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Triple Product Formula and Mass Equidistribution on Modular Curves of Level N

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Let N be a fixed integer and f be a holomorphic newform of level q, weight k and trivial nebentypus, where q is a multiple of N. In this article, we prove that the pushforward to the modular curve of level N of the mass measure of f tends weakly to the Haar measure as $qk \to \infty$. This generalizes the previous results for modular curve of level 1. The main innovation of this article is to obtain an upper bound for the local integral which cancels the convexity bound of the corresponding L-function in level aspect.

1 Introduction

Let $\Gamma_0(N)$ be the standard congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, and let $Y_0(N) = \Gamma_0(N)\backslash\mathbb{H}$ be the corresponding modular curve of level N. Let

$$d\mu(z) = \frac{\mathrm{d}x\mathrm{d}y}{y^2} \tag{1.1}$$

be the standard hyperbolic volume measure on $Y_0(N)$.

Let $f : \mathbb{H} \to \mathbb{C}$ be a holomorphic newform of weight $k \in 2\mathbb{N}$, level q and trivial nebentypus, where $N|q$. For a bounded continuous test function ϕ on $Y_0(N)$, consider the

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following pushforward of mass measure on $Y_0(N)$ associated to f:

$$\mu_f(\phi) = \int_{\Gamma_0(q)\backslash\mathbb{H}} \phi(z)|f|^2(z)y^k \frac{\mathrm{d}x\mathrm{d}y}{y^2}. \tag{1.2}$$

We will show that the measure μ_f converges weakly to $d\mu$ on $Y_0(N)$ as $qk \to \infty$. To be more precise, define

$$D_f(\phi) = \frac{\mu_f(\phi)}{\mu_f(1)} - \frac{\mu(\phi)}{\mu(1)}. \tag{1.3}$$

Theorem 1.1. Let ϕ be a fixed bounded continuous function on $Y_0(N)$ and let f traverse a sequence of holomorphic newforms of weight k and level q, where $k \in 2\mathbb{N}$ and $N|q$. Then

$$D_f(\phi) \to 0 \tag{1.4}$$

whenever $qk \to \infty$. □

Such results were first proved conditionally for the $N = q = 1, k \to \infty$ case by Sarnak in [21] and by Luo-Sarnak in [15]. Holowinsky and Soundararajan proved this case unconditionally in [7], [8], and [22]. Their work provided the basic framework for subsequent papers. Nelson in [17] and Nelson-Pitale-Saha in [18] generalized the work of Holowinsky–Soundararajan and solved the case $N = 1, qk \to \infty$. This paper is a natural successor to those papers and allows general level N.

1.1 Sketch of proof

Theorem 1.1 will follow from the spectrum decomposition result of square integrable functions on $Y_0(N)$, and the following two inequalities:

Theorem 1.2. Let ϕ be a Maass eigencuspform or an incomplete Eisenstein series. Then

$$D_f(\phi) <<_{\phi,\epsilon} log(qk)^{\epsilon} \frac{(q/\sqrt{C})^{-1+2\alpha+\epsilon}}{log(kC)^{\delta} L(f, Ad, 1)}. \tag{1.5}$$

Here $\alpha \in [0, 7/64]$ is a bound towards the Ramanujan conjecture for ϕ at primes dividing q, and $\alpha = 0$ if ϕ is an incomplete Eisenstein series. C is the finite conductor of $\pi \times \pi$. $\delta = 1/2$ or 1 according as ϕ is cuspidal or incomplete Eisenstein series. □

Theorem 1.3. Let ϕ be a Maass eigencuspform or an incomplete Eisenstein series. Then

$$D_f(\phi) <<_{\phi,\epsilon} log(qk)^{\epsilon} q_{\diamond}^{\epsilon} log(qk)^{1/12} L(f, Ad, 1)^{1/4}. \tag{1.6}$$

Here $q_{\diamond}$ is the largest integer such that $q_{\diamond}^2 | q$. □

Separately none of these two inequalities suffices to prove the main theorem due to the subtle behavior of the adjoint L-function $L(f, Ad, 1)$. But together they guarantee that $D_f(\phi) \to 0$.

The structure of this article is organized to prove Theorems 1.2 and 1.3 separately. Section 2 will be about notations and preliminary results. We will prove Theorem 1.2 for the Maass eigencuspforms in Section 3, and for the incomplete Eisenstein series in Section 4. Section 5 will be devoted to prove Theorem 1.3.

The idea to prove Theorem 1.2 is to adelize the integal $\mu_f(\phi)$ and reduce the problem to Triple product integral in the case of Maass eigencuspforms, or Rankin–Selberg integral in the case of incomplete Eisenstein series. Then Theorem 1.2 would follow from Soundararajan's weak subconvexity bound for the corresponding L-functions, and a reasonable bound for the local integrals.

The main innovation of this article is to control the local integrals for general high ramifications. We will actually consider more general situation than we need, that is, as long as two local representations have same levels, larger than the last one. This is mainly done in Section 3 and we will also introduce more details in the next subsection. The upshot here is that we can give an upper bound for these local factors which will cancel the convexity bound in general, thus proving Theorem 1.2.

On the other hand, it is relatively easier to generalize the proof in [18] for Theorem 1.3 to our case. The main difference is that there are now several cusps for $\Gamma_0(N)$, and one need to bound the Fourier coefficients of ϕ along each cusp. This is already done for Maass eigencuspforms by Iwaniec in [13]. We will deal with the case of Eisenstein series in Section 5.1. We believe such control is probably well-known or expected by experts. But as we didn't find a proper reference, we will give detailed proof in this paper.

1.2 The size of the local integral

In the case of Triple product integral, the corresponding local integral is given by

$$I_v = \int\limits_{\mathbb{Q}_p^*\backslash \mathrm{GL}_2(\mathbb{Q}_p)} \prod_{i=1}^{3} < \pi_i(g)f_i, f_i > dg. \tag{1.7}$$

Here f_i are the local new forms from local unitary representations π_i. $< \pi_i(g)f_i, f_i >$ is the corresponding matrix coefficient.

Woodbury in [23] and Nelson in [17] computed this local integral for representations with squarefree levels. In [18], Nelson, Pitale and Saha computed I_v for higher ramifications, with the assumption that $\pi_1 = \pi_2$ and π_3 is unramified. Their work is based on Lemma (3.4.2) of [16], which relates I_v to the local Rankin–Selberg integral. But this method cannot be generalized to the case when all the representations are supercuspidal, which is necessary for our purpose.

In general, the size of the local integral should reflect the convexity bound for the related $L-$function. In particular at a non-archimedean place, we expect

Conjecture 1.4. In depth aspect, the size of the local integral for representations of GL_2, whenever nonzero, should roughly be the inverse of the convexity bound for the corresponding special value of L-function, with proper normalized test vectors chosen from newforms or simple variants of newforms. □

Here by simple variants of newforms, we mean the diagonal translates of newforms, or the vectors associated to the newforms of twisted representations.

The significance of this expectation is that it relates, in depth aspect, the equidistribution type of results directly with the subconvexity type of results. In [9], we proved a power saving in the global integral, and used the size of the local integrals to get a subconvexity bound for the triple product L-function. In this article, we will prove in a different scenario an upper bound for the local integral which implies a saving in the global integral.

We already have various partial evidences for this expectation. In [9], we computed I_v in a more direct way, whenever one of the representations has higher level than the other two. We showed that the size of the local integral in that case is indeed the inverse of the convexity bound for the triple product $L-$function, while we picked the test vectors to be either newforms or diagonal translates of newforms. In [11], we computed Waldspurger's local period integral for a scenario with joint ramifications, using carefully chosen test vectors from the pool mentioned above, and verified the expectation in depth aspect. The $L-$function involved in that paper is the twisted base change $L-$function. There are also various evidences for the Rankin–Selberg integral with general levels.

In this article, we are mainly interested in the case when π_1 and π_2 have equally high level, π_3 is ramified of smaller level and all test vectors are chosen to be newforms. In this case I_v is too complicated to compute directly, but we will prove an upper bound of size as expected from Conjecture 1.4 when π_1 and π_2 are in general positions.

When we pick $\pi_1 = \pi_2$, the actual convexity bound becomes much smaller than the general case for powerful level. So it may seem that we are getting an upper bound smaller than suggested by Conjecture 1.4. I would remark here that the newforms are not the best choice when π_1 and π_2 are related. Propositions 3.6 and 3.9 in this article hint that the size of the local integral could be larger if we pick a proper old form from π_3. Similar phenomenon in the Rankin-Selberg integral is very important in the ongoing work on the subconvexity bound of the Rankin-Selberg $L-$function for related modular forms.

The basic tool in this article is the description of the Whittaker functional and the matrix coefficient for highly ramified representations as developed in [9], [10] and further refined in Section 2.7 of this article. The upshot for the local calculations in [9] and this article is that, while the exact values of the matrix coefficients for such representations are very complicated involving local epsilon factors, knowing their support and frequencies (or levels) are sometimes already enough to evaluate the local integral, as many pieces of the integral will have no contribution because of different supports or frequencies. We will also use this tool to give an upper bound for the local Rankin–Selberg integral in Proposition 4.3.

2 Notations and preliminary results

2.1 Basic Definitions

Let $\mathbb{H}$ be the upper half plane with the standard hyperbolic volume measure $d\mu = \frac{\mathrm{d}x\mathrm{d}y}{y^2}$. Let $\Delta = y^{-2}(\partial_x^2 + \partial_y^2)$ be the hyperbolic Laplacian on $\mathbb{H}$. Let GL_2^+ be the subgroup of GL_2 with positive determinants. Then GL_2^+ acts on $\mathbb{H}$ by the fractional linear transformations. Let

$$\Gamma_0(N) = \left\{\gamma \in \mathrm{SL}_2(\mathbb{Z}) | \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod N\right\}, \tag{2.1}$$

$$\Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} | n \in \mathbb{Z} \right\}. \tag{2.2}$$

Let $Y_0(N) = \Gamma_0(N)\backslash\mathbb{H}$ be the modular curve of level N.

Given a function $f : \mathbb{H} \to \mathbb{C}$ and $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+$, we denote $f|_k\alpha$ to be the function

$$z \mapsto \det(\alpha)^{k/2}(cz+d)^{-k}f(\alpha z). \tag{2.3}$$

A holomorphic cusp form of weight k and level q with trivial nebentypus is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ that satisfies $f|_k\alpha = f$ for all $\alpha \in \Gamma_0(q)$ and vanishes at all cusps of $Y_0(q)$. A holomorphic newform is a cusp form that is an eigenform of the algebra of Hecke operators and orthogonal to the oldforms. (See [3].)

A Maass cusp form ϕ of level N (and weight 0) is a $\Gamma_0(N)-$invariant eigenfunction of the hyperbolic Laplacian Δ on $\mathbb{H}$ that decays rapidly at the cusps of $\Gamma_0(N)$.

$$(\Delta + 1/4 + r^2)\phi = 0, \quad r \in \mathbb{R} \cup i(-1/2, 1/2). \tag{2.4}$$

A Maass eigencuspform is a Maass cusp form which is an eigenfunction of the Hecke operators at all finite places and also the involution $T_{-1} : \phi \mapsto [z \mapsto \phi(-\bar{z})]$.

As we will care about asymptotic behaviors, we use the notation

$$f(x,y) << _y g(x,y) \tag{2.5}$$

to indicate that there exists a positive real function $C(y)$ independent of x such that

$$|f(x,y)| \le C(y)|g(x,y)|. \tag{2.6}$$

Further if

$$f(x,y) <<_y g(x,y) <<_y f(x,y), \tag{2.7}$$

we will say $f(x,y) \asymp_y g(x,y)$.

We shall also work adelically. In general let $\mathbb{F}$ be a number field. Let v be a finite place of it and ϖ_v be a local uniformizer at v. Let O_v be the ring of integers of the local field $\mathbb{F}_v$. For an integer c, let $K_0(\varpi_v^c) \subset \mathrm{GL}_2(O_v)$ be the set of matrices which are congruent to $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ $\mathrm{mod}(\varpi_v^c)$. Similarly let $K_1(\varpi_v^c)$ denote those congruent to $\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$ $\mathrm{mod}(\varpi_v^c)$. In this article, we are mostly interested in the case when $\mathbb{F} = \mathbb{Q}$. In that case, we will let p denote both the place and the local uniformizer. But the arguments in Section 3 and 4 apply to a general number field directly.

We will say a local representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ is of level c, or $c(\pi_p) = c$, if there is a unique up to constant element which is invariant under $K_1(p^c)$. Note that for representations of trivial central character this is equivalent to the invariance by $K_0(p^c)$. For an automorphic representation π of GL_2 with trivial central character, its finite conductor is $C(\pi) = N = \prod_p p^{e_p}$, where $e_p = c(\pi_p)$.

For simplicity, we shall use the Haar measure on GL_2 over a finite place v such that $\mathrm{GL}_2(O_v)$ has volume 1. At infinite, we shall normalize the Haar measure such that the product measure is the Tamagawa measure.

