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Dual Orlicz–Brunn–Minkowski theory

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A B S T R A C T

In this paper, a dual Orlicz–Brunn–Minkowski theory is presented. An Orlicz radial sum and dual Orlicz mixed volumes are introduced. The dual Orlicz–Minkowski inequality and the dual Orlicz–Brunn–Minkowski inequality are established. The variational formula for the volume with respect to the Orlicz radial sum is proved. The equivalence between the dual Orlicz–Minkowski inequality and the dual Orlicz–Brunn–Minkowski inequality is demonstrated. Orlicz intersection bodies are defined and the Orlicz–Busemann–Petty problem is posed.

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1. Introduction

The classical Brunn–Minkowski theory, also known as the theory of mixed volumes, is the core of convex geometric analysis. It originated with Minkowski when he combined his concept of mixed volume with the Brunn–Minkowski inequality. One of Minkowski's major contributions to the theory was to show how his theory could be developed from a few basic concepts, such as support function, vector addition, and volume.

During the last two decades, the Brunn–Minkowski theory has been extended to the L_p Brunn–Minkowski theory, which combines volume and a generalized vector addition of compact convex sets introduced by Firey in the early 1960s (see [10]) and now called L_p addition. Thirty years after Fiery introduced his new L_p addition, the new L_p Brunn–Minkowski theory was born in Lutwak's papers [37,38] and has since witnessed a rapid growth. See [4,5,7,22,39,41,45,51,52] for additional references.

Recently, progress towards an Orlicz–Brunn–Minkowski theory was made by Lutwak, Yang and Zhang [43,44] and Ludwig [32]. This theory is far more general than the L_p Brunn–Minkowski theory, we refer the reader to [6,16,24,26,29,32,40,43,44,46,53, 57]. The Orlicz extension was motivated by concepts within the asymmetric L_p Brunn–Minkowski theory developed by, e.g., Ludwig [30], Haberl and Schuster [23], and Ludwig and Reitzner [33]. As part of the new Orlicz–Brunn–Minkowski theory, Lutwak, Yang and Zhang established two fundamental affine inequalities, the Orlicz Busemann–Petty centroid inequality [44] and the Orlicz Petty projection inequality [43]. The concepts of the Orlicz–Brunn–Minkowski theory are much more general than those of the L_p Brunn–Minkowski theory.

During the last three decades, convex geometric analysis has achieved important developments. Lutwak's dual Brunn–Minkowski theory, introduced in the 1970s, helped achieving a major breakthrough in the solution of the Busemann–Petty problem in the 1990s. In contrast to the Brunn–Minkowski theory, in the dual theory, convex bodies are replaced by star-shaped sets, and projections onto subspaces are replaced by intersections with subspaces. The machinery of the dual theory includes dual mixed volumes and important auxiliary bodies known as intersection bodies (see [11–13,18,27,28,35,36, 54–56]).

L_p harmonic radial combination of convex bodies were investigated by Firey (see [8,9]). Then, Lutwak extended the L_p harmonic radial combination to star bodies in [38]. If K and L are star bodies (see Section 2 for precise definition and unexplained terminology), and $a, b \geq 0$ (not both zero), then for $p \geq 1$, the L_p harmonic radial combination, $a \cdot K \mathbin{\widetilde{+}_p} b \cdot L$ is a star body and defined by (see [38])

$$\rho(a \cdot K \mathbin{\widetilde{+}_p} b \cdot L, u)^{-p} = a\rho(K, u)^{-p} + b\rho(L, u)^{-p}, \quad u \in S^{n-1}, \tag{1.1}$$

where the function ρ is the radial function of the set involved. In [38], Lutwak showed that associated with L_p harmonic radial combination there were integral representation and inequalities for the L_p dual mixed volume. Since then an L_p dual Brunn–Minkowski theory has been developed (see [20,31,33,38,42,56]). Motivated by the manner in which Lutwak, Yang and Zhang extended the L_p Brunn–Minkowski theory to the Orlicz–Brunn–Minkowski theory, we extend the dual L_p Brunn–Minkowski theory to a dual Orlicz–Brunn–Minkowski theory.

To do so, we define an Orlicz radial sum and dual Orlicz mixed volumes as extensions of dual L_p mixed volumes. Consider convex function $\phi : (-\infty, 0) \cup (0, +\infty) \to (0, \infty)$ such that $\lim_{t\to\infty} \phi(t) = 0$, $\lim_{t\to 0} \phi(t) = \infty$, and set $\phi(0) = \infty$. This means that ϕ must be increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$. We will assume that ϕ is either strictly increasing on $(-\infty, 0)$ or strictly decreasing on $(0, \infty)$ throughout this paper. Let $\mathcal{C}$ denote the class of all such convex functions ϕ. Let $\mathcal{C}^+$ be the class of convex and strictly decreasing functions $\phi : (0, \infty) \to (0, \infty)$ such that $\lim_{t\to\infty} \phi(t) = 0$, $\lim_{t\to 0} \phi(t) = \infty$, and $\phi(0) = \infty$.

For $\phi \in \mathcal{C}^+$, we define the Orlicz radial sum $K \widetilde{+}_\phi L$ of two star bodies about the origin K and L in $\mathbb{R}^n$, by

$$\rho(K \widetilde{+}_\phi L, u) = \sup\left\{t > 0 : \phi\left(\frac{\rho(K, u)}{t}\right) + \phi\left(\frac{\rho(L, u)}{t}\right) \le \phi(1)\right\}, \quad u \in S^{n-1}. \tag{1.2}$$

We will show that the Orlicz radial sum reduces to the L_p harmonic radial sum when $\phi(t) = t^{-p}$ in Section 3.

We also define the dual Orlicz mixed volume $\widetilde{V}_\phi(K, L)$, and prove the following dual Orlicz–Minkowski inequality: For $\phi \in \mathcal{C}^+$, if $K, L \in \mathcal{S}_o^n$ (see Section 2 for precise definition), then

$$\widetilde{V}_\phi(K, L) \ge V(K)\phi\left(\left(\frac{V(L)}{V(K)}\right)^{\frac{1}{n}}\right),$$

where $V(\cdot)$ is the volume function of the set involved.

In Section 5, we will establish Orlicz radial sum versions of the dual Brunn–Minkowski inequality for star bodies including the equality conditions. For example, the following dual Orlicz–Brunn–Minkowski inequality:

$$\phi(1) \ge \phi\left(\left(\frac{V(K)}{V(K \widetilde{+}_\phi L)}\right)^{\frac{1}{n}}\right) + \phi\left(\left(\frac{V(L)}{V(K \widetilde{+}_\phi L)}\right)^{\frac{1}{n}}\right), \tag{1.3}$$

with equality if and only if K and L are dilates of each other. Here $\phi \in \mathcal{C}^+$ and V denotes volume.

When $\phi(t) = t^{-p}$, with $p \ge 1$. The above dual Orlicz–Brunn–Minkowski inequality reduces to Lutwak's L_p dual Brunn–Minkowski inequality (see [38]):

$$V(K \widetilde{+}_\phi L)^{-\frac{p}{n}} \ge V(K)^{-\frac{p}{n}} + V(L)^{-\frac{p}{n}}, \tag{1.4}$$

with equality if and only if K and L are dilates of each other.

This paper is organized as follows. In Section 2, we collect some basic concepts and various facts that will be used in the proofs of our results. In Section 3, the Orlicz radial sum is introduced, and some of its basic properties are shown. The definition of the dual Orlicz mixed volume is given in Section 4. In Section 5, we establish the dual Orlicz–Brunn–Minkowski inequality and the dual Orlicz–Minkowski inequality. We view these inequalities as central to the dual Orlicz–Brunn–Minkowski Theory. In Section 6, we introduce Orlicz intersection bodies and propose the Orlicz–Busemann–Petty problem.

2. Preliminaries

The unit ball in $\mathbb{R}^n$ and its surface are denoted by B and S^{n-1}, respectively. The volume of the unit ball B is denoted by $\omega_n = \pi^{n/2}/\Gamma(1+n/2)$, where $\Gamma(\cdot)$ is the Gamma function. We write $V(K)$ for the volume of the compact set K in $\mathbb{R}^n$.

For $A \in GL(n)$ write A^t for the transpose of A and A^{-t} for the inverse of the transpose of A. Write $|A|$ for the absolute value of the determinant of A.

