The mixed L_p geominimal surface areas for multiple convex bodies *

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Abstract

In this paper, we introduce several mixed L_p geominimal surface areas for multiple convex bodies for all $-n \neq p \in \mathbb{R}$. Our definitions are motivated from an equivalent formula for the mixed p-affine surface area. Some properties, such as the affine invariance, for these mixed L_p geominimal surface areas are proved. Related inequalities, such as, Alexander-Fenchel type inequality, Santaló style inequality, affine isoperimetric inequalities, and cyclic inequalities are established. Moreover, we also study some properties and inequalities for the i-th mixed L_p geominimal surface areas for two convex bodies.

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1 Introduction

The combination of the Minkowski sum and the volume naturally leads to the mixed volume for multiple convex bodies (i.e., convex compact subsets in $\mathbb{R}^n$ with nonempty interior), which is now the very core of the Brunn-Minkowski theory of convex bodies. Numerous widely-studied functionals on convex bodies, e.g., the volume and the surface area, are special cases of the mixed volume. The mixed volume has many nice properties and important inequalities which are fundamental in applications. For instance, the Alexander-Fenchel inequality related to the mixed volume is one of the most important inequalities in convex geometry. Many fundamental geometric inequalities, such as, the Minkowski's first inequality and the Brunn-Minkowski inequality for convex bodies, can be derived from the Alexander-Fenchel inequality. Readers are referred to [29] for more details and more references regarding the mixed volume and related properties.

The rapidly developing L_p Brunn-Minkowski theory of convex bodies is a natural extension of the Brunn-Minkowski theory. Such an extension arises from the combination of the volume

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and the Firey p-sum of convex bodies for $p \geq 1$ introduced by Firey in [7] about 50 years ago. It has been thought that the L_p Brunn-Minkowski theory of convex bodies has the L_p affine surface area as its core. The L_p affine surface area was introduced by Lutwak in his seminal paper [22] about 70 years after the classical affine surface area (i.e., for $p = 1$) was first introduced by Blaschke [4] in 1923. One of the most remarkable result about the L_p affine surface area is the characterization theorem: roughly speaking, any upper (lower) semi-continuous and affine invariant (with certain homogeneous degree) valuation on convex bodies is essentially (up to a multiplicative constant) the L_p affine surface area for some $p > 0$ ($-n < p < 0$) [16, 17]. Contributions on the L_p affine surface area include [14, 20, 23, 24, 31, 32, 34, 37] among others. In particular, the L_p affine surface area plays fundamental roles in the theory of valuations (see e.g. [2, 3, 16, 17]), in approximation of convex bodies by polytopes (see e.g. [8, 15, 32]) and in the information theory of convex bodies (see e.g., [11, 26, 35, 36]). A full set of affine isoperimetric inequalities related to the L_p affine surface area has been established in [22, 37]. That is, among convex bodies with fixed volume and with centroid at the origin, the L_p affine surface area attains the maximum (minimum) at and only at origin-symmetric ellipsoids for $p > 0$ ($-n < p < 0$).

Closely related to the L_p affine surface area, the L_p geominimal surface area for $p \geq 1$ introduced in [22, 27] has many nice properties similar to those of the L_p affine surface area, such as affine invariance with the same homogeneous degrees. It is well-known that the L_p geominimal surface for $p \geq 1$ is continuous but the L_p affine surface area is only upper semi-continuous. Hence these two closely related concepts are different from each other. Moreover, the L_p geominimal surface area does not have a "nice" integral expression similar to that for the L_p affine surface area (which is essential for extending the L_p affine surface area from $p \geq 1$ to all $-n \neq p \in \mathbb{R}$). Recently, motivated by an equivalent formula for the L_p affine surface area, the L_p geominimal surface area was successfully extended to all $-n \neq p \in \mathbb{R}$ by the first author in [41]. When $p = 1$, one gets the classical geominimal surface area [27], which serves as the bridge between the affine geometry, relative geometry and Minkowski geometry (as claimed in [27]). Contributions include [22, 27, 28, 30, 41, 42] among others. Affine isoperimetric inequalities related to the L_p geominimal surface area can be found in [22, 27, 28, 41]. That is, origin-symmetric ellipsoids are the only maximizers (minimizers) of the L_p geominimal surface area for $p > 0$ (for $-n < p < 0$), among convex bodies with fixed volume and with centroid at the origin.

A well-studied concept in the literature of convex geometry is the L_p affine surface area for multiple convex bodies, named as the mixed p-affine surface area (see e.g., [19, 22, 33, 38]). The mixed p-affine surface area contains many important functionals on convex bodies as special cases, such as the L_p affine surface area and the dual mixed volume. From the information theory point of view, the mixed p-affine surface area is a very special f-divergence of the distributions associated with multiple convex bodies [36, 39], which can be used to measure the similarity between multiple convex bodies. Note that the definition of the mixed p-affine surface area is also based on a nice integral expression for the L_p affine surface area. Moreover, such a nice integral expression makes it possible to define the general mixed affine surface areas involving nonhomogenous convex and concave functions [40]. Alexander-Fenchel type inequalities and affine isoperimetric inequalities for the mixed p-affine surface area were established in [18, 19, 22, 38]. On the other hand, by using the so-called p-Petty body for $p \geq 1$, the second and the third authors propose a way to define the mixed L_p geominimal surface area for multiple convex bodies [43] for $p \geq 1$. Related Alexander-Fenchel type inequalities and affine isoperimetric inequalities were also established in [43].

In this paper, we further extend the mixed L_p geominimal surface area to all $-n \neq p \in \mathbb{R}$. Our definitions are motivated from an equivalent formula for the mixed p-affine surface area (see Proposition 3.1 and Theorem 3.1), which differ from the definition in [43]. Similar idea was used to successfully extend the L_p geominimal surface area to all $-n \neq p \in \mathbb{R}$ in [41]. Our paper is organized as follows. Section 2 is for background and notation. In Section 3, we prove Proposition 3.1 and Theorem 3.1. Hence, an equivalent formula for the mixed p-affine surface area is provided. In particular, the mixed p-affine surface area could be rewritten as the dual mixed volume of the p-curvature images of corresponding convex bodies. Section 4 is dedicated to the definition of our mixed L_p geominimal surface areas for multiple convex bodies for all $-n \neq p \in \mathbb{R}$. Properties such as affine invariance are proved. Moreover, relations between mixed L_p geominimal surface areas and the mixed p-affine surface area are discussed. Alexander-Fenchel type inequality, affine isoperimetric inequality, the Santaló style inequality, and the cyclic inequality are proved in Section 5. The i-th mixed L_p geominimal surface areas and related isoperimetric inequality are given in Section 6. We refer readers to [9, 13, 29] for more background in convex geometry.

2 Background and Notation

We will work on $\mathbb{R}^n$ with inner product $\langle \cdot, \cdot \rangle$ and the Euclidean norm $\| \cdot \|$. We use $B_2^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ and $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$ for the unit Euclidean ball and the unit sphere in $\mathbb{R}^n$, respectively. For a subset $K \subset \mathbb{R}^n$, its Hausdorff content is denoted by $|K|$. In particular, the volume of B_2^n is written as $\omega_n = |B_2^n|$.

A set $L \subset \mathbb{R}^n$ is star-convex about the origin 0 if for each $x \in L$, the line segment from 0 to x is contained in L. The radial function of L, denoted by $\rho_L : S^{n-1} \to [0, \infty)$, is defined by $\rho_L(u) = \max\{\lambda \geq 0 : \lambda u \in L\}$. If ρ_L is positive and continuous, then L is called a star-convex body about the origin. The set of all star-convex bodies about the origin is denoted by $\mathscr{S}_0$. We say that $L_1, L_2 \in \mathscr{S}_0$ are dilates of one another if there is a constant $\lambda > 0$, such that $\rho_{L_1}(u) = \lambda \rho_{L_2}(u)$ for all $u \in S^{n-1}$. The volume of $L \in \mathscr{S}_0$ can be calculated by

$$|L| = \frac{1}{n} \int_{S^{n-1}} \rho_L^n(u) d\sigma(u),$$

where σ is the spherical measure on S^{n-1}.

We say that $K \subset \mathbb{R}^n$ is a convex body if K is a compact, convex subset in $\mathbb{R}^n$ with non-empty interior. The set of all convex bodies is written as $\mathscr{K}$, and its subset $\mathscr{K}_0$ denotes the set of convex bodies containing the origin in their interiors. Similarly, we use $\mathscr{K}_c$ for the set of convex bodies with centroid at the origin. Besides the radial function, any convex body can be uniquely determined by its support function. Here, for $K \in \mathscr{K}_0$, its support function $h_K : S^{n-1} \to [0, \infty)$ is defined by $h_K(u) = \max\{\langle x, u\rangle : x \in K\}$. Associated with each $K \in \mathscr{K}_0$, one can uniquely define its polar body $K^\circ \in \mathscr{K}_0$ by

$$K^\circ = \{x \in \mathbb{R}^n : \langle x, y\rangle \le 1, \quad \forall y \in K\}.$$

It is easily verified that $(K^\circ)^\circ = K$ for all $K \in \mathscr{K}_0$. Moreover,

$$h_{K^\circ}(u)\rho_K(u) = 1 \ \ \& \ \ \rho_{K^\circ}(u)h_K(u) = 1, \quad \text{for all} \quad u \in S^{n-1}.$$

A convex body $K \in \mathscr{K}_0$ is said to have Santaló point at the origin, if K° has centroid at the origin, i.e., $K \in \mathscr{K}_s \Leftrightarrow K^\circ \in \mathscr{K}_c$. Hereafter $\mathscr{K}_s \subset \mathscr{K}_0$ denotes the set of all convex bodies with Santaló point at the origin.

The p-mixed volume, $V_p(K, L)$, of $K, L \in \mathscr{K}_0$ for $p \ge 1$ was defined in [21] by

$$\frac{n}{p}V_p(K, L) = \lim_{\varepsilon\to 0} \frac{|K +_p \varepsilon L| - |K|}{\varepsilon},$$

where $K +_p \varepsilon L$ is a convex body with support function defined by

$$\big(h_{K+_p\varepsilon L}(u)\big)^p = \big(h_K(u)\big)^p + \varepsilon\big(h_L(u)\big)^p, \quad \forall u \in S^{n-1}.$$

This sum is the well-known Firey p-sum [7], which generalizes the famous Minkowski sum (i.e., $p = 1$). Note that the Minkowski sum $\lambda K + \eta L$ for $K, L \in \mathscr{K}_0$ and for $\lambda, \eta > 0$ is defined by

$$h_{\lambda K+\eta L}(u) = \lambda h_K(u) + \eta h_L(u), \ \ \forall u \in S^{n-1}.$$

It is well-known that for all $\lambda_1, \cdots, \lambda_m > 0$ and $K_1, \cdots, K_m \in \mathscr{K}_0$ with $m \in \mathbb{N}$, one has

$$|\lambda_1 K_1 + \cdots + \lambda_m K_m| = \sum_{i_1,\cdots,i_n=1}^{m} \lambda_{i_1} \cdots \lambda_{i_n} V(K_{i_1}, \cdots, K_{i_n}).$$

The coefficient $V(K_{i_1}, \cdots, K_{i_n})$, named as the mixed volume of $K_{i_1}, \cdots, K_{i_n}$, is invariant under permutations of $K_{i_1}, \cdots, K_{i_n}$. The classical Alexander-Fenchel inequality for the mixed volume (see [15, 29]) asserts that for all $m \in \mathbb{N}$ such that $1 \le m \le n$,

$$\prod_{i=0}^{m-1} V(K_1, \cdots, K_{n-m}, \underbrace{K_{n-i}, \cdots, K_{n-i}}_{m}) \le V(K_1, \cdots, K_n)^m.$$

In particular, if $m = n$, one has the Minkowski inequality

$$V(K_1, \cdots, K_n)^n \ge |K_1| \cdots |K_n|. \tag{2.1}$$

It was proved in [21] that for each $K \in \mathscr{K}_0$, there is a positive Borel measure $S_p(K,\cdot)$ on S^{n-1} such that, for each $L \in \mathscr{K}_0$ and $p \geq 1$,

$$V_p(K,L) = \frac{1}{n}\int_{S^{n-1}} h_L(u)^p dS_p(K,u).$$

Moreover, the measure $S_p(K,\cdot)$ for $p \geq 1$ has the following form

$$dS_p(K,\cdot) = h_K(\cdot)^{1-p} dS(K,\cdot),$$

where the measure $S(K,\cdot)$ is just the classical surface area measure of K (see [1, 6]). We write $K \in \mathscr{F}_0$ if $K \in \mathscr{K}_0$ has a curvature function, namely, the measure $S(K,\cdot)$ is absolutely continuous with respect to the spherical measure σ. Hence, there is a function $f_K : S^{n-1} \to \mathbb{R}$, the *curvature function* of K, such that,

$$dS(K,u) = f_K(u) d\sigma(u).$$

The L_p *curvature function* for $K \in \mathscr{F}_0$ and $p \geq 1$, denoted by $f_p(K,u)$ (see [22]) then takes the form

$$f_p(K,u) = h_K(u)^{1-p} f_K(u).$$

We write $\mathscr{F}_c = \mathscr{F}_0 \cap \mathscr{K}_c$ and $\mathscr{F}_s = \mathscr{F}_0 \cap \mathscr{K}_s$ for convex bodies in $\mathscr{F}_0$ with centroid and the Santaló point at the origin respectively. The set of all convex bodies in $\mathscr{F}_0$ with continuous positive curvature function $f_K(\cdot)$ on S^{n-1} is denoted by $\mathscr{F}_0^+$.

The mixed p-volume for $p \geq 1$ was formally extended to $p < 1$ in [41]. Hereafter, the p-surface area measure of $K \in \mathscr{K}_0$ is

$$dS_p(K,u) = h_K(u)^{1-p} dS(K,u), \quad p \in \mathbb{R},$$

and the p-mixed volume of $K, Q \in \mathscr{K}_0$ is

$$V_p(K,Q) = \frac{1}{n}\int_{S^{n-1}} h_Q(u)^p dS_p(K,u), \quad p \in \mathbb{R}. \tag{2.2}$$

For $K \in \mathscr{K}_0$ and $L \in \mathscr{S}_0$, we let $V_p(K,L^\circ)$ be

$$V_p(K,L^\circ) = \frac{1}{n}\int_{S^{n-1}} \rho_L(u)^{-p} dS_p(K,u), \quad p \in \mathbb{R}.$$

This formula coincides with formula (2.2) if $L \in \mathscr{K}_0$. When $K \in \mathscr{F}_0$, one can define the L_p curvature function (and denoted by $f_p(K,\cdot)$) as

$$f_p(K,u) = h_K(u)^{1-p} f_K(u), \quad p \in \mathbb{R},$$

and hence the p-surface area measure can be formulated by

$$dS_p(K,u) = f_p(K,u) d\sigma(u), \quad p \in \mathbb{R}.$$

Now we can define the L_p geominimal surface area of $K \in \mathscr{K}_0$ as follows. See [27] for $p = 1$, [22] for $p > 1$ and [41] for all $-n \neq p \in \mathbb{R}$.

Definition 2.1 *Let $K \in \mathscr{K}_0$ be a convex body with the origin in its interior.*
(i). For $p = 0$, we let $\widetilde{G}_p(K) = n|K|$. For $p > 0$, the L_p geominimal surface area of K is defined by

$$\widetilde{G}_p(K) = \inf_{L \in \mathscr{K}_0} \left\{ nV_p(K,L)^{\frac{n}{n+p}} \ |L^\circ|^{\frac{p}{n+p}} \right\}.$$

(ii). For $-n \neq p < 0$, the L_p geominimal surface area of K is defined by

$$\widetilde{G}_p(K) = \sup_{L \in \mathscr{K}_0} \left\{ nV_p(K,L)^{\frac{n}{n+p}} \ |L^\circ|^{\frac{p}{n+p}} \right\}.$$

We let $\mathbf{L} = (L_1, \cdots, L_n)$ be a vector with each $L_i \subset \mathbb{R}^n$, and $\mathbf{L} \in \mathscr{S}_0^n$ means each $L_i \in \mathscr{S}_0$. We use $\mathbf{L}^\circ$ for $(L_1^\circ, \cdots, L_n^\circ)$. Similarly, we let $\mathbf{K} = (K_1, \cdots, K_n)$ and $\mathbf{K} \in \mathscr{F}_0^n$ means each $K_i \in \mathscr{F}_0$. For all $\{K_i\}_{i=1}^n \subset \mathscr{F}_0$, $\{Q_i\}_{i=1}^n \subset \mathscr{K}_0$, and $p \in \mathbb{R}$, we define

$$V_p(\mathbf{K};\mathbf{Q}) = \frac{1}{n} \int_{S^{n-1}} \prod_{i=1}^n \left[h_{Q_i}(u)^p f_p(K_i,u)\right]^{\frac{1}{n}} \, d\sigma(u). \tag{2.3}$$

When all K_i coincide with K and all Q_i coincide with Q, one can easily get

$$V_p(\mathbf{K};\mathbf{Q}) = V_p(K,Q).$$

When $L_1, \cdots, L_n \in \mathscr{S}_0$ and $p \in \mathbb{R}$, we use the following variation formula

$$V_p(\mathbf{K};\mathbf{L}^\circ) = \frac{1}{n} \int_{S^{n-1}} \prod_{i=1}^n \left[\rho_{L_i}(u)^{-p} f_p(K_i,u)\right]^{\frac{1}{n}} \, d\sigma(u), \tag{2.4}$$

which is consistent with formula (2.3) when $\mathbf{L} \in \mathscr{K}_0^n$. When all K_i coincide with K and all L_i coincide with L, one gets

$$V_p(\mathbf{K};\mathbf{L}^\circ) = V_p(K, L^\circ). \tag{2.5}$$

We use $\widetilde{V}(\mathbf{L})$ to denote the dual mixed volume of $L_1, \cdots, L_n \in \mathscr{S}_0$ (see[18]). That is,

$$\widetilde{V}(\mathbf{L}) = \widetilde{V}(L_1, \cdots, L_n) = \frac{1}{n} \int_{S^{n-1}} \prod_{i=1}^n \rho_{L_i}(u) \, d\sigma(u).$$

When $L_1 = L_2 = \cdots = L_n = L$, one has $\widetilde{V}(\mathbf{L}) = |L|$. It is easy to get the following inequality for the dual mixed volume:

$$[\widetilde{V}(\mathbf{L})]^n = [\widetilde{V}(L_1, \cdots, L_n)]^n \leq |L_1| \cdots |L_n|, \tag{2.6}$$

with equality if and only if L_i $(1 \leq i \leq n)$ are dilates of each other.