2.2 Cusps and Fourier expansions

In general for a congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$, denote $\mathcal{C}(\Gamma) = \Gamma\backslash\mathbb{Q} \cup \{\infty\}$ to be the set of cusps of $\Gamma\backslash\mathbb{H}$. Equivalently,

$$\mathcal{C}(\Gamma) = \Gamma\backslash \mathrm{SL}_2(\mathbb{Z})/\Gamma_\infty. \tag{2.8}$$

This is a finite set. Fix $\mathfrak{a} \in \mathcal{C}(\Gamma)$, let $\tau_\mathfrak{a} \in \mathrm{SL}_2(\mathbb{Z})$ be such that

$$\tau_\mathfrak{a}\infty = \mathfrak{a}. \tag{2.9}$$

Let $\Gamma_\mathfrak{a}$ be the stabilizer of $\mathfrak{a}$ in Γ. The width of the cusp $\mathfrak{a}$ is defined to be

$$d_\mathfrak{a} = [\Gamma_\infty : \tau_\mathfrak{a}^{-1}\Gamma_\mathfrak{a}\tau_\mathfrak{a}]. \tag{2.10}$$

Define

$$\sigma_\mathfrak{a} = \tau_\mathfrak{a}\begin{pmatrix} d_\mathfrak{a} & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.11}$$

It satisfies the property that

$$\sigma_\mathfrak{a}^{-1}\Gamma_\mathfrak{a}\sigma_\mathfrak{a} = \Gamma_\infty. \tag{2.12}$$

We now specify Γ to be $\Gamma_0(N)$. As in Section 3.4.1 of [18], one can consider the transitive right action of $SL_2(\mathbb{Z})$ on $\mathbb{P}^1(\mathbb{Z}/N)$:

$$[x:y]\cdot\begin{pmatrix} a & b \\ c & d \end{pmatrix} = [ax+cy : bx+dy]. \tag{2.13}$$

Note that $\Gamma_0(N)$ is the stablizer of $[0:1]$ in $SL_2(\mathbb{Z})$. As a result, the set of cusps $\mathcal{C}(\Gamma_0(N))$ can be parameterized by the set of ordered pairs

$$\{[c:d]: \ \ c|N, d\in(\mathbb{Z}/(c,N/c))^*\}. \tag{2.14}$$

If a cusp $\mathfrak{a}$ corresponds to a pair $[c:d]$ for $c|N$ and $d\in(\mathbb{Z}/(c,N/c))^*$, we will call $c_{\mathfrak{a}}=c$ the denominator of the cusp $\mathfrak{a}$. For a fixed $c|N$, let $\mathcal{C}[c]$ denote the set of cusps whose denominator is c. Then by the above parameterization,

$$\sharp\mathcal{C}[c]=\varphi((c,N/c)), \tag{2.15}$$

where φ is the Euler totient function.

The width of a cusp can also be given in terms of $c_{\mathfrak{a}}$:

$$d_{\mathfrak{a}}=\frac{[N,c_{\mathfrak{a}}^2]}{c_{\mathfrak{a}}^2}. \tag{2.16}$$

Here $[x,y]$ means the least common multiple of x and y.

Now let f be a holomorphic new form of weight k and level q. Then for any cusp $\mathfrak{a}$, f has Fourier expansion along $\mathfrak{a}$ in the following form:

$$f|_k\sigma_{\mathfrak{a}}(z)=y^{-k/2}\sum_{n\in\mathbb{N}}\frac{\lambda_{f,\mathfrak{a}}(n)}{\sqrt{n}}\kappa_f(ny)\mathrm{e}^{2\pi inx}, \tag{2.17}$$

where $\kappa_f(y)=y^{k/2}e^{-2\pi y}$ for y positive real and $\lambda_{f,\mathfrak{a}}\in\mathbb{C}$. From the Fourier expansion and Deligne's bound on the coefficients, one has the following control for y large enough:

$$|f(z)| << \mathrm{e}^{-2\pi y}. \tag{2.18}$$

For a given $c|q$, define

$$\lambda_{f,[c]}(n)=\left(\frac{1}{\varphi((c,q/c))}\sum_{\mathfrak{a}\in\mathcal{C}[c]}|\lambda_{f,\mathfrak{a}}(n)|^2\right)^{1/2}. \tag{2.19}$$

This average of Fourier coefficients is so-called factorizable, and [18] gives a bound of convolution sums for $\lambda_{f,[c]}(n)$.

For a Maass eigencuspform ϕ of level N and a fixed cusp $\mathfrak{a}$, there is a similar Fourier expansion

$$\phi(\sigma_{\mathfrak{a}} z) = \sum_{n\neq 0} \frac{\lambda_{\phi,\mathfrak{a}}(n)}{\sqrt{|n|}} \kappa_{ir}(ny) e^{2\pi i n x}, \tag{2.20}$$

where $\kappa_{ir}(y) = 2|y|^{1/2} K_{ir}(2\pi|y|)$ with K_{ir} being the standard K-Bessel function and r is as in (2.4). We have $|\kappa_{ir}(y)| \leq 1$ for all $s \in \mathbb{R} \cup i(-1/2, 1/2)$ and all $y \in \mathbb{R}^+$.

As a corollary of Theorem 3.2 of [13], we have the following result

Corollary 2.1. For a Maass eigencuspform ϕ of level N and any cusp $\mathfrak{a}$ as above, we have

$$\sum_{|n|\leq M} \lambda_{\phi,\mathfrak{a}}(n) << _{\phi} M. \tag{2.21}$$

Using Cauchy–Schwartz inequality and that $\mathcal{C}(\Gamma)$ is finite,

$$\sum_{|n|\leq M} \sum_{\mathfrak{a}\in\mathcal{C}(\Gamma)} \frac{|\lambda_{\phi,\mathfrak{a}}(n)|}{\sqrt{|n|}} << _{\phi,\epsilon} M^{1/2+\epsilon}. \tag{2.22}$$

□

2.3 Eisenstein series and the spectral theory of modular curve of level N

For a compactly supported test function h on $\mathbb{R}^+$ and a cusp $\mathfrak{a}$, the associated incomplete Eisenstein series for Γ is defined to be

$$E_{\mathfrak{a}}(z, h) = \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} h(\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)). \tag{2.23}$$

For Re(s) large enough, the Eisenstein series for Γ along cusp $\mathfrak{a}$ is defined to be

$$E_{\mathfrak{a}}(z, s) = \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s. \tag{2.24}$$

It has meromorphic continuation to the whole complex plane.

Theorem 2.2. Let Γ be a congruence subgroup and $\mathrm{Re}(s) \geq 1/2$. Then $E_{\mathfrak{a}}(z,s)$ has a unique pole at $s=1$, and

$$\mathrm{res}_{s=1} E_{\mathfrak{a}}(z,s) = Vol(\Gamma\backslash\mathbb{H})^{-1}. \tag{2.25}$$

□

According to [13], the space of square integrable functions on $\Gamma\backslash\mathbb{H}$ is spanned by the space of Maass cuspforms and the space of incomplete Eisenstein series; The latter can be further decomposed into residuals of Eisenstein series and direct integral of Eisenstein series at $\mathrm{Re}(s) = 1/2$.

For an incomplete Eisenstein series $E_{\mathfrak{a}}(z,h)$, one can get its spectrum decomposition by Mellin inversion formula. Let $\hat{h}(s) = \int_0^\infty h(y) y^{-s-1} dy$ be the Mellin transform of h. It satisfies the growth control

$$\hat{h}(s) << _{h,A} (1+|s|)^{-A} \tag{2.26}$$

for any positive A and bounded $\mathrm{Re}(s)$. Then the Mellin inversion formula claims that

$$h(y) = \frac{1}{2\pi i} \int_{(2)} \hat{h}(s) y^s \mathrm{d}s, \tag{2.27}$$

where $\int_{(2)}$ denotes the integral taken over the vertical contour from $2-i\infty$ to $2+i\infty$. Then by summing over $\sigma_{\mathfrak{a}}^{-1}\gamma$ translates for $\gamma \in \Gamma_{\mathfrak{a}}\backslash\Gamma$, we get

$$E_{\mathfrak{a}}(z,h) = \frac{1}{2\pi i} \int_{(2)} \hat{h}(s) E_{\mathfrak{a}}(z,s) \mathrm{d}s. \tag{2.28}$$

Now move the integration to the line $\mathrm{Re}(s) = 1/2$. By Cauchy's theorem, we have

$$E_{\mathfrak{a}}(z,h) = \frac{\hat{h}(1)}{Vol(\Gamma\backslash\mathbb{H})} + \frac{1}{2\pi i} \int_{(1/2)} \hat{h}(s) E_{\mathfrak{a}}(z,s) \mathrm{d}s. \tag{2.29}$$

By a change of variable, we have in general for another cusp $\mathfrak{b}$,

$$E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z,h) = \frac{\hat{h}(1)}{Vol(\Gamma\backslash\mathbb{H})} + \frac{1}{2\pi i} \int_{(1/2)} \hat{h}(s) E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z,s) \mathrm{d}s. \tag{2.30}$$

For the standard Eisenstein series $E(z,s)$, we have its Fourier expansion

$$E(z,s)=y^s+M(s)y^{1-s}+\frac{1}{\xi(2s)}\sum_{n\neq 0}\frac{\lambda_{s-1/2}}{\sqrt{|n|}}\kappa_{s-1/2}(ny)e^{2\pi inx}, \tag{2.31}$$

where $\lambda_{s-1/2}(n)=\sum_{ab=n}(a/b)^{s-1/2}$, $\kappa_{s-1/2}(y)=2|y|^{1/2}K_{s-1/2}(2\pi|y|)$, $M(s)=\xi(2s-1)/\xi(2s)$ and $\xi(s)=\pi^{-s/2}\Gamma(s/2)\zeta(s)$ is the completed Riemann zeta function.

In general, let $\mathfrak{a}$, $\mathfrak{b}$ be two cusps for $\Gamma_0(N)$. Then by Theorem 3.4 of [13],

$$E_{\mathfrak{a}}(\sigma_{\mathfrak{b}}z,s)=\delta_{\mathfrak{ab}}y^s+\varphi_{\mathfrak{ab}}(s)y^{1-s}+\sum_{n\neq 0}\varphi_{\mathfrak{ab}}(n,s)\kappa_{s-1/2}(ny)e^{2\pi inx}, \tag{2.32}$$

where $\delta_{\mathfrak{ab}}$ is the Kronecker symbol, $\varphi_{\mathfrak{ab}}(s)$ and $\varphi_{\mathfrak{ab}}(n,s)$ are defined using generalized Kloosterman sum.

Now let $\phi(z)=E_{\mathfrak{a}}(\sigma_{\mathfrak{b}}z,h)$ be an incomplete Eisenstein series. Using its spectrum decomposition and the Fourier expansion above, we have

$$\phi(z)=\sum_{n\in\mathbb{Z}}\phi_n(y)e^{2\pi inx}, \tag{2.33}$$

where

$$\phi_n(y)=\frac{1}{2\pi i}\int_{(1/2)}\hat{h}(s)\varphi_{\mathfrak{ab}}(n,s)\kappa_{s-1/2}(ny)ds, \tag{2.34}$$

for $n\neq 0$, and

$$\phi_0(y)=\frac{\hat{h}(1)}{Vol(\Gamma\backslash\mathbb{H})}+\frac{1}{2\pi i}\int_{(1/2)}\hat{h}(s)(\delta_{\mathfrak{ab}}y^s+\varphi_{\mathfrak{ab}}(s)y^{1-s})ds. \tag{2.35}$$

2.4 Associate classical modular forms to automorphic forms

Let ϕ be a modular form of weight k and level N. We can associate to it an automorphic form $\tilde{\phi}$ as follows. By strong approximation,

$$\mathrm{GL}_2(\mathbb{A})=\mathrm{GL}_2(\mathbb{Q})\mathrm{GL}_2^+(\mathbb{R})\prod_p K_0(p^{e_p}). \tag{2.36}$$

For any element $g\in\mathrm{GL}_2(\mathbb{A})$, we can then write it as

$$g=hg_\infty k,$$

where $h \in \mathrm{GL}_2(\mathbb{Q})$, $g_\infty \in \mathrm{GL}_2^+(\mathbb{R})$ and $k \in \prod_p K_0(p^{e_p})$. Then define $\tilde{\phi} : \mathrm{GL}_2(\mathbb{A}) \to \mathbb{C}$ as

$$\tilde{\phi}(g) = \phi|_{g_\infty}(i), \tag{2.37}$$

where $\phi|_{g_\infty}$ is as in (2.3).

This automorphic form $\tilde{\phi}$ is called the adelization of ϕ. It is clearly invariant under $\prod_p K_0(p^{e_p})$, and its infinity component is weight k. We won't distinguish ϕ and $\tilde{\phi}$ later on when there is no confusion.

As an example, consider the Eisenstein series $E(z,s)$ of weight 0 and level 1. When $\mathrm{Re}(s)$ is large enough, its adelization is

$$E(g,s) = \sum_{\gamma \in B(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{Q})} \Phi_s(\gamma g), \tag{2.38}$$

where $\Phi_s \in \mathrm{Ind}_B^{\mathrm{GL}_2}(|\cdot|^{s-1/2}, |\cdot|^{-s+1/2})$ is spherical satisfying

$$\Phi_s\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g\right) = |a|^s |d|^{-s} \Phi_s(g), \quad \Phi_s(1) = 1. \tag{2.39}$$

Similarly, it has meromorphic continuation to the whole complex plane.

One can further translate the Fourier expansion of $E(z,s)$ into adelic languages. For $z = x + iy$, let

$$g_z = \left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}, 1, 1, \cdots\right) \in \mathrm{GL}_2(\mathbb{A}), \tag{2.40}$$

where the first component is the component at infinity. So $E(z,s)$ can be recovered as

$$E(z,s) = E(g_z, s). \tag{2.41}$$

The general Fourier inversion theorem implies that

$$E(g_z, s) = \sum_{a \in \mathbb{Q}} \int_{\mathbb{Q}\backslash \mathbb{A}} E\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g_z, s\right) \psi(-ax) \mathrm{d}x. \tag{2.42}$$

When $a = 0$, the corresponding integral gives the constant term for the Fourier expansion. A standard unfolding technique would give (see [1])

$$\int_{\mathbb{Q}\backslash\mathbb{A}} E\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g_z, s\right) dx = \Phi_s(g_z) + \int_{\mathbb{A}} \Phi_s\left(\omega\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g_z\right) \mathrm{d}x. \tag{2.43}$$

Note that $\Phi_s(g_z) = y^s$, and the second integral gives $M(s)y^{1-s}$.

