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Accurate numerical solution for structured M-matrix algebraic Riccati equations

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ABSTRACT

This paper is concerned with a M-matrix algebraic Riccati equation (MARE) $XDX - AX - XB + C = 0$ for which A is block-diagonal and its defining matrix $W = \begin{bmatrix} B & -D \\ -C & A \end{bmatrix}$ is a nonsingular or irreducible singular M-matrix. Such an MARE can be decomposed into many coupled algebraic Riccati equations (ARES) that can be solved by the Jacobi- or Gauss–Seidel-like iteration updating scheme at the outer-loop while by a doubling algorithm in the inner loop for each coupled ARE, as first proposed by Meini (2013). The goals of this paper are two-fold. One is to resolve a critical technical detail in Meini's algorithm that was not addressed. It is about whether each ARE in the inner loop has a minimal nonnegative solution. It is proved that the defining matrix of each coupled ARE during a doubling iteration is indeed a nonsingular or irreducible singular M-matrix and, as a result, they do have minimal nonnegative solutions and a doubling algorithm is an efficient way to compute them. The other goal is to design a highly accurate implementation of the doubling algorithm for the inner loop so that all entries of the minimal nonnegative solution to the original MARE are calculated with high entrywise relative accuracies, regardless of their magnitudes. This is made possible by a novel way of constructing triplet representations for the coupled ARE during doubling iterations. Numerical examples are presented to demonstrate that the resulting algorithm can indeed deliver an entrywise relatively accurate solution.

1. Introduction

Consider the algebraic Riccati equation (ARE)

$$XDX - AX - XB + C = 0, \tag{1.1a}$$

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where $X \in \mathbb{R}^{n\times m}$ is the unknown, and the sizes of the constant coefficient matrices A, B, C, and D are determined by the partitioning:

$$W = \begin{array}{c} \\ m \\ n \end{array} \begin{array}{c} \begin{array}{cc} m & n \end{array} \\ \left[\begin{array}{cc} B & -D \\ -C & A \end{array} \right] \end{array} \in \mathbb{R}^{(m+n)\times(m+n)}. \tag{1.1b}$$

We call this matrix W in (1.1b) the *defining* matrix of ARE (1.1a).

The ARES (1.1a) that we are interested in are those when W is an M-matrix. Following [1], we will call ARE (1.1a) a *M-Matrix algebraic Riccati equation* (MARE) if W is an M-matrix. Previously, when the term MARE was first coined in [2], it was required that

$$\boxed{\text{defining matrix } W \text{ is a nonsingular or irreducible singular } M\text{-matrix,}} \tag{1.2}$$

which is often the case for ARES arising from relevant applications such as applied probability and transportation theory (see [3–9] and the references therein). It is these applications that motivated early studies on MARES by the numerical linear algebra communities. It was shown in [3,5,10] that ARE (1.1a) has a unique minimal nonnegative solution Φ, i.e., entrywise

$$\Phi \le X \quad \text{for any other nonnegative solution } X \text{ of ARE (1.1a)}$$

under assumption (1.2), making all studies on such an ARE a natural thing to do. That is why MARE was defined the way it was in [2].

In general, W just being an M-matrix is too broad to allow one to say much about the solution to the associated ARE. Guo [10] constructed an example:

$$m = n = 1, \quad W = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, \tag{1.3}$$

which is an M-matrix. The associated MARE is $1 = 0$ and thus has no solution, not to mention a minimal nonnegative solution. This simple example shows that ARE (1.1) being simply an MARE, i.e., W being an M-matrix, does not necessarily guarantee itself to have a solution. But Guo [10] made a remarkable observation. What really matters, behind the assumption (1.2) that guarantees the existence of the minimal nonnegative solution Φ, is, beyond W being an M-matrix, that there is a positive vector $\boldsymbol{u} \in \mathbb{R}^{m+n}$ such that $W\boldsymbol{u} \ge 0$. He introduced the notation of *regular* M-matrix for such an M-matrix. It is well-known that under (1.2), there exists some positive vector $\boldsymbol{u}$ such that $W\boldsymbol{u} \ge 0$. Consequently, Guo [10] expanded the set of ARES that have minimal nonnegative solutions. Later Guo and Lu [11] proved that the doubling algorithms [6,12,13] still converge at least linearly if W is a regular M-matrix and if $\text{rank}(W) \ge m + n - 1$. The latter automatically holds if (1.2) holds. Inspired by these developments, we streamline the nomenclature of ARES in connection with an M-matrix as follows.

Definition 1.1. We call ARE (1.1) an MARE if W is an M-matrix [1], a *regular* MARE if W is a regular M-matrix [10,11], a *strongly regular* MARE if W is a regular M-matrix and if $\text{rank}(W) \ge m + n - 1$, and finally a *super-regular* MARE if its defining matrix W satisfies (1.2).

Evidently, a super-regular MARE is a strongly regular MARE which is a regular MARE which is an MARE. According to [10,11], any regular MARE has a minimal nonnegative solution, and the doubling algorithms globally converge for strongly regular MARES and locally the convergence is linear (with the linear rate 1/2) or quadratic, which extended earlier results for a super-regular MARE, previously called an MARE [2].

A super-regular MAREwas really the focus of the study in the past 20 years or so. It is very well understood nowadays both theoretically and numerically. C. Guo and his collaborators completed much of the studies into the existence and basic properties of the unique minimal nonnegative solution Φ [3,4,14]. The first structure-preserving doubling algorithm (SDA) was proposed by X. Guo, Lin and Xu [6] in 2006, and it was immediately clear at that moment that SDA is much superior to Newton's method in solving MARE for the unique minimal nonnegative solution Φ. Soon after, SDA was improved by two other more efficient doubling algorithms SDA-SS [12] and ADDA [13] with ADDA provably being the best. A highly accurate implementation of ADDA was discovered first by Nguyen and Poloni [15] for a singular but irreducible W and then by Xue and Li [16] for nonsingular W as well. An entrywise relative perturbation theory for MARE was established earlier in [2,17].

In this paper, we will study super-regular MARE (1.1a) coming from multi-type queues with general customer impatience [18,19] and risk processes [20]. Evidently, existing doubling algorithms can be used to calculate its minimal nonnegative solution Φ, and, if entrywise accuracy of Φ is what is needed, we can apply ACCADDA [15,16], the highly accurate implementation of ADDA. However, for the MARE, A is a block-diagonal M-matrix, and as a result the MARE can be broken into many coupled ARES of smaller sizes, a structure that should be taken advantage of for more efficient methods. Rightly, Meini [21] did just that. She proposed an inner–outer iterative method, where the Jacobi- or Gauss–Seidel-like updating scheme is used as an outer iteration while a doubling algorithm serves as the inner iteration for each smaller-sized ARE. The method was analyzed theoretically and demonstrated numerically. However, the theoretical argument there [21] is incomplete in that it did not properly justify that the smaller-sized ARES have minimal nonnegative solutions, although it was proved that their defining matrices are indeed M-matrices which, however, are not enough as Guo's example (1.3) indicates.

We have two primary goals in this paper. One is to provide a rigorous theoretical analysis of the inner–outer iterative method in [21]. It is proved that each smaller-sized ARE in the inner doubling iterations are indeed a super-regular MARE, and thus it has a unique minimal nonnegative solution and the doubling algorithm is guaranteed convergent and the convergence is at least linear but often quadratic. Our second goal is to devise a highly accurate implementation of Meini's algorithm based on ACCADDA [15,16]. The key for making that possible is a novel way to construct entrywise accurate triplet representations of all defining M-matrices of the smaller-sized ARES during the doubling iterative process, assuming that a triplet representation for the defining matrix W of MARE (1.1a) is known *a priori*.

The paper is organized as follows. Section 2 collects necessary preliminaries on M-matrix, its accurate inverse, and super-regular MARE that will be needed later. In Section 3, we investigate the structured MARE theoretically to lay the foundation for our highly accurate algorithm in Section 5. We review the highly accurate doubling algorithm ACCADDA [15,16] with an additional output that was not in the original ACCADDA. In Section 5, we first fill in the gap of technical incompleteness we mentioned earlier and then present our highly accurate algorithm. Two numerical examples are presented in Section 6 to demonstrate that our new highly accurate algorithm can indeed deliver computed minimal nonnegative solution with nearly full entrywise relative accuracy in the working precision. We draw our conclusions in Section 7.

Notation. $\mathbb{R}^{m\times n}$ is the set of all $m\times n$ real matrices, $\mathbb{R}^n=\mathbb{R}^{n\times 1}$, and $\mathbb{R}=\mathbb{R}^1$. I_n (or simply I if its dimension is clear from the context) is the $n\times n$ identity matrix. The superscript in $\cdot^{\mathrm{T}}$ takes transpose. For $X\in\mathbb{R}^{m\times n}$, $X_{(i,j)}$ refers to its (i,j)th entry, $|X|\in\mathbb{R}^{m\times n}$ takes entrywise absolute value. Inequality $X\le Y$ means $X_{(i,j)}\le Y_{(i,j)}$ for all (i,j), and similarly for $X<Y$, $X\ge Y$, and $X>Y$. In particular, $X\ge 0$ means that X is entrywise nonnegative. For a square matrix X, denote by $\rho(X)$ its spectral radius and by $\operatorname{eig}(X)$ the set of its eigenvalues counted algebraic multiplicities; $\operatorname{diag}(X)$ is a diagonal matrix extracting the diagonal part of X, and $\operatorname{offdiag}(X)=X-\operatorname{diag}(X)$. $\mathbf{1}_n\in\mathbb{R}^n$ is the n-vector of all ones and $\mathbf{1}_{m\times n}\in\mathbb{R}^{m\times n}$ is the $m\times n$ matrix of all ones. The symbol $\mathfrak{u}$ is the unit machine roundoff.

2. Preliminaries

2.1. M-matrix

A matrix $A\in\mathbb{R}^{n\times n}$ is called a *Z-matrix* if $A_{(i,j)}\le 0$ for all $i\ne j$ [22, p. 284]. Any Z-matrix A can be written as $sI-N$ with $N\ge 0$, and it is called an *M-matrix* if $s\ge\rho(N)$, a *singular M-matrix* if $s=\rho(N)$, and a *nonsingular M-matrix* if $s>\rho(N)$.

