, which restricts to a correspondence ${}^{\le n}\mathrm{Sht}_G(h_x) = \overleftarrow{p}^{-1}(U_n)$ between U_n and U_{n+1}. Similar to the construction of $C(h_x)_{d,d'}$ in (7.1), the fundamental class of ${}^{\le n}\mathrm{Sht}_G(h_x)$ gives a map $C(h_x)_n : K_n \to K_{n+1}$. Since $C(h_x)_n \circ \iota_{n-1} = \iota_n \circ C(h_x)_{n-1}$, we have the induced map $\tau_n : C_n \to C_{n+1}$. We claim that τ_n is an isomorphism for $n > 0$. In fact, since C_n and C_{n+1} are local systems in degree r, it suffices to check that τ_n induces an isomorphism between the geometric generic stalks $C_{n,\overline{\eta}}$ and $C_{n+1,\overline{\eta}}$. By Corollary 7.6, we have an isomorphism induced from the maps γ_d for $d \in \mathcal{D}_n$ (cf. (7.11))

$$C_{n,\overline{\eta}} \cong \bigoplus_{d\in\mathcal{D}_n} \mathrm{H}_0(\mathrm{Sht}^d_{H,\overline{\eta}}).$$

By Lemma 7.8, $\tau_{n,\overline{\eta}} : C_{n,\overline{\eta}} \to C_{n+1,\overline{\eta}}$ is the same as the direct sum of the isomorphisms $t_x : \mathrm{H}_0(\mathrm{Sht}^d_{H,\overline{\eta}}) \to \mathrm{H}_0(\mathrm{Sht}^{d+d_x}_{H,\overline{\eta}})$ (the other term $q_x t_x^{-1} : \mathrm{H}_0(\mathrm{Sht}^d_{H,\overline{\eta}}) \to \mathrm{H}_0(\mathrm{Sht}^{d-d_x}_{H,\overline{\eta}})$ does not appear because $d-d_x \in \mathcal{D}_{\leq n-1}$ hence the corresponding contribution becomes zero in $C_{n+1,\overline{\eta}}$). Therefore $\tau_{n,\overline{\eta}}$ is an isomorphism, hence so is τ_n.

We claim that there exists $n_0 \geq 0$ such that for any $n \geq n_0$, the map

$$W_{\leq 2r}\mathrm{H}^{2r+1}(X^r \otimes_k \overline{k}, K_n) \longrightarrow W_{\leq 2r}\mathrm{H}^{2r+1}(X^r \otimes_k \overline{k}, K_{n+1})$$

is an isomorphism. Here $W_{\leq 2r}$ is the weight filtration using Frobenius weights. In fact, the next term in the long exact sequence is $W_{\leq 2r}\mathrm{H}^{2r+1}(X^r \otimes_k \overline{k}, C_{n+1})$, which is zero because C_{n+1} is pure of weight 0. Therefore the natural map $W_{\leq 2r}\mathrm{H}^{2r+1}(X^r \otimes_k \overline{k}, K_n) \to W_{\leq 2r}\mathrm{H}^{2r+1}(X^r \otimes_k \overline{k}, K_{n+1})$ is always surjective for $n \geq 0$, hence it has to be an isomorphism for sufficiently large n.

The triangle $K_n \to K_{n+1} \to C_{n+1} \to K_n[1]$ gives a long exact sequence

$$\begin{aligned} &\mathrm{H}^{2r}(X^r \otimes_k \overline{k}, K_n) \longrightarrow \mathrm{H}^{2r}(X^r \otimes_k \overline{k}, K_{n+1}) \longrightarrow \mathrm{H}^{2r}(X^r \otimes_k \overline{k}, C_{n+1}) \longrightarrow \\ \longrightarrow\ &W_{\leq 2r}\mathrm{H}^{2r+1}(X^r \otimes_k \overline{k}, K_n) \longrightarrow W_{\leq 2r}\mathrm{H}^{2r+1}(X^r \otimes_k \overline{k}, K_{n+1}) \end{aligned} \tag{7.17}$$

Here we are using the fact that $\mathrm{H}^{2r}(X^r \otimes_k \overline{k}, C_{n+1})$ is pure of weight $2r$ (since C_{n+1} is pure of weight 0). For $n \geq n_0$, the last map above is an isomorphism, therefore the first row of (7.17) is exact on the right.

Let $F_{\leq n}V$ be the image of $\mathrm{H}^{2r}(X^r \otimes_k \overline{k}, K_n)(r) \to \varinjlim_n \mathrm{H}^{2r}(X^r \otimes_k \overline{k}, K_n)(r) = V$. Then for $n \geq n_0$, the exactness of (7.17) implies $\mathrm{H}^{2r}(X^r \otimes_k \overline{k}, C_{n+1})(r) \twoheadrightarrow \mathrm{Gr}^F_{n+1}V$ for $n \geq n_0$. The Hecke operator $C(h_x)$ sends $F_{\leq n}V$ to $F_{\leq n+1}V$ and induces a map $\mathrm{Gr}^F_n C(h_x) : \mathrm{Gr}^F_n V \to \mathrm{Gr}^F_{n+1}V$. We have a commutative diagram for $n \geq n_0$

$$\begin{tikzcd} \mathrm{H}^{2r}(X^r \otimes_k \overline{k}, C_n)(r) \arrow[r, "\mathrm{H}^{2r}(X^r\otimes_k\overline{k},\tau_n)"] \arrow[d] & \mathrm{H}^{2r}(X^r \otimes_k \overline{k}, C_{n+1})(r) \arrow[d, two heads] \\ \mathrm{Gr}^F_n V \arrow[r, "\mathrm{Gr}^F_n C(h_x)"] & \mathrm{Gr}^F_{n+1} V \end{tikzcd}$$

The fact that $\tau_n : C_n \to C_{n+1}$ is an isomorphism implies that $\mathrm{Gr}^F_n C(h_x)$ is surjective for $n \geq n_0$. Therefore the action map

$$\mathscr{H}_x \otimes_{\mathbb{Q}} F_{\leq n_0}V = \mathbb{Q}[h_x] \otimes_{\mathbb{Q}} F_{\leq n_0}V \longrightarrow V$$

is surjective by checking the surjectivity on the associated graded. Since $F_{\leq n_0}V$ is finite-dimensional over $\mathbb{Q}_\ell$, V is finitely generated as an $\mathscr{H}_x \otimes \mathbb{Q}_\ell$-module.

(2) We have $\overline{\mathscr{H}_\ell} \subset \mathrm{End}_{\mathscr{H}_x \otimes \mathbb{Q}_\ell}(V \oplus \mathbb{Q}_\ell[\mathrm{Pic}_X(k)]^{\iota_{\mathrm{Pic}}})$. Since both V and $\mathbb{Q}_\ell[\mathrm{Pic}_X(k)]^{\iota_{\mathrm{Pic}}}$ are finitely generated $\mathscr{H}_x \otimes \mathbb{Q}_\ell$-modules by Part (1) and Lemma 4.2, $\mathrm{End}_{\mathscr{H}_x \otimes \mathbb{Q}_\ell}(V \oplus \mathbb{Q}_\ell[\mathrm{Pic}_X(k)]^{\iota_{\mathrm{Pic}}})$ is also finitely generated as an $\mathscr{H}_x \otimes \mathbb{Q}_\ell$-module. Since $\mathscr{H}_x \otimes \mathbb{Q}_\ell$ is a polynomial ring in one variable over $\mathbb{Q}_\ell$, $\overline{\mathscr{H}_\ell}$ is a finitely generated algebra over $\mathbb{Q}_\ell$ of Krull dimension at most one. Since $\overline{\mathscr{H}_\ell} \to \mathbb{Q}_\ell[\mathrm{Pic}_X(k)]^{\iota_{\mathrm{Pic}}}$ is surjective by Lemma 4.2 and $\mathbb{Q}_\ell[\mathrm{Pic}_X(k)]^{\iota_{\mathrm{Pic}}}$ has Krull dimension one, $\overline{\mathscr{H}_\ell}$ also has Krull dimension one. ☐

The map $\overline{a}_{\mathrm{Eis}} : \overline{\mathscr{H}_\ell} \to \mathbb{Q}_\ell[\mathrm{Pic}_X(k)]^{\iota_{\mathrm{Pic}}}$ is surjective by Lemma 4.2 (2). It induces a closed embedding $\mathrm{Spec}(\overline{a}_{\mathrm{Eis}}) : Z_{\mathrm{Eis},\mathbb{Q}_\ell} = \mathrm{Spec}\,\mathbb{Q}_\ell[\mathrm{Pic}_X(k)]^{\iota_{\mathrm{Pic}}} \hookrightarrow \mathrm{Spec}\,\overline{\mathscr{H}_\ell}$.

Theorem 7.14 (Cohomological spectral decomposition)**.** *(1) There is a decomposition of the reduced scheme of* $\mathrm{Spec}\,\overline{\mathscr{H}_\ell}$ *into a disjoint union*

$$\mathrm{Spec}\left(\overline{\mathscr{H}_\ell}\right)^{\mathrm{red}} = Z_{\mathrm{Eis},\mathbb{Q}_\ell} \coprod Z^r_{0,\ell} \tag{7.18}$$

where $Z^r_{0,\ell}$ *consists of a finite set of closed points. There is a unique decomposition*

$$V = V_{\mathrm{Eis}} \oplus V_0$$

into $\mathscr{H} \otimes \mathbb{Q}_\ell$*-submodules, such that* $\mathrm{Supp}(V_{\mathrm{Eis}}) \subset Z_{\mathrm{Eis},\mathbb{Q}_\ell}$ *and* $\mathrm{Supp}(V_0) = Z^r_{0,\ell}$. [4]

(2) The subspace V_0 *is finite dimensional over* $\mathbb{Q}_\ell$.

[4] When we talk about the support of a coherent module M over a Noetherian ring R, we always mean a closed subset of $\mathrm{Spec}\,R$ with the reduced scheme structure.

Proof. (1) Let $V' = L_{\le r}V$. Let $\overline{\mathcal{I}}_{\mathrm{Eis}} \subset \overline{\mathscr{H}_\ell}$ be the ideal generated by the image of $\mathcal{I}_{\mathrm{Eis}}$. By Lemma 7.13, V' is a submodule of a finitely generated module V over the noetherian ring $\overline{\mathscr{H}_\ell}$, therefore V' is also finitely generated. By Lemma 7.11, $\overline{\mathcal{I}}_{\mathrm{Eis}}V'$ is a finite-dimensional $\overline{\mathscr{H}_\ell}$-submodule of V'. Let $Z' \subset \mathrm{Spec}(\overline{\mathscr{H}_\ell})^{\mathrm{red}}$ be the finite set of closed points corresponding to the action of $\overline{\mathscr{H}_\ell}$ on $\overline{\mathcal{I}}_{\mathrm{Eis}}V'$. We claim that $\mathrm{Supp}(V')$ is contained in the union $Z_{\mathrm{Eis},\mathbb{Q}_\ell} \cup Z'$. In fact, suppose $f \in \overline{\mathscr{H}_\ell}$ lies in the defining radical ideal $\mathcal{J}$ of $Z_{\mathrm{Eis},\mathbb{Q}_\ell} \cup Z'$, then after replacing f by a power of it, we have $f \in \overline{\mathcal{I}}_{\mathrm{Eis}}$ (since $\mathcal{J}$ is contained in the radical of $\overline{\mathcal{I}}_{\mathrm{Eis}}$) and f acts on $\overline{\mathcal{I}}_{\mathrm{Eis}}V'$ by zero. Therefore f^2 acts on V' by zero, hence f lies in the radical ideal defining $\mathrm{Supp}(V')$.

By Lemma 7.10, V/V' is finite-dimensional. Let $Z'' \subset \mathrm{Spec}(\overline{\mathscr{H}_\ell})^{\mathrm{red}}$ be the support of V/V' as a $\overline{\mathscr{H}_\ell}$-module, which is a finite set. Then $\mathrm{Spec}(\overline{\mathscr{H}_\ell})^{\mathrm{red}} = \mathrm{Supp}(V) \cup Z_{\mathrm{Eis},\mathbb{Q}_\ell} = Z_{\mathrm{Eis},\mathbb{Q}_\ell} \cup Z' \cup Z''$. Let $Z^r_{0,\ell} = (Z' \cup Z'') - Z_{\mathrm{Eis},\mathbb{Q}_\ell}$, we get the desired decomposition (7.18).

According to (7.18), the finitely generated $\overline{\mathscr{H}_\ell}$-module V, viewed as a coherent sheaf on $\mathrm{Spec}\,\overline{\mathscr{H}_\ell}$, can be uniquely decomposed into

$$V = V_{\mathrm{Eis}} \oplus V_0$$

with $\mathrm{Supp}(V_{\mathrm{Eis}}) \subset Z_{\mathrm{Eis},\mathbb{Q}_\ell}$ and $\mathrm{Supp}(V_0) = Z^r_{0,\ell}$.

(2) We know that V_0 is a coherent sheaf on the scheme $\mathrm{Spec}\,\overline{\mathscr{H}_\ell}$ which is of finite type over $\mathbb{Q}_\ell$ and that $\mathrm{Supp}(V_0) = Z^r_{0,\ell}$ is finite. Therefore V_0 is finite dimensional over $\mathbb{Q}_\ell$. □

7.4.1. *The case $r = 0$.* Let us reformulate the result in Theorem 7.14 in the case $r = 0$ in terms of automorphic forms. Let $\mathcal{A} = C_c(G(F)\backslash G(\mathbb{A}_F)/K, \mathbb{Q})$ be the space of compactly supported $\mathbb{Q}$-valued unramified automorphic forms, where $K = \prod_x G(\mathcal{O}_x)$. This is a $\mathbb{Q}$-form of the $\mathbb{Q}_\ell$-vector space V for $r = 0$. Let $\mathscr{H}_{\mathrm{aut}}$ be the image of the action map $\mathscr{H} \to \mathrm{End}_{\mathbb{Q}}(\mathcal{A}) \times \mathbb{Q}[\mathrm{Pic}_X(k)]^{\iota_{\mathrm{Pic}}}$. The $\mathbb{Q}_\ell$-algebra $\mathscr{H}_{\mathrm{aut},\mathbb{Q}_\ell} := \mathscr{H}_{\mathrm{aut}} \otimes \mathbb{Q}_\ell$ is the algebra $\overline{\mathscr{H}_\ell}$ defined in Definition 7.12 for $r = 0$.

Theorem 7.14 for $r = 0$ reads

$$\mathrm{Spec}\,\mathscr{H}^{\mathrm{red}}_{\mathrm{aut},\mathbb{Q}_\ell} = Z_{\mathrm{Eis},\mathbb{Q}_\ell} \coprod Z^0_{0,\ell} \tag{7.19}$$

where $Z^0_{0,\ell}$ is a finite set of closed points. Below we will strengthen this decomposition to work over $\mathbb{Q}$, and link $Z^0_{0,\ell}$ to the set of cuspidal automorphic representations.

7.4.2. *Positivity and reducedness.* The first thing to observe is that $\mathscr{H}_{\mathrm{aut}}$ is already reduced. In fact, we may extend the Petersson inner product on $\mathcal{A}$ to a positive definitive quadratic form on $\mathcal{A}_{\mathbb{R}}$. By the $r = 0$ case of Lemma 7.2, $\mathscr{H}_{\mathrm{aut}}$ acts on $\mathcal{A}_{\mathbb{R}}$ as self-adjoint operators, its image in $\mathrm{End}(\mathcal{A})$ is therefore reduced. Since $\mathbb{Q}[\mathrm{Pic}_X(k)]^{\iota_{\mathrm{Pic}}}$ is reduced as well, we conclude that $\mathscr{H}_{\mathrm{aut}}$ is reduced.

Let $\mathcal{A}_{\mathrm{cusp}} \subset \mathcal{A}$ be the finite-dimensional $\mathbb{Q}$-vector space of cusp forms. Let $\mathscr{H}_{\mathrm{cusp}}$ be the image of $\mathscr{H}_{\mathrm{aut}}$ in $\mathrm{End}_{\mathbb{Q}}(\mathcal{A}_{\mathrm{cusp}})$. Then $\mathscr{H}_{\mathrm{cusp}}$ is a reduced artinian $\mathbb{Q}$-algebra, hence a product of fields. Let $Z_{\mathrm{cusp}} = \mathrm{Spec}\,\mathscr{H}_{\mathrm{cusp}}$. Then a point in Z_{cusp} is the same as an everywhere unramified cuspidal automorphic representation π of G in the sense of §1.2. Therefore we have a canonical isomorphism

$$\mathscr{H}_{\mathrm{cusp}} = \prod_{\pi \in Z_{\mathrm{cusp}}} E_\pi$$

where E_π is the coefficient field of π.

Lemma 7.15. *(1) There is a canonical isomorphism of $\mathbb{Q}$-algebras*

$$\mathscr{H}_{\mathrm{aut}} \cong \mathbb{Q}[\mathrm{Pic}_X(k)]^{\iota_{\mathrm{Pic}}} \times \mathscr{H}_{\mathrm{cusp}}$$

Equivalently, we have a decomposition into disjoint reduced closed subschemes

$$\mathrm{Spec}\,\mathscr{H}_{\mathrm{aut}} = Z_{\mathrm{Eis}} \coprod Z_{\mathrm{cusp}}. \tag{7.20}$$

(2) We have $Z^0_{0,\ell} = Z_{\mathrm{cusp},\mathbb{Q}_\ell}$, the base change of Z_{cusp} from $\mathbb{Q}$ to $\mathbb{Q}_\ell$.

Proof. (1) The $\mathbb{Q}$ version of Lemma 7.13 says that $\mathscr{H}_{\mathrm{aut}}$ is a finitely generated $\mathbb{Q}$-algebra, and that $\mathcal{A}$ is a finitely generated $\mathscr{H}_{\mathrm{aut}}$-module. By the same argument of Theorem 7.14, we get a decomposition

$$\mathrm{Spec}\,\mathscr{H}^{\mathrm{red}}_{\mathrm{aut}} = \mathrm{Spec}\,\mathscr{H}_{\mathrm{aut}} = Z_{\mathrm{Eis}} \coprod Z_0 \tag{7.21}$$

where Z_0 is a finite collection of closed points. Correspondingly we have a decomposition

$$\mathcal{A} = \mathcal{A}_{\mathrm{Eis}} \oplus \mathcal{A}_0$$

with $\mathrm{Supp}(\mathcal{A}_{\mathrm{Eis}}) \subset Z_{\mathrm{Eis}}$ and $\mathrm{Supp}(\mathcal{A}_0) = Z_0$. Since $\mathcal{A}_0$ is finitely generated over $\mathscr{H}_{\mathrm{aut}}$ with finite support, it is finite dimensional over $\mathbb{Q}$. Since $\mathcal{A}_0$ is finite dimensional and stable under $\mathscr{H}$, we necessarily have $\mathcal{A}_0 \subset \mathcal{A}_{\mathrm{cusp}}$ (see [16, Lemme 8.13]; in fact in our case it can be easily deduced from the $r = 0$ case of Lemma 7.8).

We claim that $\mathcal{A}_0 = \mathcal{A}_{\mathrm{cusp}}$. To show the inclusion in the other direction, it suffices to show that any cuspidal Hecke eigenform $\varphi \in \mathcal{A}_{\mathrm{cusp}} \otimes \overline{\mathbb{Q}}$ lies in $\mathcal{A}_0 \otimes \overline{\mathbb{Q}}$. Suppose this is not the case for φ, letting $\lambda : \mathscr{H} \to \overline{\mathbb{Q}}$ be the character by which $\mathscr{H}$ acts on φ, then $\lambda \notin Z_0(\overline{\mathbb{Q}})$. By (7.21), $\lambda \in Z_{\mathrm{Eis}}(\overline{\mathbb{Q}})$, which means that the action of $\mathscr{H}$ on φ factors through $\mathbb{Q}[\mathrm{Pic}_X(k)]$ via a_{Eis}, which is impossible.

Now $\mathcal{A}_0 = \mathcal{A}_{\mathrm{cusp}}$ implies that $Z_0 = \mathrm{Supp}(\mathcal{A}_0) = \mathrm{Supp}(\mathcal{A}_{\mathrm{cusp}}) = Z_{\mathrm{cusp}}$. Combining with (7.21), we get (7.20).

Part (2) follows from comparing (7.19) to the base change of (7.20) to $\mathbb{Q}_\ell$. □

7.5. **Decomposition of the Heegner–Drinfeld cycle class.** In previous subsections, we have been working with the middle-dimensional cohomology (with compact support) of $\mathrm{Sht}_G = \mathrm{Sht}_G^r$, and we established a decomposition of it as an $\mathscr{H}_{\overline{\mathbb{Q}}_\ell}$-module. Exactly the same argument works if we replace Sht_G with $\mathrm{Sht}'_G = \mathrm{Sht}'^r_G$. Instead of repeating the argument we simply state the corresponding result for Sht'_G in what follows.

Let

$$V' = \mathrm{H}_c^{2r}(\mathrm{Sht}'_G \otimes_k \overline{k}, \mathbb{Q}_\ell)(r).$$

Then V' is equipped with a $\mathbb{Q}_\ell$-valued cup product pairing

$$(\cdot,\cdot) : V' \otimes_{\mathbb{Q}_\ell} V' \longrightarrow \mathbb{Q}_\ell \tag{7.22}$$

and an action of $\mathscr{H}$ by self-adjoint operators.

Similar to Definition 7.12, we define the $\mathbb{Q}_\ell$-algebra $\overline{\mathscr{H}_\ell}'$ to be the image of the map

$$\mathscr{H} \otimes \mathbb{Q}_\ell \longrightarrow \mathrm{End}_{\mathbb{Q}_\ell}(V') \times \mathbb{Q}_\ell[\mathrm{Pic}_X(k)]^{\iota_{\mathrm{Pic}}}.$$

Theorem 7.16 (Variant of Lemma 7.13 and Theorem 7.14). *(1) For any $x \in |X|$, V' is a finitely generated $\mathscr{H}_x \otimes \mathbb{Q}_\ell$-module.*

(2) The $\mathbb{Q}_\ell$-algebra $\overline{\mathscr{H}_\ell}'$ is finitely generated over $\mathbb{Q}_\ell$ and is one-dimensional as a ring.

