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Waldspurger's period integral for newforms

by

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1. Introduction. In this paper we study Waldspurger's local period integral for newforms in new cases. Suppose that $p \neq 2$. Let π be a smooth irreducible unitary representation of GL_2 over a p-adic field $\mathbb{F}$ with central character w_π, and χ be any character over a quadratic field extension $\mathbb{E}/\mathbb{F}$ such that $w_\pi \chi|_{\mathbb{F}^\times} = 1$. For any test vectors $\varphi, \varphi' \in \pi$, denote the local Waldspurger's period integral against a character χ on $\mathbb{E}^\times$ by

$$\{\varphi, \varphi'\} = \int_{t \in \mathbb{F}^\times \backslash \mathbb{E}^\times} \Phi_{\varphi,\varphi'}(t)\chi(t)\, dt. \tag{1.1}$$

Here $\Phi_{\varphi,\varphi'}(t) = (\pi(t)\varphi, \varphi')$ is the matrix coefficient associated to φ, φ'. We are particularly concerned with the case when π is a supercuspidal representation and φ, φ' are proper translates of the newform for a fixed embedding of $\mathbb{E}$ (or equivalently φ is a fixed newform and the embedding of $\mathbb{E}$ can vary).

The explicit knowledge of the local period integral and the particular choice of test vectors have been useful in analytic number theory as well as for arithmetic geometry problems. For example, they were used to study the moments and the subconvexity bound of L-functions in [FMP17] and [Wu]. In [CST14], they were used to give general explicit Gross–Zagier and Waldspurger formulae, which is important when attacking the refined BSD conjecture. In the most recent example, we used the explicit Gross–Zagier formula in [HSY] to study the 3-part full BSD conjecture for elliptic curves related to the Sylvester conjecture. A byproduct of this paper is a special example of a local period integral which is not covered by the previous literature, and is used in [HSY] to establish the explicit Gross–Zagier formula there.

1.1. A short history of local test vectors. [GP91] was the first work to consider the test vector problem for Waldspurger's period integrals with

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ramifications. It assumes disjoint ramification, and describes a test vector in terms of invariance under proper compact subgroups. Many years later [FMP17] gave test vectors in more general situations on the GL_2 side. In particular, the case when $\mathbb{E}$ is split was solved. When $\mathbb{E}$ is a quadratic field extension, the method can deal with the range $c(\pi_\chi) > c(\pi)$ (here π_χ is the representation of GL_2 associated to χ via the Theta correspondence or Langlands correspondence, and $c(\pi)$ is the exponent of the conductor of π) and the test vectors used are diagonal translates of newforms.

The recent work [HN] provides test vectors for the complementary range $c(\pi_\chi) \leq c(\pi)$ when π is supercuspidal under mild assumptions, using a new type of test vectors called minimal vectors. A particular minimal vector is given in the Kirillov model in Lemma 2.11 below, and any single translate of this element is still considered to be a minimal vector. Such test vectors arise naturally from compact induction theory for supercuspidal representations and have better properties than the standard newforms in terms of their matrix coefficients and Whittaker functionals. For some applications however (in particular with [HSY] in mind), it is still necessary to understand the local period integral for the classical newforms. The purpose of this paper is thus to make use of [HN] to predict the local period integral for newforms.

1.2. Main results. The goal of this paper is mainly to present the method, rather than exhausting all the cases. So we restrict ourselves to a special setting to avoid lengthy discussions. We shall assume that π is associated to a character θ over a ramified extension $\mathbb{L}$ via compact induction theory with $c(\theta) = 2n$, $\theta|_{\mathbb{F}^\times} = w_\pi = 1$, $\mathbb{E} \simeq \mathbb{L}$ is also ramified, and $c(\pi_\chi) \leq c(\pi)$. We also assume $\epsilon(\pi_\mathbb{E} \times \chi) = 1$ so that $\mathrm{Hom}_{\mathbb{E}^\times}(\pi, \chi^{-1}) \neq 0$.

For simplicity we shall fix the embedding of $\mathbb{L}$ into M_2 by

$$x + y\sqrt{D'} \mapsto \begin{pmatrix} x & y \\ yD' & x \end{pmatrix}, \tag{1.2}$$

where $v(D') = 1$. Any other embedding differs by a conjugation, which is effectively equivalent to a single translate of a test vector.

Using the explicit Kirillov model for a particular minimal vector φ_0 in Lemma 2.11, we can write a newform as a sum of minimal vectors in Corollary 2.12. Correspondingly we can write the period integral for the newform as a sum of period integrals for minimal vectors in Corollary 3.1, which specializes in our setting to

$$\{\widetilde{\varphi_{\mathrm{new}}}, \widetilde{\varphi_{\mathrm{new}}}\} = \frac{1}{(q-1)q^{\lceil n/2 \rceil - 1}} \sum_{x, x' \in (\mathcal{O}_\mathbb{F}/\varpi^{\lceil n/2 \rceil}\mathcal{O}_\mathbb{F})^\times} \{\varphi_x, \varphi_{x'}\}. \tag{1.3}$$

Here

$$\widetilde{\varphi_{\mathrm{new}}} = \pi\left(\begin{pmatrix} \varpi^n & 0 \\ 0 & 1 \end{pmatrix}\right)\varphi_{\mathrm{new}}, \qquad \varphi_x = \pi\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\right)\varphi_0$$

are diagonal translates of the standard newform and φ_0 respectively. The work in [HN] computed the diagonal terms $\{\varphi_x, \varphi_x\}$ which are either 0 or some constant only depending on the associated conductors.

When there is a single diagonal term $\{\varphi_x, \varphi_x\}$ in the family which is non-vanishing, one can use multiplicity 1 of χ-equivariant functionals (as in Theorem 2.4) to easily get the following

Proposition 1.1 (Proposition 3.3). *If $\{\varphi_x, \varphi_x\} \neq 0$ for a single $x \in (\mathcal{O}_{\mathbb{F}}/\varpi^{\lceil \frac{n}{2} \rceil}\mathcal{O}_{\mathbb{F}})^\times$, then all off-diagonal terms vanish and*

$$\{\widetilde{\varphi_{\mathrm{new}}}, \widetilde{\varphi_{\mathrm{new}}}\} = \frac{1}{(q-1)q^{\lceil n/2 \rceil - 1}}\{\varphi_x, \varphi_x\}. \tag{1.4}$$

The more challenging case is when there are several non-vanishing diagonal terms. The corresponding off-diagonal terms will also be non-vanishing and have the same absolute value as the diagonal terms by Lemma 3.2. The main innovation of this paper is that we devise a way to detect the phase factor for the off-diagonal terms with relatively simple computations. Roughly speaking, the sizes of the non-vanishing off-diagonal terms are the same as those for the non-vanishing diagonal terms (computed in [HN]). The support of the integral (see Definition 3.5) can also be computed directly. By comparing the support of the integral with the size of the integral, we shall see that the integrand must be constant on that support. This constant is the phase factor and can be easily detected by taking special values.

In particular we obtain the following result.

Proposition 1.2 (Proposition 3.7). *Suppose that $\mathbb{E} \simeq \mathbb{L}$ are ramified, $c(\theta) = 2n$, $w_\pi = 1$, $\epsilon(\pi_{\mathbb{E}} \times \chi) = 1$, and $0 < c(\theta\overline{\chi}) = 2l \leq 2n$ with $n - l$ even. Then*

$$\{\widetilde{\varphi_{\mathrm{new}}}, \widetilde{\varphi_{\mathrm{new}}}\} = \frac{1}{(q-1)q^{\lceil n/2 \rceil - 1}} \frac{1}{q^{\lfloor l/2 \rfloor}}(1 + \theta\chi(\sqrt{D}))^2.$$

Remark 1.3. (1) Note that in this setting, the conductor for the associated Rankin–Selberg L-function $C(\pi \times \pi_\chi)$ equals $C(\pi_{\theta\chi})C(\pi_{\theta\overline{\chi}}) = q^{2n+2l+2}$. On the other hand, $\theta\chi(\sqrt{D}) = \pm 1$ in our setting. Thus $\{\widetilde{\varphi_{\mathrm{new}}}, \widetilde{\varphi_{\mathrm{new}}}\}$ is either 0 or asymptotically $\frac{1}{C(\pi \times \pi_\chi)^{1/4}}$ (up to a bounded power of q).

The relation of the local period integral to the convexity bound of the corresponding L-function has also been observed before for the Rankin–Selberg L-function and the triple product L-function. It can be used to study the subconvexity bound of the associated L-function or the mass equidistribution problem if desired. See [Hu18, Section 1.2] for more details.

(2) One feature used in our particular setting in Proposition 1.2 is that there are exactly two non-vanishing diagonal terms in the expansion (1.3). According to [HN], this is true in most of the other cases.

(3) Unlike the range $c(\pi_\chi) > c(\pi)$, we see here the additional obstruction to using diagonal translates of newforms.

(4) We expect the same strategy to work for more general cases.

We point out that both Proposition 3.3 and Proposition 3.7 are new cases not known before. As a motivation, we give the local period integrals coming from a special arithmetic setting, where Proposition 3.3 alone turns out to be enough. Such a result, i.e., Corollary 4.6, is one of the key ingredients used in [HSY] to discuss the 3-part full BSD conjecture.