We say that the sequence $\{\phi_i\}$, of $\phi_i \in \mathcal{C}^+$, is such that $\phi_i \to \phi_0 \in \mathcal{C}^+$ provided

$$|\phi_i - \phi_0|_I := \max_{s\in I}\bigl|\phi_i(s) - \phi_0(s)\bigr| \to 0,$$

for every compact interval $I \subset \mathbb{R}$.

The radial function $\rho_K = \rho(K,\cdot) : \mathbb{R}^n \setminus \{0\} \to [0,\infty)$, of a compact star-shaped (about the origin) $K \subset \mathbb{R}^n$, is defined by (see [14,50])

$$\rho(K,x) = \max\{\lambda \geq 0 : \lambda x \in K\}.$$

If ρ_K is positive and continuous, then K is called a star body (about the origin). Write $\mathcal{S}_o^n$ for the set of star bodies about the origin in $\mathbb{R}^n$. Two star bodies K and L are dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$. If $s > 0$, we have

$$\rho(sK,x) = s\rho(K,x), \quad \text{for all } x \in \mathbb{R}^n \setminus \{0\}. \tag{2.1}$$

More generally, from the definition of the radial function it follows immediately that for $A \in GL(n)$ the radial function of the image $AK = \{Ay : y \in K\}$ of K is given by (see [14,50])

$$\rho(AK,x) = \rho\bigl(K,A^{-1}x\bigr), \quad \text{for all } x \in \mathbb{R}^n. \tag{2.2}$$

For $K \in \mathcal{S}_o^n$, define the real numbers R_K and r_K by

$$R_K = \max_{u\in S^{n-1}} \rho_K(u) \quad \text{and} \quad r_K = \min_{u\in S^{n-1}} \rho_K(u). \tag{2.3}$$

Note that $0 < r_K < R_K < \infty$, for all $K \in \mathcal{S}_o^n$.

Obviously, for $K, L \in \mathcal{S}_o^n$,

$$K \subset L \quad \text{if and only if} \quad \rho_K \le \rho_L. \tag{2.4}$$

The radial Hausdorff metric between the star bodies K and L is

$$\widetilde{\delta}(K, L) = \max_{u \in S^{n-1}} \left|\rho_K(u) - \rho_L(u)\right|.$$

A sequence $\{K_i\}$ of star bodies is said to be convergent to K if

$$\widetilde{\delta}(K_i, K) \to 0, \quad \text{as } i \to \infty.$$

Therefore, a sequence of star bodies K_i converges to K if and only if the sequence of radial functions $\rho(K_i, \cdot)$ converges uniformly to $\rho(K, \cdot)$ (see [49, Theorem 7.9]).

The radial Minkowski addition and scalar product of sets Q and T in $\mathbb{R}^n$ is defined by (see [14,35])

$$aQ \,\widetilde{+}\, bT = \{ax \,\widetilde{+}\, by : x \in Q, y \in T\}, \quad \text{for all } a, b \in \mathbb{R}.$$

If $K, L \in \mathcal{S}_o^n$ and $a, b \ge 0$, $aK \widetilde{+} bL$ can be defined as a star body such that

$$\rho_{aK \,\widetilde{+}\, bL}(u) = a\rho_K(u) + b\rho_L(u), \quad \text{for all } u \in S^{n-1}. \tag{2.5}$$

The polar coordinate formula for the volume of a compact set K is

$$V(K) = \frac{1}{n} \int_{S^{n-1}} \rho_K^n(u)\, dS(u), \tag{2.6}$$

where S is the Lebesgue measure on S^{n-1} (i.e., the $(n-1)$-dimensional Hausdorff measure).

The first dual mixed volume, $\widetilde{V}_1(K, L)$, of $K, L \in \mathcal{S}_o^n$, is defined by

$$n\widetilde{V}_1(K, L) = \lim_{\varepsilon \to 0} \frac{V(K \,\widetilde{+}\, \varepsilon L) - V(K)}{\varepsilon} \tag{2.7}$$

In [34] Lutwak proved the following integral representation for the first dual mixed volume: If $K, L \in \mathcal{S}_o^n$, then

$$\widetilde{V}_1(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho_K^{n-1}(u)\rho_L(u)\, dS(u). \tag{2.8}$$

This integral representation, together with the Hölder inequality (see [25]), and the polar coordinate formula, immediately leads to the following dual Minkowski inequality for the first dual mixed volume (see [35]): If $K, L \in \mathcal{S}_o^n$, then

$$\widetilde{V}_1(K, L)^n \leq V(K)^{n-1} V(L), \tag{2.9}$$

with equality if and only if K and L are dilates.

From the dual Minkowski inequality, we can obtain the dual Brunn–Minkowski inequality (see [35]): If $K, L \in \mathcal{S}_o^n$ and $a, b \geq 0$ then

$$V(aK \mathbin{\widetilde{+}} bL)^{\frac{1}{n}} \leq aV(K)^{\frac{1}{n}} + bV(L)^{\frac{1}{n}}, \tag{2.10}$$

with equality if and only if K and L are dilates.

For $K, L \in \mathcal{S}_0^n$, $p \geq 1$, the L_p dual mixed volume $\widetilde{V}_{-p}(K, L)$ of K and L is defined by

$$-\frac{n}{p}\widetilde{V}_{-p}(K, L) = \lim_{\varepsilon \to 0} \frac{V(K \mathbin{\widetilde{+}_p} \varepsilon \cdot L) - V(K)}{\varepsilon}. \tag{2.11}$$

In [38] Lutwak also proved the following integral representation for the L_p dual mixed volume: For $K, L \in \mathcal{S}_0^n$, and $p \geq 1$,

$$\widetilde{V}_{-p}(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho_K^{n+p}(u) \rho_L^{-p}(u) dS(u). \tag{2.12}$$

Suppose that μ is a probability measure on a space X and $g : X \to I \subset \mathbb{R}$ is a μ-integrable function, where I is a possibly infinite interval. Jensen's inequality states that if $\phi : I \to \mathbb{R}$ is a convex function, then

$$\int_X \phi\big(g(x)\big) d\mu(x) \geq \phi\bigg(\int_X g(x) d\mu(x)\bigg). \tag{2.13}$$

If ϕ is strictly convex, equality holds if and only if $g(x)$ is constant for μ-almost all $x \in X$ (see [25]).

3. Orlicz radial sum

We first define the Orlicz radial sum.

Definition 3.1. Let $K, L \in \mathcal{S}_o^n$ with radial functions ρ_K, ρ_L, respectively. For $a, b \geq 0$ (not both zero) and $\phi \in \mathcal{C}^+$, define the Orlicz radial sum $a \cdot K \mathbin{\widetilde{+}_\phi} b \cdot L$ of K and L by

$$\rho_{a \cdot K \widetilde{+}_\phi b \cdot L}(u) = \sup\bigg\{t > 0 : a\phi\bigg(\frac{\rho_K(u)}{t}\bigg) + b\phi\bigg(\frac{\rho_L(u)}{t}\bigg) \leq \phi(1)\bigg\},$$
$$u \in S^{n-1}. \tag{3.1}$$

From (2.2) and the definition of Orlicz radial sum, we have:

Proposition 3.1. *Suppose $K, L \in \mathcal{S}_o^n$, and $a, b \geq 0$. If $\phi \in \mathcal{C}^+$, then for $A \in GL(n)$,*

$$A(a \cdot K \widetilde{+}_\phi b \cdot L) = a \cdot AK \widetilde{+}_\phi b \cdot AL. \tag{3.2}$$

Proof. For $u \in S^{n-1}$, by (2.2)

$$\begin{aligned}\rho_{a\cdot AK \widetilde{+}_\phi b\cdot AL}(u) &= \sup\left\{t > 0 : a\phi\left(\frac{\rho_{AK}(u)}{t}\right) + b\phi\left(\frac{\rho_{AL}(u)}{t}\right) \leq \phi(1)\right\} \\ &= \sup\left\{t > 0 : a\phi\left(\frac{\rho_K(A^{-1}u)}{t}\right) + b\phi\left(\frac{\rho_L(A^{-1}u)}{t}\right) \leq \phi(1)\right\} \\ &= \rho_{a\cdot K \widetilde{+}_\phi b\cdot L}\left(A^{-1}u\right) \\ &= \rho_{A(a\cdot K \widetilde{+}_\phi b\cdot L)}(u). \qquad \square\end{aligned}$$

