ON THE FREE SURFACE MOTION OF HIGHLY SUBSONIC HEAT-CONDUCTING INVISCID FLOWS

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Abstract. For the free surface problem of the highly subsonic heat-conducting inviscid flow in 2-D and 3-D, *a priori* estimates for geometric quantities of free surfaces, such as the second fundamental form and the injectivity radius of the normal exponential map, and the Sobolev norms of fluid variables are proved by investigating the coupling of the boundary geometry and the interior solutions. An interesting feature for the free surface problem studied in this paper is the loss of one more derivative than the problem of incompressible Euler equations for which a geometric approach was introduced by Christodoulou and Lindblad in [11]. Due to the loss of one more derivative and loss of symmetry of equations, the geometric approach in [11] needs to be substantially developed by exploring the interaction of large variation of temperature, heat-conduction, non-zero divergence of the fluid velocity and the evolution of free surfaces.

Keywords: geometry of free surfaces, heat-conducting fluids, loss of derivatives.
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Contents

1. Introduction

1.1. **Backgroud and motivations.** The system of PDEs describing highly subsonic heat-conducting inviscid flows is obtained by taking the low Mach number limit in compressible Euler equations with heat-conductivity in the energy equation modeling general heat-conductive compressible inviscid flows. The general procedure for this is as follows. Introduce first the Mach number, the ratio of in the flow to the sound speed in the fluid, and the inverse of Peclet number which measures the importance of heat-conduction. Both the Mach number and the inverse of Peclet number are dimensionless parameters, which are used to rescale equations. Use the change of variables (rescaling) to rewrite compressible Euler equations with heat-conductivity in the energy equation in the dimensionless way for the basic unknowns of pressure, velocity and temperature. Let the Mach number tend to zero, and then obtain the following limit system

$$(\partial_t + v^k\partial_k)v_j + \mathcal{T}\partial_j p = 0, \quad j = 1, \cdots, n \tag{1.1a}$$

$$\text{div}v = \kappa\Delta\mathcal{T}, \quad (\partial_t + v^k\partial_k)\mathcal{T} = \kappa\mathcal{T}\Delta\mathcal{T}, \tag{1.1b}$$

which describes highly subsonic heat-conducting inviscid flows. Here n is the spatial dimension, $v = (v_1, \cdots, v_n)$ is the velocity field, $\mathcal{T}$ is the the temperature, p is the pressure, $\kappa > 0$ is the (scaled) heat-conductive coefficient which is assumed to be constant for simplicity. One may refer to [1, 41, 43] for more physical background of system (1.1). For the analysis of low Mach number limit of compressible flows, one may refer to [1, 29, 30, 42].

Fluid free surface problems arise in many important physical situations naturally. Depending on the particular hypotheses on fluids, such problems can be used to model a wide range of phenomena such as water waves, shape of stars, liquid drops, vortex sheets, etc. Due to their great mathematical challenges, strong physical background and wide applications, fluid free surface problems have received much attention. In the last decades, a lot of attention has been given to the case of homogeneous, incompressible, and usually inviscid fluids with applications in oceanography through the water wave problem (cf. [2,3,5,11,13,17,31,32,34,45,50,51,57]). Many important analytic and geometric techniques are developed to solve those problems. More recently, the methods developed for this situation have been brought to bear on models of more complicated fluids.

In the present work, we study the free surface problem for system (1.1) of a highly subsonic inviscid heat-conductive flow with the following motivations: (1) to understand the role of heat-conduction played to the free surface motion for inviscid fluids, an issue seldom addressed in literatures; (2) to serve as a step to understand the behavior of free surface motions of inviscid compressible flows for low Mach number when the effect of heat-conductivity is considered; (3) to develop and extend the geometric approaches, in particular, the one introduced in [11] for the study of the free surface problem of incompressible Euler equations, to fluid free surface problems losing more derivatives than those for incompressible and isentropic Euler equations.