2. Notations and preliminary results

2.1. Notations and basics. For a real number a, let $\lfloor a \rfloor$ be the largest integer $\leq a$, and $\lceil a \rceil$ be the smallest integer $\geq a$.

Let $\mathbb{F}$ be a p-adic field with residue field of order q, uniformizer $\varpi = \varpi_\mathbb{F}$, ring of integers $\mathcal{O}_\mathbb{F}$, maximal ideal $\mathfrak{p}_\mathbb{F}$ and p-adic valuation $v_\mathbb{F}$. Let ψ be an additive character of $\mathbb{F}$. Assume that $2 \nmid q$. For $n \geq 1$, let $U_\mathbb{F}(n) = 1 + \mathfrak{p}_\mathbb{F}^n$. Let π be a supercuspidal representation of $\mathrm{GL}_2(\mathbb{F})$ with central character $w_\pi = 1$.

Let $\mathbb{L}$ be a quadratic field extension over $\mathbb{F}$. Let $e_\mathbb{L} = e(\mathbb{L}/\mathbb{F})$ be the ramification index and $v_\mathbb{L}$ be the valuation on $\mathbb{L}$. Let $\varpi_\mathbb{L}$ be a uniformizer for $\mathbb{L}$. When $\mathbb{L}$ is unramified we shall identify $\varpi_\mathbb{L}$ with $\varpi_\mathbb{F}$. Otherwise we suppose that $\varpi_\mathbb{L}^2 = \varpi_\mathbb{F}$. Let $x \mapsto \overline{x}$ be the unique non-trivial involution of $\mathbb{L}/\mathbb{F}$. Let $\psi_\mathbb{L} = \psi \circ \mathrm{Tr}_{\mathbb{L}/\mathbb{F}}$. One can make similar definitions for a possibly different quadratic field extension $\mathbb{E}$. Note that we shall assume that $\varpi_\mathbb{E}^2 = \xi \varpi_\mathbb{F}$ for $\xi \in \mathcal{O}_\mathbb{F}^\times - (\mathcal{O}_\mathbb{F}^\times)^2$ if $\mathbb{E}$ and $\mathbb{L}$ are both ramified and distinct.

For χ a multiplicative character on $\mathbb{F}^\times$, let $c(\chi)$ be the smallest integer such that χ is trivial on $1 + \mathfrak{p}_\mathbb{F}^{c(\chi)}$. Similarly, $c(\psi)$ is the smallest integer such that ψ is trivial on $\mathfrak{p}_\mathbb{F}^{c(\psi)}$. We choose ψ to be unramified, or equivalently $c(\psi) = 0$. Then $c(\psi_\mathbb{L}) = -e_\mathbb{L} + 1$. Let $c(\pi)$ be the power of the conductor of π.

When χ is a character over a quadratic extension, denote $\overline{\chi}(x) = \chi(\overline{x})$.

LEMMA 2.1. *For a multiplicative character ν over a general p-adic field $\mathbb{F}$ with $c(\nu) \geq 2$, there exists $\alpha_\nu \in \mathbb{F}^\times$ with $v_\mathbb{F}(\alpha_\nu) = -c(\nu) + c(\psi)$ such that*

$$\nu(1+u) = \psi(\alpha_\nu u) \tag{2.1}$$

for any $u \in \mathfrak{p}_\mathbb{F}^{\lceil c(\nu)/2 \rceil}$. Moreover, the element α_ν is uniquely determined modulo $\mathfrak{p}_\mathbb{F}^{-\lceil c(\nu)/2 \rceil + c(\psi)}$.

One can easily check this lemma by using the fact that $u \mapsto \nu(1+u)$ becomes an additive character for $u \in \mathfrak{p}_{\mathbb{F}}^{\lceil c(\nu)/2 \rceil}$.

REMARK 2.2. Note that Lemma 2.1 will be applied to characters over quadratic extensions $\mathbb{E}$, $\mathbb{L}$. Therefore we keep $c(\psi)$ in the formulation.

We shall also need some basic results on compact induction and matrix coefficients. In general, let G be a unimodular locally profinite group with center Z. Let $H \subset G$ be an open and closed subgroup containing Z with H/Z compact. Let ρ be an irreducible smooth representation of H with unitary central character and

$$\begin{aligned}\pi' &= c\text{-Ind}_H^G(\rho) \\ &= \left\{ f : G \to \rho \,\middle|\, \begin{array}{l} f(hg) = \rho(h) f(g)\ \forall h \in H, \\ f \text{ is smooth and compactly supported mod } H \end{array} \right\}.\end{aligned}$$

By the assumption on H/Z, ρ is automatically unitarizable, and we shall denote the unitary pairing on ρ by $\langle \cdot, \cdot \rangle_\rho$. Then one can define a unitary pairing on π' by

$$\langle \phi, \psi \rangle = \sum_{x \in H \backslash G} \langle \phi(x), \psi(x) \rangle_\rho. \tag{2.2}$$

If we let $y \in H\backslash G$ and $\{v_i\}$ be a basis for ρ, the elements

$$f_{y,v_i}(g) = \begin{cases} \rho(h) v_i & \text{if } g = hy \in Hy, \\ 0 & \text{otherwise} \end{cases} \tag{2.3}$$

form a basis for π'.

LEMMA 2.3. *For $y, z \in H\backslash G$,*

$$\langle \pi'(g) f_{y,v_i}, f_{z,v_j} \rangle = \begin{cases} \langle \rho(h) v_i, v_j \rangle_\rho & \text{if } g = z^{-1} h y \in z^{-1} H y, \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

Recall that π is a smooth irreducible representation of $\mathrm{GL}_2(\mathbb{F})$. Let $\mathbb{B}$ be a quaternion algebra over $\mathbb{F}$, and we set $\epsilon(\mathbb{B}) = 1$ resp. -1 if $\mathbb{B}$ is a matrix algebra resp. a division algebra. We also recall the multiplicity-1 result and Tunnell–Saito's epsilon-value test.

THEOREM 2.4 ([Tun83], [Sai93]). *Suppose that $w_\pi = \chi|_{\mathbb{F}^\times}$. Let $\pi^{\mathbb{B}}$ be the image of π under the Jacquet–Langlands correspondence. The space*

$$\mathrm{Hom}_{\mathbb{E}^\times}(\pi^{\mathbb{B}} \otimes \chi^{-1}, \mathbb{C})$$

is at most one-dimensional. It is non-zero if and only if

$$\epsilon(\pi_{\mathbb{E}} \times \chi^{-1}) = \chi(-1)\epsilon(\mathbb{B}). \tag{2.5}$$

Here $\pi_{\mathbb{E}}$ is the base change of π to $\mathbb{E}$.

2.2. Compact induction theory for supercuspidal representations and minimal vectors. We shall review the compact induction theory for supercuspidal representations on GL_2. For more details, see [BH06].

We shall fix the embeddings and work out everything explicitly. For $v_{\mathbb{F}}(D') = 0$ or 1, we shall refer to the following embedding of a quadratic field extension $\mathbb{L} = \mathbb{F}(\sqrt{D'})$ as the *standard embedding*:

$$x + y\sqrt{D'} \mapsto \begin{pmatrix} x & y \\ yD' & x \end{pmatrix}. \tag{2.6}$$

The supercuspidal representations are parametrized via compact induction by characters θ over some quadratic field extension $\mathbb{L}$. Twisting by a character if necessary, we may assume without loss of generality that π is twist-minimal, in the sense that $c(\pi) \leq c(\pi \otimes \nu)$ for any character ν of $\mathbb{F}^\times$. In that case we have the following quick guide.

- $c(\pi) = 2n + 1$ corresponds to $e_{\mathbb{L}} = 2$ and $c(\theta) = 2n$.
- $c(\pi) = 4n$ corresponds to $e_{\mathbb{L}} = 1$ and $c(\theta) = 2n$.
- $c(\pi) = 4n + 2$ corresponds to $e_{\mathbb{L}} = 1$ and $c(\theta) = 2n + 1$.

Definition 2.5. For $e_{\mathbb{L}} = 1, 2$, let

$$\mathfrak{A}_{e_{\mathbb{L}}} = \begin{cases} M_2(\mathcal{O}_{\mathbb{F}}) & \text{if } e_{\mathbb{L}} = 1, \\ \begin{pmatrix} \mathcal{O}_{\mathbb{F}} & \mathcal{O}_{\mathbb{F}} \\ \mathfrak{p}_{\mathbb{F}} & \mathcal{O}_{\mathbb{F}} \end{pmatrix} & \text{otherwise.} \end{cases}$$

Its Jacobson radical is given by

$$\mathcal{B}_{e_{\mathbb{L}}} = \begin{cases} \varpi M_2(\mathcal{O}_{\mathbb{F}}) & \text{if } e_{\mathbb{L}} = 1, \\ \begin{pmatrix} \mathfrak{p}_{\mathbb{F}} & \mathcal{O}_{\mathbb{F}} \\ \mathfrak{p}_{\mathbb{F}} & \mathfrak{p}_{\mathbb{F}} \end{pmatrix} & \text{otherwise.} \end{cases}$$

Define the filtrations of compact open subgroups as follows:

$$K_{\mathfrak{A}_{e_{\mathbb{L}}}}(n) = 1 + \mathcal{B}_{e_{\mathbb{L}}}^n, \qquad U_{\mathbb{L}}(n) = 1 + \varpi_{\mathbb{L}}^n \mathcal{O}_{\mathbb{L}}. \tag{2.7}$$

Note that each $K_{\mathfrak{A}_{e_{\mathbb{L}}}}(n)$ is normalized by $\mathbb{L}^\times$, which is embedded as in (2.6).