The group of nonsingular linear transformations is denoted by $GL(n)$. Its subset $SL(n)$ refers to the group of special linear transformations. For $\phi \in GL(n)$, the absolute value of the determinant, the transpose and the inverse of ϕ will be denoted by $|det(\phi)|$, ϕ^t and ϕ^{-1} respectively. For $\mathbf{K} = (K_1, \cdots, K_n) \in \mathscr{K}_0^n$, we let $\phi\mathbf{K} = (\phi K_1, \cdots, \phi K_n)$. An origin-symmetric ellipsoid $\mathscr{E} \in \mathscr{K}_0$ can be obtained from B_2^n under some $\phi \in GL(n)$, that is, $\mathscr{E} = \phi B_2^n$ for some $\phi \in GL(n)$. Note that $(\phi L)^\circ = (\phi^t)^{-1} L^\circ$ for $\phi \in GL(n)$ and $L \in \mathscr{S}_0$, then

$$\widetilde{V}((\phi\mathbf{L})^\circ) = |det(\phi)|^{-1} \widetilde{V}(\mathbf{L}^\circ). \tag{2.7}$$

3 The mixed p-affine surface area, another view

In this section, we will prove some alternative formulas for the mixed p-affine surface area. Our definition for the mixed L_p geominimal surface areas for multiple convex bodies is motivated by these alternative formulas.

Proposition 3.1 *Let $K_1, \cdots, K_n \in \mathscr{F}_0$.*
(i). For $p \geq 0$, one has

$$\begin{aligned}\inf_{\mathbf{L}\in\mathscr{S}_0^n} \left\{ nV_p(\mathbf{K};\mathbf{L}^\circ)^{\frac{n}{n+p}}\ \widetilde{V}(\mathbf{L})^{\frac{p}{n+p}} \right\} &= \inf_{\mathbf{L}\in\mathscr{S}_0^n} \left\{ nV_p(\mathbf{K};\mathbf{L}^\circ)^{\frac{n}{n+p}}\ \prod_{i=1}^n |L_i|^{\frac{p}{(n+p)n}} \right\} \\ &= \inf_{L\in\mathscr{S}_0} \left\{ nV_p(\mathbf{K};L^\circ,\cdots,L^\circ)^{\frac{n}{n+p}}\ |L|^{\frac{p}{n+p}} \right\}.\end{aligned}$$

(ii). For $-n < p < 0$, one has

$$\begin{aligned}\sup_{\mathbf{L}\in\mathscr{S}_0^n} \left\{ nV_p(\mathbf{K};\mathbf{L}^\circ)^{\frac{n}{n+p}}\ \widetilde{V}(\mathbf{L})^{\frac{p}{n+p}} \right\} &= \sup_{\mathbf{L}\in\mathscr{S}_0^n} \left\{ nV_p(\mathbf{K};\mathbf{L}^\circ)^{\frac{n}{n+p}}\ \prod_{i=1}^n |L_i|^{\frac{p}{(n+p)n}} \right\} \\ &= \sup_{L\in\mathscr{S}_0} \left\{ nV_p(\mathbf{K};L^\circ,\cdots,L^\circ)^{\frac{n}{n+p}}\ |L|^{\frac{p}{n+p}} \right\}.\end{aligned}$$

(iii). For $p < -n$, one has

$$\sup_{\mathbf{L}\in\mathscr{S}_0^n} \left\{ nV_p(\mathbf{K};\mathbf{L}^\circ)^{\frac{n}{n+p}}\ \widetilde{V}(\mathbf{L})^{\frac{p}{n+p}} \right\} = \sup_{L\in\mathscr{S}_0} \left\{ nV_p(\mathbf{K};L^\circ,\cdots,L^\circ)^{\frac{n}{n+p}}\ |L|^{\frac{p}{n+p}} \right\}.$$

Proof. First, notice that there is a sequence $\{\mathbf{L}_j\}_{j=1}^\infty \subset \mathscr{S}_0^n$, s.t.,

$$\inf_{\mathbf{L}\in\mathscr{S}_0^n} \left\{ nV_p(\mathbf{K};\mathbf{L}^\circ)^{\frac{n}{n+p}}\ \widetilde{V}(\mathbf{L})^{\frac{p}{n+p}} \right\} = \lim_{j\to\infty} \left\{ nV_p(\mathbf{K};\mathbf{L}_j^\circ)^{\frac{n}{n+p}}\ \widetilde{V}(\mathbf{L}_j)^{\frac{p}{n+p}} \right\},\quad p>0; \tag{3.8}$$

$$\sup_{\mathbf{L}\in\mathscr{S}_0^n} \left\{ nV_p(\mathbf{K};\mathbf{L}^\circ)^{\frac{n}{n+p}}\ \widetilde{V}(\mathbf{L})^{\frac{p}{n+p}} \right\} = \lim_{j\to\infty} \left\{ nV_p(\mathbf{K};\mathbf{L}_j^\circ)^{\frac{n}{n+p}}\ \widetilde{V}(\mathbf{L}_j)^{\frac{p}{n+p}} \right\},\quad -n\neq p<0. \tag{3.9}$$

For each $\mathbf{L}_j = (L_{1j}, \cdots, L_{nj}) \in \mathscr{S}_0^n$, one defines $L^j \in \mathscr{S}_0$ by $\rho_{L^j}^n(u) = \prod_{i=1}^n \rho_{L_{ij}}(u)$. Hence, for all j,

$$\widetilde{V}(\mathbf{L}_j) = \frac{1}{n}\int_{S^{n-1}} \prod_{i=1}^n \rho_{L_{ij}}\, d\sigma(u) = \frac{1}{n}\int_{S^{n-1}} \rho_{L^j}(u)^n\, d\sigma(u) = |L^j|. \tag{3.10}$$

Moreover, by formula (2.4), one has,

$$\begin{aligned}V_p(\mathbf{K};\mathbf{L}_j^\circ) &= \frac{1}{n}\int_{S^{n-1}} \prod_{i=1}^n [\rho_{L_{ij}}(u)^{-p}]^{\frac{1}{n}} \prod_{i=1}^n [f_p(K_i,u)]^{\frac{1}{n}}\, d\sigma(u) \\ &= \frac{1}{n}\int_{S^{n-1}} \rho_{L^j}(u)^{-p} \prod_{i=1}^n [f_p(K_i,u)]^{\frac{1}{n}}\, d\sigma(u) = V_p(\mathbf{K};(L^j)^\circ,\cdots,(L^j)^\circ).\end{aligned} \tag{3.11}$$

(i). The case of $p = 0$ is clear. Let $p \in (0, \infty)$, then

$$\begin{aligned}\inf_{\mathbf{L}\in\mathscr{S}_0^n}\left\{nV_p(\mathbf{K};\mathbf{L}^\circ)^{\frac{n}{n+p}}\ \widetilde{V}(\mathbf{L})^{\frac{p}{n+p}}\right\} &\le \inf_{\mathbf{L}\in\mathscr{S}_0^n}\left\{nV_p(\mathbf{K};\mathbf{L}^\circ)^{\frac{n}{n+p}}\ \prod_{i=1}^n |L_i|^{\frac{p}{(n+p)n}}\right\}\\ &\le \inf_{L\in\mathscr{S}_0}\left\{nV_p(\mathbf{K};L^\circ,\cdots,L^\circ)^{\frac{n}{n+p}}\ |L|^{\frac{p}{n+p}}\right\},\end{aligned}$$

where the first inequality follows from inequality (2.6) and the second inequality is due to $(L,\cdots,L)\in\mathscr{S}_0^n$. On the other hand, formulas (3.8), (3.10) and (3.11) imply that

$$\begin{aligned}\inf_{\mathbf{L}\in\mathscr{S}_0^n}\left\{nV_p(\mathbf{K};\mathbf{L}^\circ)^{\frac{n}{n+p}}\ \widetilde{V}(\mathbf{L})^{\frac{p}{n+p}}\right\} &= \lim_{j\to\infty}\left\{nV_p(\mathbf{K};\mathbf{L}_j^\circ)^{\frac{n}{n+p}}\ \widetilde{V}(\mathbf{L}_j)^{\frac{p}{n+p}}\right\}\\ &= \lim_{j\to\infty}\left\{nV_p(\mathbf{K};(L^j)^\circ,\cdots,(L^j)^\circ)^{\frac{n}{n+p}}\ |L^j|^{\frac{p}{n+p}}\right\}\\ &\ge \inf_{L\in\mathscr{S}_0}\left\{nV_p(\mathbf{K};L^\circ,\cdots,L^\circ)^{\frac{n}{n+p}}\ |L|^{\frac{p}{n+p}}\right\}.\end{aligned}$$

(ii). Let $p\in(-n,0)$, then $\frac{p}{n+p}<0$. By formulas (3.9), (3.10) and (3.11), inequality (2.6), and $(L,\cdots,L)\in\mathscr{S}_0^n$, one has

$$\begin{aligned}\sup_{\mathbf{L}\in\mathscr{S}_0^n}\left\{nV_p(\mathbf{K};\mathbf{L}^\circ)^{\frac{n}{n+p}}\ \widetilde{V}(\mathbf{L})^{\frac{p}{n+p}}\right\} &\ge \sup_{\mathbf{L}\in\mathscr{S}_0^n}\left\{nV_p(\mathbf{K};\mathbf{L}^\circ)^{\frac{n}{n+p}}\ \prod_{i=1}^n |L_i|^{\frac{p}{(n+p)n}}\right\}\\ &\ge \sup_{L\in\mathscr{S}_0}\left\{nV_p(\mathbf{K};L^\circ,\cdots,L^\circ)^{\frac{n}{n+p}}\ |L|^{\frac{p}{n+p}}\right\}\\ &\ge \lim_{j\to\infty}\left\{nV_p(\mathbf{K};(L^j)^\circ,\cdots,(L^j)^\circ)^{\frac{n}{n+p}}\ |L^j|^{\frac{p}{n+p}}\right\}\\ &= \sup_{\mathbf{L}\in\mathscr{S}_0^n}\left\{nV_p(\mathbf{K};\mathbf{L}^\circ)^{\frac{n}{n+p}}\ \widetilde{V}(\mathbf{L})^{\frac{p}{n+p}}\right\}.\end{aligned}$$

(iii). Let $p<-n$. Due to $(L,\cdots,L)\in\mathscr{S}_0^n$, and formulas (3.9), (3.10) and (3.11), one has

$$\begin{aligned}\sup_{\mathbf{L}\in\mathscr{S}_0^n}\left\{nV_p(\mathbf{K};\mathbf{L}^\circ)^{\frac{n}{n+p}}\ \widetilde{V}(\mathbf{L})^{\frac{p}{n+p}}\right\} &\ge \sup_{L\in\mathscr{S}_0}\left\{nV_p(\mathbf{K};L^\circ,\cdots,L^\circ)^{\frac{n}{n+p}}\ |L|^{\frac{p}{n+p}}\right\}\\ &\ge \lim_{j\to\infty}\left\{nV_p(\mathbf{K};(L^j)^\circ,\cdots,(L^j)^\circ)^{\frac{n}{n+p}}\ |L^j|^{\frac{p}{n+p}}\right\}\\ &= \sup_{\mathbf{L}\in\mathscr{S}_0^n}\left\{nV_p(\mathbf{K};\mathbf{L}^\circ)^{\frac{n}{n+p}}\ \widetilde{V}(\mathbf{L})^{\frac{p}{n+p}}\right\}.\end{aligned}$$

Remark. For $\mathbf{K}\in\mathscr{F}_0^n$ and for $p<-n$, due to $(L,\cdots,L)\in\mathscr{S}_0^n$, one can prove that

$$\begin{aligned}\sup_{\mathbf{L}\in\mathscr{S}_0^n}\left\{nV_p(\mathbf{K};\mathbf{L}^\circ)^{\frac{n}{n+p}}\ \prod_{i=1}^n |L_i|^{\frac{p}{(n+p)n}}\right\} &\ge \sup_{L\in\mathscr{S}_0}\left\{nV_p(\mathbf{K};L^\circ,\cdots,L^\circ)^{\frac{n}{n+p}}\ |L|^{\frac{p}{n+p}}\right\}\\ &= \sup_{\mathbf{L}\in\mathscr{S}_0^n}\left\{nV_p(\mathbf{K};\mathbf{L}^\circ)^{\frac{n}{n+p}}\ \widetilde{V}(\mathbf{L})^{\frac{p}{n+p}}\right\}. \qquad (3.12)\end{aligned}$$

However, $\exists \mathbf{K} \in \mathscr{F}_0^n$, s.t. "$>$" holds (see the remark after Definition 3.1 for more details).

In literature, the mixed p-affine surface area of convex bodies $K_1, \cdots, K_n \in \mathscr{F}_0^+$, denoted by $as_p(K_1, \cdots, K_n)$, was defined by [22, 38]

$$as_p(K_1, \cdots, K_n) = \int_{S^{n-1}} \prod_{i=1}^{n} [f_p(K_i, u)]^{\frac{1}{n+p}} d\sigma(u). \tag{3.13}$$

When all K_i coincide with $K \in \mathscr{F}_0^+$, one gets, by Theorem 3.1 in [41],

$$as_p(K_1, \cdots, K_n) = as_p(K) = \begin{cases} \inf_{L \in \mathscr{S}_0} \left\{ nV_p(K, L^\circ)^{\frac{n}{n+p}} \ |L|^{\frac{p}{n+p}} \right\}, & p \geq 0; \\ \sup_{L \in \mathscr{S}_0} \left\{ nV_p(K, L^\circ)^{\frac{n}{n+p}} \ |L|^{\frac{p}{n+p}} \right\}, & -n \neq p < 0. \end{cases}$$

Theorem 3.1 *Let $K_1, \cdots, K_n \in \mathscr{F}_0^+$.*
(i). For $p \geq 0$, one has

$$as_p(K_1, \cdots, K_n) = \inf_{L \in \mathscr{S}_0} \left\{ nV_p(\mathbf{K}; L^\circ, \cdots, L^\circ)^{\frac{n}{n+p}} \ |L|^{\frac{p}{n+p}} \right\}.$$

(ii). For $-n \neq p < 0$, one has

$$as_p(K_1, \cdots, K_n) = \sup_{L \in \mathscr{S}_0} \left\{ nV_p(\mathbf{K}; L^\circ, \cdots, L^\circ)^{\frac{n}{n+p}} \ |L|^{\frac{p}{n+p}} \right\}.$$

Proof. First, notice that for all $-n \neq p \in \mathbb{R}$,

$$as_p(K_1, \cdots, K_n) = nV_p(\mathbf{K}; L_0^\circ, \cdots, L_0^\circ)^{\frac{n}{n+p}} \ |L_0|^{\frac{p}{n+p}}, \tag{3.14}$$

where $L_0 \in \mathscr{S}_0$ is defined by

$$\rho_{L_0}^n = \left(\prod_{i=1}^{n} f_p(K_i, u) \right)^{\frac{1}{n+p}} > 0, \quad \forall u \in S^{n-1}.$$

(i). Clearly it holds for $p = 0$. Let $p \in (0, \infty)$, then $\frac{n}{n+p} \in (0, 1)$. Employing Hölder inequality (see [10]) to formula (3.13), one has, for all $L \in \mathscr{S}_0$,

$$\begin{aligned} as_p(K_1, \cdots, K_n) &= \int_{S^{n-1}} \left(\prod_{i=1}^{n} [\rho_L(u)^{-p} f_p(K_i, u)]^{\frac{1}{n}} \right)^{\frac{n}{n+p}} \left(\rho_L^n(u) \right)^{\frac{p}{n+p}} d\sigma(u) \\ &\leq \left(\int_{S^{n-1}} \prod_{i=1}^{n} [\rho_L(u)^{-p} f_p(K_i, u)]^{\frac{1}{n}} d\sigma(u) \right)^{\frac{n}{n+p}} \left(\int_{S^{n-1}} \rho_L^n(u) d\sigma(u) \right)^{\frac{p}{n+p}} \\ &= nV_p(\mathbf{K}; L^\circ, \cdots, L^\circ)^{\frac{n}{n+p}} \ |L|^{\frac{p}{n+p}}. \end{aligned}$$

Taking infimum over $L \in \mathscr{S}_0$ and together with formula (3.14), one gets

$$as_p(K_1, \cdots, K_n) \leq \inf_{L\in\mathscr{S}_0} \left\{ nV_p(\mathbf{K}; L^\circ, \cdots, L^\circ)^{\frac{n}{n+p}} \ |L|^{\frac{p}{n+p}} \right\} \leq as_p(K_1, \cdots, K_n).$$

(ii). Note that $-n \neq p < 0$ implies $\frac{n}{n+p} > 1$ or $\frac{n}{n+p} < 0$. Employing Hölder inequality (see [10]) to formula (3.13), one has, for all $L \in \mathscr{S}_0$,

$$\begin{aligned} as_p(K_1, \cdots, K_n) &= \int_{S^{n-1}} \left(\prod_{i=1}^n [\rho_L(u)^{-p} f_p(K_i, u)]^{\frac{1}{n}} \right)^{\frac{n}{n+p}} \left(\rho_L^n(u) \right)^{\frac{p}{n+p}} d\sigma(u) \\ &\geq \left(\int_{S^{n-1}} \prod_{i=1}^n [\rho_L(u)^{-p} f_p(K_i, u)]^{\frac{1}{n}} d\sigma(u) \right)^{\frac{n}{n+p}} \left(\int_{S^{n-1}} \rho_L^n(u) d\sigma(u) \right)^{\frac{p}{n+p}} \\ &= nV_p(\mathbf{K}; L^\circ, \cdots, L^\circ)^{\frac{n}{n+p}} \ |L|^{\frac{p}{n+p}}. \end{aligned}$$

Taking supremum over $L \in \mathscr{S}_0$ and together with formula (3.14), one gets

$$as_p(K_1, \cdots, K_n) \geq \sup_{L\in\mathscr{S}_0} \left\{ nV_p(\mathbf{K}; L^\circ, \cdots, L^\circ)^{\frac{n}{n+p}} \ |L|^{\frac{p}{n+p}} \right\} \geq as_p(K_1, \cdots, K_n).$$

Motivated by Proposition 3.1 and Theorem 3.1, the mixed p-affine surface area for $K_1, \cdots, K_n \in \mathscr{F}_0$ may be defined as follows.

