J. Math. Anal. Appl. 422 (2015) 1247–1263

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

L_p mixed geominimal surface area ☆

Baocheng Zhu [a,b], Jiazu Zhou [a,c,*], Wenxue Xu [a]

[a] School of Mathematics and Statistics, Southwest University, Chongqing 400715, China
[b] Department of Mathematics, Hubei University for Nationalities, Enshi, Hubei, 445000, China
[c] College of Mathematical Science, Kaili University, Kaili, Guizhou, 556000, China

ARTICLE INFO

Article history:
Received 13 April 2014
Available online 28 September 2014
Submitted by Richard M. Aron

Keywords:
L_p geominimal surface area
L_p mixed geominimal surface area
L_p affine surface area
Affine isoperimetric inequality

ABSTRACT

This paper deals with L_p geominimal surface area and its extension to L_p mixed geominimal surface area. We give an integral formula of L_p geominimal surface area by the p-Petty body and introduce the concept of L_p mixed geominimal surface area which is a natural extension of L_p geominimal surface area. Some inequalities, such as, analogues of Alexandrov–Fenchel inequalities, Blaschke–Santaló inequalities, and affine isoperimetric inequalities for L_p mixed geominimal surface areas are obtained.

1. Introduction

The geominimal surface area was first introduced by Petty [27] more than three decades ago. Since then this seminal concept and its L_p extensions introduced by Lutwak [19,21], have been served as bridges connecting affine differential geometry, relative differential geometry and Minkowski geometry. The basic theory concerning geominimal surface area is developed, and a close connection is established between this theory and affine differential geometry in [27]. In [21], Lutwak demonstrated that there were natural extensions of affine and geominimal surface areas in the Brunn–Minkowski–Firey theory. It motivates extensions of some known inequalities for affine surface area and geominimal surface areas to L_p affine surface area and L_p geominimal surface areas, respectively. These new inequalities of L_p type ($p > 1$) are stronger than their classical counterparts.

Both affine surface area and geominimal surface area are unimodular affine invariant functionals of convex hypersurfaces. Isoperimetric inequalities for geominimal surface area are closely related to many isoperimetric inequalities for affine surface area and clarify the equality conditions of many of inequalities.

☆ Supported in part by National Natural Science Foundation of China (No. 11271302 and No. 11401486), the Fundamental Research Funds for the Central Universities (No. XDJK2014C164) and Postdoctoral Research Funds of Chongqing (No. Xm2014025).
* Corresponding author at: School of Mathematics and Statistics, Southwest University, Chongqing 400715, China.
E-mail addresses: zhubaocheng814@163.com (B. Zhu), zhoujz@swu.edu.cn (J. Zhou), xwxjk@163.com (W. Xu).

http://dx.doi.org/10.1016/j.jmaa.2014.09.035
0022-247X/

The classical affine surface area was introduced by Blaschke in 1923 [1]. The L_p affine surface area was first generalized by Lutwak for $p > 1$ in [21]. Since then, considerable attention has been paid to the L_p affine surface area. The L_p affine surface area is one important concept and plays crucial roles in L_p Brunn–Minkowski theory initialized by Lutwak (cf. [2,3,6,5,8,12,13,16,19–23,25,41]). The L_p affine surface area has been further extended to all $p \in \mathbb{R}$ via geometric interpretations (cf. [26,32,35,37,38]). It was witnessed that the L_p affine surface area is related closely to the theory of valuation (cf. [15,16]), the information theory of convex bodies and the approximation of convex bodies by polytopes (cf. [17,32]). Furthermore, the L_p affine surface area has been extended to L_p mixed affine surface area by its integral expression (cf. [34,37]), and to more general mixed affine surface area [38].

Unlike the L_p affine surface area, L_p geominimal surface area has no nice integral expression. This leads to a big obstacle on extending the L_p geominimal surface area. Recently, Ye [39] introduced the L_p geominimal surface area for all $-n \neq p < 1$, which extends the classical geominimal surface area ($p = 1$) by Petty and the L_p geominimal surface area ($p > 1$) by Lutwak. In [39], Ye extended the L_p geominimal surface area by his equivalent formula of the L_p affine surface area. Then he obtained some affine isoperimetric inequalities and the Santaló style inequality for all $p \in \mathbb{R}$. There are several papers on L_p geominimal surface area, see e.g., [19,21,39,40,42,43].

The authors extended the Petty's theory of L_p geominimal surface area with the information on the general L_p affine surface area to any convex body in [43]. From the Petty's theory for the L_p version (see Theorem 3.1), we can know easily that the isoperimetric inequalities for L_p geominimal surface area are stronger than the ones for L_p affine surface area. We note that all above isoperimetric inequalities are part of the L_p Brunn–Minkowski theory which has applications in analysis (cf. [4,9,10,24]). In general, all these analogue concepts and results can be viewed as parts of the L_p valuation theory (cf. [14,17,15,16,30,31, 33–36]).

In this paper, we provide an integral formula for L_p geominimal surface area by p-Petty body (see Proposition 3.1). Moreover, motivated by ideas and results achieved by Lutwak, Yang and Zhang and others in affine geometry, we define the L_p mixed geominimal surface area. Then we establish some new L_p affine isoperimetric inequalities. Our paper is organized as follows. In Section 2 we provide the necessary background, such as definitions and known results which will be needed. Section 3 includes the basic theory of L_p geominimal surface area. In Section 4, we give the integral definition of L_p geominimal surface area, and introduce the L_p mixed geominimal surface area and prove some important properties, such as affine invariant properties. We also obtain analogues of Alexandrov–Fenchel inequalities, Blaschke–Santaló inequalities, and affine isoperimetric inequalities for L_p mixed geominimal surface areas. Finally, we investigate the ith L_p mixed geominimal surface areas and obtain analogues of Blaschke–Santaló and affine isoperimetric inequalities in Section 5.

2. Preliminaries and notions

In this section, we collect some basic well-known facts that we will use in the proofs of our results. For more references about the Brunn–Minkowski theory, see [7] and [28].

Let $\mathcal{K}^n$ denote the set of convex bodies, that is, compact, convex subsets with non-empty interiors in $\mathbb{R}^n$. For the set of convex bodies containing the origin in their interior and the set of convex bodies whose centroids lie at the origin, we write $\mathcal{K}_o^n$ and $\mathcal{K}_c^n$, respectively. The unit ball in $\mathbb{R}^n$ and its surface are denoted by B and S^{n-1}, respectively. The volume of the unit ball B is denoted by $\omega_n = \pi^{n/2}/\Gamma(1 + n/2)$.