When $a \neq 0$, the corresponding integral is related to the global Whittaker functional

$$\int_{\mathbb{Q}\backslash\mathbb{A}} E\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g, s\right) \psi(-ax)dx = W\left(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix} g\right), \tag{2.44}$$

where

$$W(g) = \int_{\mathbb{A}} \Phi_s\left(\omega\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right) \psi(-x)\, \mathrm{d}x, \tag{2.45}$$

and $\omega = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$. The integral (2.45) is directly a product of local integrals, which in turn are exactly the local Whittaker functional of induced representations. See Section 2.7 for more details on Whittaker model. We write

$$W(g) = \prod_v W_v(g), \tag{2.46}$$

and we can normalize the local Whittaker functionals so that

$$W_\infty(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix} g_z) = \frac{1}{\xi(2s)} \kappa_{s-1/2}(ay)\mathrm{e}^{2\pi iax}, \tag{2.47}$$

and

$$W_p(1) = 1 \tag{2.48}$$

for all finite prime p.

If Φ_s is spherical at all finite places, then $W_p(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix})$ is not zero only if a is integral locally. As a result, the summation in the Fourier expansion (2.42) is actually only over integers. Then comparing (2.42) with (2.31), we get

$$\lambda_{s-1/2}(n) = |n|^{1/2} \prod_{p|n} W_p \left(\begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix} \right). \tag{2.49}$$

Under this identification, the fact that

$$\lambda_{s-1/2}(n) = \sum_{ab=n} (\frac{a}{b})^{s-1/2} \tag{2.50}$$

would follow directly from the well-known formula for the Whittaker function associated to the spherical element in $\mathrm{Ind}_B^{\mathrm{GL}_2}(|\cdot|^{s-1/2}, |\cdot|^{-s+1/2})$:

$$W_p \left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) = |a|^{1/2} \frac{p^{(v(a)+1)(s-1/2)} - p^{-(v(a)+1)(s-1/2)}}{p^{(s-1/2)} - p^{-(s-1/2)}}. \tag{2.51}$$

2.5 Integrals at non-archimedean places

Let ψ be a fixed additive character of $\mathbb{A}$. Without loss of generality, we will always assume that ψ is unramified at any finite place. Let $p = |\varpi_v|_v^{-1}$. A multiplicative character χ of $O_v{}^*$ or $\mathbb{F}_v^*$ is of level c, or $c(\chi) = c$, if c is the minimal integer such that $\chi(1+\varpi_v^c O_v) = 1$.

Lemma 2.3. Let $m \in \mathbb{F}_v$ such that $v(m) = -j < 0$, and μ be a character of O_v^* with $c(\mu) = k > 0$. Then

$$\left| \int_{v(x)=0} \psi_v(mx)\mu^{-1}(x) \mathrm{d}^*x \right| = \begin{cases} \sqrt{\frac{p}{(p-1)^2 p^{k-1}}}, & \text{if } j = k; \\ 0, & \text{otherwise.} \end{cases} \tag{2.52}$$

□

This is just a variant of the classical result on Gauss sum.

Lemma 2.4. Let μ and η both be multiplicative characters of O_v^* and $j \in \mathbb{Z}$. Suppose that $c(\mu) = i > 0$.

(1) If $0 < j \leq i - 2$, then

$$\int_{v(x)=0} \mu(1 + \varpi^j x)\eta(x) \mathrm{d}^*x$$

is not zero only if $c(\eta) = i - j$. If $j = i - 1$, it is not zero only if $c(\eta) = 0$ or 1, and

$$\int_{v(x)=0} \mu(1 + \varpi^{i-1}x)\mathrm{d}^*x = -\frac{1}{p-1},$$

(2) When $i > 1$,

$$\int_{v(x)=0, x \notin -1+\varpi O_v} \mu(1 + x)\eta(x)\mathrm{d}^*x$$

is not zero only if $c(\eta) = i$. When $i = 1$, it is not zero only if $c(\eta) = 0$ or 1, and

$$\int_{v(x)=0, x \notin -1+\varpi O_v} \mu(1 + x)\mathrm{d}^*x = -\frac{1}{p-1},$$

(3) When $j < 0$,

$$\int_{v(x)=0} \mu(1 + \varpi^{j}x)\eta(x)\mathrm{d}^*x$$

is not zero only if $c(\eta) = i$. □

Proof. If $c(\eta)$ is greater than claimed, then the integral is zero by a simple change of variable. So we just need to show that when the level of η is less than claimed, the integral is also zero. For conciseness we will only prove part (1), as the other two parts are very similar.

In particular let $0 < j < i$. Suppose first that $i - j \geq 2$. We can split the domain of the integral into intervals of form $x = a + \varpi^{i-j-1}u$ where $a \in (O_v/\varpi^{i-j-1}O_v)^*$ is fixed and $u \in O_v$.Then η is constant on each such intervals by the condition on its level. On the other hand,

$$\mu(1 + \varpi^j x) = \mu(1 + \varpi^j a + \varpi^{i-1}u) = \mu(1 + \varpi^j a)\mu(1 + \frac{\varpi^{i-1}u}{1 + \varpi^j a}), \tag{2.53}$$

and by a change of variable,

$$\int_{v(x)=0} \mu(1 + \varpi^j x)\eta(x)\mathrm{d}^*x = \frac{1}{(p-1)p^{i-j-2}} \sum_{a \in (O_v/\varpi^{i-j-1}O_v)^*} \int_{u \in O_v} \mu(1 + \varpi^j a)\eta(a)\mu(1 + \varpi^{i-1}u)\mathrm{d}u. \tag{2.54}$$

The main observation here is that $\mu(1 + \varpi^{i-1}u)$ as a function of u is a nontrivial additive character. So each integral in u will give 0.

When $i-j=1$, let $c(\eta)=0<1$. Then

$$\int_{v(x)=0} \mu(1+\varpi^{i-1}x)\mathrm{d}^*x = -\frac{1}{p-1} \tag{2.55}$$

■

Now we record some basic facts about integrals on $\mathrm{GL}_2(\mathbb{F}_v)$ when v is finite.

Lemma 2.5. For every positive integer c,

$$\mathrm{GL}_2(\mathbb{F}_v) = \coprod_{0\le i\le c} B\begin{pmatrix} 1 & 0 \\ \varpi_v^i & 1 \end{pmatrix} K_1(\varpi_v^c).$$

Here B is the Borel subgroup of GL_2. □

We normalize the Haar measure on $\mathrm{GL}_2(\mathbb{F}_v)$ such that $K_v=\mathrm{GL}_2(O_v)$ has volume 1. Then we have the following easy result.

Lemma 2.6. Locally let f be a $K_1(\varpi_v^c)$−invariant function, on which the center acts trivially. Then

$$\int_{F_v^*\backslash \mathrm{GL}_2(\mathbb{F}_v)} f(g)\mathrm{d}g = \sum_{0\le i\le c} A_i \int_{\mathbb{F}_v^*\backslash B(\mathbb{F}_v)} f\left(b\begin{pmatrix} 1 & 0 \\ \varpi_v^i & 1 \end{pmatrix}\right)\mathrm{d}b. \tag{2.56}$$

Here db is the left Haar measure on $\mathbb{F}_v^*\backslash B(\mathbb{F}_v)$, and

$$A_0 = \frac{p}{p+1},\quad A_c = \frac{1}{(p+1)p^{c-1}},\quad \text{and } A_i = \frac{p-1}{(p+1)p^i} \text{ for } 0<i<c.$$ □

2.6 Triple product formula

Let π_i, $i=1,2,3$ be three unitary cuspidal automorphic representations with central characters w_{π_i}. Suppose that

$$\prod_i w_{\pi_i} = 1. \tag{2.57}$$

Denote $\Pi=\pi_1\otimes\pi_2\otimes\pi_3$. Then one can associate the triple product L-function $L(\Pi,s)$ to Π. (See [4] and [19].) It has usual properties like analytic continuation and functional equation. In particular, there exist local epsilon factors $\epsilon_v(\Pi_v, \psi_v, s)$ and global epsilon factor $\epsilon(\Pi, s) = \prod_v \epsilon(\Pi_v, \psi_v, s)$ such that

$$L(\Pi, 1-s) = \epsilon(\Pi, s) L(\check{\Pi}, s). \tag{2.58}$$

With the assumption that $\prod_i w_{\pi_i} = 1$, we have

$$\Pi \cong \check{\Pi}.$$

The special values of local epsilon factors $\epsilon_v(\Pi_v, \psi_v, 1/2)$ are actually independent of ψ_v and always take value ± 1. For simplicity, we will write

$$\epsilon_v(\Pi_v, 1/2) = \epsilon_v(\Pi_v, \psi_v, 1/2).$$

For any place v, there is a unique (up to isomorphism) division algebra $\mathbb{D}_v$. Then Prasad proved in [20] the following theorem about the dimension of the space of local trilinear forms:

Theorem 2.7.

(1) $\dim \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{F}_v)}(\Pi_v, \mathbb{C}) \leq 1$, with the equality if and only if $\epsilon_v(\Pi_v, 1/2) = 1$.
(2) $\dim \mathrm{Hom}_{\mathbb{D}_v}(\Pi_v^{\mathbb{D}_v}, \mathbb{C}) \leq 1$, with the equality if and only if $\epsilon_v(\Pi_v, 1/2) = -1$.

Here $\Pi_v^{\mathbb{D}_v}$ is the image of Π_v under Jacquet-Langlands correspondence. □

This motivated the following result which is conjectured by Jacquet and later on proved by Harris and Kudla in [5] and [6]:

Theorem 2.8.

$$\{L(\Pi, 1/2) \neq 0\} \Longleftrightarrow \left\{ \begin{array}{c} \text{there exist } \mathbb{D} \text{ and } f_i \in \pi_i^{\mathbb{D}} \text{ s.t.} \\ \int\limits_{Z_{\mathbb{A}}\mathbb{D}^*(\mathbb{F})\backslash \mathbb{D}^*(\mathbb{A})} f_1(g) f_2(g) f_3(g) \mathrm{d}g \neq 0 \end{array} \right\}. \qquad \square$$

Here the quaternion algebra $\mathbb{D}$ is uniquely determined by the local epsilon factors as in Prasad's criterion. This result hints that

$$\int\limits_{Z_{\mathbb{A}}\mathbb{D}^*(\mathbb{F})\backslash \mathbb{D}^*(\mathbb{A})} f_1(g) f_2(g) f_3(g) \mathrm{d}g$$

could be a potential integral representation of $L(\Pi, 1/2)$. Later on there are a lot of works on explicitly relating both sides. In particular one can see Ichino's work in [12]. We only need a special version here.

$$\left| \int_{Z_{\mathbb{A}}\mathbb{D}^*(\mathbb{F})\backslash\mathbb{D}^*(\mathbb{A})} f_1(g)f_2(g)f_3(g)\mathrm{d}g \right|^2 = \frac{\zeta_{\mathbb{F}}^2(2)L(\Pi, 1/2)}{8L(\Pi, Ad, 1)} \prod_v I_v^0(f_{1,v}, f_{2,v}, f_{3,v}), \tag{2.59}$$

where

$$I_v^0(f_{1,v}, f_{2,v}, f_{3,v}) = \frac{L_v(\Pi_v, Ad, 1)}{\zeta_v^2(2)L_v(\Pi_v, 1/2)} I_v(f_{1,v}, f_{2,v}, f_{3,v}), \tag{2.60}$$

and

$$I_v(f_{1,v}, f_{2,v}, f_{3,v}) = \int_{\mathbb{F}_v^*\backslash\mathbb{D}^*(\mathbb{F}_v)} \prod_{i=1}^{3} < \pi_i^{\mathbb{D}}(g)f_{i,v}, f_{i,v} > \mathrm{d}g. \tag{2.61}$$

We will however be mainly interested in the case when $\mathbb{D}$ is the matrix algebra. If $\mathbb{D}$ turns out to be a division algebra according to the conditions on local epsilon factors, that means the triple product integral on the GL_2 side will be zero automatically.

2.7 Whittaker model for some highly ramified representations

By saying π is highly ramified, we mean π is either supercuspidal, or induced from two ramified characters (instead of unramified special representation). This subsection is purely local, so we will suppress the subscript v for all notations.

Let π be a local irreducible (generic) representation of GL_2. Let ψ be a fixed unramified additive character. Then there is a unique realization of π in the space of functions W on GL_2 such that

$$W\left(\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} g\right) = \psi(n)W(g). \tag{2.62}$$

When π is unitary, one can define a unitary pairing on π using the Whittaker model:

$$< W_1, W_2 >= \int_{\mathbb{F}^*} W_1\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}\right) \overline{W_2\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}\right)} \mathrm{d}^*\alpha. \tag{2.63}$$

Let π be a supercuspidal representation first. The Kirillov model of π is a unique realization on the space of Schwartz functions $S(\mathbb{F}^*)$ such that

$$\pi\left(\begin{pmatrix} a_1 & m \\ 0 & a_2 \end{pmatrix}\right)\varphi(x) = w_\pi(a_2)\psi(ma_2^{-1}x)\varphi(a_1a_2^{-1}x), \tag{2.64}$$

where w_π is the central character for π. Let W_φ be the Whittaker function associated to φ. Then they are related by

$$\varphi(\alpha) = W_\varphi\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}\right),$$

$$W_\varphi(g) = \pi(g)\varphi(1).$$

When π is unitary, one can define the $G-$invariant unitary pairing in the Kirillov model by

$$< f_1, f_2 >= \int_{\mathbb{F}^*} f_1(x)\overline{f_2}(x)\mathrm{d}^*x. \tag{2.65}$$

This is consistent with the unitary pairing defined above using the Whittaker model. By Bruhat decomposition, one just has to know the action of $\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ to understand the whole group action.

For χ a character of O_v^*, define

$$\mathbf{1}_{\chi,n}(x) = \begin{cases} \chi(u), & \text{if } x = u\varpi^n \text{ for } u \in O_v^*; \\ 0, & \text{otherwise.} \end{cases}$$

Roughly speaking, it's the character χ supported at $v(x) = n$. We can then describe the action of $\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on $\mathbf{1}_{\chi,n}$ explicitly according to [14]:

$$\pi(\omega)\mathbf{1}_{\chi,n} = C_{\chi w_0^{-1}} z_0^{-n} \mathbf{1}_{\chi^{-1}w_0, -n+n_{\chi^{-1}}}. \tag{2.66}$$

Here $z_0 = w(\varpi)$ and $w_0 = w_\pi|_{O_v^*}$. n_χ is an integer decided by the representation π and the character χ (and independent of n). It's well known that $n_\chi \leq -2$ for any χ. Further we have $c = c(\pi) = -n_1$. The local new form is simply $\mathbf{1}_{1,0}$.