Definition 3.1 *Let $K_1, \cdots, K_n \in \mathscr{F}_0$.*
(i). For $p = 0$, we let

$$as_0(K_1, \cdots, K_n) = \int_{S^{n-1}} \prod_{i=1}^n [f_{K_i}(u) h_{K_i}(u)]^{\frac{1}{n}} d\sigma(u).$$

For $p > 0$, we define $as_p(K_1, \cdots, K_n)$ by

$$as_p(K_1, \cdots, K_n) = \inf_{L\in\mathscr{S}_0} \left\{ nV_p(\mathbf{K}; L^\circ, \cdots, L^\circ)^{\frac{n}{n+p}} \ |L|^{\frac{p}{n+p}} \right\}.$$

(ii). For $-n \neq p < 0$, we define $as_p(K_1, \cdots, K_n)$ by

$$as_p(K_1, \cdots, K_n) = \sup_{L\in\mathscr{S}_0} \left\{ nV_p(\mathbf{K}; L^\circ, \cdots, L^\circ)^{\frac{n}{n+p}} \ |L|^{\frac{p}{n+p}} \right\}.$$

Moreover, for $p < -n$, one can define another mixed p-affine surface area for $\mathbf{K} \in \mathscr{F}_0^n$ as

$$as_p^{(1)}(K_1, \cdots, K_n) = \sup_{\mathbf{L}\in\mathscr{S}_0^n} \left\{ nV_p(\mathbf{K}; \mathbf{L}^\circ)^{\frac{n}{n+p}} \ \prod_{i=1}^n |L_i|^{\frac{p}{(n+p)n}} \right\}.$$

Remark. For $\mathbf{K} \in \mathscr{F}_0^n$ and for $p < -n$, inequality (3.12) implies

$$as_p(K_1, \cdots, K_n) \ \leq \ as_p^{(1)}(K_1, \cdots, K_n).$$

In general, one *cannot* expect equality in the above inequality. To see this, suppose that $K_1, \cdots, K_n \in \mathscr{F}_0^+$. Proposition 2.1 in [38] and Theorem 3.1 imply

$$as_p(K_1, \cdots, K_n) \leq as_p(K_1)^{\frac{1}{n}} \cdots as_p(K_n)^{\frac{1}{n}},$$

with equality if and only if there are constants $\lambda_1, \lambda_2, \cdots, \lambda_n > 0$ such that $\lambda_i f_p(K_i, u) = \lambda_j f_p(K_j, u)$ for all $u \in S^{n-1}$. Now suppose that there is *no constant* $\lambda > 0$ satisfying $f_p(K_1, u) = \lambda f_p(K_2, u)$ almost everywhere with respect to the spherical measure σ, then

$$as_p(K_1, \cdots, K_n) < as_p(K_1)^{\frac{1}{n}} \cdots as_p(K_n)^{\frac{1}{n}}.$$

On the other hand, by Hölder inequality (see [10]), one gets

$$V_p(\mathbf{K}; \mathbf{L}^\circ) \leq \prod_{i=1}^n \left[V_p(K_i, L_i^\circ)\right]^{\frac{1}{n}}.$$

Note that $p < -n$ implies $\frac{n}{n+p} < 0$. Therefore,

$$\begin{aligned} as_p^{(1)}(K_1, \cdots, K_n) &= \sup_{\mathbf{L} \in \mathscr{S}_0^n} \left\{ nV_p(\mathbf{K}; \mathbf{L}^\circ)^{\frac{n}{n+p}} \ \prod_{i=1}^n |L_i|^{\frac{p}{(n+p)n}} \right\} \\ &\geq \prod_{i=1}^n \sup_{L_i \in \mathscr{S}_0} \left[nV_p(K_i, L_i^\circ)^{\frac{n}{n+p}} |L_i|^{\frac{p}{n+p}} \right]^{\frac{1}{n}} \\ &= as_p(K_1)^{\frac{1}{n}} \cdots as_p(K_n)^{\frac{1}{n}} > as_p(K_1, \cdots, K_n). \end{aligned}$$

The p-curvature image of K for $-n \neq p \in \mathbb{R}$ [22, 41] is denoted by $\Lambda_p K$ and defined by

$$f_p(K, u) = \frac{\omega_n}{|\Lambda_p K|} \rho_{\Lambda_p K}(u)^{n+p}, \quad \forall u \in S^{n-1}.$$

Proposition 3.2 *For $K_1, \cdots, K_n \in \mathscr{F}_0^+$ and for $-n \neq p \in \mathbb{R}$,*

$$as_p(K_1, \cdots, K_n)^{n+p} = \frac{n^{n+p} \omega_n^n}{|\Lambda_p K_1| \cdots |\Lambda_p K_n|} \widetilde{V}(\Lambda_p K_1, \cdots, \Lambda_p K_n)^{n+p}.$$

Proof. Note that formula (3.14) still holds if one changes L_0 to λL_0 for any $\lambda > 0$. In particular, we choose L_0 to be

$$\left[\rho_{L_0}(u)\right]^n = \rho_{\Lambda_p K_1}(u) \cdots \rho_{\Lambda_p K_n}(u) = \left(\prod_{i=1}^n \frac{|\Lambda_p K_i|}{\omega_n} \cdot \prod_{i=1}^n f_p(K_i, u) \right)^{\frac{1}{n+p}}, \quad \forall u \in S^{n-1}.$$

Combining with formula (3.13), one has

$$\begin{aligned} n\widetilde{V}(\Lambda_p K_1, \cdots, \Lambda_p K_n) &= \int_{S^{n-1}} \rho_{\Lambda_p K_1}(u) \cdots \rho_{\Lambda_p K_n}(u)\, d\sigma(u) \\ &= \int_{S^{n-1}} \left(\prod_{i=1}^{n} \frac{|\Lambda_p K_i|}{\omega_n} \cdot \prod_{i=1}^{n} f_p(K_i, u) \right)^{\frac{1}{n+p}} d\sigma(u) \\ &= \left(\prod_{i=1}^{n} \frac{|\Lambda_p K_i|}{\omega_n} \right)^{\frac{1}{n+p}} \int_{S^{n-1}} \prod_{i=1}^{n} \left[f_p(K_i, u) \right]^{\frac{1}{n+p}} d\sigma(u) \\ &= \left(\prod_{i=1}^{n} \frac{|\Lambda_p K_i|}{\omega_n} \right)^{\frac{1}{n+p}} as_p(K_1, \cdots, K_n). \end{aligned}$$

This concludes the proof of Proposition 3.2.

4 Mixed L_p geominimal surface areas

We now introduce several mixed L_p geominimal surface areas for all $-n \neq p \in \mathbb{R}$.

Definition 4.1 *Let $K_1, \cdots, K_n \in \mathscr{F}_0$.*
(i). For $p = 0$, we let

$$G_0^{(\alpha)}(K_1, \cdots, K_n) = \int_{S^{n-1}} \prod_{i=1}^{n} [f_{K_i}(u) h_{K_i}(u)]^{\frac{1}{n}} d\sigma(u), \quad \alpha = 1, 2, 3.$$

For $p > 0$, we define the mixed L_p geominimal surface areas for $\mathbf{K} \in \mathscr{F}_0^n$ as:

$$\begin{aligned} G_p^{(1)}(K_1, \cdots, K_n) &= \inf_{L \in \mathscr{K}_0} \left\{ nV_p(\mathbf{K}; L, \cdots, L)^{\frac{n}{n+p}} \ |L^\circ|^{\frac{p}{n+p}} \right\}, \\ G_p^{(2)}(K_1, \cdots, K_n) &= \inf_{\mathbf{L} \in \mathscr{K}_0^n} \left\{ nV_p(\mathbf{K}; \mathbf{L})^{\frac{n}{n+p}} \ \prod_{i=1}^{n} |L_i^\circ|^{\frac{p}{(n+p)n}} \right\}, \\ G_p^{(3)}(K_1, \cdots, K_n) &= \inf_{\mathbf{L} \in \mathscr{K}_0^n} \left\{ nV_p(\mathbf{K}; \mathbf{L})^{\frac{n}{n+p}} \ \widetilde{V}(\mathbf{L}^\circ)^{\frac{p}{n+p}} \right\}. \end{aligned}$$

(ii). For $-n \neq p < 0$, we define the mixed L_p geominimal surface areas for $\mathbf{K} \in \mathscr{F}_0^n$ as:

$$\begin{aligned} G_p^{(1)}(K_1, \cdots, K_n) &= \sup_{L \in \mathscr{K}_0} \left\{ nV_p(\mathbf{K}; L, \cdots, L)^{\frac{n}{n+p}} \ |L^\circ|^{\frac{p}{n+p}} \right\}, \\ G_p^{(2)}(K_1, \cdots, K_n) &= \sup_{\mathbf{L} \in \mathscr{K}_0^n} \left\{ nV_p(\mathbf{K}; \mathbf{L})^{\frac{n}{n+p}} \ \prod_{i=1}^{n} |L_i^\circ|^{\frac{p}{(n+p)n}} \right\}, \\ G_p^{(3)}(K_1, \cdots, K_n) &= \sup_{\mathbf{L} \in \mathscr{K}_0^n} \left\{ nV_p(\mathbf{K}; \mathbf{L})^{\frac{n}{n+p}} \ \widetilde{V}(\mathbf{L}^\circ)^{\frac{p}{n+p}} \right\}. \end{aligned}$$

We now prove that the mixed L_p geominimal surface areas defined in Definition 4.1 are all affine invariant.

Proposition 4.1 *If $K_1, \cdots, K_n \in \mathscr{F}_0$ and $\phi \in GL(n)$, then for all $\alpha = 1, 2, 3$, and for all $-n \neq p \in \mathbb{R}$, one has*

$$G_p^{(\alpha)}(\phi K_1, \cdots, \phi K_n) = |det(\phi)|^{\frac{n-p}{n+p}} G_p^{(\alpha)}(K_1, \cdots, K_n).$$

In particular, $G_p^{(\alpha)}(K_1, \cdots, K_n)$ for all $\alpha = 1, 2, 3$ are affine invariant, namely if $\phi \in SL(n)$, then $G_p^{(\alpha)}(\phi K_1, \cdots, \phi K_n) = G_p^{(\alpha)}(K_1, \cdots, K_n)$.

Proof. Let $\phi \in GL(n)$ and $v = \frac{\phi^{-t}(u)}{\|\phi^{-t}(u)\|}$ for any $u \in S^{n-1}$. Then for all $K \in \mathscr{K}_0$

$$h_K(u) = \max_{x \in K} \langle x, u \rangle = \max_{x \in K} \langle \phi x, \phi^{-t}(u) \rangle = \max_{x \in K} \|\phi^{-t}(u)\| \langle \phi x, v \rangle = \|\phi^{-t}(u)\| h_{\phi K}(v).$$

Hence, for all $u \in S^{n-1}$,

$$\frac{h_{\phi Q_i}(v)}{h_{\phi K_i}(v)} = \frac{h_{Q_i}(u)}{h_{K_i}(u)}. \tag{4.15}$$

On the other hand, $\frac{1}{n} h_K(u) f_K(u) d\sigma(u)$ is the volume element for K and then

$$h_{\phi K}(v) f_{\phi K}(v) d\sigma(v) = |det(\phi)| h_K(u) f_K(u) d\sigma(u).$$

This further implies

$$\prod_{i=1}^n [h_{\phi K_i}(v) f_{\phi K_i}(v)]^{\frac{1}{n}} d\sigma(v) = |det(\phi)| \prod_{i=1}^n [h_{K_i}(u) f_{K_i}(u)]^{\frac{1}{n}} d\sigma(u).$$

Combining with formulas (2.3) and (4.15), one has for $L_1, \cdots, L_n \in \mathscr{K}_0$,

$$\begin{aligned}
V_p(\phi \mathbf{K}; \phi \mathbf{L}) &= \frac{1}{n} \int_{S^{n-1}} \prod_{i=1}^n [h_{\phi L_i}(v)^p f_p(\phi K_i, v)]^{\frac{1}{n}} \ d\sigma(v) \\
&= \frac{1}{n} \int_{S^{n-1}} \prod_{i=1}^n \left(\frac{h_{\phi L_i}(v)}{h_{\phi K_i}(v)} \right)^{\frac{p}{n}} \prod_{i=1}^n [h_{\phi K_i}(v) f_{\phi K_i}(v)]^{\frac{1}{n}} \ d\sigma(v) \\
&= |det(\phi)| \cdot \frac{1}{n} \int_{S^{n-1}} \prod_{i=1}^n \left(\frac{h_{L_i}(u)}{h_{K_i}(u)} \right)^{\frac{p}{n}} \prod_{i=1}^n [h_{K_i}(u) f_{K_i}(u)]^{\frac{1}{n}} \ d\sigma(u) \\
&= |det(\phi)| V_p(\mathbf{K}; \mathbf{L}).
\end{aligned}$$

Letting $L_1 = \cdots = L_n = L$ in formula (2.7), one gets for $p \geq 0$,

$$\begin{aligned}
G_p^{(1)}(\phi K_1, \cdots, \phi K_n) &= \inf_{L \in \mathscr{K}_0} \left\{ n V_p(\phi \mathbf{K}; \phi L, \cdots, \phi L)^{\frac{n}{n+p}} \ |(\phi L)^\circ|^{\frac{p}{n+p}} \right\} \\
&= |det(\phi)|^{\frac{n}{n+p}} |det(\phi)|^{-\frac{p}{n+p}} \inf_{L \in \mathscr{K}_0} \left\{ n V_p(\mathbf{K}; L, \cdots, L)^{\frac{n}{n+p}} \ |L^\circ|^{\frac{p}{n+p}} \right\} \\
&= |det(\phi)|^{\frac{n-p}{n+p}} G_p^{(1)}(K_1, \cdots, K_n).
\end{aligned}$$

Similarly, for $-n \neq p < 0$,

$$\begin{aligned} G_p^{(1)}(\phi K_1, \cdots, \phi K_n) &= \sup_{L \in \mathscr{K}_0} \left\{ nV_p(\phi \mathbf{K}; \phi L, \cdots, \phi L)^{\frac{n}{n+p}} \ |(\phi L)^\circ|^{\frac{p}{n+p}} \right\} \\ &= |det(\phi)|^{\frac{n-p}{n+p}} G_p^{(1)}(K_1, \cdots, K_n). \end{aligned}$$

The proofs for $G_p^{(2)}(\phi K_1, \cdots, \phi K_n)$ and $G_p^{(3)}(\phi K_1, \cdots, \phi K_n)$ follow the same line.

Proposition 4.2 *Let $K_1, \cdots, K_n \in \mathscr{F}_0$.*
(i). For $p \geq 0$, one has

$$G_p^{(1)}(K_1, \cdots, K_n) \geq G_p^{(2)}(K_1, \cdots, K_n) \geq G_p^{(3)}(K_1, \cdots, K_n).$$

(ii). For $-n < p < 0$, one has

$$G_p^{(1)}(K_1, \cdots, K_n) \leq G_p^{(2)}(K_1, \cdots, K_n) \leq G_p^{(3)}(K_1, \cdots, K_n).$$

(iii). For $p < -n$, one has

$$G_p^{(1)}(K_1, \cdots, K_n) \leq G_p^{(3)}(K_1, \cdots, K_n) \leq G_p^{(2)}(K_1, \cdots, K_n).$$

Proof. (i). It is clear for $p = 0$. Let $p > 0$, then $(L, \cdots, L) \in \mathscr{K}_0^n$ and inequality (2.6) imply

$$\begin{aligned} \inf_{L \in \mathscr{K}_0} \left\{ nV_p(\mathbf{K}; L, \cdots, L)^{\frac{n}{n+p}} \ |L^\circ|^{\frac{p}{n+p}} \right\} &\geq \inf_{\mathbf{L} \in \mathscr{K}_0^n} \left\{ nV_p(\mathbf{K}; \mathbf{L})^{\frac{n}{n+p}} \ \prod_{i=1}^n |L_i^\circ|^{\frac{p}{(n+p)n}} \right\} \\ &\geq \inf_{\mathbf{L} \in \mathscr{K}_0^n} \left\{ nV_p(\mathbf{K}; \mathbf{L})^{\frac{n}{n+p}} \ \widetilde{V}(\mathbf{L}^\circ)^{\frac{p}{n+p}} \right\}. \end{aligned}$$

According to Definition 4.1, one gets the desired conclusion.

(ii). Let $p \in (-n, 0)$, then $\frac{p}{n+p} < 0$. By $(L, \cdots, L) \in \mathscr{K}_0^n$ and inequality (2.6), one has

$$\begin{aligned} \sup_{L \in \mathscr{K}_0} \left\{ nV_p(\mathbf{K}; L, \cdots, L)^{\frac{n}{n+p}} \ |L^\circ|^{\frac{p}{n+p}} \right\} &\leq \sup_{\mathbf{L} \in \mathscr{K}_0^n} \left\{ nV_p(\mathbf{K}; \mathbf{L})^{\frac{n}{n+p}} \ \prod_{i=1}^n |L_i^\circ|^{\frac{p}{(n+p)n}} \right\} \\ &\leq \sup_{\mathbf{L} \in \mathscr{K}_0^n} \left\{ nV_p(\mathbf{K}; \mathbf{L})^{\frac{n}{n+p}} \ \widetilde{V}(\mathbf{L}^\circ)^{\frac{p}{n+p}} \right\}. \end{aligned}$$

According to Definition 4.1, one gets the desired conclusion.

(iii). Let $p \in (-\infty, -n)$, then $\frac{p}{n+p} > 0$. From inequality (2.6), one has

$$\sup_{\mathbf{L} \in \mathscr{K}_0^n} \left\{ nV_p(\mathbf{K}; \mathbf{L})^{\frac{n}{n+p}} \ \widetilde{V}(\mathbf{L}^\circ)^{\frac{p}{n+p}} \right\} \leq \sup_{\mathbf{L} \in \mathscr{K}_0^n} \left\{ nV_p(\mathbf{K}; \mathbf{L})^{\frac{n}{n+p}} \ \prod_{i=1}^n |L_i^\circ|^{\frac{p}{(n+p)n}} \right\},$$

which implies $G_p^{(3)}(K_1, \cdots, K_n) \leq G_p^{(2)}(K_1, \cdots, K_n)$. Due to $(L, \cdots, L) \in \mathscr{K}_0^n$, one has

$$\sup_{L \in \mathscr{K}_0} \left\{ nV_p(\mathbf{K}; L, \cdots, L)^{\frac{n}{n+p}} \ |L^\circ|^{\frac{p}{n+p}} \right\} \leq \sup_{\mathbf{L} \in \mathscr{K}_0^n} \left\{ nV_p(\mathbf{K}; \mathbf{L})^{\frac{n}{n+p}} \ \widetilde{V}(\mathbf{L}^\circ)^{\frac{p}{n+p}} \right\},$$

and hence $G_p^{(1)}(K_1, \cdots, K_n) \leq G_p^{(3)}(K_1, \cdots, K_n)$. The desired result then follows.