Denote

$$J = \mathbb{L}^\times K_{\mathfrak{A}_{e_{\mathbb{L}}}}(\lfloor c(\theta)/2 \rfloor),$$
$$J^1 = U_{\mathbb{L}}(1) K_{\mathfrak{A}_{e_{\mathbb{L}}}}(\lfloor c(\theta)/2 \rfloor),$$
$$H^1 = U_{\mathbb{L}}(1) K_{\mathfrak{A}_{e_{\mathbb{L}}}}(\lceil c(\theta)/2 \rceil).$$

Then θ on $\mathbb{L}^\times$ can extended to a character $\tilde{\theta}$ on H^1 by

$$\tilde{\theta}(l(1 + x)) = \theta(l)\psi \circ \mathrm{Tr}(\alpha_\theta x), \tag{2.8}$$

where $l \in \mathbb{L}^\times$, $1+x \in K_{\mathfrak{A}_{e_{\mathbb{L}}}}(\lceil c(\theta)/2 \rceil)$ and $\alpha_\theta \in \mathbb{L}^\times \subset M_2(\mathbb{F})$ is associated to θ by Lemma 2.1 under the fixed embedding.

When $c(\theta)$ is even, $H^1 = J^1$ and $\tilde{\theta}$ can be further extended to J by the same formula. In this case denote $\Lambda = \tilde{\theta}$ and $\pi = c\text{-Ind}_J^G \Lambda$.

When $c(\theta)$ is odd, J^1/H^1 is a two-dimensional vector space over the residue field. This case only occurs when $c(\pi) = 4n+2$ as listed above. For simplicity consider the case $n \geq 1$. Then there exists a q-dimensional representation Λ of J such that $\Lambda|_{H^1}$ is a multiple of $\tilde{\theta}$, and $\Lambda|_{\mathbb{L}^\times} = \oplus\theta\nu$ where ν is over $\mathbb{L}^\times$, $c(\nu) = 1$ and $\nu|_{\mathbb{F}^\times} = 1$. More specifically, let B^1 be any intermediate group between J^1 and H^1 such that B^1/H^1 gives a polarization of J^1/H^1 under the pairing given by

$$(1+x, 1+y) \mapsto \psi \circ \mathrm{Tr}(\alpha_\theta[x,y]).$$

Then $\tilde{\theta}$ can be extended to B^1 by the same formula (2.8) and $\Lambda|_{J^1} = \mathrm{Ind}_{B^1}^{J^1} \tilde{\theta}$. Again $\pi = c\text{-Ind}_J^G \Lambda$ in this case.

In the case $J^1 = H^1$, we take $B^1 = J$ for uniformity. In either case, we have $w_\pi = \theta|_{\mathbb{F}^\times}$. We recall the following result which comes naturally from compact induction theory in [BH06] and was made explicit in [HN].

Lemma 2.6. *There exists a unique (up to a constant) element $\varphi_0 \in \pi$ such that B^1 acts on it by $\tilde{\theta}$ (a Type 1 minimal vector in the terminology of* [HN]). *We call any single translate $\pi(g)\varphi_0$ a* minimal vector.

Lemma 2.7. *Let $\Phi_{\varphi_0} = \Phi_{\varphi_0,\varphi_0}$ be the matrix coefficient associated to a minimal vector φ_0 as above. Then Φ_{φ_0} is supported on J, and*

$$\Phi_{\varphi_0}(bx) = \Phi_{\varphi_0}(xb) = \tilde{\theta}(b)\Phi_{\varphi_0}(x) \quad \text{for any } b \in B^1. \tag{2.9}$$

Furthermore when $\dim \Lambda \neq 1$, *$\Phi_{\varphi_0}|_{J^1}$ is supported only on B^1.*

Remark 2.8. (1) Note that φ_0 is basically f_{1,v_i} as in (2.3) for the coset representative $1 \in J \backslash G$. Lemma 2.7 follows immediately from the definition of φ_0 and Lemma 2.3.

(2) The conjugated group gB^1g^{-1} acts on $\pi(g)\varphi_0$ by the conjugated character $\tilde{\theta}^g$.

(3) When $c(\theta)$ is even, the set $\{\pi(g)\varphi_0 \mid g \in G/J\}$ forms an orthonormal basis of π. When $c(\theta)$ is odd, any orthonormal basis of Λ can give rise to an orthonormal basis of π in a similar fashion. See Type 1 and Type 2 minimal vectors in [HN] as two examples.

2.3. Local Langlands correspondence and compact induction. Here we describe the relation between the compact induction parametrization and the local Langlands correspondence. See [BH06, Section 34] for more details.

For a field extension $\mathbb{L}/\mathbb{F}$ and an additive character ψ over $\mathbb{F}$, let $\lambda_{\mathbb{L}/\mathbb{F}}(\psi)$ be the Langlands λ-function in [Lan]. When $\mathbb{L}/\mathbb{F}$ is a quadratic field extension, let $\eta_{\mathbb{L}/\mathbb{F}}$ be the associated quadratic character. By [Lan, Lemma 5.1], for $\psi_\beta(x) = \psi(\beta x)$ we have

$$\lambda_{\mathbb{L}/\mathbb{F}}(\psi_\beta) = \eta_{\mathbb{L}/\mathbb{F}}(\beta)\lambda_{\mathbb{L}/\mathbb{F}}(\psi). \tag{2.10}$$

Definition 2.9. (1) If $\mathbb{L}$ is inert, define Δ_θ to be the unique unramified character of $\mathbb{L}^\times$ of order 2.

(2) If $\mathbb{L}$ is ramified and θ is a character over $\mathbb{L}$ with $c(\theta) > 0$ even, associate α_θ to θ as in Lemma 2.1. Then define Δ_θ to be the unique level 1 character of $\mathbb{L}^\times$ such that

$$\Delta_\theta|_{\mathbb{F}^\times} = \eta_{\mathbb{L}/\mathbb{F}}, \qquad \Delta_\theta(\varpi_\mathbb{L}) = \eta_{\mathbb{L}/\mathbb{F}}(\varpi_\mathbb{L}^{c(\theta)-1}\alpha_\theta)\lambda_{\mathbb{L}/\mathbb{F}}^{c(\theta)-1}(\psi). \tag{2.11}$$

Note that in [BH06], ψ is chosen to be of level 1. We have adapted the formula there to our choice of ψ using (2.10). The definition is also independent of the choice of $\varpi_\mathbb{L}$.

Theorem 2.10. *If π is associated by compact induction to a character θ over a quadratic extension $\mathbb{L}$, then its associated Deligne–Weil representation by local Langlands correspondence is $\sigma = \mathrm{Ind}_\mathbb{L}^\mathbb{F}(\Theta)$, where $\Theta = \theta\Delta_\theta^{-1}$, or equivalently $\theta = \Theta\Delta_\Theta$.*

Note here that Θ and θ always differ by a level ≤ 1 character, so α_Θ can be chosen to be the same as α_θ in Lemma 2.1 and $\Delta_\Theta = \Delta_\theta$.

2.4. Using minimal vectors for Waldspurger's period integral. Now we review the local Waldspurger's period integral for minimal vectors, in the setting $e_\mathbb{L} = 2$ and $\mathbb{E} \simeq \mathbb{L}$. For more details and proofs, see [HN], and in particular its appendix for explicit treatment. Recall that

$$\{\varphi, \varphi'\} = \int\limits_{t\in\mathbb{F}^\times\backslash\mathbb{E}^\times} \Phi_{\varphi,\varphi'}(t)\chi(t)\,dt.$$

When $\varphi' = \varphi$, we use the notation $I(\varphi, \chi) = \{\varphi, \varphi\}$, which also keeps track of χ.

According to [Hu, Section 2.4], by twisting π and χ if necessary, we can assume without loss of generality that $c(w_\pi) \leq 1$. As a result we can further assume that α_θ is imaginary (i.e. $\overline{\alpha_\theta} = -\alpha_\theta$) for the purpose of defining the character $\widetilde{\theta}$ in (2.8). We shall make these two assumptions from now on.

For simplicity we pick

$$D' = \frac{1}{\alpha_\theta^2\varpi_\mathbb{L}^{2c(\theta)}},$$

and identify $\frac{1}{\alpha_\theta\varpi_\mathbb{L}^{c(\theta)}}$ with $\sqrt{D'}$ and $\mathbb{L}$ with $\mathbb{F}(\sqrt{D'})$. With this choice, we have $v_\mathbb{F}(D') = 0$ if $e_\mathbb{L} = 1$ and $v_\mathbb{F}(D') = 1$ if $e_\mathbb{L} = 2$. By the standard embedding

(2.6), we have

$$\alpha_\theta = \frac{1}{\varpi_{\mathbb{L}}^{c(\theta)}}\frac{1}{\sqrt{D'}} \mapsto \frac{1}{\varpi^{c(\theta)/e_{\mathbb{L}}}}\begin{pmatrix} 0 & 1/D' \\ 1 & 0 \end{pmatrix}. \tag{2.12}$$

This choice is not essential. A different choice will result in a slightly different formulation, for example, in (2.17), but the final results are similar.