Fluid free surface problems have been receiving much attention due to their physical importance and mathematical challenges. For incompressible inviscid flows, the local-in-time well-posedness in Sobolev spaces was obtained first in [50,51] for the irrotational case, and then in [2,3,5,11,13,17,34, 45, 57] for substantial progresses including the cases without irrotational assumptions, finite depth water waves, lower regularities, uniform estimates with respect to surface tension, etc; the global or almost global existence for water waves was achieved first in [19, 52, 53], and then in [4, 7, 16, 25, 26] for recent developments on this topic and the other global well-posedness for related problems; and the singularity formation was proved in [9, 15, 54]. One may refer to the survey [32] for more references. For compressible inviscid flows, the local-in-time well-posedness of smooth solutions was established for liquids in [35, 49] (see also [12] for zero surface tension limits); while for gases with physical vacuum singularity, the related results can be found in [14,27,39] for the local-in-time theories, and in [20, 21, 40, 55, 56] for the global-in-time ones. Most of the above results are either for incompressible or isentropic fluids without taking the effect of heat-conductivity into account. In many physical situations, the heat-conductivity is an important driving force to motions of fluid free surfaces, for example, for a gaseous star. As noted in [33], the heat-conductivity plays a crucial role to driving the evolution of a star in the phase of secular evolution, while the viscosity plays much less role. It is imperative and necessary in general to understand the role played by heat-conductivities to the evolution of fluid free surfaces. As far as we know, however, there have been no results on the free surface problem of heat-conductive inviscid flows, though some results are available for viscous and heat-conductive ones, for example, in [44], where the viscosity plays an essential role to the regularity of solutions in the analysis. When the effect of heat-conductivities is taken into account for inviscid fluids, the analysis becomes significantly difficult due to the strong coupling among the large variations of temperature, heat-conduction, entropy, non-zero divergence of velocity and evolutions of free surfaces, and it is challenging to obtain the regularity of free surfaces since we cannot take advantage of the smoothing effect of the viscosity as did in [44]. This motivates us to understand the role of heat-conduction played to the fluid free surface motion without viscosity.

When the heat-conductivity is taken into account for compressible flows, unlike the problems without heat-conductivity which have been extensively studied, the limit flow of the low Mach number is no longer incompressible (cf. [1]), and the problems become difficult and subtle with rich and interesting phenomena, as observed in [1, 41, 43]. In [1], the low Mach number limit was rigorously justified in $\mathbb{R}^n$ domains without boundaries or periodic domains. (See also [24, 28] for the related results.) Even in the 1-D case without boundaries, the low Mach number asymptotic behavior is quite interesting, the solution tends to a nonlinear diffusion wave of (1.1) whose motion is driven by the variation of the temperature due to the heat-conductivity. However, if the heat-conductivity is not considered in such a case, the limit flow becomes trivial constant states. For free surface problems, the only available result on the low Mach number limit is recent in [36] for isentropic flows where the limit flow is described by the incompressible Euler equations without considering the effect of heat-conductivity. In order to investigate the asymptotic behavior of solutions to low Mach number problems of free surfaces for heat-conducting flows, it is important to gain a good understanding of solutions to limit flows which may be used as the leading order approximation. This motivates us to study the free surface problem of (1.1).

A very interesting feature for the free surface problem of equations (1.1) is the loss of one more derivative than the ones of incompressible and isentropic Euler equations, which we will address later. It is usually quite challenging to deal with PDE problems of losing derivatives, for example, with the problems of Riemannian manifold embedding and Prandtl equations. This third motivation of this work is to develop new analytic and geometric approaches for free surface problems of losing derivatives. In particular, we want to develop and extend the geometric approach, which was introduced in [11] for the study of the free surface problem for incompressible Euler equations, and adopted in [36, 38] for isentropic flows, to problems losing more derivatives, and expect the extended approach can be applied to other fluid free surface problems, which are losing more derivatives than those for incompressible and isentropic fluids.