For $K \in \mathcal{K}_o^n$, its support function $h_K = h(K, \cdot) : \mathbb{R}^n \setminus \{0\} \to [0, \infty)$ is defined by $x \in \mathbb{R}^n \setminus \{0\}$, $h(K, x) = \max\{\langle x, y\rangle : y \in K\}$, where $\langle \cdot, \cdot\rangle$ is the standard inner product on $\mathbb{R}^n$. Associated with each $K \in \mathcal{K}_o^n$, one can uniquely define its polar body $K^* \in \mathcal{K}_o^n$ by

$$K^* = \{x \in \mathbb{R}^n : \langle x, y\rangle \leq 1,\ \forall y \in K\}.$$

It is easily verified that $Q^{**} = Q$ if $Q \in \mathcal{K}_o^n$. Let l_x is the line through the origin containing $x \in \mathbb{R}^n \setminus \{o\}$. A set L in $\mathbb{R}^n$ with origin $o \in L$ is star-shaped at o if $L \cap l_u$ is a closed line segment for each $u \in S^{n-1}$. The radial function $\rho_K = \rho(K, \cdot) : \mathbb{R}^n \setminus \{0\} \to [0, \infty)$, of a compact star-shaped about the origin $K \subset \mathbb{R}^n$, is defined by $\rho(K, x) = \max\{\lambda \geq 0 : \lambda x \in K\}$. If ρ_K is positive and continuous, then K is called a star body about the origin. Write $\mathcal{S}_o^n$ for the set of star bodies in $\mathbb{R}^n$. Two star bodies K and L are dilates of one another if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

According to definitions of the polar body for convex body, the support function and radial function, it follows that, for $K \in \mathcal{K}_o^n$

$$h_{K^*}(u)\rho_K(u) = 1, \qquad \rho_{K^*}(u)h_K(u) = 1, \quad \text{for all } u \in S^{n-1}.$$

For real $p \geq 1$, $\lambda, \mu \geq 0$ (not both zero), the Firey linear combination $\lambda \cdot K +_p \mu \cdot L$ of $K, L \in \mathcal{K}_o^n$ is defined by [5]

$$h(\lambda \cdot K +_p \mu \cdot L, \cdot)^p = \lambda h(K, \cdot)^p + \mu h(L, \cdot)^p.$$

For $p \geq 1$, the L_p mixed volume, $V_p(K, L)$, of $K, L \in \mathcal{K}_o^n$, is defined in [20] by

$$\frac{n}{p} V_p(K, L) = \lim_{\varepsilon \to 0} \frac{V(K +_p \varepsilon \cdot L) - V(K)}{\varepsilon}, \tag{2.1}$$

where $V(K)$ denotes the volume of K. There is a polar coordinate formula for volume is

$$V(K) = \frac{1}{n} \int_{S^{n-1}} \rho_K^n(u) dS(u).$$

In [20], Lutwak proved that for each $K \in \mathcal{K}_o^n$, there is a positive Borel measure $S_p(K, \cdot)$ on S^{n-1} such that

$$V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} h(L, u)^p dS_p(K, u), \tag{2.2}$$

for each $L \in \mathcal{K}_o^n$. The L_p surface area measure $S_p(K, \cdot)$ is absolutely continuous with respect to the surface area $S(K, \cdot)$ of K, and has Radon–Nikodym derivative

$$dS_p(K, \cdot)/dS(K, \cdot) = h(K, \cdot)^{1-p}. \tag{2.3}$$

It follows from (2.3) that the measure $S_1(K, \cdot)$ is just the classical surface area measure $S(K, \cdot)$ of K.

From formula (2.2), it follows immediately that for each $K \in \mathcal{K}_o^n$,

$$V_p(K, K) = V(K).$$

For the constant $\lambda > 0$, since $S(\lambda K, \cdot) = \lambda^{n-1} S(K, \cdot)$, by (2.3) we have $S_p(\lambda K, \cdot) = \lambda^{n-p} S_p(K, \cdot)$. Thus together with formula (2.2), we obtain:

$$V_p(\lambda K, L) = \lambda^{n-p} V_p(K, L),$$

and

$$V_p(K, \lambda L) = \lambda^p V_p(K, L). \tag{2.4}$$

The L_p Minkowski inequality was given by Lutwak in [21]: If $K, L \in \mathcal{K}_o^n$ and $p \geq 1$, then

$$V_p(K, L)^n \geq V(K)^{n-p} V(L)^p.$$

With equality for $p = 1$ if and only if K and L are homothetic, and for $p > 1$ if and only if K and L are dilates.

Together with the L_p Minkowski inequality, an immediate consequence is [21]:

Lemma 2.1. *If $K, L \in \mathcal{K}_o^n$, and for all $Q \in \mathcal{K}_o^n$,*

$$V_p(K, Q) = V_p(L, Q),$$

then $K = L$ for $n \neq p > 1$; $K = L + x$ for $p = 1$ and $x \in \mathbb{R}^n$.

Let $GL(n)$ and $SL(n)$ denote the group of nonsingular linear transformations and special linear transformations, respectively. We write $|\det(\phi)|$, ϕ^t and ϕ^{-1} for the absolute value of the determinant, the transpose and the inverse of linear transform ϕ, respectively.

In [21], Lutwak proved: For $K, L \in \mathcal{K}_o^n$, and $\varepsilon \geq 0$. If $p \geq 1$ and $\phi \in GL(n)$, then

$$\phi(K +_p \varepsilon \cdot L) = \phi K +_p \varepsilon \cdot \phi L.$$

Since $V(\phi K) = |\det(\phi)| V(K)$, for all $K \in \mathcal{K}^n$, and $\phi \in GL(n)$, it follows from (2.1) that:

Proposition 2.1. *If $p \geq 1$, and $K, L \in \mathcal{K}_o^n$, then for $\phi \in GL(n)$,*

$$V_p(\phi K, \phi L) = \left|\det(\phi)\right| V_p(K, L).$$

3. L_p affine and geominimal surface area

A convex body $K \in \mathcal{K}_o^n$ is said to have a L_p curvature function $f_p(K, \cdot) : S^{n-1} \to \mathbb{R}$, if its L_p surface area measure $S_p(K, \cdot)$ is absolutely continuous with respect to spherical Lebesgue measure S, and

$$dS_p(K, \cdot)/dS = f_p(K, \cdot). \tag{3.1}$$

For $p = 1$ and $u \in S^{n-1}$, $f_1(K, u) = f(K, u)$ is just the curvature function of K at u. i.e., the reciprocal of the Gauss curvature $G_K(x)$ at this point $x \in \partial K$, the smooth boundary of K, that has $u = \nu_K(x)$ as its outer normal.

By (3.1) and (2.3), we know that

$$f_p(K, u) = h_K^{1-p}(u) f(K, u), \tag{3.2}$$

for a convex body K in $\mathbb{R}^n$ and $u \in S^{n-1}$.

Let $\mathcal{F}_o^n$, $\mathcal{F}_c^n$ denote sets of bodies in $\mathcal{K}_o^n$, $\mathcal{K}_c^n$, with L_p curvature functions, respectively.