(3) There is a decomposition of the reduced scheme of $\mathrm{Spec}\,\overline{\mathscr{H}_\ell}'$ into a disjoint union

$$\mathrm{Spec}\left(\overline{\mathscr{H}_\ell}'\right)^{\mathrm{red}} = Z_{\mathrm{Eis},\mathbb{Q}_\ell} \coprod Z'^r_{0,\ell} \tag{7.23}$$

where $Z'^r_{0,\ell}$ consists of a finite set of closed points. There is a unique decomposition

$$V' = V'_{\mathrm{Eis}} \oplus V'_0$$

into $\mathscr{H} \otimes \mathbb{Q}_\ell$-submodules, such that $\mathrm{Supp}(V'_{\mathrm{Eis}}) \subset Z_{\mathrm{Eis},\mathbb{Q}_\ell}$ and $\mathrm{Supp}(V'_0) = Z'^r_{0,\ell}$.

(4) The subspace V'_0 is finite dimensional over $\mathbb{Q}_\ell$.

We may further decompose $V'_{\overline{\mathbb{Q}}_\ell} := V' \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell$ according to points in $Z'^r_{0,\ell}(\overline{\mathbb{Q}}_\ell)$. A point in $Z'^r_{0,\ell}(\overline{\mathbb{Q}}_\ell)$ is a maximal ideal $\mathfrak{m} \subset \mathscr{H}_{\overline{\mathbb{Q}}_\ell}$, or equivalently a ring homomorphism $\mathscr{H} \to \overline{\mathbb{Q}}_\ell$ whose kernel is $\mathfrak{m}$. We have a decomposition

$$V'_{\overline{\mathbb{Q}}_\ell} = V'_{\mathrm{Eis},\overline{\mathbb{Q}}_\ell} \oplus \Big(\bigoplus_{\mathfrak{m} \in Z'^r_{0,\ell}(\overline{\mathbb{Q}}_\ell)} V'_{\mathfrak{m}}\Big). \tag{7.24}$$

Then $V'_{\mathfrak{m}}$ is characterized as the largest $\overline{\mathbb{Q}}_\ell$-subspace of $V'_{\overline{\mathbb{Q}}_\ell}$ on which the action of $\mathfrak{m}$ is locally nilpotent. By Theorem 7.16, $V'_{\mathfrak{m}}$ turns out to be the localization of V' at the maximal ideal $\mathfrak{m}$, hence our notation $V'_{\mathfrak{m}}$ is consistent with the standard notation used in commutative algebra.

We may decompose the cycle class $\mathrm{cl}(\theta^\mu_*[\mathrm{Sht}^\mu_T]) \in V'_{\overline{\mathbb{Q}}_\ell}$ according to the decomposition (7.24)

$$\mathrm{cl}(\theta^\mu_*[\mathrm{Sht}^\mu_T]) = [\mathrm{Sht}_T]_{\mathrm{Eis}} + \sum_{\mathfrak{m} \in Z^r_{0,\ell}(\overline{\mathbb{Q}}_\ell)} [\mathrm{Sht}_T]_{\mathfrak{m}} \tag{7.25}$$

where $[\mathrm{Sht}_T]_{\mathrm{Eis}} \in V'_{\mathrm{Eis}}$ and $[\mathrm{Sht}_T]_{\mathfrak{m}} \in V'_{\mathfrak{m}}$.

Corollary 7.17. *(1) The decomposition (7.24) is an orthogonal decomposition under the cup product pairing (7.22) on V'.*

(2) For any $f \in \mathscr{H}$, we have

$$\mathbb{I}_r(f) = ([\mathrm{Sht}_T]_{\mathrm{Eis}}, f * [\mathrm{Sht}_T]_{\mathrm{Eis}}) + \sum_{\mathfrak{m} \in Z'^{r}_{0,\ell}(\overline{\mathbb{Q}}_\ell)} \mathbb{I}_r(\mathfrak{m}, f) \tag{7.26}$$

where

$$\mathbb{I}_r(\mathfrak{m}, f) := ([\mathrm{Sht}_T]_{\mathfrak{m}}, f * [\mathrm{Sht}_T]_{\mathfrak{m}}).$$

Proof. The orthogonality of the decomposition (7.24) follows from the self-adjointness of $\mathscr{H}$ with respect to the cup product pairing, i.e., variant of Lemma 7.2 for Sht'_G. The formula (7.26) then follows from the orthogonality of the terms in the decomposition (7.25). □

Part 3. The comparison

8. Comparison for most Hecke functions

The goal of this section is to prove the key identity (1.9) for most Hecke functions. More precisely, we will prove the following theorem.

Theorem 8.1. *Let D be an effective divisor on X of degree $d \geq \max\{2g'-1, 2g\}$. Then for any $u \in \mathbb{P}^1(F) - \{1\}$ we have*

$$(\log q)^{-r} \mathbb{J}_r(u, h_D) = \mathbb{I}_r(u, h_D). \tag{8.1}$$

In particular, we have

$$(\log q)^{-r} \mathbb{J}_r(h_D) = \mathbb{I}_r(h_D). \tag{8.2}$$

For the definition of $\mathbb{J}_r(u, h_D)$ and $\mathbb{I}_r(u, h_D)$, see (2.16) and (6.11) respectively.

8.1. Direct image of $f_{\mathcal{M}}$.

8.1.1. *The local system $L(\rho_i)$.* Let $j : X_d^\circ \subset X_d \subset \widehat{X}_d$ be the locus of multiplicity-free divisors. Taking the preimage of X_d° under the branched cover $X'^d \to X^d \to X_d$, we get an étale Galois cover

$$u : X'^{d,\circ} \longrightarrow X^{d,\circ} \longrightarrow X_d^\circ$$

with Galois group $\Gamma_d := \{\pm 1\}^d \rtimes S_d$. For $0 \leq i \leq d$, let χ_i be the character $\{\pm 1\}^d \to \{\pm 1\}$ that is nontrivial on the first i factors and trivial on the rest. Let $S_{i,d-i} \cong S_i \times S_{d-i}$ be the subgroup of S_d stabilizing $\{1, 2, \cdots, i\} \subset \{1, \cdots, d\}$. Then χ_i extends to the subgroup $\Gamma_d(i) = \{\pm 1\}^d \rtimes S_{i,d-i}$ of Γ_d with the trivial representation on the $S_{i,d-i}$-factor. The induced representation

$$\rho_i = \mathrm{Ind}_{\Gamma_d(i)}^{\Gamma_d}(\chi_i \boxtimes \mathbf{1}) \tag{8.3}$$

is an irreducible representation of Γ_d. This representation gives rise to an irreducible local system $L(\rho_i)$ on X_d°. Let $K_i := j_{!*}(L(\rho_i)[d])[-d]$ be the middle extension of $L(\rho_i)$ (see [3, 2.1.7]). Then K_i is a shifted simple perverse sheaf on $\widehat{X}_d$.

Proposition 8.2. *Suppose $d \geq 2g'-1$. Then we have a canonical isomorphism of shifted perverse sheaves*

$$\mathbf{R}f_{\mathcal{M},*}\mathbb{Q}_\ell \cong \bigoplus_{i,j=0}^{d} (K_i \boxtimes K_j)|_{\mathcal{A}_d}. \tag{8.4}$$

Here $K_i \boxtimes K_j$ lives on $\widehat{X}_d \times_{\mathrm{Pic}_X^d} \widehat{X}_d$, which contains $\mathcal{A}_d$ as an open subscheme.

Proof. By Proposition 6.1(4), $f_{\mathcal{M}}$ is the restriction of $\widehat{\nu}_d \times \widehat{\nu}_d : \widehat{X}'_d \times_{\mathrm{Pic}_X^d} \widehat{X}'_d \to \widehat{X}_d \times_{\mathrm{Pic}_X^d} \widehat{X}_d$, where $\widehat{\nu}_d : \widehat{X}'_d \to \widehat{X}_d$ is the norm map. By Proposition 6.1(3), $\widehat{\nu}_d$ is also proper. Therefore by the Künneth formula, it suffices to show that

$$\mathbf{R}\widehat{\nu}_{d,*}\mathbb{Q}_\ell \cong \bigoplus_{i=0}^{d} K_i. \tag{8.5}$$

We claim that $\widehat{\nu}_d$ is a small map (see [11, 6.2]). In fact the only positive dimension fibers are over the zero section $\mathrm{Pic}^d_X \hookrightarrow \widehat{X}_d$, which has codimension $d-g+1$. On the other hand, the restriction of $\widehat{\nu}_d$ to the zero section is the norm map $\mathrm{Pic}^d_{X'} \to \mathrm{Pic}^d_X$, which has fiber dimension $g-1$. The condition $d \geq 2g'-1 \geq 3g-2$ implies $d-g+1 \geq 2(g-1)+1$, therefore $\widehat{\nu}_d$ is a small map.

Now $\widehat{\nu}_d$ is proper, small with smooth and geometrically irreducible source, $\mathbf{R}\widehat{\nu}_{d,*}\mathbb{Q}_\ell$ is the middle extension of its restriction to any dense open subset of $\widehat{X}_d$ (see [11, Theorem at the end of 6.2]). In particular, $\mathbf{R}\widehat{\nu}_{d,*}\mathbb{Q}_\ell$ is the middle extension of its restriction to X°_d. It remains to show

$$\mathbf{R}\widehat{\nu}_{d,*}\mathbb{Q}_\ell|_{X^\circ_d} \cong \bigoplus_{i=0}^{d} L(\rho_i). \tag{8.6}$$

Let $\nu^\circ_d : X'^\circ_d = \nu_d^{-1}(X^\circ_d) \to X^\circ_d$ be the restriction of $\nu_d : X'_d \to X_d$ over X°_d. Then $\mathbf{R}\nu^\circ_{d,*}\mathbb{Q}_\ell$ is the local system on X°_d associated with the representation $\mathrm{Ind}^{\Gamma_d}_{S_d}\mathbb{Q}_\ell = \mathbb{Q}_\ell[\Gamma_d/S_d]$ of Γ_d. A basis $\{\mathbf{1}_\varepsilon\}$ of $\mathbb{Q}_\ell[\Gamma_d/S_d]$ is given by the indicator functions of the S_d-coset of $\varepsilon \in \{\pm 1\}^d$. For any character $\chi : \{\pm 1\}^d \to \{\pm 1\}$, let $\mathbf{1}_\chi := \sum_\varepsilon \chi(\varepsilon)\mathbf{1}_\varepsilon \in \mathbb{Q}_\ell[\Gamma_d/S_d]$. For the character χ_i considered in §8.1.1, $\mathbf{1}_{\chi_i}$ is invariant under $S_{i,d-i}$, and therefore we have a Γ_d-equivariant embedding $\rho_i = \mathrm{Ind}^{\Gamma_d}_{\Gamma_d(i)}(\chi_i \boxtimes \mathbf{1}) \hookrightarrow \mathbb{Q}_\ell[\Gamma_d/S_d]$. Checking total dimensions we conclude that

$$\mathbb{Q}_\ell[\Gamma_d/S_d] \cong \bigoplus_{i=0}^{d} \rho_i.$$

This gives a canonical isomorphism of local systems $\mathbf{R}\nu^\circ_{d,*}\mathbb{Q}_\ell \cong \oplus_{i=0}^d L(\rho_i)$, which is (8.6). $\square$

In §6.2.3, we have defined a self-correspondence $\mathcal{H} = \mathrm{Hk}^1_{\mathcal{M},d}$ of $\mathcal{M}_d$ over $\mathcal{A}_d$. Recall that $\mathcal{A}^\diamondsuit_d \subset \mathcal{A}_d$ is the open subscheme $\widehat{X}_d \times_{\mathrm{Pic}^d_X} X_d$, and $\mathcal{M}^\diamondsuit_d$ and $\mathcal{H}^\diamondsuit$ are the restrictions of $\mathcal{M}_d$ and $\mathcal{H}$ to $\mathcal{A}^\diamondsuit_d$. Recall that $[\mathcal{H}^\diamondsuit] \in \mathrm{Ch}_{2d-g+1}(\mathcal{H})_\mathbb{Q}$ is the fundamental cycle of the closure of $\mathcal{H}^\diamondsuit$.

Proposition 8.3. *Suppose $d \geq 2g'-1$. Then the action $f_{\mathcal{M},!}[\mathcal{H}^\diamondsuit]$ on $\mathbf{R}f_{\mathcal{M}*}\mathbb{Q}_\ell$ preserves each direct summand $K_i \boxtimes K_j$ under the decomposition* (8.4)*, and acts on $K_i \boxtimes K_j$ by the scalar $(d-2j)$.*

Proof. By Proposition 8.2, $\mathbf{R}f_{\mathcal{M}*}\mathbb{Q}_\ell$ is a shifted perverse sheaf all of whose simple constituents have full support. Therefore it suffices to prove the same statement after restricting to any dense open subset $U \subset \mathcal{A}_d$. We work with $U = \mathcal{A}^\diamondsuit_d$.

Recall $\mathcal{H}$ is indeed a self-correspondence of $\mathcal{M}_d$ over $\widetilde{\mathcal{A}}_d$ (see §6.2.2):

$$\begin{tikzcd} & \mathcal{H} \arrow[dl, "\gamma_0"'] \arrow[dr, "\gamma_1"] & \\ \mathcal{M}_d \arrow[dr, "\widetilde{f}_\mathcal{M}"'] & & \mathcal{M}_d \arrow[dl, "\widetilde{f}_\mathcal{M}"] \\ & \widetilde{\mathcal{A}}_d & \end{tikzcd} \tag{8.7}$$

By Lemma 6.3, the diagram (8.7) restricted to $\widetilde{\mathcal{A}}^\diamondsuit_d$ (the preimage of $\mathcal{A}^\diamondsuit_d$ in $\widetilde{\mathcal{A}}_d$) is obtained from the following correspondence via base change along the second projection $\mathrm{pr}_2 : \widetilde{\mathcal{A}}^\diamondsuit_d \cong \widehat{X}'_d \times_{\mathrm{Pic}^d_X} X_d \to X_d$ which is smooth

$$\begin{tikzcd} & I'_d \arrow[dl, "\mathrm{pr}"'] \arrow[dr, "q"] & \\ X'_d \arrow[dr, "\nu_d"'] & & X'_d \arrow[dl, "\nu_d"] \\ & X_d & \end{tikzcd}$$

Here for (D,y) in the universal divisor $I'_d \subset X'_d \times X'$, $\mathrm{pr}(D,y) = D$ and $q(D,y) = D - y + \sigma(y)$.

Let $T_d := \nu_{d,!}[I'_d] : \mathbf{R}\nu_{d,*}\mathbb{Q}_\ell \to \mathbf{R}\nu_{d,*}\mathbb{Q}_\ell$ be the operator on $\mathbf{R}\nu_{d,*}\mathbb{Q}_\ell$ induced from the cohomological correspondence between the constant sheaf $\mathbb{Q}_\ell$ on X'_d and itself given by the fundamental class of I'_d. Under the isomorphism $\mathbf{R}\widetilde{f}_{\mathcal{M},!}\mathbb{Q}_\ell|_{\mathcal{A}_d^\diamondsuit} \cong \mathrm{pr}_2^*\mathbf{R}\nu_{d,*}\mathbb{Q}_\ell$, the action of $\widetilde{f}_{\mathcal{M},!}[\mathcal{H}^\diamondsuit]$ is the pullback along the smooth map pr_2 of the action of $T_d = \nu_{d,!}[I'_d]$. Therefore it suffices to show that T_d preserves the decomposition (8.5) (restricted to X_d), and acts on each K_j by the scalar $(d-2j)$.

Since $\mathbf{R}\nu_{d,*}\mathbb{Q}_\ell$ is the middle extension of the local system $L = \oplus_{j=0}^d L(\rho_j)$ on X_d°, it suffices to calculate the action of T_d on L, or rather calculate its action over a geometric generic point $\eta \in X_d$. Write $\eta = x_1 + x_2 + \cdots + x_d$ and name the two points in X' over x_i by x_i^+ and x_i^- (in one of the two ways). The fiber $\nu_d^{-1}(\eta)$ consists of points ξ_ε where $\varepsilon \in \{\pm\}^r$, and $\xi_\varepsilon = \sum_{i=1}^d x_i^{\varepsilon_i}$. As in the proof of Proposition 8.2, we may identify the stalk L_η with $\mathbb{Q}_\ell[\Gamma_d/S_d] = \mathrm{Span}\{\mathbf{1}_\varepsilon; \varepsilon \in \{\pm\}^r\}$ (we identify $\{\pm\}$ with $\{\pm 1\}$). Now we denote $\mathbf{1}_\varepsilon$ formally by the monomial $x_1^{\varepsilon_1}\cdots x_d^{\varepsilon_d}$. The stalk $L(\rho_j)_\eta$ has a basis given by $\{P_\delta\}$, where

$$P_\delta := \prod_{i=1}^d (x_i^+ + \delta_i x_i^-)$$

and δ runs over those elements $\delta = (\delta_1, \cdots, \delta_d) \in \{\pm\}^d$ with exactly i minuses. The action of T_d on L_η turns each monomial basis element $x_1^{\varepsilon_1}\cdots x_d^{\varepsilon_d}$ into $\sum_{t=1}^d x_1^{\varepsilon_1}\cdots x_t^{-\varepsilon_t}\cdots x_d^{\varepsilon_d}$. Therefore, T_d is a derivation in the following sense: for any linear form ℓ_i in x_i^+ and x_i^-, we have

$$T_d \prod_{i=1}^d \ell_i = (T_d\ell_1)\cdot \ell_2 \cdots \ell_d + \ell_1(T_d\ell_2)\ell_3\cdots\ell_d + \cdots + \ell_1\cdots\ell_{d-1}(T_d\ell_d).$$

Also $T_d(x_i^+ + x_i^-) = x_i^+ + x_i^-$ and $T_d(x_i^+ - x_i^-) = -(x_i^+ - x_i^-)$. From these we easily calculate that $T_dP_\delta = (d - 2|\delta|)P_\delta$ where $|\delta|$ is the number of minuses in δ. Since $L(\rho_j)_\eta$ is the span of P_δ with $|\delta| = j$, it is exactly the eigenspace of T_d with eigenvalue $(d-2j)$. This finishes the proof. □

Combining Theorem 6.5, (6.11) with Proposition 8.3, we get

Corollary 8.4. *Suppose $d \geq \max\{2g'-1, 2g\}$. Let $D \in X_d(k)$. Then*

$$\mathbb{I}_r(u, h_D) = \begin{cases} \sum_{i,j=0}^d (d-2j)^r \operatorname{Tr}(\mathrm{Frob}_a, (K_i)_{\overline{a}} \otimes (K_j)_{\overline{a}}) & u = \mathrm{inv}_D(a), a \in \mathcal{A}_D(k) \\ 0 & \textit{otherwise.} \end{cases}$$

8.2. **Direct image of $f_{\mathcal{N}_{\underline{d}}}$.** Recall the moduli space $\mathcal{N}_{\underline{d}}$ defined in §3.2.2 for $\underline{d} \in \Sigma_d$. It carries a local system $L_{\underline{d}}$, see §3.3.1.

Proposition 8.5. *Let $d \geq 2g'-1$ and $\underline{d} \in \Sigma_d$. Then there is a canonical isomorphism*

$$\mathbf{R}f_{\mathcal{N}_{\underline{d}},*}L_{\underline{d}} \cong (K_{d_{11}} \boxtimes K_{d_{12}})|_{\mathcal{A}_d}. \tag{8.8}$$

Proof. The condition $d \geq 2g'-1$ does not imply that $f_{\mathcal{N}_{\underline{d}}}$ is small. Nevertheless we shall show that the complex $K_{\underline{d}} := \mathbf{R}f_{\mathcal{N}_{\underline{d}},*}L_{\underline{d}}$ is the middle extension from its restriction to $\mathcal{B} := X_d \times_{\mathrm{Pic}_X^d} X_d \subset \mathcal{A}_d$. By Proposition 3.1(2), $\mathcal{N}_{\underline{d}}$ is smooth hence $L_{\underline{d}}[\dim \mathcal{N}_{\underline{d}}]$ is Verdier self-dual up to a Tate twist. By Proposition 3.1(3), $f_{\mathcal{N}_{\underline{d}}}$ is proper, hence the complex $K_{\underline{d}}[\dim \mathcal{N}_{\underline{d}}]$ is also Verdier self-dual up to a Tate twist. The morphism $f_{\mathcal{N}_{\underline{d}}}$ is finite over the open stratum $\mathcal{B}$, therefore $K_{\underline{d}}|_{\mathcal{B}}$ is concentrated in degree 0. The complement $\mathcal{A}_d - \mathcal{B}$ is the disjoint union of $\mathcal{C} = \{0\} \times X_d$ and $\mathcal{C}' = X_d \times \{0\}$. We compute the restriction $K_{\underline{d}}|_{\mathcal{C}}$.

When $d_{11} < d_{22}$, by the last condition in the definition of $\mathcal{N}_{\underline{d}}$, φ_{22} is allowed to be zero but φ_{11} is not. The fiber of $f_{\mathcal{N}_{\underline{d}}}$ over a point $(0, D) \in \mathcal{C}$ is of the form $X_{d_{11}} \times \mathrm{add}^{-1}_{d_{12},d_{21}}(D)$, where $\mathrm{add}_{j,d-j} : X_j \times X_{d-j} \to X_d$ is the addition map. We have $(K_{\underline{d}})_{(0,D)} = \mathrm{H}^*(X_{d_{11}} \otimes_k \overline{k}, L_{d_{11}}) \otimes M$ where $M = \mathrm{H}^0(\mathrm{add}^{-1}_{d_{12},d_{21}}(D) \otimes_k \overline{k}, L_{d_{12}})$ is a finite-dimensional vector space. We have $\mathrm{H}^*(X_{d_{11}} \otimes_k \overline{k}, L_{d_{11}}) \cong \bigwedge^{d_{11}}(\mathrm{H}^1(X \otimes_k \overline{k}, L_{X'/X}))[-d_{11}]$ which is concentrated in degree d_{11}, and is zero for $d_{11} > 2g-2$. Therefore $(K_{\underline{d}})_{(0,D)}$ is concentrated in some degree $\leq 2g-2$, which is smaller than $\mathrm{codim}_{\mathcal{A}_d}\mathcal{C} = d - g + 1$.