The results in Theorem 2.1 are either well-known [22] or can be proved straightforwardly. For item (e), the reader is referred to [23, Lemma 2.5].

Theorem 2.1.

(a) *If A is a nonsingular M-matrix and B is Z-matrix satisfying $B\ge A$, then B is a nonsingular M-matrix.*
(b) *If A is an irreducible singular M-matrix and B is Z-matrix satisfying $B\ge A$, then B is a nonsingular or irreducible singular M-matrix. If also $B\ne A$, then B is a nonsingular M-matrix.*
(c) *If A is a Z-matrix and if $A\boldsymbol{u}\ge 0$ for some $\boldsymbol{u}>0$, then A is an M-matrix.*
(d) *If A is a Z-matrix and if $A\boldsymbol{u}>0$ for some $\boldsymbol{u}>0$, then A is a nonsingular M-matrix.*
(e) *Let $A\in\mathbb{R}^{n\times n}$ be a nonsingular or irreducible singular M-matrix, conformally partitioned as*

$$A=\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix},$$

where A_{11} and A_{22} are square matrices. Then A_{11} and A_{22} are nonsingular M-matrices, and their Schur complements

$$A_{22}-A_{21}A_{11}^{-1}A_{12},\quad A_{11}-A_{12}A_{22}^{-1}A_{21}$$

are nonsingular M-matrices if A is a nonsingular M-matrix, or irreducible singular M-matrices if A is an irreducible singular M-matrix.

2.2. Accurate inverses of an M-matrix

The key ingredient in recent work [15,16] to achieve high entrywise relative accuracy is the GTH-like algorithm for inverting a nonsingular M-matrix due to Alfa, Xue, and Ye [24]. They proposed to represent a nonsingular M-matrix A by the so-called *triplet representation* which can determine A^{-1} to high entrywise relative accuracy. Specifically, a triplet representation $(\operatorname{offdiag}(A),\boldsymbol{u},\boldsymbol{v})$ of the M-matrix $A\in\mathbb{R}^{n\times n}$ consists of $\operatorname{offdiag}(A)$ which is obtained by simply resetting the diagonal part of A to 0, $0<\boldsymbol{u}\in\mathbb{R}^n$, and $\boldsymbol{v}=A\boldsymbol{u}\ge 0$. Often for convenience, we will not distinguish A from its triplet representation and write

$$A=(\operatorname{offdiag}(A),\boldsymbol{u},\boldsymbol{v}).$$

It is proved [25] that if all entries of offdiag(A), $\boldsymbol{u}$, and $\boldsymbol{v}$ are known to high entrywise relative accuracy, then all entries of A^{-1} are determined to a comparable high entrywise relative accuracy, or equivalently the solution $\boldsymbol{x}$ to $A\boldsymbol{x} = \boldsymbol{b}$ for any $\boldsymbol{b} \geq 0$ is determined to a comparable high entrywise relative accuracy. Numerically, the GTH-like algorithm of Alfa, Xue, and Ye [24], using the idea in [26], computes the LU decomposition $A = LU$, via the Gaussian elimination without pivoting and without any cancellation[4] and, consequently, L and U are computed with high entrywise relative accuracy. Moreover, the diagonal entries of L are all 1 and its off-diagonal entries are non-positive, U has positive diagonal entries and non-positive off-diagonal entries. These properties of L and U ensure that the solution $\boldsymbol{x}$ of $A\boldsymbol{x} = \boldsymbol{b} \geq 0$ can be computed to the claimed accuracy, without any cancellation. For more details, the reader is referred to [1].

2.3. Properties of MARE

We will summarize important results for a super-regular MARE (1.1). They are mostly due to [3,5,10] (see also [1]). Since W is assumed a nonsingular or irreducible singular M-matrix, there exist $\boldsymbol{u}_1 \in \mathbb{R}^m$ and $\boldsymbol{u}_2 \in \mathbb{R}^n$ such that

$$\boldsymbol{u}_1 > 0,\ \boldsymbol{u}_2 > 0,\ \begin{bmatrix} \hat{\boldsymbol{u}}_1 \\ \hat{\boldsymbol{u}}_2 \end{bmatrix} := W \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \end{bmatrix} \geq 0, \tag{2.1a}$$

where $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ can be chosen to satisfy [22]

$$\hat{\boldsymbol{u}}_1 > 0,\ \hat{\boldsymbol{u}}_2 > 0, \quad \text{if } W \text{ is a nonsingular } M\text{-matrix}; \tag{2.1b}$$

$$\hat{\boldsymbol{u}}_1 = 0,\ \hat{\boldsymbol{u}}_2 = 0, \quad \text{if } W \text{ is an irreducible singular } M\text{-matrix}. \tag{2.1c}$$

It is well-known that ARE (1.1) is equivalent to [1,27]

$$H \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} M, \tag{2.2a}$$

where $M = B - DX$ and

$$H = \begin{bmatrix} I_m & \\ & -I_n \end{bmatrix} W = \begin{bmatrix} B & -D \\ C & -A \end{bmatrix}. \tag{2.2b}$$

Eq. (2.2a) is an eigenvalue problem, seeking an invariant subspace of H. Denote the set of the eigenvalues, counted algebraic multiplicities, of H by

$$\mathrm{eig}(H) = \{\lambda_1, \dots, \lambda_{m+n}\}, \tag{2.3}$$

where λ_i for $1 \leq i \leq m+n$ are ordered by their nonincreasing real parts, i.e., $\Re(\lambda_j) \leq \Re(\lambda_i)$ for $i < j$.

Theorem 2.2 (*[3,5,10]*). *Suppose that (1.1) is a super-regular MARE, i.e., W in (1.1b) is a nonsingular or an irreducible singular M-matrix.*

(a) *λ_m and λ_{m+1} are real, $\Re(\lambda_{m+2}) < 0 < \Re(\lambda_{m-1})$, and*

$$\Re(\lambda_{m+n}) \leq \cdots \leq \Re(\lambda_{m+2}) \leq \lambda_{m+1} \leq 0 \leq \lambda_m \leq \Re(\lambda_{m-1}) \leq \cdots \leq \Re(\lambda_1). \tag{2.4}$$

In particular, this implies $\lambda_{m+1} < 0 < \lambda_m$ if W is nonsingular.

(b) MARE (1.1) *has a unique minimal nonnegative solution Φ. Moreover,*

$$\mathrm{eig}(B - D\Phi) = \{\lambda_1, \dots, \lambda_m\},\ \ \mathrm{eig}(A - \Phi D) = \{-\lambda_{m+1}, \dots, -\lambda_{m+n}\}.$$

(c) *If W is* irreducible, *then $\Phi > 0$, and $A - \Phi D$ and $B - D\Phi$ are* irreducible *M-matrices.*

(d) *If W is* nonsingular, *then $A - \Phi D$ and $B - D\Phi$ are* nonsingular *M-matrices.*

(e) *$\Phi \boldsymbol{u}_1 \leq \boldsymbol{u}_2$. Moreover, $\Phi \boldsymbol{u}_1 < \boldsymbol{u}_2$ if W is nonsingular.*

(f) *H has a unique m-dimensional eigenspace associated with its eigenvalues in $\mathbb{C}_{0+} := \{z \in \mathbb{C} : \Re(z) \geq 0\}$, and $\begin{bmatrix} I_m \\ \Phi \end{bmatrix}$ is a basis matrix of the eigenspace.*

3. The structured MARE

The type of MARE coming from multi-type queues with general customer impatience [18,19] and risk processes [20] has an additional block diagonal structure in A. In this section, we will analyze such MARE, inspired by Meini [21]. Specifically, consider MARE:

$$XDX - AX - XB + C = 0, \tag{3.1a}$$

[4] By cancellation we mean any subtraction of a real number from another one of the same sign.

where A is a $K \times K$ block diagonal matrix:

$$A = \begin{array}{c} \\ n_1 \\ n_2 \\ \vdots \\ n_K \end{array} \begin{array}{c} \begin{array}{cccc} n_1 & n_2 & \cdots & n_K \end{array} \\ \begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_K \end{bmatrix} \end{array} \in \mathbb{R}^{n\times n}, \quad n = \sum_{i=1}^{K} n_i, \tag{3.1b}$$

and, as before, $B \in \mathbb{R}^{m\times m}$, $C \in \mathbb{R}^{n\times m}$, and $D \in \mathbb{R}^{m\times n}$. Assume $K \geq 2$ since MARE (3.1) with $K = 1$ reduces to the one that has been well-studied.

Correspondingly, we partition C, D and the unknown X as

$$C = \begin{array}{c} \\ n_1 \\ n_2 \\ \vdots \\ n_K \end{array} \begin{array}{c} m \\ \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_K \end{bmatrix} \end{array}, \quad D = m \begin{array}{c} \begin{array}{cccc} n_1 & n_2 & \cdots & n_K \end{array} \\ \begin{bmatrix} D_1 & D_2 & \cdots & D_K \end{bmatrix} \end{array}, \quad X = \begin{array}{c} \\ n_1 \\ n_2 \\ \vdots \\ n_K \end{array} \begin{array}{c} m \\ \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_K \end{bmatrix} \end{array}. \tag{3.2}$$

The structured MARE (3.1) can be equivalently turned into a system of coupled matrix Riccati equations in X_j:

$$X_jD_jX_j - A_jX_j - X_jB_j(X) + C_j = 0 \quad \text{for } 1 \leq j \leq K, \tag{3.3}$$

where

$$B_j(X) = B - \sum_{i\neq j} D_iX_i \in \mathbb{R}^{m\times m} \quad \text{for } 1 \leq j \leq K. \tag{3.4}$$

It can be seen that $B_j(X) - D_jX_j = B - DX$. Let for $1 \leq j \leq K$

$$W_j(X) = \begin{bmatrix} B_j(X) & -D_j \\ -C_j & A_j \end{bmatrix}, \quad H_j(X) = \begin{bmatrix} I_m & \\ & -I_{n_j} \end{bmatrix} W_j(X) = \begin{bmatrix} B_j(X) & -D_j \\ C_j & -A_j \end{bmatrix}. \tag{3.5}$$

We will still formally denote the defining coefficient matrix of MARE (3.1) by the matrix W of (1.1b):

$$W = \begin{bmatrix} B & -D \\ -C & A \end{bmatrix}.$$

For MARE (3.1) originally arising from the aforementioned applications, W is an irreducible singular M-matrix. As we mentioned before, the argument in Meini [21] is incomplete in the sense that although all involved $W_j(\cdot)$ during the doubling iterations was indeed proven to be an M-matrix, that alone is not enough to guarantee that the associated MARE (3.3) has a minimal nonnegative solution. One of our two goals is to remove this incompleteness. Specifically, we will show that each $W_j(\cdot)$ during the doubling iterations is a nonsingular or irreducible singular M-matrix, and thus each MARE (3.3) is super-regular and has a minimal nonnegative solution which can be found efficiently by any of the doubling algorithms for MARE [1].