Since $K, L \in \mathcal{S}_o^n$, $0 < \rho_K < \infty$ and $0 < \rho_L < \infty$, hence $\frac{\rho_K}{t} \to 0$ and $\frac{\rho_L}{t} \to 0$ as $t \to \infty$. By this and the assumption that ϕ is strictly decreasing in $(0, \infty)$, the function

$$t \mapsto a\phi\left(\frac{\rho_K}{t}\right) + b\phi\left(\frac{\rho_L}{t}\right)$$

is strictly increasing in $(0, \infty)$. It is also continuous. Thus, we have:

Lemma 3.1. *Suppose $K, L \in \mathcal{S}_o^n$ and $u \in S^{n-1}$. If $\phi \in \mathcal{C}^+$, then*

$$a\phi\left(\frac{\rho_K(u)}{t}\right) + b\phi\left(\frac{\rho_L(u)}{t}\right) = \phi(1)$$

if and only if

$$\rho_{a\cdot K \widetilde{+}_\phi b\cdot L}(u) = t.$$

Remark 3.1. When $\phi(t) = t^{-p}$, with $p \geq 1$, it is easy to show that the Orlicz radial sum reduces to Lutwak's L_p harmonic radial combination (see [38]):

$$\rho(a \cdot K \widetilde{+}_\phi b \cdot L, u)^{-p} = a\rho(K, u)^{-p} + b\rho(L, u)^{-p}, \quad u \in S^{n-1}.$$

If $K, L \in \mathcal{S}_o^n$, let $R = \max\{R_K, R_L\}$ and $r = \min\{r_K, r_L\}$. For $a, b \geq 0$, let $C = \max\{a, b\}$ and $c = a + b$. Since ϕ is continuous and strictly deceasing in $(0, \infty)$, hence the inverse ϕ^{-1} is also continuous and deceasing in $(0, \infty)$.

Lemma 3.2. *Suppose $K, L \in \mathcal{S}_o^n$. If $\phi \in \mathcal{C}^+$, then*

$$\frac{r}{\phi^{-1}(\frac{\phi(1)}{2C})} \le \rho_{a\cdot K \,\widetilde{+}_\phi\, b\cdot L}(u) \le \frac{R}{\phi^{-1}(\frac{\phi(1)}{c})},$$

for all $u \in S^{n-1}$.

Proof. Suppose $u \in S^{n-1}$ and let $\rho_{a\cdot K \,\widetilde{+}_\phi\, b\cdot L}(u) = t$. By Lemma 3.1 and the fact that ϕ is strictly deceasing on $(0,\infty)$, we have

$$\begin{aligned} \phi(1) &= a\phi\left(\frac{\rho_K(u)}{t}\right) + b\phi\left(\frac{\rho_L(u)}{t}\right) \\ &\le C\phi\left(\frac{r_K}{t}\right) + C\phi\left(\frac{r_L}{t}\right) \\ &\le 2C\phi\left(\frac{r}{t}\right). \end{aligned} \tag{3.3}$$

Since the inverse ϕ^{-1} of ϕ is strictly deceasing on $(0,\infty)$, we have the lower bound for $\rho_{a\cdot K \,\widetilde{+}_\phi\, b\cdot L}(u)$:

$$t \ge \frac{r}{\phi^{-1}(\frac{\phi(1)}{2C})}.$$

On the other hand, from Eq. (3.3), together with the convexity and the strictly decreasing on $(0,\infty)$ of ϕ, we have

$$\begin{aligned} \frac{\phi(1)}{a+b} &= \frac{a}{a+b}\phi\left(\frac{\rho_K(u)}{t}\right) + \frac{b}{a+b}\phi\left(\frac{\rho_L(u)}{t}\right) \\ &\ge \frac{a}{a+b}\phi\left(\frac{R_K}{t}\right) + \frac{b}{a+b}\phi\left(\frac{R_L}{t}\right) \\ &\ge \phi\left(\frac{\frac{a}{a+b}R_K + \frac{b}{a+b}R_L}{t}\right) \\ &\ge \phi\left(\frac{R}{t}\right). \end{aligned}$$

Then we obtain the upper estimate:

$$t \le \frac{R}{\phi^{-1}(\frac{\phi(1)}{c})}. \qquad \square$$

We now prove that the Orlicz radial sum of two star bodies is again a star body.

Lemma 3.3. *Suppose $\phi \in \mathcal{C}^+$ and $a, b \ge 0$ (not both zero). If $K, L \in \mathcal{S}_o^n$, then $a \cdot K \,\widetilde{+}_\phi\, b \cdot L \in \mathcal{S}_o^n$.*

Proof. Let $u_0 \in S^{n-1}$, for any subsequence $\{u_i\} \subset S^{n-1}$ such that $u_i \to u_0$ as $i \to \infty$, we need to show

$$\rho_{a\cdot K \,\widetilde{+}_\phi\, b\cdot L}(u_i) \to \rho_{a\cdot K \,\widetilde{+}_\phi\, b\cdot L}(u_0), \quad \text{as } i \to \infty.$$

Let

$$\rho_{a\cdot K \,\widetilde{+}_\phi\, b\cdot L}(u_i) = t_i.$$

Then Lemma 3.2 gives

$$\frac{r}{\phi^{-1}(\frac{\phi(1)}{2C})} \le t_i \le \frac{R}{\phi^{-1}(\frac{\phi(1)}{c})}.$$

Since $K, L \in \mathcal{S}_o^n$, we have $0 < r_K \le R_K < \infty$, $0 < r_L \le R_L < \infty$. Thus, there exist λ, μ such that $0 < \lambda \le t_i \le \mu < \infty$, for all i. To show that the bounded sequence $\{t_i\}$ converges to $\rho_{a\cdot K \,\widetilde{+}_\phi\, b\cdot L}(u_0)$, we show that every convergent subsequence of $\{t_i\}$ converges to $\rho_{a\cdot K \,\widetilde{+}_\phi\, b\cdot L}(u_0)$. Denote an arbitrary convergent subsequence of $\{t_i\}$ by $\{t_i\}$ as well, and suppose that for this subsequence

$$t_i \to t_0.$$

It is clear that $\lambda \le t_0 \le \mu$. Lemma 3.1 and the fact $\rho_{a\cdot K \,\widetilde{+}_\phi\, b\cdot L}(u_i) = t_i$ show that

$$a\phi\left(\frac{\rho_K(u_i)}{t_i}\right) + b\phi\left(\frac{\rho_L(u_i)}{t_i}\right) = \phi(1).$$

Since ρ_K and ρ_L are continuous on S^{n-1}, together with the continuity of ϕ and $t_i \to t_0$, it follows that

$$a\phi\left(\frac{\rho_K(u_0)}{t_0}\right) + b\phi\left(\frac{\rho_L(u_0)}{t_0}\right) = \phi(1).$$

By Lemma 3.1, we have

$$t_0 = \rho_{a\cdot K \,\widetilde{+}_\phi\, b\cdot L}(u_0).$$

This shows

$$\rho_{a\cdot K \,\widetilde{+}_\phi\, b\cdot L}(u_i) \to \rho_{a\cdot K \,\widetilde{+}_\phi\, b\cdot L}(u_0).$$

Therefore, the continuity of $\rho_{a\cdot K \,\widetilde{+}_\phi\, b\cdot L}$ is proved and $a \cdot K \,\widetilde{+}_\phi\, b \cdot L \in \mathcal{S}_o^n$. □

From the definition of the Orlicz radial sum (3.1), for $\sigma > 0$, we have

$$\begin{aligned}\rho_{a\cdot(\sigma K)\,\widetilde{+}_\phi\,b\cdot(\sigma L)}(u) &= \sup\left\{t>0: a\phi\left(\frac{\rho_{\sigma K}(u)}{t}\right)+b\phi\left(\frac{\rho_{\sigma L}(u)}{t}\right)\le\phi(1)\right\}\\ &= \sup\left\{\sigma t>0: a\phi\left(\frac{\rho_{K}(u)}{t}\right)+b\phi\left(\frac{\rho_{L}(u)}{t}\right)\le\phi(1)\right\}\\ &= \sigma\rho_{a\cdot K\,\widetilde{+}_\phi\,b\cdot L}(u).\end{aligned}$$

This gives that

$$\rho_{a\cdot(\sigma K)\,\widetilde{+}_\phi\,b\cdot(\sigma L)}(u)=\sigma\rho_{a\cdot K\,\widetilde{+}_\phi\,b\cdot L}(u). \tag{3.4}$$

Next, we show that the Orlicz radial sum $\widetilde{+}_\phi : \mathcal{S}_o^n \to \mathcal{S}_o^n$ is continuous.