In [21], Lutwak defined the L_p affine surface area as follows: For $K \in \mathcal{F}_o^n$ and $p \geq 1$, the L_p affine surface area, $\Omega_p(K)$, of K is defined by

$$\Omega_p(K) = \int_{S^{n-1}} f_p(K, u)^{\frac{n}{n+p}} dS(u).$$

Furthermore, Lutwak introduced the concept of L_p mixed affine surface area: For $p \geq 1$, the L_p mixed affine surface area, $\Omega_p(K_1, \ldots, K_n)$, of $K_1, \ldots, K_n \in \mathcal{F}_o^n$ is defined by

$$\Omega_p(K_1, \ldots, K_n) = \int_{S^{n-1}} \left[f_p(K_1, u) \cdots f_p(K_n, u)\right]^{\frac{1}{n+p}} dS(u).$$

For $K \in \mathcal{K}_o^n$ and $p \geq 1$, the L_p geominimal surface area, $G_p(K)$, is defined in [21] by

$$\omega_n^{p/n} G_p(K) = \inf\{nV_p(K, Q)V\left(Q^*\right)^{p/n} : Q \in \mathcal{K}_o^n\}.$$

Associated with L_p geominimal surface area, Lutwak [21] proved the following L_p affine isoperimetric inequality.

Theorem A. *If $p \geq 1$, and $K \in \mathcal{K}_o^n$, then*

$$G_p(K)^n \leq n^n \omega_n^p V(K)^{n-p}, \tag{3.3}$$

with equality if and only if K is an ellipsoid.

Call a body $K \in \mathcal{F}_o^n$ is of p-elliptic type if the function $f_p(K, \cdot)^{\frac{1}{n+p}}$ is the support function of a convex body in $\mathcal{K}_o^n$; i.e., K is of p-elliptic type if there exists a body $Q \in \mathcal{K}_o^n$ such that

$$f_p(K, \cdot) = h(Q, \cdot)^{-(n+p)}.$$

In [21], Lutwak defined

$$\mathcal{V}_p^n = \left\{K \in \mathcal{F}_o^n : \text{there exists a } Q \in \mathcal{K}_o^n \text{ with } f_p(K, \cdot) = h(Q, \cdot)^{-(n+p)}\right\}.$$

The Petty's theory of L_p geominimal surface area with the information on the general L_p affine surface area was proved by Lutwak [21].

Theorem B. *If $p \geq 1$, and $K \in \mathcal{F}_o^n$, then*

$$\Omega_p(K)^{n+p} \leq (n\omega_n)^p G_p(K)^n, \tag{3.4}$$

with equality if and only if $K \in \mathcal{V}_p^n$.

The case $p = 1$ of inequality (3.4) was proved by Petty [27] for $K \in \mathcal{F}_o^n$ and extended by Lutwak [21] to $K \in \mathcal{K}_o^n$. The equality condition of Lutwak's extension for $K \in \mathcal{K}_o^n$ is proved by Rolf Schneider recently in [29]. The equality condition for (3.4) was only known under the additional assumption that $K \in \mathcal{F}_o^n$. Lutwak proved the inequality (3.4) for $K \in \mathcal{K}_o^n$ and $p \geq 1$ without the equality condition. In [43], the authors proved the inequality (3.4) for any convex body $K \in \mathcal{K}_o^n$ with the equality condition as follows.

Theorem 3.1. *If $p \geq 1$, and $K \in \mathcal{K}_o^n$, then*

$$\Omega_p(K)^{n+p} \leq (n\omega_n)^p G_p(K)^n,$$

with equality if and only if $K \in \mathcal{V}_p^n$.

Therefore, we may say the isoperimetric inequalities for L_p geominimal surface area are stronger than the ones for L_p affine surface area by Theorem 3.1. The main tool of the extension of Theorem 3.1 is the following p-Petty body.

For $K \in \mathcal{K}^n$, there exists a unique point $s(K)$ in the interior of K, called the Santaló point of K, such that

$$V\big(\big(-s(K)+K\big)^*\big) = \min\big\{V\big((-x+K)^*\big) : x \in \operatorname{int} K\big\},$$

or equivalently, as the unique $s(K) \in K$, such that

$$\int_{S^{n-1}} uh\big(-s(K)+K,u\big)^{-(n+1)}dS(u) = 0.$$

Let $\mathcal{K}_s^n$ denote the set of convex bodies having their Santaló point at the origin. Thus, we have

$$K \in \mathcal{K}_s^n \quad \text{if and only if} \quad K^* \in \mathcal{K}_c^n.$$

Let

$$\mathcal{T}^n = \big\{T \in \mathcal{K}^n : s(T) = o, V\big(T^*\big) = \omega_n\big\}.$$

We need the following Lutwak' result, which is the Proposition 3.3 in [21].

Lemma 3.1. *(and Definition.) For each $K \in \mathcal{K}_o^n$, and $p \geq 1$, there exists a unique body $T_pK \in \mathcal{T}^n$ with $G_p(K) = nV_p(K, T_pK)$.*

The unique body T_pK is called the p-Petty body of K. When $p = 1$, the subscript will often be suppressed and defined by Petty [27].

By Lemma 3.1, (2.2) and (3.1), we have the following integral formula of $G_p(K)$.

Proposition 3.1. *For each $K \in \mathcal{F}_0^n$, there exists a unique convex body $T = T_pK \in \mathcal{T}^n$ with*

$$G_p(K) = \int_{S^{n-1}} h_T^p(u)f_p(K,u)dS(u).$$

4. The L_p mixed geominimal surface area

Motivated by the definition of L_p mixed affine surface area of Lutwak, we now define the L_p mixed geominimal surface area, $G_p(K_1, \ldots, K_n)$, of $K_1, \ldots, K_n \in \mathcal{F}_o^n$ for $p \geq 1$ as follow:

Definition 4.1. For each $K_i \in \mathcal{F}_o^n$ and $p \geq 1$, there exists a unique convex body (Petty body of K_i) $T_i = T_pK_i \in \mathcal{T}^n$ $(i = 1, \ldots, n)$ with

$$G_p(K_1, \ldots, K_n) = \int_{S^{n-1}} \big[h_{T_1}^p(u)f_p(K_1,u)\cdots h_{T_n}^p(u)f_p(K_n,u)\big]^{\frac{1}{n}}dS(u).$$

Let $g_p(K_i,u) = h_{T_i}^p(u)f_p(K_i,u)$. Then, the $G_p(K_1, \ldots, K_n)$ can be written as follows:

$$G_p(K_1, \ldots, K_n) = \int_{S^{n-1}} \big[g_p(K_1,u)\cdots g_p(K_n,u)\big]^{\frac{1}{n}}dS(u). \tag{4.1}$$

To prove the L_p mixed geominimal surface area is affine invariant, we will need the following propositions.

Proposition 4.1. *Suppose $K \in \mathcal{K}_o^n$. If $p \geq 1$ and $\phi \in GL(n)$, then*

$$G_p(\phi K) = \left|\det(\phi)\right|^{\frac{n-p}{n}} G_p(K).$$

The $SL(n)$ case of Proposition 4.1 is due to Lutwak [21]. One can easily find the degree of the homogeneous factor follow from $SL(n)$ case.