For the ease of future reference, we will call MARE (3.1) a *structured* MARE if its defining W is an M-matrix. Our study in this paper is for the case W is a nonsingular or irreducible singular M-matrix, i.e., (3.1) is also a super-regular MARE. Then it has a unique minimal nonnegative solution $\Phi \in \mathbb{R}^{n\times m}$ which is partitioned, similarly to X in (3.2), as

$$\Phi = \begin{array}{c} \\ n_1 \\ n_2 \\ \vdots \\ n_K \end{array} \begin{array}{c} m \\ \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_K \end{bmatrix} \end{array} \in \mathbb{R}^{n\times m}. \tag{3.6}$$

Evidently, for each $1 \leq j \leq K$, Φ_j is a nonnegative solution of

$$X_jD_jX_j - A_jX_j - X_jB_j(\Phi) + C_j = 0. \tag{3.7}$$

Theorem 3.1 contains our main result in this section.

Theorem 3.1. *Suppose that MARE (3.1) is super-regular, i.e., W is a nonsingular or irreducible singular M-matrix.*

(a) *$B - D\Phi$ is an M-matrix, and each $A_j - \Phi_jD_j$ is a nonsingular M-matrix for $1 \leq j \leq K$.*
(b) *$\mathrm{eig}(H_j(\Phi))$ is the multi-set union of $\mathrm{eig}(B - D\Phi)$ and $\mathrm{eig}(-(A_j - \Phi_jD_j))$, and*

$$\mathrm{eig}(B - D\Phi) \cap \mathrm{eig}(-(A_j - \Phi_jD_j)) = \emptyset. \tag{3.8}$$

Thus $H_j(\Phi)$ has exactly n_j eigenvalues in the open left-half plane given by $\mathrm{eig}(-(A_j - \Phi_j D_j))$ *and the other m eigenvalues are in the closed right-half plane given by* $\mathrm{eig}(B - D\Phi)$. *Moreover, if $H_j(\Phi)$ has an eigenvalue on the imaginary axis, then that eigenvalue is 0 and it is a simple eigenvalue.*

(c) *Each (3.7) is a super-regular MARE, i.e., $W_j(\Phi)$ is a nonsingular or irreducible singular M-matrix, and Φ_j is the unique minimal nonnegative solution to MARE (3.7).*

Proof. The first claim in item (a) is due to Theorem 2.2(c,d). By Theorem 2.2(c), we know that $A - \Phi D$ is either a nonsingular or irreducible singular M-matrix, and, therefore, each $A_j - \Phi_j D_j$ is a nonsingular M-matrix by Theorem 2.1(e).

Because Φ_j is a solution to (3.7), it can be verified that

$$H_j(\Phi)\begin{bmatrix} I & 0 \\ \Phi_j & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ \Phi_j & I \end{bmatrix}\begin{bmatrix} B - D\Phi & -D_j \\ 0 & -(A_j - \Phi_j D_j) \end{bmatrix}. \tag{3.9}$$

In verifying (3.9), we used the fact $B_j(\Phi) - D_j\Phi_j = B - D\Phi$. That $\mathrm{eig}(H_j(\Phi))$ is the multi-set union of $\mathrm{eig}(B - D\Phi)$ and $\mathrm{eig}(-(A_j - \Phi_j D_j))$ is a straightforward consequence of (3.9). Since $A_j - \Phi_j D_j$ is a nonsingular M-matrix and thus its eigenvalues are in the open right-half plane. So (3.8) holds. This completes the proof of item (b).

Since W is a nonsingular or irreducible singular M-matrix, we have (2.1a). Partition positive vector $\boldsymbol{u}_2 \in \mathbb{R}^n$ and nonnegative vector $\hat{\boldsymbol{u}}_2 \in \mathbb{R}^n$ as

$$\boldsymbol{u}_2 = \begin{matrix} n_1 \\ n_2 \\ \vdots \\ n_K \end{matrix}\begin{bmatrix} \boldsymbol{u}_{2,1} \\ \boldsymbol{u}_{2,2} \\ \vdots \\ \boldsymbol{u}_{2,K} \end{bmatrix}, \quad \hat{\boldsymbol{u}}_2 = \begin{matrix} n_1 \\ n_2 \\ \vdots \\ n_K \end{matrix}\begin{bmatrix} \hat{\boldsymbol{u}}_{2,1} \\ \hat{\boldsymbol{u}}_{2,2} \\ \vdots \\ \hat{\boldsymbol{u}}_{2,K} \end{bmatrix}. \tag{3.10}$$

Expand (2.1a) to get

$$B\boldsymbol{u}_1 - \sum_{i=1}^{K} D_i \boldsymbol{u}_{2,i} = \hat{\boldsymbol{u}}_1, \tag{3.11a}$$

$$-C_j\boldsymbol{u}_1 + A_j\boldsymbol{u}_{2,j} = \hat{\boldsymbol{u}}_{2,j} \quad \text{for } j = 1, 2, \ldots, K. \tag{3.11b}$$

Since $\Phi\boldsymbol{u}_1 \le \boldsymbol{u}_2$ by Theorem 2.2(e), we have

$$\boldsymbol{u}_{2,j} - \Phi_j\boldsymbol{u}_1 \ge 0 \quad \text{for } j = 1, 2, \ldots, K. \tag{3.12}$$

Combining (3.11a) and (3.12), we get

$$\begin{aligned} B\boldsymbol{u}_1 - D_j\boldsymbol{u}_{2,j} - \sum_{i\neq j} D_i\Phi_i\boldsymbol{u}_1 &= \hat{\boldsymbol{u}}_1 + \sum_{i\neq j} D_i\boldsymbol{u}_{2,i} - \sum_{i\neq j} D_i\Phi_i\boldsymbol{u}_1 \\ &= \hat{\boldsymbol{u}}_1 + \sum_{i\neq j} D_i(\boldsymbol{u}_{2,i} - \Phi_i\boldsymbol{u}_1) \ge 0. \end{aligned} \tag{3.13}$$

Thus for $j = 1, 2, \ldots, K$

$$\begin{aligned} W_j(\Phi)\begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_{2,j} \end{bmatrix} &= \begin{bmatrix} B_j(\Phi) & -D_j \\ -C_j & A_j \end{bmatrix}\begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_{2,j} \end{bmatrix} \\ &= \begin{bmatrix} B\boldsymbol{u}_1 - D_j\boldsymbol{u}_{2,j} - \sum_{i\neq j} D_i\Phi_i\boldsymbol{u}_1 \\ -C_j\boldsymbol{u}_1 + A_j\boldsymbol{u}_{2,j} \end{bmatrix} \\ &= \begin{bmatrix} \hat{\boldsymbol{u}}_1 + \sum_{i\neq j} D_i(\boldsymbol{u}_{2,i} - \Phi_i\boldsymbol{u}_1) \\ \hat{\boldsymbol{u}}_{2,j} \end{bmatrix} \ge 0. \end{aligned} \tag{3.14}$$

Because $W_j(\Phi)$ is a Z-matrix, $W_j(\Phi)$ is an M-matrix by Theorem 2.1(c).

If W is nonsingular, then $\hat{\boldsymbol{u}}_1 > 0$, $\hat{\boldsymbol{u}}_2 > 0$ and $\Phi\boldsymbol{u}_1 < \boldsymbol{u}_2$ by Theorem 2.2(e). Consequently, the inequalities in (3.12) and (3.13) are strict, and so is (3.14), which means $W_j(\Phi)$ is a nonsingular M-matrix by Theorem 2.1(d).

Consider now that W is an irreducible singular M-matrix. We know that all A_i for $1 \le i \le K$ are nonsingular M-matrices by Theorem 2.1(e). The Schur complement of $\mathrm{diag}(A_2, \ldots, A_K)$ in W

$$\begin{bmatrix} B - \sum_{i=2}^{K} D_i A_i^{-1} C_i & -D_1 \\ -C_1 & A_1 \end{bmatrix}$$

is also an irreducible singular M-matrix by Theorem 2.1(e). By Theorem 2.2(c), all $\Phi_i > 0$, and therefore

$$\left\{(s,t) \,:\, s \neq t,\, \left[B - \sum_{i=2}^{K} D_i A_i^{-1} C_i\right]_{(s,t)} < 0\right\} \subseteq \left\{(s,t) \,:\, s \neq t,\, \left[B - \sum_{i=2}^{K} D_i \Phi_i\right]_{(s,t)} < 0\right\},$$

implying that

$$W_1(\Phi) = \begin{bmatrix} B_1(\Phi) & -D_1 \\ -C_1 & A_1 \end{bmatrix} \quad \text{is irreducible.}$$

Moments ago, we showed $W_1(\Phi)$ is an M-matrix. Thus $W_1(\Phi)$ is a nonsingular or irreducible singular M-matrix. For $W_j(\Phi)$ with $j > 1$, we permute symmetrically W to

$$\begin{bmatrix} B & D_j & -\widehat{D}_j \\ -C_j & A_j & \\ -\widehat{C}_j & & \widehat{A}_j \end{bmatrix},$$

where $\widehat{C}_j$ and $\widehat{D}_j$ are obtained from C and D with their jth block removed, and $\widehat{A}_j$ from A with its jth block row and column removed. Now use the same proof we had for $W_1(\Phi)$ to conclude that $W_j(\Phi)$ is a nonsingular or irreducible singular M-matrix.