1.2. **The problem and mathematical challenge.** We consider the following free surface problem of system (1.1) in spatial dimensions $n = 2$ and $n = 3$:

$$(\partial_t + v^k\partial_k)v_j + \mathcal{T}\partial_j p = 0, \quad j = 1, \cdots, n \qquad \text{in } \mathscr{D}, \tag{1.2a}$$

$$\mathrm{div} v = \kappa\Delta\mathcal{T}, \quad (\partial_t + v^k\partial_k)\mathcal{T} = \kappa\mathcal{T}\Delta\mathcal{T}, \qquad \text{in } \mathscr{D}. \tag{1.2b}$$

Given a simply connected bounded domain $\mathscr{D}_0 \subset \mathbb{R}^n$ and initial velocity and temperature $(v_0, \mathcal{T}_0)$ satisfying $\mathrm{div} v_0 = \kappa\Delta\mathcal{T}_0$, the problem is to find a set $\mathscr{D} \subset [0, T] \times \mathbb{R}^n$, a vector field v of fluid velocity and scalar functions p of pressure and $\mathcal{T}$ of temperature solving (1.2) and satisfying the initial conditions:

$$\mathscr{D}_0 = \{x : \ (0, x) \in \mathscr{D}\} \ \text{ and } \ (v, \mathcal{T}) = (v_0, \mathcal{T}_0) \ \text{ on } \ \{0\} \times \mathscr{D}_0, \tag{1.3}$$

and the boundary conditions :

$$p = 0, \quad \mathcal{T} = \mathcal{T}_b \ \text{ and } \ \Big(\partial_t + v^k\partial_k\Big)\big|_{\partial\mathscr{D}} \in \mathbf{T}(\partial\mathscr{D}), \tag{1.4}$$

where $\mathcal{T}_b$ is a positive constant, $\mathbf{T}(\partial\mathscr{D})$ is the tangent space to the boundary $\partial\mathscr{D}$. Let $\mathscr{D}_t = \{x \in \mathbb{R}^n : \ (t, x) \in \mathscr{D}\}$ for $t \in [0, T]$, then the last boundary condition is equivalent to saying that the free surface $\partial\mathscr{D}_t$ moves with the fluid: $v \cdot \mathcal{N} = \varpi$, where $\mathcal{N}$ and ϖ are the exterior unit normal to and the normal velocity of $\partial\mathscr{D}_t$, respectively.

A purpose of the work is to investigate the evolution of the geometry of free surfaces and prove *a priori* estimates of Sobolev norms of fluid variables for problem (1.2)-(1.4) when the initial data satisfies

$$\min_{\partial\mathscr{D}_0}(-\partial_{\mathcal{N}}p) > 0,$$

which implies, as we will prove, that for some $T > 0$ and $0 \le t \le T$,

$$-\partial_{\mathcal{N}}p \ge \epsilon_b > 0 \quad \text{on} \quad \partial\mathscr{D}_t, \tag{1.5}$$

where $\partial_{\mathcal{N}} = \mathcal{N}^j\partial_j$, and $\epsilon_b = 2^{-1}\min_{\partial\mathscr{D}_0}(-\partial_{\mathcal{N}}p)$. (1.5) is a natural stability condition called the *physical condition* or the *Taylor sign condition* for an incompressible inviscid fluid in literatures (cf. [11, 13, 17, 31, 34, 45, 50, 51, 57]), excluding the possibility of the Rayleigh-Taylor type instability (cf. [17]). Since system (1.2) keeps unchanged when a constant is added to p, the condition $p = 0$ on $\partial\mathscr{D}_t$ is equivalent to that of p being a constant. Therefore, the boundary conditions $p = 0$ and $\mathcal{T} = \mathcal{T}_b$ on $\partial\mathscr{D}_t$ are to match the exterior media with the constant pressure and temperature. The boundary condition $p = 0$ on $\partial\mathscr{D}_t$ is commonly used for incompressible flows without surface tensions in literatures (cf. [9, 11, 13, 17, 19, 22, 34, 45, 50, 53, 57] and references therein).