When $d_{11} \geq d_{22}$, φ_{11} may be zero but φ_{22} is nonzero. The fiber of $f_{\mathcal{N}_{\underline{d}}}$ over a point $(0,D) \in \mathcal{C}$ is of the form $X_{d_{22}} \times \mathrm{add}^{-1}_{d_{12},d_{21}}(D)$. For $(D_{22}, D_{12}, D_{21}) \in X_{d_{22}} \times \mathrm{add}^{-1}_{d_{12},d_{21}}(D)$, its image in $\mathrm{Pic}^{d_{11}}_X$ is $\mathcal{O}_X(D - D_{22})$, therefore the restriction of $L_{d_{11}}$ to $f^{-1}_{\mathcal{N}_{\underline{d}}}(0, D)$ is isomorphic to $L^{-1}_{d_{22}}$ on the $X_{d_{22}}$ factor. Therefore $(K_{\underline{d}})_{(0,D)} = \mathrm{H}^*(X_{d_{22}} \otimes_k \overline{k}, L^{-1}_{d_{22}}) \otimes \mathrm{H}^0(\mathrm{add}^{-1}_{d_{12},d_{21}}(D) \otimes_k \overline{k}, L_{d_{12}})$, which is again concentrated in some degree $\leq 2g - 2 < \mathrm{codim}_{\mathcal{A}_d}\mathcal{C} = d - g + 1$.

Same argument shows that the stalks of $K_{\underline{d}}$ over $\mathcal{C}'$ are concentrated in some degree $\leq 2g-2 < \mathrm{codim}_{\mathcal{A}_d}\mathcal{C}' = d - g + 1$. Using Verdier self-duality of $K_{\underline{d}}[\dim \mathcal{N}_{\underline{d}}]$, we conclude that $K_{\underline{d}}$ is the middle extension from its restriction to $\mathcal{B}$.

By Proposition 3.1(3) and the Kunneth formula, we have

$$K_{\underline{d}}|_{\mathcal{B}} \cong \mathrm{add}_{d_{11},d_{22},*}(L_{d_{11}} \boxtimes \mathbb{Q}_\ell) \boxtimes \mathrm{add}_{d_{12},d_{21},*}(L_{d_{12}} \boxtimes \mathbb{Q}_\ell).$$

To prove the proposition, it suffices to give a canonical isomorphism

$$\mathrm{add}_{j,d-j,*}(L_j \boxtimes \mathbb{Q}_\ell) \cong K_j|_{X_d} \tag{8.9}$$

for every $0 \leq j \leq d$. Both sides of (8.9) are middle extensions from X°_d, we only need to give an isomorphism between their restrictions to X°_d. Over X°_j, the local system L_j is given by the representation $\pi_1(X^\circ_j) \to \pi_1(X)^j \rtimes S_j \twoheadrightarrow \mathrm{Gal}(X'/X)^j \rtimes S_j \cong \{\pm 1\}^j \rtimes S_j \to \{\pm 1\}$ which is nontrivial on each factor $\mathrm{Gal}(X'/X)$ and trivial on the S_j-factor. The finite étale cover $\mathrm{add}^\circ_{j,d-j} : (X_j \times X_{d-j})^\circ \to X^\circ_d$ (restriction of $\mathrm{add}_{j,d-j}$ to X°_d) is the quotient $X^{d,\circ}/S_{j,d-j}$ where $S_{j,d-j} \subset S_d$ is the subgroup defined in §8.1.1. Therefore the local system $\mathrm{add}^\circ_{j,d-j,*}(L_j \boxtimes \mathbb{Q}_\ell)$ corresponds to the representation ρ_j of Γ_d, and $\mathrm{add}^\circ_{j,d-j,*}(L_j \boxtimes \mathbb{Q}_\ell) \cong L(\rho_j)$ as local systems over X°_d. This completes the proof of (8.9), and the proposition is proved. □

Combining Propositions 8.2 and 8.5, we get

Corollary 8.6. *Assume $d \geq 2g' - 1$. Then there is a canonical isomorphism*

$$\mathbf{R}f_{\mathcal{M},*}\mathbb{Q}_\ell \cong \bigoplus_{\underline{d} \in \Sigma_d} \mathbf{R}f_{\mathcal{N}_{\underline{d}},*}L_{\underline{d}}$$

such that the (i,j)-grading of the LHS appearing in (8.4) *corresponds to the (d_{11}, d_{12})-grading on the RHS.*

8.3. **Proof of Theorem 8.1.** By Corollary 3.3 and (2.16), both $\mathbb{J}_r(u, h_D)$ and $\mathbb{I}_r(u, h_D)$ vanish when u is not of the form $\mathrm{inv}_D(a)$ for $a \in \mathcal{A}_D(k)$. We only need to prove (8.1) when $u = \mathrm{inv}_D(a)$ for $a \in \mathcal{A}_D(k)$. In this case we have

$$\begin{aligned}
(\log q)^{-r}\mathbb{J}_r(u, h_D) &= \sum_{\underline{d} \in \Sigma_d} (2d_{12} - d)^r \,\mathrm{Tr}\left(\mathrm{Frob}_a, \left(\mathbf{R}f_{\mathcal{N}_{\underline{d}},*}L_{\underline{d}}\right)_{\overline{a}}\right) \quad \text{(Corollary 3.3)} \\
&= \sum_{d_{11},d_{12}=0}^{d} (2d_{12} - d)^r \,\mathrm{Tr}\left(\mathrm{Frob}_a, (K_{d_{11}})_{\overline{a}} \otimes (K_{d_{12}})_{\overline{a}}\right) \quad \text{(Prop. 8.5)} \\
&= \sum_{i,j=0}^{d} (d - 2j)^r \,\mathrm{Tr}\left(\mathrm{Frob}_a, (K_i)_{\overline{a}} \otimes (K_j)_{\overline{a}}\right) \quad (r \text{ is even}) \\
&= \mathbb{I}_r(u, h_D) \quad \text{(Corollary 8.4)}
\end{aligned}$$

Therefore (8.1) is proved. By (2.14) and (6.10), (8.1) implies (8.2).

9. Proof of the main theorems

In this section we complete the proofs of our main results stated in the Introduction.

9.1. **The identity $(\log q)^{-r}\mathbb{J}_r(f) = \mathbb{I}_r(f)$ for all Hecke functions.** By Theorem 8.1, we have $(\log q)^{-r}\mathbb{J}_r(f) = \mathbb{I}_r(f)$ for all $f = h_D$ where D is an effective divisor with $\deg(D) \geq \max\{2g' - 1, 2g\}$. Our goal in this subsection is to show by some algebraic manipulations that this identity holds for all $f \in \mathscr{H}$.

We first fix a place $x \in |X|$. Recall the Satake transform identifies $\mathscr{H}_x = \mathbb{Q}[h_x]$ with the subalgebra of $\mathbb{Q}[t^{\pm 1}_x]$ generated by $h_x = t_x + q_x t^{-1}_x$. For $n \geq 0$, we have $\mathrm{Sat}_x(h_{nx}) = t^n_x + q_x t^{n-2}_x + \cdots + q^{n-1}_x t^{-n+2}_x + q^n_x t^{-n}_x$.

Lemma 9.1. *Let E be any field containing $\mathbb{Q}$. Let I be a nonzero ideal of $\mathscr{H}_{x,E} := \mathscr{H}_x \otimes_{\mathbb{Q}} E$ and let m be a positive integer. Then $I + \mathrm{Span}_E\{h_{mx}, h_{(m+1)x}, \cdots\} = \mathscr{H}_{x,E}$.*

Proof. Let $t = q_x^{-1/2} t_x$. Then $h_{nx} = q_x^{n/2} T_n$ where $T_n = t^n + t^{n-2} + \cdots + t^{2-n} + t^{-n}$ for any $n \geq 0$. It suffices to show that $I + \mathrm{Span}_E\{T_m, T_{m+1}, \cdots\} = \mathscr{H}_{x,E}$.

Let $\pi : \mathscr{H}_{x,E} \to \mathscr{H}_{x,E}/I$ be the quotient map. Let $\mathscr{H}_{m,E} \subset \mathscr{H}_{x,E}$ be the E-span of $t^n + t^{-n}$ for $n \geq m$. Note that $T_n - T_{n-2} = t^n + t^{-n}$, therefore it suffices to show that $\pi(\mathscr{H}_{m,E}) = \mathscr{H}_{x,E}/I$ for all m. To show this, it suffices to show the same statement after base change from E to an algebraic closure $\overline{E}$. From now on we use the notation $\mathscr{H}_x, I$ and $\mathscr{H}_m$ to denote their base changes to $\overline{E}$.

To show that $\pi(\mathscr{H}_m) = \mathscr{H}_x/I$, we take any nonzero linear function $\ell : \mathscr{H}_x/I \to \overline{E}$. We only need to show that $\ell(\pi(t^n + t^{-n})) \neq 0$ for some $n \geq m$. We prove this by contradiction: suppose $\ell(\pi(t^n + t^{-n})) = 0$ for all $n \geq m$.

Let $\nu : \mathbb{G}_m \to \mathbb{A}^1 = \mathrm{Spec}\, \mathscr{H}_x$ be the morphism given by $t \mapsto T = t + t^{-1}$. This is the quotient by the involution $\sigma(t) = t^{-1}$. Consider the finite subscheme $Z = \mathrm{Spec}(\mathscr{H}_x/I)$ and its preimage $\widetilde{Z} = \nu^{-1}(Z)$ in $\mathbb{G}_m$. We have $\mathcal{O}_Z = \mathscr{H}_x/I = \mathcal{O}_{\widetilde{Z}}^{\sigma} \subset \mathcal{O}_{\widetilde{Z}}$. One can uniquely extend ℓ to a σ-invariant linear function $\widetilde{\ell} : \mathcal{O}_{\widetilde{Z}} \to \overline{E}$. Note that $\mathcal{O}_{\widetilde{Z}}$ is a product of the form $\overline{E}[t]/(t-z)^{d_z}$ for a finite set of points $z \in \overline{E}^{\times}$, and that $z \in \widetilde{Z}$ if and only if $\sigma(z) = z^{-1} \in \widetilde{Z}$. Any linear function $\widetilde{\ell}$ on $\mathcal{O}_{\widetilde{Z}}$, when pulled back to $\mathcal{O}_{\mathbb{G}_m} = \overline{E}[t, t^{-1}]$, takes the form

$$\overline{E}[t, t^{-1}] \ni f \longmapsto \sum_{z \in \widetilde{Z}} (D_z f)(z)$$

with $D_z = \sum_{j \geq 0} c_j(z)(t\frac{d}{dt})^j$ (finitely many terms) a differential operator on $\mathbb{G}_m$ with constant coefficients $c_j(z)$ depending on z. The σ-invariance of $\widetilde{\ell}$ is equivalent to

$$c_j(z) = (-1)^j c_j(z^{-1}), \quad \text{for all } z \in \widetilde{Z} \text{ and } j. \tag{9.1}$$

Evaluating at $f = t^n + t^{-n}$, we get that

$$\ell(\pi(t^n + t^{-n})) = \sum_{z \in \widetilde{Z}} P_z(n) z^n + P_z(-n) z^{-n}$$

where $P_z(T) = \sum_j c_j(z) T^j \in \overline{E}[T]$ is a polynomial depending on z. The symmetry (9.1) implies $P_z(T) = P_{z^{-1}}(-T)$. Using this symmetry, we may collect the terms corresponding to z and z^{-1} and re-organize the sum above as

$$\ell(\pi(t^n + t^{-n})) = 2 \sum_{z \in \widetilde{Z}} P_z(n) z^n = 0, \quad \text{for all } n \geq m.$$

By linear independence of $\phi_{a,z} : n \mapsto n^a z^n$ as functions on $\{m, m+1, m+2, \cdots\}$, we see that all polynomials $P_z(T)$ are identically zero. Hence $\widetilde{\ell} = 0$ and $\ell = 0$, which is a contradiction! $\square$

Theorem 9.2. *For any $f \in \mathscr{H}$, we have the identity*

$$(\log q)^{-r} \mathbb{J}_r(f) = \mathbb{I}_r(f).$$

Proof. Let $\widetilde{\mathscr{H}_\ell}$ be the image of $\mathscr{H} \otimes \mathbb{Q}_\ell$ in $\mathrm{End}_{\mathbb{Q}_\ell}(V') \times \mathrm{End}_{\mathbb{Q}_\ell}(\mathcal{A} \otimes \mathbb{Q}_\ell) \times \mathbb{Q}_\ell[\mathrm{Pic}_X(k)]^{\iota_{\mathrm{Pic}}}$. Denote the quotient map $\mathscr{H} \otimes \mathbb{Q}_\ell \twoheadrightarrow \widetilde{\mathscr{H}_\ell}$ by a. Then for any $x \in |X|$, $\widetilde{\mathscr{H}_\ell} \subset \mathrm{End}_{\mathscr{H}_x \otimes \mathbb{Q}_\ell}(V' \oplus \mathcal{A} \otimes \mathbb{Q}_\ell \oplus \mathbb{Q}_\ell[\mathrm{Pic}_X(k)]^{\iota_{\mathrm{Pic}}})$. The latter being finitely generated over $\mathscr{H}_x \otimes \mathbb{Q}_\ell$ by Lemma 7.13 (or rather, the analogous assertion for V'), $\widetilde{\mathscr{H}_\ell}$ is also a finitely generated $\mathscr{H}_x \otimes \mathbb{Q}_\ell$-module, and hence a finitely generated $\mathbb{Q}_\ell$-algebra. Clearly for $f \in \mathscr{H}$, $\mathbb{I}_r(f)$ and $\mathbb{J}_r(f)$ only depend on the image of f in $\widetilde{\mathscr{H}_\ell}$. Let $\mathscr{H}^\dagger \subset \mathscr{H}$ be the linear span of the functions h_D for effective divisors D such that $\deg D \geq \max\{2g' - 1, 2g\}$. By Theorem 8.1, we have $(\log q)^{-r} \mathbb{J}_r(f) = \mathbb{I}_r(f)$ for all $f \in \mathscr{H}^\dagger$. Therefore it suffices to show that the composition $\mathscr{H}^\dagger \otimes \mathbb{Q}_\ell \to \mathscr{H} \otimes \mathbb{Q}_\ell \xrightarrow{a} \widetilde{\mathscr{H}_\ell}$ is surjective.

Since $\widetilde{\mathscr{H}_\ell}$ is finitely generated as an algebra, there exists a finite set $S \subset |X|$ such that $\{a(h_x)\}_{x \in S}$ generate $\widetilde{\mathscr{H}_\ell}$. We may enlarge S and assume that S contains all places with degree $\leq \max\{2g' - 1, 2g\}$. Let $y \in |X| - S$, then for any $f \in \mathscr{H}_S = \otimes_{x \in S} \mathscr{H}_x$, we have $f h_y \in \mathscr{H}^\dagger$.

Therefore $a(\mathscr{H}^\dagger \otimes \mathbb{Q}_\ell) \supset a(\mathscr{H}_S \otimes \mathbb{Q}_\ell)a(h_y) = \widetilde{\mathscr{H}_\ell}a(h_y)$. In other words, $a(\mathscr{H}^\dagger \otimes \mathbb{Q}_\ell)$ contains the ideal I generated by the $a(h_y)$ for $y \notin S$.

We claim that the quotient $\widetilde{\mathscr{H}_\ell}/I$ is finite-dimensional over $\mathbb{Q}_\ell$. Since $\widetilde{\mathscr{H}_\ell}$ is finitely generated over $\mathbb{Q}_\ell$, it suffices to show that $\mathrm{Spec}(\widetilde{\mathscr{H}_\ell}/I)$ is finite. Combining Theorem 7.16 and (7.19), $\mathrm{Spec}\,\widetilde{\mathscr{H}_\ell} = \mathrm{Spec}\,\widetilde{\mathscr{H}_\ell}' \cup \mathrm{Spec}\,\mathscr{H}_{\mathrm{aut},\mathbb{Q}_\ell} = Z_{\mathrm{Eis},\mathbb{Q}_\ell} \cup Z'^r_{0,\ell} \cup Z^0_{0,\ell}$. Let $\sigma : \widetilde{\mathscr{H}_\ell}/I \to \overline{\mathbb{Q}}_\ell$ be a $\overline{\mathbb{Q}}_\ell$-point of $\mathrm{Spec}(\widetilde{\mathscr{H}_\ell}/I)$. If σ lies in $Z_{\mathrm{Eis},\mathbb{Q}_\ell}$, then the composition $\mathscr{H} \to \widetilde{\mathscr{H}_\ell}/I \xrightarrow{\sigma} \overline{\mathbb{Q}}_\ell$ factors as $\mathscr{H} \xrightarrow{\mathrm{Sat}} \mathbb{Q}[\mathrm{Pic}_X(k)] \xrightarrow{\chi} \overline{\mathbb{Q}}_\ell$ for some character $\chi : \mathrm{Pic}_X(k) \to \overline{\mathbb{Q}}_\ell^\times$. Since h_y vanishes in $\widetilde{\mathscr{H}_\ell}/I$ for any $y \notin S$, we have $\chi(\mathrm{Sat}(h_y)) = \chi(t_y) + q_y\chi(t_y^{-1}) = 0$ for all $y \notin S$, which implies that $\chi(t_y) = \pm(-q_y)^{1/2}$ for all $y \notin S$. Let $\chi' : \mathrm{Pic}_X(k) \to \overline{\mathbb{Q}}_\ell^\times$ be the character $\chi' = \chi \cdot q^{-\deg/2}$. Then χ' is a character with finite image satisfying $\chi'(t_y) = \pm\sqrt{-1}$ for all but finitely y. This contradicts Chebotarev density since there should be a positive density of y such that $\chi'(t_y) = 1$. Therefore $\mathrm{Spec}(\widetilde{\mathscr{H}_\ell}/I)$ is disjoint from $Z_{\mathrm{Eis},\mathbb{Q}_\ell}$ hence $\mathrm{Spec}(\widetilde{\mathscr{H}_\ell}/I)^{\mathrm{red}} \subset Z'^r_{0,\ell} \cup Z^0_{0,\ell}$, hence finite.

Let $\overline{a} : \mathscr{H} \otimes \mathbb{Q}_\ell \xrightarrow{a} \widetilde{\mathscr{H}_\ell} \to \widetilde{\mathscr{H}_\ell}/I$ be the quotient map. For each $x \in |X|$, consider the surjective ring homomorphism $\mathscr{H}_x \otimes \mathbb{Q}_\ell \to \overline{a}(\mathscr{H}_x \otimes \mathbb{Q}_\ell)$. Note that $\mathscr{H}^\dagger \cap \mathscr{H}_x$ is spanned by elements of the form h_{nx} for $n\deg(x) \geq \max\{2g'-1, 2g\}$. Since $\overline{a}(\mathscr{H}_x \otimes \mathbb{Q}_\ell) \subset \widetilde{\mathscr{H}_\ell}/I$ is finite-dimensional over $\mathbb{Q}_\ell$, Lemma 9.1 implies that $(\mathscr{H}^\dagger \cap \mathscr{H}_x) \otimes \mathbb{Q}_\ell \to \overline{a}(\mathscr{H}_x \otimes \mathbb{Q}_\ell)$ is surjective. Therefore $\overline{a}(\mathscr{H}^\dagger \otimes \mathbb{Q}_\ell)$ contains $\overline{a}(\mathscr{H}_x \otimes \mathbb{Q}_\ell)$ for all $x \in |X|$. Since $\overline{a}$ is surjective, $\overline{a}(\mathscr{H}_x \otimes \mathbb{Q}_\ell)$ (all $x \in |X|$) generate the image $\widetilde{\mathscr{H}_\ell}/I$ as an algebra, hence $\overline{a}(\mathscr{H}^\dagger \otimes \mathbb{Q}_\ell) = \widetilde{\mathscr{H}_\ell}/I$. Since $a(\mathscr{H}^\dagger \otimes \mathbb{Q}_\ell)$ already contains I, we conclude that $a(\mathscr{H}^\dagger \otimes \mathbb{Q}_\ell) = \widetilde{\mathscr{H}_\ell}$. □

9.1.1. *Proof of Theorem 1.8.* Apply Theorem 9.2 to the unit function $h = \mathbf{1}_K$, we get

$$(\theta^\mu_*[\mathrm{Sht}^\mu_T], \quad \theta^\mu_*[\mathrm{Sht}^\mu_T])_{\mathrm{Sht}'^r_G} = (\log q)^{-r}\mathbb{J}_r(\mathbf{1}_K).$$

We then apply Corollary 2.5 to write the RHS using the r-th derivative of $L(\eta, s)$, as desired.

Remark 9.3. Let $r = 0$. Note that Sht^μ_T, resp. Sht'^r_G, is the constant groupoid $\mathrm{Bun}_T(k)$, resp. $\mathrm{Bun}_G(k)$. We write $\theta_*[\mathrm{Bun}_T(k)]$ for $\theta^\mu_*[\mathrm{Sht}^\mu_T]$, as an element in $C^\infty_c(\mathrm{Bun}_G(k), \mathbb{Q})$. The analogous statement of Theorem 1.8 should be

$$\langle\theta_*[\mathrm{Bun}_T(k)], \quad \theta_*[\mathrm{Bun}_T(k)]\rangle_{\mathrm{Bun}_G(k)} = 4L(\eta, 0) + q - 2. \tag{9.2}$$

Here the left side $\langle -, - \rangle_{\mathrm{Bun}_G(k)}$ is the inner product on $C^\infty_c(\mathrm{Bun}_G(k), \mathbb{Q})$ defined such that the characteristic functions $\{\mathbf{1}_{[\mathcal{E}]}\}_{\mathcal{E} \in \mathrm{Bun}_G(k)}$ are orthogonal to each other and that

$$\langle \mathbf{1}_{[\mathcal{E}]}, \mathbf{1}_{[\mathcal{E}]}\rangle_{\mathrm{Bun}_G(k)} = \frac{1}{\#\,\mathrm{Aut}(\mathcal{E})}.$$

The equality (9.2) can be proved directly. We leave the detail to the reader.