In summary, MARE (3.7) is super-regular, and thus has a minimal nonnegative solution by Theorem 2.2. That solution, denoted by $\widehat{\Phi}_j$, can be uniquely characterized by that $\begin{bmatrix} I \\ \widehat{\Phi}_j \end{bmatrix}$ is the basis matrix for the eigenspace of $H_j(\Phi)$ associated with its m right most eigenvalues of $H_j(\Phi)$, given by $\operatorname{eig}(B - D\Phi)$. That eigenspace is unique by item (b) we just proved. On the other hand, it follows from (3.9) that $\begin{bmatrix} I \\ \Phi_j \end{bmatrix}$ is the basis matrix for the same eigenspace. Therefore $\widehat{\Phi}_j = \Phi_j$, as expected. □

Theorem 3.2. *Assume (1.2). Given $U = [U_1^{\mathrm{T}}, U_2^{\mathrm{T}}, \dots, U_K^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{n\times m}$ partitioned in the same way as X in (3.2), if $0 \le U_i \le \Phi_i$ for $1 \le i \le K$, then*

$$X_jD_jX_j - A_jX_j - X_jB_j(U) + C_j = 0 \quad \text{for } 1 \le j \le K$$

are super-regular MAREs and thus each has a unique minimal nonnegative solution.

Proof. Recall (3.5). The condition of the theorem implies $B_j(U) \ge B_j(\Phi)$ and thus $W_j(U) \ge W_j(\Phi)$. $W_j(\Phi)$ is a nonsingular or irreducible singular M-matrix by Theorem 3.1(c), and, hence, $W_j(U)$ is a nonsingular or irreducible singular M-matrix by Theorem 2.1(a,b). The proof is completed. □

The next theorem is about the monotonicity in the minimal nonnegative solution of super-regular MARE. Besides MARE (1.1), consider MARE

$$\widetilde{X}\widetilde{D}\widetilde{X} - \widetilde{A}\widetilde{X} - \widetilde{X}\widetilde{B} + \widetilde{C} = 0, \tag{3.15}$$

where $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$, and $\widetilde{D}$ have the same sizes as A, B, C, and D of (1.1). Denote by $\widetilde{W}$ the corresponding defining coefficient matrix of (3.15).

Theorem 3.3. *Suppose that both ARE (1.1) and (3.15) are super-regular, and let Φ and $\widetilde{\Phi}$ be their minimal nonnegative solutions, respectively. If $\widetilde{W} \ge W$, then $\widetilde{\Phi} \le \Phi$.*

Proof. Split A and B as $A = D_A - N_A$ and $B = D_B - N_B$, where $D_A = \operatorname{diag}(A)$ and $D_B = \operatorname{diag}(B)$. The following iterative scheme

$$Z_0 = 0,$$
$$D_AZ_{k+1} + Z_{k+1}D_B = N_AZ_k + Z_kN_B + Z_kDZ_k + C_k \text{ for } k \ge 0$$

produces a sequence $\{Z_k\}_{k=0}^{\infty}$ that monotonically converges to the minimal nonnegative solution Φ of (1.1) [3, Theorem 2.3]. The same idea applied to (3.15) yields a sequence $\{\widetilde{Z}_k\}_{k=0}^{\infty}$ that monotonically converges to $\widetilde{\Phi}$. Inductively, it is not hard to show $\widetilde{Z}_k \le Z_k$ for all k, which leads to the desired conclusion. □

4. Doubling algorithm – accADDA

In this section, we outline the doubling algorithm, ADDA [13] and its highly accurate implementation accADDA [16] (see also [1]) for super-regular MARE (1.1). ADDA starts by picking parameters α and β that satisfy

$$0 \le \alpha \le \alpha_{\text{opt}} := \left(\max_i [A]_{(i,i)}\right)^{-1},\ 0 \le \beta \le \beta_{\text{opt}} := \left(\max_i [B]_{(i,i)}\right)^{-1}, \tag{4.1a}$$

$$\max\{\alpha, \beta\} \ne 0. \tag{4.1b}$$

Often we take $\alpha = \alpha_{\text{opt}}$ and $\beta = \beta_{\text{opt}}$ for the fastest convergence [13, Theorem 3.3]. Then it computes $E_0 \in \mathbb{R}^{m\times m}$, $F_0 \in \mathbb{R}^{n\times n}$, $Z_0 \in \mathbb{R}^{n\times m}$ and $Y_0 \in \mathbb{R}^{m\times n}$ by solving

$$\begin{bmatrix} \alpha B + I & -\beta D \\ -\alpha C & \beta A + I \end{bmatrix} \begin{bmatrix} E_0 & Y_0 \\ Z_0 & F_0 \end{bmatrix} = \begin{bmatrix} I - \beta B & \alpha D \\ \beta C & I - \alpha A \end{bmatrix}, \tag{4.2}$$

which is followed by the doubling iteration: for $k = 0, 1, \dots,$

$$E_{k+1} = E_k(I_m - Y_kZ_k)^{-1}E_k, \tag{4.3a}$$

$$F_{k+1} = F_k(I_n - Z_kY_k)^{-1}F_k, \tag{4.3b}$$

$$Z_{k+1} = Z_k + F_k(I_n - Z_kY_k)^{-1}Z_kE_k \tag{4.3c}$$

$$= Z_k + F_kZ_k(I_m - Y_kZ_k)^{-1}E_k, \tag{4.3d}$$

$$Y_{k+1} = Y_k + E_k(I_m - Y_kZ_k)^{-1}Y_iF_k \tag{4.3e}$$

$$= Y_k + E_kY_k(I_n - Z_kY_k)^{-1}F_k. \tag{4.3f}$$

A detailed derivation of the formulas (4.2) and (4.3) can be found in [1, pp. 20–21]. The alternative expression (4.3d) *vs.* (4.3c) and (4.3f) *vs.* (4.3e) can be useful at implementation, especially when either $m \ll n$ or $n \ll m$.

With α and β satisfying (4.1), Z_k is monotonically increasing and converges to Φ quadratically, except in the case when H has a double eigenvalue 0 coming from a 2×2 Jordan block, for which the convergence is only linear with the linear rate 1/2. For more detailed statements of ADDA's convergence, the reader is referred to [1, Theorem 6.3].

A highly accurate implementation of ADDA was discovered first by Nguyen and Poloni [15] for a singular but irreducible M-matrix W and then by Xue and Li [16] for a nonsingular M-matrix W. The key part in implementation is the computations of the inverses of provably nonsingular M-matrices

$$\begin{bmatrix} \alpha B + I & -\beta D \\ -\alpha C & \beta A + I \end{bmatrix}, \quad I - Z_kY_k, \quad I - Y_kZ_k, \tag{4.4}$$

to almost full entrywise relative accuracy by the GTH-like algorithm [24,25]. It is made possible by a novel way, especially for the case when W is a nonsingular M-matrix [16], to find triplet representations for $I - Z_kY_k$ and $I - Y_kZ_k$ to nearly full entrywise relative accuracy during the iterative process.

Xue and Li [16] started by assuming a triplet representation

$$W = \left(\text{offdiag}(W), \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \end{bmatrix}, \begin{bmatrix} \hat{\boldsymbol{u}}_1 \\ \hat{\boldsymbol{u}}_2 \end{bmatrix}\right) \tag{4.5a}$$

of W is known to almost full entrywise relative accuracy, where

$$\begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \end{bmatrix} > 0, \quad \begin{bmatrix} \hat{\boldsymbol{u}}_1 \\ \hat{\boldsymbol{u}}_2 \end{bmatrix} = W \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \end{bmatrix} \geq 0. \tag{4.5b}$$

To accurately invert the first M-matrix in (4.4), we have the following lemma.

Theorem 4.1 (*[16]*).

(a) *If $\alpha = 0$ but $\beta > 0$, then*

$$\begin{bmatrix} I_m & -\beta D \\ 0 & I_n + \beta A \end{bmatrix}^{-1} = \begin{bmatrix} I_m & \beta D(I + \beta A)^{-1} \\ 0 & (I_n + \beta A)^{-1} \end{bmatrix},$$

and a triplet representation for $I_n + \beta A$ can be read off from

$$(I_n + \beta A)\boldsymbol{u}_2 = \boldsymbol{u}_2 + \beta(C\boldsymbol{u}_1 + \hat{\boldsymbol{u}}_2).$$

(b) *If $\alpha > 0$ and $\beta = 0$, then*

$$\begin{bmatrix} \alpha B + I_m & 0 \\ -\alpha C & I_n \end{bmatrix}^{-1} = \begin{bmatrix} (\alpha B + I_m)^{-1} & 0 \\ \alpha C(\alpha B + I_m)^{-1} & I_n \end{bmatrix}$$

and a triplet representation for $\alpha B + I_m$ can be read off from

$$(\alpha B + I_m)\boldsymbol{u}_1 = \alpha(\hat{\boldsymbol{u}}_1 + D\boldsymbol{u}_2) + \boldsymbol{u}_1.$$

(c) *If $\alpha > 0$ and $\beta > 0$, then*

$$\begin{bmatrix} \alpha B + I_m & -\beta D \\ -\alpha C & \beta A + I_n \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_1/\alpha \\ \boldsymbol{u}_2/\beta \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{u}}_1 + \boldsymbol{u}_1/\alpha \\ \hat{\boldsymbol{u}}_2 + \boldsymbol{u}_2/\beta \end{bmatrix},$$

which yields a triplet representation for $\begin{bmatrix} \alpha B + I_m & -\beta D \\ -\alpha C & \beta A + I_n \end{bmatrix}$ *immediately.*

Algorithm 4.1 Highly Accurate ADDA for MARE (1.1)

Input: Strongly regular MARE (1.1), vectors $\boldsymbol{u}_1,\ \boldsymbol{u}_2$ and $\hat{\boldsymbol{u}}_1,\ \hat{\boldsymbol{u}}_2$ that satisfy (4.5b);
Output: the minimal nonnegative solution Φ and $\boldsymbol{z} = \boldsymbol{u}_2 - \Phi \boldsymbol{u}_1$.