The relation $\omega^2 = -\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ implies

$$n_\chi = n_{\chi^{-1}w_0^{-1}},\ \ C_\chi C_{\chi^{-1}w_0^{-1}} = w_0(-1)z_0^{n_\chi}. \tag{2.67}$$

When π is unitary, we have

$$|C_\chi| = 1. \tag{2.68}$$

One can easily show this by using the fact that

$$< \pi(\omega)\mathbf{1}_{\chi,0}, \pi(\omega)\mathbf{1}_{\chi,0} >=< \mathbf{1}_{\chi,0}, \mathbf{1}_{\chi,0} > . \tag{2.69}$$

It is essentially proved in [10] that

Proposition 2.9. Suppose that π is a supercuspidal representation with $c = c(\pi)$ and $c(w_\pi) \leq 1$. If $p \neq 2$ and χ is a level i character, then we have

$$n_\chi = \min\{-c, -2i\}.$$

When $p = 2$ or $c(w_\pi) > 1$, we have the same statement, except when $c \geq 4$ is an even integer and $i = c/2$. In that case, we only have $n_\chi \geq -c$. □

Now let π be a unitary induced representation $\pi(\mu_1, \mu_2)$, where $c(\mu_1) = c(\mu_2) = k > 0$. Let $c = c(\pi) = 2k$. Then by the classical results in [2], there exists a new form in the model of induced representation which is right $K_1(\varpi^c)$−invariant and supported on

$$B\begin{pmatrix}1 & 0\\ \varpi^k & 1\end{pmatrix}K_1(\varpi^c),$$

where B is the Borel subgroup. From now on let W be the Whittaker function associated to this new form. It's normalized so that $W(1) = 1$.

For an induced representation of GL_2, one can compute its Whittaker functional by the following formula:

$$W(g) = \int_{m\in\mathbb{F}} \varphi(\omega\begin{pmatrix}1 & m\\ 0 & 1\end{pmatrix}g)\psi(-m)\mathrm{d}m, \tag{2.70}$$

where $\varphi \in \pi$ and $\omega = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$.

Definition 2.10. Denote

$$W^{(i)}(\alpha) = W\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \varpi^i & 1\end{pmatrix}\right). \tag{2.71}$$

□

Let

$$C_0 = \int_{u\in O_v{}^*} \mu_1(-\varpi^k)\mu_2(-\varpi^{-k}u)\psi(-\varpi^{-k}u)\mathrm{d}u. \tag{2.72}$$

In [9], we gave the following formulae to compute $W^{(i)}$:

If $i < k$, then

$$W^{(i)}(\alpha) = C_0^{-1}\int_{u\in O_v{}^*} \mu_1\left(-\frac{\varpi^i}{u}\right)\mu_2(\alpha\varpi^{-i}(1-\varpi^{k-i}u))\psi(\alpha\varpi^{-i}(1-\varpi^{k-i}u))p^{i-k-v(\alpha)/2}\mathrm{d}u. \tag{2.73}$$

If $k < i \leq c$, then

$$W^{(i)}(\alpha) = C_0^{-1}\int_{u\in O_v{}^*} \mu_1\left(-\frac{\varpi^k}{1+u\varpi^{i-k}}\right)\mu_2(-\varpi^{-k}\alpha u)p^{-v(\alpha)/2}\psi(-\varpi^{-k}\alpha u)\mathrm{d}u. \tag{2.74}$$

If $i = k$,

$$W^{(k)}(\alpha) = C_0^{-1}\int_{v(u)\leq -k,\, u\notin\varpi^{-k}(-1+\varpi O_v)} \mu_1\left(-\frac{\varpi^k}{1+u\varpi^k}\right)\mu_2(-\alpha u)\left|\frac{\varpi^k}{\alpha u(1+u\varpi^k)}\right|^{1/2}\psi(-\alpha u)p^{-v(\alpha)}\mathrm{d}u. \tag{2.75}$$

Definition 2.11. We will say that a function $f(x)$ consists of level i components, if we can write

$$f(x) = \sum_{c(\chi)=i}\sum_{n\in\mathbb{Z}} a_{\chi,n}\mathbf{1}_{\chi,n}, \tag{2.76}$$

where χ's are characters of $O_v{}^*$.

By L^2 norm of a sequence of numbers $\{a_i\}$, we mean

$$\left(\sum |a_i|^2\right)^{1/2}. \tag{2.77}$$

We will say that $f(x)$ consists of level i components with L^2 norm h, if $f(x)$ consists of level i components and the sequence of coefficients $\{a_{\chi,n}\}$ is of L^2 norm h. □

Proposition 2.12. Let π be a supercuspidal representation of level c, or induced representation $\pi(\mu_1,\mu_2)$ where $c(\mu_1)=c(\mu_2)=k=c/2$. Let W be the normalized Whittaker function for a new form of π, and $W^{(i)}$ be as in Definition 2.10.

(1) $W^{(c)}(\alpha)=\mathbf{1}_{1,0}(\alpha)$.

(2) For $i=c-1>1$, $W^{(c-1)}(\alpha)$ is supported only at $v(\alpha)=0$, consisting of level 1 components with L^2 norm $\sqrt{\frac{p(p-2)}{(p-1)^2}}$, and also level 0 component with coefficient being $-\frac{1}{p-1}$.

(3) In general for $0\leq i<c-1$, $i\neq c/2$, $W^{(i)}(\alpha)$ is supported only at $v(\alpha)=\min\{0,2i-c\}$, consisting of level $c-i$ components with L^2 norm 1.

(4) When $i=k>1$, $W^{(c/2)}$ is supported at $v(\alpha)\geq 0$, consisting of level $c/2$ components with L^2 norm 1.

When $i=k=1$, $W^{(1)}(\alpha)$ consists of level 0 component at $v(\alpha)=0$ with coefficient being $-\frac{1}{p-1}$, and level 1 components at $v(\alpha)\geq 0$ with L^2 norm $\sqrt{\frac{p(p-2)}{(p-1)^2}}$. □

Remark 2.13. In part (2), the coefficients for level 1 components together with level 0 components have L^2 norm 1. When π is supercuspidal, one can actually get that $W^{(i)}(\alpha)$ consists of level $c-i$ components with coefficients of absolute value $\sqrt{\frac{p}{(p-1)^2p^{c-i-1}}}$. Counting the number of level $c-i$ characters, one can see that this is consistent with the L^2 norm as claimed. This is however not necessarily true for the induced representations. □

Proof. Let π be a supercuspidal representation first. Then the only difference of this result from Corollary 2.18 in [9] is the claim about the coefficients. Part (2) can be easily proved using Lemma 2.3 and (2.68). Part (3) and (4) follow simply from the invariance of the unitary pairing (2.63).

Now let $\pi=\pi(\mu_1,\mu_2)$ where $c(\mu_1)=c(\mu_2)=k=c/2$. The proof refines that of Lemma 4.2 in [9] by using Lemma 2.4 above. For conciseness we will only prove part (4) when $i=k>1$. So

$$W^{(k)}(\alpha)=C_0^{-1}\int\limits_{v(u)\leq -k,u\notin\varpi^{-k}(-1+\varpi O_v)}\mu_1(-\frac{\varpi^k}{1+u\varpi^k})\mu_2(-\alpha u)\psi(-\alpha u)p^{-\frac{1}{2}v(\alpha)+v(u)}\mathrm{d}u. \tag{2.78}$$

Let χ be an multiplicative unitary character.

$$\int\limits_{v(\alpha)\text{ fixed}} W^{(k)}(\alpha)\chi(\alpha)\mathrm{d}^*\alpha = C_0^{-1} \int\limits_{v(u)\leq -k, u\notin \varpi^{-k}(-1+\varpi O_v)} \left(\int\limits_{v(\alpha)\text{ fixed}} \chi(\alpha u)\mu_2(-\alpha u)\psi(-\alpha u)d^*\alpha \right)$$
$$\mu_1(-\frac{\varpi^k}{1+u\varpi^k})\chi^{-1}(u)p^{-\frac{1}{2}v(\alpha)+v(u)}\mathrm{d}u. \tag{2.79}$$

For each fixed $v(u)$, the integral $\int\limits_{v(\alpha)\text{ fixed}} \chi(\alpha u)\mu_2(-\alpha u)\psi(-\alpha u)d^*\alpha$ is actually independent of u by a change of variable. Then by Lemma 2.4, the integral in u for fixed $v(u)$ is not zero only if $c(\chi) = k$.

Then as functions in α, $\chi(\alpha u)\mu_2(-\alpha u)$ is of level $\leq k$, $\psi(-\alpha u)$ is of level $-v(\alpha) - v(u) \geq k - v(\alpha)$. Then we need $v(\alpha) \geq 0$ for the integral $\int\limits_{v(\alpha)\text{ fixed}} \chi(\alpha u)\mu_2(-\alpha u)\psi(-\alpha u)d^*\alpha$ ever to be nonzero.

The claim about the L^2 norm of the coefficients follows from the invariance of the unitary pairing (2.63) as in the supercuspidal case. ■

We can say something more for the case $i = c/2$. When π is supercuspidal, $W^{(i)}$ is supported at $v(\alpha) \geq 0$, and will vanish for sufficiently large $v(\alpha)$. This is however not true for the induced representations.

Lemma 2.14.

$$W^{(k)}(a) <<_k p^{(\alpha-1/2)v(a)}v(a). \tag{2.80}$$

Here α is a bound towards Ramanujan conjecture and we can pick $\alpha \leq 7/64$. □

Proof. Let χ be a character of $\mathbb{F}_v^*$ which is trivial on a fixed uniformizer. As in the proof above,

$$\int\limits_{v(a)\text{ fixed}} W^{(k)}(a)\chi(a)\mathrm{d}^*a = C_0^{-1} \int\limits_{v(u)\leq -k, u\notin \varpi^{-k}(-1+\varpi O_v)} \left(\int\limits_{v(a)\text{ fixed}} \chi(au)\mu_2(-au)\psi(-au)\mathrm{d}^*a \right) \mu_1$$
$$\left(-\frac{\varpi^k}{1+u\varpi^k}\right)\chi^{-1}(u)p^{-\frac{1}{2}v(a)+v(u)}\mathrm{d}u. \tag{2.81}$$

When $v(a) > 2k$, we can further separate the integral above into two parts:

$$I_1 = \int_{-2k\le v(u)\le -k, u\notin \varpi^{-k}(-1+\varpi O_v)} \left(\int_{v(a)\text{ fixed}} \chi(au)\mu_2(-au)\mathrm{d}^*a \right)\mu_1\left(-\frac{\varpi^k}{1+u\varpi^k}\right)\chi^{-1}(u)p^{-\frac{1}{2}v(a)+v(u)}\mathrm{d}u, \tag{2.82}$$

and

$$I_2 = \int_{v(u)<-2k} \left(\int_{v(a)\text{ fixed}} \chi(au)\mu_2(-au)\psi(-au)\mathrm{d}^*a \right) \mu_1(-\frac{1}{u})\chi^{-1}(u)p^{-\frac{1}{2}v(a)+v(u)}\mathrm{d}u. \tag{2.83}$$

Note that the first integral will be non-zero only if χ is essentially μ_2^{-1}. (This means they are identical on units, but can differ on a uniformizer.) Then it's clear that

$$\int_{v(a)\text{ fixed}} \chi(au)\mu_2(-au)\mathrm{d}^*a <<_k p^{\alpha v(a)}, \quad I_1 <<_k p^{(\alpha-\frac{1}{2})v(a)}. \tag{2.84}$$

The second integral is non-zero only if χ is essentially μ_1^{-1}. If $\mu = \frac{\mu_1}{\mu_2}$ is unramified, then the non-zero contribution comes from $v(u) \ge -v(a) - 1$, and

$$I_2 <<_k p^{(\alpha-1/2)v(a)}v(a). \tag{2.85}$$

When $c(\mu) = j$, then the non-zero contribution comes from $v(u) = -v(a) - j$ (so that $\psi(au)$ is of level j). Then

$$I_2 <<_k p^{(\alpha-\frac{1}{2})v(a)}. \tag{2.86}$$

Combining the bounds for I_1 I_2 and the restriction for χ, the claim in the lemma is then clear. ■

3 The first inequality when testing on Cusp forms

In this section, we will prove Theorem 1.2 when ϕ is a Maass eigencuspform. In this case $\mu(\phi) = 0$ directly. So we just need to prove the same inequality for $\mu_f(\phi)$.

The idea is to adelize ϕ, f and $f' = \bar{f}y^k$. Then $\mu_f(\phi)$ becomes an automorphic integral as in the triple product formula. Using (2.59), it would be enough to apply the weak subconvexity bound for the triple product L-function (see [22]), and give a reasonable upper bound for the normalized local integrals. As the weight aspect was already solved in the previous works, we will focus only on the level aspect here.

We will start with necessary tools to give upper bound for the local integrals. Section 3.1–3.3 will be purely local, so we will omit subscript v without confusion.

3.1 Matrix coefficient for highly ramified representations at non-archimedean places

Let π be a supercuspidal representation of level c or of form $\pi(\mu_1,\mu_2)$ where $c(\mu_1)=c(\mu_2)=k=c/2$. Let φ be a new form for π which is invariant under $K_1(\varpi^c)$. Let

$$\Phi(g) = <\pi(g)\varphi,\varphi> \tag{3.1}$$

be the matrix coefficient associated to φ. It is bi-$K_1(\varpi^c)-$invariant. But we will only make use of the right $K_1(\varpi^c)-$invariance now. By Lemma 2.5, to understand $\Phi(g)$, it will be enough to understand $\Phi\left(\begin{pmatrix}a & m\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \varpi^i & 1\end{pmatrix}\right)$ for $0\leq i\leq c$. Let $p=|\varpi|^{-1}$. The following result is a refinement of Lemma 4.2 in [9].