Note that when $e_{\mathbb{L}} = 2$, $c(\theta)$ must be even when $\theta|_{\mathbb{F}^\times} = 1$. Assume that $\mathbb{E} = \mathbb{F}(\sqrt{D})$ for $v_{\mathbb{F}}(D) = 0, 1$ is also embedded as

$$x + y\sqrt{D} \mapsto \begin{pmatrix} x & y \\ yD & x \end{pmatrix}. \tag{2.13}$$

In [HN], test vectors of the form $\pi(g)\varphi_0$ were used to study $\{\varphi, \varphi\}$ for general combinations of π, $\mathbb{E}$ and χ. For the purpose of this paper we shall only review the case when $\mathbb{L} \simeq \mathbb{E}$ are ramified, $w_\pi = 1$, $c(\pi) = 2n + 1$ is odd and $c(\pi_\chi) \le c(\pi)$. We shall normalize the Haar measure on $\mathbb{E}^\times$ so that $\mathrm{Vol}(\mathcal{O}_{\mathbb{F}}^\times \setminus \mathcal{O}_{\mathbb{E}}^\times) = 1$. Then $\mathrm{Vol}(\mathbb{F}^\times \setminus \mathbb{E}^\times) = 2$.

In this case we use test vectors of the form

$$\varphi = \pi\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}\right)\varphi_0$$

for some $u \in \mathcal{O}_{\mathbb{F}}$ and $v \in \mathcal{O}_{\mathbb{F}}^\times$. Recall that when $e_{\mathbb{L}} = 2$,

$$K_{\mathfrak{A}_{e_{\mathbb{L}}}}(n) = 1 + \begin{pmatrix} \mathfrak{p}_{\mathbb{F}}^{\lceil n/2 \rceil} & \mathfrak{p}_{\mathbb{F}}^{\lfloor n/2 \rfloor} \\ \mathfrak{p}_{\mathbb{F}}^{\lfloor n/2 \rfloor + 1} & \mathfrak{p}_{\mathbb{F}}^{\lceil n/2 \rceil} \end{pmatrix},$$

and $J = \mathbb{L}^\times K_{\mathfrak{A}_{e_{\mathbb{L}}}}(n)$ acts on φ_0 by a character, so we can assume that $v \in (\mathcal{O}_{\mathbb{F}}/\varpi^{\lceil n/2 \rceil}\mathcal{O}_{\mathbb{F}})^\times$.

There are two situations depending on $\min\{c(\theta\chi), c(\theta\overline{\chi})\}$. Note that for the embedding of $\mathbb{E}$ fixed above, we have

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x & y \\ yD & x \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x & -y \\ -yD & x \end{pmatrix} \tag{2.14}$$

and thus

$$I(\varphi, \chi) = I\left(\pi\left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right)\varphi, \overline{\chi}\right). \tag{2.15}$$

So we shall always assume that $c(\theta\overline{\chi}) \le c(\theta\chi)$.

The first situation is when $c(\theta\overline{\chi}) = 0$; then the Tunnell–Saito test requires $\theta\overline{\chi}$ to be trivial for $I(\varphi, \chi)$ to be non-zero (see [HN, Section A.5]). In that case we can take $u = 0$. Then according to [HN, Section B.3] there is a unique v mod $\varpi^{\lceil n/2 \rceil}$ such that $I(\varphi, \chi) \neq 0$, and for this v we have

$$I(\varphi, \chi) = \mathrm{Vol}(\mathbb{F}^\times \setminus \mathbb{E}^\times) = 2. \tag{2.16}$$

The second situation is when $0 < c(\theta_{\overline{\chi}}) = 2l \le 2n$. In this case, we may use Lemma 2.1 to associate with $\theta_{\overline{\chi}}$ an element $\alpha_{\theta_{\overline{\chi}}} \in \mathbb{F}^\times$. By [HN, Section B.2.3], φ would be a test vector if v, u are solutions of the quadratic equation

$$\frac{D}{D'}v^2 - \left(2\varpi^n\alpha_{\theta_{\overline{\chi}}}\sqrt{D} - 2\sqrt{\frac{D}{D'}}\right)v + (1 - Du^2) \equiv 0 \bmod \varpi^{n-\lfloor l/2\rfloor}. \tag{2.17}$$

This implies that for fixed u, the discriminant of the equation,

$$\Delta(u) = 4\varpi^n\alpha_{\theta_{\overline{\chi}}}\sqrt{D}\left(\varpi^n\alpha_{\theta_{\overline{\chi}}}\sqrt{D} - 2\sqrt{\frac{D}{D'}}\right) + 4\frac{D}{D'}Du^2, \tag{2.18}$$

has to be a square mod $\varpi^{n-\lfloor l/2\rfloor}$. When $n-l$ is even, we can pick $u=0$ directly. Whether $\Delta(0)$ is a square is consistent with Tunnell–Saito's test. When $\Delta(0)$ is indeed a square, we get two solutions of v mod $\varpi^{\lceil n/2\rceil}$. For each of these solutions we have

$$I(\varphi,\chi) = \frac{1}{q^{\lfloor l/2\rfloor}}. \tag{2.19}$$

Now if $n-l$ is odd, $v_{\mathbb{F}}(\Delta(0)) = n-l$ is odd, thus $\Delta(0)$ can never be a square. We need to pick u such that $v_{\mathbb{F}}(u) = \frac{n-l-1}{2}$ and $\Delta(0) + 4\frac{D}{D'}Du^2$ can be of higher evaluation and a square. Whether this is possible is again consistent with Tunnell–Saito's test. In this case it is possible to get more or less solutions of v mod $\varpi^{\lceil n/2\rceil}$. For each solution we have

$$I(\varphi,\chi) = \frac{1}{q^{\lfloor l/2\rfloor}}. \tag{2.20}$$

2.5. Kirillov model for minimal vectors. Here we describe the minimal vectors explicitly in the Kirillov model. For this purpose, we choose the special intermediate subgroup $B^1 = U_{\mathbb{L}}(1)K_{\mathfrak{A}_2}(2n+1)$ in the case $e_{\mathbb{L}} = 1$ and $c(\pi) = 4n+2$. Recall that we choose D' such that (see (2.12))

$$\alpha_\theta = \frac{1}{\varpi_{\mathbb{L}}^{c(\theta)}}\frac{1}{\sqrt{D'}} \mapsto \frac{1}{\varpi^{c(\theta)/e_{\mathbb{L}}}}\begin{pmatrix}0 & 1/D'\\ 1 & 0\end{pmatrix}. \tag{2.21}$$

From this, one can explicitly write down the character $\tilde{\theta}$ as in (2.8) and the matrix coefficient for the associated minimal vector as in Lemmas 2.6 and 2.7. We define the intertwining operator from π to its Whittaker model by

$$\varphi \mapsto W_\varphi(g) = \int_{\mathbb{F}} \Phi_{\varphi,\varphi_0}\left(\begin{pmatrix}\varpi^{\lfloor c(\pi)/2\rfloor} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}g\right)\psi(-n)\,dn. \tag{2.22}$$

A particular minimal vector was given in the Kirillov model in [HN] under this operator.

LEMMA 2.11. *Up to a constant multiple, the minimal vector φ_0 associated to B^1 and $\tilde{\theta}$ above is given in the Kirillov model by the following:*

(1) *When* $c(\pi)=4n$, $\varphi_0=\mathrm{Char}(\varpi^{-2n}U_{\mathbb{F}}(n))$.
(2) *When* $c(\pi)=2n+1$, $\varphi_0=\mathrm{Char}(\varpi^{-n}U_{\mathbb{F}}(\lceil n/2\rceil))$.
(3) *When* $c(\pi)=4n+2$, $\varphi_0=\mathrm{Char}(\varpi^{-2n-1}U_{\mathbb{F}}(n+1))$.

COROLLARY 2.12. *The newform φ_{new} is related to φ_0 by the formula*

$$\varphi_{\mathrm{new}}=\frac{1}{\sqrt{(q-1)q^{\lceil c(\theta)/(2e_{\mathbb{L}})\rceil-1}}}\sum_{x\in(\mathcal{O}_{\mathbb{F}}/\varpi^{\lceil c(\theta)/(2e_{\mathbb{L}})\rceil}\mathcal{O}_{\mathbb{F}})^\times}\pi\left(\begin{pmatrix}\varpi^{-c(\theta)/e_{\mathbb{L}}}x & 0\\ 0 & 1\end{pmatrix}\right)\varphi_0. \tag{2.23}$$

Here φ_0 and φ_{new} are both L^2-normalized.