9.2. **Proof of Theorem 1.6.** The theorem was formulated as an equality in $E_{\pi,\lambda}$, but for the proof we shall extend scalars from $\mathbb{Q}_\ell$ to $\overline{\mathbb{Q}}_\ell$, and use the decomposition (7.24) instead. For any embedding $\iota : E_\pi \hookrightarrow \overline{\mathbb{Q}}_\ell$, we have the point $\mathfrak{m}(\pi, \iota) \in Z_{\mathrm{cusp}}(\overline{\mathbb{Q}}_\ell)$ corresponding to the homomorphism $\mathscr{H} \xrightarrow{\lambda_\pi} E_\pi \xrightarrow{\iota} \overline{\mathbb{Q}}_\ell$. To prove the theorem, it suffices to showing that for all embeddings $\iota : E_\pi \hookrightarrow \overline{\mathbb{Q}}_\ell$, we have an identity in $\overline{\mathbb{Q}}_\ell$

$$\frac{|\omega_X|}{2(\log q)^r}\iota(\mathscr{L}^{(r)}(\pi_{F'}, 1/2)) = \Big([\mathrm{Sht}^\mu_T]_{\mathfrak{m}(\pi,\iota)}, [\mathrm{Sht}^\mu_T]_{\mathfrak{m}(\pi,\iota)}\Big)_{\overline{\mathbb{Q}}_\ell}$$

where $(\cdot,\cdot)_{\overline{\mathbb{Q}}_\ell}$ is the $\overline{\mathbb{Q}}_\ell$-bilinear extension of the cup product pairing (7.22) on V'. In other words, for any everywhere unramified cuspidal automorphic $\overline{\mathbb{Q}}_\ell$-representation π of $G(\mathbb{A}_F)$ that corresponds to the homomorphism $\lambda_\pi : \mathscr{H}_{\overline{\mathbb{Q}}_\ell} \to \overline{\mathbb{Q}}_\ell$, we need to show

$$\frac{|\omega_X|}{2(\log q)^r}\mathscr{L}^{(r)}(\pi_{F'}, 1/2) = \Big([\mathrm{Sht}^\mu_T]_{\mathfrak{m}_\pi}, [\mathrm{Sht}^\mu_T]_{\mathfrak{m}_\pi}\Big)_{\overline{\mathbb{Q}}_\ell} \tag{9.3}$$

where $\mathfrak{m}_\pi = \ker(\lambda_\pi)$ is the maximal ideal of $\mathscr{H}_{\overline{\mathbb{Q}}_\ell}$, and $[\mathrm{Sht}^\mu_T]_{\mathfrak{m}_\pi}$ is understood to be zero if $\mathfrak{m}_\pi \notin Z'^r_{0,\ell}(\overline{\mathbb{Q}}_\ell)$.

As in the proof of Theorem 9.2, let $\widetilde{\mathscr{H}_\ell}$ be the image of $\mathscr{H}_{\mathbb{Q}_\ell}$ in $\mathrm{End}_{\mathbb{Q}_\ell}(V') \times \mathrm{End}_{\mathbb{Q}_\ell}(\mathcal{A} \otimes \mathbb{Q}_\ell) \times \mathbb{Q}_\ell[\mathrm{Pic}_X(k)]^{\iota_{\mathrm{Pic}}}$. By Theorem 7.16 and (7.19), we may write $\mathrm{Spec}\, \widetilde{\mathscr{H}_\ell}$ as a disjoint union of closed subsets

$$\mathrm{Spec}\, \widetilde{\mathscr{H}_\ell}^{\mathrm{red}} = Z_{\mathrm{Eis},\mathbb{Q}_\ell} \coprod \widetilde{Z}_{0,\ell}. \tag{9.4}$$

where $\widetilde{Z}_{0,\ell} = Z'^r_{0,\ell} \cup Z^0_{0,\ell}$ is a finite collection of closed points. This gives a product decomposition of the ring $\widetilde{\mathscr{H}_\ell}$

$$\widetilde{\mathscr{H}_\ell} = \widetilde{\mathscr{H}_{\ell,\mathrm{Eis}}} \times \widetilde{\mathscr{H}_{\ell,0}} \tag{9.5}$$

with $\mathrm{Spec}\, \widetilde{\mathscr{H}}^{\mathrm{red}}_{\ell,\mathrm{Eis}} = Z_{\mathrm{Eis},\mathbb{Q}_\ell}$ and $\mathrm{Spec}\, \widetilde{\mathscr{H}}^{\mathrm{red}}_{\ell,0} = \widetilde{Z}_{0,\ell}$. For any element $h \in \widetilde{\mathscr{H}_{\ell,0}}$, we view it as the element $(0,h) \in \widetilde{\mathscr{H}_\ell}$. By Corollary 7.17 we have for any $h \in \widetilde{\mathscr{H}_{\ell,0}}$

$$\mathbb{I}_r(h) \quad = \sum_{\mathfrak{m} \in Z'^r_{0,\ell}(\overline{\mathbb{Q}}_\ell)} \Big([\mathrm{Sht}_T]_{\mathfrak{m}}, h * [\mathrm{Sht}_T]_{\mathfrak{m}} \Big) \tag{9.6}$$

Extending by linearity, the above formula also holds for all $h \in \widetilde{\mathscr{H}_{\ell,0}} \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell$. Note that the linear function $h \mapsto ([\mathrm{Sht}_T]_{\mathfrak{m}}, h * [\mathrm{Sht}_T]_{\mathfrak{m}})$ on $\widetilde{\mathscr{H}_{\ell,0}} \otimes \overline{\mathbb{Q}}_\ell$ factors through the localization $\widetilde{\mathscr{H}_{\ell,0}} \otimes \overline{\mathbb{Q}}_\ell \to (\widetilde{\mathscr{H}_{\ell,0}} \otimes \overline{\mathbb{Q}}_\ell)_{\mathfrak{m}}$ (viewing $\mathfrak{m}$ as a maximal ideal of $\widetilde{\mathscr{H}_{\ell,0}} \otimes \overline{\mathbb{Q}}_\ell$).

On the other hand, let $\widetilde{\mathcal{I}}_{\mathrm{Eis}}$ be the ideal of $\widetilde{\mathscr{H}_\ell}$ generated by the image of $\mathcal{I}_{\mathrm{Eis}}$. We have $(0,h) \in \widetilde{\mathcal{I}}_{Eis}$. By Theorem 4.7, we have for any $h \in \widetilde{\mathscr{H}_{\ell,0}} \otimes \overline{\mathbb{Q}}_\ell$

$$\begin{aligned} \mathbb{J}_r(h) \quad &= \sum_{\pi \in Z_{\mathrm{cusp}}(\overline{\mathbb{Q}}_\ell)} \frac{d^r}{ds^r}\Big|_{s=0} \mathbb{J}_\pi((0,h)) && (9.7) \\ &= \sum_{\pi \in Z_{\mathrm{cusp}}(\overline{\mathbb{Q}}_\ell)} \frac{|\omega_X|}{2} \lambda_\pi(h) \mathscr{L}^{(r)}(\pi_{F'}, 1/2). && (9.8) \end{aligned}$$

By Lemma 7.15(3), $Z_{\mathrm{cusp}}(\overline{\mathbb{Q}}_\ell) = Z^0_{0,\ell}(\overline{\mathbb{Q}}_\ell)$, hence can be viewed as a subset of $\widetilde{Z}_{0,\ell}$. Comparing the RHS of (9.6) and (9.7), and using Theorem 9.2, we get for any $h \in \widetilde{\mathscr{H}_{\ell,0}} \otimes \overline{\mathbb{Q}}_\ell$

$$\sum_{\mathfrak{m} \in Z^r_{0,\ell}(\overline{\mathbb{Q}}_\ell)} \Big([\mathrm{Sht}_T]_{\mathfrak{m}}, h * \mathrm{Sht}_T]_{\mathfrak{m}} \Big) = \sum_{\pi \in Z_{\mathrm{cusp}}(\overline{\mathbb{Q}}_\ell)} \frac{|\omega_X|}{2(\log q)^r} \lambda_\pi(h) \mathscr{L}^{(r)}(\pi_{F'}, 1/2). \tag{9.9}$$

Since $\widetilde{\mathscr{H}_{\ell,0}} \otimes \overline{\mathbb{Q}}_\ell$ is an artinian algebra, we have a canonical decomposition into local artinian algebras

$$\widetilde{\mathscr{H}_{\ell,0}} \otimes \overline{\mathbb{Q}}_\ell \cong \prod_{\mathfrak{m} \in \widetilde{Z}_{0,\ell}(\overline{\mathbb{Q}}_\ell)} (\widetilde{\mathscr{H}_{\ell,0}} \otimes \overline{\mathbb{Q}}_\ell)_{\mathfrak{m}}. \tag{9.10}$$

As linear functions on $\widetilde{\mathscr{H}_{\ell,0}} \otimes \overline{\mathbb{Q}}_\ell$, the $\mathfrak{m}$-summand of the left side of (9.9) factors through $(\widetilde{\mathscr{H}_{\ell,0}} \otimes \overline{\mathbb{Q}}_\ell)_{\mathfrak{m}}$ while the π-summand of the right side of (9.9) factors through $(\widetilde{\mathscr{H}_{\ell,0}} \otimes \overline{\mathbb{Q}}_\ell)_{\mathfrak{m}_\pi}$. By the decomposition (9.10), we conclude that

- If $\mathfrak{m} \in Z'^r_{0,\ell}(\overline{\mathbb{Q}}_\ell) - Z_{\mathrm{cusp}}(\overline{\mathbb{Q}}_\ell)$, then for any $h \in \widetilde{\mathscr{H}_{\ell,0}} \otimes \overline{\mathbb{Q}}_\ell$,
$$([\mathrm{Sht}_T]_{\mathfrak{m}}, \quad h * [\mathrm{Sht}_T]_{\mathfrak{m}})_{\overline{\mathbb{Q}}_\ell} = 0.$$
- If $\mathfrak{m} \in Z'^r_{0,\ell}(\overline{\mathbb{Q}}_\ell) \cap Z_{\mathrm{cusp}}(\overline{\mathbb{Q}}_\ell)$, i.e., there is a (necessarily unique) $\pi \in Z_{\mathrm{cusp}}(\overline{\mathbb{Q}}_\ell)$ such that $\mathfrak{m} = \mathfrak{m}_\pi$, then for any $h \in \widetilde{\mathscr{H}_{\ell,0}} \otimes \overline{\mathbb{Q}}_\ell$ we have
$$\lambda_\pi(h) \frac{|\omega_X|}{2(\log q)^r} \mathscr{L}^{(r)}(\pi_{F'}, 1/2) = ([\mathrm{Sht}_T]_{\mathfrak{m}_\pi}, \quad h * [\mathrm{Sht}_T]_{\mathfrak{m}_\pi})_{\overline{\mathbb{Q}}_\ell}.$$
In particular, taking $h = 1$ we get
$$\frac{|\omega_X|}{2(\log q)^r} \mathscr{L}^{(r)}(\pi_{F'}, 1/2) = ([\mathrm{Sht}_T]_{\mathfrak{m}_\pi}, \quad [\mathrm{Sht}_T]_{\mathfrak{m}_\pi})_{\overline{\mathbb{Q}}_\ell}.$$
- If $\pi \in Z_{\mathrm{cusp}}(\overline{\mathbb{Q}}_\ell) - Z'^r_{0,\ell}(\overline{\mathbb{Q}}_\ell)$, then
$$\mathscr{L}^{(r)}(\pi_{F'}, 1/2) = 0.$$

These together imply (9.3), which finishes the proof of Theorem 1.6.

9.3. **The Chow group version of the main theorem.** In §1.4 we defined an $\mathscr{H}$-module W equipped with a perfect symmetric bilinear pairing $(\cdot,\cdot)$. Recall that $\widetilde{W}$ is the $\mathscr{H}$-submodule of $\mathrm{Ch}_{c,r}(\mathrm{Sht}'^r_G)_{\mathbb{Q}}$ generated by $\theta^\mu_*[\mathrm{Sht}^\mu_T]$, and W is by definition the quotient of $\widetilde{W}$ by the kernel $\widetilde{W}_0$ of the intersection pairing.

Corollary 9.4 (of Theorem 9.2)**.** *The action of $\mathscr{H}$ on W factors through $\mathscr{H}_{\mathrm{aut}}$. In particular, W is a cyclic $\mathscr{H}_{\mathrm{aut}}$-module, hence finitely generated module over $\mathscr{H}_x$ for any $x \in |X|$.*

Proof. Suppose $f \in \mathscr{H}$ is in the kernel of $\mathscr{H} \to \mathscr{H}_{\mathrm{aut}}$, then $\mathbb{J}_r(f) = 0$ hence $\mathbb{I}_r(f) = 0$ by Theorem 9.2. In particular, for any $h \in \mathscr{H}$, we have $\mathbb{J}_r(hf) = 0$. Therefore $\langle h * \theta^\mu_*[\mathrm{Sht}^\mu_T], f * \theta^\mu_*[\mathrm{Sht}^\mu_T]\rangle = \mathbb{I}_r(hf) = 0$. This implies that $f * \theta^\mu_*[\mathrm{Sht}^\mu_T] \in \widetilde{W}_0$, hence $f * \theta^\mu_*[\mathrm{Sht}^\mu_T]$ is zero in W, i.e., f acts as zero on W. □

9.3.1. *Proof of Theorem 1.1.* By the decomposition (7.20), we have an orthogonal decomposition

$$W = W_{\mathrm{Eis}} \oplus W_{\mathrm{cusp}}$$

with $\mathrm{Supp}(W_{\mathrm{Eis}}) \subset Z_{\mathrm{Eis}}$ and $\mathrm{Supp}(W_{\mathrm{cusp}}) \subset Z_{\mathrm{cusp}}$. Since W_{cusp} is a finitely generated $\mathscr{H}_{\mathrm{aut}}$-module with finite support, it is finite-dimensional over $\mathbb{Q}$. By Lemma 7.15, Z_{cusp} is the set of unramified cuspidal automorphic representations in $\mathcal{A}$, which implies the finer decomposition (1.5). Since W is a cyclic $\mathscr{H}_{\mathrm{aut}}$-module, we have $\dim_{E_\pi} W_\pi \le \dim_{E_\pi} \mathscr{H}_{\mathrm{aut},\pi} = 1$ by the decomposition in Lemma 7.15(1).

9.3.2. *Proof of Theorem 1.2.* Pick any place λ of E_π over ℓ, then by the compatibility of the intersection pairing and the cup product pairing under the cycle class map, we have

$$([\mathrm{Sht}^\mu_T]_\pi, \quad [\mathrm{Sht}^\mu_T]_\pi)_\pi = ([\mathrm{Sht}^\mu_T]_{\pi,\lambda}, \quad [\mathrm{Sht}^\mu_T]_{\pi,\lambda})_{\pi,\lambda}$$

both as elements in the local field $E_{\pi,\lambda}$. Therefore Theorem 1.2 follows from Theorem 1.6.

Appendix A. Results from intersection theory

In this appendix, we use Roman letters X, Y, V, W, etc to denote algebraic stacks over a field k. In particular, X does not mean an algebraic curve. All algebraic stacks we consider are locally of finite type over k.

A.1. Rational Chow groups for Deligne–Mumford stacks.

A.1.1. *Generalities about intersection theory on stacks.* We refer to [15] for the definition of the Chow group $\mathrm{Ch}_*(X)$ of an algebraic stack X over k.

For a Deligne–Mumford stack of finite type over k, the rational Chow group $\mathrm{Ch}_*(X)_{\mathbb{Q}}$ can be defined in a more naive way using $\mathbb{Q}$-coefficient cycles modulo rational equivalence, see [23].

A.1.2. *Chow group of proper cycles.* Let X be a Deligne–Mumford stack locally of finite type over k. Let $Z_{c,i}(X)_{\mathbb{Q}}$ denote the $\mathbb{Q}$-vector space spanned by irreducible i-dimensional closed substacks $Z \subset X$ that are *proper over* k. Let $\mathrm{Ch}_{c,i}(X)_{\mathbb{Q}}$ be the quotient of $Z_{c,i}(X)_{\mathbb{Q}}$ modulo rational equivalence which comes from rational functions on cycles which are proper over k. Equivalently, $\mathrm{Ch}_{c,i}(X)_{\mathbb{Q}} = \varinjlim_{Y \subset X} \mathrm{Ch}_i(Y)_{\mathbb{Q}}$ where Y runs over closed substacks of X that are proper over k, partially ordered by inclusion.

From the definition, we see that if X is exhausted by open substacks $X_1 \subset X_2 \subset \cdots$, then we have

$$\mathrm{Ch}_{c,i}(X)_{\mathbb{Q}} \cong \varinjlim_n \mathrm{Ch}_{c,i}(X_n)_{\mathbb{Q}}.$$

A.1.3. *The degree map.* When X is a Deligne–Mumford stack, we have a degree map

$$\deg : \mathrm{Ch}_{c,0}(X)_{\mathbb{Q}} \longrightarrow \mathbb{Q}.$$

Suppose $x \in X$ is a closed point with residue field k_x and automorphism group $\mathrm{Aut}(x)$ (a finite group scheme over k_x). Let $|\mathrm{Aut}(x)|_{k_x}$ be the order of $\mathrm{Aut}(x)$ as a finite group scheme over k_x. Let $[x] \in \mathrm{Ch}_{c,0}(X)_{\mathbb{Q}}$ be the cycle class of the closed point x. Then

$$\deg([x]) = [k_x : k]/|\mathrm{Aut}(x)|_{k_x}.$$

A.1.4. *Intersection pairing.* For the rest of §A.1, we assume that X is a smooth separated Deligne–Mumford stack, locally of finite type over k with pure dimension n. There is an intersection product

$$(-)\cdot_X(-):\mathrm{Ch}_{c,i}(X)_{\mathbb{Q}}\times\mathrm{Ch}_{c,j}(X)_{\mathbb{Q}}\longrightarrow\mathrm{Ch}_{c,i+j-n}(X)_{\mathbb{Q}}$$

defined as follows. For closed substacks Y_1 and Y_2 of X that are proper over k, the refined Gysin map attached to the regular local immersion $\Delta:X\to X\times X$ gives an intersection product

$$\begin{array}{rcl}\mathrm{Ch}_i(Y_1)_{\mathbb{Q}}\times\mathrm{Ch}_j(Y_2)_{\mathbb{Q}} & \longrightarrow & \mathrm{Ch}_{i+j-n}(Y_1\cap Y_2)_{\mathbb{Q}}\longrightarrow\mathrm{Ch}_{c,i+j-n}(X)_{\mathbb{Q}}\\ (\zeta_1,\zeta_2) & \longmapsto & \Delta^!(\zeta_1\times\zeta_2)\end{array}$$

Note that $Y_1\cap Y_2=Y_1\times_X Y_2\to Y_1$ is proper, hence $Y_1\cap Y_2$ is proper over k. Taking direct limits for Y_1 and Y_2, we get the intersection product on $\mathrm{Ch}_{c,*}(X)_{\mathbb{Q}}$.

Composing with the degree map, we get an intersection pairing

$$\langle\cdot,\cdot\rangle_X:\mathrm{Ch}_{c,j}(X)_{\mathbb{Q}}\times\mathrm{Ch}_{c,n-j}(X)_{\mathbb{Q}}\longrightarrow\mathbb{Q}\tag{A.1}$$

defined as

$$\langle\zeta_1,\zeta_2\rangle_X=\deg(\zeta_1\cdot_X\zeta_2),\quad \zeta_1\in\mathrm{Ch}_{c,j}(X)_{\mathbb{Q}},\zeta_2\in\mathrm{Ch}_{c,n-j}(X)_{\mathbb{Q}}.$$

A.1.5. *The cycle class map.* For any closed substack $Y\subset X$ that is proper over k, we have the usual cycle class map into the ℓ-adic (Borel-Moore) homology of Y

$$\mathrm{cl}_Y:\mathrm{Ch}_j(Y)_{\mathbb{Q}}\longrightarrow\mathrm{H}^{\mathrm{BM}}_{2j}(Y\otimes_k\overline{k},\mathbb{Q}_\ell)(-j)\cong\mathrm{H}_{2j}(Y\otimes_k\overline{k},\mathbb{Q}_\ell)(-j).$$

Composing with the proper map $i:Y\hookrightarrow X$ we get

$$\mathrm{cl}_{Y,X}:\mathrm{Ch}_j(Y)_{\mathbb{Q}}\xrightarrow{\mathrm{cl}_Y}\mathrm{H}_{2j}(Y\otimes_k\overline{k},\mathbb{Q}_\ell)(-j)\xrightarrow{i_*}\mathrm{H}_{2j}(X\otimes_k\overline{k},\mathbb{Q}_\ell)(-j)\cong\mathrm{H}^{2n-2j}_c(X\otimes_k\overline{k},\mathbb{Q}_\ell)(n-j).\tag{A.2}$$

where the last isomorphism is the Poincaré duality for X. Taking inductive limit over all such proper Y, we get a cycle class map for proper cycles on X

$$\mathrm{cl}_X:\mathrm{Ch}_{c,j}(X)_{\mathbb{Q}}=\varinjlim_{Y}\mathrm{Ch}_j(Y)_{\mathbb{Q}}\xrightarrow{\varinjlim\mathrm{cl}_{Y,X}}\mathrm{H}^{2n-2j}_c(X\otimes_k\overline{k},\mathbb{Q}_\ell)(n-j).$$

This map intertwines the intersection pairing (A.1) with the cup product pairing

$$\mathrm{H}^{2j}_c(X\otimes_k\overline{k},\mathbb{Q}_\ell)(j)\times\mathrm{H}^{2n-2j}_c(X\otimes_k\overline{k},\mathbb{Q}_\ell)(n-j)\xrightarrow{\cup}\mathrm{H}^{2n}_c(X\otimes_k\overline{k},\mathbb{Q}_\ell)(n)\xrightarrow{\cap[X]}\mathbb{Q}_\ell.$$

A.1.6. *A ring of correspondences.* Let

$${}_c\mathrm{Ch}_n(X\times X)_{\mathbb{Q}}=\varinjlim_{Z\subset X\times X,\mathrm{pr}_1:Z\longrightarrow X\text{ is proper}}\mathrm{Ch}_n(Z)_{\mathbb{Q}}.$$

For closed substacks $Z_1,Z_2\subset X\times X$ that are proper over X via the first projections, we have a bilinear map

$$\begin{array}{rcl}\mathrm{Ch}_n(Z_1)_{\mathbb{Q}}\times\mathrm{Ch}_n(Z_2)_{\mathbb{Q}} & \longrightarrow & \mathrm{Ch}_n((Z_1\times X)\cap(X\times Z_2))_{\mathbb{Q}}\xrightarrow{\mathrm{pr}_{13*}}{}_c\mathrm{Ch}_n(X\times X)_{\mathbb{Q}}\\ (\rho_1,\rho_2) & \longmapsto & \rho_1*\rho_2:=\mathrm{pr}_{13*}\big((\rho_1\times[X])\cdot_{X^3}([X]\times\rho_2)\big).\end{array}$$

Note that $(Z_1\times X)\cap(X\times Z_2)=Z_1\times_{\mathrm{pr}_2,X,\mathrm{pr}_1}Z_2$ is proper over Z_1, hence is proper over X via the first projection. Taking direct limit over such Z_1 and Z_2, we get a convolutiton product

$$(-)*(-):{}_c\mathrm{Ch}_n(X\times X)_{\mathbb{Q}}\times{}_c\mathrm{Ch}_n(X\times X)_{\mathbb{Q}}\longrightarrow{}_c\mathrm{Ch}_n(X\times X)_{\mathbb{Q}}.$$

This gives ${}_c\mathrm{Ch}_n(X\times X)_{\mathbb{Q}}$ the structure of an associative $\mathbb{Q}$-algebra.