Proposition 4.3 *Let $K_1,\cdots,K_n\in\mathscr{F}_0$.*
(i). For $p\geq 0$,

$$as_p(K_1,\cdots,K_n)\leq G_p^{(\alpha)}(K_1,\cdots,K_n),\quad \alpha=1,2,3.$$

(ii). For $-n<p<0$,

$$as_p(K_1,\cdots,K_n)\geq G_p^{(\alpha)}(K_1,\cdots,K_n),\quad \alpha=1,2,3.$$

(iii). For $p<-n$, one has

$$\begin{aligned} as_p(K_1,\cdots,K_n) &\geq G_p^{(\alpha)}(K_1,\cdots,K_n),\quad \alpha=1,3;\\ as_p^{(1)}(K_1,\cdots,K_n) &\geq G_p^{(\alpha)}(K_1,\cdots,K_n),\quad \alpha=1,2,3.\end{aligned}$$

Proof. (i). The case of $p=0$ holds trivially. We only prove the case $p>0$. By Proposition 3.1, Definition 3.1 and $\mathscr{K}_0\subset\mathscr{S}_0$, one can get

$$\begin{aligned} as_p(K_1,\cdots,K_n) &= \inf_{\mathbf{L}\in\mathscr{S}_0^n}\left\{nV_p(\mathbf{K};\mathbf{L}^\circ)^{\frac{n}{n+p}}\ \widetilde{V}(\mathbf{L})^{\frac{p}{n+p}}\right\}\\ &\leq \inf_{\mathbf{L}\in\mathscr{K}_0^n}\left\{nV_p(\mathbf{K};\mathbf{L}^\circ)^{\frac{n}{n+p}}\ \widetilde{V}(\mathbf{L})^{\frac{p}{n+p}}\right\}=G_p^{(3)}(K_1,\cdots,K_n).\end{aligned}$$

Hence, Proposition 4.2 implies that, for $K_1,\cdots,K_n\in\mathscr{F}_0$ and for all $p>0$,

$$as_p(K_1,\cdots,K_n)\leq G_p^{(3)}(K_1,\cdots,K_n)\leq G_p^{(2)}(K_1,\cdots,K_n)\leq G_p^{(1)}(K_1,\cdots,K_n).$$

(ii). Let $-n<p<0$. Propositions 3.1 and 4.2, Definition 3.1 and $\mathscr{K}_0\subset\mathscr{S}_0$ imply that

$$\begin{aligned} as_p(K_1,\cdots,K_n) &= \sup_{\mathbf{L}\in\mathscr{S}_0^n}\left\{nV_p(\mathbf{K};\mathbf{L}^\circ)^{\frac{n}{n+p}}\widetilde{V}(\mathbf{L})^{\frac{p}{n+p}}\right\}\geq \sup_{\mathbf{L}\in\mathscr{K}_0^n}\left\{nV_p(\mathbf{K};\mathbf{L}^\circ)^{\frac{n}{n+p}}\widetilde{V}(\mathbf{L})^{\frac{p}{n+p}}\right\}\\ &= G_p^{(3)}(K_1,\cdots,K_n)\geq G_p^{(2)}(K_1,\cdots,K_n)\geq G_p^{(1)}(K_1,\cdots,K_n).\end{aligned}$$

(iii). Let $p<-n$. Propositions 3.1 and 4.2, Definition 3.1 and $\mathscr{K}_0\subset\mathscr{S}_0$ imply that

$$\begin{aligned} as_p(K_1,\cdots,K_n) &= \sup_{\mathbf{L}\in\mathscr{S}_0^n}\left\{nV_p(\mathbf{K};\mathbf{L}^\circ)^{\frac{n}{n+p}}\widetilde{V}(\mathbf{L})^{\frac{p}{n+p}}\right\}\geq \sup_{\mathbf{L}\in\mathscr{K}_0^n}\left\{nV_p(\mathbf{K};\mathbf{L}^\circ)^{\frac{n}{n+p}}\widetilde{V}(\mathbf{L})^{\frac{p}{n+p}}\right\}\\ &= G_p^{(3)}(K_1,\cdots,K_n)\geq G_p^{(1)}(K_1,\cdots,K_n).\end{aligned}$$

Similarly, Proposition 4.2, Definition 3.1 and $\mathscr{K}_0\subset\mathscr{S}_0$ imply that

$$\begin{aligned} as_p^{(1)}(K_1,\cdots,K_n) &= \sup_{\mathbf{L}\in\mathscr{S}_0^n}\left\{nV_p(\mathbf{K};\mathbf{L}^\circ)^{\frac{n}{n+p}}\prod_{i=1}^n|L_i|^{\frac{p}{(n+p)n}}\right\}\\ &\geq \sup_{\mathbf{L}\in\mathscr{K}_0^n}\left\{nV_p(\mathbf{K};\mathbf{L}^\circ)^{\frac{n}{n+p}}\prod_{i=1}^n|L_i|^{\frac{p}{(n+p)n}}\right\}=G_p^{(2)}(K_1,\cdots,K_n)\\ &\geq \max\{G_p^{(1)}(K_1,\cdots,K_n),G_p^{(3)}(K_1,\cdots,K_n)\},\end{aligned}$$

as desired. This concludes the proof of Proposition 4.3.

Let $\mathscr{V}_p$ for $-n \neq p \in \mathbb{R}$ be the subset of $\mathscr{F}_0^+$ [22, 41] defined as

$$\mathscr{V}_p = \{K \in \mathscr{F}_0^+ : \exists Q \in \mathscr{K}_0 \quad with \quad f_p(K,u) = h_Q(u)^{-(n+p)}, \forall u \in S^{n-1}\}.$$

Note that $\mathscr{V}_p \neq \emptyset$ for all $-n \neq p \in \mathbb{R}$ as $B_2^n \in \mathscr{V}_p$.

Theorem 4.1 *Let $K_1, \cdots, K_n \in \mathscr{V}_p$. Then for $-n \neq p \in \mathbb{R}$,*

$$G_p^{(3)}(K_1, \cdots, K_n) = as_p(K_1, \cdots, K_n).$$

Remark. It is easily checked that for all $\mathbf{K} \in \mathscr{F}_0^n$ and for all $\alpha = 1, 2, 3$,

$$G_0^{(\alpha)}(K_1, \cdots, K_n) = as_0(K_1, \cdots, K_n) = \int_{S^{n-1}} \prod_{i=1}^n [f_{K_i}(u) h_{K_i}(u)]^{\frac{1}{n}} d\sigma(u).$$

Proof. Let $K_i \in \mathscr{V}_p$ and $p > 0$. Proposition 3.3 in [41] asserts that, for all $-n \neq p \in \mathbb{R}$, $K_i \in \mathscr{V}_p$ implies $\Lambda_p K_i \in \mathscr{K}_0$. Thus, by Propositions 3.2 and 4.3, we have

$$\begin{aligned} G_p^{(3)}(K_1, \cdots, K_n) &\geq as_p(K_1, \cdots, K_n) \\ &= n\left(\frac{\omega_n}{|\Lambda_p K_1|^{\frac{1}{n}} \cdots |\Lambda_p K_n|^{\frac{1}{n}}}\right)^{\frac{n}{n+p}} \widetilde{V}(\Lambda_p K_1, \cdots, \Lambda_p K_n) \\ &= nV_p(\mathbf{K}; (\Lambda_p K_1)^\circ, \cdots, (\Lambda_p K_n)^\circ)^{\frac{n}{n+p}} \widetilde{V}(\Lambda_p K_1, \cdots, \Lambda_p K_n)^{\frac{p}{n+p}} \\ &\geq \inf_{\mathbf{L} \in \mathscr{K}_0^n} \left\{ nV_p(\mathbf{K}; \mathbf{L}^\circ)^{\frac{n}{n+p}} \widetilde{V}(\mathbf{L})^{\frac{p}{n+p}} \right\} = G_p^{(3)}(K_1, \cdots, K_n). \end{aligned}$$

Hence, if $p > 0$, $G_p^{(3)}(K_1, \cdots, K_n) = as_p(K_1, \cdots, K_n)$ for all $K_1, \cdots, K_n \in \mathscr{V}_p$. Now let $-n \neq p < 0$. As above, one has

$$\begin{aligned} G_p^{(3)}(K_1, \cdots, K_n) &\leq as_p(K_1, \cdots, K_n) \\ &= nV_p(\mathbf{K}; (\Lambda_p K_1)^\circ, \cdots, (\Lambda_p K_n)^\circ)^{\frac{n}{n+p}} \widetilde{V}(\Lambda_p K_1, \cdots, \Lambda_p K_n)^{\frac{p}{n+p}} \\ &\leq \sup_{\mathbf{L} \in \mathscr{K}_0^n} \left\{ nV_p(\mathbf{K}; \mathbf{L}^\circ)^{\frac{n}{n+p}} \widetilde{V}(\mathbf{L})^{\frac{p}{n+p}} \right\} = G_p^{(3)}(K_1, \cdots, K_n). \end{aligned}$$

Hence, if $-n \neq p < 0$, $G_p^{(3)}(K_1, \cdots, K_n) = as_p(K_1, \cdots, K_n)$ for all $K_1, \cdots, K_n \in \mathscr{V}_p$.

Define the subset $\mathscr{V}_{p,n}$ of $(\mathscr{F}_0^+)^n$ for $-n \neq p \in \mathbb{R}$ as

$$\mathscr{V}_{p,n} = \left\{ \mathbf{K} \in (\mathscr{F}_0^+)^n : \exists Q \in \mathscr{K}_0 \ s.t. \ \prod_{i=1}^n [f_p(K_i, u)]^{\frac{1}{n}} = h_Q(u)^{-(n+p)}, \forall u \in S^{n-1} \right\}.$$

Theorem 4.2 *Let* $(K_1, \cdots, K_n) \in \mathscr{V}_{p,n}$ *with* $p \neq 0, -n$.
(i). For $p > -n$ *and* $\alpha = 1, 2, 3$,

$$G_p^{(\alpha)}(K_1, \cdots, K_n) = as_p(K_1, \cdots, K_n).$$

(ii). For $p < -n$ *and* $\alpha = 1, 3$,

$$G_p^{(\alpha)}(K_1, \cdots, K_n) = as_p(K_1, \cdots, K_n).$$

Proof. Note that $(K_1, \cdots, K_n) \in \mathscr{V}_{p,n}$ for $-n \neq p \in \mathbb{R}$ implies $L_0 \in \mathscr{K}_0$ with

$$\big[\rho_{L_0}(u)\big]^n = \left(\prod_{i=1}^n f_p(K_i, u)\right)^{\frac{1}{n+p}}, \quad \forall u \in S^{n-1}.$$

(i). We first prove the case of $p > 0$. Proposition 4.3 and formula (3.14) imply

$$\begin{aligned} G_p^{(1)}(K_1, \cdots, K_n) &\geq as_p(K_1, \cdots, K_n) = nV_p(\mathbf{K}; L_0^\circ, \cdots, L_0^\circ)^{\frac{n}{n+p}} |L_0|^{\frac{p}{n+p}} \\ &\geq \inf_{L \in \mathscr{K}_0} \left\{ nV_p(\mathbf{K}; L^\circ, \cdots, L^\circ)^{\frac{n}{n+p}} |L|^{\frac{p}{n+p}} \right\} = G_p^{(1)}(K_1, \cdots, K_n), \end{aligned}$$

where the second inequality follows from $L_0 \in \mathscr{K}_0$. Hence, for $\mathbf{K} \in \mathscr{V}_{p,n}$ and for $p > 0$, $G_p^{(1)}(K_1, \cdots, K_n) = as_p(K_1, \cdots, K_n)$. Combining with Propositions 4.2 and 4.3, one has

$$\begin{aligned} G_p^{(1)}(K_1, \cdots, K_n) &\geq G_p^{(2)}(K_1, \cdots, K_n) \geq G_p^{(3)}(K_1, \cdots, K_n) \\ &\geq as_p(K_1, \cdots, K_n) = G_p^{(1)}(K_1, \cdots, K_n), \end{aligned}$$

as desired. Similarly for $-n < p < 0$, Propositions 4.2 and 4.3, and formula (3.14) imply

$$\begin{aligned} G_p^{(1)}(K_1, \cdots, K_n) &\leq G_p^{(2)}(K_1, \cdots, K_n) \leq G_p^{(3)}(K_1, \cdots, K_n) \\ &\leq as_p(K_1, \cdots, K_n) = nV_p(\mathbf{K}; L_0^\circ, \cdots, L_0^\circ)^{\frac{n}{n+p}} |L_0|^{\frac{p}{n+p}} \\ &\leq \sup_{L \in \mathscr{K}_0} \left\{ nV_p(\mathbf{K}; L^\circ, \cdots, L^\circ)^{\frac{n}{n+p}} |L|^{\frac{p}{n+p}} \right\} = G_p^{(1)}(K_1, \cdots, K_n). \end{aligned}$$

(ii). Similarly for $p < -n$, Propositions 4.2 and 4.3, and formula (3.14) imply

$$\begin{aligned} G_p^{(1)}(K_1, \cdots, K_n) &\leq G_p^{(3)}(K_1, \cdots, K_n) \leq as_p(K_1, \cdots, K_n) \\ &= nV_p(\mathbf{K}; L_0^\circ, \cdots, L_0^\circ)^{\frac{n}{n+p}} |L_0|^{\frac{p}{n+p}} \\ &\leq \sup_{L \in \mathscr{K}_0} \left\{ nV_p(\mathbf{K}; L^\circ, \cdots, L^\circ)^{\frac{n}{n+p}} |L|^{\frac{p}{n+p}} \right\} = G_p^{(1)}(K_1, \cdots, K_n). \end{aligned}$$

Hence, for $p < -n$ and for $\alpha = 1, 3$, $G_p^{(\alpha)}(K_1, \cdots, K_n) = as_p(K_1, \cdots, K_n)$.

Remark. Note that if all $K_i = K \in \mathscr{F}_0$, Definitions 4.1 and 2.1 together with formula (2.5) imply $G_p^{(1)}(K, \cdots, K) = \widetilde{G}_p(K)$ for all $-n \neq p \in \mathbb{R}$. Moreover, if $K \in \mathscr{V}_p$, then $(K, \cdots, K) \in \mathscr{V}_{p,n}$. Therefore, for $K \in \mathscr{V}_p$,

$$\begin{aligned} G_p^{(\alpha)}(K, \cdots, K) &= \widetilde{G}_p(K) = as_p(K), \quad for \ -n \neq p \in \mathbb{R} \text{ and } \alpha = 1, 3; \\ G_p^{(2)}(K, \cdots, K) &= \widetilde{G}_p(K) = as_p(K), \quad for \ p > -n. \end{aligned}$$

In particular, as $(B_2^n, \cdots, B_2^n) \in \mathscr{V}_{p,n}$, one gets,

$$\begin{aligned} G_p^{(\alpha)}(B_2^n, \cdots, B_2^n) &= \widetilde{G}_p(B_2^n) = n|B_2^n|, \quad for \ -n \neq p \in \mathbb{R} \ and \ \alpha = 1, 3; \\ G_p^{(2)}(B_2^n, \cdots, B_2^n) &= \widetilde{G}_p(B_2^n) = n|B_2^n|, \quad for \ p > -n. \end{aligned}$$

5 Inequalities for L_p mixed geominimal surface areas

In this section, we will prove the Alexander-Fenchel type inequality, the affine isoperimetric inequality, the Santaló style inequality, and the cyclic inequality.

5.1 The Alexander-Fenchel type inequality

The classical Alexander-Fenchel inequality for the mixed volume (see [15, 29]) is fundamental in applications. It has been extended to the mixed p-affine surface area [18, 19, 22, 38]. Here, we prove the Alexander-Fenchel type inequality for L_p mixed geominimal surface areas.

Theorem 5.1 *Let $K_1, \cdots, K_n \in \mathscr{F}_0$. For $1 \leq m \leq n$ and for $\alpha = 1, 2$, one has,*

$$\left[G_p^{(\alpha)}(K_1, \cdots, K_n)\right]^m \leq \prod_{i=0}^{m-1} G_p^{(\alpha)}(K_1, \cdots, K_{n-m}, \underbrace{K_{n-i}, \cdots, K_{n-i}}_{m}), \quad -n < p < 0.$$

In particular, if $m = n$, then for $-n < p < 0$ and for $\alpha = 1, 2$,

$$\left[G_p^{(\alpha)}(K_1, \cdots, K_n)\right]^n \leq \widetilde{G}_p(K_1) \cdots \widetilde{G}_p(K_n).$$

Proof. By Hölder's inequality (see [10]), one has

$$\begin{aligned} V_p(\mathbf{K}; \mathbf{L})^m &= \left(\frac{1}{n} \int_{S^{n-1}} H_0(u) H_1(u) \cdots H_m(u) d\sigma(u)\right)^m \\ &\leq \prod_{i=0}^{m-1} \left(\frac{1}{n} \int_{S^{n-1}} H_0(u) [H_{i+1}(u)]^m d\sigma(u)\right) \\ &= \prod_{i=0}^{m-1} V_p(K_1, \cdots, K_{n-m}, \underbrace{K_{n-i}, \cdots, K_{n-i}}_{m}; L_1, \cdots, L_{n-m}, \underbrace{L_{n-i}, \cdots, L_{n-i}}_{m}), \end{aligned} \tag{5.16}$$

where for $i = 0, \cdots, m-1$, we let

$$\begin{aligned} H_0(u) &= [h^p_{L_1}(u) f_p(K_1, u) \cdots h^p_{L_{n-m}}(u) f_p(K_{n-m}, u)]^{\frac{1}{n}}, \\ H_{i+1}(u) &= [h^p_{L_{n-i}}(u) f_p(K_{n-i}, u)]^{\frac{1}{n}}. \end{aligned}$$

Note that if $-n < p < 0$, then $\frac{n}{n+p} > 0$. Therefore,

$$\begin{aligned} \big[G^{(2)}_p(K_1, \cdots, K_n)\big]^m &= \sup_{\mathbf{L} \in \mathscr{K}_0^n} \big\{ nV_p(\mathbf{K}; \mathbf{L})^{\frac{n}{n+p}} \prod_{i=1}^n |L_i^\circ|^{\frac{p}{n(n+p)}} \big\}^m \\ &\le \prod_{i=0}^{m-1} \sup_{L_i \in \mathscr{K}_0} \big[nV_p(K_1, \cdots, K_{n-m}, \underbrace{K_{n-i}, \cdots, K_{n-i}}_{m}; L_1, \cdots, L_{n-m}, \underbrace{L_{n-i}, \cdots, L_{n-i}}_{m})^{\frac{n}{n+p}} \\ &\qquad \times |L_{n-i}^\circ|^{\frac{mp}{n(n+p)}} \prod_{i=1}^{n-m} |L_i^\circ|^{\frac{p}{n(n+p)}} \big] \\ &= \prod_{i=0}^{m-1} G^{(2)}_p(K_1, \cdots, K_{n-m}, \underbrace{K_{n-i}, \cdots, K_{n-i}}_{m}). \end{aligned} \tag{5.17}$$

The case of $\alpha = 1$ follows directly by letting all $L_i = L$ in inequalities (5.16) and (5.17).