1: $\alpha = \left(\max_i [A]_{(i,i)}\right)^{-1}$, $\beta = \left(\max_j [B]_{(j,j)}\right)^{-1}$, $k = -1$;
2: compute E_0, F_0, Z_0 and Y_0 according to (4.2) by the GTH-like algorithm using the triplet representation provided by Theorem 4.1;
3: compute $\boldsymbol{w}_1^{(0)}$ and $\boldsymbol{w}_2^{(0)}$ according to (4.8) by the GTH-like algorithm;
4: **repeat**
5: $k = k + 1$;
6: compute $\boldsymbol{v}_1^{(k)}$ and $\boldsymbol{v}_2^{(k)}$ according to (4.9) and generate the triplet representations for $I - Y_k Z_k$ and $I - Z_k Y_k$ as in (4.7);
7: compute $E_{k+1}, F_{k+1}, Z_{k+1}$ and Y_{k+1} according to (4.3) by the GTH-like algorithm using the triplet representations for $I - Y_k Z_k$ and $I - Z_k Y_k$;
8: compute $\boldsymbol{w}_1^{(k+1)}$ and $\boldsymbol{w}_2^{(k+1)}$ according to (4.9c) and (4.9d) (reuse $E_k(I - Y_k Z_k)^{-1}$ and $F_k(I - Z_k Y_k)^{-1}$ that appear in implementing line 8 to reduce work);
9: **until** convergence;
10: **return** the last $Z_k \approx \Phi$, and $\boldsymbol{z}_k = \boldsymbol{w}_2^{(k)} + F_k \boldsymbol{u}_2 \approx \boldsymbol{z}$.

To accurately invert the second and third M-matrices in (4.4), Xue and Li [16] introduced auxiliary vectors

$$\begin{bmatrix} \boldsymbol{w}_1^{(k)} \\ \boldsymbol{w}_2^{(k)} \end{bmatrix} := \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \end{bmatrix} - \begin{bmatrix} E_k & Y_k \\ Z_k & F_k \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \end{bmatrix}, \tag{4.6}$$

which are provably nonnegative and can be computed, not in the way as defined in (4.6), but recursively according to the following theorem.

Theorem 4.2 ([16]). *The triplet representations for $I_m - Y_k Z_k$ and $I - Y_k Z_k$ are given by*

$$I_m - Y_k Z_k = \left(\mathrm{offdiag}(I_m - Y_k Z_k), \boldsymbol{u}_1, \boldsymbol{v}_1^{(k)}\right), \tag{4.7a}$$

$$I_n - Z_k Y_k = \left(\mathrm{offdiag}(I_n - Z_k Y_k), \boldsymbol{u}_2, \boldsymbol{v}_2^{(k)}\right), \tag{4.7b}$$

where $\boldsymbol{v}_1^{(k)}$ and $\boldsymbol{v}_2^{(k)}$ are computed recursively as follows: solving

$$\begin{bmatrix} \alpha B + I_m & -\beta D \\ -\alpha C & \beta A + I_n \end{bmatrix} \begin{bmatrix} \boldsymbol{w}_1^{(0)} \\ \boldsymbol{w}_2^{(0)} \end{bmatrix} = (\alpha + \beta) \begin{bmatrix} \hat{\boldsymbol{u}}_1 \\ \hat{\boldsymbol{u}}_2 \end{bmatrix}, \tag{4.8}$$

for $\boldsymbol{w}_1^{(0)}$ and $\boldsymbol{w}_2^{(0)}$, and for $k = 0, 1, 2, \ldots$

$$\boldsymbol{v}_1^{(k)} = \boldsymbol{w}_1^{(k)} + E_k \boldsymbol{u}_1 + Y_k(F_k \boldsymbol{u}_2 + \boldsymbol{w}_2^{(k)}) \geq 0, \tag{4.9a}$$

$$\boldsymbol{v}_2^{(k)} = \boldsymbol{w}_2^{(k)} + F_k \boldsymbol{u}_2 + Z_k(E_k \boldsymbol{u}_1 + \boldsymbol{w}_1^{(k)}) \geq 0, \tag{4.9b}$$

$$\boldsymbol{w}_1^{(k+1)} = \boldsymbol{w}_1^{(k)} + E_k(I - Y_k Z_k)^{-1}[\boldsymbol{w}_1^{(k)} + Y_k \boldsymbol{w}_2^{(k)}], \tag{4.9c}$$

$$\boldsymbol{w}_2^{(k+1)} = \boldsymbol{w}_2^{(k)} + F_k(I - Z_k Y_k)^{-1}[Z_k \boldsymbol{w}_1^{(k)} + \boldsymbol{w}_2^{(k)}]. \tag{4.9d}$$

With the help of these triplet representations in Theorems 4.1 and 4.2 all three nonsingular M-matrices in (4.4) can now be inverted in a cancellation-free way, leading to ACCADDA of [16]. However, in using ACCADDA later in Section 5 to solve each smaller-sized super-regular MARE (3.3) highly accurately, upon fixing $B_j(X)$, we will need to be able to compute the vector

$$\boldsymbol{z} := \boldsymbol{u}_2 - \Phi \boldsymbol{u}_1 \tag{4.10}$$

with high entrywise relative accuracy. This expression cannot be straightforwardly used to fulfill the task because, if computed as $\boldsymbol{z} \approx \boldsymbol{u}_2 - Z_k \boldsymbol{u}_1$, potential cancellations will likely destroy entrywise relative accuracy in some of the computed entries of $\boldsymbol{z}$. We have to do something different. The next lemma is essentially [28, Lemma 5.1] which is stated for triplet representations.

Lemma 4.1. *Let $\boldsymbol{z}_k = \boldsymbol{w}_2^{(k)} + F_k \boldsymbol{u}_2$. Then $\boldsymbol{z}_k = \boldsymbol{u}_2 - Z_k \boldsymbol{u}_1$, and, as a result,*

$$\boldsymbol{z} = \boldsymbol{u}_2 - \Phi \boldsymbol{u}_1 = \lim_{k \to \infty} (\boldsymbol{u}_2 - Z_k \boldsymbol{u}_1) = \lim_{k \to \infty} \boldsymbol{z}_k. \tag{4.11}$$

Proof. It follows from (4.6) that $\boldsymbol{w}_2^{(k)} = \boldsymbol{u}_2 - Z_k \boldsymbol{u}_1 - F_k \boldsymbol{u}_2$, and thus $\boldsymbol{z}_k = \boldsymbol{u}_2 - Z_k \boldsymbol{u}_1$. Letting k go to ∞ yields (4.11). □

Algorithm 4.1 summarizes ACCADDA of [16], except also returning a highly accurate approximation to $\boldsymbol{z} = \boldsymbol{u}_2 - \Phi\boldsymbol{u}_1$ as the limit of $\boldsymbol{z}_k$ in Lemma 4.1.

The stopping criterion we use decide when to stop the iteration: Lines 4 – 9 of Algorithm 4.1, is based on the entrywise relative residual (ERRes), to be defined in a moment, that reflects the non-negativeness property of the equation and is born out of the first order asymptotic error analysis in [16] for MARE (1.1). Split A and B as

$$A = D_A - N_A, \qquad D_A = \operatorname{diag}(A), \tag{4.12}$$

$$B = D_B - N_B, \qquad D_B = \operatorname{diag}(A). \tag{4.13}$$

Rearrange MARE (1.1) to get

$$\mathscr{R}_L(X) := XDX + N_AX + XN_B + C = D_AX + XD_B =: \mathscr{R}_R(X).$$

An important outcome of such an arrangement is that there is no cancellation in evaluating both $\mathscr{R}_L(X)$ and $\mathscr{R}_R(X)$. Consider now a nonnegative $\widetilde{\Phi}$ as an approximation to Φ, and write

$$\mathscr{R}_L(\widetilde{\Phi}) = \mathscr{R}_R(\widetilde{\Phi}) + E,$$

where $E = \widetilde{\Phi}D\widetilde{\Phi} - A\widetilde{\Phi} - \widetilde{\Phi}B + C$ which is the usual residual. Ideally we would like to have $E = 0$, but that is unlikely in practice. ERRes is defined as

$$\mathtt{ERRes}(\widetilde{\Phi}) = \max_{i,j} \frac{|\mathscr{R}_L(\widetilde{\Phi}) - \mathscr{R}_R(\widetilde{\Phi})|_{(i,j)}}{|\mathscr{R}_R(\widetilde{\Phi})|_{(i,j)}}, \tag{4.14}$$

where, as a convention, 0/0 is treated as 0. Xue and Li [16, Theorem 4.2] show that some multiple of ERRes by a constant factor can serve, up to the first order, as an upper bound on the *entrywise relative error* (ERErr),

$$\mathtt{ERErr}(\widetilde{\Phi}) = \max_{i,j} \frac{|(\widetilde{\Phi} - \Phi)_{(i,j)}|}{\Phi_{(i,j)}}. \tag{4.15}$$

Tiny ERErr guarantees that all entries of Φ, large or small in magnitude, are well approximated. Finally, we stop the iteration: Lines 4 – 9 of Algorithm 4.1, if

$$\mathtt{ERRes}(Z_k) \le \mathtt{rtol}, \tag{4.16}$$

where rtol is a given relative tolerance.

5. Highly accurate algorithm

For the structured MARE (3.1), Meini [21] designed an iterative method, based on the Jacobi- or Gauss–Seidel-like updating. Algorithm 5.1 outlines the framework of Meini's algorithm. There are two imminent issues to address. First, does ARE (5.2) have a minimal nonnegative solution? Second, if it does, how is it computed?