Proposition 3.1. Let Φ be as mentioned above.

(i) For $c-1\leq i\leq c$, $\Phi\left(\begin{pmatrix}a & m\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \varpi^i & 1\end{pmatrix}\right)$ is supported on $v(a)=0$ and $v(m)\geq -1$. On the support, we have

$$\Phi\left(\begin{pmatrix}a & m\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \varpi^i & 1\end{pmatrix}\right)=\begin{cases}1, & \text{if } v(m)\geq 0 \text{ and } i=c;\\ -\frac{1}{p-1}, & \text{if } v(m)=-1 \text{ and } i=c;\\ -\frac{1}{p-1}, & \text{if } v(m)\geq 0 \text{ and } i=c-1.\end{cases} \tag{3.2}$$

When $v(a)=0$, $v(m)=-1$ and $i=c-1>1$, $\Phi\left(\begin{pmatrix}a & m\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \varpi^{c-1} & 1\end{pmatrix}\right)$ consists of level 1 components with L^2 norm $\frac{p\sqrt{p-2}}{(p-1)^2}$, and also level 0 component with coefficient $\frac{1}{(p-1)^2}$.

(ii) For $0\leq i<c-1$, $i\neq c/2$, $\Phi\left(\begin{pmatrix}a & m\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \varpi^i & 1\end{pmatrix}\right)$ is supported on $v(a)=\min\{0,2i-c\}$, $v(m)=i-c$. As a function in a it consists of level $c-i$ components with L^2 norm $\sqrt{\frac{p}{(p-1)^2p^{c-i-1}}}$.

(iii) When c is even and $i=c/2>1$, $\Phi\left(\begin{pmatrix}a & m\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \varpi^i & 1\end{pmatrix}\right)$ is supported on $v(a)\geq 0$, $v(m)=-c/2$. As a function in a it consists of level $c-i$ components with L^2 norm $\sqrt{\frac{p}{(p-1)^2p^{c/2-1}}}$.

When $i = c/2 = 1$, $\Phi\left(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^i & 1 \end{pmatrix}\right)$ is supported on $v(a) \geq 0$, $v(m) \geq -1$. When $v(m) \geq 0$, its value is as in (i). When $v(m) = -1$, as a function in a it consists of level 0 component at $v(a) = 0$ with coefficient $\frac{1}{(p-1)^2}$, and level 1 components at $v(a) \geq 0$ with L^2 norm $\frac{p\sqrt{p-2}}{(p-1)^2}$. □

Remark 3.2. The second part of (iii) looks like a combination of (i) and the first part of (iii). This case would not make essential difference for, for example, the bound for the local triple product integral in Proposition 3.5 and 3.7. □

Proof. By definition,

$$\Phi\left(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^i & 1 \end{pmatrix}\right) = \int_{\mathbb{F}_v^*} \pi\left(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^i & 1 \end{pmatrix}\right) W\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}\right) \overline{W\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}\right)} \mathrm{d}^*\alpha$$
$$= \int_{\mathbb{F}_v^*} \psi(m\alpha) W^{(i)}(a\alpha)\overline{W^{(c)}}(\alpha)\mathrm{d}^*\alpha$$
$$= \int_{v(\alpha)=0} \psi(m\alpha) W^{(i)}(a\alpha)\mathrm{d}^*\alpha. \tag{3.3}$$

To get a non-zero value for Φ, we just need a level 0 component supported at $v(\alpha) = 0$ for $\psi(m\alpha)W^{(i)}(a\alpha)$. Then the claims follow from Proposition 2.12. For conciseness we will only prove part (ii) here.

Let $i < c-1, i \neq c/2$. According to part (3) of Proposition 2.12, $W^{(i)}(x)$ is supported at $v(x) = \min\{0, 2i - c\}$. So (3.3) is not zero only if $v(a) = \min\{0, 2i - c\}$. We further know that $W^{(i)}(a\alpha)$ consists only of level $c - i$ characters in α. Then to get level 0 component for the product $\psi(m\alpha)W^{(i)}(a\alpha)$ at $v(\alpha) = 0$, we need $v(m) = i - c$.

To make things simpler, suppose $i > c/2$ so that $W^{(i)}(x)$ is supported at $v(x) = 0$. (The case when $i < c/2$ is very similar.) If we write

$$W^{(i)}(x) = \sum_{c(\chi)=c-i} a_\chi \chi(x) \tag{3.4}$$

for $x \in O_v^*$, then

$$\Phi\left(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^i & 1 \end{pmatrix}\right) = \int_{v(\alpha)=0} \psi(m\alpha)W^{(i)}(a\alpha)d^*\alpha = \sum_{c(\chi)=c-i} a_\chi \left(\int_{v(\alpha)=0} \psi(m\alpha)\chi(\alpha)d^*\alpha\right)\chi(a). \tag{3.5}$$

So as a function in a, $\Phi\left(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^i & 1 \end{pmatrix}\right)$ consists of level $c-i$ components with L^2 norm $\sqrt{\frac{p}{(p-1)^2p^{c-i-1}}}$. This is because the sequence $\{a_\chi\}$ is of L^2 norm 1 and

$$\left|\int_{v(\alpha)=0} \psi(m\alpha)\chi(\alpha)\mathrm{d}^*\alpha\right| = \sqrt{\frac{p}{(p-1)^2p^{c-i-1}}}$$

for all χ of level $c-i$ at $v(m)=i-c$.

One can prove the other parts similarly. In particular (i) follows from (1), (2) of Proposition 2.12, and (iii) follows from (4) of Proposition 2.12. ■

Corollary 3.3. Let $\widetilde{\Phi}$ be the matrix coefficient associated to $\pi\left(\begin{pmatrix} \varpi^{-n} & 0 \\ 0 & 1 \end{pmatrix}\right)\varphi$, where φ is a new form. Then $\widetilde{\Phi}$ is right $K_1(\varpi^{c+n})$–invariant, and $\widetilde{\Phi}\left(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^i & 1 \end{pmatrix}\right)$ is supported at $v(m)\geq -n-1$ for $i=c+n$ or $c+n-1$, and $v(m)=i-2n-c$ for $i<c+n-1$. □

Proof. Let Φ be the matrix coefficient associated to the new form as in Proposition 3.1. Then

$$\widetilde{\Phi}\left(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^i & 1 \end{pmatrix}\right) = \Phi\left(\begin{pmatrix} a & m\varpi^n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^{i-n} & 1 \end{pmatrix}\right). \tag{3.6}$$

When $i\geq n$, we can use Proposition 3.1 directly to get $\widetilde{\Phi}$. When $i<n$, we have

$$\begin{pmatrix} a & m\varpi^n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^{i-n} & 1 \end{pmatrix}$$
$$= \varpi^{i-n}\begin{pmatrix} a\varpi^{-2i+2n} & a(\varpi^{-i+n}-\varpi^{-2i+2n})+m\varpi^n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1+\varpi^{-i+n} \\ 0 & 1 \end{pmatrix}. \tag{3.7}$$

By Proposition 3.1, $\widetilde{\Phi}$ is nonzero only when $v(a\varpi^{-2i+2n})=-c$ and $v(a(\varpi^{-i+n}-\varpi^{-2i+2n})+m\varpi^n)=-c$. Note that $v(a(\varpi^{-i+n}-\varpi^{-2i+2n}))=-c+i-n<-c$ in this case, which forces $v(m)=i-2n-c$. ■

Remark 3.4. Using the same proof, one can get more detailed descriptions of matrix coefficients of old forms as in Proposition 3.1. It is straightforward and we will omit details here. □

3.2 Bound for the local triple product integral I

Proposition 3.5. Let π_i for $i = 1, 2, 3$ be three representations of $\mathrm{GL}_2(\mathbb{Q}_v)$ with trivial central characters. Suppose that $c(\pi_1) = c_1 > 1$ and $c(\pi_2) = c(\pi_3) = c > c_1$. Let Φ_i be the normalized matrix coefficients associated to the new forms of π_i. Then

$$|I_v| = \left| \int_{\mathbb{Q}_v^* \backslash \mathrm{GL}_2(\mathbb{Q}_v)} \prod_{i=1}^{3} \Phi_i(g)\mathrm{d}g \right| \leq \frac{2}{(p+1)p^{c-1}} \frac{p^3 - 2p^2 + 1}{(p-1)^3} \leq 4p^{-c}. \tag{3.8}$$

□

Proof. By Lemma 2.5 and 2.6, we can decompose the integral as

$$\sum_{i=0}^{c} A_i \int_{a,m} \prod_{j=1}^{3} \Phi_j \left(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \varpi^i & 1 \end{pmatrix} \right) |a|^{-1} \mathrm{d}^* a dm. \tag{3.9}$$

By our assumption on π_i, Proposition 3.1 holds for Φ_i. In particular when $i < c - 1$, the support of Φ_1 is disjoint with the support of Φ_2 or Φ_3. So one only need to consider $i = c$ or $c - 1$ in (3.9). When $i = c$, the corresponding integral is

$$\int_{v(a)=0, v(m) \geq -1} \Phi_1 \left(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix} \right) \Phi_2 \left(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix} \right) \Phi_3 \left(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix} \right) \mathrm{d}^* a dm$$
$$= 1 + (p-1)(-\frac{1}{(p-1)^3}) = \frac{p^2 - 2p}{(p-1)^2}. \tag{3.10}$$

When $i = c - 1$ and $v(m) \geq 0$,

$$\int_{v(a)=0, v(m) \geq 0} \Phi_1 \left(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix} \right) \Phi_2 \left(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \varpi^{c-1} & 1 \end{pmatrix} \right) \Phi_3 \left(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \varpi^{c-1} & 1 \end{pmatrix} \right) \mathrm{d}^* a dm$$
$$= \frac{1}{(p-1)^2}. \tag{3.11}$$

When $i = c - 1$ and $v(m) = -1$,

$$\int_{v(a)=0, v(m) = -1} \Phi_1 \left(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix} \right) \Phi_2 \left(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \varpi^{c-1} & 1 \end{pmatrix} \right) \Phi_3 \left(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \varpi^{c-1} & 1 \end{pmatrix} \right) \mathrm{d}^* a dm \tag{3.12}$$

$$= \int_{v(a)=0, v(m) = -1} -\frac{1}{p-1} \Phi_2 \left(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \varpi^{c-1} & 1 \end{pmatrix} \right) \Phi_3 \left(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \varpi^{c-1} & 1 \end{pmatrix} \right) \mathrm{d}^* a dm.$$

For $i = 2, 3$, $\Phi_i\left(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^{c-1} & 1 \end{pmatrix}\right)$ as a function in a consists of level 1 components with L^2 norm $\frac{p\sqrt{p-2}}{(p-1)^2}$, and also level 0 component with coefficient $\frac{1}{(p-1)^2}$. For fixed m, suppose that

$$\Phi_2\left(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^{c-1} & 1 \end{pmatrix}\right) = \sum_{c(\chi)=1} a_\chi \chi(a) + \frac{1}{(p-1)^2}, \tag{3.13}$$

and

$$\Phi_3\left(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^{c-1} & 1 \end{pmatrix}\right) = \sum_{c(\chi)=1} b_\chi \chi(a) + \frac{1}{(p-1)^2}. \tag{3.14}$$

Then the coefficient of the level 0 component of the product $\Phi_2\Phi_3$ is

$$\sum_{\chi \text{ of level } 1} a_\chi b_{\chi^{-1}} + \frac{1}{(p-1)^4} \leq \frac{p^2(p-2)}{(p-1)^4} + \frac{1}{(p-1)^4} = \frac{p^3-2p^2+1}{(p-1)^4}. \tag{3.15}$$

Here we have used Cauchy–Schwartz inequality. Then if we integrate in a first for (3.12), we can get

$$\left|\int_{v(a)=0, v(m)=-1} -\frac{1}{p-1}\Phi_2\left(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^{c-1} & 1 \end{pmatrix}\right)\Phi_3\left(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^{c-1} & 1 \end{pmatrix}\right) \mathrm{d}^*a dm\right| \leq \frac{p^3-2p^2+1}{(p-1)^4}. \tag{3.16}$$

Now put all pieces together, we have

$$|I_v| \leq \frac{1}{(p+1)p^{c-1}}\frac{p^2-2p}{(p-1)^2} + \frac{p-1}{(p+1)p^{c-1}}[\frac{1}{(p-1)^2} + \frac{p^3-2p^2+1}{(p-1)^4}] \leq \frac{2}{(p+1)p^{c-1}}\frac{p^3-2p^2+1}{(p-1)^3}. \tag{3.17}$$

Lastly

$$\frac{2}{(p+1)p^{c-1}}\frac{p^3-2p^2+1}{(p-1)^3} \leq 4p^{-c} \tag{3.18}$$

as $p \geq 2$. ■

Proposition 3.6. Let $\pi_i, i = 1, 2, 3$ and Φ_2 Φ_3 be as in Proposition 3.5. Let Φ_1 be the matrix coefficient associated to an old form $\pi_1\left(\begin{pmatrix} \varpi^{-n} & 0 \\ 0 & 1 \end{pmatrix}\right)f_1$, where f_1 is still a new form. Suppose that $c_1 + 2n < c$. Then

$$|I_v| = \left| \int\limits_{\mathbb{Q}_v^* \backslash \mathrm{GL}_2(\mathbb{Q}_v)} \prod_{i=1}^{3} \Phi_i(g) \mathrm{d}g \right| \leq \frac{4}{p^c} p^n. \tag{3.19}$$

□

Proof. In this case we apply Corollary 3.3 to Φ_1. By the assumption that $c_1 + 2n < c$, the support of Φ_1 will still be disjoint from the support of Φ_2 and Φ_3 unless $i \geq c - n - 1$. On the common support, the value of Φ_1 is easy to write down by the assumption. Then one can bound the local integral similarly as in the proof above. ■

3.3 Bound for the local triple product integral II

In this subsection we consider the case when π_1 is an unramified special representation, that is, an unramified twist of Steinberg representation. Note that for the ramified quadratic twist of Steinberg representation, we can pick the new form similarly as in the previous subsection, and the integral can be bounded similarly.