Theorem 5.2 *Let $K_1, \cdots, K_n \in \mathscr{F}_0$.*
(i). For $p \ge 0$,

$$[G^{(3)}_p(K_1, \cdots, K_n)]^n \le [G^{(2)}_p(K_1, \cdots, K_n)]^n \le \widetilde{G}_p(K_1) \cdots \widetilde{G}_p(K_n). \tag{5.18}$$

(ii). For $p < -n$,

$$[G^{(2)}_p(K_1, \cdots, K_n)]^n \ge \widetilde{G}_p(K_1) \cdots \widetilde{G}_p(K_n). \tag{5.19}$$

Proof. (i). Let $p \ge 0$, then $\frac{n}{n+p} > 0$. Inequality (5.16) implies that

$$V_p(\mathbf{K}; \mathbf{L})^n \le \prod_{i=1}^n V_p(K_i, L_i).$$

Combining with Proposition 4.2, Definition 4.1, and Definition 2.1, one has

$$\begin{aligned} \big[G^{(3)}_p(K_1, \cdots, K_n)\big]^n &\le \big[G^{(2)}_p(K_1, \cdots, K_n)\big]^n \\ &= \inf_{\mathbf{L} \in \mathscr{K}_0^n} \big\{ nV_p(\mathbf{K}; \mathbf{L})^{\frac{n}{n+p}} \prod_{i=1}^n |L_i^\circ|^{\frac{p}{n(n+p)}} \big\}^n \\ &\le \inf_{\mathbf{L} \in \mathscr{K}_0^n} \prod_{i=1}^n \big[nV_p(K_i, L_i)^{\frac{n}{n+p}} |L_i^\circ|^{\frac{p}{n+p}} \big] \\ &= \prod_{i=1}^n \inf_{L_i \in \mathscr{K}_0} \big[nV_p(K_i, L_i)^{\frac{n}{n+p}} |L_i^\circ|^{\frac{p}{n+p}} \big] \\ &= \widetilde{G}_p(K_1) \cdots \widetilde{G}_p(K_n). \end{aligned}$$

(ii). Let $p<-n$, then $\frac{n}{n+p}<0$. By Definition 4.1 and Definition 2.1, one has

$$\begin{aligned}\big[G_p^{(2)}(K_1,\cdots,K_n)\big]^n &= \sup_{\mathbf{L}\in\mathscr{K}_0^n}\big\{nV_p(\mathbf{K};\mathbf{L})^{\frac{n}{n+p}}\prod_{i=1}^n|L_i^\circ|^{\frac{p}{n(n+p)}}\big\}^n\\ &\geq \prod_{i=1}^n\sup_{L_i\in\mathscr{K}_0}\big[nV_p(K_i,L_i)^{\frac{n}{n+p}}|L_i^\circ|^{\frac{p}{n+p}}\big]=\widetilde{G}_p(K_1)\cdots\widetilde{G}_p(K_n).\end{aligned}$$

5.2 The affine isoperimetric inequality

We will prove the affine isoperimetric inequality and the Santaló type inequality for the mixed L_p geominimal surface areas.

Theorem 5.3 *Let $K_1,\cdots,K_n\in\mathscr{F}_c$ be convex bodies in $\mathscr{F}_0$ with centroid at the origin.*
(i). For $p\geq 0$ and $\alpha=2,3$,

$$\left(\frac{G_p^{(\alpha)}(K_1,\cdots,K_n)}{G_p^{(\alpha)}(B_2^n,\cdots,B_2^n)}\right)^n\leq\min\left\{\left(\frac{|K_1|}{|B_2^n|}\cdots\frac{|K_n|}{|B_2^n|}\right)^{\frac{n-p}{n+p}},\left(\frac{|K_1^\circ|}{|B_2^n|}\cdots\frac{|K_n^\circ|}{|B_2^n|}\right)^{\frac{p-n}{n+p}}\right\},$$

with equality if all $K_1,\cdots,K_n$ are ellipsoids that are dilates of one another.
(ii). For $0\leq p\leq n$ and $\alpha=2,3$,

$$\left(\frac{G_p^{(\alpha)}(K_1,\cdots,K_n)}{G_p^{(\alpha)}(B_2^n,\cdots,B_2^n)}\right)^{n+p}\leq\min\left\{\frac{V(K_1,\cdots,K_n)}{V(B_2^n,\cdots,B_2^n)},\ \frac{\widetilde{V}(B_2^n,\cdots,B_2^n)}{\widetilde{V}(K_1^\circ,\cdots,K_n^\circ)}\right\}^{n-p},$$

with equality if all $K_1,\cdots,K_n$ are ellipsoids that are dilates of one another. In particular,

$$G_n^{(\alpha)}(K_1,\cdots,K_n)\leq G_n^{(\alpha)}(B_2^n,\cdots,B_2^n).$$

(iii). For $p>n$ and $\alpha=2,3$,

$$\left(\frac{G_p^{(\alpha)}(K_1,\cdots,K_n)}{G_p^{(\alpha)}(B_2^n,\cdots,B_2^n)}\right)^{n+p}\leq\min\left\{\frac{V(K_1^\circ,\cdots,K_n^\circ)}{V(B_2^n,\cdots,B_2^n)},\ \frac{\widetilde{V}(B_2^n,\cdots,B_2^n)}{\widetilde{V}(K_1,\cdots,K_n)}\right\}^{p-n},$$

with equality if all $K_1,\cdots,K_n$ are ellipsoids that are dilates of one another.
(iv). For $p<-n$,

$$G_p^{(2)}(K_1,\cdots,K_n)\geq n\omega_n^{\frac{2n}{n+p}}\big[\widetilde{V}(K_1^\circ,\cdots,K_n^\circ)\big]^{\frac{p-n}{n+p}}.$$

Proof. Note that, for all $p\geq 0$ and all $K\in\mathscr{F}_c$, one has (see Theorem 4.2 in [41])

$$\frac{\widetilde{G}_p(K)}{\widetilde{G}_p(B_2^n)}\leq\min\left\{\left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n+p}},\ \left(\frac{|K^\circ|}{|B_2^n|}\right)^{\frac{p-n}{n+p}}\right\}. \tag{5.20}$$

(i). Recall that $G_p^{(\alpha)}(B_2^n,\cdots,B_2^n)=\widetilde{G}_p(B_2^n)$ for all $\alpha=2,3$ and $p\geq 0$. By inequalities (5.18) and (5.20), one gets that for all $p\geq 0$ and $\alpha=2,3$,

$$\left(\frac{G_p^{(\alpha)}(K_1,\cdots,K_n)}{G_p^{(\alpha)}(B_2^n,\cdots,B_2^n)}\right)^n\leq\frac{\widetilde{G}_p(K_1)}{\widetilde{G}_p(B_2^n)}\cdots\frac{\widetilde{G}_p(K_n)}{\widetilde{G}_p(B_2^n)}$$
$$\leq\min\left\{\left(\frac{|K_1|}{|B_2^n|}\cdots\frac{|K_n|}{|B_2^n|}\right)^{\frac{n-p}{n+p}},\left(\frac{|K_1^\circ|}{|B_2^n|}\cdots\frac{|K_n^\circ|}{|B_2^n|}\right)^{\frac{p-n}{n+p}}\right\}.\qquad(5.21)$$

Clearly, equality holds if all $K_1,\cdots,K_n$ are ellipsoids that are dilates of one another.
(ii). If $0\leq p\leq n$, then $\frac{n-p}{n+p}\geq 0$ and $\frac{p-n}{n+p}\leq 0$. Note that $V(B_2^n,\cdots,B_2^n)=|B_2^n|$. Combining inequality (2.1) with inequality (5.21), one has, for $\alpha=2,3$

$$\frac{G_p^{(\alpha)}(K_1,\cdots,K_n)}{G_p^{(\alpha)}(B_2^n,\cdots,B_2^n)}\leq\left(\frac{V(K_1,\cdots,K_n)}{V(B_2^n,\cdots,B_2^n)}\right)^{\frac{n-p}{n+p}}.$$

Also note that $\widetilde{V}(B_2^n,\cdots,B_2^n)=|B_2^n|$. Combining inequality (5.21) with inequality (2.6), one gets for $\alpha=2,3$

$$\frac{G_p^{(\alpha)}(K_1,\cdots,K_n)}{G_p^{(\alpha)}(B_2^n,\cdots,B_2^n)}\leq\left(\frac{\widetilde{V}(K_1^\circ,\cdots,K_n^\circ)}{\widetilde{V}(B_2^n,\cdots,B_2^n)}\right)^{\frac{p-n}{n+p}}.$$

Clearly, equality holds if all $K_1,\cdots,K_n$ are ellipsoids that are dilates of one another.
(iii). If $p>n$, then $\frac{n-p}{n+p}<0$ and $\frac{p-n}{n+p}>0$. By inequalities (5.21) and (2.6), for $\alpha=2,3$

$$\frac{G_p^{(\alpha)}(K_1,\cdots,K_n)}{G_p^{(\alpha)}(B_2^n,\cdots,B_2^n)}\leq\left(\frac{\widetilde{V}(K_1,\cdots,K_n)}{\widetilde{V}(B_2^n,\cdots,B_2^n)}\right)^{\frac{n-p}{n+p}}.$$

Combining inequality (2.1) with inequality (5.21), one has, for $\alpha=2,3$

$$\frac{G_p^{(\alpha)}(K_1,\cdots,K_n)}{G_p^{(\alpha)}(B_2^n,\cdots,B_2^n)}\leq\left(\frac{V(K_1^\circ,\cdots,K_n^\circ)}{V(B_2^n,\cdots,B_2^n)}\right)^{\frac{p-n}{n+p}}.$$

Clearly, equality holds if all $K_1,\cdots,K_n$ are ellipsoids that are dilates of one another.
(iv). Note that $\widetilde{G}_p(B_2^n)=n|B_2^n|=n\omega_n$. From Theorem 4.2 in [41] and inequality (5.19), one gets that for all $p<-n$,

$$\left[G_p^{(2)}(K_1,\cdots,K_n)\right]^n\geq\widetilde{G}_p(K_1)\cdots\widetilde{G}_p(K_n)\geq\left(\frac{|K_1^\circ|}{|B_2^n|}\cdots\frac{|K_n^\circ|}{|B_2^n|}\right)^{\frac{p-n}{n+p}}(n\omega_n)^n.$$

The desired result follows from inequality (2.6).

Remark. When $\mathbf{K}\in\mathscr{F}_c^n$ with $V(K_1,\cdots,K_n)=|B_2^n|$ or with $\widetilde{V}(K_1^\circ,\cdots,K_n^\circ)=|B_2^n|$, then Theorem 5.3 implies that for all $p\in(0,n)$,

$$G_p^{(\alpha)}(K_1,\cdots,K_n)\leq G_p^{(\alpha)}(B_2^n,\cdots,B_2^n),\quad\alpha=2,3.$$

That is, the mixed L_p geominimal surface area $G_p^{(\alpha)}(K_1, \cdots, K_n)$ with $\alpha = 2, 3$ attains the maximum at original-symmetric ellipsoids that are dilates of each other. While for $p > n$, the mixed L_p geominimal surface area $G_p^{(\alpha)}(K_1, \cdots, K_n)$ with $\alpha = 2, 3$ attains its maximum at original-symmetric ellipsoids that are dilates of each other among $\mathbf{K} \in \mathscr{F}_c^n$ such that either $V(K_1^\circ, \cdots, K_n^\circ) = |B_2^n|$ or $\widetilde{V}(K_1, \cdots, K_n) = |B_2^n|$. Although the condition $K_1, \cdots, K_n \in \mathscr{F}_c$ is used in Theorems 5.3 and 5.4, and Corollary 5.1, such a condition can be replaced by the following more general condition: some K_i are in $\mathscr{F}_c$ but others are in $\mathscr{F}_s$.

Corollary 5.1 *Let $\mathscr{E}$ be an origin-symmetric ellipsoid and $K_1, \cdots, K_n \in \mathscr{F}_c$.*
(i). For $0 \le p < n$ and $K_1, \cdots, K_n \subset \mathscr{E}$, one has for $\alpha = 2, 3$

$$G_p^{(\alpha)}(K_1, \cdots, K_n) \le \widetilde{G}_p(\mathscr{E}).$$

For $p = n$, the inequality holds for all $K_i \in \mathscr{F}_c$ by (ii) of Theorem 5.3.
(ii). For $p > n$ and $\mathscr{E} \subset K_1, \cdots, K_n$, one has for $\alpha = 2, 3$

$$G_p^{(\alpha)}(K_1, \cdots, K_n) \le \widetilde{G}_p(\mathscr{E}).$$

(iii). For $p < -n$ and $K_1, \cdots, K_n \subset \mathscr{E}$, one has

$$G_p^{(2)}(K_1, \cdots, K_n) \ge \widetilde{G}_p(\mathscr{E}).$$

Proof. By Proposition 4.1, it is enough to prove the proposition for $\mathscr{E} = B_2^n$.
(i). For $0 \le p < n$, one has $\frac{n-p}{n+p} > 0$ and hence $\left(\frac{|K_i|}{|B_2^n|}\right)^{\frac{n-p}{n+p}} \le 1$ as $K_i \subset B_2^n$. Combining with inequality (5.21), one has for $\alpha = 2, 3$

$$\left(\frac{G_p^{(\alpha)}(K_1, \cdots, K_n)}{G_p^{(\alpha)}(B_2^n, \cdots, B_2^n)}\right)^n \le \left(\frac{|K_1|}{|B_2^n|} \cdots \frac{|K_n|}{|B_2^n|}\right)^{\frac{n-p}{n+p}} \le 1,$$

and the desired inequality holds.
(ii). The condition $p > n$ implies $\frac{n-p}{n+p} < 0$ and hence $\left(\frac{|K_i|}{|B_2^n|}\right)^{\frac{n-p}{n+p}} \le 1$ as $B_2^n \subset K_i$. Combining with inequality (5.21), one has for $\alpha = 2, 3$

$$\left(\frac{G_p^{(\alpha)}(K_1, \cdots, K_n)}{G_p^{(\alpha)}(B_2^n, \cdots, B_2^n)}\right)^n \le \left(\frac{|K_1|}{|B_2^n|} \cdots \frac{|K_n|}{|B_2^n|}\right)^{\frac{n-p}{n+p}} \le 1,$$

and the desired inequality holds.
(iii). For $p < -n$, by Corollary 4.1 in [41] and inequality (5.19), one has, for $K_i \subset B_2^n$

$$G_p^{(2)}(K_1, \cdots, K_n) \ge \left[\widetilde{G}_p(K_1) \cdots \widetilde{G}_p(K_n)\right]^{1/n} \ge \widetilde{G}_p(B_2^n),$$

and the desired inequality holds.

The celebrated Blaschke-Santaló inequality states that, for all $K \in \mathscr{K}_c$ (or $K \in \mathscr{K}_s$), $|K||K^\circ| \leq |B_2^n|^2$ with equality if and only if K is an origin-symmetric ellipsoid. For the lower bound, Bourgain and Milman proved the following inverse Santaló inequality [5] (see also [12, 25]): there is a constant $c > 0$, such that, $c^n|B_2^n|^2 \leq |K||K^\circ|$ for all $K \in \mathscr{K}_c$ (or $K \in \mathscr{K}_s$). The next theorem provides a Santaló type inequality for L_p mixed Goeminimal surface areas.

Theorem 5.4 *Let $K_1, \cdots, K_n \in \mathscr{F}_c$.*
(i). For $p \geq 0$ and $\alpha = 2, 3$,

$$G_p^{(\alpha)}(K_1, \cdots, K_n) G_p^{(\alpha)}(K_1^\circ, \cdots, K_n^\circ) \leq [G_p^{(\alpha)}(B_2^n, \cdots, B_2^n)]^2.$$

Equality holds if $K_1, \cdots, K_n$ are ellipsoids that are dilates to each other.
(ii). For $p < -n$

$$G_p^{(2)}(K_1, \cdots, K_n) G_p^{(2)}(K_1^\circ, \cdots, K_n^\circ) \geq c^n [\widetilde{G}_p(B_2^n)]^2.$$

where c is the universal constant from the Bourgain-Milman inverse Santaló inequality.

Proof. (i). From inequality (5.18), we have for all $p \geq 0$ and $\alpha = 2, 3$

$$\left[G_p^{(\alpha)}(K_1, \cdots, K_n) G_p^{(\alpha)}(K_1^\circ, \cdots, K_n^\circ)\right]^n \leq \widetilde{G}_p(K_1)\widetilde{G}_p(K_1^\circ) \cdots \widetilde{G}_p(K_n)\widetilde{G}_p(K_n^\circ).$$

Theorem 4.1 in [41] implies that for $p \geq 0$ and $\alpha = 2, 3$

$$\left[G_p^{(\alpha)}(K_1, \cdots, K_n) G_p^{(\alpha)}(K_1^\circ, \cdots, K_n^\circ)\right]^n \leq \left[\widetilde{G}_p(B_2^n)\right]^{2n} = [G_p^{(\alpha)}(B_2^n, \cdots, B_2^n)]^{2n},$$

as desired. Clearly, equality holds if all $K_1, \cdots, K_n$ are origin-symmetric ellipsoids that are dilates of each other.
(ii). From inequality (5.19), we have for $p < -n$,

$$\left[G_p^{(2)}(K_1, \cdots, K_n) G_p^{(2)}(K_1^\circ, \cdots, K_n^\circ)\right]^n \geq \widetilde{G}_p(K_1)\widetilde{G}_p(K_1^\circ) \cdots \widetilde{G}_p(K_n)\widetilde{G}_p(K_n^\circ).$$

Theorem 4.1 in [41] implies that for all $p < -n$

$$G_p^{(2)}(K_1, \cdots, K_n) G_p^{(2)}(K_1^\circ, \cdots, K_n^\circ) \geq c^n \left[\widetilde{G}_p(B_2^n)\right]^2.$$

Let $S_p(K)$ denote the p-surface area of K for all $p \in \mathbb{R}$. That is, $S_p(K) = nV_p(K, B_2^n)$. In particular, $S_p(B_2^n) = n|B_2^n| = G_p^{(\alpha)}(B_2^n, \cdots, B_2^n)$ for all $\alpha = 1, 2, 3$ if $p > -n$, and for all $\alpha = 1, 3$ if $p < -n$.