Proposition 4.2. *If $p \geq 1$ and $K \in \mathcal{K}_o^n$, then for $\phi \in GL(n)$,*

$$\left|\det(\phi)\right|^{\frac{1}{n}} T_p \phi K = \phi T_p K.$$

Proof. From the definition of T_p and Proposition 4.1,

$$nV_p(K, T_pK) = G_p(K) = \left|\det(\phi)\right|^{\frac{p-n}{n}} G_p(\phi K) = \left|\det(\phi)\right|^{\frac{p-n}{n}} nV_p(\phi K, T_p\phi K).$$

By Proposition 2.1, and (2.4),

$$\begin{aligned} V_p(K, T_pK) &= \left|\det(\phi)\right|^{\frac{p-n}{n}} V_p(\phi K, T_p \phi K) \\ &= \left|\det(\phi)\right|^{-1} V_p\left(\phi K, \left|\det(\phi)\right|^{\frac{1}{n}} T_p \phi K\right) \\ &= V_p\left(K, \phi^{-1}\left(\left|\det(\phi)\right|^{\frac{1}{n}} T_p \phi K\right)\right). \end{aligned}$$

The uniqueness part of Lemma 3.1 shows that $T_pK = \phi^{-1}(|\det(\phi)|^{\frac{1}{n}} T_p\phi K)$, which is the desired result. □

The case $\phi \in SL(n)$ of Proposition 4.2 is due to Lutwak [21]. The case $p = 1$ and $\phi \in SL(n)$ of this proposition is due to Petty [27].

We now prove that the L_p mixed geominimal surface area is affine invariant.

Proposition 4.3. *If $p \geq 1$, and $K_1, \dots, K_n \in \mathcal{F}_o^n$, then for $\phi \in GL(n)$,*

$$G_p(\phi K_1, \dots, \phi K_n) = \left|\det(\phi)\right|^{\frac{n-p}{n}} G_p(K_1, \dots, K_n).$$

In particular, if $\phi \in SL(n)$, then $G_p(K_1, \dots, K_n)$ is affine invariant, that is,

$$G_p(\phi K_1, \dots, \phi K_n) = G_p(K_1, \dots, K_n).$$

Proof. Since $K \in \mathcal{F}_o^n$, for $\phi \in GL(n)$ and any $u \in S^{n-1}$, there exists a unique $x \in \partial K$ such that $u = \nu_K(x)$ and $f(K, u) = \frac{1}{G_K(x)}$. By Lemma 12 in [32],

$$f(K, u) = \frac{1}{G_K(x)} = \frac{f(\phi K, v)}{\det^2(\phi)\|\phi^{-t}(u)\|^{n+1}}, \tag{4.2}$$

where $v = \frac{\phi^{-t}(u)}{\|\phi^{-t}(u)\|} \in S^{n-1}$.

On the other hand,

$$h_K(u) = \langle x, u\rangle = \left\langle \phi x, \phi^{-t}(u)\right\rangle = \left\|\phi^{-t}(u)\right\| \langle \phi x, v\rangle = \left\|\phi^{-t}(u)\right\| h_{\phi K}(v). \tag{4.3}$$

By formula (3.2), for $p \geq 1$, we have

$$f_p(K,u) = \frac{f(\phi K, v) h_{\phi K}^{1-p}(v) \|\phi^{-t}(u)\|^{1-p}}{\det^2(\phi)\|\phi^{-t}(u)\|^{n+1}} = \frac{f_p(\phi K, v)}{\det^2(\phi)\|\phi^{-t}(u)\|^{n+p}}. \tag{4.4}$$

Lemma 10 and its proof in [32] show that

$$f(\phi K, v) dS(v) = \left|\det(\phi)\right| \left\|\phi^{-t}(u)\right\| f(K,u) dS(u).$$

Together with (4.1), we obtain:

$$\left\|\phi^{-t}(u)\right\|^{-n} dS(u) = \left|\det(\phi)\right| dS(v). \tag{4.5}$$

By the (4.3), (4.4) and Proposition 4.2, we have

$$\begin{aligned} g_p(K_i,u) &= h_{T_p K_i}^p(u) f_p(K_i,u) \\ &= \left\|\phi^{-t}(u)\right\|^p h_{\phi T_p K_i}^p(v) \frac{f_p(\phi K_i, v)}{\det^2(\phi)\|\phi^{-t}(u)\|^{n+p}} \\ &= \left|\det(\phi)\right|^{\frac{p-2n}{n}} \frac{h_{T_p \phi K_i}^p(v) f_p(\phi K_i, v)}{\|\phi^{-t}(u)\|^n} \\ &= \left|\det(\phi)\right|^{\frac{p-2n}{n}} \frac{g_p(\phi K_i, v)}{\|\phi^{-t}(u)\|^n}. \end{aligned}$$

This together with (4.5) yield:

$$\begin{aligned} G_p(K_1,\ldots,K_n) &= \int_{S^{n-1}} \left[g_p(K_1,u)\cdots g_p(K_n,u)\right]^{1/(n)} dS(u) \\ &= \left|\det(\phi)\right|^{\frac{p-2n}{n}} \int_{S^{n-1}} \frac{[g_p(\phi K_1, v)\cdots g_p(\phi K_n, v)]^{\frac{1}{n}}}{\|\phi^{-t}(u)\|^n} dS(u) \\ &= \left|\det(\phi)\right|^{\frac{p-n}{n}} \int_{S^{n-1}} \left[g_p(\phi K_1, v)\cdots g_p(\phi K_n, v)\right]^{\frac{1}{n}} dS(v) \\ &= \left|\det(\phi)\right|^{\frac{p-n}{n}} G_p(\phi K_1,\ldots,\phi K_n). \end{aligned}$$

This complete the proof. □

The classical Alexandrov–Fenchel inequalities for mixed volumes (cf. [17,28]) can be written as

$$\prod_{i=0}^{m-1} V(K_1,\ldots,K_{n-m},\underbrace{K_{n-i},\ldots,K_{n-i}}_{m}) \leq V(K_1,\ldots,K_n)^m.$$

The following inequalities are the analogous Alexandrov–Fenchel inequalities for L_p mixed geominimal surface area.

Theorem 4.1. *If $n \neq p > 1$, and $K_1,\ldots,K_n \in \mathcal{F}_o^n$, then for $1 \leq m \leq n$*

$$G_p(K_1,\ldots,K_n)^m \leq \prod_{i=0}^{m-1} G_p(K_1,\ldots,K_{n-m},\underbrace{K_{n-i},\ldots,K_{n-i}}_{m}).$$

Equality holds if the K_j are dilates of each other for $j = n-m+1,\ldots,n$. If $m = 1$ equality holds trivially.

In particular, if $m = n$, then

$$G_p(K_1, \dots, K_n)^n \le G_p(K_1) \cdots G_p(K_n), \tag{4.6}$$

with equality if the K_i are dilates of each other.