For a closed substack $Z\subset X\times X$ such that pr_1 is proper, and a closed substack $Y\subset X$ which is proper over k, we have a bilinear map

$$\begin{array}{rcl}\mathrm{Ch}_n(Z)_{\mathbb{Q}}\times\mathrm{Ch}_i(Y)_{\mathbb{Q}} & \longrightarrow & \mathrm{Ch}_i(Z\cap(Y\times X))_{\mathbb{Q}}\xrightarrow{\mathrm{pr}_{2*}}\mathrm{Ch}_{c,i}(X)_{\mathbb{Q}}\\ (\rho,\zeta) & \longmapsto & \rho*\zeta:=\mathrm{pr}_{2*}\big(\rho\cdot_{X\times X}(\zeta\times[X])\big)\end{array}$$

Note here $Z\cap(Y\times X)=Z\times_{\mathrm{pr}_1,X}Y$ is proper over Y, hence is itself proper over k. Taking direct limit over such Z and Y, we get a bilinear map

$${}_c\mathrm{Ch}_n(X\times X)_{\mathbb{Q}}\times\mathrm{Ch}_{c,i}(X)_{\mathbb{Q}}\longrightarrow\mathrm{Ch}_{c,i}(X)_{\mathbb{Q}}.$$

This defines an action of the $\mathbb{Q}$-algebra ${}_c\mathrm{Ch}_n(X\times X)_{\mathbb{Q}}$ on $\mathrm{Ch}_{c,i}(X)_{\mathbb{Q}}$.

A.2. **Graded K_0' and Chow groups for Deligne–Mumford stacks.**

A.2.1. *A naive filtration on $K_0'(X)_{\mathbb{Q}}$.* For an algebraic stack X over k, let $\mathrm{Coh}(X)$ be the abelian category of coherent $\mathcal{O}_X$-modules on X. Let $K_0'(X)$ denote the Grothendieck group of $\mathrm{Coh}(X)$.

Let $\mathrm{Coh}(X)_{\leq n}$ be the full subcategory of coherent sheaves of $\mathcal{O}_X$-modules with support dimension $\leq n$. We define $K_0'(X)^{\mathrm{naive}}_{\mathbb{Q},\leq n}$ to be the the image of $K_0(\mathrm{Coh}(X)_{\leq n})_{\mathbb{Q}} \to K_0'(X)_{\mathbb{Q}}$. They give an increasing filtration on $K_0'(X)_{\mathbb{Q}}$. This is not yet the correct filtration to put on $K_0'(X)_{\mathbb{Q}}$, but let us first review the case where X is a scheme.

Let X be a scheme of finite type over k. Recall from [7, §15.1.5] that there is a natural graded map $\phi_X : \mathrm{Ch}_*(X)_{\mathbb{Q}} \to \mathrm{Gr}^{\mathrm{naive}}_* K_0'(X)_{\mathbb{Q}}$ sending the class of an irreducible subvariety $V \subset X$ of dimension n to the image of $\mathcal{O}_V$ in $\mathrm{Gr}^{\mathrm{naive}}_n K_0'(X)_{\mathbb{Q}}$. This map is in fact an isomorphism, with inverse $\psi_X : \mathrm{Gr}^{\mathrm{naive}}_* K_0'(X)_{\mathbb{Q}} \to \mathrm{Ch}_*(X)_{\mathbb{Q}}$ given by the leading term of the Riemann–Roch map $\tau_X : K_0'(X)_{\mathbb{Q}} \to \mathrm{Ch}_*(X)_{\mathbb{Q}}$. For details, see [7, Theorem 18.3, and proof of Corollary 18.3.2]. These results also hold for algebraic spaces X over k by Gillet [8].

A.2.2. A naive attempt to generalize the map ψ_X to stacks is the following. Let $Z_n(X)_{\mathbb{Q}}$ be the naive cycle group of X, namely the $\mathbb{Q}$-vector space with a basis given by integral closed substacks $V \subset X$ of dimension n.

We define a linear map $\mathrm{supp}_X : K_0(\mathrm{Coh}(X)_{\leq n})_{\mathbb{Q}} \to Z_n(X)_{\mathbb{Q}}$ sending a coherent sheaf $\mathcal{F}$ to $\sum_V m_V(\mathcal{F})[V]$, where V runs over all integral substacks of X of dimension n and $m_V(\mathcal{F})$ is the length of $\mathcal{F}$ at the generic point of V.

Clearly this map kills the image of $K_0(\mathrm{Coh}(X)_{\leq n-1})_{\mathbb{Q}}$ but what is not clear is whether or not the composition $K_0(\mathrm{Coh}(X)_{\leq n})_{\mathbb{Q}} \xrightarrow{\mathrm{supp}_X} Z_n(X)_{\mathbb{Q}} \to \mathrm{Ch}_n(X)_{\mathbb{Q}}$ factors through $K_0'(X)^{\mathrm{naive}}_{\mathbb{Q},\leq n}$. For this reason we will look for another filtration on $K_0'(X)_{\mathbb{Q}}$. [5]

When X is an algebraic space, the map supp_X does induce a map $\mathrm{Gr}^{\mathrm{naive}}_n K_0'(X)_{\mathbb{Q}} \to \mathrm{Ch}_n(X)_{\mathbb{Q}}$, and it is the same as the map ψ_X, the top term of the Riemann–Roch map.

A.2.3. *Another filtration on $K_0'(X)_{\mathbb{Q}}$.* Now we define another filtration on $K_0'(X)_{\mathbb{Q}}$ when X is a Deligne–Mumford stack satisfying the following condition.

Definition A.1. Let X be a Deligne–Mumford stack over k. A finite flat surjective map $U \to X$ from an algebraic space U of finite type over k is called a *finite flat presentation* of X. We say that X admits a finite flat presentation if such a map $U \to X$ exists.

We define $K_0'(X)_{\mathbb{Q},\leq n}$ to be the subset of elements $\alpha \in K_0'(X)_{\mathbb{Q}}$ such that there exists a finite flat presentation $\pi : U \to X$ such that $\pi^*\alpha \in K_0'(U)^{\mathrm{naive}}_{\mathbb{Q},\leq n}$.

We claim that $K_0'(X)_{\mathbb{Q},\leq n}$ is a $\mathbb{Q}$-linear subspace of $K_0'(X)_{\mathbb{Q}}$. In fact, for any two elements $\alpha_1, \alpha_2 \in K_0'(X)_{\mathbb{Q},\leq n}$, we find finite flat presentations $\pi_i : U_i \to X$ such that $\pi_i^*\alpha_i \in K_0'(U_i)^{\mathrm{naive}}_{\mathbb{Q},\leq n}$ for $i = 1, 2$. Then the pullback of the sum $\alpha_1 + \alpha_2$ to the finite flat presentation $U_1 \times_X U_2 \to X$ lies in $K_0'(U_1 \times_X U_2)^{\mathrm{naive}}_{\mathbb{Q},\leq n}$.

By this definition, $K_0'(X)_{\mathbb{Q},\leq n}$ may not be zero for $n < 0$. For any negative n, $K_0'(X)_{\mathbb{Q},\leq n}$ consists of those classes that vanish when pulled back to some finite flat presentation $U \to X$.

Lemma A.2. *When X is an algebraic space of finite type over k, the filtration $K_0'(X)_{\mathbb{Q},\leq n}$ is the same as the naive one $K_0'(X)^{\mathrm{naive}}_{\mathbb{Q},\leq n}$.*

Proof. To see this, it suffices to show that for a finite flat surjective map $\pi : U \to X$ of algebraic spaces over k, and an element $\alpha \in K_0'(X)_{\mathbb{Q}}$, if $\pi^*\alpha \in K_0'(U)^{\mathrm{naive}}_{\mathbb{Q},\leq n}$, then $\alpha \in K_0'(X)^{\mathrm{naive}}_{\mathbb{Q},\leq n}$. In fact, suppose $\alpha \in K_0'(X)^{\mathrm{naive}}_{\mathbb{Q},\leq m}$ for some $m > n$, let α_m be its image in $\mathrm{Gr}^{\mathrm{naive}}_m K_0'(X)_{\mathbb{Q}}$. Since the composition $\pi_*\pi^* : \mathrm{Ch}_m(X)_{\mathbb{Q}} \to \mathrm{Ch}_m(U)_{\mathbb{Q}} \to \mathrm{Ch}_m(X)_{\mathbb{Q}}$ is the multiplication by $\deg(\pi) \neq 0$ on each connected component, it is an isomorphism hence $\pi^* : \mathrm{Ch}_m(X)_{\mathbb{Q}} \to \mathrm{Ch}_m(U)_{\mathbb{Q}}$ is injective. By the compatibility between the isomorphism $\psi_X : \mathrm{Gr}^{\mathrm{naive}}_m K_0'(X)_{\mathbb{Q}} \cong \mathrm{Ch}_m(X)_{\mathbb{Q}}$ and flat pullback, the map $\pi^* : \mathrm{Gr}_m K_0'(X)_{\mathbb{Q}} \to \mathrm{Gr}_m K_0'(U)_{\mathbb{Q}}$ is also injective. Now $\pi^*(\alpha_m) = 0 \in \mathrm{Gr}^{\mathrm{naive}}_m K_0'(U)_{\mathbb{Q}}$ because $m > n$, we see that $\alpha_m = 0$, i.e., $\alpha \in K_0'(X)^{\mathrm{naive}}_{\mathbb{Q},\leq m-1}$. Repeating the argument we see that α has to lie in $K_0'(X)^{\mathrm{naive}}_{\mathbb{Q},\leq n}$. □

[5] Our definition in §A.2.3 may still seem naive to experts, but it suffices for our applications. We wonder if there is a way to put a natural λ-structure on $K_0'(X)_{\mathbb{Q}}$ when X is a Deligne–Mumford stack, and then one may define a filtration on it using eigenvalues of the Adams operations.

For a Deligne–Mumford stack X that admits a finite flat presentation, we denote by $\mathrm{Gr}_n K_0'(X)_{\mathbb{Q}}$ the associated graded of $K_0'(X)_{\mathbb{Q}}$ with respect to the filtration $K_0'(X)_{\mathbb{Q},\le n}$. We always have $K_0'(X)^{\mathrm{naive}}_{\mathbb{Q},\le n} \subset K_0'(X)_{\mathbb{Q},\le n}$, but the inclusion can be strict. For example, when X is the classifying space of a finite group G, we have $K_0'(X)_{\mathbb{Q}} = R_k(G)_{\mathbb{Q}}$ is the k-representation ring of G with $\mathbb{Q}$-coefficients. Any element $\alpha \in R_k(G)_{\mathbb{Q}}$ with virtual dimension 0 vanishes when pulled back along the finite flat map $\operatorname{Spec} k \to X$, therefore $K_0'(X)_{\mathbb{Q},\le -1} \subset K_0'(X)_{\mathbb{Q}}$ is the augmentation ideal of classes of virtual degree 0, and $\mathrm{Gr}_0 K_0'(X)_{\mathbb{Q}} = \mathbb{Q}$.

A.2.4. *Functoriality under flat pullback.* The filtration $K_0'(X)_{\mathbb{Q},\le n}$ is functorial under flat pullback. Suppose $f : X \to Y$ is a flat map of relative dimension d between Deligne–Mumford stacks that admit finite flat presentations, then $f^* : K_0'(Y)_{\mathbb{Q}} \to K_0'(X)_{\mathbb{Q}}$ is defined. Let $\alpha \in K_0'(Y)_{\mathbb{Q},\le n}$. We claim that $f^*\alpha \in K_0'(X)_{\mathbb{Q},\le n+d}$. In fact, choose a finite flat presentation $\pi : V \to Y$ such that $\pi^*\alpha \in K_0'(V)^{\mathrm{naive}}_{\mathbb{Q},\le n}$. Let $W = V \times_Y X$, then $\pi' : W \to X$ is representable, finite flat and surjective. Although W itself may not be an algebraic space, we may take any finite flat presentation $\sigma : U \to X$ and let $U' := W \times_X U$. Then U' is an algebraic space and $\xi : U' = W \times_X U \to X$ is a finite flat presentation. The map $f' : U' \to W \to V$ is flat of relative dimension d between algebraic spaces, hence $f'^*\pi^*\alpha \in K_0'(U')^{\mathrm{naive}}_{\mathbb{Q},\le d+n}$. Since $f'^*\pi^*\alpha = \xi^* f^*\alpha$, we see that $f^*\alpha \in K_0'(X)_{\mathbb{Q},\le n+d}$.

As a particular case of the above discussion, we have

Lemma A.3. *Let X be a Deligne–Mumford stack that admits a finite flat presentation. Let $\alpha \in K_0'(X)_{\mathbb{Q},\le n}$. Then for any finite flat representable map $f : X' \to X$, where X' is a Deligne–Mumford stack (which automatically admits a finite flat presentation), $f^*\alpha \in K_0'(X')_{\mathbb{Q},\le n}$.*

A.2.5. *Functoriality under proper pushforward.* The filtration $K_0'(X)_{\mathbb{Q},\le n}$ is also functorial under proper representable pushforward. Suppose $f : X \to Y$ is a proper representable map of Deligne–Mumford stacks that admit finite flat presentations. Suppose $\alpha \in K_0'(X)_{\mathbb{Q},\le n}$, we claim that $f_*\alpha \in K_0'(Y)_{\mathbb{Q},\le n}$. Let $\pi : V \to Y$ be a finite flat presentation. Let $\sigma : U = X \times_Y V \to X$ be the corresponding finite flat presentation of X (U is an algebraic space because f is representable). Then $f' : U \to V$ is a proper map of algebraic spaces. By Lemma A.3, $\sigma^*\alpha \in K_0'(U)_{\mathbb{Q},\le n} = K_0'(U)^{\mathrm{naive}}_{\mathbb{Q},\le n}$, therefore $\pi^* f_*\alpha = f'_*\sigma^*\alpha \in K_0'(V)^{\mathrm{naive}}_{\mathbb{Q},\le n}$, hence $f_*\alpha \in K_0'(Y)_{\mathbb{Q},\le n}$.

A.2.6. For a Deligne–Mumford stack X that admits a finite flat presentation, we now define a graded map $\psi_X : \mathrm{Gr}_* K_0'(X)_{\mathbb{Q}} \to \mathrm{Ch}_*(X)_{\mathbb{Q}}$ extending the same-named map for algebraic spaces X.

We may assume X is connected for otherwise both sides break up into direct summands indexed by the connected components of X and we can define ψ_X for each component. Let $\pi : U \to X$ be a finite flat presentation of constant degree d. For $\alpha \in K_0'(X)_{\mathbb{Q},\le n}$, we know from Lemma A.3 that $\pi^*\alpha \in K_0'(U)_{\mathbb{Q},\le n}$. Then we define

$$\psi_X(\alpha) := \frac{1}{d}\pi_*\psi_U(\pi^*\alpha) \in \mathrm{Ch}_n(X)_{\mathbb{Q}}.$$

It is easy to check that thus defined ψ_X is independent of the choice of the finite flat presentation U by dominating two finite flat presentations by their Cartesian product over X.

A.2.7. The definition of ψ_X is compatible with the support map supp_n in the sense that the following diagram is commutative when X is a Deligne–Mumford stack admitting a finite flat presentation

$$\begin{tikzcd}
K_0(\mathrm{Coh}(X)_{\le n})_{\mathbb{Q}} \arrow[r] \arrow[d, "\mathrm{supp}_X"] & K_0'(X)^{\mathrm{naive}}_{\mathbb{Q},\le n} \arrow[r] & K_0'(X)_{\mathbb{Q},\le n} \arrow[d, "\psi_X"] \\
Z_n(X)_{\mathbb{Q}} \arrow[rr] & & \mathrm{Ch}_n(X)_{\mathbb{Q}}
\end{tikzcd}$$

A.2.8. *Compatibility with the Gysin map.* We need a compatibility result of ψ_X and the refined Gysin map. Consider a Cartesian diagram of algebraic stacks

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g"] & Y' \arrow[d, "h"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{A.3}$$

satisfying the following conditions

(1) The stack X' is a Deligne–Mumford stack that admits a finite flat presentation.

(2) The morphism f can be factored as $X \xrightarrow{i} P \xrightarrow{p} Y$, where i is a regular local immersion of pure codimension e , and p is a smooth relative Deligne–Mumford type morphism of pure relative dimension $e - d$.

Remark A.4. Let X and Y be smooth Deligne–Mumford stacks, and $f : X \to Y$ is any morphism. Then we may factor f as $X \xrightarrow{(\mathrm{id},f)} X \times Y \xrightarrow{\mathrm{pr}_Y} Y$, which is the composition of a regular local immersion with a smooth morphism of Deligne–Mumford type. In this case any f always satisfies the condition (2).

A.2.9. In the situation of §A.2.8, the refined Gysin map [15, Theorem 2.1.12(xi), and end of p.529] is defined

$$f^! : \mathrm{Ch}_*(Y')_{\mathbb{Q}} \longrightarrow \mathrm{Ch}_{*-d}(X')_{\mathbb{Q}}.$$

We also have a map

$$f^* : K_0'(Y') \longrightarrow K_0'(X') \tag{A.4}$$

defined using derived pullback of coherent sheaves. Let $\mathcal{F}$ be a coherent sheaf on Y'. Then the derived tensor product $f'^{-1}\mathcal{F} \overset{\mathbf{L}}{\otimes}_{(fg)^{-1}\mathcal{O}_Y} g^{-1}\mathcal{O}_X$ has cohomology sheaves only in a bounded range because for a regular local immersion it can be computed locally by a Koszul complex. Then the alternating sum

$$f^*[\mathcal{F}] = \sum_i (-1)^i [\mathrm{Tor}_i^{(fg)^{-1}\mathcal{O}_Y}(f'^{-1}\mathcal{F}, g^{-1}\mathcal{O}_X)]$$

is a well-defined element in $K_0'(X')$. We then extend this definition by linearity to obtain the map f^* in (A.4).

Proposition A.5. *In the situation of* (A.3)*, assume all conditions in §A.2.8 are satisfied. Let $n \geq 0$ be an integer. We have*

(1) The map f^* sends $K_0'(Y')^{\mathrm{naive}}_{\mathbb{Q},\leq n}$ to $K_0'(X')_{\mathbb{Q},\leq n-d}$, and hence induces

$$\mathrm{Gr}_n^{\mathrm{naive}} f^* : \mathrm{Gr}_n^{\mathrm{naive}} K_0'(Y')_{\mathbb{Q}} \longrightarrow \mathrm{Gr}_{n-d} K_0'(X')_{\mathbb{Q}}.$$

(2) The following diagram is commutative

$$\begin{tikzcd} K_0(\mathrm{Coh}(Y')_{\leq n})_{\mathbb{Q}} \arrow[r] \arrow[d, "\mathrm{supp}_{Y'}"] & \mathrm{Gr}_n^{\mathrm{naive}} K_0'(Y')_{\mathbb{Q}} \arrow[r, "\mathrm{Gr}_n^{\mathrm{naive}} f^*"] & \mathrm{Gr}_{n-d} K_0'(X')_{\mathbb{Q}} \arrow[d, "\psi_{X'}"] \\ Z_n(Y')_{\mathbb{Q}} \arrow[r] & \mathrm{Ch}_n(Y')_{\mathbb{Q}} \arrow[r, "f^!"] & \mathrm{Ch}_{n-d}(X')_{\mathbb{Q}} \end{tikzcd} \tag{A.5}$$

(3) If Y' is also a Deligne–Mumford stack that admits a finite flat presentation, then f^* sends $K_0'(Y')_{\mathbb{Q},\leq n}$ to $K_0'(X')_{\mathbb{Q},\leq n-d}$, and we have a commutative diagram

$$\begin{tikzcd} \mathrm{Gr}_n K_0'(Y')_{\mathbb{Q}} \arrow[r, "\mathrm{Gr}_n f^*"] \arrow[d, "\psi_{Y'}"] & \mathrm{Gr}_{n-d} K_0'(X')_{\mathbb{Q}} \arrow[d, "\psi_{X'}"] \\ \mathrm{Ch}_n(Y')_{\mathbb{Q}} \arrow[r, "f^!"] & \mathrm{Ch}_{n-d}(X')_{\mathbb{Q}} \end{tikzcd}$$

Proof. (1) and (2). Write $f = p\circ i : X \xrightarrow{i} P \xrightarrow{p} Y$ as in condition (2) in §A.2.8. Let $P' = P\times_Y Y'$. For the smooth morphism p of relative dimension $e-d$, p^* sends $\mathrm{Coh}(Y')_{\le n}$ to $\mathrm{Coh}(P')_{\le n+e-d}$. Then we have a commutative diagram

$$\begin{tikzcd} K_0'(\mathrm{Coh}(Y')_{\le n})_{\mathbb{Q}} \arrow[r, "p^*"] \arrow[d, "\mathrm{supp}_{Y'}"] & K_0'(\mathrm{Coh}(P')_{\le n+e-d})_{\mathbb{Q}} \arrow[d, "\mathrm{supp}_{P'}"] \\ Z_n(Y')_{\mathbb{Q}} \arrow[r, "p^*"] & Z_{n+e-d}(P')_{\mathbb{Q}} \end{tikzcd} \tag{A.6}$$

Therefore to prove (1) and (2) we may replace $f : X \to Y$ with $i : X \to P$ hence reducing to the case f is a regular local immersion of pure codimension d.