Consider the first issue. For simplicity, we will assume each time (5.2) is exactly solved, i.e., $\Phi_j^{(\nu+1)}$ is exact. Although this is a strong assumption that is impossibly to satisfy in actually numerical computations because of rounding errors, it is not unreasonable to assume in the analysis phase. Let

$$W_j^{(\nu)} = \begin{bmatrix} B_j^{(\nu)} & -D_j \\ -C_j & A_j \end{bmatrix} \tag{5.3}$$

be the defining matrix of ARE (5.2). In justifying ARE (5.2) has a minimal nonnegative solution, Meini [21, Theorem 7] shows that $W_j^{(\nu)}$ is an M-matrix, and that is not enough as we commented before. In what follows, we will show that each MARE (5.2) is super-regular.

Theorem 5.1 is our major convergence theorem for Algorithm 5.1. Although it bears much similarity to the results in [21, Theorems 8 and 9], for example, the inequalities in (5.4) and the convergence relation (5.5) have appeared in [21, Theorems 8 and 9], albeit not being properly justified. We would like to emphasize two major differences: (i) Theorem 5.1 also covers the case that W is nonsingular, and (ii) more importantly, it shows that each MARE (5.2) is a super-regular MARE and thus has a minimum nonnegative solution indeed [10].

Theorem 5.1. *Assume (1.2). The following statements hold for Algorithm 5.1.*

(a) *Each MARE (5.2) is super-regular, i.e., $W_j^{(\nu)}$ is a nonsingular or irreducible singular M-matrix, and thus it has a minimal nonnegative solution, denoted by $\Phi_j^{(\nu+1)}$.*
(b) *The doubling algorithm, ADDA as described by (4.1)–(4.3), on MARE (5.2) converges to $\Phi_j^{(\nu+1)}$.*

Moreover, for each $j = 1, \ldots, K$,

$$0 \le \Phi_j^{(\nu)} \le \Phi_j^{(\nu+1)} \le \Phi_j \quad \text{for } \nu \ge 0, \tag{5.4}$$

Algorithm 5.1 Jacobi–/Gauss–Seidel-like iteration for structured MARE (3.1) [21]

Input: structured super-regular MARE (3.1);
Output: the minimal nonnegative solution Φ of MARE.

$\nu = 0$, $\Phi_j^{(\nu)} = 0 \in \mathbb{R}^{n_j \times m}$ for $1 \le j \le K$;
repeat
  for $j = 1, 2, \dots, K$ **do**
    compute

$$B_j^{(\nu)} = \begin{cases} B - \sum_{i \neq j} D_j \Phi_j^{(\nu)}, & \text{for Jacobi-like updating,} \\ B - \sum_{i=1}^{j-1} D_i \Phi_i^{(\nu+1)} - \sum_{i=j+1}^{K} D_i \Phi_i^{(\nu)}, & \text{for Gauss-Seidel-like updating;} \end{cases} \tag{5.1}$$

    solve

$$X_j D_j X_j - A_j X_j - X_j B_j^{(\nu)} + C_j = 0 \tag{5.2}$$

    for its minimal nonnegative solution, denoted by $\Phi_j^{(\nu+1)}$;
  end for
  $\nu = \nu + 1$;
until convergence
return $\Phi^{(\nu)} := \left[(\Phi_1^{(\nu)})^{\mathrm{T}}, (\Phi_2^{(\nu)})^{\mathrm{T}}, \dots, (\Phi_K^{(\nu)})^{\mathrm{T}}\right]^{\mathrm{T}}$ as an approximation to Φ.

and

$$\lim_{\nu \to \infty} \Phi_j^{(\nu)} = \Phi_j. \tag{5.5}$$

Proof. Item (b) is a direct consequence of item (a), thanks to [13]. The relation (5.5) is a direct consequence of (5.4). Let $\Phi_{*j} = \lim_{\nu \to \infty} \Phi_j^{(\nu)}$ and let $\Phi_* = [\Phi_{*1}^{\mathrm{T}}, \Phi_{*2}^{\mathrm{T}}, \dots, \Phi_{*2}^{\mathrm{T}}]^{\mathrm{T}}$. Evidently, Φ_* is a nonnegative solution of the structured MARE (3.1) and $\Phi_* \le \Phi$. Hence $\Phi_* = \Phi$ because Φ is minimal, implying (5.5).

It remains to show that each ARE (5.2) is a super-regular MARE and that (5.4) holds. We only provide a proof for the Jacobi-like updating. A proof for the Gauss–Seidel-like updating can be given in the same way.

We will prove the claim by induction. Initially, $\Phi_j^{(0)} = 0$ for $1 \le j \le K$. Then $B_j^{(0)} = B \ge B_j(\Phi)$, and thus $W_j^{(0)} \ge W_j(\Phi)$. Noticing that $W_j^{(0)}$ is a Z-matrix and that $W_j(\Phi)$ is a nonsingular or irreducible singular M-matrix, we conclude that $W_j^{(0)}$ is a nonsingular or irreducible singular M-matrix by Theorem 2.1(a,b), i.e., MARE (5.2) for $j = 0$ is super-regular and thus has a minimal nonnegative solution $\Phi_j^{(1)}$. By Theorem 3.3, we have $\Phi_j^{(1)} \le \Phi_j$. Therefore, $\Phi_j^{(0)} = 0 \le \Phi_j^{(1)} \le \Phi_j$ for $1 \le j \le K$.

Assume that, for $\nu = \ell$, (5.4) holds and ARE (5.2) is a super-regular MARE. We will prove them for $\nu = \ell + 1$ as well. Since $\Phi_j^{(\ell+1)} \le \Phi_j$, we have $W_j^{(\ell+1)} \ge W_j(\Phi)$ and thus $W_j^{(\ell+1)}$ is a nonsingular or irreducible singular M-matrix for $1 \le j \le K$, which means that each (5.2) is a super-regular MARE for $\nu = \ell + 1$ and for $j = 1, \dots, K$ and thus has a minimal nonnegative solution $\Phi_j^{(\ell+2)}$ that satisfies $\Phi_j^{(\ell+2)} \le \Phi_j$ by Theorem 3.3. Also $W_j^{(\ell)} \ge W_j^{(\ell+1)}$ because $\Phi_j^{(\ell)} \le \Phi_j^{(\ell+1)}$, and hence again by Theorem 3.3 we get $\Phi_j^{(\ell+1)} \le \Phi_j^{(\ell+2)}$ as well. Together, we have $\Phi_j^{(\ell)} \le \Phi_j^{(\ell+1)} \le \Phi_j$. This completes the induction step. □

Return to the second issue: how to solve MARE (5.2) which, by Theorem 5.1, is a super-regular MARE. Naturally, we will use a doubling algorithm [6,12,13], as Meini [21] did. Since ADDA is fastest among all doubling algorithms [13] and also we would like the computed minimal nonnegative solution to have high entrywise relative accuracy, we will use Algorithm 4.1 for each MARE (5.2). In order to do that, we must be able to calculate an accurate triplet representation for $W_j^{(\nu)}$.

To begin with, we assume to have an accurate triplet representation as in (4.5) for W. In what follows, we will explain how to construct an accurate triplet representation for each $W_j^{(\nu)}$ along the way.

Apply ADDA to (5.2) to generate the sequence

$$\left\{E_{j;k}^{(\nu)}, F_{j;k}^{(\nu)}, Z_{j;k}^{(\nu)}, Y_{j;k}^{(\nu)}\right\}_{\nu=0}^{\infty} \quad \text{for } 1 \le j \le K.$$

It is known that [11,13]

$$0 \le Z_{j;k-1}^{(\nu)} \le Z_{j;k}^{(\nu)} \le \Phi_j^{(\nu+1)}, \quad \lim_{k \to \infty} Z_{j;k}^{(\nu)} = \Phi_j^{(\nu+1)}.$$

for $\nu \ge 0$ and $j = 1, \dots, K$. The following theorem establishes a triplet representation for the matrix $W_j^{(\nu)}$ of (5.3).

Theorem 5.2. *Assume (1.2) and the triplet representation as in (4.5) for W, where $\boldsymbol{u}_2$ and $\hat{\boldsymbol{u}}_2$ are partitioned as in (3.10). Then, for $\nu \geq 0$ and $j = 1, 2, \dots, K$,*

$$W_j^{(\nu)} \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_{2,j} \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{u}}_j^{(\nu)} \\ \hat{\boldsymbol{u}}_{2,j} \end{bmatrix}, \tag{5.6a}$$

where

$$\hat{\boldsymbol{u}}_j^{(\nu)} = \begin{cases} \hat{\boldsymbol{u}}_1 + \sum_{i \neq j} D_i(\boldsymbol{u}_{2,i} - \Phi_i^{(\nu)} \boldsymbol{u}_1), & \textit{for Jacobi-like updating}, \\ \hat{\boldsymbol{u}}_1 + \sum_{i=1}^{j-1} D_i(\boldsymbol{u}_{2,i} - \Phi_i^{(\nu+1)} \boldsymbol{u}_1) & \\ \quad + \sum_{i=j+1}^{K} D_i(\boldsymbol{u}_{2,i} - \Phi_i^{(\nu)} \boldsymbol{u}_1), & \textit{for Gauss-Seidel-like updating}. \end{cases} \tag{5.6b}$$

Moreover, $\hat{\boldsymbol{u}}_j^{(\nu)} \geq \hat{\boldsymbol{u}}_1 \geq 0$ for $j = 1, 2, \dots, K$.