Note that system (1.2) is reduced to the usual incompressible Euler equations if the heat-conductive coefficient $\kappa = 0$ or temperature $\mathcal{T}$ is constant, for which a geometric approach was introduced in [11] to study the free surface problem without assuming that the flow is irrotational. The approach lays in the coupling of the free surface geometry and interior solutions, that is, the bounds for fluid variables and those for geometric quantities of free surfaces, such as the L^∞-bounds for the second fundamental form and lower bound for the injective radius of the normal exponential map, are bounded together. The bounds for the geometric quantities are not only needed to bound Sobolev norms of fluid variables, but also vital to understanding the evolution of the geometry of free surfaces, for example, in the study of the formation of singularities, such as the curvature blow-up or the self-intersection. It should be noted that the singularities such as the splash singularity or splat singularity in [9, 15] and wave crests in [54] all occur on free surfaces, corresponding to a zero injective radius of the normal exponential map and an unbounded second fundamental form, respectively. We adopt the geometric approach introduced in [11] to study problem (1.2)-(1.4) for a highly subsonic heat-conductive flow. However, thorny difficulties arise in extending the analysis in [11] to the problem (1.2)-(1.4) we study here, including loss of symmetries of equations (no conservation laws), loss of one more derivative, the strong coupling of the large variation of temperature, heat-conduction, the non-zero divergence of the velocity filed, and the evolution of free surfaces.

To put it in perspective, we set $\rho = \mathcal{T}^{-1}$ and $D_t = \partial_t + v^k\partial_k$ and write system (1.2) as:

$$D_t\rho + \rho\,\mathrm{div}v = 0, \qquad \rho D_t v + \partial p = 0, \tag{1.6}$$

which looks like the isentropic Euler equations. However, the pressure p in (1.6) is not a function of ρ. Instead, it depends on ρ and v non-locally, determined by

$$\mathrm{div}(\mathcal{T}\nabla p) = -(\partial_i v^j)\partial_j v^i - D_t\mathrm{div}v \quad \text{in} \quad \mathscr{D}_t, \qquad p = 0 \quad \text{on} \quad \partial\mathscr{D}_t. \tag{1.7}$$

When $\mathcal{T}$ is a constant, for example, $\mathcal{T} = 1$, (1.7) reduces to

$$\Delta p = -(\partial_i v^j)\partial_j v^i \quad \text{in} \quad \mathscr{D}_t, \qquad p = 0 \quad \text{on} \quad \partial\mathscr{D}_t. \tag{1.8}$$

This is how the pressure p is determined in the homogeneous incompressible case studied in [11]. For the isentropic fluids modelled by (1.6) for $p = p(\rho)$ with $p'(\rho) > 0$, studied in [35, 36], the equation corresponding to (1.7) reads

$$\Delta h(\rho) = -(\partial_i v^j)\partial_j v^i, \quad \text{where} \quad h(\rho) = \int^\rho s^{-1}p'(s)ds. \tag{1.9}$$

It should be noted that only the first derivatives of the velocity appear on the right hand sides of equations (1.8) and (1.9). Because the regularity of the boundary enters to highest order estimates, it is essential to use the fact that the free surface is the level set of pressure p to build up the the boundary regularity. A crucial point in [11] for this is to use the Dirichlet problem (1.8), by which one gains the regularity: one derivative of v in the interior gives two derivatives of p, which gives a gain of one time derivative of v by use of the equation $D_t v + \partial p = 0$ in the incompressible case. However, the second space-time derivative of the velocity field, $D_t\mathrm{div}v$, appears in (1.7). This is one of reasons that one more derivative is lost compared with the problem studied in [11]. From the Dirichlet problem (1.7), it is clear that one has to estimate the material derivative of the velocity field, $D_t\mathrm{div}v$, which leads us to consider

$$\begin{aligned} &D_t\mathrm{div}v - \kappa\mathcal{T}\Delta\mathrm{div}v \\ &\quad = (\mathrm{div}v)^2 + 2\kappa(\partial\mathcal{T})\cdot\partial\mathrm{div}v - \kappa\Big((\Delta v^k)\partial_k\mathcal{T} + 2(\partial v^k)\cdot\partial\partial_k\mathcal{T}\Big) \qquad \text{in } \mathscr{D}_t, \qquad (1.10\text{a}) \\ &\mathrm{div}v = 0 \qquad \text{on } \partial\mathscr{D}_t. \qquad (1.10\text{b}) \end{aligned}$$

Note that $\mathcal{T}$ appears as the coefficients in the evolution equation for $\mathrm{div}v$, and the dependence of $\mathcal{T}$ on $\mathrm{div}v$ is non-local given by

$$\mathcal{T} = \mathcal{T}_b + \kappa^{-1}\Delta^{-1}\mathrm{div}v, \qquad (1.11)$$

where Δ^{-1} is for the zero Dirichlet boundary condition. The right hand side of (1.10a) suggests the complicated coupling of the velocity field, the variation of temperature and the non-zero divergence.