Let $\alpha \in K_0(\mathrm{Coh}(Y')_{\le n})_{\mathbb{Q}}$. Then there exists a closed n-dimensional closed substack $Y'' \subset Y'$ such that α is in the image of $K_0'(Y'')_{\mathbb{Q}}$. We may replace Y' with Y'' and replace X' with $X'' := X' \times_{Y'} Y''$. It suffices to prove the statements (1)(2) for X'' and Y'' for then we may pushforward along the closed immersion $X'' \hookrightarrow X'$ to get the desired statements for X' and Y'. Therefore we may assume that $\dim Y' = n$.

The construction of the deformation to the normal cone can be extended to our situation, see [15, p.529]. Let N_f be the normal bundle of the regular local immersion f. Then the normal cone $C_{X'}Y'$ for the morphism $f' : X' \to Y'$ is a closed substack of g^*N_f. We denote the total space of the deformation by $M^\circ_{X'}Y'$. This is a stack over $\mathbb{P}^1$ whose restriction to $\mathbb{A}^1$ is $Y' \times \mathbb{A}^1$ and whose fiber over ∞ is the normal cone $C_{X'}Y'$. Let $i_\infty : C_{X'}Y' \hookrightarrow M^\circ_{X'}Y'$ be the inclusion of the fiber over ∞. We have the specialization map for K-groups

$$\mathrm{Sp} : K_0'(Y')_{\mathbb{Q}} \xrightarrow{\mathrm{pr}_{Y'}^*} K_0'(Y' \times \mathbb{A}^1)_{\mathbb{Q}} \cong K_0'(M^\circ_{X'}Y')_{\mathbb{Q}}/K_0'(C_{X'}Y')_{\mathbb{Q}} \xrightarrow{i_\infty^*} K_0'(C_{X'}Y')_{\mathbb{Q}}.$$

Similarly, we also have a specialization map for the naive cycle groups

$$\mathrm{Sp} : Z_n(Y')_{\mathbb{Q}} \xrightarrow{\mathrm{pr}_{Y'}^*} Z_{n+1}(Y' \times \mathbb{A}^1)_{\mathbb{Q}} \xrightarrow{\sim} Z_{n+1}(M^\circ_{X'}Y')_{\mathbb{Q}} \xrightarrow{i_\infty^!} Z_n(C_{X'}Y')_{\mathbb{Q}}.$$

Here we are using the fact that $n = \dim Y' = \dim C_{X'}Y' = \dim M^\circ_{X'}Y' - 1$, and $Z_*(-)_{\mathbb{Q}}$ is the naive cycle group. For any n-dimensional integral closed substack $V \subset Y'$, $\mathrm{Sp}([V])$ is the class of the cone $C_{X'\cap V}V \subset C_{X'}Y'$.

The diagram (A.5) can be decomposed into two diagrams

$$\begin{tikzcd} K_0'(Y')_{\mathbb{Q}} \arrow[r, "\mathrm{Sp}"] \arrow[d, "\mathrm{supp}_{Y'}"] & K_0'(C_{X'}Y')_{\mathbb{Q}} \arrow[r, dashed, "s^*"] \arrow[d, "\mathrm{supp}_{C_{X'}Y'}"] & K_0'(X')_{\mathbb{Q},\le n-d} \arrow[d, "\psi_{X'}"] \\ Z_n(Y')_{\mathbb{Q}} \arrow[r, "\mathrm{Sp}"] & Z_n(C_{X'}Y')_{\mathbb{Q}} \arrow[r, "s^!"] & \mathrm{Ch}_{n-d}(X')_{\mathbb{Q}} \end{tikzcd}$$

The dotted arrow is conditional on showing that the image of s^* lands $K_0'(X')_{\mathbb{Q},\le n-d}$. The left square above is commutative: since we are checking an equality of top-dimensional cycles, we may pass to a smooth atlas and reduce the problem to the case of schemes for which the statement is easy. Therefore it remains to show that the image of s^* lands $K_0(X')_{\mathbb{Q},\le n-d}$, and that the right square is commutative. Since $C_{X'}Y' \subset g^*N_f$, it suffices to replace $C_{X'}Y'$ by g^*N_f and prove the same original statements (1) and (2), but without assuming that $\dim g^*N_f = n$. In other words, we have reduced the problem to the following special situation

$$\begin{aligned} &X' = X,\, Y' = Y \text{ is a vector bundle of rank } d \text{ over } X, \\ &g = \mathrm{id}_X, h = \mathrm{id}_Y \text{ and } f = s \text{ is the inclusion of the zero section.} \end{aligned} \tag{A.7}$$

In this case, let $\pi : U \to X$ be a finite flat presentation, let Y_U be the vector bundle Y base changed to U. Then U and Y_U are both algebraic spaces. Let $s_U : U \hookrightarrow Y_U$ be the inclusion of the zero section and let $\sigma : Y_U \to Y$ be the projection. For any $\alpha \in K_0'(Y)^{\mathrm{naive}}_{\mathbb{Q},\le n}$, we have $\pi^*s^*\alpha = s_U^*\sigma^*\alpha \in K_0'(U)_{\mathbb{Q}}$. We have $\sigma^*\alpha \in K_0'(Y_U)^{\mathrm{naive}}_{\mathbb{Q},\le n}$. In the case of the regular embedding of algebraic spaces $s_U : U \hookrightarrow Y_U$, s_U^* sends $K_0'(Y_U)_{\mathbb{Q},\le n}$ to $K_0'(U)_{\mathbb{Q},\le n-d}$ by the compatibility of the Riemann–Roch map with the Gysin map ([7, Theorem 18.3(4)]). Therefore $\pi^*s^*\alpha = s_U^*\sigma^*\alpha \in K_0'(U)_{\mathbb{Q},\le n-d}$, hence $s^*\alpha \in K_0'(X)_{\mathbb{Q},\le n-d}$.

We finally check the commutativity of (A.5) in the special case (A.7). For any $\alpha \in K_0'(\mathrm{Coh}(Y)_{\leq n})_{\mathbb{Q}}$, we need to check that $\delta = s^! \mathrm{supp}_Y(\alpha) - \psi_X(s^*\alpha) \in \mathrm{Ch}_{n-d}(X)_{\mathbb{Q}}$ is zero. Since $\pi_*\pi^* : \mathrm{Ch}_{n-d}(X)_{\mathbb{Q}} \to \mathrm{Ch}_{n-d}(U)_{\mathbb{Q}} \to \mathrm{Ch}_{n-d}(X)_{\mathbb{Q}}$ is the multiplication by $\deg(\pi)$ on each component of X, in particular it is an isomorphism and π^* is injective. Therefore it suffices to check that $\pi^*\delta = 0 \in \mathrm{Ch}_{n-d}(U)_{\mathbb{Q}}$. Since $\pi^*\delta = s_U^! \mathrm{supp}_{Y_U}(\sigma^*\alpha) - \psi_U s_U^*(\sigma^*\alpha)$, we reduce to the situation of $s_U : U \hookrightarrow Y_U$, a regular embedding of algebraic spaces. In this case, the equality $s_U^! \mathrm{supp}_U = \psi_U s_U^*$ follows from the compatibility of the Riemann–Roch map with the Gysin map ([7, Theorem 18.3(4)]).

(3) Let $\alpha \in K_0'(Y')_{\mathbb{Q},\leq n}$. Then for some finite flat presentation $\pi_V : V \to Y'$, $\pi_V^*\alpha \in K_0'(V)^{\mathrm{naive}}_{\mathbb{Q},\leq n}$. Let $W = X' \times_{Y'} V = X \times_Y V$, and let $f'' : W \to V$ be the projection. Then we have a Cartesian diagram as in (A.3) with the top row replaced by $f'' : W \to V$. Since $\pi_W : W \to X'$ is a finite flat surjective map (W may not be an algebraic space because we are not assuming that f is representable), $\pi_W^* : \mathrm{Ch}_{n-d}(X')_{\mathbb{Q}} \to \mathrm{Ch}_{n-d}(W)_{\mathbb{Q}}$ is injective. Therefore, in order to show that $f^*\alpha \in K_0'(X')_{\mathbb{Q},\leq n-d}$ and that $\psi_{X'} f^*\alpha - f^! \psi_{Y'}\alpha = 0$ in $\mathrm{Ch}_{n-d}(X')_{\mathbb{Q}}$, it suffices to show that $\pi_W^* f^*\alpha = f^*\pi_V^*\alpha \in K_0'(W)_{\mathbb{Q},\leq n-d}$ and that $\pi_W^*(\psi_{X'} f^*\alpha - f^!\psi_{Y'}\alpha) = \psi_W f^*(\pi_V^*\alpha) - f^!\psi_V(\pi_V^*\alpha)$ is zero in $\mathrm{Ch}_{n-d}(W)_{\mathbb{Q}}$. Therefore we have reduced to the case where $Y' = V$ is an algebraic space. In this case $K_0'(Y')_{\mathbb{Q},\leq n} = K_0'(Y')^{\mathrm{naive}}_{\mathbb{Q},\leq n}$, and the statements follows from (1)(2). $\square$

By applying Proposition A.5 to the diagonal map $X \to X \times X$ (and taking g, h to be the identity maps), we get the following result, which is not used in the paper.

Corollary A.6. *Let X be a smooth Deligne–Mumford stack that admits a finite flat presentation, then the map ψ_X is a graded ring homomorphism.*

A.2.10. *The case of proper intersection.* There is another situation where an analog of Proposition A.5 can be easily proved. We consider a Cartesian diagram as in (A.3) satisfying the following conditions

(1) X' is a Deligne–Mumford stack, and h (hence g) is representable.
(2) The normal cone stack of f is a vector bundle stack (see [2, Definition 1.9]) of some constant virtual rank d.
(3) There exists a commutative diagram

$$\begin{tikzcd} U \arrow[r, hook, "i"] \arrow[d, "u"] & V \arrow[d, "v"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{A.8}$$

where U and V are schemes locally of finite type over k, u and v are smooth surjective and i is a regular local immersion.
(4) We have $\dim Y' = n$ and $\dim X' = n - d$.

Remark A.7. Suppose X and Y are smooth stacks over k. Pick any smooth surjective $W \twoheadrightarrow Y$ where W is a smooth scheme, and let $u : U \to X \times_Y W$ be any smooth surjective map from a smooth scheme U. Take $V = U \times W$, then $i = (\mathrm{id}, \mathrm{pr}_W \circ u) : U \to V = U \times W$ is a regular local immersion. Therefore, in this case, f satisfies the condition (3) above.

If f satisfies the condition (2) above, the refined Gysin map is defined (see [15, End of p.529 and footnote]). We only consider the top degree Gysin map

$$f^! : \mathrm{Ch}_n(Y')_{\mathbb{Q}} \longrightarrow \mathrm{Ch}_{n-d}(X')_{\mathbb{Q}}$$

On the other hand, derived pullback by f^* gives

$$f^* : K_0'(Y') \longrightarrow K_0'(X')$$

as in (A.4). Here the boundedness of Tor can be checked by passing to a smooth cover of X', and we may use the diagram (A.8) to reduce to the case where f is a regular local immersion, where Tor-boundedness can be proved by using the Koszul resolution.

Lemma A.8. *Under the assumptions of §A.2.10, we have a commutative diagram*

$$\begin{tikzcd} K_0'(Y')_{\mathbb{Q}} \arrow[r, "f^*"] \arrow[d, "\mathrm{supp}_{Y'}"'] & K_0'(X')_{\mathbb{Q}} \arrow[d, "\mathrm{supp}_{X'}"] \\ Z_n(Y')_{\mathbb{Q}} \arrow[r, "f^!"] & Z_{n-d}(X')_{\mathbb{Q}} \end{tikzcd}$$

Proof. The statement we would like to prove is an equality of top-dimensional cycles in X'. Such an equality can be checked after pulling back along a smooth surjective morphism $X'' \to X'$. We shall use this observation to reduce the general case to the case where all members of the diagram are algebraic spaces and that f is a regular embedding.

Let $i : U \to V$ be a regular local immersion of schemes as in Condition (3) of §A.2.10 that covers $f : X \to Y$. By passing to connected components of U and V, we may assume that the maps u, v and i in (A.8) have pure (co)dimension. Let $U' = X' \times_X U$ and $V' = Y' \times_Y V$, then we have a diagram where all three squares and the outer square are Cartesian

$$\begin{tikzcd} X' \arrow[d, "g"] \arrow[rrr, bend left, "f'"] & U' \arrow[l, "u'"'] \arrow[r, "i'"] \arrow[d] & V' \arrow[r, "v'"] \arrow[d] & Y' \arrow[d, "h"] \\ X \arrow[rrr, bend right, "f"'] & U \arrow[l, "u"'] \arrow[r, "i"] & V \arrow[r, "v"] & Y \end{tikzcd}$$

Let $\alpha \in K_0'(Y')$. To show $\mathrm{supp}_{X'}(f^*\alpha) - f^!\mathrm{supp}_{Y'}(\alpha) = 0 \in Z_{n-d}(X')_{\mathbb{Q}}$, it suffices to show its pullback to U' is zero. We have

$$\begin{aligned} u'^*(\mathrm{supp}_{X'}(f'^*\alpha) - f^!\mathrm{supp}_{Y'}(\alpha)) &= \mathrm{supp}_{U'}(u^*f^*\alpha) - u^!f^!\mathrm{supp}_{Y'}(\alpha) \\ &= \mathrm{supp}_{U'}(i^*v^*\alpha) - i^!v^!\mathrm{supp}_{Y'}(\alpha). \end{aligned} \tag{A.9}$$

Since v is smooth and representable, we have $v^!\mathrm{supp}_{Y'}(\alpha) = \mathrm{supp}_{V'}(v'^*\alpha)$. Letting $\beta = v'^*\alpha \in K_0'(V')$, we get

$$\mathrm{supp}_{U'}(i^*v^*\alpha) - i^!v^!\mathrm{supp}_{Y'}(\alpha) = \mathrm{supp}_{U'}(i^*\beta) - i^!\mathrm{supp}_{V'}(\beta).$$

To show the LHS of (A.9) is zero, we only need to show that $\mathrm{supp}_{U'}(i^*\beta) - i^!\mathrm{supp}_{V'}(\beta) = 0$. Therefore we have reduced to the following situation:

X and Y are schemes and f is a regular local immersion.

In this case, X' and Y' are also algebraic spaces by the representability of h and g. In this case we have $\mathrm{supp}_{X'} = \psi_{X'}$ and $\mathrm{supp}_{Y'} = \psi_{Y'}$. The identity $\mathrm{supp}_{X'}(f^*\alpha) = \psi_{X'}(f^*\alpha) = f^!\psi_{Y'}(\alpha) = f^!\mathrm{supp}_{Y'}(\alpha)$ follows from the compatibility of the Riemann–Roch map with the Gysin map ([7, Theorem 18.3(4)]). □

A.3. **The octahedron lemma.** We consider the following commutative diagram of algebraic stacks over k

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d] & X \arrow[d] & B \arrow[l] \arrow[d] \\ U \arrow[r] & S & V \arrow[l] \\ C \arrow[r] \arrow[u] & Y \arrow[u] & D \arrow[l] \arrow[u, "d"'] \end{tikzcd} \tag{A.10}$$

Attached to this diagram we may form the fiber product of each row

$$A \times_X B \longrightarrow U \times_S V \xleftarrow{\gamma} C \times_Y D \tag{A.11}$$

and the fiber product of each column

$$C \times_U A \xrightarrow{\alpha} Y \times_S X \leftarrow D \times_V B \tag{A.12}$$

We form the fiber products of the maps in (A.11) and in (A.12):

$$\begin{tikzcd}
(C \times_Y D) \times_{(U \times_S V)} (A \times_X B) \arrow[r] \arrow[d] & A \times_X B \arrow[d] \\
C \times_Y D \arrow[r, "\delta"] & U \times_S V
\end{tikzcd} \tag{A.13}$$

$$\begin{tikzcd}
(C \times_U A) \times_{(Y \times_S X)} (D \times_V B) \arrow[r] \arrow[d] & D \times_V B \arrow[d] \\
C \times_U A \arrow[r, "\alpha"] & Y \times_S X
\end{tikzcd} \tag{A.14}$$

Finally we introduce another stack N as the fiber product

$$\begin{tikzcd}
N \arrow[r] \arrow[d] & A \times B \times C \times D \arrow[d] \\
X \times_S Y \times_S U \times_S V \arrow[r] & (X \times_S U) \times (X \times_S V) \times (Y \times_S U) \times (Y \times_S V) =: R
\end{tikzcd} \tag{A.15}$$

Lemma A.9. *There is a canonical isomorphism of stacks that appear in the northwest corners of* (A.13) *and* (A.14)

$$(C \times_Y D) \times_{(U \times_S V)} (A \times_X B) \cong N \cong (C \times_U A) \times_{(Y \times_S X)} (D \times_V B) \tag{A.16}$$

Proof. For the first isomorphism, we consider the diagram (to shorten notation, we use $\cdot$ instead of $\times$)

$$\begin{tikzcd}
& (C \cdot_Y D) \cdot_{(U \cdot_S V)} (A \cdot_X B) \arrow[r] \arrow[d] & (C \cdot_Y D) \cdot (A \cdot_X B) \arrow[r] \arrow[d] & A \cdot B \cdot C \cdot D \arrow[d] \\
& Y \cdot_S X \cdot_S U \cdot_S V \arrow[r] \arrow[dl] & (Y \cdot_S U \cdot_S V) \cdot (X \cdot_S U \cdot_S V) \arrow[r] \arrow[dl] \arrow[d] & R \arrow[d] \\
U \cdot_S V \arrow[r, "\Delta"] & (U \cdot_S V)^2 & Y \cdot X \arrow[r, "\Delta_Y \cdot \Delta_X"] & Y^2 \cdot X^2
\end{tikzcd} \tag{A.17}$$

Here all the squares are Cartesian. The upper two squares combined give the square in (A.15). This shows that the LHS of (A.16) is canonically isomorphic to N.

For the second isomorphism, we argue in the same way using the following diagram instead

$$\begin{tikzcd}
& (C \cdot_U A) \cdot_{(Y \cdot_S X)} (D \cdot_V B) \arrow[r] \arrow[d] & (C \cdot_U A) \cdot (D \cdot_V B) \arrow[r] \arrow[d] & A \cdot B \cdot C \cdot D \arrow[d] \\
& Y \cdot_S X \cdot_S U \cdot_S V \arrow[r] \arrow[dl] & (Y \cdot_S X \cdot_S U) \cdot (Y \cdot_S X \cdot_S V) \arrow[r] \arrow[dl] \arrow[d] & R \arrow[d] \\
Y \cdot_S X \arrow[r, "\Delta"] & (Y \cdot_S X)^2 & U \cdot V \arrow[r, "\Delta_U \cdot \Delta_V"] & U^2 \cdot V^2
\end{tikzcd} \tag{A.18}$$

□

There is a way to label the vertices of the barycentric subdivision of an octahedron by the stacks introduced above. We consider an octahedron with a north pole, a south pole and a square as the equator. We put S at the south pole. The four vertices of the equator labelled with A, B, D and C clockwisely. The barycenters of the four lower faces are labeled by U, V, X, Y so that their adjacency relation with the vertices labelled by A, B, C, D is consistent with the diagram (A.10). At the barycenters of the four upper faces we put the fiber products: e.g., for the triangle with bottom edge labeled by A, B, we put $A \times_X B$ at the barycenter of this triangle. Finally we put N at the north pole.

Theorem A.10 (The Octahedron Lemma)**.** *Suppose we are in the above situation. Suppose further that*

(1) The algebraic stacks A, C, D, U, V, X, Y and S (everybody except B, B for bad) are smooth and equidimensional over k. We denote $\dim A$ *by d_A, etc.*
(2) The fiber products $U \times_S V$, $Y \times_S X$, $C \times_Y D$ and $C \times_U A$ have expected dimensions $d_U + d_V - d_S$, etc.
(3) Each of the Cartesian squares

$$\begin{tikzcd} A \times_X B \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r, "a"] & X \end{tikzcd} \tag{A.19}$$

$$\begin{tikzcd} D \times_V B \arrow[r] \arrow[d] & B \arrow[d] \\ D \arrow[r, "d"] & V \end{tikzcd} \tag{A.20}$$

satisfies either the conditions in §A.2.8 or the conditions in §A.2.10.
(4) The Cartesian squares (A.13) *and* (A.14) *satisfy the conditions in §A.2.8.*

Let $n = d_A + d_B + d_C + d_D - d_U - d_V - d_X - d_Y + d_S$. Then

$$\delta^! a^! [B] = \alpha^! d^! [B] \in \mathrm{Ch}_n(N).$$

Proof. Since U, S and V are smooth and pure dimensional, and $U \times_S V$ has the expected dimension, it is a local complete intersection and we have

$$\mathcal{O}_{U \times_S V} \cong \mathcal{O}_U \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_S} \mathcal{O}_V$$

Here we implicitly pullback the sheaves $\mathcal{O}_U, \mathcal{O}_V$ and $\mathcal{O}_S$ to $U \times_S V$ using the plain sheaf pullback. Similar argument shows that the usual structure sheaves $\mathcal{O}_{Y \times_S X}, \mathcal{O}_{C \times_Y D}$ and $\mathcal{O}_{C \times_U A}$ coincide with the corresponding derived tensor products.

We now show a derived version of the isomorphism (A.16). We equip each member of the diagrams (A.13), (A.14) and (A.15) with the derived structure sheaves, starting from the usual structure sheaves of A, B, C, D, X, Y, U, V and S. For N, we use (A.15) to equip it with the derived structure sheaf

$$\mathcal{O}_N^{\mathrm{der}} := (\mathcal{O}_X \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_S} \mathcal{O}_Y \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_S} \mathcal{O}_U \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_S} \mathcal{O}_V) \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_R^{\mathrm{der}}} (\mathcal{O}_A \boxtimes \mathcal{O}_B \boxtimes \mathcal{O}_C \boxtimes \mathcal{O}_D)$$

where $\mathcal{O}_R^{\mathrm{der}}$ is the derived structure sheaf $(\mathcal{O}_X \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_S} \mathcal{O}_U) \boxtimes \cdots \boxtimes (\mathcal{O}_Y \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_S} \mathcal{O}_V)$ on $R = (X \times_S U) \times \cdots \times (Y \times_S V)$. To make sense of this derived tensor product over $\mathcal{O}_R^{\mathrm{der}}$, we need to work with dg categories of coherent complexes rather than the derived category.