Proof. By Lemma 2.11 one can uniformly write

$$\varphi_0=\sqrt{(q-1)q^{\lceil c(\theta)/(2e_{\mathbb{L}})\rceil-1}}\,\mathrm{Char}\left(\varpi^{-c(\theta)/e_{\mathbb{L}}}U_{\mathbb{F}}\left(\left\lceil\frac{c(\theta)}{2e_{\mathbb{L}}}\right\rceil\right)\right). \tag{2.24}$$

The coefficient comes from the L^2-normalization of φ_0, as

$$\mathrm{Vol}\left(U_{\mathbb{F}}\left(\left\lceil\frac{c(\theta)}{2e_{\mathbb{L}}}\right\rceil\right)\right)=\frac{1}{(q-1)q^{\lceil c(\theta)/(2e_{\mathbb{L}})\rceil-1}}.$$

Recall that for φ in the Kirillov model and $x\in\mathbb{F}^\times$, we have

$$\pi\left(\begin{pmatrix}x & 0\\ 0 & 1\end{pmatrix}\right)\varphi(y)=\varphi(xy).$$

Thus the right-hand side equals

$$\sum_{x\in(\mathcal{O}_{\mathbb{F}}/\varpi^{\lceil c(\theta)/(2e_{\mathbb{L}})\rceil}\mathcal{O}_{\mathbb{F}})^\times}\mathrm{Char}\left(x^{-1}U_{\mathbb{F}}\left(\left\lceil\frac{c(\theta)}{2e_{\mathbb{L}}}\right\rceil\right)\right)=\mathrm{Char}(\mathcal{O}_{\mathbb{F}}^\times).$$

Then one just has to recall that $\varphi_{\mathrm{new}}=\mathrm{Char}(\mathcal{O}_{\mathbb{F}}^\times)$ in the Kirillov model. ∎

3. Waldspurger's period integral using newforms. In the last section we reviewed the local Waldspurger's period integral for minimal vectors. In this section we show how to work out this period integral for newforms. Using the relation between the newform and the minimal vectors in Corollary 2.12, and the bilinearity of Waldspurger's period integral, we can write the integral for the newform as a sum of integrals for the minimal vectors. When there is a single non-vanishing term, we get the integral very easily in Section 3.1, which turns out to be enough for the special example in Section 4. We further illustrate how to evaluate the off-diagonal terms in Section 3.2 in more general cases for possible future applications.

As in the introduction, we denote

$$\{\varphi_1, \varphi_2\} = \int_{t\in\mathbb{F}^\times\backslash\mathbb{E}^\times} (\pi(t)\varphi_1, \varphi_2)\chi(t)\,dt \tag{3.1}$$

for the embedding of $\mathbb{E}$ as in (2.13). Using bilinearity and Corollary 2.12, we immediately obtain

COROLLARY 3.1. *Let*

$$\widetilde{\varphi_{\mathrm{new}}} = \pi\left(\begin{pmatrix} \varpi^{c(\theta)/e_{\mathbb{L}}} & 0 \\ 0 & 1 \end{pmatrix}\right)\varphi_{\mathrm{new}}, \qquad \varphi_x = \pi\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\right)\varphi_0.$$

Then

$$\begin{aligned}\{\widetilde{\varphi_{\mathrm{new}}}, \widetilde{\varphi_{\mathrm{new}}}\} \\ = \frac{1}{(q-1)q^{\lceil c(\theta)/(2e_{\mathbb{L}})\rceil-1}} \sum_{x,x'\in(\mathcal{O}_{\mathbb{F}}/\varpi^{\lceil c(\theta)/(2e_{\mathbb{L}})\rceil}\mathcal{O}_{\mathbb{F}})^\times} \{\varphi_x, \varphi_{x'}\}.\end{aligned} \tag{3.2}$$

LEMMA 3.2.

(1) *If* $\{\varphi_x, \varphi_x\} = 0$, *then* $\{\varphi_{x'}, \varphi_x\} = \{\varphi_x, \varphi_{x'}\} = 0$ *for any* x'.
(2) *If* $|\{\varphi_x, \varphi_x\}| = |\{\varphi_{x'}, \varphi_{x'}\}|$, *then* $|\{\varphi_x, \varphi_{x'}\}| = |\{\varphi_x, \varphi_x\}|$.

Proof. For any non-trivial functional $\mathcal{F} \in \mathrm{Hom}_{\mathbb{E}^\times}(\pi\otimes\chi, \mathbb{C})$, we have

$$\{\varphi_1, \varphi_2\} = C\mathcal{F}(\varphi_1)\overline{\mathcal{F}(\varphi_2)}$$

for some non-zero C independent of the test vectors, as $\dim \mathrm{Hom}_{\mathbb{E}^\times}(\pi\otimes\chi, \mathbb{C}) \le 1$. Then

$$\begin{aligned}&|\{\varphi_x, \varphi_x\}| = |C\mathcal{F}(\varphi_x)^2|, \qquad |\{\varphi_{x'}, \varphi_{x'}\}| = |C\mathcal{F}(\varphi_{x'})^2|, \\ &|\{\varphi_x, \varphi_{x'}\}| = |C\mathcal{F}(\varphi_x)\overline{\mathcal{F}(\varphi_{x'})}|.\end{aligned}$$

Now the results are clear. ∎

Note that the diagonal terms where $x = x'$ are already known by the results of [HN] recalled in Section 2.4.

3.1. The special case

PROPOSITION 3.3. *If* $\{\varphi_x, \varphi_x\} \neq 0$ *for a single* $x \in (\mathcal{O}_{\mathbb{F}}/\varpi^{\lceil c(\theta)/(2e_{\mathbb{L}})\rceil}\mathcal{O}_{\mathbb{F}})^\times$, *then all off-diagonal terms vanish and*

$$\{\widetilde{\varphi_{\mathrm{new}}}, \widetilde{\varphi_{\mathrm{new}}}\} = \frac{1}{(q-1)q^{\lceil c(\theta)/(2e_{\mathbb{L}})\rceil-1}}\{\varphi_x, \varphi_x\}. \tag{3.3}$$

Proof. It follows from Lemma 3.2 that all the off-diagonal terms vanish, and only a single diagonal term is non-vanishing. The proposition now follows directly from Corollary 3.1. ∎

REMARK 3.4. This case happens mostly when $c(\theta\chi) \le 1$ or $c(\theta\overline{\chi}) \le 1$. But there are other possibilities, as we shall see in the special example in Section 4.

3.2. The general cases. Now we explain how to work out the off-diagonal terms in Corollary 3.1 in more general cases.

DEFINITION 3.5. The *support* of the local Waldspurger's period integral $\{\varphi, \varphi'\}$ is the set $\mathbb{E}^\times \cap \operatorname{Supp} \Phi_{\varphi,\varphi'}$.

The main idea is as follows. On the one hand, the size of an off-diagonal term is given by Lemma 3.2, so it remains to figure out its phase. On the other hand, we shall find the support of the integral in Lemma 3.6 using the fact that the off-diagonal term corresponds to different solutions for (2.17). We shall see that the volume of the support is exactly the predicted size of the off-diagonal term. This forces the integrand to be constant (with absolute value 1) on the support of the integral, since the integrand is absolutely bounded by 1. Then one can easily detect the phase by looking at the value of the integrand at any point in the support of the integral.

For simplicity, however, we stay in the setting where $\mathbb{E} \simeq \mathbb{L}$ are ramified, $0 < c(\theta\overline{\chi}) = 2l \le 2n$. We further assume that $n - l$ is even. By Section 2.4, we can pick $u = 0$, and there exist two solutions v, v' mod $\varpi^{\lceil n/2 \rceil}$ to (2.17), while the diagonal terms are $1/q^{\lfloor l/2 \rfloor}$ for both solutions.

According to Lemma 3.2, we can write $\{\varphi_v, \varphi_{v'}\} = \gamma\{\varphi_v, \varphi_v\}$ for some phase factor γ with $|\gamma| = 1$. Then

$$\begin{aligned}(3.4)\qquad & I(\widetilde{\varphi_{\text{new}}}, \chi) \\ &= \frac{1}{(q-1)q^{\lceil c(\theta)/(2e_{\mathbb{L}}) \rceil - 1}} (\{\varphi_v, \varphi_v\} + \{\varphi_v, \varphi_{v'}\} + \{\varphi_{v'}, \varphi_v\} + \{\varphi_{v'}, \varphi_{v'}\}) \\ &= \frac{1}{(q-1)q^{\lceil c(\theta)/(2e_{\mathbb{L}}) \rceil - 1}} \frac{1}{q^{\lfloor l/2 \rfloor}} (1+\gamma)(1+\overline{\gamma}).\end{aligned}$$

To study γ, we first study the support of the integral. Recall that the minimal vector φ_v only depends on v mod $\varpi^{\lceil n/2 \rceil}$. By Hensel's lemma we can actually assume that v, v' satisfy the exact equality in (2.17) with $u = 0$, i.e.,

$$(3.5)\qquad \frac{D}{D'}v^2 - \left(2\varpi^n \alpha_{\theta\overline{\chi}}\sqrt{D} - 2\sqrt{\frac{D}{D'}}\right)v + 1 = 0.$$

Let $k = \left(\begin{smallmatrix} v & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $k' = \left(\begin{smallmatrix} v' & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then for $t = \left(\begin{smallmatrix} a & b \\ bD & a \end{smallmatrix}\right)$,

$$k'^{-1}tk = \begin{pmatrix} vv'^{-1}a & v'^{-1}b \\ vbD & a \end{pmatrix},$$

and

$$\{\varphi_v, \varphi_{v'}\} = \int\limits_{\mathbb{F}^\times\backslash\mathbb{E}^\times} \Phi_{\varphi_0}(k'^{-1}tk)\chi(t)\,dt. \tag{3.6}$$

LEMMA 3.6. *For the setting as above, we have* $vv'D = D'$, *and* $v_{\mathbb{F}}(v/v'-1) = (n-l)/2$. *The support of the integral is* $v_{\mathbb{F}}(b) = 0, v_{\mathbb{F}}(a) \geq \lceil (l+1)/2 \rceil$.