Lemma 3.4. *Suppose $\phi \in \mathcal{C}^+$. If $K_i, L_i \in \mathcal{S}_o^n$ and $K_i \to K \in \mathcal{S}_o^n, L_i \to L \in \mathcal{S}_o^n$, as $i \to \infty$, then*

$$a\cdot K_i\,\widetilde{+}_\phi\,b\cdot L_i \to a\cdot K\,\widetilde{+}_\phi\,b\cdot L, \quad \text{as } i\to\infty$$

for all a and b.

Proof. Suppose $u \in S^{n-1}$. We will show that

$$\rho_{a\cdot K_i\,\widetilde{+}_\phi\,b\cdot L_i}(u)\to\rho_{a\cdot K\,\widetilde{+}_\phi\,b\cdot L}(u), \quad \text{as } i\to\infty. \tag{3.5}$$

Let

$$\rho_{a\cdot K_i\,\widetilde{+}_\phi\,b\cdot L_i}(u)=t_i.$$

Write $R_i = \max\{R_{K_i}, R_{L_i}\}$ and $r_i = \min\{r_{K_i}, r_{L_i}\}$. Then Lemma 3.2 gives

$$\frac{r_i}{\phi^{-1}(\frac{\phi(1)}{2C})}\le t_i\le\frac{R_i}{\phi^{-1}(\frac{\phi(1)}{c})}.$$

Since $K_i \to K \in \mathcal{S}_o^n$ and $L_i \to L \in \mathcal{S}_o^n$, we have $R_{K_i} \to R_K < \infty$, $R_{L_i} \to R_L < \infty$, and $r_{K_i} \to r_K > 0$, $r_{L_i} \to r_L > 0$. By the fact that the functions $R = \max\{R_K, R_L\}$ and $r = \min\{r_K, r_L\}$ are continuous, we have $R_i \to R < \infty$, $r_i \to r > 0$. Thus, there exist λ, μ such that

$$0<\lambda\le t_i\le\mu<\infty, \quad \text{for all } i. \tag{3.6}$$

To show that the bounded sequence $\{t_i\}$ converges to $\rho_{a\cdot K\,\widetilde{+}_\phi\,b\cdot L}(u)$, we show that every convergent subsequence of $\{t_i\}$ converges to $\rho_{a\cdot K\,\widetilde{+}_\phi\,b\cdot L}(u)$. Denote an arbitrary convergent subsequence of $\{t_i\}$ by $\{t_i\}$ as well, and suppose that for this subsequence we have

$$t_i \to t_0.$$

It is clear that $\lambda \leq t_0 \leq \mu$. Let $\widetilde{K}_i = t_i^{-1} K_i$ and $\widetilde{L}_i = t_i^{-1} L_i$. Since $K_i \to K$, $L_i \to L$, and $t_i^{-1} \to t_0^{-1}$, we have $t_i^{-1} K_i \to t_0^{-1} K$ and $t_i^{-1} L_i \to t_0^{-1} L$.

Now (3.4), and the fact $\rho_{a \cdot K_i \,\widetilde{+}_\phi\, b \cdot L_i}(u) = t_i$, show that $\rho_{a \cdot \widetilde{K}_i \,\widetilde{+}_\phi\, b \cdot \widetilde{L}_i}(u) = 1$. That is,

$$a\phi\big(\rho_{\widetilde{K}_i}(u)\big) + b\phi\big(\rho_{\widetilde{L}_i}(u)\big) = \phi(1), \quad \text{for all } i.$$

Since $t_i^{-1} K_i \to t_0^{-1} K$ and $t_i^{-1} L_i \to t_0^{-1} L$, together with the continuity of ϕ, and (2.1), it follows that

$$a\phi\left(\frac{\rho_K(u)}{t_0}\right) + b\phi\left(\frac{\rho_L(u)}{t_0}\right) = \phi(1).$$

By Lemma 3.1, we have

$$t_0 = \rho_{a \cdot K \,\widetilde{+}_\phi\, b \cdot L}(u).$$

This shows

$$\rho_{a \cdot K_i \,\widetilde{+}_\phi\, b \cdot L_i}(u) \to \rho_{a \cdot K \,\widetilde{+}_\phi\, b \cdot L}(u).$$

Now the pointwise convergence (3.5) has been proved.

We will show that the convergence (3.5) is uniform for any $u_0 \in S^{n-1}$.

Assume that $\rho_{a \cdot K_i \,\widetilde{+}_\phi\, b \cdot L_i}$ does not converge uniformly to $\rho_{a \cdot K \,\widetilde{+}_\phi\, b \cdot L}$. Then, there exists a $\delta_0 > 0$ such that, for $i \geq N_0 \in \mathbb{N}$,

$$\big|\rho_{a \cdot K_i \,\widetilde{+}_\phi\, b \cdot L_i}(u_i) - \rho_{a \cdot K \,\widetilde{+}_\phi\, b \cdot L}(u_i)\big| \geq \delta_0. \tag{3.7}$$

Since S^{n-1} is compact, for some $u_0 \in S^{n-1}$, there exists a subsequence $\{u_i\} \subset S^{n-1}$ such that $u_i \to u_0$ as $i \to \infty$.

From Lemma 3.2, there exist an $N_0 \in \mathbb{N}$ and positive λ, μ such that (3.6) holds for $i \geq N_0$. Then, there exists a positive s_0 such that

$$\rho_{a \cdot K_i \,\widetilde{+}_\phi\, b \cdot L_i}(u_i) \to s_0.$$

From (3.7), we have

$$\big|s_0 - \rho_{a \cdot K \,\widetilde{+}_\phi\, b \cdot L}(u_0)\big| \geq \delta_0.$$

This implies

$$s_0 \neq \rho_{a\cdot K \,\widetilde{+}_\phi\, b\cdot L}(u_0). \tag{3.8}$$

Let $s_i = \rho_{a\cdot K_i \,\widetilde{+}_\phi\, b\cdot L_i}(u_i)$. By Lemma 3.1, we have

$$a\phi\left(\frac{\rho_{K_i}(u_i)}{s_i}\right) + b\phi\left(\frac{\rho_{L_i}(u_i)}{s_i}\right) = \phi(1).$$

This, together with the facts that $K_i \to K, L_i \to L$ and $s_i \to s_0$, gives

$$a\phi\left(\frac{\rho_K(u_0)}{s_0}\right) + b\phi\left(\frac{\rho_L(u_0)}{s_0}\right) = \phi(1).$$

By Lemma 3.1 again, we have

$$s_0 = \rho_{a\cdot K \,\widetilde{+}_\phi\, b\cdot L}(u_0).$$

This contradicts to (3.8). Therefore,

$$\rho_{a\cdot K_i \,\widetilde{+}_\phi\, b\cdot L_i} \to \rho_{a\cdot K \,\widetilde{+}_\phi\, b\cdot L}$$

uniformly on S^{n-1} and hence

$$a \cdot K_i \,\widetilde{+}_\phi\, b \cdot L_i \to a \cdot K \,\widetilde{+}_\phi\, b \cdot L. \qquad \square$$

We will see that the Orlicz radial sum $\widetilde{+}_\phi$ is continuous in a and b.

Lemma 3.5. *Suppose $\phi \in \mathcal{C}^+$. If $a_i, b_i \geq 0$ and $a_i \to a, b_i \to b$, as $i \to \infty$, then*

$$a_i \cdot K \,\widetilde{+}_\phi\, b_i \cdot L \to a \cdot K \,\widetilde{+}_\phi\, b \cdot L, \quad \text{as } i \to \infty,$$

for all $K, L \in \mathcal{S}_o^n$.