Proof. We only provide a proof for the Jacobi-like updating. The case for the Gauss–Seidel-like updating can be handled in the same way. By (3.11a), we have

$$B\boldsymbol{u}_1 - D_j \boldsymbol{u}_{2,j} = \hat{\boldsymbol{u}}_1 + \sum_{i \neq j} D_i \boldsymbol{u}_{2,i}.$$

Thus

$$\begin{aligned} W_j^{(\nu)} \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_{2,j} \end{bmatrix} &= \begin{bmatrix} B\boldsymbol{u}_1 - \sum_{i \neq j} D_i \Phi_i^{(\nu)} \boldsymbol{u}_1 - D_j \boldsymbol{u}_{2,j} \\ -C_j \boldsymbol{u}_1 + A_j \boldsymbol{u}_{2,j} \end{bmatrix} \\ &= \begin{bmatrix} \hat{\boldsymbol{u}}_1 + \sum_{i \neq j} D_i(\boldsymbol{u}_{2,i} - \Phi_i^{(\nu)} \boldsymbol{u}_1) \\ \hat{\boldsymbol{u}}_{2,j} \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{u}}_j^{(\nu)} \\ \hat{\boldsymbol{u}}_{2,j} \end{bmatrix}, \end{aligned}$$

where $\hat{\boldsymbol{u}}_j^{(\nu)}$ is the one in (5.6b) for the Jacobi-like updating. By Theorem 5.1, we know $\Phi_j^{(\nu)} \leq \Phi_j^{(\nu+1)} \leq \Phi_j$ and thus $\boldsymbol{u}_{2,i} - \Phi_i^{(\nu)} \boldsymbol{u}_1 \geq 0$, which implies $\hat{\boldsymbol{u}}_j^{(\nu)} \geq 0$. □

Although (5.6) immediately yields a triplet representation

$$W_j^{(\nu)} = \left(\operatorname{offdiag}(W_j^{(\nu)}), \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_{2,j} \end{bmatrix}, \begin{bmatrix} \hat{\boldsymbol{u}}_j^{(\nu)} \\ \hat{\boldsymbol{u}}_{2,j} \end{bmatrix} \right),$$

it cannot be used for computation because of the cancellations in $\boldsymbol{u}_{2,i} - \Phi_i^{(\nu)} \boldsymbol{u}_1$. We need to find an alternative formula to calculate $\hat{\boldsymbol{u}}_j^{(\nu)}$ without any cancellation. The trick lies in Lemma 4.1. As in Section 4, we define

$$\begin{bmatrix} \boldsymbol{w}_{j;1}^{(\nu;k)} \\ \boldsymbol{w}_{j;2}^{(\nu;k)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_{2,j} \end{bmatrix} - \begin{bmatrix} E_{j;k}^{(\nu)} & Y_{j;k}^{(\nu)} \\ Z_{j;k}^{(\nu)} & F_{j;k}^{(\nu)} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_{2,j} \end{bmatrix}. \tag{5.7}$$

They are nonnegative vectors and can be computed recursively without any cancellation. By Lemma 4.1, we have

$$\boldsymbol{z}_i^{(\nu)} := \boldsymbol{u}_{2,i} - \Phi_i^{(\nu)} \boldsymbol{u}_1 = \lim_{k \to \infty} \left(\boldsymbol{w}_{i;2}^{(\nu;k)} + F_{i;k}^{(\nu)} \boldsymbol{u}_{2,i} \right). \tag{5.8}$$

Now we are ready to present our highly accurate algorithm – Algorithm 5.2 – to solve structured MARE (3.1). The stopping criterion for the loop: lines 3–11 is the entrywise relative residual (4.14) introduced in [16], and more details will be given in the next section. At line 8, the diagonal entries of $B_i^{(\nu)}$ should be calculated accurately by $B_i^{(\nu)} \boldsymbol{u}_1 = D_j \boldsymbol{u}_{2,j} + \hat{\boldsymbol{u}}_j^{(\nu)}$ without cancellation.

6. Numerical examples

In this section, we will present two numerical examples to compare four possible combinations:

either Jacobi-like or Gauss–Seidel-like updating (Algorithm 5.1) (6.1a)

at the outer-loop together with

either ADDA (iteration (4.3) as is) or accADDA (Algorithm 4.1) (6.1b)

at the inner-loop. Our numerical results will showcase the superior performance of Algorithm 5.2 in delivering entrywise accuracy in computed minimal nonnegative solutions.

Algorithm 5.2 Highly Accurate Method to Solve Structured MARE (3.1)

Input: structured super-regular MARE (3.1) with an accurate triplet representation as in (4.5) for W;
Output: minimal nonnegative solution Φ.

```
Φ_0^(0) = 0 ∈ R^{n_j×m} and z_i^(0) = u_{2,i} for i = 1, 2, ···, K;
ν = −1;
repeat
    ν = ν + 1;
    for j = 1, 2, ..., K do
        compute offdiag(B_j^(ν)) according to (5.1);
        compute u_i^(ν) according to (5.6b), where u_{2,i} − Φ_i^(ℓ) u_1 is replaced by z_i^(ℓ) and ℓ = ν or ν + 1 as the formulas call for;
        compute the diagonal entries of B_i^(ν) according to (5.6);
        call Algorithm 4.1 with inputs: W_j^(ν), [u_1; u_{2,j}] and [û_j^(ν); û_{2,j}] to output Φ_j^(ν+1) and z_j^(ν+1);
    end for
until convergence;
return the last Φ_j^(ν+1) as an approximation to Φ_j for j = 1, ···, K.
```

To gauge accuracy in a computed solution $\widetilde{\Phi}$, we will use four measures. The first one is the *normalized residual* (NRes)

$$\mathtt{NRes}(\widetilde{\Phi}) = \frac{\|\widetilde{\Phi}D\widetilde{\Phi} - A\widetilde{\Phi} - \widetilde{\Phi}B + C\|_{\mathrm{F}}}{\|\widetilde{\Phi}\|_{\mathrm{F}}(\|\widetilde{\Phi}\|_1\|D\|_1 + \|A\|_1 + \|B\|_1) + \|C\|_1}, \tag{6.2}$$

a commonly used legacy measure because it is computationally available, where $\|\cdot\|_{\mathrm{F}}$ and $\|\cdot\|_1$ are the matrix Frobenius norm and the ℓ_1 operator norm, respectively. The use of ℓ_1-operator norm, instead of the spectral norm $\|\cdot\|_2$, is inconsequential but for computational convenience. In general, some multiple of NRes by a constant factor, called the *condition number*, can serve, up to the first order, as an upper bound on the *normalized error* in norm (NErr) in $\widetilde{\Phi}$:

$$\mathtt{NErr}(\widetilde{\Phi}) := \frac{\|\widetilde{\Phi} - \Phi\|_{\mathrm{F}}}{\|\Phi\|_{\mathrm{F}}}. \tag{6.3}$$

Such a bound is good only in telling relative errors in the larger entries of $\widetilde{\Phi}$, those of $O(\|\widetilde{\Phi}\|_1)$ in magnitude. In the case where there is a wide variation in magnitude among the entries of Φ the bound is not able to yield any meaningful information on how accurate the smaller entries are. NErr is the second measure.

The third and fourth measures are the entrywise relative residual $\mathtt{ERRes}(\widetilde{\Phi})$ in (4.14) and the *entrywise relative error* $\mathtt{ERErr}(\widetilde{\Phi})$ in (4.15), discussed in Section 4. Tiny ERErr guarantees that all entries of Φ, large or small in magnitude, are well approximated, unlike NErr in (6.3) which only guarantees that large entries of Φ.

Both NErr and ERErr are not available in actual computations because Φ is unknown in the first place. In our tests, we compute an "exact" Φ using MATLAB's variable-precision floating-point arithmetic vpa with digits(100) so that we can report both ERErr and NErr for testing purpose.

All tests are done by MATLAB with unit machine roundoff $\mathfrak{u} = 2^{-53} \approx 1.11\times10^{-16}$. We will use $\mathtt{ERRes} \le \mathtt{rtol} = 10^{-16}$ as the stopping criteria for the inner and outer iteration in Algorithm 5.2 to prevent the iterations from being terminated prematurely, lest the eventually computed minimal nonnegative solutions may not achieve entrywise relative accuracy as they deserve. However, $\mathtt{rtol} = 10^{-16}$ is too stringent, especially for the use of the plain ADDA. In general, we suggest to use rtol around 10^{-12} to 10^{-14} for Algorithm 5.2. For Algorithm 5.1 combined with the plain ADDA to solve MARE (5.2), one should really use $\mathtt{NRes} \le \mathtt{tol}$ with tol about $10^{-15} \sim 10^{-16}$.

Example 6.1. This is the first example in Meini [21] (see also [19, Example 5.2]), describing a MMAP[K]/PH[K]/1 queue with adaptive Poisson source and background traffic. We let the accepted probability be the same as the rejected probability. Specifically, the size parameters are $m = 8$, $n = 200$, $n_i = 5$, $K = 40$, and

$$A = \mathrm{diag}(A_1, \dots, A_K) \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{m\times m}, C \in \mathbb{R}^{n\times m}, D \in \mathbb{R}^{m\times n}.$$

Example details are available in an MATLAB m-file upon request. Using MATLAB's vpa, we computed the "exact" Φ and found

$$2.5817 \times 10^{-15} \le \Phi_{(i,j)} \le 9.9977 \times 10^{-1},$$

which suggests that the smallest entry could lose all of its significant decimal digits entirely in the worst case by the plain ADDA. Fig. 6.1 plots the convergence histories by the four possible combinations from (6.1). It can be observed that