We claim that under the isomorphisms between both sides of (A.16) and N, their derived structure sheaves are also quasi-isomorphic to each other. In fact we simply put derived structures sheaves on each vertex of the diagram (A.17). Since the upper two squares combined give the square in (A.15), transitivity of the derived tensor product gives a quasi-isomorphism

$$\mathcal{O}_N^{\mathrm{der}} \cong (\mathcal{O}_C \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_Y} \mathcal{O}_D) \overset{\mathbf{L}}{\otimes}_{(\mathcal{O}_U \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_S} \mathcal{O}_V)} (\mathcal{O}_A \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_X} \mathcal{O}_B) \tag{A.21}$$

Similarly, by considering the diagram (A.18), we get a quasi-isomorphism

$$\mathcal{O}_N^{\mathrm{der}} \cong (\mathcal{O}_C \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_U} \mathcal{O}_A) \overset{\mathbf{L}}{\otimes}_{(\mathcal{O}_Y \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_S} \mathcal{O}_X)} (\mathcal{O}_D \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_V} \mathcal{O}_B) \tag{A.22}$$

Combing the isomorphisms (A.21) and (A.22), and using the fact that $U \times_S V$, $Y \times_S X$, $C \times_Y D$ and $C \times_U A$ need not be derived, we get an isomorphism of coherent complexes on N

$$\mathcal{O}_{C \times_Y D} \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_{U \times_S V}} (\mathcal{O}_A \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_X} \mathcal{O}_B) \cong \mathcal{O}_{C \times_U A} \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_{Y \times_S X}} (\mathcal{O}_D \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_V} \mathcal{O}_B)$$

These are bounded complexes because the diagrams (A.19), (A.20), (A.13) and (A.14) satisfy the conditions in §A.2.8 or §A.2.10. Taking classes in $K_0'(N)_{\mathbb{Q}}$ we get

$$\delta^* a^* \mathcal{O}_B = \alpha^* d^* \mathcal{O}_B \in K_0'(N)_{\mathbb{Q}} \tag{A.23}$$

Here a^*, d^*, α^* and δ^* are the derived pullbacks maps between K_0'-groups defined using the relevant Cartesian diagrams. Now we apply Proposition A.5 to the diagrams (A.19), (A.20), (A.13) and (A.14), to conclude that both sides of (A.23) lie in $K_0'(N)_{\mathbb{Q},\le n}$ (where n is the expected dimension of N). In case (A.19) or (A.20) satisfies §A.2.10 instead of §A.2.8, the corresponding statement $K_0'(B)_{\mathbb{Q},\le d_B} \to K_0'(A\times_X B)_{\mathbb{Q},\le d_A+d_B-d_X}$ or $K_0'(B)_{\mathbb{Q},\le d_B} \to K_0'(D\times_V B)_{\mathbb{Q},\le d_D+d_B-d_V}$ is automatic for dimension reasons.

Now we finish the proof. We treat only the case where (A.19) satisfies the conditions in §A.2.8 and (A.20) satisfies the conditions in §A.20. This is the case which we actually use in the main body of the paper, and the other cases can be treated in the same way.

Let $\overline{\delta^*a^*\mathcal{O}_B}$ and $\overline{\alpha^*d^*\mathcal{O}_B}$ denote their images in $\mathrm{Gr}_n K_0'(N)_{\mathbb{Q}}$. Similarly let $\overline{a^*\mathcal{O}_B} \in \mathrm{Gr}_{d_A+d_B-d_X} K_0'(A\times_X B)_{\mathbb{Q}}$ be the images of $a^*\mathcal{O}_B$. Applying Proposition A.5 three times and Lemma A.8 once we get

$$\begin{aligned}
&\delta^! a^! [B] = \delta^! a^! \mathrm{supp}_B(\mathcal{O}_B)\\
=\;& \delta^! \psi_{A\times_X B}(\overline{a^*\mathcal{O}_B}) \quad \text{(Prop A.5(2) applied to (A.19))}\\
=\;& \psi_N(\overline{\delta^*a^*\mathcal{O}_B}) \quad \text{(Prop A.5(3) applied to (A.13))}\\
=\;& \psi_N(\overline{\alpha^*d^*\mathcal{O}_B}) \quad \text{(A.23)}\\
=\;& \alpha^! \mathrm{supp}_{D\times_V B}(d^*\mathcal{O}_B) \quad \text{(Prop A.5(2) applied to (A.14))}\\
=\;& \alpha^! d^! \mathrm{supp}_B(\mathcal{O}_B) \quad \text{(Lemma A.8 applied to (A.20))}\\
=\;& \alpha^! d^! [B].
\end{aligned}$$

□

A.4. **A Lefschetz trace formula.** In this subsection, we will assume:

- *All sheaf-theoretic functors are derived functors.*

A.4.1. *Cohomological correspondences.* We first review some basic definitions and properties of cohomological correspondences following [22]. Consider a diagram of algebraic stacks over k

$$X \xleftarrow{\overleftarrow{c}} C \xrightarrow{\overrightarrow{c}} Y \tag{A.24}$$

We call C together with the maps $\overleftarrow{c}$ and $\overrightarrow{c}$ a correspondence between X and Y.

Let $\mathcal{F} \in D_c^b(X)$ and $\mathcal{G} \in D_c^b(Y)$ be $\mathbb{Q}_\ell$-complexes of sheaves. A *cohomological correspondence between $\mathcal{F}$ and $\mathcal{G}$ supported on C* is a map

$$\zeta : \overleftarrow{c}^*\mathcal{F} \longrightarrow \overrightarrow{c}^!\mathcal{G}$$

in $D_c^b(C)$.

Suppose we have a map of correspondences

$$\begin{tikzcd}
X \arrow[d, "f"] & C \arrow[l, "\overleftarrow{c}"'] \arrow[r, "\overrightarrow{c}"] \arrow[d, "h"] & Y \arrow[d, "g"] \\
S & B \arrow[l, "\overleftarrow{b}"'] \arrow[r, "\overrightarrow{b}"] & T
\end{tikzcd}$$

where $\overleftarrow{c}$ and $\overleftarrow{b}$ are proper, then we have an induced map between the group of cohomological correspondences supported on C and on B (see [22, §1.1.6(a)])

$$h_! : \mathrm{Hom}_C(\overleftarrow{c}^*\mathcal{F}, \overrightarrow{c}^!\mathcal{G}) \longrightarrow \mathrm{Hom}_B(\overleftarrow{b}^* f_!\mathcal{F}, \overrightarrow{b}^! g_!\mathcal{G})$$

In particular, if $S = B = T$ and $\overleftarrow{b} = \overrightarrow{b} = \mathrm{id}_S$, then $\zeta \in \mathrm{Hom}_C(\overleftarrow{c}^*\mathcal{F}, \overrightarrow{c}^!\mathcal{G})$ induces a map $h_!\zeta$ between $f_!\mathcal{F}$ and $g_!\mathcal{G}$ given by the composition

$$h_!\zeta : f_!\mathcal{F} \longrightarrow f_!\overleftarrow{c}_!\overleftarrow{c}^*\mathcal{F} \xrightarrow{f_!\overleftarrow{c}_!(\zeta)} f_!\overleftarrow{c}_!\overrightarrow{c}^!\mathcal{G} = g_!\overrightarrow{c}_!\overrightarrow{c}^!\mathcal{G} \longrightarrow g_!\mathcal{G}. \tag{A.25}$$

When $S = T$, B the diagonal of S, $X = Y$ and $f = g$, we call C a self-correspondence of X over S. In this case, for a cohomological correspondence ζ between $\mathcal{F}$ and $\mathcal{G}$ supported on C, we also use $f_!\zeta$ to denote $h_!\zeta \in \mathrm{Hom}_S(f_!\mathcal{F}, f_!\mathcal{G})$ defined above.

A.4.2. *Fixed locus and the trace map.* Suppose in the diagram (A.24) we have $X = Y$. We denote X by M. Define the fixed point locus $\mathrm{Fix}(C)$ of C by the Cartesian diagram

$$\begin{tikzcd} \mathrm{Fix}(C) \arrow[r] \arrow[d] & C \arrow[d, "(\overleftarrow{c}, \overrightarrow{c})"] \\ M \arrow[r, "\Delta"] & M \times M \end{tikzcd}$$

For any $\mathcal{F} \in D^b_c(M)$, there is a natural trace map (see [22, Eqn(1.2)])

$$\tau_C : \mathrm{Hom}(\overleftarrow{c}^*\mathcal{F}, \overrightarrow{c}^!\mathcal{F}) \longrightarrow \mathrm{H}^{\mathrm{BM}}_0(\mathrm{Fix}(C) \otimes_k \overline{k})$$

In other words, for a cohomological self-correspondence ζ of $\mathcal{F}$ supported on C, there is a well-defined Borel-Moore homology class $\tau_C(\zeta) \in \mathrm{H}^{\mathrm{BM}}_0(\mathrm{Fix}(C) \otimes_k \overline{k})$.

A.4.3. In the situation of §A.4.2, we further assume that both C and M are Deligne–Mumford stacks, M is smooth and separated over k of pure dimension n, and that $\mathcal{F} = \mathbb{Q}_{\ell,M}$ is the constant sheaf on M.

Using Poincaré duality for M, a cohomological self-correspondence of the constant sheaf $\mathbb{Q}_{\ell,M}$ supported on C is the same as a map

$$\mathbb{Q}_{\ell,C} = \overleftarrow{c}^*\mathbb{Q}_{\ell,M} \longrightarrow \overrightarrow{c}^!\mathbb{Q}_{\ell,M} \cong \overrightarrow{c}^!\mathbb{D}_M[-2n](-n) \cong \mathbb{D}_C[-2n](-n)$$

Over $C \otimes_k \overline{k}$, this is the same thing as an element in $\mathrm{H}^{\mathrm{BM}}_{2n}(C \otimes_k \overline{k})(-n)$. In this case, the trace map τ_C becomes the map

$$\overline{\tau}_C : \mathrm{H}^{\mathrm{BM}}_{2n}(C \otimes_k \overline{k})(-n) \longrightarrow \mathrm{H}^{\mathrm{BM}}_0(\mathrm{Fix}(C) \otimes_k \overline{k}).$$

On the other hand, we have the cycle class map

$$\mathrm{cl}_C : \mathrm{Ch}_n(C)_{\mathbb{Q}} \longrightarrow \mathrm{H}^{\mathrm{BM}}_{2n}(C \otimes_k \overline{k})(-n) = \mathrm{Hom}(\overleftarrow{c}^*\mathbb{Q}_{\ell,M}, \overrightarrow{c}^!\mathbb{Q}_{\ell,M}).$$

Therefore, any cycle $\zeta \in \mathrm{Ch}_n(C)_{\mathbb{Q}}$ gives a cohomological self-correspondence of the constant sheaf $\mathbb{Q}_{\ell,M}$ supported on C. We will use the same notation ζ to denote the cohomological self-correspondence induced by it. Since $\Delta_M : M \to M \times M$ is a regular local immersion of pure codimension n, we have the refined Gysin map

$$\Delta^!_M : \mathrm{Ch}_n(C)_{\mathbb{Q}} \longrightarrow \mathrm{Ch}_0(\mathrm{Fix}(C))_{\mathbb{Q}}.$$

Lemma A.11. *Under the assumptions of §A.4.3, we have a commutative diagram*

$$\begin{tikzcd} \mathrm{Ch}_n(C)_{\mathbb{Q}} \arrow[r, "\Delta^!_M"] \arrow[d, "\mathrm{cl}_C"] & \mathrm{Ch}_0(\mathrm{Fix}(C))_{\mathbb{Q}} \arrow[d, "\mathrm{cl}_{\mathrm{Fix}(C)}"] \\ \mathrm{H}^{\mathrm{BM}}_{2n}(C \otimes_k \overline{k})(-n) \arrow[r, "\overline{\tau}_C"] & \mathrm{H}^{\mathrm{BM}}_0(\mathrm{Fix}(C) \otimes_k \overline{k}) \end{tikzcd}$$

Proof. Let us base change to $\overline{k}$ and keep the same notation for M, C etc. Tracing through the definition of τ_C, we see that it is the same as the cap product with the relative cycle class of $\Delta(M)$ in $\mathrm{H}^{2n}(M \times M, M \times M - \Delta(M))(n)$. Then the lemma follows from [7, Theorem 19.2]. Note that [7, Theorem 19.2] is for schemes over $\mathbb{C}$ but the argument there works in our situation as well, using the construction of the deformation to the normal cone for Deligne-Mumford stacks in [14, p.489]. □

A.4.4. *Intersection with the graph of Frobenius.* Suppose we are given a self-correspondence C of M over S

$$\begin{tikzcd} & C \arrow[dl, "\overleftarrow{c}"'] \arrow[dr, "\overrightarrow{c}"] \arrow[dd, "h"] & \\ M \arrow[dr, "f"'] & & M \arrow[dl, "f"] \\ & S & \end{tikzcd}$$

satisfying

- k is a finite field;
- S is a scheme over k;
- M is a smooth and separated Deligne–Mumford stack over k of pure dimension n;
- $f : M \to S$ is proper;
- $\overleftarrow{c} : C \to M$ is representable and proper.

We define Sht_C by the Cartesian diagram

$$\begin{tikzcd} \mathrm{Sht}_C \arrow[r] \arrow[d] & C \arrow[d, "(\overleftarrow{c}, \overrightarrow{c})"] \\ M \arrow[r, "(\mathrm{id}, \mathrm{Fr}_M)"] & M \times M \end{tikzcd} \tag{A.26}$$

Here the notation Sht_C suggests that in applications Sht_C will be a kind of moduli of Shtukas. We denote the image of the fundamental class $[M]$ under $(\mathrm{id}, \mathrm{Fr}_M)_*$ by $\Gamma(\mathrm{Fr}_M)$. Since $(\mathrm{id}, \mathrm{Fr}_M)$ is a regular immersion of pure codimension n, the refined Gysin map

$$(\mathrm{id}, \mathrm{Fr}_M)^! : \mathrm{Ch}_n(C)_{\mathbb{Q}} \longrightarrow \mathrm{Ch}_0(\mathrm{Sht}_C)_{\mathbb{Q}}$$

is defined. In particular, for $\zeta \in \mathrm{Ch}_n(C)_{\mathbb{Q}}$, we get a 0-cycle

$$(\mathrm{id}, \mathrm{Fr}_M)^! \zeta \in \mathrm{Ch}_0(\mathrm{Sht}_C)_{\mathbb{Q}}.$$

A.4.5. Since $C \to M \times_S M$, while $(\mathrm{id}, \mathrm{Fr}_M) : M \to M \times M$ covers the similar map $(\mathrm{id}, \mathrm{Fr}_S) : S \to S \times S$, the map $\mathrm{Sht}_C \to S$ factors through the discrete set $S(k)$, viewed as a discrete closed subscheme of S. Since $\mathrm{Sht}_C \to S(k)$, we get a decomposition of Sht_C into open and closed subschemes

$$\mathrm{Sht}_C = \coprod_{s \in S(k)} \mathrm{Sht}_C(s).$$

Therefore

$$\mathrm{Ch}_0(\mathrm{Sht}_C)_{\mathbb{Q}} = \bigoplus_{s \in S(k)} \mathrm{Ch}_0(\mathrm{Sht}_C(s))_{\mathbb{Q}}.$$

For $\zeta \in \mathrm{Ch}_n(C)_{\mathbb{Q}}$, the 0-cycle $\zeta \cdot_{M \times M} \Gamma(\mathrm{Fr}_M)$ can be written uniquely as the sum of 0-cycles

$$((\mathrm{id}, \mathrm{Fr}_M)^! \zeta)_s \in \mathrm{Ch}_0(\mathrm{Sht}_C(s))_{\mathbb{Q}}, \quad \forall s \in S(k). \tag{A.27}$$

Each $\mathrm{Sht}_C(s) = \Gamma(\mathrm{Fr}_{M_s}) \times_{M_s \times M_s} C_s$. Since $\overleftarrow{c} : C_s \to M_s$ is proper and M_s is separated (because f is proper), $C_s \to M_s \times M_s$ is proper, therefore $\mathrm{Sht}_C(s)$ is proper over $\Gamma(\mathrm{Fr}_{M_s})$, hence it is itself proper over k because $\Gamma(\mathrm{Fr}_{M_s}) \cong M_s$ is proper over k. Therefore the degree map $\deg : \mathrm{Ch}_0(\mathrm{Sht}_C(s))_{\mathbb{Q}} \to \mathbb{Q}$ is defined, we get an intersection number indexed by $s \in S(k)$:

$$\langle \zeta, \Gamma(\mathrm{Fr}_M) \rangle_s := \deg((\mathrm{id}, \mathrm{Fr}_M)^! \zeta)_s \in \mathbb{Q}.$$

The main result of this subsection is the following.

Proposition A.12. *Assume all conditions in §A.4.4 are satisfied. Let $\zeta \in \mathrm{Ch}_n(C)_{\mathbb{Q}}$. Then for all $s \in S(k)$, we have*

$$\langle \zeta, \Gamma(\mathrm{Fr}_M) \rangle_s = \mathrm{Tr}\left((f_! \mathrm{cl}_C(\zeta))_s \circ \mathrm{Frob}_s, (f_! \mathbb{Q}_\ell)_{\overline{s}} \right). \tag{A.28}$$

Here $f_! \mathrm{cl}_C(\zeta) := h_! \mathrm{cl}_C(\zeta)$ is the endomorphism of $f_! \mathbb{Q}_\ell$ induced by the cohomological correspondence $\mathrm{cl}_C(\zeta)$ supported on C, and $(f_! \mathrm{cl}_C(\zeta))_s$ is its action on the geometric stalk $(f_! \mathbb{Q}_\ell)_{\overline{s}}$.

Proof. Let $'C = C$ but viewed as a self-correspondence of M via the following maps

$$M \xleftarrow{\overleftarrow{'c} = \mathrm{Fr}_M \circ \overleftarrow{c}} {'C} = C \xrightarrow{\overrightarrow{'c} = \overrightarrow{c}} M$$

However, $'C$ is no longer a self-correspondence of M over S. Instead, it maps to the Frobenius graph of S:

$$\begin{tikzcd} M \arrow[d, "f"] & {'C} \arrow[l, "\overleftarrow{'c}"'] \arrow[r, "\overrightarrow{'c}"] \arrow[d, "'h"] & M \arrow[d, "f"] \\ S & {'S} \arrow[l, "\mathrm{Fr}_S"'] \arrow[r, "\mathrm{id}"] & S \end{tikzcd} \tag{A.29}$$

Here $'S = S$ but viewed as a self-correspondence of S via $(\mathrm{Fr}_S, \mathrm{id}) : {'S} \to S \times S$. The map $'h : {'C} \to {'S}$ is simply the original map $h : C \to S$.

We have the following diagram where both squares are Cartesian and the top square is (A.26)

$$\begin{tikzcd}
\mathrm{Sht}_C \arrow[r] \arrow[d] & {'C} = C \arrow[d, "(\overleftarrow{c}, \overrightarrow{c})"] \\
M \arrow[r, "(\mathrm{id}, \mathrm{Fr}_M)"] \arrow[d, "\mathrm{Fr}_M"] & M \times M \arrow[d, "(\mathrm{Fr}_M, \mathrm{id})"] \\
M \arrow[r, "\Delta_M"] & M \times M
\end{tikzcd} \tag{A.30}$$

Therefore the outer square is also Cartesian, i.e., there is a canonical isomorphism $\mathrm{Fix}({'C}) = \mathrm{Sht}_C$.

For $\zeta \in \mathrm{Ch}_n(C)_{\mathbb{Q}} = \mathrm{Ch}_n({'C})_{\mathbb{Q}}$, we may also view it as a cohomological self-correspondence of $\mathbb{Q}_{\ell,M}$ supported on $'C$. We denote it by $'\zeta \in \mathrm{Ch}_n({'C})_{\mathbb{Q}}$ to emphasize that it is supported on C'. We claim that

$$(\mathrm{id}, \mathrm{Fr}_M)^! \zeta = \Delta_M^!({'\zeta}) \in \mathrm{Ch}_0(\mathrm{Sht}_C)_{\mathbb{Q}}.$$

In fact, this is a very special case of the Excess Intersection Formula [7, Theorem 6.3] applied to the diagram (A.30) where both $(\mathrm{id}, \mathrm{Fr}_M)$ and Δ_M are regular immersions of the same codimension. In particular, taking the degree of the s components, we have

$$\langle {'\zeta}, \Delta_*[M] \rangle_s = \langle \zeta, \Gamma(\mathrm{Fr}_M) \rangle_s \quad \text{for all } s \in S(k). \tag{A.31}$$

By [22, Prop. 1.2.5] applied to the proper map (A.29) between correspondences, we get a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(\overleftarrow{'c}^* \mathbb{Q}_{\ell,M}, \overrightarrow{'c}^! \mathbb{Q}_{\ell,M}) \arrow[r, "\tau_{'C}"] \arrow[d, "'h_!(-)"] & \mathrm{H}_0^{\mathrm{BM}}(\mathrm{Fix}({'C}) \otimes_k \overline{k}) \arrow[r, equal] \arrow[d] & \bigoplus_{s \in S(k)} \mathrm{H}_0^{\mathrm{BM}}(\mathrm{Sht}_C(s) \otimes_k \overline{k}) \arrow[d, "\deg"] \\
\mathrm{Hom}(\mathrm{Fr}_S^* f_! \mathbb{Q}_{\ell,M}, f_! \mathbb{Q}_{\ell,M}) \arrow[r, "\tau_{'S}"] & \mathrm{H}_0^{\mathrm{BM}}(S(k) \otimes_k \overline{k}) \arrow[r, equal] & \bigoplus_{s \in S(k)} \mathbb{Q}_\ell
\end{tikzcd}$$

Combining the with the commutative diagram in Lemma A.11 applied to $'C$, we get a commutative diagram

$$\begin{tikzcd}
\mathrm{Ch}_n({'C})_{\mathbb{Q}} \arrow[r, "\Delta_M^!"] \arrow[d, "'h_! \circ \mathrm{cl}_{'C}"] & \mathrm{Ch}_0(\mathrm{Fix}({'C}))_{\mathbb{Q}} \arrow[r, equal] \arrow[d] & \bigoplus_{s \in S(k)} \mathrm{Ch}_0(\mathrm{Sht}_C(s)) \arrow[d, "\deg"] \\
\mathrm{Hom}(\mathrm{Fr}_S^* f_! \mathbb{Q}_{\ell,M}, f_! \mathbb{Q}_{\ell,M}) \arrow[r, "\tau_{'S}"] & \mathrm{H}_0^{\mathrm{BM}}(S(k) \otimes_k \overline{k}) \arrow[r, equal] & \bigoplus_{s \in S(k)} \mathbb{Q}_\ell
\end{tikzcd} \tag{A.32}$$

Applying (A.32) to $'\zeta$, and using (A.31), we get that for all $s \in S(k)$

$$\tau_{'S}\big({'h_!} \mathrm{cl}_{'C}({'\zeta})\big)_s = \langle {'\zeta}, \Delta_*[M] \rangle_s = \langle \zeta, \Gamma(\mathrm{Fr}_M) \rangle_s. \tag{A.33}$$

Here $\tau_{'S}(-)_s \in \mathbb{Q}_\ell$ denotes the s-component of the class $\tau_{'S}(-) \in \mathrm{H}_0^{\mathrm{BM}}(S(k) \otimes_k \overline{k}) = \oplus_{s \in S(k)} \mathbb{Q}_\ell$.