Proof. Suppose $u \in S^{n-1}$ and $K, L \in \mathcal{S}_o^n$. We will show that

$$\rho_{a_i\cdot K \,\widetilde{+}_\phi\, b_i\cdot L}(u) \to \rho_{a\cdot K \,\widetilde{+}_\phi\, b\cdot L}(u), \quad \text{as } i \to \infty. \tag{3.9}$$

Let

$$\rho_{a_i\cdot K \,\widetilde{+}_\phi\, b_i\cdot L}(u) = t_i.$$

Then Lemma 3.2 gives

$$\frac{r}{\phi^{-1}(\frac{\phi(1)}{2C_i})} \leq t_i \leq \frac{R}{\phi^{-1}(\frac{\phi(1)}{c_i})}.$$

Since $a_i \to a, b_i \to b$ and the facts that the functions $C_i = \max\{a_i, b_i\}$ and $c_i = a_i + b_i$ are continuous, we have $C_i \to C$ and $c_i \to c$. Since the inverse ϕ^{-1} of ϕ is also continuous and deceasing in $(0, \infty)$, there exist λ, μ such that $0 < \lambda \le t_i \le \mu < \infty$, for all i. To show that the bound sequence $\{t_i\}$ converges to $\rho_{a\cdot K \,\tilde{+}_\phi\, b\cdot L}(u)$, we show that every convergent subsequence of $\{t_i\}$ converges to $\rho_{a\cdot K \,\tilde{+}_\phi\, b\cdot L}(u)$. Denote an arbitrary convergent subsequence of $\{t_i\}$ by $\{t_i\}$ as well, and suppose that for this subsequence we have

$$t_i \to t_0.$$

It is clear that $0 < \lambda \le t_0 \le \mu$. Since $\rho_{a_i\cdot K \,\tilde{+}_\phi\, b_i\cdot L}(u) = t_i$, that is,

$$a_i\phi\left(\frac{\rho_K(u)}{t_i}\right) + b_i\phi\left(\frac{\rho_L(u)}{t_i}\right) = \phi(1) \quad \text{for all } i.$$

Since $a_i \to a$ and $b_i \to b$, together with the continuity of ϕ, and $t_i \to t_0$ it follows that

$$a\phi\left(\frac{\rho_K(u)}{t_0}\right) + b\phi\left(\frac{\rho_L(u)}{t_0}\right) = \phi(1).$$

By Lemma 3.1, we have

$$t_0 = \rho_{a\cdot K \,\tilde{+}_\phi\, b\cdot L}(u).$$

This shows

$$\rho_{a_i\cdot K \,\tilde{+}_\phi\, b_i\cdot L}(u) \to \rho_{a\cdot K \,\tilde{+}_\phi\, b\cdot L}(u).$$

Now the pointwise convergence (3.9) has been proved.

To show the convergence (3.9) is uniform on S^{n-1}, we assume that $\rho_{a_i\cdot K \,\tilde{+}_\phi\, b_i\cdot L}$ does not converge uniformly to $\rho_{a\cdot K \,\tilde{+}_\phi\, b\cdot L}$. Then, there exist a positive δ_0 and an $N_0 \in \mathbb{N}$ such that, for $i \ge N_0$,

$$\left|\rho_{a_i\cdot K \,\tilde{+}_\phi\, b_i\cdot L}(u_i) - \rho_{a\cdot K \,\tilde{+}_\phi\, b\cdot L}(u_i)\right| \ge \delta_0. \tag{3.10}$$

Since S^{n-1} is compact, for $u_0 \in S^{n-1}$, there exists a subsequence $\{u_i\} \subset S^{n-1}$ such that $u_i \to u_0$ as $i \to \infty$.

From Lemma 3.2, there exist an $N_0 \in \mathbb{N}$ and positive λ, μ such that, for $i \ge N_0$,

$$0 < \lambda \le \rho_{a_i\cdot K \,\tilde{+}_\phi\, b_i\cdot L}(u_i) \le \mu < \infty.$$

Then, there exists a positive s_0 such that, for $i \ge N_0$,

$$\rho_{a_i\cdot K \,\tilde{+}_\phi\, b_i\cdot L}(u_i) \to s_0.$$

From (3.10), we have

$$\left|s_0 - \rho_{a\cdot K \,\widetilde{+}_\phi\, b\cdot L}(u_0)\right| \geq \delta_0.$$

This implies

$$s_0 \neq \rho_{a\cdot K \,\widetilde{+}_\phi\, b\cdot L}(u_0). \tag{3.11}$$

Let $s_i = \rho_{a_i\cdot K \,\widetilde{+}_\phi\, b_i\cdot L}(u_i)$. By Lemma 3.1, we have

$$a_i\phi\left(\frac{\rho_K(u_i)}{s_i}\right) + b_i\phi\left(\frac{\rho_L(u_i)}{s_i}\right) = \phi(1).$$

This, together with the facts that $a_i \to a$, $b_i \to b$ and $s_i \to s_0$, gives

$$a\phi\left(\frac{\rho_K(u_0)}{s_0}\right) + b\phi\left(\frac{\rho_L(u_0)}{s_0}\right) = \phi(1).$$

By Lemma 3.1 again, we have

$$s_0 = \rho_{a\cdot K \,\widetilde{+}_\phi\, b\cdot L}(u_0).$$

This contradicts to (3.11). Therefore,

$$\rho_{a_i\cdot K \,\widetilde{+}_\phi\, b_i\cdot L} \to \rho_{a\cdot K \,\widetilde{+}_\phi\, b\cdot L}$$

uniformly on S^{n-1} and

$$a_i \cdot K \,\widetilde{+}_\phi\, b_i \cdot L \to a \cdot K \widetilde{+}_\phi b \cdot L. \qquad \square$$

The following lemma shows that the Orlicz radial sum and the radial Minkowski addition are closely related.

Lemma 3.6. *Let $K, L \in \mathcal{S}_o^n$. For $0 < \lambda < 1$, if $\phi \in \mathcal{C}^+$, then*

$$(1-\lambda) \cdot K \,\widetilde{+}_\phi\, \lambda \cdot L \subseteq (1-\lambda)K \,\widetilde{+}\, \lambda L. \tag{3.12}$$

If ϕ is strictly convex, equality holds if and only if K and L are dilates of each other.

Proof. Let $K_\lambda = (1-\lambda) \cdot K \,\widetilde{+}_\phi\, \lambda \cdot L$. By Lemma 3.1 and convexity of ϕ, we have

$$\begin{aligned}\phi(1) &= (1-\lambda)\phi\left(\frac{\rho_K(u)}{\rho_{K_\lambda}(u)}\right) + \lambda\phi\left(\frac{\rho_L(u)}{\rho_{K_\lambda}(u)}\right) \\ &\geq \phi\left(\frac{(1-\lambda)\rho_K(u) + \lambda\rho_L(u)}{\rho_{K_\lambda}(u)}\right).\end{aligned} \tag{3.13}$$

Since ϕ is strictly decreasing on $(0,\infty)$, then we have

$$(1-\lambda)\rho_K(u)+\lambda\rho_L(u)\geq\rho_{K_\lambda}(u).$$

This is,

$$\rho_{(1-\lambda)K\,\tilde{+}\,\lambda L}(u)\geq\rho_{K_\lambda}(u).$$

By (2.4), we obtain the desired inclusion. From the equality condition in Jensen's inequality (2.13), if ϕ is strictly convex, then equation holds in (3.13) if and only if K and L are dilates of each other. □

4. Dual Orlicz mixed volume

Let us introduce the dual Orlicz mixed volume.

Definition 4.1. For $\phi\in\mathcal{C}^+$, we define the dual Orlicz mixed volume $\widetilde{V}_\phi(K,L)$ by

$$\widetilde{V}_\phi(K,L)=\frac{1}{n}\int\limits_{S^{n-1}}\phi\left(\frac{\rho_L(u)}{\rho_K(u)}\right)\rho_K^n(u)dS(u),\tag{4.1}$$

for all $K,L\in\mathcal{S}_o^n$.

Remark 4.1. When $\phi(t)=t^{-p}$, with $p\geq 1$. The dual Orlicz mixed volume reduces to Lutwak's L_p dual mixed volume (see [38]):

$$\widetilde{V}_{-p}(K,L)=\frac{1}{n}\int\limits_{S^{n-1}}\rho_K^{n+p}(u)\rho_L^{-p}(u)dS(u),$$

for all $K,L\in\mathcal{S}_o^n$.

We denote the right derivative of a real-valued function f by f_r'. For $\phi\in\mathcal{C}^+$, there is $\phi_r'(1)<0$ because ϕ is convex and strictly decreasing.

Lemma 4.1. *Let $\phi\in\mathcal{C}^+$ and $K,L\in\mathcal{S}_o^n$. Then*

$$\lim_{\varepsilon\to 0^+}\frac{\rho_{K\,\tilde{+}_\phi\,\varepsilon\cdot L}(u)-\rho_K(u)}{\varepsilon}=\frac{\rho_K(u)}{\phi_r'(1)}\phi\left(\frac{\rho_L(u)}{\rho_K(u)}\right),\tag{4.2}$$

uniformly for all $u\in S^{n-1}$.