Proof. Let $H_0(u) = [g_p(K_1, u) \cdots g_p(K_{n-m}, u)]^{\frac{1}{n}}$ and $H_{i+1}(u) = [g_p(K_{n-i}, u)]^{\frac{1}{n}}$ for $i = 0, \dots, m-1$. By Hölder's inequality (cf. [11])

$$\begin{aligned} G_p(K_1, \dots, K_n) &= \int_{S^{n-1}} \left[g_p(K_1, u) \cdots g_p(K_n, u)\right]^{\frac{1}{n}} dS(u) \\ &= \int_{S^{n-1}} H_0(u) H_1(u) \cdots H_m(u) dS(u) \\ &\le \prod_{i=0}^{m-1} \left(\int_{S^{n-1}} H_0 H_{i+1}(u)^m dS(u) \right)^{\frac{1}{m}} \\ &= \prod_{i=0}^{m-1} G_p^{\frac{1}{m}}(K_1, \dots, K_{n-m}, \underbrace{K_{n-i}, \dots, K_{n-i}}_{m}). \end{aligned}$$

The equality in Hölder's inequality holds if and only if $H_0(u) H_{i+1}^m(u) = c_{ij}^m H_0(u) H_{j+1}^m(u)$ for some $c_{ij} > 0$ and all $0 \le i \ne j \le m-1$. This is equivalent to $h^p_{T_p K_{n-i}}(u) f_p(K_{n-i}, u) = c_{ij} h^p_{T_p K_{n-j}}(u) f_p(K_{n-j}, u)$. From Proposition 4.2, $T_p K = T_p(\lambda K)$ for a constant λ. Thus, the equality holds if K_{n-i} and K_{n-j} are dilates of each other. □

Let $V(K_1, \dots, K_n)$ be the mixed volume of $K_1, \dots, K_n \in \mathcal{K}^n$. Then the Minkowski inequality for mixed volume is

$$V(K_1, \dots, K_n)^n \ge V(K_1) \cdots V(K_n), \tag{4.7}$$

with equality if and only if K_i $(1 \le i \le n)$ are homothetic.

The analogous Minkowski inequality for dual mixed volume $\widetilde{V}(K_1, \dots, K_n)$, introduced by Lutwak in [18], is

$$\widetilde{V}(K_1, \dots, K_n)^n \le V(K_1) \cdots V(K_n), \tag{4.8}$$

with equality if and only if K_i $(1 \le i \le n)$ are dilates of one another.

Now, we are in the position to prove affine isoperimetric inequalities for L_p mixed geominimal surface areas.

Theorem 4.2. *Let $K_i \in \mathcal{F}_o^n$, $1 \le i \le n$.*

(i) *For $p \ge 1$,*

$$\left(\frac{G_p(K_1, \dots, K_n)}{G_p(B, \dots, B)} \right)^n \le \left(\frac{V(K_1)}{V(B)} \cdots \frac{V(K_n)}{V(B)} \right)^{\frac{n-p}{n}},$$

with equality if the K_i are ellipsoids that are dilates of each other.

(ii) *For* $1 \leq p \leq n$,

$$\frac{G_p(K_1,\ldots,K_n)}{G_p(B,\ldots,B)} \leq \left(\frac{V(K_1,\ldots,K_n)}{V(B,\ldots,B)}\right)^{\frac{n-p}{n}},$$

with equality if the K_i *are ellipsoids that are dilates of each other.*
In particular, for $p = n$

$$G_p(K_1,\ldots,K_n) \leq G_p(B,\ldots,B),$$

with equality if the K_i *are ellipsoids that are dilates of each other.*

(iii) *For* $p \geq n$,

$$\frac{G_p(K_1,\ldots,K_n)}{G_p(B,\ldots,B)} \leq \left(\frac{\widetilde{V}(K_1,\ldots,K_n)}{\widetilde{V}(B,\ldots,B)}\right)^{\frac{n-p}{n}},$$

with equality if the K_i *are ellipsoids that are dilates of each other.*

Proof. (i) By the Theorem A, we have $G_p(B) = n\omega_n$, then $G_p(B,\ldots,B) = G_p(B) = n\omega_n$. By the inequality (4.6) and (3.3), one gets that for all $p \geq 1$,

$$\left(\frac{G_p(K_1,\ldots,K_n)}{G_p(B,\ldots,B)}\right)^n \leq \frac{G_p(K_1)}{G_p(B)}\cdots\frac{G_p(K_n)}{G_p(B)} \leq \left(\frac{V(K_1)}{V(B)}\cdots\frac{V(K_n)}{V(B)}\right)^{\frac{n-p}{n}}. \tag{4.9}$$

Equality holds for the L_p isoperimetric inequality (3.3) if and only if K_i are all ellipsoids, equality holds in inequality (4.6) if the K_i are dilates of one another. Thus, equality holds in (4.9) if $K_1,\cdots,K_n$ are dilated ellipsoids of each other.

(ii) If $1 \leq p \leq n$, then $\frac{n-p}{n} \geq 0$. By the inequality (4.7), one gets

$$\left[V(K_1)\cdots V(K_n)\right]^{\frac{n-p}{n}} \leq \left[V(K_1,\ldots,K_n)^n\right]^{\frac{n-p}{n}}.$$

Since $V(B,\ldots,B) = V(B)$, one gets together with (4.9)

$$\frac{G_p(K_1,\ldots,K_n)}{G_p(B,\ldots,B)} \leq \left(\frac{V(K_1,\ldots,K_n)}{V(B,\ldots,B)}\right)^{\frac{n-p}{n}},$$

with equality if the K_i are ellipsoids that are dilates of each other.

(iii) Since $p > n$ implies $\frac{n-p}{n} < 0$. Thus

$$\left[V(K_1)\cdots V(K_n)\right]^{\frac{n-p}{n}} \leq \left[\widetilde{V}(K_1,\ldots,K_n)^n\right]^{\frac{n-p}{n}}.$$

By inequality (4.9) and $\widetilde{V}(B,\ldots,B) = V(B)$, one gets

$$\frac{G_p(K_1,\ldots,K_n)}{G_p(B,\ldots,B)} \leq \left(\frac{\widetilde{V}(K_1,\ldots,K_n)}{\widetilde{V}(B,\ldots,B)}\right)^{\frac{n-p}{n}}.$$

The equality condition can get from the equality condition in inequality (4.9) and (4.8). □

Corollary 4.1. *If $K_i \in \mathcal{F}_o^n$ are convex bodies in $\mathcal{K}_o^n$ with positive absolutely continuous L_p curvature functions,*

(i) *For $1 \le p \le n$,*

$$G_p(K_1, \dots, K_n)^n \le n^n \omega_n^p V(K_1, \dots, K_n)^{n-p},$$

with equality if the K_i are ellipsoids that are dilates of each other.

(ii) *For $p > n$*

$$G_p(K_1, \dots, K_n)^n \le n^n \omega_n^p \widetilde{V}(K_1, \dots, K_n)^{n-p},$$

with equality if the K_i are ellipsoids that are dilates of each other.