Next we would like to express $\tau_{'S}\big({'h_!} \mathrm{cl}_{'C}({'\zeta})\big)_s$ as a trace. The argument works more generally when $\mathbb{Q}_{\ell,M}$ is replaced with any $\mathcal{F} \in D_c^b(M)$ and $\mathrm{cl}_C(\zeta)$ replaced with any cohomological self-correspondence $\eta : \overleftarrow{c}^* \mathcal{F} \to \overrightarrow{c}^! \mathcal{F}$ supported on C. So we will work in this generality. For any $\mathcal{F} \in D_c^b(M)$ we have a canonical isomorphism $\Phi_{\mathcal{F}} : \mathrm{Fr}_M^* \mathcal{F} \xrightarrow{\sim} \mathcal{F}$ whose restriction to the geometric stalk at $x \in M(k)$ is given by the *geometric* Frobenius Frob_x acting on $\mathcal{F}_{\overline{x}}$. Similar remark applies to complexes on S. Using η we define a cohomological self-correspondence $'\eta$ of $\mathcal{F}$ supported on $'C$ as the composition

$$'\eta : \overleftarrow{'c}^* \mathcal{F} = \overleftarrow{c}^* \mathrm{Fr}_M^* \mathcal{F} \xrightarrow{\overleftarrow{c}^* \Phi_{\mathcal{F}}} \overleftarrow{c}^* \mathcal{F} \xrightarrow{\eta} \overrightarrow{c}^! \mathcal{F} = \overrightarrow{'c}^! \mathcal{F}.$$

On the other hand we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Fr}_S^* f_!\mathcal{F} \arrow[r, "adj."', "\sim"] \arrow[dr, "\Phi_{f_!\mathcal{F}}"', "\sim"] & f_! \mathrm{Fr}_M^* \mathcal{F} \arrow[r, "adj."] \arrow[d, "f_!\Phi_{\mathcal{F}}", "\wr"'] & h_! \overleftarrow{c}^* \mathrm{Fr}_M^* \mathcal{F} \arrow[r, equal] \arrow[d, "h_!\overleftarrow{c}^*\Phi_{\mathcal{F}}", "\wr"'] & h_! \overleftarrow{c}^* \mathcal{F} \arrow[r] \arrow[d, "h_!('\eta)"] & f_!\mathcal{F} \arrow[d, equal] \\
 & f_!\mathcal{F} \arrow[r, "adj."] & h_! \overleftarrow{c}^* \mathcal{F} \arrow[r, "h_!(\eta)"] & h_! \overrightarrow{c}^! \mathcal{F} \arrow[r, "adj."] & f_!\mathcal{F}
\end{tikzcd} \tag{A.34}$$

Here the arrows indexed by "adj." are induced from adjunctions, using the properness of $\overleftarrow{c}$. The middle square is commutative by the definition of $'\eta$, and the right square is commutative by design. The composition of the top row in (A.34) is by definition the push-forward $'h_!'\eta$ as a cohomological self-correspondence of $f_!\mathcal{F}$ supported on $'S$; the composition of the bottom row in (A.34) is by definition the push-forward $h_!\eta$ as a cohomological self-correspondence of $f_!\mathcal{F}$ supported on the diagonal S. Therefore, (A.34) shows that $'h_!'\eta$ may be written as the composition

$$'h_!'\eta : \mathrm{Fr}_S^* f_!\mathcal{F} \xrightarrow{\Phi_{f_!\mathcal{F}}} f_!\mathcal{F} \xrightarrow{h_!\eta} f_!\mathcal{F}. \tag{A.35}$$

For any cohomological self-correspondence ξ of $\mathcal{G} \in D_c^b(S)$ supported on the graph of Frobenius $'S$, i.e., $\xi : \mathrm{Fr}_S^* \mathcal{G} \to \mathcal{G}$, the trace $\mathcal{T}_{'S}(\xi)_s$ at $s \in S(k)$ is simply given by the trace of ξ_s acting on the geometric stalk $\mathcal{G}_{\overline{s}}$: this is because the Frobenius map is contracting at its fixed points, so the local term for the correspondence supported on its graph is the naive local term (a very special case of the main result in [22, Theorem 2.1.3]). Applying this observation to $\xi = 'h_!'\eta$ we get

$$\begin{aligned}
\mathcal{T}_{'S}('h_!'\eta)_s &= \mathrm{Tr}\left(('h_!'\eta)_s, (f_!\mathcal{F})_{\overline{s}}\right) \\
&= \mathrm{Tr}\left((h_!\eta)_s \circ \mathrm{Frob}_s, (f_!\mathcal{F})_{\overline{s}}\right) \quad \text{by (A.35).}
\end{aligned} \tag{A.36}$$

Now apply (A.36) to $\mathcal{F} = \mathbb{Q}_{\ell,M}$, $\eta = \mathrm{cl}_C(\zeta)$ and note that $'\eta = \mathrm{cl}_{'C}('\zeta)$. Then (A.36) gives

$$\mathcal{T}_{'S}('h_!\mathrm{cl}_{'C}('\zeta))_s = \mathrm{Tr}\left((f_!\mathrm{cl}_C(\zeta))_s \circ \mathrm{Frob}_s, (f_!\mathbb{Q}_{\ell,M})_{\overline{s}}\right). \tag{A.37}$$

Combining (A.37) with (A.33) we get the desired formula (A.28). □

Appendix B. Super-positivity of L-values

In this appendix we show the positivity of all derivatives of certain L-functions (suitably corrected by their epsilon factors), assuming the Riemann hypothesis. The result is unconditional in the function field case since the Riemann hypothesis is known to hold.

It is well-known that the positivity of the leading coefficient of such L-function is implied by the Riemann hypothesis. We have not seen the positivity of non-leading terms in the literature and we provisionally call such phenomenon "super-positivity".

B.1. **The product expansion of an entire function.** We recall the (canonical) product expansion of an entire function following [1, §5.2.3, §5.3.2]. Let $\phi(s)$ be an entire function in the variable $s \in \mathbb{C}$. Let m be the vanishing order of ϕ at $s = 0$. List all the nonzero roots of ϕ as $\alpha_1, \alpha_2, ..., \alpha_i, ...$ (multiple roots being repeated) indexed by a subset I of $\mathbb{Z}_{>0}$, such that $|\alpha_1| \leq |\alpha_2| \leq ....$ Let E_n be the elementary Weierstrass function

$$E_n(u) = \begin{cases} (1-u), & n = 0; \\ (1-u)\, e^{u + \frac{1}{2}u^2 + \cdots + \frac{1}{n}u^n}, & n \geq 1. \end{cases}$$

An entire function ϕ is said to have finite genus if it can be written as an absolutely convergent product

$$\phi(s) = s^m\, e^{h(s)} \prod_{i \in \mathbb{Z}} E_n\left(\frac{s}{\alpha_i}\right) \tag{B.1}$$

for a polynomial $h(s) \in \mathbb{C}[s]$ and an integer $n \geq 0$. The product (B.1) is unique if we further demand that n is the smallest possible integer, which is characterized as the smallest $n \in \mathbb{Z}_{\geq 0}$

such that

$$\sum_{i\in I}\frac{1}{\left|\alpha_i\right|^{n+1}}<\infty. \tag{B.2}$$

The genus $g(\phi)$ of such ϕ is then defined to be

$$g(\phi):=\max\{\deg(h),\, n\}.$$

The order $\rho(\phi)$ of an entire function ϕ is defined as the smallest real number $\rho\in[0,\infty]$ with the following property: for every $\epsilon>0$, there is a constant C_ϵ such that

$$\left|\phi(s)\right|\le e^{|s|^{\rho+\epsilon}},\quad \text{when } |s|\ge C_\epsilon.$$

If ϕ is a non-constant entire function, an equivalent definition is

$$\rho(\phi)=\limsup_{r\longrightarrow\infty}\frac{\log\log\|\phi\|_{\infty,B_r}}{r}$$

where $\|\phi\|_{\infty,B_r}$ is the supremum norm of the function ϕ on the disc B_r of radius r. If the order of ϕ is finite, then Hadamard theorem [1, §5.3.2] asserts that the function ϕ has finite genus and

$$g(\phi)\le\rho(\phi)\le g(\phi)+1. \tag{B.3}$$

In particular, an entire function of finite order admits a product expansion of the form (B.1).

Proposition B.1. *Let $\phi(s)$ be an entire function with the following properties*

(1) It has a functional equation $\phi(-s)=\pm\phi(s)$.

(2) For $s\in\mathbb{R}$ such that $s\gg 0$, we have $\phi(s)\in\mathbb{R}_{>0}$.

(3) The order $\rho(\phi)$ of $\phi(s)$ is at most 1*.*

(4) (RH) *The only zeros of $\phi(s)$ lie on the imaginary axis* $\mathrm{Re}(s)=0$*.*

Then we have for all $r\ge 0$,

$$\phi^{(r)}(0):=\frac{d}{ds}\Big|_{s=0}\phi(s)\ge 0.$$

Moreover, if $\phi(s)$ is not a constant function, we have

$$\phi^{(r_0)}(0)\neq 0\Longrightarrow\phi^{(r_0+2i)}(0)\neq 0,\quad \textit{for all } r_0 \textit{ and } i\in\mathbb{Z}_{\ge 0}.$$

Proof. By the functional equation, if α is a root of ϕ, so is $-\alpha$ with the same multiplicity. Therefore we may list all nonzero roots as $\{\alpha_i\}_{i\in\mathbb{Z}\setminus\{0\}}$ such that

$$\alpha_{-i}=-\alpha_i,\quad\text{and}\quad |\alpha_1|\le|\alpha_2|\le\ldots.$$

If ϕ has only finitely many roots the sequence terminates at a finite number.

Since the order $\rho(\phi)\le 1$, by (B.3) we have $g(\phi)\le 1$. Hence we may write ϕ as a product

$$\phi(s)=s^m\,e^{h(s)}\prod_{i=1}^{\infty}E_1\left(\frac{s}{\alpha_i}\right)E_1\left(-\frac{s}{\alpha_i}\right),$$

where m is the vanishing order at $s=0$. Note that it is possible that $g(\phi)=0$, in which case one still has a product expansion using E_1 by the convergence of (B.2).

By the functional equation, we conclude that $h(s)=h$ is a constant.

By the condition (4)(RH), all roots α_i are purely imaginary, and hence $\overline{\alpha}_i=\alpha_{-i}$. We have

$$\begin{aligned}\phi(s)&=s^m\,e^h\prod_{i=1}^{\infty}E_1\left(\frac{s}{\alpha_i}\right)E_1\left(\frac{s}{\overline{\alpha}_i}\right)\\&=s^m\,e^h\prod_{i=1}^{\infty}\left(1+\frac{s^2}{\alpha_i\overline{\alpha}_i}\right).\end{aligned}$$

By the condition (2), the leading coefficient e^h is a positive real number. Then the desired assertion follows from the product above. $\square$

B.2. **Super-positivity.** Let F be a global field (i.e., a number field, or the function field of a connected smooth projective curve over a finite field $\mathbb{F}_q$). Let $\mathbb{A}$ be the ring of adèles of F. Let π be an irreducible cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A})$. Let $L(\pi, s)$ be the complete (standard) L-function associated to π [9]. We have a functional equation

$$L(\pi, s) = \epsilon(\pi, s)L(\widetilde{\pi}, 1-s),$$

where $\widetilde{\pi}$ denotes the contragredient of π, and

$$\epsilon(\pi, s) = \epsilon(\pi, 1/2)N_\pi^{s-1/2}$$

for some positive real number N_π. Define

$$\Lambda(\pi, s) = N_\pi^{-\frac{(s-1/2)}{2}} L(\pi, s),$$

and

$$\Lambda^{(r)}(\pi, 1/2) := \frac{d}{ds}\Big|_{s=1/2} \Lambda(\pi, s).$$

Theorem B.2. *Let π be a nontrivial cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A})$. Assume that it is self-dual:*

$$\pi \simeq \widetilde{\pi}.$$

Assume that, if F is a number field, the Riemann hypothesis holds for $L(\pi, s)$, that is, all the roots of $L(\pi, s)$ have real parts equal to $1/2$.

(1) For all $r \in \mathbb{Z}_{\geq 0}$, we have

$$\Lambda^{(r)}(\pi, 1/2) \geq 0.$$

(2) If $\Lambda(\pi, s)$ is not a constant function, we have

$$\Lambda^{(r_0)}(\pi, 1/2) \neq 0 \Longrightarrow \Lambda^{(r_0+2i)}(\pi, 1/2) \neq 0, \quad \textit{for all } i \in \mathbb{Z}_{\geq 0}.$$

Proof. We consider

$$\lambda(\pi, s) := \Lambda(\pi, s + 1/2).$$

Since π is cuspidal and nontrivial, its standard L-function $L(\pi, s)$ is entire in $s \in \mathbb{C}$. By the equality $\epsilon(\pi, s)\epsilon(\widetilde{\pi}, 1-s) = 1$ and the self-duality $\pi \simeq \widetilde{\pi}$ we deduce

$$1 = \epsilon(\pi, 1/2)\epsilon(\pi, 1-1/2) = \epsilon(\pi, 1/2)^2.$$

Hence $\epsilon(\pi, 1/2) = \pm 1$, and we have a functional equation

$$\lambda(\pi, s) = \pm\lambda(\pi, -s). \tag{B.4}$$

We apply Proposition B.1 to the entire function $\lambda(\pi, s)$. The function $L(\pi, s)$ is entire of order one, and so is $\lambda(\pi, s)$. In the function field case, the condition (4)(RH) is known by the theorem of Deligne on Weil conjecture, and of Drinfeld and L. Lafforgue on the global Langlands correspondence. It remains to verify the condition (2) for $\lambda(\pi, s)$. This follows from the following lemma. □

The local L-factor $L(\pi_v, s)$ is of the form $\frac{1}{P_{\pi_v}(q_v^{-s})}$ where P_{π_v} is a polynomial with constant term equal to one when v is non-archimedean, and a product of functions of the form $\Gamma_\mathbb{C}(s+\alpha)$, or $\Gamma_\mathbb{R}(s+\alpha)$, where $\alpha \in \mathbb{C}$, and

$$\Gamma_\mathbb{C}(s) = 2(2\pi)^{-s}\Gamma(s), \quad \Gamma_\mathbb{R}(s) = \pi^{-s/2}\Gamma(s/2),$$

when v is archimedean. We say that $L(\pi_v, s)$ has real coefficients if the polynomial P_{π_v} has real coefficients when v is non-archimedean, and the factor $\Gamma_{F_v}(s+\alpha)$ in $L(\pi_v, s)$ has real α or the pair $\Gamma_{F_v}(s+\alpha)$ and $\Gamma_{F_v}(s+\overline{\alpha})$ show up simultaneously when v is archimedean. In particular, if $L(\pi_v, s)$ has real coefficients, it takes positive real values when s is real and sufficiently large.

Lemma B.3. *Let π_v be unitary and self-dual. Then $L(\pi_v, s)$ has real coefficients.*

Proof. We suppress the index v in the notation and write F for a local field. Let π be irreducible admissible representation of $\mathrm{GL}_n(F)$. It suffices to show that, if π is unitary, then we have

$$\overline{L(\pi,\overline{s})}=L(\widetilde{\pi},s). \tag{B.5}$$

Let $\mathscr{Z}_\pi$ be the space of local zeta integrals, i.e., the meromorphic continuation of

$$Z(\Phi,s,f)=\int_{\mathrm{GL}_n(F)} f(g)\Phi(g)|g|^{s+\frac{n-1}{2}}\,dg$$

where f runs over all matrix coefficients of π, and Φ runs over all Bruhat–Schwartz functions on $\mathrm{Mat}_n(F)$ (a certain subspace, stable under complex conjugation, if F is archimedean, cf. [9, §8]). We recall that from [9, Theorem 3.3, 8.7] that the Euler factor $L(\pi,s)$ is uniquely determined by the space $\mathscr{Z}_\pi$ (for instance, it is a certain normalized generator of the $\mathbb{C}[q^s,q^{-s}]$-module $\mathscr{Z}_\pi$ if F is non-archimedean).

Let $\mathscr{C}_\pi$ be the space of matrix coefficients of π, i.e., the space consisting of all linear combinations of functions on $\mathrm{GL}_n(F)$: $g\mapsto(\pi(g)u,v)$ where $u\in\pi, v\in\widetilde{\pi}$ and $(\cdot,\cdot):\pi\times\widetilde{\pi}\to\mathbb{C}$ is the canonical bilinear pairing. We remark that the involution $g\mapsto g^{-1}$ induces an isomorphism between $\mathscr{C}_\pi$ with $\mathscr{C}_{\widetilde{\pi}}$.

To show (B.5), it now suffices to show that, if π is unitary, the complex conjugation induces an isomorphism between $\mathscr{C}_\pi$ and $\mathscr{C}_{\widetilde{\pi}}$. Let $\langle\cdot,\cdot\rangle:\pi\times\pi\to\mathbb{C}$ be a non-degenerate *Hermitian* pairing invariant under $\mathrm{GL}_n(F)$. Then the space $\mathscr{C}_\pi$ consists of all functions $f_{u,v}: g\mapsto\langle\pi(g)u,v\rangle, u,v\in\pi$. Under complex conjugation we have $\overline{f_{u,v}}(g)=\overline{\langle\pi(g)u,v\rangle}=\langle v,\pi(g)u\rangle=\langle\pi(g^{-1})v,u\rangle=f_{v,u}(g^{-1})$. This function belongs to $\mathscr{C}_{\widetilde{\pi}}$ by the remark at the end of the previous paragraph. This clearly shows that the complex conjugation induces the desired isomorphism. ☐

Remark B.4. In the case of a function field, we have a simpler proof of Theorem B.2. The function $L(\pi,s)$ is a polynomial in q^{-s} of degree denoted by d. Then the function $\lambda(\pi,s)$ is of the form

$$\lambda(\pi,s)=q^{ds/2}\prod_{i=1}^{d}\big(1-\alpha_i q^{-s}\big), \tag{B.6}$$

where all the roots α_i satisfy $|\alpha_i|=1$. By the functional equation (B.4), if α is a root in (B.6), so is $\alpha^{-1}=\overline{\alpha}$. We divide all roots not equal to ±1 into pairs $\alpha_1^{\pm1},\alpha_2^{\pm1},...,\alpha_m^{\pm1}$ (some of them may repeat). Consider

$$\begin{aligned}A_i(s)&=q^s\big(1-\alpha_i q^{-s}\big)\big(1-\alpha_i^{-1}q^{-s}\big)\\&=q^s+q^{-s}-\alpha_i-\overline{\alpha}_i\\&=\big(2-\alpha_i-\overline{\alpha}_i\big)+2\sum_{j\geq1}\frac{(s\log q)^{2j}}{j!}.\end{aligned}$$

From $|\alpha_i|=1$ and $\alpha_i\neq1$ it follows that $A_i(s)$ has strictly positive coefficients at all even degrees. Now let a (resp., b) be the multiplicity of the root 1 (resp., -1). We then have

$$\lambda(\pi,s)=\big(q^{s/2}-q^{-s/2}\big)^a\,\big(q^{s/2}+q^{-s/2}\big)^b\prod_{i=1}^{m}A_i(s),\quad 2m+a+b=d.$$

The desired assertions follow immediately from this product expansion.

Remark B.5. In the statement of the theorem, we excludes the trivial representation. In this case the complete L-function has a pole at $s=1$. If we replace $\Lambda(\pi,s)$ by $s(s-1)\Lambda(\pi,s)$, the theorem still holds by the same proof. Moreover, if $F=\mathbb{Q}$, we have the Riemann zeta function, and the super-positivity is known without assuming the Riemann hypothesis, by Pólya [4]. The super-positivity also holds when the L-function is "positive definite" as defined by Sarnak in [20]. One of such examples is the weight 12 cusp form with q-expansion $\Delta=q\prod_{n\geq1}(1-q^n)^{24}$. More recently, Goldfeld and Huang in [10] prove that there are infinitely many classical holomorphic cusp forms (Hecke eigenforms) on $\mathrm{SL}_2(\mathbb{Z})$ whose L-functions satisfy super-positivity.

Remark B.6. The positivity of the central value is known for the standard L-function attached to a symplectic cuspidal representation of $\mathrm{GL}_n(\mathbb{A})$ by [17].

Remark B.7. The positivity of the first derivative is known for the L-function appearing in the Gross–Zagier formula in [12], [25], for example the L-function of an elliptic curve over $\mathbb{Q}$.

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