Proof. Suppose $\varepsilon>0$, $K,L\in\mathcal{S}_o^n$, and $u\in S^{n-1}$. Let

$$t(\varepsilon)=\rho_{K\,\tilde{+}_\phi\,\varepsilon\cdot L}(u).$$

Then, by Lemma 3.5, we have

$$t(\varepsilon) \to \rho_K(u) \quad \text{as } \varepsilon \to 0. \tag{4.3}$$

By Lemma 3.1, we have

$$\phi\left(\frac{\rho_K(u)}{t(\varepsilon)}\right) + \varepsilon\phi\left(\frac{\rho_L(u)}{t(\varepsilon)}\right) = \phi(1).$$

Then

$$\frac{\rho_K(u)}{t(\varepsilon)} = \phi^{-1}\left(\phi(1) - \varepsilon\phi\left(\frac{\rho_L(u)}{t(\varepsilon)}\right)\right).$$

Let

$$s = \phi^{-1}\left(\phi(1) - \varepsilon\phi\left(\frac{\rho_L(u)}{t(\varepsilon)}\right)\right) \tag{4.4}$$

and note that $s \to 1^+$ as $\varepsilon \to 0^+$. Thus

$$\frac{t(\varepsilon) - \rho_K(u)}{t(\varepsilon)} = 1 - \frac{\rho_K(u)}{t(\varepsilon)} = 1 - s. \tag{4.5}$$

Combining (4.5) and Lemma 3.5, we obtain

$$\begin{aligned}
\lim_{\varepsilon\to 0^+} \frac{\rho_{K\,\widetilde{+}_\phi\,\varepsilon\cdot L}(u) - \rho_K(u)}{\varepsilon} &= \lim_{\varepsilon\to 0^+} \frac{t(\varepsilon)}{\varepsilon} \cdot \frac{t(\varepsilon) - \rho_K(u)}{t(\varepsilon)} \\
&= \lim_{\varepsilon\to 0^+} t(\varepsilon) \cdot \phi\left(\frac{\rho_L(u)}{t(\varepsilon)}\right) \cdot \frac{\frac{t(\varepsilon)-\rho_K(u)}{t(\varepsilon)}}{\phi(1) - (\phi(1) - \varepsilon\phi(\frac{\rho_L(u)}{t(\varepsilon)}))} \\
&= \rho_K(u) \cdot \phi\left(\frac{\rho_L(u)}{\rho_K(u)}\right) \cdot \lim_{s\to 1^+} \frac{1-s}{\phi(1) - \phi(s)} \\
&= \frac{\rho_K(u)}{\phi_r'(1)} \phi\left(\frac{\rho_L(u)}{\rho_K(u)}\right).
\end{aligned} \tag{4.6}$$

Then the pointwise limit (4.2) has been proved.

Moreover, the convergence is uniform for any $u \in S^{n-1}$. Indeed, by (4.4) and (4.6), it suffices to recall that by Lemma 3.5,

$$\lim_{\varepsilon\to 0^+} \rho_{K\,\widetilde{+}_\phi\,\varepsilon\cdot L}(u) = \rho_K(u),$$

uniformly for $u \in S^{n-1}$. □

We are ready to derive the variational formula of volume for the Orlicz radial sum.

Theorem 4.1. *Let $\phi \in \mathcal{C}^+$ and $K, L \in \mathcal{S}_o^n$. Then*

$$\lim_{\varepsilon \to 0^+} \frac{V(K \widetilde{+}_\phi \varepsilon \cdot L) - V(K)}{\varepsilon} = \frac{1}{\phi_r'(1)} \int_{S^{n-1}} \phi\left(\frac{\rho_L(u)}{\rho_K(u)}\right) \rho_K^n(u) dS(u).$$

Proof. Suppose $\varepsilon > 0$, $K, L \in \mathcal{S}_o^n$, and $u \in S^{n-1}$. By Lemma 4.1, it follows that

$$\begin{aligned} \lim_{\varepsilon \to 0^+} \frac{\rho^n_{K \widetilde{+}_\phi \varepsilon \cdot L}(u) - \rho^n_K(u)}{\varepsilon} &= n\rho^{n-1}_{K \widetilde{+}_\phi \varepsilon \cdot L}(u)|_{\varepsilon = 0^+} \cdot \lim_{\varepsilon \to 0^+} \frac{\rho_{K \widetilde{+}_\phi \varepsilon \cdot L}(u) - \rho_K(u)}{\varepsilon} \\ &= \frac{n\rho_K^n(u)}{\phi_r'(1)} \phi\left(\frac{\rho_L(u)}{\rho_K(u)}\right), \end{aligned}$$

uniformly on S^{n-1}.

Hence

$$\begin{aligned} \lim_{\varepsilon \to 0^+} \frac{V(K \widetilde{+}_\phi \varepsilon \cdot L) - V(K)}{\varepsilon} &= \lim_{\varepsilon \to 0^+} \left(\frac{1}{n} \int_{S^{n-1}} \frac{\rho^n_{K \widetilde{+}_\phi \varepsilon \cdot L}(u) - \rho^n_K(u)}{\varepsilon} dS(u) \right) \\ &= \frac{1}{n} \int_{S^{n-1}} \lim_{\varepsilon \to 0^+} \frac{\rho^n_{K \widetilde{+}_\phi \varepsilon \cdot L}(u) - \rho^n_K(u)}{\varepsilon} dS(u) \\ &= \frac{1}{\phi_r'(1)} \int_{S^{n-1}} \phi\left(\frac{\rho_L(u)}{\rho_K(u)}\right) \rho_K^n(u) dS(u). \end{aligned}$$

We complete the proof of Theorem 4.1. □

From the definition (4.1) and the variational formula of volume in Theorem 4.1, we have

$$\frac{n}{\phi_r'(1)} \widetilde{V}_\phi(K, L) = \lim_{\varepsilon \to 0^+} \frac{V(K \widetilde{+}_\phi \varepsilon \cdot L) - V(K)}{\varepsilon}. \tag{4.7}$$

An immediate consequence of Proposition 3.1 and (4.7), is the invariance of the Orlicz dual mixed volume under simultaneous unimodular centro-affine transformations:

Proposition 4.1. *If $\phi \in \mathcal{C}^+$, and $K, L \in \mathcal{S}_o^n$, then for $A \in SL(n)$,*

$$\widetilde{V}_\phi(AK, AL) = \widetilde{V}_\phi(K, L).$$

Proof. From Proposition 3.1 and (4.7), we have, for $A \in SL(n)$,

$$\begin{aligned}\widetilde{V}_\phi(AK, AL) &= \frac{\phi_r'(1)}{n} \lim_{\varepsilon \to 0^+} \frac{V(AK \,\widetilde{+}_\phi\, \varepsilon \cdot AL) - V(AK)}{\varepsilon} \\ &= \frac{\phi_r'(1)}{n} \lim_{\varepsilon \to 0^+} \frac{V(A(K \,\widetilde{+}_\phi\, \varepsilon \cdot L)) - V(K)}{\varepsilon} \\ &= \frac{\phi_r'(1)}{n} \lim_{\varepsilon \to 0^+} \frac{V(K \,\widetilde{+}_\phi\, \varepsilon \cdot L) - V(K)}{\varepsilon} \\ &= \widetilde{V}_\phi(K, L). \qquad \square\end{aligned}$$

5. Dual Orlicz–Brunn–Minkowski inequalities

For $K \in \mathcal{S}_o^n$, it will be convenient to use the volume-normalized dual conical measure V_K^* defined by (see [44])

$$V(K)dV_K^* = \frac{1}{n}\rho_K^n dS, \tag{5.1}$$

where S is the Lebesgue measure on S^{n-1}. We shall use the fact that the volume-normalized dual conical measure V_K^* is a probability measure on S^{n-1}.

We now establish the following dual Orlicz–Minkowski inequality:

Theorem 5.1. *Suppose $\phi \in \mathcal{C}^+$. Let $K, L \in \mathcal{S}_o^n$, then*

$$\widetilde{V}_\phi(K, L) \geq V(K)\phi\left(\left(\frac{V(L)}{V(K)}\right)^{\frac{1}{n}}\right), \tag{5.2}$$

the equality holds if and only if K and L are dilates of each other.