Proof. According to (3.5), v and v' satisfy

$$vv' = D'/D, \qquad v + v' = 2(\varpi^n \alpha_{\theta\overline{\chi}}\sqrt{D'} - 1)\sqrt{D'/D},$$

so the first result is direct. For the second result, we note that

$$\left(\frac{v}{v'} - 1\right)^2 = \frac{(v+v')^2 - 4vv'}{v'^2} = \frac{D'}{D}\,\frac{4\varpi^n\alpha_{\theta\overline{\chi}}\sqrt{D'}(\varpi^n\alpha_{\theta\overline{\chi}}\sqrt{D'} - 2)}{v'^2}. \tag{3.7}$$

Thus $v_{\mathbb{F}}((v/v'-1)^2) = n - l$ and $v_{\mathbb{F}}(v/v'-1) = (n-l)/2$. Now for

$$k'^{-1}tk \in J = \mathbb{L}^\times K_{\mathfrak{A}_{e_{\mathbb{L}}}}(n),$$

there are two cases to consider: either $v_{\mathbb{F}}(b) = 0$, $v_{\mathbb{F}}(a) > 0$ or $v_{\mathbb{F}}(a) = 0$, $v_{\mathbb{F}}(b) \geq 0$. In the first case, since

$$\varpi_{\mathbb{L}}\mathcal{B}^n_{e_{\mathbb{L}}} = \begin{pmatrix} \mathfrak{p}_{\mathbb{F}}^{\lceil (n+1)/2\rceil} & \mathfrak{p}_{\mathbb{F}}^{\lfloor (n+1)/2\rfloor} \\ \mathfrak{p}_{\mathbb{F}}^{\lfloor (n+1)/2\rfloor+1} & \mathfrak{p}_{\mathbb{F}}^{\lceil (n+1)/2\rceil} \end{pmatrix},$$

we must have

$$\begin{aligned} a\left(\frac{v}{v'} - 1\right) &\equiv 0 \bmod \varpi^{\lceil (n+1)/2\rceil}, \\ vbD - v'^{-1}bD' &\equiv 0 \bmod \varpi^{\lfloor (n+1)/2\rfloor+1}. \end{aligned} \tag{3.8}$$

The second relation is automatic as $vv' = D'/D$. From the first relation and the computation for $v_{\mathbb{F}}(v/v'-1)$ above, we get $v_{\mathbb{F}}(a) \geq \lceil (l+1)/2\rceil$. A similar argument shows that $k'^{-1}tk \in \mathbb{L}^\times K_{\mathfrak{A}_{e_{\mathbb{L}}}}(n)$ is impossible when $v_{\mathbb{F}}(a) = 0$ and $v_{\mathbb{F}}(b) \geq 0$. ∎

PROPOSITION 3.7. *Suppose that* $\mathbb{E} \simeq \mathbb{L}$ *are ramified*, $c(\theta) = 2n$, $w_\pi = 1$, $\epsilon(\pi_{\mathbb{E}} \times \chi) = 1$, *and* $0 < c(\theta\overline{\chi}) = 2l \leq 2n$ *with* $n - l$ *even. Then*

$$I(\widetilde{\varphi_{\text{new}}}, \chi) = \frac{1}{(q-1)q^{\lceil c(\theta)/(2e_{\mathbb{L}})\rceil - 1}}\,\frac{1}{q^{\lfloor l/2\rfloor}}(1 + \theta\chi(\sqrt{D}))^2.$$

Proof. We already know that $|\{\varphi_v, \varphi_{v'}\}| = 1/q^{\lfloor l/2\rfloor}$. By the previous lemma one can check that the support of the integral also has volume $1/q^{\lfloor l/2\rfloor}$, while the integrand satisfies $|\langle \pi(t)\varphi_v, \varphi_{v'}\rangle\chi(t)| \leq 1$. Then $\langle \pi(t)\varphi_v, \varphi_{v'}\rangle\chi(t)$ must be some constant γ on the whole support with $|\gamma| = 1$. To detect this constant we just have to take $t = \left(\begin{smallmatrix} 0 & 1 \\ D & 0 \end{smallmatrix}\right)$. Then

$$\gamma = \Phi_{\varphi_0}\left(\begin{pmatrix} 0 & v'^{-1} \\ vD & 0 \end{pmatrix}\right)\chi(\sqrt{D}) = \theta(v'^{-1}\sqrt{D'})\chi(\sqrt{D}) = \theta\chi(\sqrt{D}). \tag{3.9}$$

In the last equality we have used $\theta|_{\mathbb{F}^\times} = 1$ so that

$$\theta(v'^{-1}\sqrt{D'}) = \theta\left(v'^{-1}\frac{\sqrt{D'}}{\sqrt{D}}\sqrt{D}\right) = \theta(\sqrt{D}).$$

Note that $\theta\chi(\sqrt{D}) = \pm 1$. The claim now follows from (3.4). ∎

4. A special example from arithmetic geometry. Now we specialize to the case required in the proof of [HSY, Theorem 4.3]. We shall first review the global arithmetic setting, and use subscripts to indicate local components.

For an integer $n \neq 0$, let E_n be the elliptic curve defined by the affine equation $x^3 + y^3 = n$. Then E_n has complex multiplication by the field $K = \mathbb{Q}(\sqrt{-3})$. For a prime $p \equiv 4, 7 \bmod 9$, the elliptic curve E_p is closely related to the well-known Sylvester conjecture. Let π be the irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ corresponding to E_9, and π_3 the 3-adic local component of π. It is well-known that π (and hence π_3) is unitary with trivial central character. Since the elliptic curve E_9 and π have conductor 3^5, we see that $c(\pi_3) = 5$. Let $\chi : \mathrm{Gal}(\bar{K}/K) \to \mathcal{O}_K^\times$ be the character given by $\chi(\sigma) = (\sqrt[3]{3p})^{\sigma-1}$. We also view χ as a Hecke character on $\mathbb{A}_K^\times$ by the Artin map. From [HSY, Proposition 2.4], $c(\chi_3) = 4$. We embed K into $\mathrm{M}_2(\mathbb{Q})$ as in [HSY, Section 2] which linearly extends the map

$$\sqrt{-3} \mapsto \begin{pmatrix} 4p + 17 + 72/p & -8p/9 - 4 - 18/p \\ 18p + 72 + 288/p & -4p - 17 - 72/p \end{pmatrix} =: \begin{pmatrix} a & 3^{-2}b \\ 3^3c & -a \end{pmatrix} \tag{4.1}$$

with $3 \parallel a$ if $p \equiv 4 \bmod 9$, $9 \parallel a$ if $p \equiv 7 \bmod 9$, $b \equiv p \bmod 9$ and $c \equiv -1 \bmod 9$. Then $\mathrm{N}_{K/\mathbb{Q}}(\sqrt{-3}) = -a^2 - 3bc = 3$. Assume that f_3 is the standard newform of π_3. The normalized Waldspurger's period integral

$$\beta_3^0(f_3, f_3) = \int\limits_{t \in \mathbb{Q}_3^\times \backslash K_3^\times} \frac{(\pi(t)f_3, f_3)}{(f_3, f_3)}\chi_3(t)\, dt \tag{4.2}$$

appears in the proof of the explicit Gross–Zagier formula for E_p in [HSY], and it is explicitly computed in

PROPOSITION 4.1. *Suppose* $\mathrm{Vol}(\mathbb{Z}_3^\times \setminus \mathcal{O}_{K,3}^\times) = 1$ *so that* $\mathrm{Vol}(\mathbb{Q}_3^\times \setminus K_3^\times) = 2$. *Then*

$$\beta_3^0(f_3, f_3) = \begin{cases} 1 & \text{if } p \equiv 7 \bmod 9, \\ 1/2 & \text{if } p \equiv 4 \bmod 9. \end{cases} \tag{4.3}$$

The rest of the section is devoted to the proof of the above proposition. Note here that K is embedded differently from the fixed embedding used in (2.13). We choose the notation β_3^0 to reflect this difference, and to be consistent with the notation in [HSY]. We shall work out the relation of these two embeddings later on.

To apply the results of the previous sections to compute (4.2), we take $\varpi = 3 = q$, $D = -3$, $K_3 \simeq \mathbb{E} \simeq \mathbb{L} \simeq \mathbb{Q}(\sqrt{-3})_3$, $c(\theta_3) = c(\chi_3) = 4$, $n = 2$. By Lemma 2.11 we have the minimal vector $\varphi_0 = \mathrm{Char}(\varpi^{-2}U_{\mathbb{F}}(1))$ in the Kirillov model. As seen from previous sections, more accurate information is needed.

4.1. Local characters associated to arithmetic information. First of all we make use of the arithmetic information to give the local characters explicitly.