Proposition 3.7. Let π_1 be an unramified special representation and $c(\pi_2) = c(\pi_3) = c > 1$. Suppose that they all have trivial central characters. Let Φ_i be the normalized matrix coefficients associated to the new forms of π_i. Then

$$|I_v| = | \int\limits_{\mathbb{Q}_v^* \backslash \mathrm{GL}_2(\mathbb{Q}_v)} \prod_{i=1}^{3} \Phi_i(g) \mathrm{d}g| \leq 4p^{-c}. \tag{3.20}$$

□

Proof. Before we start, we first recall the matrix coefficient for unramified special representation from [23].

Let $\sigma_n = \begin{pmatrix} \varpi^n & 0 \\ 0 & 1 \end{pmatrix}$, and $\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Lemma 3.8. Let $\pi = \sigma(\chi|\cdot|^{1/2}, \chi|\cdot|^{-1/2})$ be an unramified special unitary representation of GL_2 with χ unramified. It has a normalized $K_1(\varpi)$–invariant new form. The associated matrix coefficient Φ for this new form is bi-$K_1(\varpi)$–invariant and can be given in the following table for double $K_1(\varpi)-$cosets:

g	1	ω	σ_n	$\omega\sigma_n$	$\sigma_n\omega$	$\omega\sigma_n\omega$
$\Phi(g)$	1	$-p^{-1}$	$\chi^n p^{-n}$	$-\chi^n p^{1-n}$	$-\chi^n p^{-1-n}$	$\chi^n p^{-n}$

In this table $n \geq 1$. □

As in the last subsection, we can split the integral I_v as

$$I_v = \sum_{i=0}^{c} A_i \int_{a,m} \prod_{j=1}^{3} \Phi_j \left(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \varpi^i & 1 \end{pmatrix} \right) |a|^{-1} \mathrm{d}^* a dm. \tag{3.21}$$

We can use the support of Φ_2 and Φ_3 to simplify the calculations. In particular, we put all the information necessary for the integral into the following table.

Cases	A_i	Φ_1	$\int \Phi_2\Phi_3 \mathrm{d}^* a$	$\lvert a\rvert^{-1} dm$
$i = c$, $v(a) = 0$, $v(m) \geq 0$	$\frac{1}{(p+1)p^{c-1}}$	1	1	1
$i = c$, $v(a) = 0$, $v(m) = -1$	$\frac{1}{(p+1)p^{c-1}}$	$-p^{-1}$	$\frac{1}{(p-1)^2}$	$p-1$
$i = c-1$, $v(a) = 0$, $v(m) \geq 0$	$\frac{p-1}{(p+1)p^{c-1}}$	1	$\frac{1}{(p-1)^2}$	1
$i = c-1$, $v(a) = 0$, $v(m) = -1$	$\frac{p-1}{(p+1)p^{c-1}}$	$-p^{-1}$	bdd by $\frac{p^3-2p^2+1}{(p-1)^4}$	$p-1$
$c/2 < i < c-1$, $v(a) = 0$, $v(m) = i - c$	$p^{-i}\frac{p-1}{(p+1)}$	$-p^{1+2i-2c}$	bdd by $\frac{p}{(p-1)^2 p^{c-1-i}}$	$(p-1)p^{c-i-1}$
$0 < i < c/2$, $v(a) = 2i - c$, $v(m) = i - c$	$p^{-i}\frac{p-1}{(p+1)}$	$-\chi^{2i-c}p^{1-c}$	bdd by $\frac{p}{(p-1)^2 p^{c-1-i}}$	$p^{i-1}(p-1)$
$i = 0$, $v(a) = -c$, $v(m) = -c$	$\frac{p}{(p+1)}$	$\lvert\Phi_1\rvert \leq p^{1-c}$	bdd by $\frac{p}{(p-1)^2 p^{c-1}}$	$1 - p^{-1}$

The column for Φ_1 is a reformulation of the results of Lemma 3.8 in terms of double $B - K_1(\varpi)$ cosets on the support of Φ_2. For $\int \Phi_2\Phi_3 \mathrm{d}^* a$ we have used Proposition 3.1, and Cauchy–Schwartz inequality whenever we know only L^2 norms of Φ_2 and Φ_3.

The first observation is that the contribution to the local integral from ith piece for $0 < i < c-1$ is always bounded by

$$\frac{p^{2-2c}}{p+1}p^i, \tag{3.22}$$

which is a geometric sequence whose sum can be easily bounded. We didn't list the piece $i = c/2$ in the table, as Proposition 3.1 claims $v(a) \geq 0$ instead of an equality in that case. But whenever there is a contribution to the local integral coming from $v(a) > 0$, one can check that $|\Phi_1| = |-\chi^{v(a)}p^{1-c}p^{-v(a)}| = p^{1-c}p^{-v(a)}$ while the Haar measure $|a|^{-1}\mathrm{d}m$ gives an additional $p^{v(a)}$. So the above bound still holds for the piece $i = c/2$.

The case when $i = 0$ is slightly more complicated.When $i = 0$, $v(a) = v(m) = -c$,

$$\Phi_1\left(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\right) = \begin{cases} -\chi^{-c}p^{1-c}, & \text{if } v(a+m) > -c; \\ \chi^{-c}p^{-c}, & \text{if } v(a+m) = -c. \end{cases} \tag{3.23}$$

In either case $|\Phi_1| \leq p^{1-c}$. As a result of this,

$$\begin{aligned} &|A_0 \int_{v(a)=v(m)=-c} \prod_{i=1}^{3} \Phi_i\left(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\right)|a|^{-1}\mathrm{d}^*a dm| \\ &\leq \frac{p}{p+1}p^{1-c}\int_{v(a)=0} |\Phi_2\Phi_3\left(\begin{pmatrix} a & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\right)||a|^{-1}\mathrm{d}^*a dm \\ &\leq \frac{p}{p^2-1}p^{2-2c}. \end{aligned} \tag{3.24}$$

Combining all the pieces, we can get

$$\begin{aligned} |I_v| \leq &\frac{1}{(p+1)p^{c-1}}[1-\frac{1}{p(p-1)}] + \frac{p-1}{(p+1)p^{c-1}}[\frac{1}{(p-1)^2} + p^{-1}\frac{p^3-2p^2+1}{(p-1)^3}] \\ &+ \sum_{i=1}^{c-2}\frac{p^{2-2c}}{p+1}p^i + \frac{p}{p^2-1}p^{2-2c} \\ &\leq 4p^{-c}. \end{aligned} \tag{3.25}$$

■

Proposition 3.9. Let $\pi_i, i = 1, 2, 3$ and Φ_2 Φ_3 be as in Proposition 3.7. Let Φ_1 be the matrix coefficient associated to an old form $\pi_1\left(\begin{pmatrix}\varpi^{-n} & 0\\ 0 & 1\end{pmatrix}\right)f_1$, where f_1 is still a new form of π_1. Suppose that $1 + 2n < c$. Then

$$|I_v| = |\int_{\mathbb{Q}_v^*\backslash \mathrm{GL}_2(\mathbb{Q}_v)} \prod_{i=1}^{3} \Phi_i(g)\mathrm{d}g| \leq \frac{4}{p^c}p^n. \tag{3.26}$$

□

Proof. As in Corollary 3.3, one can describe the matrix coefficient of an old form using the matrix coefficient of the new form in Lemma 3.8. Then one can bound the local integral as in Proposition 3.7. We shall skip the technical details here. ■

3.4 Proof of Theorem 1.2 when ϕ is a Maass eigencuspform

For the cases considered in Proposition 3.5 and 3.7, we can also get a bound for the normalized local integrals

$$I_v^0 \leq 10^5 p^{-c}. \tag{3.27}$$

We are not very careful on bounding the normalizing L-factors, as any fixed constant multiple will eventually be bounded by $(q/\sqrt{C})^{-1+2\alpha+\epsilon}$ for any $\epsilon > 0$. Note that this bound is actually better than the bound obtained in Corollary 2.8 of [18], as the bound towards Ramanujan Conjecture contributes to the local integral in their cases.

When ϕ is an old form, we need to use Proposition 3.6 and Proposition 3.9, and there is an extra factor p^n in the local upper bounds. But when we take a product, such extra factors will be controlled by N, which is the fixed level of ϕ. So they can be ignored when we discuss the asymptotic behavior of f. Note that we haven't consider the case when the local component of ϕ is an old form from an unramified representation. For this case one can either compute an upper bound as in Proposition 3.9, or change the local triple product integral into Rankin–Selberg integral as in [18], and we will give an upper bound for the local Rankin–Selberg integral (though for tempered datum) in the next section. So we will not consider this case in detail here.

From this point on we can use the same argument as in [18] to prove Theorem 1.2. So we will be very brief. Suppose that after adelization, f belongs to an automorphic cuspidal representation π. f' also belongs to π, while its component at infinity is of weight $-k$. Similarly suppose that the adelization of ϕ belongs to π_ϕ.

Recall that C is the finite conductor of $\pi \times \pi$. The conductor of $\pi \times \pi \times \pi_\phi$ is then $\asymp_\phi C^2k^4$. The argument of [22] implies that

$$L(\pi_\phi \times \pi \times \pi, 1/2) << \frac{\sqrt{C}k}{log(Ck)^{1-\epsilon}}. \tag{3.28}$$

Then combining the local bounds (including Corollary 2.8 of [18]) with this weak subconvexity bound into the Triple product formula will prove the claim in Theorem 1.2.

4 Proof of Theorem 1.2 when testing on an incomplete Eisenstein series

In this section, we shall prove Theorem 1.2 when $\phi = E_\mathfrak{a}(z, \Psi)$ is an incomplete Eisenstein series of level N. Here $\mathfrak{a}$ is a cusp for $\Gamma_0(N)$ and Ψ is a compactly supported function on $\mathbb{R}^+$.

According to (2.29),

$$E_\mathfrak{a}(z, \Psi) = \frac{\hat{\Psi}(1)}{Vol(Y_0(N))} + \frac{1}{2\pi i}\int_{(1/2)} \hat{\Psi}(s)E_\mathfrak{a}(z, s)\mathrm{d}s. \tag{4.1}$$

So

$$\frac{\mu_f(E_\mathfrak{a}(z, \Psi))}{\mu_f(1)} = \frac{\hat{\Psi}(1)}{Vol(Y_0(N))} + \frac{1}{2\pi i\mu_f(1)}\int_{(1/2)} \hat{\Psi}(s)\mu_f(E_\mathfrak{a}(z, s))\mathrm{d}s. \tag{4.2}$$

On the other hand,

$$\begin{aligned}\mu(E_\mathfrak{a}(z, \Psi)) &= \int_{\Gamma_0(N)\backslash\mathbb{H}} \sum_{\gamma\in\Gamma_0(N)_\mathfrak{a}\backslash\Gamma_0(N)} \Psi(\mathrm{Im}(\sigma_\mathfrak{a}^{-1}\gamma z))\frac{\mathrm{d}x\mathrm{d}y}{y^2} = \int_{\Gamma_0(N)_\mathfrak{a}\backslash\mathbb{H}} \Psi(\mathrm{Im}(\sigma_\mathfrak{a}^{-1}z))\frac{\mathrm{d}x\mathrm{d}y}{y^2} \\ &= \int_{\Gamma_\infty\backslash\mathbb{H}} \Psi(\mathrm{Im}(z))\frac{\mathrm{d}x\mathrm{d}y}{y^2}\end{aligned} \tag{4.3}$$

In the last equality, we have made a change of variable $\sigma_\mathfrak{a}^{-1}z \to z$, and used that

$$\sigma_\mathfrak{a}^{-1}\Gamma_0(N)_\mathfrak{a}\sigma_\mathfrak{a} = \Gamma_\infty. \tag{4.4}$$

Then

$$\frac{\mu(E_\mathfrak{a}(z, \Psi))}{\mu(1)} = \frac{1}{\mu(1)}\int_{x=-1/2}^{1/2}\int_{y=0}^{\infty} \Psi(y)\frac{\mathrm{d}x\mathrm{d}y}{y^2} = \frac{\hat{\Psi}(1)}{Vol(Y_0(N))}. \tag{4.5}$$

So to prove Theorem 1.2 for an incomplete Eisenstein series, it would be enough to show the same inequality for

$$\mu_f(E_{\mathfrak{a}}(z,s)) \tag{4.6}$$

with uniform implied constant for all s on the line $\mathrm{Re}(s) = 1/2$.

The idea is then similar as in the case of Maass eigencuspforms. We shall adelize $f\,f'$ and $E_{\mathfrak{a}}(z,s)$. Then $\mu_f(E_{\mathfrak{a}}(z,s))$ will be essentially the Rankin–Selberg integral. The result will follow from the weak subconvexity bound for the Rankin-Selberg L-function and a reasonable upper bound for the local integrals.

4.1 Adelization of $E_{\mathfrak{a}}(z,s)$

We first put $E_{\mathfrak{a}}(z,s)$ into the adelic framework. The process should be standard so we skip the proof here.

Suppose that $\tau_{\mathfrak{a}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $c|N$ and $d \in (\mathbb{Z}/(c,N/c))^*$, as in Section 2.2. For simplicity, let $c' = (c, N/c)$. Recall that $N_\diamond$ is the largest integer such that $N_\diamond^2|N$.

Definition 4.1. Define $V_E(N)$ to be space spanned by Φ_s, where Φ_s runs over new forms or old forms of level dividing N in $\mathrm{Ind}_B^{\mathrm{GL}_2}(\chi|\cdot|^{s-1/2}, \chi^{-1}|\cdot|^{-s+1/2})$, and χ runs over idelic lifts of Dirichlet characters of level dividing $N_\diamond$. Define the operator *Eis* to be

$$Eis : \Phi_s \mapsto \sum_{\gamma\in B(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{Q})} \Phi_s(\gamma g). \tag{4.7}$$

□

Lemma 4.2. The adelization of $(\frac{N}{cc'})^s E_{\mathfrak{a}}(z,s)$ belongs to $Eis(V_E(N))$. □

The way we are going to use this result is as follows. The factor $(\frac{N}{cc'})^s$ is easily controlled on $\mathrm{Re}(s) = 1/2$. To prove certain asymptotic property for $E_{\mathfrak{a}}(z,s)$, it would then be enough to prove the same asymptotic property for a basis of $Eis(V_E(N))$. We shall choose a basis that facilitates our calculations.