Proof. Since the volume-normalized dual conical measure V_K^* defined by (5.1) is a probability measure on S^{n-1}, then by Jensen's inequality (2.13), the integral formulas of dual mixed volume (2.8), dual Minkowski inequality (2.9), and the fact that ϕ is decreasing, we obtain

$$\begin{aligned}\frac{\widetilde{V}_\phi(K, L)}{V(K)} &= \frac{1}{nV(K)} \int\limits_{S^{n-1}} \phi\left(\frac{\rho_L(u)}{\rho_K(u)}\right)\rho_K^n(u)dS(u) \\ &\geq \phi\left(\frac{1}{nV(K)} \int\limits_{S^{n-1}} \frac{\rho_L(u)}{\rho_K(u)}\rho_K^n(u)dS(u)\right) \\ &= \phi\left(\frac{\widetilde{V}_1(K, L)}{V(K)}\right)\end{aligned}$$

$$\geq \phi\left(\frac{V(K)^{\frac{n-1}{n}}V(L)^{\frac{1}{n}}}{V(K)}\right)$$
$$= \phi\left(\left(\frac{V(L)}{V(K)}\right)^{\frac{1}{n}}\right).$$

This gives the desired inequality.

Suppose that equality holds in (5.2). Since ϕ is strictly decreasing, we have equality in the dual Minkowski inequality. So there is $\gamma \geq 0$ such that $L = \gamma K$ and hence

$$\rho_L(u) = \gamma \rho_K(u),$$

for all $u \in S^{n-1}$.

Conversely, when $L = \gamma K$, by (4.1), we have

$$\widetilde{V}_\phi(K, L) = V(K)\phi(\gamma) = V(K)\phi\left(\left(\frac{V(L)}{V(K)}\right)^{\frac{1}{n}}\right). \qquad \square$$

Remark 5.1. When $\phi(t) = t^{-p}$, with $p \geq 1$. The dual Orlicz–Minkowski inequality (5.2) reduces to Lutwak's L_p dual Minkowski inequality for the L_p dual mixed volume (see [38]):

$$\widetilde{V}_{-p}(K, L)^n \geq V(K)^{n+p}V(L)^{-p}, \tag{5.3}$$

with equality if and only if K and L are dilates of each other.

The following uniqueness is a direct consequence of the dual Orlicz–Minkowski inequality (5.2).

Corollary 5.1. *Suppose $\phi \in \mathcal{C}^+$, and $\mathscr{M} \subset \mathcal{S}_o^n$ such that $K, L \in \mathscr{M}$. If*

$$\widetilde{V}_\phi(M, K) = \widetilde{V}_\phi(M, L), \quad \text{for all } M \in \mathscr{M}, \tag{5.4}$$

or

$$\frac{\widetilde{V}_\phi(K, M)}{V(K)} = \frac{\widetilde{V}_\phi(L, M)}{V(L)}, \quad \text{for all } M \in \mathscr{M}, \tag{5.5}$$

then $K = L$.

Proof. Suppose (5.4) holds. If we take K for M, then from the definition (4.1) and the formulas (2.6), we obtain

$$\phi(1)V(K) = \widetilde{V}_\phi(K, K) = \widetilde{V}_\phi(K, L).$$

Hence, from the dual Orlicz–Minkowski inequality (5.2) we have

$$\phi(1) \geq \phi\left(\left(\frac{V(L)}{V(K)}\right)^{\frac{1}{n}}\right),$$

with equality if and only if K and L are dilates of each other. Since ϕ is strictly decreasing on $(0, \infty)$, we have

$$V(L) \geq V(K),$$

with equality if and only if K and L are dilates of each other. If we take L for M we similarly have $V(L) \leq V(K)$. Hence, $V(K) = V(L)$ and from the equality conditions we can conclude that K and L are dilates of each other. However, since they have the same volume they must be equal.

Next, suppose (5.5) holds. If we take K for M, then from the definition (4.1) and the formulas (2.6), we obtain

$$\phi(1) = \frac{\widetilde{V}_\phi(K, K)}{V(K)} = \frac{\widetilde{V}_\phi(L, K)}{V(L)}.$$

Then, from the dual Orlicz–Minkowski inequality (5.2) we have

$$\phi(1) \geq \phi\left(\left(\frac{V(K)}{V(L)}\right)^{\frac{1}{n}}\right),$$

with equality if and only if K and L are dilates of each other. Since ϕ is strictly decreasing on $(0, \infty)$, we have

$$V(K) \geq V(L),$$

with equality if and only if K and L are dilates of each other. If we take L for M we similarly have $V(K) \leq V(L)$. Hence, $V(K) = V(L)$ and from the equality conditions we can conclude that K and L are dilates of each other. However, since they have the same volume they must be equal. □

We derive the dual Orlicz–Brunn–Minkowski inequality as follows:

Theorem 5.2. *Suppose $K, L \in \mathcal{S}_o^n$, and $a, b > 0$. If $\phi \in \mathcal{C}^+$, then*

$$\phi(1) \geq a\phi\left(\left(\frac{V(K)}{V(a \cdot K \,\widetilde{+}_\phi\, b \cdot L)}\right)^{\frac{1}{n}}\right) + b\phi\left(\left(\frac{V(L)}{V(a \cdot K \,\widetilde{+}_\phi\, b \cdot L)}\right)^{\frac{1}{n}}\right), \tag{5.6}$$

with equality if and only if K and L are dilates of each other.

Proof. Let $K_\phi = a \cdot K \widetilde{+}_\phi b \cdot L$. From the formulas (2.6), Lemma 3.1 and the dual Orlicz–Minkowski inequality (5.2), it follows that

$$\begin{aligned}
\phi(1) &= \frac{1}{nV(K_\phi)} \int\limits_{S^{n-1}} \phi(1)\rho^n_{K_\phi}(u)dS(u) \\
&= \frac{1}{nV(K_\phi)} \int\limits_{S^{n-1}} \left[a\phi\left(\frac{\rho_K(u)}{\rho_{K_\phi}(u)}\right) + b\phi\left(\frac{\rho_L(u)}{\rho_{K_\phi}(u)}\right)\right]\rho^n_{K_\phi}(u)dS(u) \\
&= \frac{a}{nV(K_\phi)} \int\limits_{S^{n-1}} \phi\left(\frac{\rho_K(u)}{\rho_{K_\phi}(u)}\right)\rho^n_{K_\phi}(u)dS(u) \\
&\quad + \frac{b}{nV(K_\phi)} \int\limits_{S^{n-1}} \phi\left(\frac{\rho_L(u)}{\rho_{K_\phi}(u)}\right)\rho^n_{K_\phi}(u)dS(u) \\
&= \frac{a}{V(K_\phi)}\widetilde{V}_\phi(K_\phi, K) + \frac{b}{V(K_\phi)}\widetilde{V}_\phi(K_\phi, L) \\
&\geq a\phi\left(\left(\frac{V(K)}{V(K_\phi)}\right)^{\frac{1}{n}}\right) + b\phi\left(\left(\frac{V(L)}{V(K_\phi)}\right)^{\frac{1}{n}}\right),
\end{aligned}$$

with equality if and only if K and L are dilates of each other. We get the desired dual Orlicz–Brunn–Minkowski inequality (5.6). □

Remark 5.2. When $\phi(t) = t^{-p}$, with $p \geq 1$, the dual Orlicz–Brunn–Minkowski inequality reduces to Lutwak's L_p dual Brunn–Minkowski inequality (see [38]):

$$V(a \cdot K \widetilde{+}_\phi b \cdot L)^{-\frac{p}{n}} \geq aV(K)^{-\frac{p}{n}} + bV(L)^{-\frac{p}{n}}, \tag{5.7}$$

with equality if and only if K and L are dilates of each other.

Corollary 5.2. *For all $K, L \in \mathcal{S}^n_o$ and $0 < \lambda < 1$. If $\phi \in \mathcal{C}^+$ and $V(K) = V(L)$, then*

$$V\big((1-\lambda) \cdot K \widetilde{+}_\phi \lambda \cdot L\big) \leq V(K), \tag{5.8}$$

equality holds if and only if $K = L$.

Proof. By Lemma 3.6 and dual Brunn–Minkowski inequality (2.10), we have

$$\begin{aligned}
V\big((1-\lambda) \cdot K \widetilde{+}_\phi \lambda \cdot L\big)^{\frac{1}{n}} &\leq V\big((1-\lambda)K \widetilde{+}_\phi \lambda L\big)^{\frac{1}{n}} \\
&\leq (1-\lambda)V(K)^{\frac{1}{n}} + \lambda V(L)^{\frac{1}{n}}.
\end{aligned}$$

Thus, $V((1-\lambda) \cdot K \widetilde{+}_\phi \lambda \cdot L) \leq V(K)$ if $V(K) = V(L)$. The equality condition can be obtained from equality condition of the dual Brunn–Minkowski inequality (2.10). □

We now derive the equivalence between the dual Orlicz–Minkowski inequality (5.2) and the dual Orlicz–Brunn–Minkowski inequality (5.6). We have proved the dual Orlicz–Brunn–Minkowski inequality (5.6) by the dual Orlicz–Minkowski inequality (5.2). Thus, we only need to prove the dual Orlicz–Minkowski inequality (5.2) by the dual Orlicz–Brunn–Minkowski inequality (5.6).