5. The ith L_p mixed geominimal surface area

In this section, we will investigate the ith L_p mixed geominimal surface area. For $K, L \in \mathcal{F}_o^n$, $p \ge 1$ and $i \in \mathbb{R}$, we define ith L_p mixed geominimal surface area, $G_{p,i}(K, L)$, of K, L as

$$G_{p,i}(K, L) = \int_{S^{n-1}} g_p(K, u)^{\frac{n-i}{n}} g_p(L, u)^{\frac{i}{n}} dS(u). \tag{5.1}$$

By the Lemma 3.1, we have

$$G_p(B) = nV_p(B, T_p B),$$

since

$$G_p(B) = n\omega_n = nV_p(B, B).$$

Thus, the above two equations and the uniqueness part of Lemma 3.1 shows that

$$T_p B = B.$$

Let $L = B$ and write

$$G_{p,i}(K, B) = G_{p,i}(K). \tag{5.2}$$

By (3.1) we get $f_p(B, \cdot) = 1$, which together with (5.1), (5.2) and $h_{T_p B} = h_B = 1$ yield

$$G_{p,i}(K) = \int_{S^{n-1}} g_p(K, u)^{\frac{n-i}{n}} dS(u).$$

By (4.1), (5.1) and (5.2), we have:

$$G_{p,0}(K) = G_p(K), \qquad G_{p,i}(K, K) = G_p(K), \tag{5.3}$$

$$G_{p,0}(K, L) = G_p(K), \qquad G_{p,n}(K, L) = G_p(L). \tag{5.4}$$

We obtain the following cyclic inequality for the ith L_p mixed geominimal surface area.

Theorem 5.1. *For $K, L \in \mathcal{F}_o^n$, $n \neq p \geq 1$, $i, j, k \in \mathbb{R}$ and $i < j < k$, we have*

$$G_{p,i}(K,L)^{k-j} G_{p,k}(K,L)^{j-i} \geq G_{p,j}(K,L)^{k-i}, \tag{5.5}$$

with equality if K and L are dilates of each other.

Proof. From definition (5.1) and Hölder's inequality, it follows that for $p \geq 1$,

$$\begin{aligned}
G_{p,i}(K,L)^{\frac{k-j}{k-i}} G_{p,k}(K,L)^{\frac{j-i}{k-i}} &= \left[\int_{S^{n-1}} g_p(K,u)^{\frac{n-i}{n}} g_p(L,u)^{\frac{i}{n}} dS(u)\right]^{\frac{k-j}{k-i}} \\
&\quad \times \left[\int_{S^{n-1}} g_p(K,u)^{\frac{n-k}{n}} f_p(L,u)^{\frac{k}{n}} dS(u)\right]^{\frac{j-i}{k-i}} \\
&= \left\{\int_{S^{n-1}} \left[g_p(K,u)^{\frac{(n-i)(k-j)}{n(k-i)}} g_p(L,u)^{\frac{i(k-j)}{n(k-i)}}\right]^{\frac{k-i}{k-j}} dS(u)\right\}^{\frac{k-j}{k-i}} \\
&\quad \times \left\{\int_{S^{n-1}} \left[g_p(K,u)^{\frac{(n-k)(j-i)}{n(k-i)}} g_p(L,u)^{\frac{k(j-i)}{n(k-i)}}\right]^{\frac{k-i}{j-i}} dS(u)\right\}^{\frac{j-i}{k-i}} \\
&\geq \int_{S^{n-1}} g_p(K,u)^{\frac{n-j}{n}} g_p(L,u)^{\frac{j}{n}} dS(u).
\end{aligned}$$

That is,

$$\begin{aligned}
G_{p,i}(K,L)^{\frac{k-j}{k-i}} G_{p,k}(K,L)^{\frac{j-i}{k-i}} &\geq \int_{S^{n-1}} g_p(K,u)^{\frac{n-j}{n}} g_p(L,u)^{\frac{j}{n}} dS(u) \\
&= G_{p,j}(K,L).
\end{aligned}$$

We obtain the inequality (5.5). According to the condition of equality in Hölder's inequality, the equality holds in (5.5) if and only if for any $u \in S^{n-1}$,

$$\frac{g_p(K,u)^{\frac{n-i}{n}} g_p(L,u)^{\frac{i}{n}}}{g_p(K,u)^{\frac{n-k}{n}} g_p(L,u)^{\frac{k}{n}}}$$

is a constant, that is, $g_p(K,u)/g_p(L,u)$ is a constant for any $u \in S^{n-1}$. By the same argument in the proof of Theorem 4.1, we conclude that equality holds if K and L are dilates of each other. □

Letting $L = B$ in Theorem 5.1 and using (5.2), we immediately obtain:

Corollary 5.1. *If $K \in \mathcal{F}_o^n, n \neq p \geq 1, i, j, k \in \mathbb{R}$ and $i < j < k$, then*

$$G_{p,i}(K)^{k-j} G_{p,k}(K)^{j-i} \geq G_{p,j}(K)^{k-i},$$

with equality if K is a ball centered at the origin.

We then derive the Minkowski inequality for the ith L_p mixed geominimal surface area:

Theorem 5.2. *If $K, L \in \mathcal{F}_o^n$, $n \neq p \geq 1$, $i \in \mathbb{R}$, then*
for $i < 0$ or $i > n$,

$$G_{p,i}(K,L)^n \geq G_p(K)^{n-i} G_p(L)^i, \tag{5.6}$$

for $0 < i < n$,

$$G_{p,i}(K,L)^n \leq G_p(K)^{n-i} G_p(L)^i. \tag{5.7}$$

Each inequality holds as an equality if K and L are dilates of each other. For $i = 0$ or $i = n$, (5.6) (or (5.7)) is identical.

Proof. (i) For $i < 0$, let $(i, j, k) = (i, 0, n)$ in Theorem 5.1, we obtain:

$$G_{p,i}(K,L)^n G_{p,n}(K,L)^{-i} \geq G_{p,0}(K,L)^{n-i},$$

with equality if K and L are dilates of each other.

From (5.4), we can get the

$$G_{p,i}(K,L)^n \geq G_p(K)^{n-i} G_p(L)^i,$$

with equality if K and L are dilates of each other.

(ii) For $i > n$, let $(i, j, k) = (0, n, i)$ in Theorem 5.1, we obtain:

$$G_{p,0}(K,L)^{i-n} G_{p,i}(K,L)^n \geq G_{p,n}(K,L)^i,$$

with equality if K and L are dilates of each other.

From (5.4), we can also get the inequality (5.6).

(iii) For $0 < i < n$, let $(i, j, k) = (0, i, n)$ in Theorem 5.1, we obtain:

$$G_{p,0}(K,L)^{n-i} G_{p,n}(K,L)^i \geq G_{p,i}(K,L)^n,$$

with equality if K and L are dilates of each other.

From (5.4), we can get the inequality (5.7).