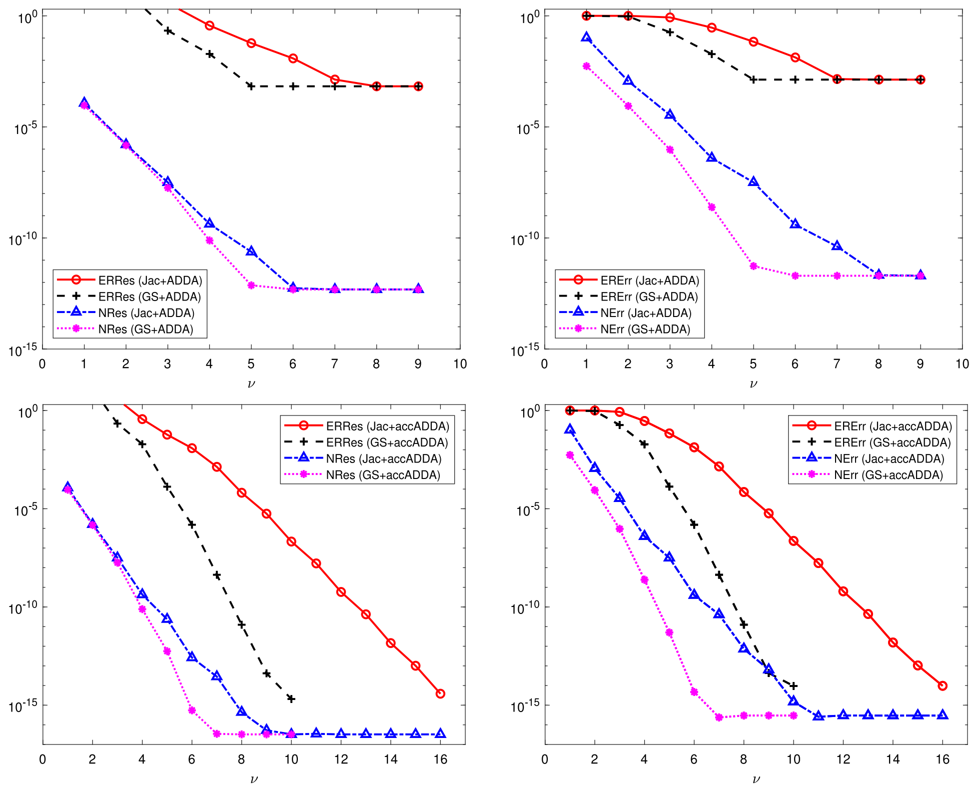

Fig. 6.1. Example 6.1. Top two plots are for Algorithm 5.1 with plain ADDA, and they clearly show that both normalized residual (`NRes`) and normalized error (`NErr`) can be driven down to an acceptable error level, but the entrywise relative residual (`ERRes`) and entrywise relative error (`ERErr`) can only go down to about 10^{-3}. The bottom two plots are for Algorithm 5.2, i.e., Algorithm 5.1 with accADDA, and they clearly demonstrate that with accADDA high entrywise relative accuracy of 15 correct significant decimal digits can be achieved for all entries, large and small, of Φ.

- normalized residual `NRes` and normalized error `NErr` can be driven down to 10^{-12} with ADDA and even more impressively down to 10^{-16} with accADDA;
- for entrywise error measures, both `ERRes` and `ERErr` refuse to move down below 10^{-3} with ADDA, implying some entries of the computed Φ have only about 3 correct significant decimal digits, but they all go down to 10^{-15} with accADDA, implying all entries of the computed Φ have at least 15 correct significant decimal digits.
- For the same accuracy, the Gauss–Seidel-like updating is faster than the Jacobi-like updating, for example, for the same level $O(10^{-15})$ of `ERRes` and `ERErr`, the Gauss–Seidel-like updating uses 6 fewer outer loops with accADDA. $\diamond$

Example 6.2. This is a made-up example (3.1) with $n = 100$, $K = 4$, $m = nK = 400$, and

$$B = \begin{bmatrix} 10 & -1 & & \\ & 10 & \ddots & \\ & & \ddots & -1 \\ -1 & & & 10 \end{bmatrix} \in \mathbb{R}^{n\times n},\ A_0 = \begin{bmatrix} 4 & -1 & & \\ & 4 & \ddots & \\ & & \ddots & -1 \\ -1 & & & 4 \end{bmatrix} \in \mathbb{R}^{m\times m},$$

$$C_0 = \begin{bmatrix} 1 & 1 & & \\ & 1 & \ddots & \\ & & \ddots & 1 \\ 1 & & & 1 \end{bmatrix} \in \mathbb{R}^{n\times n},\ C = \begin{bmatrix} C_0 \\ C_0 \\ C_0 \\ C_0 \end{bmatrix} \in \mathbb{R}^{m\times n},\ D_0 = \frac{1}{2}C_0,$$

$$A = \operatorname{diag}(A_0, A_0, A_0, A_0) \in \mathbb{R}^{m\times m},\ D = [D_0, D_0, D_0, D_0] \in \mathbb{R}^{n\times m}$$

The associated W is a nonsingular M-matrix. In fact, $W\mathbf{1}_{m+n} = [5\mathbf{1}_m^{\mathrm{T}}, \mathbf{1}_n^{\mathrm{T}}]^{\mathrm{T}}$. Again using MATLAB's `vpa`, we computed the "exact" Φ and found

$$2.662 \times 10^{-40} \le \Phi_{(i,j)} \le 8.4220 \times 10^{-2},$$

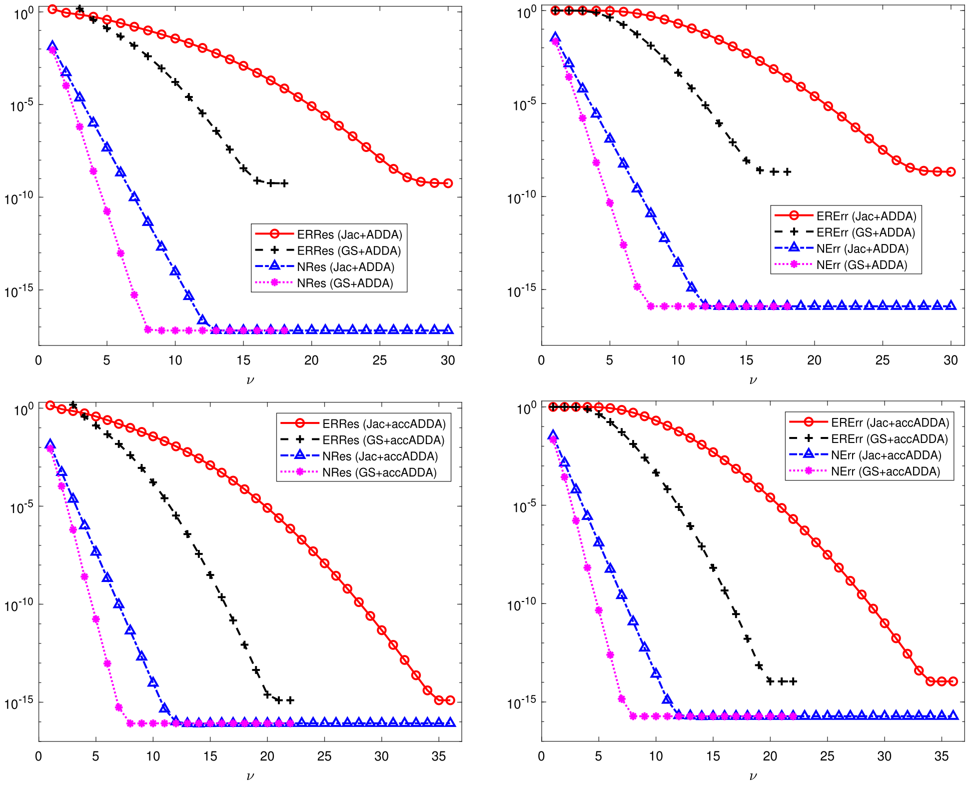

Fig. 6.2. Example 6.2. Top two plots are for Algorithm 5.1 with plain ADDA, and they clearly show that both normalized residual (NRes) and normalized error (NErr) can be driven down to an acceptable error level $O(10^{-16})$, but the entrywise relative residual (ERRes) and entrywise relative error (ERErr) can only go down to $O(10^{-9})$. The bottom two plots are for Algorithm 5.2, i.e., Algorithm 5.1 with accADDA, and they clearly demonstrate that with accADDA high entrywise relative accuracy of 15 correct significant decimal digits can be achieved to all entries, large and small, of Φ.

which suggests that the smallest entries could lose all of its significant decimal digits entirely in the worst case by the plain ADDA. Fig. 6.2 plots the convergence histories by the four possible combinations from (6.1). It is still surprising that all entries of Φ are computed to at least 9 correct significant decimal digits despite of the smallest entries of Φ are in the order of $O(10^{-40})$. The reason is as follows. All the matrices (cf. (4.4)) to be inverted during ADDA iterations are nonsingular M-matrices and for this example, they are all strictly diagonally dominant due to $\boldsymbol{u} = \mathbf{1}_{m+n}$. For such nonsingular M-matrices, the straightforward Gaussian elimination can still produce very accurate entrywise solutions, even though not so accurate as by the GTH-like algorithm [24]. In Example 6.1, the entrywise relative accuracy with ADDA is not as impressive as in this example even though the smallest entries of Φ there are about $O(10^{-15})$. This is because W in Example 6.1 is singular and, as a result, some of nonsingular M-matrices (cf. (4.4)) during ADDA iterations approach to singular or nearly singular M-matrices. ◇

7. Conclusions

We investigated a special type of M-matrix algebraic Riccati equation $XDX - AX - XB + C = 0$ for which A is block-diagonal as in (3.1). Such an MARE arises from multi-type queues with general customer impatience [18,19] and risk processes [20], and it can be broken into many coupled AREs of smaller sizes to allow the use of the Jacobi- or Gauss–Seidel-like updating scheme for its numerical solution in an inner- and outer-iterative fashion, where the inner iteration is about many coupled AREs that will be solved by a doubling algorithm [1]. This is exactly what Meini [21] proposed. In this paper, we started by proving that each smaller ARE during the doubling iterations is indeed an MARE whose defining matrix is a nonsingular or irreducible singular M-matrix, and thus it has a minimal nonnegative solution that can be efficiently computed by the doubling algorithm. Our second contribution is a creative way of constructing a triplet representation for the defining matrices of all smaller AREs during the doubling iterations so that the highly accurate implementations of doubling algorithms due to Nguyen and Poloni [15] and Xue and Li [16] can be applied to deliver all entries of the computed minimal nonnegative solution to the original MARE with high entrywise relative accuracies, regardless of their magnitudes. This is important to the applications where the entries of the minimal nonnegative solution represent probabilities and often tiny probabilities can carry important practical significance. Numerical examples are presented to demonstrate that our highly accurate algorithm can indeed deliver solutions with high entrywise relative accuracies as they deserve.

We also streamline the nomenclature of ARE in connection with an M-matrix in Definition 1.1.

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