Proof. For $\varepsilon \geq 0$, let $K_\varepsilon = K \mathbin{\widetilde{+}_\phi} \varepsilon \cdot L$. By the dual Orlicz–Brunn–Minkowski inequality, the following function

$$G(\varepsilon) = \phi\left(\left(\frac{V(K)}{V(K_\varepsilon)}\right)^{\frac{1}{n}}\right) + \varepsilon\phi\left(\left(\frac{V(L)}{V(K_\varepsilon)}\right)^{\frac{1}{n}}\right) - \phi(1) \tag{5.9}$$

is non-positive. Then by Lemma 3.5,

$$\begin{aligned}
\lim_{\varepsilon\to 0^+}\frac{G(\varepsilon)-G(0)}{\varepsilon} &= \lim_{\varepsilon\to 0^+}\frac{\phi((\frac{V(K)}{V(K_\varepsilon)})^{\frac{1}{n}}) + \varepsilon\phi((\frac{V(L)}{V(K_\varepsilon)})^{\frac{1}{n}}) - \phi(1)}{\varepsilon} \\
&= \lim_{\varepsilon\to 0^+}\frac{\phi((\frac{V(K)}{V(K_\varepsilon)})^{\frac{1}{n}}) - \phi(1)}{\varepsilon} + \phi\left(\left(\frac{V(L)}{V(K)}\right)^{\frac{1}{n}}\right) \\
&= \lim_{\varepsilon\to 0^+}\frac{\phi((\frac{V(K)}{V(K_\varepsilon)})^{\frac{1}{n}}) - \phi(1)}{(\frac{V(K)}{V(K_\varepsilon)})^{\frac{1}{n}} - 1} \cdot \lim_{\varepsilon\to 0^+}\frac{(\frac{V(K)}{V(K_\varepsilon)})^{\frac{1}{n}} - 1}{\varepsilon} \\
&\quad + \phi\left(\left(\frac{V(L)}{V(K)}\right)^{\frac{1}{n}}\right).
\end{aligned} \tag{5.10}$$

Let $s = (\frac{V(K)}{V(K_\varepsilon)})^{\frac{1}{n}}$ and note that $s \to 1^+$ as $\varepsilon \to 0^+$. Consequently,

$$\lim_{\varepsilon\to 0^+}\frac{\phi((\frac{V(K)}{V(K_\varepsilon)})^{\frac{1}{n}}) - \phi(1)}{(\frac{V(K)}{V(K_\varepsilon)})^{\frac{1}{n}} - 1} = \lim_{s\to 1^+}\frac{\phi(s)-\phi(1)}{s-1} = \phi_r'(1). \tag{5.11}$$

By Lemma 3.5 and (4.7), we have

$$\begin{aligned}
\lim_{\varepsilon\to 0^+}\frac{(\frac{V(K)}{V(K_\varepsilon)})^{\frac{1}{n}} - 1}{\varepsilon} &= -\lim_{\varepsilon\to 0^+}\frac{V(K_\varepsilon)^{\frac{1}{n}} - V(K)^{\frac{1}{n}}}{\varepsilon} \cdot \lim_{\varepsilon\to 0^+} V(K_\varepsilon)^{-\frac{1}{n}} \\
&= -\frac{1}{n}V(K)^{\frac{1}{n}-1} \cdot \lim_{\varepsilon\to 0^+}\frac{V(K_\varepsilon)-V(K)}{\varepsilon} \cdot V(K)^{-\frac{1}{n}} \\
&= -\frac{1}{\phi_r'(1)}\frac{\widetilde{V}_\phi(K,L)}{V(K)}.
\end{aligned} \tag{5.12}$$

From (5.10), (5.11), (5.12) and $G(\varepsilon)$ is non-positive, it follows that

$$\lim_{\varepsilon\to 0^+}\frac{G(\varepsilon)-G(0)}{\varepsilon} = -\frac{\widetilde{V}_\phi(K,L)}{V(K)} + \phi\left(\left(\frac{V(L)}{V(K)}\right)^{\frac{1}{n}}\right) \leq 0. \tag{5.13}$$

From the definition of function $G(\varepsilon)$, we have $G(0) = 0$. Therefore, the equality holds in (5.13) if and only if $G(\varepsilon) = G(0) = 0$, this implies that equality can be obtained from the equality condition of dual Orlicz–Brunn–Minkowski inequality. $\square$

6. Orlicz intersection body

The celebrated Busemann–Petty problem asks the following: if K and L are origin-symmetric n-dimensional convex bodies such that any $(n-1)$-dimensional volume of a central hyperplane section of K is less than that of the corresponding section of L, does it follow that the volume of K is less than the volume of L?

Busemann and Petty proved that the answer is positive if K is a ball. The answer is affirmative when $n \leq 4$ and negative when $n \geq 5$. Lutwak introduced intersection bodies in [35] and showed that the answer for the Busemann–Petty problem is affirmative for a given dimension if and only if every symmetric convex body is an intersection body. Gardner proved that the answer for Busemann–Petty problem is affirmative for $n = 3$ by using Lutwak's result in [13]. Zhang proved that the answer for the Busemann–Petty problem is affirmative when $n = 4$ in [55]. Later a unified proof was given in [15]. See [1–3, 11–13,15,17,19,47,48,55] for more historical details about the Busemann–Petty problem.

The intersection body IK of a star body K, introduced by Lutwak [35], is a fundamental concept in the dual Brunn–Minkowski theory. In this section, the notion of Orlicz intersection body is introduced.

For $\phi \in \mathcal{C}$, we define the Orlicz intersection body of a star body K as the body whose radial function is given by

$$\rho^{-1}_{I_\phi K}(x) = \sup\left\{\lambda > 0 : \int\limits_{y \in K} \phi\left(\frac{x \cdot y}{\lambda}\right) dy \leq 1\right\}. \tag{6.1}$$

When $\phi_p(t) = |t|^{-p}$, with $0 < p < 1$, then

$$I_{\phi_p} K = I_p K,$$

where $I_p K$ is the L_p intersection body, whose radial function is given by

$$\rho^p_{I_p K}(x) = \int\limits_K |x \cdot y|^{-p} dy, \quad x \in \mathbb{R}^n. \tag{6.2}$$

If K is a convex polytope in $\mathbb{R}^n$ with the origin in its interior, then this L_p intersection body is consistent with the L_p intersection body in [21]. There is a different constant between the L_p intersection body defined in (6.2) and the following p-intersection body in [20].

$$\rho(I_p K, u)^p = \frac{1}{\Gamma(1-p)} \int\limits_K |x \cdot u|^{-p} dx; \quad 0 < p < 1,\ u \in S^{n-1}. \tag{6.3}$$

In [20], Haberl also showed that: if $p \uparrow 1$, then the p-intersection body $I_pK \to IK$. Here IK is the classical intersection body whose radial function in the direction $u \in S^{n-1}$ is equal to the $(n-1)$-dimensional volume of the section of K by $u^\perp$, the hyperplane orthogonal to u. Therefore, for $u \in S^{n-1}$,

$$\rho(IK, u) = \mathrm{vol}_{n-1}\big(K \cap u^\perp\big),$$

where $\mathrm{vol}_{n-1}(\cdot)$ denotes the $(n-1)$-dimensional volume.

The celebrated Busemann–Petty problem can be stated that whether the implication

$$IK \subset IL \quad \Rightarrow \quad V(K) \le V(L)$$

holds for arbitrary origin-symmetric convex bodies K, L.

For $\phi \in \mathcal{C}$, the Orlicz analogue of this question asks: Does $I_\phi K \subset I_\phi L$ for origin-symmetric convex bodies K, L imply $V(K) \le V(L)$?

Conjecture. *Let K be an Orlicz intersection body of a star body and L be a star body in $\mathbb{R}^n$. If*

$$I_\phi K \subset I_\phi L,$$

then

$$V(K) \le V(L) \quad for\ \phi \in \mathcal{C},$$

with equality only if $K = L$.

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