Recall that $K = \mathbb{Q}(\sqrt{-3})$ is an imaginary quadratic field and $\mathcal{O}_K = \mathbb{Z}[\omega]$ is its ring of integers with $\omega = (-1+\sqrt{-3})/2$. Let $\Theta : K^\times \setminus \mathbb{A}_K^\times \to \mathbb{C}^\times$ be the unitary Hecke character associated to the base-changed CM elliptic curve $E_{9/K}$. Then Θ has conductor $9\mathcal{O}_K$. For any place v of K, let Θ_v be the local component of Θ at v. Then Θ_v is the character used to construct the local Weil–Deligne representation in Theorem 2.10. We denote by Θ_3 the 3-adic local component of Θ. Then π_3 is the local representation of $\mathrm{GL}_2(\mathbb{Q}_3)$ corresponding to Θ_3. Note

$$\mathcal{O}_{K,3}^\times/(1+9\mathcal{O}_{K,3}) \simeq \langle \pm 1\rangle^{\mathbb{Z}/2\mathbb{Z}} \times \langle 1+\sqrt{-3}\rangle^{\mathbb{Z}/3\mathbb{Z}} \times \langle 1-\sqrt{-3}\rangle^{\mathbb{Z}/3\mathbb{Z}} \times \langle 1+3\sqrt{-3}\rangle^{\mathbb{Z}/3\mathbb{Z}}.$$

LEMMA 4.2. *We have $c(\Theta_3) = 4$. The values of Θ_3 are given explicitly by*

$$\Theta_3(-1) = -1, \quad \Theta_3(1+\sqrt{-3}) = \frac{-1-\sqrt{-3}}{2}, \quad \Theta_3(\sqrt{-3}) = i,$$
$$\Theta_3(1-\sqrt{-3}) = \frac{-1+\sqrt{-3}}{2}, \quad \Theta_3(1+3\sqrt{-3}) = \frac{-1+\sqrt{-3}}{2}.$$

Proof. It is well-known that $\Theta_\infty(x) = \|x\|/x$ (see for example [Sil94, Chapter II, Theorem 9.2] and note that we normalize it to make it unitary) and Θ is unramified when $3 \nmid v$. Note

$$\Theta_\infty(-1)\Theta_3(-1) = 1, \quad \Theta_\infty(-1) = -1.$$

So $\Theta_3(-1) = -1$.

Let $\mathfrak{p} = (a)$ be a prime of K relatively prime to 6, with the unique generator $a \equiv 2 \bmod 3$. By [Sil94, Chapter II, Example 10.6], we have

$$\Theta(\mathfrak{p}) = -\mathbb{N}_{\mathfrak{p}}^{-1/2}\overline{\left(\frac{-3}{a}\right)_6}\, a.$$

Here $\left(\frac{\cdot}{a}\right)_6$ is the sixth power residue symbol and $\mathbb{N}_\mathfrak{p}$ is the norm of $\mathfrak{p}$. If $\mathfrak{p} = (5)$, then

$$\Theta_5(5) = -\overline{\left(\frac{-3}{5}\right)_6} = -1.$$

From

$$\Theta_\infty(10)\Theta_2(10)\Theta_3(10)\Theta_5(10) = 1,$$

we get $\Theta_2(2) = -1$. Since Θ_2 is unramified and $1 + \sqrt{-3}$ is another uniformizer of the ideal (2), we see $\Theta_2(1 \pm \sqrt{-3}) = \Theta_2(2) = -1$. From

$$\Theta_\infty(1 \pm \sqrt{-3})\Theta_2(1 \pm \sqrt{-3})\Theta_3(1 \pm \sqrt{-3}) = 1, \quad \Theta_\infty(1 \pm \sqrt{-3}) = \frac{2}{1 \pm \sqrt{-3}},$$

we get

$$\Theta_3(1 \pm \sqrt{-3}) = \frac{-1 \mp \sqrt{-3}}{2}.$$

The other evaluations can be obtained in a similar way. ■

The local character χ_3 has conductor $\mathbb{Z}_3^\times(1 + 9\mathcal{O}_{K,3})$, and hence it is in fact a character of the quotient group $\mathcal{O}_{K,3}^\times/\mathbb{Z}_3^\times(1 + 9\mathcal{O}_{K,3})$. Note that

$$\mathcal{O}_{K,3}^\times/\mathbb{Z}_3^\times(1 + 9\mathcal{O}_{K,3}) \simeq \langle 1 + \sqrt{-3}\rangle^{\mathbb{Z}/3\mathbb{Z}} \times \langle 1 + 3\sqrt{-3}\rangle^{\mathbb{Z}/3\mathbb{Z}}.$$

We have the following:

LEMMA 4.3. *$c(\chi_3) = 4$ and $\chi_3|_{\mathbb{Q}_3^\times} = 1$. The values of χ_3 are given explicitly by the following table:*

p mod 9	$\chi_3(1 + \sqrt{-3})$	$\chi_3(1 + 3\sqrt{-3})$	$\chi_3(\sqrt{-3})$
4	ω	ω	1
7	ω^2	ω	1

Proof. This follows directly from the explicit local class field theory. We note that all the elements in $1 + 9\mathcal{O}_{K,3}$ are cubes in K_3. Hence for any $t \in K_3^\times$,

$$\chi_3(t) = \left(\sqrt[3]{3p}\right)^{\sigma_t - 1} = \left(\frac{t, 3p}{K_3; 3}\right) = \begin{cases} \left(\frac{t, 12}{K_3; 3}\right), & p \equiv 4 \bmod 9, \\ \left(\frac{t, 21}{K_3; 3}\right), & p \equiv 7 \bmod 9. \end{cases}$$

Recall that σ_t is the image of t under the Artin map, and $\left(\frac{\cdot,\cdot}{K_3;3}\right)$ denotes the third Hilbert symbol over $K_3^\times$ as in [HSY]. Using the local and global principle, it is straightforward to compute the values of χ_3 in the above table.

For example when $p \equiv 4 \bmod 9$,

$$\begin{aligned}\chi_3(1+\sqrt{-3}) &= \left(\frac{1+\sqrt{-3},3}{K_3;3}\right)\left(\frac{1+\sqrt{-3},4}{K_3;3}\right)\\ &= \left(\frac{1+\sqrt{-3},3}{K_2;3}\right)^{-1}\left(\frac{1+\sqrt{-3},4}{K_2;3}\right)^{-1} = \omega,\end{aligned}$$

where the second equality uses the fact that the product of the Hilbert symbol at all places equals 1 and the Hilbert symbol we consider is unramified outside 2 and 3; the third equality uses the formula in [Neu99, Chapter V, Proposition 3.4]. ∎

4.2. Local period integral. From Section 2.4, we see that the test vector issue for Waldspurger's period integral is closely related to $c(\theta_3\overline{\chi}_3)$ or $c(\theta_3\chi_3)$ and some further details like $\alpha_{\theta_3\overline{\chi}_3}$. We can work out these by using Lemmas 4.2 and 4.3, and the relation between θ_3 and Θ_3 as in Theorem 2.10.

Corollary 4.4. *If* $p \equiv 4 \bmod 9$*, then the local character* $\Theta_3\overline{\chi}_3$ *is given explicitly by*

$$\Theta_3\overline{\chi}_3(-1) = -1, \quad \Theta_3\overline{\chi}_3(1+\sqrt{-3}) = \omega,$$
$$\Theta_3\overline{\chi}_3(1-\sqrt{-3}) = \omega^2, \quad \Theta_3\overline{\chi}_3(1+3\sqrt{-3}) = 1, \quad \Theta_3\overline{\chi}_3(\sqrt{-3}) = i.$$

If $p \equiv 7 \bmod 9$*, the local character* $\Theta_3\overline{\chi}_3$ *is given explicitly by*

$$\Theta_3\overline{\chi}_3(-1) = -1, \quad \Theta_3\overline{\chi}_3(1+\sqrt{-3}) = 1,$$
$$\Theta_3\overline{\chi}_3(1-\sqrt{-3}) = 1, \quad \Theta_3\overline{\chi}_3(1+3\sqrt{-3}) = 1, \quad \Theta_3\overline{\chi}_3(\sqrt{-3}) = i.$$

Now we can prove the following key lemma in our special case.

Lemma 4.5. *When* $p \equiv 7 \bmod 9$*, we have* $\theta_3\overline{\chi}_3 = 1$*. When* $p \equiv 4 \bmod 9$*, we have* $c(\theta_3\overline{\chi}_3) = 2$ *and* $\alpha_{\theta_3\overline{\chi}_3} = \frac{1}{3\sqrt{-3}}$.

Proof. Let ψ_3 be the additive character such that $\psi_3(x) = e^{2\pi i\iota(x)}$ where $\iota : \mathbb{Q}_3 \to \mathbb{Q}_3/\mathbb{Z}_3 \subset \mathbb{Q}/\mathbb{Z}$ is given by $x \mapsto -x \bmod \mathbb{Z}_3$, which is compatible with the choice in [CST14]. Let $\psi_{K_3}(x) = \psi_3 \circ \mathrm{Tr}_{K_3/\mathbb{Q}_3}(x)$ be the additive character of K_3.