(iv) For $i = 0$ (or $i = n$), by (5.4), one can see (5.6) (or (5.7)) is identical. □

Let $L = B$ in Theorem 5.2, $G_p(B) = n\omega_n$ and (5.2) will lead to the following:

Corollary 5.2. *If $K \in \mathcal{F}_o^n$, $n \neq p \geq 1$, $i \in \mathbb{R}$, then*
for $i < 0$ or $i > n$,

$$G_{p,i}(K)^n \geq (n\omega_n)^i G_p(K)^{n-i}; \tag{5.8}$$

for $0 < i < n$,

$$G_{p,i}(K)^n \leq (n\omega_n)^i G_p(K)^{n-i}. \tag{5.9}$$

Each inequality holds as an equality if K is a ball centered at the origin. For $i = 0$ or $i = n$, (5.8) (or (5.9)) is identical.

One of the most important inequalities in convex geometry is the Blaschke–Santaló inequality about polar body (cf. [22,27,28]): If $K \in \mathcal{K}_c^n$, then

$$V(K)V\big(K^*\big) \le \omega_n^2, \tag{5.10}$$

where the equality holds if and only if K is an ellipsoid. Recently, In [8] Haberl and Schuster showed that there is an interesting asymmetric L_p version of (5.10).

In [42], we also proved the following Blaschke–Santaló inequality for the L_p geominimal surface area: If $K \in \mathcal{K}_c^n$ and $1 \le p < n$, then

$$G_p(K)G_p\big(K^*\big) \le (n\omega_n)^2, \tag{5.11}$$

with equality if and only if K is an ellipsoid.

In [21], Lutwak proved the following Proposition: If $p \ge 1$, and $K \in \mathcal{K}_o^n$, then

$$\omega_n\left(\frac{G_p(K)^n}{n^nV(K)^{n-p}}\right)^{\frac{1}{p}} \le V(K)V\big(K^*\big). \tag{5.12}$$

Inequality (5.12), for K and K^*, immediately yields: If $p \ge 1$, and $K \in \mathcal{K}_o^n$, then

$$G_p(K)G_p\big(K^*\big) \le \frac{n^2V(K)^{(n+p)/n}V(K^*)^{(n+p)/n}}{\omega_n^{2p/n}}. \tag{5.13}$$

The Blaschke–Santaló inequality, in conjunction with inequality (5.13), gives the generalized consequence of (5.11): If $K \in \mathcal{K}_c^n$ and $p \ge 1$, then

$$G_p(K)G_p\big(K^*\big) \le (n\omega_n)^2, \tag{5.14}$$

with equality if and only if K is an ellipsoid.

As the extension of inequality (5.14) we obtain an analogue of Blaschke–Santaló inequality for the ith L_p mixed geominimal surface area.

Theorem 5.3. *If $K, L \in \mathcal{F}_c^n$, $n \ne p \ge 1$, $i \in \mathbb{R}$, and $0 \le i \le n$, then*

$$G_{p,i}(K,L)G_{p,i}\big(K^*,L^*\big) \le (n\omega_n)^2. \tag{5.15}$$

The equality holds for $0 < i < n$ if K and L are dilated ellipsoids of each other. The inequality holds as an equality for $i = 0$ (or $i = n$) if K (or L) is an ellipsoid.

Proof. For $0 < i < n$, via (5.7) and (5.14), we obtain

$$\begin{aligned} G_{p,i}(K,L)^nG_{p,i}\big(K^*,L^*\big)^n &\le \big[G_p(K)G_p\big(K^*\big)\big]^{n-i}\big[G_p(L)G_p\big(L^*\big)\big]^i \\ &\le (n\omega_n)^{2n}. \end{aligned}$$

That is,

$$G_{p,i}(K,L)G_{p,i}\big(K^*,L^*\big) \le (n\omega_n)^2.$$

The equality holds if K and L are dilated ellipsoids of each other.

For $i = 0$ (or $i = n$), from (5.4) and inequality (5.7), the inequality (5.15) is obviously true, and with equality if K (or L) is an ellipsoid. □

Recall the classical isoperimetric inequality:

$$\left(\frac{\mathrm{Area}(K)}{\mathrm{Area}(B)}\right)^n \geq \left(\frac{V(K)}{V(B)}\right)^{n-1},$$

the equality holds if and only if K is a ball. Here Area$(\cdot)$ denotes the general surface area.

We now establish generalized isoperimetric inequalities for $G_{p,i}(K)$.

Theorem 5.4. *If $K \in \mathcal{F}_o^n$, then*

(i) *If $p \geq 1$ and $0 \leq i \leq n$,*

$$\frac{G_{p,i}(K)}{G_{p,i}(B)} \leq \left(\frac{V(K)}{V(B)}\right)^{\frac{(n-p)(n-i)}{n^2}},$$

with equality if K is a ball.

(ii) *If $p \geq 1$ and $i \geq n$,*

$$\frac{G_{p,i}(K)}{G_{p,i}(B)} \geq \left(\frac{V(K)}{V(B)}\right)^{\frac{(n-p)(n-i)}{n^2}},$$

with equality if K is a ball.

Proof. (i) For $i = 0$, by (5.3), we have

$$\frac{G_p(K)}{G_p(B)} \leq \left(\frac{V(K)}{V(B)}\right)^{\frac{n-p}{n}}.$$

This is Lutwak's inequality (3.3).

For $i = n$, by (5.2), (5.3) and (5.4), the equality holds trivially.

For $0 < i < n$, the inequality (5.9) gives

$$\left(\frac{G_{p,i}(K)}{G_{p,i}(B)}\right)^n \leq \left(\frac{G_p(K)}{G_p(B)}\right)^{n-i},$$

with equality if K is a ball.

Since $G_{p,i}(B) = G_p(B) = n\omega_n$, we obtain the following isoperimetric inequality as a consequence of the L_p isoperimetric inequality (3.3).

$$\frac{G_{p,i}(K)}{G_{p,i}(B)} \leq \left(\frac{G_p(K)}{G_p(B)}\right)^{\frac{n-i}{n}} \leq \left(\frac{V(K)}{V(B)}\right)^{\frac{(n-p)(n-i)}{n^2}},$$

with equality if K is a ball.

(ii) For $i = n$, by (5.2), (5.3) and (5.4), the equality holds trivially. We now prove the case $i > n$. The inequality (5.8) gives

$$\left(\frac{G_{p,i}(K)}{G_{p,i}(B)}\right)^n \geq \left(\frac{G_p(K)}{G_p(B)}\right)^{n-i}. \tag{5.16}$$

Hence for $i > n$ and $p \geq 1$, the L_p affine isoperimetric inequality (3.3) implies that

$$\frac{G_{p,i}(K)}{G_{p,i}(B)} \geq \left(\frac{G_p(K)}{G_p(B)}\right)^{\frac{n-i}{n}} \geq \left(\frac{V(K)}{V(B)}\right)^{\frac{(n-p)(n-i)}{n^2}},$$

with equality if K is a ball. □

Corollary 5.3. *If $K \in \mathcal{F}_c^n$, then*

(i) *If $p \geq 1$ and $0 \leq i \leq n$,*

$$G_{p,i}(K)G_{p,i}(K^*) \leq (n\omega_n)^2$$

with equality if K is a ball.