Corollary 5.2 *Let $K_1, \cdots, K_n \in \mathscr{F}_0$.*
(i). For $p \geq 0$, one has

$$\frac{G_p^{(\alpha)}(K_1, \cdots, K_n)}{G_p^{(\alpha)}(B_2^n, \cdots, B_2^n)} \leq \prod_{i=1}^n \left(\frac{S_p(K_i)}{S_p(B_2^n)}\right)^{\frac{1}{n+p}}, \quad \alpha = 1, 2, 3.$$

Equality holds if all K_i are balls with center at the origin.
(ii). For $p \in (-\infty, -n)$, one has

$$\frac{G_p^{(\alpha)}(K_1, \cdots, K_n)}{G_p^{(\alpha)}(B_2^n, \cdots, B_2^n)} \geq \prod_{i=1}^n \left(\frac{S_p(K_i)}{S_p(B_2^n)} \right)^{\frac{1}{n+p}}, \quad \alpha = 1, 3.$$

Equality holds if all K_i are balls with center at the origin. For $\alpha = 2$, one has

$$\frac{G_p^{(2)}(K_1, \cdots, K_n)}{n|B_2^n|} \geq \prod_{i=1}^n \left(\frac{S_p(K_i)}{S_p(B_2^n)} \right)^{\frac{1}{n+p}}.$$

Remark. Combining with Proposition 4.3, one has

$$\begin{aligned} \frac{as_p(K_1, \cdots, K_n)}{as_p(B_2^n, \cdots, B_2^n)} &\leq \prod_{i=1}^n \left(\frac{S_p(K_i)}{S_p(B_2^n)} \right)^{\frac{1}{n+p}}, \quad for\ p \geq 0; \\ \frac{as_p^{(1)}(K_1, \cdots, K_n)}{as_p(B_2^n, \cdots, B_2^n)} &\geq \frac{as_p(K_1, \cdots, K_n)}{as_p(B_2^n, \cdots, B_2^n)} \geq \prod_{i=1}^n \left(\frac{S_p(K_i)}{S_p(B_2^n)} \right)^{\frac{1}{n+p}}, \quad for\ p < -n. \end{aligned}$$

Proof. Let $p \geq 0$. Definition 4.1, inequality (5.16) and Proposition 4.2 imply that

$$\begin{aligned} G_p^{(3)}(K_1, \cdots, K_n) &\leq G_p^{(2)}(K_1, \cdots, K_n) \leq G_p^{(1)}(K_1, \cdots, K_n) \\ &= \inf_{L \in \mathscr{K}_0} \left\{ nV_p(\mathbf{K}; L, \cdots, L)^{\frac{n}{n+p}} |L^\circ|^{\frac{p}{n+p}} \right\} \\ &\leq nV_p(\mathbf{K}; B_2^n, \cdots, B_2^n)^{\frac{n}{n+p}} |B_2^n|^{\frac{p}{n+p}} \\ &\leq n|B_2^n|^{\frac{p}{n+p}} \left[\prod_{i=1}^n V_p(K_i, B_2^n) \right]^{\frac{1}{n+p}} = (n|B_2^n|)^{\frac{p}{n+p}} \left[\prod_{i=1}^n S_p(K_i) \right]^{\frac{1}{n+p}}. \end{aligned}$$

The desired inequality follows from dividing by $G_p^{(\alpha)}(B_2^n, \cdots, B_2^n) = n|B_2^n| = S_p(B_2^n)$ (with $\alpha = 1, 2, 3$). Equality holds if all K_i are balls with center at the origin. Similarly, for $p < -n$,

$$\begin{aligned} G_p^{(3)}(K_1, \cdots, K_n) &\geq G_p^{(1)}(K_1, \cdots, K_n) = \sup_{L \in \mathscr{K}_0} \left\{ nV_p(\mathbf{K}; L, \cdots, L)^{\frac{n}{n+p}} |L^\circ|^{\frac{p}{n+p}} \right\} \\ &\geq nV_p(\mathbf{K}; B_2^n, \cdots, B_2^n)^{\frac{n}{n+p}} |B_2^n|^{\frac{p}{n+p}} \geq n|B_2^n|^{\frac{p}{n+p}} \left[\prod_{i=1}^n V_p(K_i, B_2^n) \right]^{\frac{1}{n+p}} \\ &= (n|B_2^n|)^{\frac{p}{n+p}} \left[\prod_{i=1}^n S_p(K_i) \right]^{\frac{1}{n+p}}, \end{aligned}$$

where the second inequality is due to $\frac{n}{n+p} < 0$. Dividing by $G_p^{(\alpha)}(B_2^n, \cdots, B_2^n)$ (with $\alpha = 1, 3$), we get the desired inequality. Equality holds if all K_i are balls with center at the origin. For

$\alpha = 2$ and $p < -n$, the above inequality together with Proposition 4.2 imply

$$G_p^{(2)}(K_1, \cdots, K_n) \geq (n|B_2^n|)^{\frac{p}{n+p}} \left[\prod_{i=1}^n S_p(K_i)\right]^{\frac{1}{n+p}},$$

and the desired inequality follows by dividing by $n|B_2^n| = S_p(B_2^n)$.

5.3 The cyclic inequality

We now prove cyclic inequalities for L_p mixed geominimal surface areas.

Theorem 5.5 *Let $K_1, \cdots, K_n \in \mathscr{F}_0$ and $\alpha = 1, 2, 3$.*

(i). If $-n < t < 0 < r < s$ or $-n < s < 0 < r < t$, then

$$G_r^{(\alpha)}(K_1, \cdots, K_n) \leq [G_s^{(\alpha)}(K_1, \cdots, K_n)]^{\frac{(t-r)(n+s)}{(t-s)(n+r)}} [G_t^{(\alpha)}(K_1, \cdots, K_n)]^{\frac{(r-s)(n+t)}{(t-s)(n+r)}}.$$

(ii). If $-n < t < r < s < 0$ or $-n < s < r < t < 0$, then

$$G_r^{(\alpha)}(K_1, \cdots, K_n) \leq [G_s^{(\alpha)}(K_1, \cdots, K_n)]^{\frac{(t-r)(n+s)}{(t-s)(n+r)}} [G_t^{(\alpha)}(K_1, \cdots, K_n)]^{\frac{(r-s)(n+t)}{(t-s)(n+r)}}.$$

(iii). If $t < r < -n < s < 0$ or $s < r < -n < t < 0$, then

$$G_r^{(\alpha)}(K_1, \cdots, K_n) \geq [G_s^{(\alpha)}(K_1, \cdots, K_n)]^{\frac{(t-r)(n+s)}{(t-s)(n+r)}} [G_t^{(\alpha)}(K_1, \cdots, K_n)]^{\frac{(r-s)(n+t)}{(t-s)(n+r)}}.$$

Proof. We only prove the case of $\alpha = 2$ and the cases $\alpha = 1, 3$ follow along the same lines. Let $\mathbf{K} \in \mathscr{F}_0^n$ and $\mathbf{L} \in \mathscr{K}_0^n$. Let $r, s, t \in \mathbb{R}$ such that $0 < \frac{t-r}{t-s} < 1$. By formula (2.4) and Hölder inequality (see [10]),

$$\begin{aligned}
nV_r(\mathbf{K}; \mathbf{L}) &= \int_{S^{n-1}} \prod_{i=1}^n [h_{L_i}^r(u) f_r(K_i, u)]^{\frac{1}{n}} d\sigma(u) \\
&= \int_{S^{n-1}} \left(\prod_{i=1}^n [h_{L_i}^s(u) f_s(K_i, u)]^{\frac{1}{n}}\right)^{\frac{t-r}{t-s}} \left(\prod_{i=1}^n [h_{L_i}^t(u) f_t(K_i, u)]^{\frac{1}{n}}\right)^{\frac{r-s}{t-s}} d\sigma(u) \\
&\leq \left(\int_{S^{n-1}} \prod_{i=1}^n [h_{L_i}^s(u) f_s(K_i, u)]^{\frac{1}{n}} d\sigma(u)\right)^{\frac{t-r}{t-s}} \left(\int_{S^{n-1}} \prod_{i=1}^n [h_{L_i}^t(u) f_t(K_i, u)]^{\frac{1}{n}} d\sigma(u)\right)^{\frac{r-s}{t-s}} \\
&= [nV_s(\mathbf{K}; \mathbf{L})]^{\frac{t-r}{t-s}} [nV_t(\mathbf{K}; \mathbf{L})]^{\frac{r-s}{t-s}}. \qquad (5.22)
\end{aligned}$$

(i). Suppose that $-n < t < 0 < r < s$, which implies $0 < \frac{t-r}{t-s} < 1$. For simplicity, we let $\lambda = \frac{(r-s)(n+t)}{(t-s)(n+r)}$, and in this case $\lambda > 0$. We also let $\mathbf{L}_1 = (L_{11}, \cdots, L_{n1}) \in \mathscr{K}_0^n$. Then,

$$[G_t^{(2)}(K_1, \cdots, K_n)]^\lambda = \left\{\sup_{\mathbf{L}_1 \in \mathscr{K}_0^n} \left[nV_t(\mathbf{K}; \mathbf{L}_1)^{\frac{n}{n+t}} \prod_{i=1}^n |L_{i1}^\circ|^{\frac{t}{n(n+t)}}\right]\right\}^\lambda = \sup_{\mathbf{L}_1 \in \mathscr{K}_0^n} \left[nV_t(\mathbf{K}; \mathbf{L}_1)^{\frac{n}{n+t}} \prod_{i=1}^n |L_{i1}^\circ|^{\frac{t}{n(n+t)}}\right]^\lambda.$$

By inequality (5.22) and $\frac{n}{n+r}>0$, one has, for all $\mathbf{L}=(L_1,\cdots,L_n)\in\mathscr{K}_0^n$,

$$\begin{aligned} G_r^{(2)}(K_1,\cdots,K_n) &\le n\big[V_r(\mathbf{K};\mathbf{L})\big]^{\frac{n}{n+r}}\prod_{i=1}^n|L_i^\circ|^{\frac{r}{n(n+r)}} \\ &\le \left[nV_s(\mathbf{K};\mathbf{L})^{\frac{n}{n+s}}\prod_{i=1}^n|L_i^\circ|^{\frac{s}{n(n+s)}}\right]^{1-\lambda}\left[nV_t(\mathbf{K};\mathbf{L})^{\frac{n}{n+t}}\prod_{i=1}^n|L_i^\circ|^{\frac{t}{n(n+t)}}\right]^{\lambda} \\ &\le \left[nV_s(\mathbf{K};\mathbf{L})^{\frac{n}{n+s}}\prod_{i=1}^n|L_i^\circ|^{\frac{s}{n(n+s)}}\right]^{1-\lambda}\sup_{\mathbf{L}_1\in\mathscr{K}_0^n}\left[nV_t(\mathbf{K};\mathbf{L}_1)^{\frac{n}{n+t}}\prod_{i=1}^n|L_{i1}^\circ|^{\frac{t}{n(n+t)}}\right]^{\lambda} \\ &= \left[nV_s(\mathbf{K};\mathbf{L})^{\frac{n}{n+s}}\prod_{i=1}^n|L_i^\circ|^{\frac{s}{n(n+s)}}\right]^{1-\lambda}[G_t^{(2)}(K_1,\cdots,K_n)]^{\lambda}. \end{aligned} \tag{5.23}$$

Taking infimum over $\mathbf{L}\in\mathscr{K}_0^n$ in inequality (5.23), we have

$$\begin{aligned} G_r^{(2)}(K_1,\cdots,K_n) &\le \inf_{\mathbf{L}\in\mathscr{K}_0^n}\left[nV_s(\mathbf{K};\mathbf{L})^{\frac{n}{n+s}}\prod_{i=1}^n|L_i^\circ|^{\frac{s}{n(n+s)}}\right]^{1-\lambda}[G_t^{(2)}(K_1,\cdots,K_n)]^{\lambda} \\ &= [G_s^{(2)}(K_1,\cdots,K_n)]^{1-\lambda}\,[G_t^{(2)}(K_1,\cdots,K_n)]^{\lambda}, \end{aligned}$$

where the equality follows from $1-\lambda=\frac{(r-s)(n+t)}{(t-s)(n+r)}>0$ and then

$$[G_s^{(2)}(K_1,\cdots,K_n)]^{1-\lambda}=\left\{\inf_{\mathbf{L}\in\mathscr{K}_0^n}\left[nV_s(\mathbf{K};\mathbf{L})^{\frac{n}{n+s}}\prod_{i=1}^n|L_i^\circ|^{\frac{s}{n(n+s)}}\right]\right\}^{1-\lambda}=\inf_{\mathbf{L}\in\mathscr{K}_0^n}\left[nV_s(\mathbf{K};\mathbf{L})^{\frac{n}{n+s}}\prod_{i=1}^n|L_i^\circ|^{\frac{s}{n(n+s)}}\right]^{1-\lambda}.$$

The case of $-n<s<0<r<t$ follows immediately by switching the roles of t and s.

(ii). Suppose that $-n<t<r<s<0$, which clearly implies $0<\frac{t-r}{t-s}<1$. We let $\lambda=\frac{(r-s)(n+t)}{(t-s)(n+r)}$ and in this case $\lambda>0$. Therefore, we have,

$$[G_t^{(2)}(K_1,\cdots,K_n)]^{\lambda}=\left\{\sup_{\mathbf{L}\in\mathscr{K}_0^n}\left[nV_t(\mathbf{K};\mathbf{L})^{\frac{n}{n+t}}\prod_{i=1}^n|L_i^\circ|^{\frac{t}{n(n+t)}}\right]\right\}^{\lambda}=\sup_{\mathbf{L}\in\mathscr{K}_0^n}\left[nV_t(\mathbf{K};\mathbf{L})^{\frac{n}{n+t}}\prod_{i=1}^n|L_i^\circ|^{\frac{t}{n(n+t)}}\right]^{\lambda}.$$

Similarly, due to $1-\lambda=\frac{(t-r)(n+s)}{(t-s)(n+r)}>0$, one has

$$[G_s^{(2)}(K_1,\cdots,K_n)]^{1-\lambda}=\sup_{\mathbf{L}\in\mathscr{K}_0^n}\left[nV_s(\mathbf{K};\mathbf{L})^{\frac{n}{n+s}}\prod_{i=1}^n|L_i^\circ|^{\frac{s}{n(n+s)}}\right]^{1-\lambda}.$$

By inequality (5.22) and $\frac{n}{n+r}>0$, we have, for all $\mathbf{L}\in\mathscr{K}_0^n$,

$$nV_r(\mathbf{K};\mathbf{L})^{\frac{n}{n+r}}\prod_{i=1}^n|L_i^\circ|^{\frac{r}{n(n+r)}}\le\left[nV_s(\mathbf{K};\mathbf{L})^{\frac{n}{n+s}}\prod_{i=1}^n|L_i^\circ|^{\frac{s}{n(n+s)}}\right]^{1-\lambda}\left[nV_t(\mathbf{K};\mathbf{L})^{\frac{n}{n+t}}\prod_{i=1}^n|L_i^\circ|^{\frac{t}{n(n+t)}}\right]^{\lambda}.$$

The desired inequality follows by taking supremum over $\mathbf{L} \in \mathscr{K}_0^n$. The case of $-n < s < r < t < 0$ follows immediately by switching the roles of t and s.

(iii). Suppose that $t < r < -n < s < 0$, which clearly implies $0 < \frac{t-r}{t-s} < 1$. Let $\lambda = \frac{(r-s)(n+t)}{(t-s)(n+r)}$ and in this case $\lambda > 1$. Then,

$$\big[G_t^{(2)}(K_1, \cdots, K_n)\big]^{\lambda} = \left\{\sup_{\mathbf{L} \in \mathscr{K}_0^n} \left[nV_t(\mathbf{K}; \mathbf{L})^{\frac{n}{n+t}} \prod_{i=1}^{n} |L_i^\circ|^{\frac{t}{n(n+t)}}\right]\right\}^{\lambda} = \sup_{\mathbf{L} \in \mathscr{K}_0^n} \left[nV_t(\mathbf{K}; \mathbf{L})^{\frac{n}{n+t}} \prod_{i=1}^{n} |L_i^\circ|^{\frac{t}{n(n+t)}}\right]^{\lambda}.$$

Similarly, due to $1 - \lambda = \frac{(t-r)(n+s)}{(t-s)(n+r)} < 0$, one has

$$\big[G_s^{(2)}(K_1, \cdots, K_n)\big]^{1-\lambda} = \left\{\sup_{\mathbf{L}_1 \in \mathscr{K}_0^n} \left[nV_s(\mathbf{K}; \mathbf{L}_1)^{\frac{n}{n+s}} \prod_{i=1}^{n} |L_{i1}^\circ|^{\frac{s}{n(n+s)}}\right]\right\}^{1-\lambda} = \inf_{\mathbf{L}_1 \in \mathscr{K}_0^n} \left[nV_s(\mathbf{K}; \mathbf{L}_1)^{\frac{n}{n+s}} \prod_{i=1}^{n} |L_{i1}^\circ|^{\frac{s}{n(n+s)}}\right]^{1-\lambda}.$$

By inequality (5.22) and $\frac{n}{n+r} < 0$, we have, for all $\mathbf{L} \in \mathscr{K}_0^n$,

$$\begin{aligned} G_r^{(2)}(K_1, \cdots, K_n) &\geq nV_r(\mathbf{K}; \mathbf{L})^{\frac{n}{n+r}} \prod_{i=1}^{n} |L_i^\circ|^{\frac{r}{n(n+r)}} \\ &\geq \left[nV_s(\mathbf{K}; \mathbf{L})^{\frac{n}{n+s}} \prod_{i=1}^{n} |L_i^\circ|^{\frac{s}{n(n+s)}}\right]^{1-\lambda} \left[nV_t(\mathbf{K}; \mathbf{L})^{\frac{n}{n+t}} \prod_{i=1}^{n} |L_i^\circ|^{\frac{t}{n(n+t)}}\right]^{\lambda} \\ &\geq \left[nV_t(\mathbf{K}; \mathbf{L})^{\frac{n}{n+t}} \prod_{i=1}^{n} |L_i^\circ|^{\frac{t}{n(n+t)}}\right]^{\lambda} \inf_{\mathbf{L}_1 \in \mathscr{K}_0^n} \left[nV_s(\mathbf{K}; \mathbf{L}_1)^{\frac{n}{n+s}} \prod_{i=1}^{n} |L_{i1}^\circ|^{\frac{s}{n(n+s)}}\right]^{1-\lambda} \\ &= \left[nV_t(\mathbf{K}; \mathbf{L})^{\frac{n}{n+t}} \prod_{i=1}^{n} |L_i^\circ|^{\frac{t}{n(n+t)}}\right]^{\lambda} [G_s^{(2)}(K_1, \cdots, K_n)]^{1-\lambda}. \end{aligned}$$

The desired inequality follows by taking supremum over $\mathbf{L} \in \mathscr{K}_0^n$. The case of $s < r < -n < 0 < t$ follows immediately by switching the roles of t and s.