Recall that α_{Θ_3} is the number associated to Θ_3 as in Lemma 2.1 so that

$$\Theta_3(1+x) = \psi_{K_3}(\alpha_{\Theta_3}x)$$

for any x satisfying $v_{K_3}(x) \geq c(\Theta_3)/2 = 2$. By the definition of ψ_{K_3} and Lemma 4.2, we know that $\alpha_{\Theta_3} = 1/(9\sqrt{-3})$. Now let η_3 be the quadratic character associated to the quadratic field extension $K_3/\mathbb{Q}_3$. Then by [BH06, Proposition 34.3], $\lambda_{K_3/\mathbb{Q}_3}(\psi') = \tau(\eta_3, \psi_3')/\sqrt{3} = -i$, where $\tau(\eta_3, \psi_3')$ is the Gauss sum and $\psi_3'(x) = \psi_3(x/3)$ is the additive character of level 1. By [Lan, Lemma 5.1], $\lambda_{K_3/\mathbb{Q}_3}(\psi_3) = \eta_3(3)\lambda_{K_3/\mathbb{Q}_3}(\psi_3') = -i$. Then Δ_{θ_3} is the unique level 1 character of K_3 such that $\Delta_{\theta_3}|_{\mathbb{Z}_3^\times} = \eta_3$ and

$$\Delta_{\theta_3}(\sqrt{-3}) = \eta((\sqrt{-3})^3\alpha_{\Theta_3})\lambda_{K_3/\mathbb{Q}_3}(\psi_3)^3 = -i.$$

Recall that $\theta_3 = \Theta_3\Delta_{\theta_3}$. Then by Corollary 4.4 we can easily check that:

(1) If $p \equiv 7 \bmod 9$, then $\theta_3\overline{\chi}_3$ is the trivial character.
(2) If $p \equiv 4 \bmod 9$, then $\theta_3\overline{\chi}_3$ is of level 2 and by definition we can choose $\alpha_{\theta_3\overline{\chi}_3} = 1/(3\sqrt{-3})$. ∎

Proof of Proposition 4.1. We may assume f_3 is L^2-normalized. To evaluate f_3 for the embedding in (4.1) is equivalent to using the standard embedding (2.13) of $\mathbb{E}$ and a different translate of the newform. In particular the embedding in (4.1) is conjugate to the standard embedding by

$$(4.4)\qquad \begin{pmatrix} a & 3^{-2}b \\ 3^3c & -a \end{pmatrix} = \begin{pmatrix} -9c & a/3 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 \\ D & 0 \end{pmatrix} \begin{pmatrix} -9c & a/3 \\ 0 & 1 \end{pmatrix}.$$

Thus

(4.5)

$$\begin{aligned}\beta_3^0(f_3,f_3) &= \int\limits_{\mathbb{F}^\times\backslash\mathbb{E}^\times} \left(\pi_3\left(\begin{pmatrix} -9c & a/3 \\ 0 & 1 \end{pmatrix}^{-1} t \begin{pmatrix} -9c & a/3 \\ 0 & 1 \end{pmatrix}\right) f_3, f_3\right)\chi(t)\,dt \\ &= \int\limits_{\mathbb{F}^\times\backslash\mathbb{E}^\times} \left(\pi_3\left(t\begin{pmatrix} -9c & a/3 \\ 0 & 1 \end{pmatrix}\right) f_3, \pi_3\left(\begin{pmatrix} -9c & a/3 \\ 0 & 1 \end{pmatrix}\right) f_3\right)\chi(t)\,dt,\end{aligned}$$

which is by definition

$$\left\{\pi_3\left(\begin{pmatrix} -9c & a/3 \\ 0 & 1 \end{pmatrix}\right) f_3, \pi_3\left(\begin{pmatrix} -9c & a/3 \\ 0 & 1 \end{pmatrix}\right) f_3\right\}$$

for the bilinear pairing as in (3.1) and the standard embedding as in (2.13). Note that by Corollary 2.12,

$$\pi_3\left(\begin{pmatrix} -9c & a/3 \\ 0 & 1 \end{pmatrix}\right) f_3 = \frac{1}{\sqrt{2}} \sum_{x\in(\mathbb{Z}_3/3\mathbb{Z}_3)^\times} \pi_3\left(\begin{pmatrix} 1 & a/3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\right)\varphi_0$$

where φ_0 is the minimal test vector.

Now there are two cases. If $p \equiv 7 \bmod 9$, then $9 \parallel a$ and by Lemma 2.7 the action of $\left(\begin{smallmatrix} 1 & a/3 \\ 0 & 1 \end{smallmatrix}\right)$ on $\varphi_x = \pi_3\left(\left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right)\right)\varphi_0$ is by a simple unitary character which is canceled since we take a dual pair. By Lemma 4.5, we have $c(\theta_3\overline{\chi}_3) = 0$ in this case. According to the $c(\theta\overline{\chi}) = 0$ case in Section 2.4, we have a unique $x \bmod \varpi$ for which $\{\varphi_x, \varphi_x\}$ is non-vanishing. By Proposition 3.3 there are no off-diagonal terms, and we have

$$(4.6)\qquad \beta_3^0(f_3,f_3) = \frac{1}{(q-1)q^{\lceil c(\theta_3)/(2e_{\mathbb{L}})\rceil - 1}}\{\varphi_x,\varphi_x\} = \frac{1}{2}\cdot 2 = 1.$$

If $p \equiv 4 \bmod 9$, we have $3 \parallel a$ and $u = a/3$. By Lemma 4.5, $c(\theta_3\overline{\chi}_3) = 2$; this is the case $l = 1$ and $n - l = 1$ odd in Section 2.4. By the choice in Section 2.4,

$$D' = \frac{1}{\alpha_{\theta_3}^2 \varpi_{\mathbb{L}}^{2c(\theta_3)}} = -3.$$

By Lemma 4.5, $\alpha_{\theta_3\chi_3^{-1}} = 1/(3\sqrt{-3})$ in this case, and we have

$$\begin{aligned}\Delta(u) &= 4\varpi^n \alpha_{\theta_3\overline{\chi}_3}\sqrt{D}\left(\varpi^n \alpha_{\theta_3\overline{\chi}_3}\sqrt{D} - 2\sqrt{\frac{D}{D'}}\right) + 4\frac{D}{D'}Du^2 \\ &\equiv 4\cdot 9\cdot \frac{1}{3\sqrt{-3}}\cdot\sqrt{-3}\cdot(-2) + 4\cdot(-3)\frac{a^2}{9} \bmod \varpi^2 \\ &\equiv -8\cdot 3 - 4\cdot 3 \bmod \varpi^2 \equiv 0 \bmod \varpi^2.\end{aligned} \tag{4.7}$$

$\Delta(u)$ is indeed congruent to a square. Then we can get a unique solution of $v \bmod \varpi$ from (2.17), and again by Proposition 3.3,

$$\beta_3^0(f_3, f_3) = \frac{1}{(q-1)q^{\lceil c(\theta_3)/(2e_{\mathbb{L}})\rceil - 1}}\,\frac{1}{q^{\lfloor l/2\rfloor}} = \frac{1}{2}. \ \blacksquare \tag{4.8}$$

Let f' be the admissible test vector of (π, χ) as defined in [CST14, Definition 1.4]. By definition, the 3-adic local vector f_3' is a χ_3^{-1}-eigenvector under the action of $K_3^\times$. The following corollary is directly used in [HSY].

COROLLARY 4.6. *For the admissible test vector f_3' and the newform f_3 we have*

$$\frac{\beta_3^0(f_3', f_3')}{\beta_3^0(f_3, f_3)} = \begin{cases} 2 & \text{if } p \equiv 7 \bmod 9, \\ 4 & \text{if } p \equiv 4 \bmod 9. \end{cases}$$

Proof. Keep the normalization of the volumes as in Proposition 4.1. By definition of f', we have $\beta_3^0(f_3', f_3') = \mathrm{Vol}(\mathbb{Q}_3^\times \setminus K_3^\times) = 2$. Then the corollary follows from Proposition 4.1. ∎

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References

[BH06] C. Bushnell and G. Henniart, *The Local Langlands Conjecture for* GL(2), Springer, Berlin, 2006.

[CST14] L. Cai, J. Shu, and Y. Tian, *Explicit Gross–Zagier and Waldspurger formulae*, Algebra Number Theory 8 (2014), 2523–2572.

[FMP17] D. File, K. Martin, and A. Pitale, *Test vectors and central L-values for* GL(2), Algebra Number Theory 11 (2017), 253–318.

[GP91] B. H. Gross and D. Prasad, *Test vectors for linear forms*, Math. Ann. 291 (1991), 343–355.

[Hu] Y. Hu, *Test vectors for Waldspurger's period integral and application to the mass equidistribution on nonsplit torus*, arXiv:1608.0610v1 (2016).

[Hu18] Y. Hu, *Triple product formula and mass equidistribution on modular curves of level N*, Int. Math. Res. Notices 2018, 2899–2943.

[HN] Y. Hu and P. D. Nelson, *New test vector for Waldspurger's period integral*, arXiv:1810.11564 (2018).

[HSY] Y. Hu, J. Shu, and H. Yin, *An explicit Gross–Zagier formula related to the Sylvester conjecture*, Trans. Amer. Math. Soc. 372 (2019), 6905–6925.

[Lan] R. P. Langlands, *On the functional equation of the artin L-functions*, unpublished note; https://publications.ias.edu/sites/default/files/a-ps.pdf.

[Neu99] J. Neukirch, *Algebraic Number Theory*, Grundlehren Math. Wiss. 322, Springer, Berlin, 1999.

[Sai93] H. Saito, *On Tunnell's formula for characters of* GL(2), Compos. Math. 85 (1993), 99–108.

[Sil94] J. H. Silverman, *Advanced Topics in the Arithmetic of Elliptic Curves*, Grad. Texts in Math. 151, Springer, New York, 1994.

[Tun83] J. B. Tunnell, *Local ϵ-factors and characters of* GL(2), Amer. J. Math. 105 (1983), 1277–1307.

[Wu] H. Wu, *Subconvex bounds for compact toric integrals*, arXiv:1604.01902 (2016).

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