(ii) *If $p \geq 1$ and $i \geq n$,*

$$G_{p,i}(K)G_{p,i}(K^*) \geq (n\omega_n)^2$$

with equality if K is a ball.

Proof. (i) The inequality $G_{p,i}(K)G_{p,i}(K^*) \leq (n\omega_n)^2$ follows from Theorem 5.3 with $L = B$.

(ii) By inequalities (5.16) and (5.14), one has for all $i \geq n$

$$\left(\frac{G_{p,i}(K)G_{p,i}(K^*)}{G_{p,i}(B)^2}\right)^n \geq \left(\frac{G_p(K)G_p(K^*)}{G_p(B)^2}\right)^{n-i} \geq 1,$$

or equivalently, $G_{p,i}(K)G_{p,i}(K^*) \geq (n\omega_n)^2$, with equality if K is a ball. □

Acknowledgments

The authors would like to thank the anonymous referees, whose invaluable comments and suggestions have directly led to an improvement of the original manuscript.

References

[1] W. Blaschke, Vorlesungen über Differentialgeometrie II, Affine Differentialgeometrie, Springer-Verlag, Berlin, 1923.
[2] W. Chen, L_p Minkowski problem with not necessarily positive data, Adv. Math. 201 (2006) 77–89.
[3] K. Chou, X. Wang, The L_p Minkowski problem and the Minkowski problem in centroaffine geometry, Adv. Math. 205 (2006) 33–83.
[4] A. Cianchi, E. Lutwak, D. Yang, G. Zhang, Affine Moser–Trudinger and Morrey–Sobolev inequalities, Calc. Var. Partial Differential Equations 36 (2009) 419–436.
[5] W.J. Firey, p-Means of convex bodies, Math. Scand. 10 (1962) 17–24.
[6] B. Fleury, O. Guédon, G. Paouris, A stability result for mean width of L_p centroid bodies, Adv. Math. 214 (2007) 865–877.
[7] R.J. Gardner, Geometric Tomography, Cambridge Univ. Press, Cambridge, 1995.
[8] C. Haberl, F.E. Schuster, General L_p affine isoperimetric inequalities, J. Differential Geom. 83 (2009) 1–26.
[9] C. Haberl, F.E. Schuster, Asymmetric affine L_p Sobolev inequalities, J. Funct. Anal. 257 (2009) 641–658.
[10] C. Haberl, F. Schuster, J. Xiao, An asymmetric affine Pólya–Szegö principle, Math. Ann. 352 (2012) 517–542.
[11] G.H. Hardy, J.E. Littlewood, G. Pólya, Inequalities, Cambridge Univ. Press, London, 1934.
[12] C. Hu, X. Ma, C. Shen, On the Christoffel–Minkowski problem of Fiery's p-sum, Calc. Var. Partial Differential Equations 21 (2004) 137–155.
[13] M. Ludwig, Ellipsoids and matrix-valued valuations, Duke Math. J. 119 (2003) 159–188.
[14] M. Ludwig, General affine surface areas, Adv. Math. 224 (2010) 2346–2360.
[15] M. Ludwig, M. Reitzner, A characterization of affine surface area, Adv. Math. 147 (1999) 138–172.
[16] M. Ludwig, M. Reitzner, A classification of $SL(n)$ invariant valuations, Ann. Math. 172 (2010) 1223–1271.
[17] M. Ludwig, C. Schütt, E. Werner, Approximation of the Euclidean ball by polytopes, Studia Math. 173 (2006) 1–18.

[18] E. Lutwak, Dual mixed volumes, Pacific J. Math. 58 (1975) 531–538.
[19] E. Lutwak, Extended affine surface area, Adv. Math. 85 (1991) 39–68.
[20] E. Lutwak, The Brunn–Minkowski–Firey theory I: mixed volumes and the Minkowski problem, J. Differential Geom. 38 (1993) 131–150.
[21] E. Lutwak, The Brunn–Minkowski–Firey theory II: affine and geominimal surface areas, Adv. Math. 118 (1996) 244–294.
[22] E. Lutwak, G. Zhang, Blaschke–Santaló inequalities, J. Differential Geom. 47 (1997) 1–16.
[23] E. Lutwak, D. Yang, G. Zhang, L_p affine isoperimetric inequalities, J. Differential Geom. 56 (2000) 111–132.
[24] E. Lutwak, D. Yang, G. Zhang, Sharp affine L_p Sobolev inequalities, J. Differential Geom. 62 (2002) 17–38.
[25] E. Lutwak, D. Yang, G. Zhang, On the L_p Minkowski problem, Trans. Amer. Math. Soc. 356 (2004) 4359–4370.
[26] M. Meyer, E. Werner, On the p-affine surface area, Adv. Math. 152 (2000) 288–313.
[27] C.M. Petty, Geominimal surface area, Geom. Dedicata 3 (1974) 77–97.
[28] R. Schneider, Convex Bodies: The Brunn–Minkowski Theory, Cambridge Univ. Press, Cambridge, 1993.
[29] R. Schneider, Affine surface area and convex bodies of elliptic type, to appear.
[30] F.E. Schuster, Volume inequalities and additive maps of convex bodies, Mathematika 53 (2006) 211–234.
[31] F.E. Schuster, T. Wannerer, $GL(n)$ contravariant Minkowski valuations, Trans. Amer. Math. Soc. 364 (2012) 815–826.
[32] C. Schütt, E. Werner, Surface bodies and p-affine surface area, Adv. Math. 187 (2004) 98–145.
[33] A. Stancu, E. Werner, New higher-order equiaffine invariants, Israel J. Math. 171 (2009) 221–235.
[34] W. Wang, G. Leng, L_p-mixed affine surface area, J. Math. Anal. Appl. 335 (2007) 341–354.
[35] E. Werner, On L_p affine surface areas, Indiana Univ. Math. J. 56 (2007) 2305–2323.
[36] E. Werner, D. Ye, New L_p affine isoperimetric inequalities, Adv. Math. 218 (2008) 762–780.
[37] E. Werner, D. Ye, Inequalities for mixed p-affine surface area, Math. Ann. 347 (2010) 703–737.
[38] D. Ye, Inequalities for general mixed affine surface areas, J. Lond. Math. Soc. 85 (2012) 101–120.
[39] D. Ye, On the L_p geominimal surface area and related inequalities, Int. Math. Res. Not. (2014).
[40] D. Ye, B. Zhu, J. Zhou, The mixed L_p geominimal surface areas for multiple convex bodies, arXiv preprint, arXiv:1311.5180, 2013.
[41] J. Yuan, S. Lv, G. Leng, The p-affine surface area, Math. Inequal. Appl. 3 (2007) 693–702.
[42] B. Zhu, N. Li, J. Zhou, Isoperimetric inequalities for L_p geominimal surface area, Glasg. Math. J. 53 (2011) 717–726.
[43] B. Zhu, J. Zhou, W. Xu, Affine isoperimetric inequalities for L_p geominimal surface area, in: Proceedings of ICMSC 2014, Springer, 2014.