Remark. Note that the statement of Theorem 5.5 does not include the cases of $s = 0$ or $r = 0$ or $t = 0$. However, from the proof of Theorem 5.5, one can easily see that cyclic inequalities still hold for (only) one of r, s, t equal to 0.

The monotonicity of $\left(\frac{\widetilde{G}_p(K)}{n|K|}\right)^{\frac{n+p}{p}}$ was proved in [41]. Here we prove similar results for L_p mixed geominimal surface areas.

Theorem 5.6 *Let $K_1, \cdots, K_n \in \mathscr{F}_0$ be such that $G_0^{(\alpha)}(K_1, \cdots, K_n) > 0$. The L_p mixed geominimal surface areas are monotone increasing in the following sense: for $0 < q < p$, or $-n < q < 0 < p$, or $-n < q < p < 0$, or $q < p < -n$,*

$$\left[\frac{G_q^{(\alpha)}(K_1, \cdots, K_n)}{G_0^{(\alpha)}(K_1, \cdots, K_n)}\right]^{\frac{n+q}{q}} \leq \left[\frac{G_p^{(\alpha)}(K_1, \cdots, K_n)}{G_0^{(\alpha)}(K_1, \cdots, K_n)}\right]^{\frac{n+p}{p}}, \quad \textit{for all } \alpha = 1, 2, 3.$$

Proof. *Case 1:* $0 < q < p$. Employing (i) of Theorem 5.5 to $t = 0, r = q$ and $s = p$, then

$$G_q^{(\alpha)}(K_1, \cdots, K_n) \le \left[G_p^{(\alpha)}(K_1, \cdots, K_n)\right]^{\frac{q(n+p)}{p(n+q)}} \left[G_0^{(\alpha)}(K_1, \cdots, K_n)\right]^{\frac{(p-q)n}{p(n+q)}}.$$

We divide both sides of the inequality by $G_0^{(\alpha)}(K_1, \cdots, K_n)$ and get

$$\frac{G_q^{(\alpha)}(K_1, \cdots, K_n)}{G_0^{(\alpha)}(K_1, \cdots, K_n)} \le \left[\frac{G_p^{(\alpha)}(K_1, \cdots, K_n)}{G_0^{(\alpha)}(K_1, \cdots, K_n)}\right]^{\frac{q(n+p)}{p(n+q)}}.$$

The desired inequality follows by taking the power $\frac{n+q}{q} > 0$ from both sides.

Case 2: $-n < q < 0 < p$. Employing (i) of Theorem 5.5 to $r = 0, t = q$ and $s = p$, then

$$G_0^{(\alpha)}(K_1, \cdots, K_n) \le \left[G_p^{(\alpha)}(K_1, \cdots, K_n)\right]^{\frac{q(n+p)}{n(q-p)}} \left[G_q^{(\alpha)}(K_1, \cdots, K_n)\right]^{\frac{(n+q)p}{n(p-q)}}.$$

We divide both sides of the inequality by $G_0^{(\alpha)}(K_1, \cdots, K_n)$ and get

$$1 \le \left[\frac{G_p^{(\alpha)}(K_1, \cdots, K_n)}{G_0^{(\alpha)}(K_1, \cdots, K_n)}\right]^{\frac{q(n+p)}{n(q-p)}} \left[\frac{G_q^{(\alpha)}(K_1, \cdots, K_n)}{G_0^{(\alpha)}(K_1, \cdots, K_n)}\right]^{\frac{(n+q)p}{n(p-q)}}.$$

The desired inequality follows by taking the power $\frac{(q-p)n}{pq} > 0$ from both sides.

Case 3: $-n < q < p < 0$. Employing (ii) of Theorem 5.5 to $s = 0, t = q$ and $r = p$, then

$$G_p^{(\alpha)}(K_1, \cdots, K_n) \le \left[G_0^{(\alpha)}(K_1, \cdots, K_n)\right]^{\frac{n(q-p)}{q(n+p)}} \left[G_q^{(\alpha)}(K_1, \cdots, K_n)\right]^{\frac{p(n+q)}{q(n+p)}}.$$

Dividing both sides of the inequality by $[G_0^{(\alpha)}(K_1, \cdots, K_n)]^{\frac{n(q-p)}{q(n+p)}}$ and as $-n < p < 0$, we get,

$$\left[\frac{G_q^{(\alpha)}(K_1, \cdots, K_n)}{G_0^{(\alpha)}(K_1, \cdots, K_n)}\right]^{\frac{n+q}{q}} \le \left[\frac{G_p^{(\alpha)}(K_1, \cdots, K_n)}{G_0^{(\alpha)}(K_1, \cdots, K_n)}\right]^{\frac{n+p}{p}}.$$

Case 4: $q < p < -n$. Employing (iii) of Theorem 5.5 to $s = 0, t = q$ and $r = p$, then

$$G_p^{(\alpha)}(K_1, \cdots, K_n) \ge [G_0^{(\alpha)}(K_1, \cdots, K_n)]^{\frac{n(q-p)}{q(n+p)}} [G_q^{(\alpha)}(K_1, \cdots, K_n)]^{\frac{p(n+q)}{q(n+p)}}.$$

Dividing both sides of the inequality by $[G_0^{(\alpha)}(K_1, \cdots, K_n)]^{\frac{n(q-p)}{q(n+p)}}$ and as $p < -n$, we get,

$$\left[\frac{G_q^{(\alpha)}(K_1, \cdots, K_n)}{G_0^{(\alpha)}(K_1, \cdots, K_n)}\right]^{\frac{n+q}{q}} \le \left[\frac{G_p^{(\alpha)}(K_1, \cdots, K_n)}{G_0^{(\alpha)}(K_1, \cdots, K_n)}\right]^{\frac{n+p}{p}}.$$

6 The i-th mixed L_p geominimal surface areas

This section dedicates to the i-th mixed L_p geominimal surface areas, in particular, its related (affine) isoperimetric inequalities. Let $K, L \in \mathscr{F}_0^+$ and $Q_1, Q_2 \in \mathscr{K}_0$, we define $V_{p,i}(K, L; Q_1, Q_2)$ for all $i \in \mathbb{R}$ and all $p \in \mathbb{R}$ as

$$nV_{p,i}(K, L; Q_1, Q_2) = \int_{S^{n-1}} [h_{Q_1}^p(u) f_p(K, u)]^{\frac{n-i}{n}} [h_{Q_2}^p(u) f_p(L, u)]^{\frac{i}{n}} d\sigma(u). \tag{6.24}$$

When $Q_1, Q_2 \in \mathscr{S}_0$, we use the variation formula for $V_{p,i}(K, L; Q_1^\circ, Q_2^\circ)$ as

$$nV_{p,i}(K, L; Q_1^\circ, Q_2^\circ) = \int_{S^{n-1}} [\rho_{Q_1}^{-p}(u) f_p(K, u)]^{\frac{n-i}{n}} [\rho_{Q_2}^{-p}(u) f_p(L, u)]^{\frac{i}{n}} d\sigma(u).$$

We also define $\widetilde{V}_i(Q_1, Q_2)$ for all $i \in \mathbb{R}$ as follows:

$$n\widetilde{V}_i(Q_1, Q_2) = \int_{S^{n-1}} [\rho_{Q_1}(u)]^{n-i} [\rho_{Q_2}(u)]^i \, d\sigma(u).$$

By Hölder's inequality (see [10]), one has,

$$[\widetilde{V}_i(Q_1, Q_2)]^n \leq |Q_1|^{n-i} |Q_2|^i, \quad if \ 0 < i < n; \tag{6.25}$$

$$[\widetilde{V}_i(Q_1, Q_2)]^n \geq |Q_1|^{n-i} |Q_2|^i, \quad if \ i < 0 \ or \ i > n. \tag{6.26}$$

Equality holds in each inequality if and only if Q_1 and Q_2 are dilates of each other. Equality always holds in (6.25) and (6.26) for $i = 0$ or $i = n$.

For $K, L \in \mathscr{F}_0^+$, the i-th mixed p-affine surface area [19, 33, 38] can be formulated as

$$as_{p,i}(K, L) = \int_{S^{n-1}} [f_p(K, u)]^{\frac{n-i}{n+p}} [f_p(L, u)]^{\frac{i}{n+p}} \, d\sigma(u), \quad -n \neq p \in \mathbb{R}, \quad i \in \mathbb{R}. \tag{6.27}$$

The i-th mixed p-affine surface area contains many functionals as its special cases, such as the L_p affine surface area and the p-surface area (i.e., $i = -p$ and $L = B_2^n$). Related properties and (affine) isoperimetric inequalities can be found in [38]. The following proposition provides an equivalent formula for the i-th mixed p-affine surface area.

Proposition 6.1 *Let $K, L \in \mathscr{F}_0^+$.*
(i). For $p \geq 0$,

$$\begin{aligned} as_{p,i}(K, L) &= \inf_{Q \in \mathscr{S}_0} \left\{ n[V_{p,i}(K, L; Q^\circ, Q^\circ)]^{\frac{n}{n+p}} |Q|^{\frac{p}{n+p}} \right\} \\ &= \inf_{\{Q_1, Q_2 \in \mathscr{S}_0\}} \left\{ n[V_{p,i}(K, L; Q_1^\circ, Q_2^\circ)]^{\frac{n}{n+p}} \widetilde{V}_i(Q_1, Q_2)^{\frac{p}{n+p}} \right\}. \end{aligned}$$

(ii). For $-n \neq p < 0$,

$$\begin{aligned} as_{p,i}(K,L) &= \sup_{Q\in\mathscr{S}_0}\Big\{n[V_{p,i}(K,L;Q^\circ,Q^\circ)]^{\frac{n}{n+p}}|Q|^{\frac{p}{n+p}}\Big\} \\ &= \sup_{\{Q_1,Q_2\in\mathscr{S}_0\}}\Big\{n[V_{p,i}(K,L;Q_1^\circ,Q_2^\circ)]^{\frac{n}{n+p}}\widetilde{V}_i(Q_1,Q_2)^{\frac{p}{n+p}}\Big\}. \end{aligned}$$

Proof. First, notice that for all $-n \neq p \in \mathbb{R}$ and $i \in \mathbb{R}$, we have

$$as_{p,i}(K,L) = n[V_{p,i}(K,L;Q_0^\circ,Q_0^\circ)]^{\frac{n}{n+p}}[\widetilde{V}_i(Q_0,Q_0)]^{\frac{p}{n+p}}, \tag{6.28}$$

where $Q_0 \in \mathscr{S}_0$ is defined by $\rho_{Q_0}(u) = [f_p(K,u)^{\frac{n-i}{n}} f_p(L,u)^{\frac{i}{n}}]^{\frac{1}{n+p}}, \forall u \in S^{n-1}$.

(i). Clearly it holds for $p=0$. Let $p \in (0,\infty)$, then $\frac{n}{n+p} \in (0,1)$. Employing Hölder inequality (see [10]) to formula (6.27), one has, for all $K,L \in \mathscr{F}_0^+$, for all $Q_1,Q_2 \in \mathscr{S}_0$, and for all $i \in \mathbb{R}$,

$$\begin{aligned} as_{p,i}(K,L) &= \int_{S^{n-1}}\Big([\rho_{Q_1}(u)^{-p}f_p(K,u)]^{\frac{n-i}{n}}[\rho_{Q_2}(u)^{-p}f_p(L,u)]^{\frac{i}{n}}\Big)^{\frac{n}{n+p}}\Big(\rho_{Q_1}^{n-i}(u)\rho_{Q_2}^{i}(u)\Big)^{\frac{p}{n+p}}d\sigma(u) \\ &\leq n[V_{p,i}(K,L;Q_1^\circ,Q_2^\circ)]^{\frac{n}{n+p}}[\widetilde{V}_i(Q_1,Q_2)]^{\frac{p}{n+p}}. \end{aligned}$$

Taking infimum over $Q_1,Q_2 \in \mathscr{S}_0$ and together with formula (6.28), one has

$$\begin{aligned} as_{p,i}(K,L) &\leq \inf_{\{Q_1,Q_2\in\mathscr{S}_0\}}\Big\{n[V_{p,i}(K,L;Q_1^\circ,Q_2^\circ)]^{\frac{n}{n+p}}\widetilde{V}_i(Q_1,Q_2)^{\frac{p}{n+p}}\Big\} \\ &\leq \inf_{Q\in\mathscr{S}_0}\Big\{n[V_{p,i}(K,L;Q^\circ,Q^\circ)]^{\frac{n}{n+p}}|Q|^{\frac{p}{n+p}}\Big\} \leq as_{p,i}(K,L). \end{aligned}$$

(ii). Note that $-n \neq p < 0$ implies $\frac{n}{n+p} > 1$ or $\frac{n}{n+p} < 0$. Employing Hölder inequality (see [10]) to formula (6.27), one has, for all $K,L \in \mathscr{F}_0^+$, for all $Q_1,Q_2 \in \mathscr{S}_0$, and for all $i \in \mathbb{R}$,

$$\begin{aligned} as_{p,i}(K,L) &= \int_{S^{n-1}}\Big([\rho_{Q_1}(u)^{-p}f_p(K,u)]^{\frac{n-i}{n}}[\rho_{Q_2}(u)^{-p}f_p(L,u)]^{\frac{i}{n}}\Big)^{\frac{n}{n+p}}\Big(\rho_{Q_1}^{n-i}(u)\rho_{Q_2}^{i}(u)\Big)^{\frac{p}{n+p}}d\sigma(u) \\ &\geq n[V_{p,i}(K,L;Q_1^\circ,Q_2^\circ)]^{\frac{n}{n+p}}[\widetilde{V}_i(Q_1,Q_2)]^{\frac{p}{n+p}}. \end{aligned}$$

Taking supremum over $Q_1,Q_2 \in \mathscr{S}_0$ and together with formula (6.28), one has

$$\begin{aligned} as_{p,i}(K,L) &\geq \sup_{\{Q_1,Q_2\in\mathscr{S}_0\}}\Big\{n[V_{p,i}(K,L;Q_1^\circ,Q_2^\circ)]^{\frac{n}{n+p}}\widetilde{V}_i(Q_1,Q_2)^{\frac{p}{n+p}}\Big\} \\ &\geq \sup_{Q\in\mathscr{S}_0}\Big\{n[V_{p,i}(K,L;Q^\circ,Q^\circ)]^{\frac{n}{n+p}}|Q|^{\frac{p}{n+p}}\Big\} \geq as_{p,i}(K,L). \end{aligned}$$

Remark. By inequality (6.25) and Proposition 6.1, one has, for $p \geq 0$ and $0 < i < n$,

$$\begin{aligned} as_{p,i}(K,L) &= \inf_{\{Q_1,Q_2\in\mathscr{S}_0\}} \left\{ nV_{p,i}(K,L;Q_1^\circ,Q_2^\circ)^{\frac{n}{n+p}} \ \widetilde{V}_i(Q_1,Q_2)^{\frac{p}{n+p}} \right\} \\ &\leq \inf_{\{Q_1,Q_2\in\mathscr{S}_0\}} \left\{ nV_{p,i}(K,L;Q_1^\circ,Q_2^\circ)^{\frac{n}{n+p}} \ |Q_1|^{\frac{p(n-i)}{n(n+p)}} |Q_2|^{\frac{pi}{n(n+p)}} \right\} \\ &\leq \inf_{Q\in\mathscr{S}_0} \left\{ nV_{p,i}(K,L;Q^\circ,Q^\circ)^{\frac{n}{n+p}} \ |Q|^{\frac{p}{n+p}} \right\} = as_{p,i}(K,L). \end{aligned}$$

Similarly, for $-n < p < 0$ with $0 < i < n$, or $p < -n$ with $i > n$ (or $i < 0$), one has

$$as_{p,i}(K,L) = \sup_{\{Q_1,Q_2\in\mathscr{S}_0\}} \left\{ n[V_{p,i}(K,L;Q_1^\circ,Q_2^\circ)]^{\frac{n}{n+p}} |Q_1|^{\frac{p(n-i)}{n(n+p)}} |Q_2|^{\frac{pi}{n(n+p)}} \right\}.$$

Motivated by Proposition 6.1, we define the i-th mixed L_p geominimal surface areas as follows:

Definition 6.1 *Let $K,\ L \in \mathscr{F}_0^+$, and $\alpha = 1,2,3$.*
(i). For $p = 0$, we let

$$G_{0,i}^{(\alpha)}(K,\ L) = \int_{S^{n-1}} [h_K(u)f_K(u)]^{\frac{n-i}{n}} [h_L(u)f_L(u)]^{\frac{i}{n}} d\sigma(u).$$

For $p > 0$,

$$\begin{aligned} G_{p,i}^{(1)}(K,L) &= \inf_{Q\in\mathscr{K}_0} \left\{ n[V_{p,i}(K,L;Q,Q)]^{\frac{n}{n+p}} |Q^\circ|^{\frac{p}{n+p}} \right\}; \\ G_{p,i}^{(2)}(K,L) &= \inf_{\{Q_1,Q_2\in\mathscr{K}_0\}} \left\{ n[V_{p,i}(K,L;Q_1,Q_2)]^{\frac{n}{n+p}} |Q_1^\circ|^{\frac{p(n-i)}{n(n+p)}} |Q_2^\circ|^{\frac{pi}{n(n+p)}} \right\}; \\ G_{p,i}^{(3)}(K,L) &= \inf_{\{Q_1,Q_2\in\mathscr{K}_0\}} \left\{ n[V_{p,i}(K,L;Q_1,Q_2)]^{\frac{n}{n+p}} \widetilde{V}_i(Q_1^\circ,Q_2^\circ)^{\frac{p}{n+p}} \right\}. \end{aligned}$$

(ii). For $-n \neq p < 0$,

$$\begin{aligned} G_{p,i}^{(1)}(K,L) &= \sup_{Q\in\mathscr{K}_0} \left\{ n[V_{p,i}(K,L;Q,Q)]^{\frac{n}{n+p}} |Q^\circ|^{\frac{p}{n+p}} \right\}; \\ G_{p,i}^{(2)}(K,L) &= \sup_{\{Q_1,Q_2\in\mathscr{K}_0\}} \left\{ n[V_{p,i}(K,L;Q_1,Q_2)]^{\frac{n}{n+p}} |Q_1^\circ|^{\frac{p(n-i)}{n(n+p)}} |Q_2^\circ|^{\frac{pi}{n(n+p)}} \right\}; \\ G_{p,i}^{(3)}(K,L) &= \sup_{\{Q_1,Q_2\in\mathscr{K}_0\}} \left\{ n[V_{p,i}(K,L;Q_1,Q_2)]^{\frac{n}{n+p}} \widetilde{V}_i(Q_1^\circ,Q_2^\circ)^{\frac{p}{n+p}} \right\}. \end{aligned}$$

Clearly, $G_{p,i}^{(\alpha)}(K,L) = G_{p,n-i}^{(\alpha)}(L,K)$ for all $i \in \mathbb{R}$. Moreover,

$$G_{p,0}^{(\alpha)}(K,L) = \widetilde{G}_p(K), \quad \& \quad G_{p,n}^{(\alpha)}(K,L) = \widetilde{G}_p(L). \tag{6.29}$$

When $L = B_2^n$, we will write $G_{p,i}^{(\alpha)}(K)$ as $G_{p,i}^{(\alpha)}(K, B_2^n)$ for all $\alpha = 1, 2, 3$.