4.2 Bound for the local Rankin–Selberg integral

In this subsection we will give a reasonable upper bound for the following local integral of Rankin–Selberg integral at finite places:

$$J_p(s) = \int_{NZ\backslash \mathrm{GL}_2} W_{f,p}(g) W_{f',p}(g) \Phi_{s,p}(g) \mathrm{d}g, \tag{4.8}$$

where Φ_s will run over a basis of $V_E(N)$, W_f is the Whittaker functional associated to f and a fixed additive character ψ, and $W_{f'}$ is associated to f' and ψ^-.

We will pick a basis for $V_E(N)$ as follows: The local component of Φ_s at p is either spherical, or supported on $B\begin{pmatrix}1 & 0\\ p^i & 1\end{pmatrix}K_0(p^{e_p})$ for $i < e_p$. Note that the case when the local component is spherical is already covered in [18]. So we only need to consider the latter case here.

Proposition 4.3. Suppose that π_i for $i = 1, 2$ have trivial central characters and $c(\pi_1) = c(\pi_2) = c \geq 2$. Let W_1 be the Whittaker functional associated to a newform of π_1 and the additive character ψ. Let W_2 be the Whittaker functional associated to a newform of π_2 and the additive character ψ^-. Let Φ_s be a function from $\mathrm{Ind}_B^{\mathrm{GL}_2}(\chi|\cdot|^{s-1/2}, \chi^{-1}|\cdot|^{-s+1/2})$, supported on $B\begin{pmatrix}1 & 0\\ p^i & 1\end{pmatrix}K_0(p^{e_p})$ for $i < e_p$. Suppose that $W_1(1) = W_2(1) = \Phi_s\left(\begin{pmatrix}1 & 0\\ p^i & 1\end{pmatrix}\right) = 1$ and $c > 2e_p$. Then for $\mathrm{Re}(s) = 1/2$,

$$|J_p(s)| = \left| \int_{NZ\backslash \mathrm{GL}_2} W_1(g) W_2(g) \Phi_s(g) \mathrm{d}g \right| \leq \frac{p-1}{p+1} p^{-c/2}. \tag{4.9}$$

□

Remark 4.4. By the theory of newforms and oldforms in [2], it would be automatic that $c(\chi) \leq \min\{i, e_p - i\}$. □

Remark 4.5. As we will only care about asymptotic behaviors, the assumption that $c > 2e_p$ is reasonable. □

Proof. First note that

$$B\begin{pmatrix}1 & 0\\ p^i & 1\end{pmatrix}K_0(p^{e_p}) = B\begin{pmatrix}1 & 0\\ p^i & 1\end{pmatrix}K_0(p^j) \tag{4.10}$$

for any $j > e_p > i$.

As Φ_s is supported on $B\begin{pmatrix}1 & 0\\ p^i & 1\end{pmatrix}K_0(p^{e_p}) = B\begin{pmatrix}1 & 0\\ p^i & 1\end{pmatrix}K_0(p^c)$ for $i < e_p$, we have directly that

$$J_p(s) = \frac{p-1}{(p+1)p^i}\int W_1\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ p^i & 1\end{pmatrix}\right)W_2\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ p^i & 1\end{pmatrix}\right)\chi(\alpha)|\alpha|^{s-1}\mathrm{d}^*\alpha. \quad (4.11)$$

Now we can apply part (3) of Proposition 2.12, which implies that both $W_1^{(i)}(\alpha)$ and $W_2^{(i)}(\alpha)$ are supported at $v(\alpha) = 2i - c$, consisting of level $c - i$ components with L^2 norm 1. Then one just has to apply Cauchy—Schwartz inequality, and the easy fact that

$$||\alpha|^{s-1}| = p^i p^{-c/2} \quad (4.12)$$

when $v(\alpha) = 2i - c$ and $\mathrm{Re}(s) = 1/2$. ■

One can argue similarly from here on to prove Theorem 1.2 as in [18].

5 The second inequality

5.1 Fourier coefficient of an Eisenstein series of level N

We first study the asymptotic behavior for the Fourier coefficients of a level N Eisenstein series $E_{\mathfrak{a}}(\sigma_{\mathfrak{b}}z, s)$ by using its adelization discussed in the previous section.

Let $\tau(n)$ be the function counting the divisors of n.

Lemma 5.1. For $\phi = E_{\mathfrak{a}}(\sigma_{\mathfrak{b}}z, s)$ and $\mathrm{Re}(s) = 1/2$, recall we can write its Fourier expansion as

$$\phi = \delta_{\mathfrak{a}\mathfrak{b}}y^s + \varphi_{\mathfrak{a}\mathfrak{b}}(s)y^{1-s} + \frac{1}{\xi(2s)}\sum_{n\neq 0}\frac{\lambda_{\phi,s}(n)}{\sqrt{|n|}}\kappa_{s-1/2}(ny)\mathrm{e}^{2\pi inx}. \quad (5.1)$$

Then

(1) $\varphi_{\mathfrak{a}\mathfrak{b}}(s) = O(1)$ as $s = 1/2 + it \to \infty$.
(2) $|\lambda_{\phi,s}(n)| <<_N \tau(n)$ as $n \to \infty$. □

Proof. We shall only prove part (2) here. Part (1) follows immediately from the unitarity of the scattering matrix in [13], or a similar consideration as below.

When $N = 1$,

$$|\lambda_{\phi,s}(n)| = |\sum_{ab=n}(\frac{a}{b})^{s-1/2}| \leq \tau(n). \tag{5.2}$$

So the claim is clear in this case. In general, we can adelize our Eisenstein series, and prove a similar result for the Whittaker functional for a basis in $Eis(V_E(N))$.

According to Lemma 4.2,

$$(\frac{N}{cc'})^s E_{\mathfrak{a}}(z,s) = \sum_{\Phi_s} a_{\Phi_s} \sum_{\gamma\in B(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{Q})} \Phi_s(\gamma g_z) \tag{5.3}$$

for proper coefficients a_{Φ_s}, and Φ_s runs over a basis of $V_E(N)$. Then

$$\left(\frac{N}{cc'}\right)^s E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z,s) = \sum_{\Phi_s} a_{\Phi_s} \sum_{\gamma\in B(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{Q})} \Phi_s(\gamma g'_z). \tag{5.4}$$

Here $g'_z = \left(\begin{pmatrix} y & x \\ 0 & 1\end{pmatrix}, \sigma_{\mathfrak{b}}^{-1}, \sigma_{\mathfrak{b}}^{-1}, \sigma_{\mathfrak{b}}^{-1}, \cdots\right)$, and we have used that each $\sum_{\gamma\in B(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{Q})} \Phi_s(\gamma g)$ is left $\mathrm{GL}_2(\mathbb{Q})-$ invariant.

Using the Fourier expansion for adelic Eisenstein series as in subsection 2.4, and comparing it with (5.1), we have

$$\left(\frac{N}{cc'}\right)^s \frac{1}{\xi(2s)} \frac{\lambda_{\phi,s}(n)}{\sqrt{|n|}} \kappa_{s-1/2}(ny) e^{2\pi i n x} = \sum_{\Phi_s} a_{\Phi_s} W\left(\begin{pmatrix} n & 0 \\ 0 & 1\end{pmatrix} g'_z\right). \tag{5.5}$$

Note that $(\frac{N}{cc'})^s$ and a_{Φ_s} are negligible for asymptotic behavior. For all $p \nmid N$, $\Phi_{s,p}$ is spherical and $\sigma_{\mathfrak{b}}^{-1}$ belongs to the local maximal compact subgroup.

Recall that we normalized the local Whittaker functional such that

$$W_\infty\left(\begin{pmatrix} n & 0 \\ 0 & 1\end{pmatrix} g_z\right) = \frac{1}{\xi(2s)} \kappa_{s-1/2}(ny) e^{2\pi i n x}, \tag{5.6}$$

and

$$W_p(1) = 1 \tag{5.7}$$

for all finite prime p.

Write $n = \prod_{p|n} p^{n_p}$. It would then be enough to show that

$$W_p\left(\begin{pmatrix} n & 0 \\ 0 & 1\end{pmatrix} \sigma_{\mathfrak{b}}^{-1}\right) <<_N p^{-\frac{1}{2}n_p} \tau(p^{n_p}) \tag{5.8}$$

for Whittaker functions of the local component of Φ_s which runs over a basis of $V_E(N)$. We will pick the basis as follows: the local component of Φ_s is either a new form, or a translate of new form by $\begin{pmatrix} p^{-j} & 0 \\ 0 & 1 \end{pmatrix}$.

For locally unramified representations, the local component of Φ_s is then a $\begin{pmatrix} p^{-j} & 0 \\ 0 & 1 \end{pmatrix}$ translate of a spherical element. Let $W_{p,0}$ be the Whittaker functional of the spherical element as given by (2.51). Then it's clear that

$$|W_{p,0}\left(\begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}\right)| \leq p^{-\frac{1}{2}n_p}\tau(p^{n_p}). \tag{5.9}$$

By the Iwasawa decomposition, the translate by $\sigma_{\mathfrak{b}}^{-1}\begin{pmatrix} p^{-j} & 0 \\ 0 & 1 \end{pmatrix}$ amount to a fixed shift in the valuation for n_p. So

$$|W_{p,0}\left(\begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}\sigma_{\mathfrak{b}}^{-1}\begin{pmatrix} p^{-j} & 0 \\ 0 & 1 \end{pmatrix}\right)| <<_N p^{-\frac{1}{2}n_p}\tau(p^{n_p}). \tag{5.10}$$

Now let Φ_s belong to a representation induced from two ramified Hecke characters, and let $W_{p,0}$ be the local Whittaker functional associated to the new form of the corresponding local representation. Again the Iwasawa decomposition (more precisely Lemma 2.5) implies that the translate by $\sigma_{\mathfrak{b}}^{-1}\begin{pmatrix} p^{-j} & 0 \\ 0 & 1 \end{pmatrix}$ will give a fixed shift in n_p locally and also decide which double $B-\ K_0(p^{e_p})$ coset that $\begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}\sigma_{\mathfrak{b}}^{-1}\begin{pmatrix} p^{-j} & 0 \\ 0 & 1 \end{pmatrix}$ belongs to. This means that we only have to care about the asymptotic behavior of $W_{p,0}^{(i)}$ for some fixed i. Then the Lemma follows from Proposition 2.12 and Lemma 2.14. (Note that we can pick $\alpha = 0$ for a unitary Eisenstein series in Lemma 2.14.) ■

5.2 Proof of Theorem 1.3

Theorem 1.3 turns out to be easier to generalize. We shall briefly follow the proof in [7] [17] [18], then focus on the difference.

Let ϕ be either a Maass eigencuspform or an incomplete Eisenstein series of level N. Let f be a holomorphic newform of weight $k \in 2\mathbb{N}$ and level q, where $N|q$. Let $Y \geq 1$ be a parameter to be chosen later, and let $h \in C_c^\infty(\mathbb{R}^+)$ be a compactly supported everywhere nonnegative test function whose Mellin transform is $\hat{h}$ and $\hat{h}(1) = \mu(1)$. Let h_Y be the function $y \mapsto h(Yy)$.

Test μ_f on $E(z,h_Y)\phi$ and apply (2.29) to $E(z,h_Y)$. We will then get

$$Y\mu_f(\phi)=\mu_f(E(z,h_Y)\phi(z))-\frac{1}{2\pi i}\int_{(1/2)}\hat{h}(s)Y^s\mu_f(E(z,s)\phi(z))\mathrm{d}s. \tag{5.11}$$

The same argument as in [17] implies that

$$\frac{1}{2\pi i}\int_{(1/2)}\hat{h}(s)Y^s\mu_f(E(z,s)\phi(z))\mathrm{d}s<<_\phi Y^{1/2}\mu_f(1), \tag{5.12}$$

as the only information about ϕ used is its rapid decay along cusps. Then a standard unfolding technique gives

$$\begin{aligned}\mu_f(E(z,h_Y)\phi(z))&=\sum_{\tau\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})/\Gamma_0(q)}\int_{\tau^{-1}\Gamma_\infty\tau\cap\Gamma_0(q)\backslash\mathbb{H}}h_Y(\mathrm{Im}(\tau z))\phi(z)|f|^2(z)y^k\frac{\mathrm{d}x\mathrm{d}y}{y^2}\\&=\sum_{\tau\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})/\Gamma_0(q)}\int_{\Gamma_\infty\cap\tau\Gamma_0(q)\tau^{-1}\backslash\mathbb{H}}h_Y(\mathrm{Im}(z))\phi(\tau^{-1}z)|f|^2(\tau^{-1}z)\,\mathrm{Im}(\tau^{-1}z)^k\frac{\mathrm{d}x\mathrm{d}y}{y^2}\\&=\sum_{\mathfrak{a}\in\mathcal{C}}\int_{y=0}^{\infty}\int_{x=0}^{d_\mathfrak{a}}h_Y(\mathrm{Im}(z))\phi(\tau_\mathfrak{a}z)|f|^2(\tau_\mathfrak{a}z)\,\mathrm{Im}(\tau_\mathfrak{a}z)^k\frac{\mathrm{d}x\mathrm{d}y}{y^2}\\&=\sum_{\mathfrak{a}\in\mathcal{C}}\int_{y=0}^{\infty}\int_{x=0}^{1}h_Y(d_\mathfrak{a}\,\mathrm{Im}(z))\phi(\tau_\mathfrak{a}\begin{pmatrix}d_\mathfrak{a}&0\\0&1\end{pmatrix}z)|f|^2(\sigma_\mathfrak{a}z)\,\mathrm{Im}(\sigma_\mathfrak{a}z)^k\frac{\mathrm{d}x\mathrm{d}y}{y^2}.\end{aligned} \tag{5.13}$$

Here $\mathfrak{a}$ is considered as a cusp for $\Gamma_0(q)$. Let $d'_\mathfrak{a}$ and $\sigma'_\mathfrak{a}$ be the width and scaling matrix for $\mathfrak{a}$ when considered as a cusp for $\Gamma_0(N)$. If $N=1$, $d'_\mathfrak{a}=1$, we have a single Fourier expansion of ϕ along cusps. But in general $d'_\mathfrak{a}$ may not be 1, and we can have several Fourier expansions along different cusps. This is the difference between our case and previous papers.

Suppose that we have the Fourier expansion

$$\phi(\sigma'_\mathfrak{a}z)=\sum_{l\in\mathbb{Z}}\phi_l(y)\mathrm{e}^{2\pi ilx}. \tag{5.14}$$

Let $\tilde{d}_\mathfrak{a}=d_\mathfrak{a}/d'_\mathfrak{a}$. Set

$$\mathcal{S}_0=\sum_{\mathfrak{a}\in\mathcal{C}}\int_{y=0}^{\infty}h_Y(\mathrm{d}_\mathfrak{a}y)\int_{x=0}^{1}\phi_0(\tilde{d}_\mathfrak{a}y)|f|^2(\sigma_\mathfrak{a}z)\,\mathrm{Im}(\sigma_\mathfrak{a}z)^k\frac{\mathrm{d}x\mathrm{d}y}{y^2},$$

$$\mathcal{S}_{0,Y^{1+\epsilon}} = \sum_{\mathfrak{a}\in\mathcal{C}} \int_{y=0}^{\infty} h_Y(\mathrm{d}_{\mathfrak{a}}y) \int_{x=0}^{1} \sum_{0<|l|<Y^{1+\epsilon}} \phi_l(\tilde{d}_{\mathfrak{a}}y)|f|^2(\sigma_{\mathfrak{a}}z)\,\mathrm{Im}(\sigma_{\mathfrak{a}}z)^k e^{2\pi i l \tilde{d}_{\mathfrak{a}}x} \frac{\mathrm{d}x\mathrm{d}y}{y^2},$$

$$\mathcal{S}_{\geq Y^{1+\epsilon}} = \sum_{\mathfrak{a}\in\mathcal{C}} \int_{y=0}^{\infty} h_Y(\mathrm{d}_{\mathfrak{a}}y) \int_{x=0}^{1} \sum_{|l|>Y^{1+\epsilon}} \phi_l(\tilde{d}_{\mathfrak{a}}y)|f|^2(\sigma_{\mathfrak{a}}z)\,\mathrm{Im}(\sigma_{\mathfrak{a}}z)^k e^{2\pi i l \tilde{d}_{\mathfrak{a}}x} \frac{\mathrm{d}x\mathrm{d}y}{y^2}.$$

So

$$\mu_f(E(z,h_Y)\phi(z)) = \mathcal{S}_0 + \mathcal{S}_{0,Y^{1+\epsilon}} + \mathcal{S}_{\geq Y^{1+\epsilon}}. \tag{5.15}$$

Lemma 5.2. (the main term) $\mathcal{S}_0$ is 0 when ϕ is a Maass eigencuspform. If ϕ is an incomplete Eisenstein series,

$$\mathcal{S}_0 = Y\mu_f(1)\left(\frac{\mu(\phi)}{\mu(1)} + O_\phi\left(\frac{1+R_f(qk)}{Y^{1/2}}\right)\right), \tag{5.16}$$

where

$$R_f(x) = \frac{x^{-1/2}}{L(f,Ad,1)} \int_{\mathbb{R}} \left|\frac{L(f,Ad,1/2+it)}{(1+|t|)^{10}}\right| \mathrm{d}t \tag{5.17}$$

is independent of ϕ. □

Proof. The first part is clear. For an incomplete Eisenstein series ϕ of level 1, this is Lemma 3.6 of [17]. The argument there used that for $y \asymp 1/Y$,

$$\phi_0(y) = \mu(\phi)/\mu(1) + O_\phi(Y^{-1/2}). \tag{5.18}$$

By (2.35) and (4.5) we know that for $\phi = E_{\mathfrak{a}}(\sigma_{\mathfrak{b}}z, \Psi)$ with Ψ compactly supported on $\mathbb{R}^*_+$,

$$\phi_0(y) = \mu(\phi)/\mu(1) + \frac{1}{2\pi i} \int_{(1/2)} \hat{\Psi}(s)(\delta_{\mathfrak{ab}}y^s + \varphi_{\mathfrak{ab}}(s)y^{1-s})\mathrm{d}s. \tag{5.19}$$

If ϕ is of level 1, (5.18) follows from that $\hat{\Psi}$ is rapidly decreasing and $\varphi_{\mathfrak{ab}}(s) = M(s)$ is always of norm 1 on $\mathrm{Re}(s) = 1/2$. In general it follows from part (1) of Lemma 5.1. The rest arguments would be the same as in [17]. ■

Lemma 5.3. (Trivial error term)

$$\mathcal{S}_{\geq Y^{1+\epsilon}} <<_{\phi,\epsilon} Y^{-10}\mu_f(1). \tag{5.20}$$

□

Proof. The original proof in [17] made use of a bound for the sum of Fourier coefficients for ϕ, which in our case follows directly from Corollary 2.1 and Lemma 5.1. ■

Now we consider the main error term $\mathcal{S}_{0,Y^{1+\epsilon}}$. Recall by (2.17),

$$f|_k\sigma_{\mathfrak{a}}(z) = y^{-k/2}\sum_{n\in\mathbb{N}}\frac{\lambda_{\mathfrak{a}}(n)}{\sqrt{n}}\kappa_f(ny)\mathrm{e}^{2\pi inx} \tag{5.21}$$

for any cusp $\mathfrak{a}$ and $\kappa_f(y) = y^{k/2}\mathrm{e}^{-2\pi y}$. Note that $|f|^2(\sigma_{\mathfrak{a}}z)\,\mathrm{Im}(\sigma_{\mathfrak{a}}z)^k = |f|_k\sigma_{\mathfrak{a}}|^2(z)y^k$. Then

$$|f|^2(\sigma_{\mathfrak{a}}z)\,\mathrm{Im}(\sigma_{\mathfrak{a}}z)^k = \sum_{m,n\in\mathbb{N}}\frac{\lambda_{\mathfrak{a}}(n)\overline{\lambda_{\mathfrak{a}}(m)}}{\sqrt{nm}}\kappa_f(ny)\kappa_f(my)\mathrm{e}^{2\pi i(n-m)x}. \tag{5.22}$$

We first focus on the case when ϕ is a Maass eigencuspform, then we can write ϕ_l more explicitly as

$$\phi_l(y) = \frac{\lambda_{\phi,\mathfrak{a}}(l)}{\sqrt{l}}\kappa_{ir}(ly). \tag{5.23}$$

Define

$$I_\phi(l,n,x) = (mn)^{-1/2}\int_0^\infty h(xy)\kappa_{ir}(ly)\kappa_f(my)\kappa_f(ny)\frac{\mathrm{d}y}{y^2}, m = n+l. \tag{5.24}$$

Then

$$\begin{aligned}\mathcal{S}_{0,Y^{1+\epsilon}} &= \sum_{\mathfrak{a}\in\mathcal{C}}\int_{y=0}^\infty h_Y(d_{\mathfrak{a}}y)\int_{x=0}^1\sum_{0<|l|<Y^{1+\epsilon}}\phi_l(\tilde{d}_{\mathfrak{a}}y)|f|^2(\sigma_{\mathfrak{a}}z)\,\mathrm{Im}(\sigma_{\mathfrak{a}}z)^k e^{2\pi il\tilde{d}_{\mathfrak{a}}x}\frac{\mathrm{d}x\mathrm{d}y}{y^2} \\ &= \sum_{\mathfrak{a}\in\mathcal{C}}\sum_{0<|l|<Y^{1+\epsilon}}\frac{\lambda_{\phi,\mathfrak{a}}(l)}{\sqrt{l}}\sum_{n\in\mathbb{N},m=n+\tilde{d}_{\mathfrak{a}}l}\lambda_{\mathfrak{a}}(n)\overline{\lambda_{\mathfrak{a}}(m)}I_\phi(\tilde{d}_{\mathfrak{a}}l,n,d_{\mathfrak{a}}Y)\end{aligned} \tag{5.25}$$

For simplicity, let $d_c = \frac{[q, c_{\mathfrak{a}}^2]}{c_{\mathfrak{a}}^2} = d_{\mathfrak{a}}$ for $\mathfrak{a} \in \mathcal{C}[c]$. When $N = 1$, we will get

$$|\mathcal{S}_{0,Y^{1+\epsilon}}| = \left|\sum_{\mathfrak{a}\in\mathcal{C}} \sum_{0<|l|<Y^{1+\epsilon}} \frac{\lambda_\phi(l)}{\sqrt{l}} \sum_{n\in\mathbb{N}, m=n+d_{\mathfrak{a}}l} \lambda_{\mathfrak{a}}(n)\overline{\lambda_{\mathfrak{a}}(m)} I_\phi(d_{\mathfrak{a}}l, n, d_{\mathfrak{a}}Y)\right| \tag{5.26}$$
$$= \sum_{0<|l|<Y^{1+\epsilon}} \left|\sum_{c|q} \frac{\lambda_\phi(l)}{\sqrt{l}} \sum_{n\in\mathbb{N}, m=n+d_c l} I_\phi(d_c l, n, d_c Y) \sum_{\mathfrak{a}\in\mathcal{C}[c]} \lambda_{\mathfrak{a}}(n)\overline{\lambda_{\mathfrak{a}}(m)}\right|$$
$$\leq \sum_{0<|l|<Y^{1+\epsilon}} \sum_{c|q} \sharp\mathcal{C}[c] \frac{|\lambda_\phi(l)|}{\sqrt{l}} \sum_{n\in\mathbb{N}, m=n+d_c l} |I_\phi(d_c l, n, d_c Y)||\lambda_{[c]}(n)\lambda_{[c]}(m)|.$$

Here $\lambda_{[c]}(n)$ is as defined in (2.19), and the last inequality follows simply from Cauchy–Schwartz inequality. Then it is proven in [17], [18] that

$$I_\phi(l, n, x) <<_A \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \max\left\{1, \frac{\max\{m, n\}}{xk}\right\}^{-A} \tag{5.27}$$

for every $A > 0$,

$$\sum_{0<|l|<Y^{1+\epsilon}} \frac{|\lambda_\phi(l)|}{\sqrt{|l|}} <<_{\phi,\epsilon} Y^{1/2+2\epsilon}, \tag{5.28}$$

and a bound of shifted convolution sum

$$\sum_{n\in\mathbb{N}, m=n+l, \max\{m,n\}\leq x} |\lambda_{[c]}(n)\lambda_{[c]}(m)| <<_\epsilon q_\diamond^\epsilon \log\log(e^e q)^{O(1)} \frac{x \prod_{p\leq x}(1+2|\lambda_f(p)|/p)}{\log(ex)^{2-\epsilon}}. \tag{5.29}$$

Combining all these bounds together with Deligne's bound $|\lambda_f(p)| \leq 2$, and taking Y as in [18] will prove Theorem 1.3 for $N = 1$ case.

In general for our case, the first issue is that $\lambda_{\phi,\mathfrak{a}}(l)$ could be different for different cusps. But there is no harm to be a little loose as there are only finitely many fixed cusps for $\Gamma_0(N)$. Denote

$$\lambda_{\phi,+}(l) = \sum_{\text{cusps for } \Gamma_0(N)} |\lambda_{\phi,\mathfrak{a}}(l)|. \tag{5.30}$$

Then by Corollary 2.1

$$\sum_{0<|l|<Y^{1+\epsilon}} \frac{|\lambda_{\phi,+}(l)|}{\sqrt{|l|}} <<_{\phi,\epsilon} Y^{1/2+2\epsilon}, \tag{5.31}$$

which is the analogue of (5.28). Note that for fixed $c|q$, $d'_{\mathfrak{a}}$ will also be the same for all $\mathfrak{a} \in \mathcal{C}[c]$. Then

$$|\mathcal{S}_{0,Y^{1+\epsilon}}| \leq \sum_{0<|l|<Y^{1+\epsilon}} \sum_{c|q} \frac{|\lambda_{\phi,+}(l)|}{\sqrt{|l|}} \sum_{n\in\mathbb{N}, m=n+\tilde{d}_{\mathfrak{a}}l} |I_\phi(\tilde{d}_c l, n, d_c Y)| \sum_{\mathfrak{a}\in\mathcal{C}[c]} |\lambda_{\mathfrak{a}}(n)\overline{\lambda_{\mathfrak{a}}(m)}| \tag{5.32}$$

$$\leq \sum_{0<|l|<Y^{1+\epsilon}} \sum_{c|q} \sharp\mathcal{C}[c] \frac{|\lambda_{\phi,+}(l)|}{\sqrt{|l|}} \sum_{n\in\mathbb{N}, m=n+\tilde{d}_c l} |I_\phi(\tilde{d}_c l, n, d_c Y)\lambda_{[c]}(n)\lambda_{[c]}(m)|.$$

Note that $\tilde{d}_c = \frac{d_c}{d'_c}$ differs from d_c by d'_c which is clearly bounded and negligible for asymptotic behavior. Then one can argue similarly from this point on to prove Theorem 1.3 as in [18]. We will not give further details. The key point here is a control for the Fourier coefficients of a Maass eigencuspform of level N as in Corollary 2.1.

When ϕ is an incomplete Eisenstein series, one can decompose it into residue spectrum and continuous spectrums as in (2.29), and proceed as in the Maass eigencuspform case. The key point will again be a control of the Fourier coefficients for an Eisenstein series of level N, which follows directly from Lemma 5.1.

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