The i-th mixed L_p geominimal surface areas have many properties similar to the mixed L_p geominimal surface areas discussed in Section 4. For instance, the i-th mixed L_p geominimal surface areas are all affine invariant. Moreover, for $K, L \in \mathscr{F}_0^+$, $i \in \mathbb{R}$, and $\alpha = 1, 3$, one has

$$G_{p,i}^{(\alpha)}(K, L) \geq as_{p,i}(K, L) \ \ if \ p \geq 0 \ \ and \ \ G_{p,i}^{(\alpha)}(K, L) \leq as_{p,i}(K, L) \ \ if \ \ -n \neq p < 0.$$

Equality holds if $K, L \in \mathscr{F}_0^+$ satisfy the following property: if $\exists Q \in \mathscr{K}_0$ s.t.

$$[f_p(K, u)]^{\frac{n-i}{n}} [f_p(L, u)]^{\frac{i}{n}} = h_Q(u)^{-(n+p)}, \quad \forall u \in S^{n-1}.$$

In particular, one gets for all $i \in \mathbb{R}$, $-n \neq p \in \mathbb{R}$ and $\alpha = 1, 3$,

$$G_{p,i}^{(\alpha)}(B_2^n, B_2^n) \ = \ as_{p,i}(B_2^n, B_2^n) = n|B_2^n|.$$

Theorem 6.1 *Let $K, L \in \mathscr{F}_0^+$. Suppose that $i, j, k \in \mathbb{R}$ satisfy $i < j < k$. Then, for $-n < p \leq 0$ and $\alpha = 1, 2$, we have*

$$G_{p,j}^{(\alpha)}(K, L)^{k-i} \leq G_{p,i}^{(\alpha)}(K, L)^{k-j} G_{p,k}^{(\alpha)}(K, L)^{j-i}.$$

In particular, by letting $L = B_2^n$, we get

$$G_{p,j}^{(\alpha)}(K)^{k-i} \leq G_{p,i}^{(\alpha)}(K)^{k-j} G_{p,k}^{(\alpha)}(K)^{j-i}.$$

Proof. Let $K, L \in \mathscr{F}_0^+$ and $Q_1, Q_2 \in \mathscr{K}_0$. From formula (6.24) and Hölder's inequality, it follows that for all $p \neq -n$, and for $i < j < k$ (which clearly implies $0 < \frac{k-j}{k-i} < 1$),

$$\begin{aligned}
V_{p,j}(K, L; Q_1, Q_2) \ &= \ \frac{1}{n} \int_{S^{n-1}} [h_{Q_1}^p(u) f_p(K, u)]^{\frac{n-j}{n}} [h_{Q_2}^p(u) f_p(L, u)]^{\frac{j}{n}} d\sigma(u) \\
&\leq \ \left\{ \frac{1}{n} \int_{S^{n-1}} [h_{Q_1}^p(u) f_p(K, u)]^{\frac{n-i}{n}} [h_{Q_2}^p(u) f_p(L, u)]^{\frac{i}{n}} d\sigma(u) \right\}^{\frac{k-j}{k-i}} \\
&\quad \times \left\{ \frac{1}{n} \int_{S^{n-1}} [h_{Q_1}^p(u) f_p(K, u)]^{\frac{n-k}{n}} [h_{Q_2}^p(u) f_p(L, u)]^{\frac{k}{n}} d\sigma(u) \right\}^{\frac{j-i}{k-i}} \\
&= \ V_{p,i}(K, L; Q_1, Q_2)^{\frac{k-j}{k-i}} V_{p,k}(K, L; Q_1, Q_2)^{\frac{j-i}{k-i}}.
\end{aligned} \tag{6.30}$$

Note that $k - i > 0, k - j > 0$ and $j - i > 0$. Then, inequality (6.30) implies that for $-n < p \leq 0$

$$\begin{aligned}
G_{p,j}^{(2)}(K, L)^{k-i} \ &= \ \sup_{\{Q_1, Q_2 \in \mathscr{K}_0\}} \left\{ n[V_{p,j}(K, L; Q_1, Q_2)]^{\frac{n}{n+p}} \ |Q_1^\circ|^{\frac{p(n-j)}{n(n+p)}} |Q_2^\circ|^{\frac{pj}{n(n+p)}} \right\}^{k-i} \\
&\leq \ \sup_{\{Q_1, Q_2 \in \mathscr{K}_0\}} \left\{ n[V_{p,i}(K, L; Q_1, Q_2)]^{\frac{n}{n+p}} \ |Q_1^\circ|^{\frac{p(n-i)}{n(n+p)}} |Q_2^\circ|^{\frac{pi}{n(n+p)}} \right\}^{k-j} \\
&\quad \times \sup_{\{Q_1, Q_2 \in \mathscr{K}_0\}} \left\{ n[V_{p,k}(K, L; Q_1, Q_2)]^{\frac{n}{n+p}} \ |Q_1^\circ|^{\frac{p(n-k)}{n(n+p)}} |Q_2^\circ|^{\frac{pk}{n(n+p)}} \right\}^{j-i} \\
&= \ G_{p,i}^{(2)}(K, L)^{k-j} G_{p,k}^{(2)}(K, L)^{j-i}.
\end{aligned}$$

The case $\alpha = 1$ follows the same line by letting $Q_1 = Q_2$.

Remark. Let $-n < p \le 0$ and $\alpha = 1, 2$. For $0 < i < n$, let $(i, j, k) = (0, i, n)$ in Theorem 6.1, by formula (6.29), we have

$$G_{p,i}^{(\alpha)}(K, L)^n \le G_{p,0}^{(\alpha)}(K, L)^{n-i} G_{p,n}^{(\alpha)}(K, L)^i = \widetilde{G}_p(K)^{n-i} \widetilde{G}_p(L)^i.$$

The equality always holds if $i = 0$ or $i = n$. Also note that for $-n < p \le 0$, $\widetilde{G}_p(B_2^n) = n|B_2^n|$. By letting $L = B_2^n$, one has

$$G_{p,i}^{(\alpha)}(K)^n \quad \le \quad (n\omega_n)^i \widetilde{G}_p(K)^{n-i}, \quad 0 \le i \le n.$$

Similarly, for $i < 0$, let $(i, j, k) = (i, 0, n)$ in Theorem 6.1, then

$$G_{p,i}^{(\alpha)}(K, L)^n G_{p,n}^{(\alpha)}(K, L)^{-i} \ge G_{p,0}^{(\alpha)}(K, L)^{n-i} \Longleftrightarrow G_{p,i}^{(\alpha)}(K, L)^n \ge \widetilde{G}_p(K)^{n-i} \widetilde{G}_p(L)^i.$$

For $i > n$, let $(i, j, k) = (0, n, i)$ in Theorem 6.1, then

$$G_{p,0}^{(\alpha)}(K, L)^{i-n} G_{p,i}^{(\alpha)}(K, L)^n \ge G_{p,n}^{(\alpha)}(K, L)^i \Longleftrightarrow G_{p,i}^{(\alpha)}(K, L)^n \ge \widetilde{G}_p(K)^{n-i} \widetilde{G}_p(L)^i.$$

By letting $L = B_2^n$, we get

$$G_{p,i}^{(\alpha)}(K)^n \quad \ge \quad (n\omega_n)^i \widetilde{G}_p(K)^{n-i}, \quad i < 0 \text{ or } i > n. \tag{6.31}$$

Theorem 6.2 *Let K, L be convex bodies with continuous positive curvature functions and with centroid (or Santaló point) at the origin.*
(i). Let $p \ge 0$ and $0 \le i \le n$, then for all $\alpha = 2, 3$

$$\frac{G_{p,i}^{(\alpha)}(K, L)}{n|B_2^n|} \le \min\left\{\left(\frac{|K|}{|B_2^n|}\right)^{\frac{(n-p)(n-i)}{n(n+p)}}, \left(\frac{|K^\circ|}{|B_2^n|}\right)^{\frac{(p-n)(n-i)}{n(n+p)}}\right\} \times \min\left\{\left(\frac{|L|}{|B_2^n|}\right)^{\frac{(n-p)i}{n(n+p)}}, \left(\frac{|L^\circ|}{|B_2^n|}\right)^{\frac{(p-n)i}{n(n+p)}}\right\}.$$

Moreover, $G_{p,i}^{(\alpha)}(K, L) G_{p,i}^{(\alpha)}(K^\circ, L^\circ) \le (n\omega_n)^2$.
(ii). Let $-n < p < 0$ and $i \le 0$, then for all $\alpha = 1, 2$

$$\frac{G_{p,i}^{(\alpha)}(K)}{n|B_2^n|} \ge \left(\frac{|K|}{|B_2^n|}\right)^{\frac{(n-p)(n-i)}{n(n+p)}}.$$

Moreover, $G_{p,i}^{(\alpha)}(K) G_{p,i}^{(\alpha)}(K^\circ) \ge c^{n-i} (n\omega_n)^2$ with $c > 0$ a universal constant.
(iii). Let $p < -n$ and $0 \le i \le n$, then

$$\frac{G_{p,i}^{(2)}(K, L)}{n|B_2^n|} \ge \left(\frac{|K^\circ|}{|B_2^n|}\right)^{\frac{(p-n)(n-i)}{n(n+p)}} \left(\frac{|L^\circ|}{|B_2^n|}\right)^{\frac{(p-n)i}{n(n+p)}}.$$

Moreover, $G_{p,i}^{(2)}(K, L) G_{p,i}^{(2)}(K^\circ, L^\circ) \ge c^n (n\omega_n)^2$ with $c > 0$ a universal constant.

Proof. (i). The case of $p=0$ is clear. Let $p>0$ and $0\le i\le n$. Hölder's inequality implies

$$\begin{aligned}V_{p,i}(K,L;Q_1,Q_2)^n &= \left(\int_{S^{n-1}}[h^p_{Q_1}(u)f_p(K,u)]^{\frac{n-i}{n}}[h^p_{Q_2}(u)f_p(L,u)]^{\frac{i}{n}}d\sigma(u)\right)^n\\ &\le \left(\int_{S^{n-1}}h^p_{Q_1}(u)f_p(K,u)d\sigma(u)\right)^{n-i}\left(\int_{S^{n-1}}h^p_{Q_2}(u)f_p(L,u)d\sigma(u)\right)^{i}\\ &= V_p(K,Q_1)^{n-i}V_p(L,Q_2)^i. \qquad (6.32)\end{aligned}$$

Note that $\frac{n}{n+p}>0$. Definitions 6.1 and 2.1 together with inequality (6.32) imply

$$\begin{aligned}[G^{(3)}_{p,i}(K,L)]^n &\le [G^{(2)}_{p,i}(K,L)]^n=\inf_{\{Q_1,Q_2\in\mathscr{K}_0\}}\left\{n[V_{p,i}(K,L;Q_1,Q_2)]^{\frac{n}{n+p}}|Q_1^\circ|^{\frac{p(n-i)}{n(n+p)}}|Q_2^\circ|^{\frac{pi}{n(n+p)}}\right\}^n\\ &\le \inf_{Q_1\in\mathscr{K}_0}\left\{n[V_p(K,Q_1)]^{\frac{n}{n+p}}|Q_1^\circ|^{\frac{p}{n+p}}\right\}^{n-i}\inf_{Q_2\in\mathscr{K}_0}\left\{n[V_p(K,Q_2)]^{\frac{n}{n+p}}|Q_2^\circ|^{\frac{p}{n+p}}\right\}^{i}\\ &= \widetilde{G}_p(K)^{n-i}\widetilde{G}_p(L)^i. \qquad (6.33)\end{aligned}$$

Combining with inequalities (5.20) and (6.33), one gets, for $0\le i\le n$, $p\ge 0$ and $\alpha=2,3$,

$$\begin{aligned}\frac{G^{(\alpha)}_{p,i}(K,L)}{n|B_2^n|} &\le \left(\frac{\widetilde{G}_p(K)}{\widetilde{G}_p(B_2^n)}\right)^{\frac{n-i}{n}}\left(\frac{\widetilde{G}_p(L)}{\widetilde{G}_p(B_2^n)}\right)^{\frac{i}{n}}\\ &\le \min\left\{\left(\frac{|K|}{|B_2^n|}\right)^{\frac{(n-p)(n-i)}{n(n+p)}}, \left(\frac{|K^\circ|}{|B_2^n|}\right)^{\frac{(p-n)(n-i)}{n(n+p)}}\right\}\times\min\left\{\left(\frac{|L|}{|B_2^n|}\right)^{\frac{(n-p)i}{n(n+p)}}, \left(\frac{|L^\circ|}{|B_2^n|}\right)^{\frac{(p-n)i}{n(n+p)}}\right\}.\end{aligned}$$

Recall that $\widetilde{G}_p(K)\widetilde{G}_p(K^\circ)\le(n\omega_n)^2$ for all $K\in\mathscr{K}_c$ and $p\ge 0$ [41]. Combining with inequality (6.33), we have, for $0\le i\le n$, $p\ge 0$ and $\alpha=2,3$,

$$G^{(\alpha)}_{p,i}(K,L)G^{(\alpha)}_{p,i}(K^\circ,L^\circ)\le[\widetilde{G}_p(K)\widetilde{G}_p(K^\circ)]^{\frac{n-i}{n}}[\widetilde{G}_p(L)\widetilde{G}_p(L^\circ)]^{\frac{i}{n}}\le(n\omega_n)^2.$$

(ii). Let $-n<p<0$ and $i\le 0$. Recall the affine isoperimetric inequality for the L_p geominimal surface area in [41] for $-n<p<0$

$$\frac{\widetilde{G}_p(K)}{\widetilde{G}_p(B_2^n)}\ge\left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n+p}}.$$

Combining with inequality (6.31), one gets, for $i\le 0$ and $\alpha=1,2$,

$$\frac{G^{(\alpha)}_{p,i}(K)}{n|B_2^n|}\ge\left(\frac{\widetilde{G}_p(K)}{\widetilde{G}_p(B_2^n)}\right)^{\frac{n-i}{n}}\ge\left(\frac{|K|}{|B_2^n|}\right)^{\frac{(n-p)(n-i)}{n(n+p)}}.$$

Recall that $\widetilde{G}_p(K)\widetilde{G}_p(K^\circ) \geq c^n(n\omega_n)^2$ for all $K \in \mathscr{K}_c$ and $-n < p < 0$ [41]. Combining with inequality (6.31), we have, for $i \leq 0$ and $-n < p < 0$

$$G_{p,i}^{(\alpha)}(K)G_{p,i}^{(\alpha)}(K^\circ) \geq [\widetilde{G}_p(K)\widetilde{G}_p(K^\circ)]^{\frac{n-i}{n}}[n\omega_n]^{\frac{2i}{n}} \geq c^{(n-i)}(n\omega_n)^2.$$

(iii). Let $0 \leq i \leq n$ and $p < -n$, which implies $\frac{n}{n+p} < 0$. Combining with formula (6.32), Definition 6.1 and Definition 2.1, one has

$$\begin{aligned}
[G_{p,i}^{(2)}(K,L)]^n &= \sup_{\{Q_1,Q_2\in\mathscr{K}_0\}} \left\{ n[V_{p,i}(K,L;Q_1,Q_2)]^{\frac{n}{n+p}}|Q_1^\circ|^{\frac{p(n-i)}{n(n+p)}}|Q_2^\circ|^{\frac{pi}{n(n+p)}} \right\}^n \\
&\geq \sup_{Q_1\in\mathscr{K}_0} \left\{ n[V_p(K,Q_1)]^{\frac{n}{n+p}}|Q_1^\circ|^{\frac{p}{n+p}} \right\}^{n-i} \sup_{Q_2\in\mathscr{K}_0} \left\{ n[V_p(K,Q_2)]^{\frac{n}{n+p}}|Q_2^\circ|^{\frac{p}{n+p}} \right\}^{i} \\
&= \widetilde{G}_p(K)^{n-i}\widetilde{G}_p(L)^i. \qquad (6.34)
\end{aligned}$$

Recall the affine isoperimetric inequality for the L_p geominimal surface area in [41] for $p < -n$

$$\frac{\widetilde{G}_p(K)}{\widetilde{G}_p(B_2^n)} \geq \left(\frac{|K^\circ|}{|B_2^n|}\right)^{\frac{p-n}{n+p}}.$$

Combining with inequality (6.34), one has, for $0 \leq i \leq n$ and $p < -n$,

$$\frac{G_{p,i}^{(2)}(K,L)}{n|B_2^n|} \geq \left(\frac{\widetilde{G}_p(K)}{\widetilde{G}_p(B_2^n)}\right)^{\frac{n-i}{n}} \left(\frac{\widetilde{G}_p(K)}{\widetilde{G}_p(B_2^n)}\right)^{\frac{i}{n}} \geq \left(\frac{|K^\circ|}{|B_2^n|}\right)^{\frac{(p-n)(n-i)}{n(n+p)}} \left(\frac{|L^\circ|}{|B_2^n|}\right)^{\frac{(p-n)i}{n(n+p)}}.$$

Recall that for all $K \in \mathscr{K}_c$ and $p < -n$ [41], $\widetilde{G}_p(K)\widetilde{G}_p(K^\circ) \geq c^n(n\omega_n)^2$. Combining with inequality (6.34), we have, for $0 \leq i \leq n$ and $p < -n$

$$G_{p,i}^{(2)}(K,L)G_{p,i}^{(2)}(K^\circ,L^\circ) \geq [\widetilde{G}_p(K)\widetilde{G}_p(K^\circ)]^{\frac{n-i}{n}}[\widetilde{G}_p(L)\widetilde{G}_p(L^\circ)]^{\frac{i}{n}} \geq c^n(n\omega_n)^2.$$

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