There is a beautiful geometric structure for inviscid incompressible motions by viewing equations as geodesic flows on the group of volume-preserving diffeomorphisms, as pointed out in a 1966 seminal paper [6]. This point of view has been adopted and developed in the study of incompressible Euler equations, for example, in [8, 18, 48] for the fixed domain problems, and in [37, 45–47] for the free surface ones. This is based on the conservation of the energy: $\int_{\mathscr{D}_t} 2^{-1}|v|^2dx$, which is a fundamental property of inviscid incompressible fluid motions. Indeed, the conservation of the energy for the incompressible Euler equations implies that there is a variational structure with the Lagrangian action: $\mathcal{A} = \int\int_{\mathscr{D}_t} 2^{-1}|v|^2dxdt$, which allows one to systematically derive conserved quantities by means of classical Noether's theorem. Roughly speaking, if an action $\mathcal{A}$ is invariant under a continuous group of transformations, then solutions v satisfy conservation laws determined by these invariants. It should be noted that the variational structure also exists for the compressible Euler equations, as discussed in [10]. However, there is no such a basic energy law for our problem (1.2)-(1.4). Indeed, we do not have any conservative quantity. This is attributed to the loss of the symmetry of equations, since the equations are the limit ones derived by letting the Mach number tend to zero, and some conservation laws provided by physics get lost. Therefore, we cannot adopt the point of view as done in [45] to derive the higher order energy bounds for the problem studied here. Instead, we adopt a more elementary approach of dealing with the coupling of the boundary geometry and interior solutions. It should be noted that the major tools for the study of problems related to water waves modeled by free surface problems of incompressible Euler equations with a constant gravity rely heavily on Fourier analysis, pseudo-differential operators and analysis on singular integral operators, which are of quite different nature from the approach adopted in the present work.

Notations. Throughout the rest of paper, C will denote a universal constant unless stated otherwise, which can change from one inequality to another. Also we use $C(\beta)$ and $C_k(\beta)$ ($k = 0, 1, 2, \cdots$) to denote certain positive constants depending continuously on quantity β.

1.3. **Main results.** Motivated by [11], we define a higher order energy functional consisting of a boundary part and an interior part. To define the boundary integral, we need to project the equations to the tangent space of the boundary.

Definition 1.1. *The orthogonal projection* Π *to the tangent space of the boundary of a* $(0,r)$ *tensor* α *is defined to be the projection of each component along the normal:*

$$(\Pi\alpha)_{i_1\cdots i_r} = \Pi_{i_1}^{j_1}\cdots\Pi_{i_r}^{j_r}\alpha_{j_1\cdots j_r}, \quad \text{where} \quad \Pi_i^j = \delta_i^j - \mathcal{N}_i\mathcal{N}^j.$$

The tangential derivative of the boundary is defined by $\bar{\partial}_i = \Pi_i^j\partial_j$*, and the second fundamental form of the boundary is defined by* $\theta_{ij} = \bar{\partial}_i\mathcal{N}_j$*.*

As in [11], we also need a positive definite quadratic form $\mathcal{Q}(\alpha,\beta)$ for tensors α and β of the same order which is the inner product of the tangential components when restricted to the boundary, and $\mathcal{Q}(\alpha,\alpha)$ increases to the norm $|\alpha|^2$ in the interior.

Definition 1.2. *Let* ι_0 *be the injectivity radius of the normal exponential map of* $\partial\mathscr{D}_t$*, i.e., the largest number such that the map*

$$\partial\mathscr{D}_t\times(-\iota_0,\iota_0) \ \to\ \{x\in\mathbb{R}^n:\ \mathrm{dist}(x,\partial\mathscr{D}_t)<\iota_0\}: \quad (\bar{x},t)\ \mapsto\ x=\bar{x}+\iota\mathcal{N}(\bar{x})$$

is an injection.

Definition 1.3. *Let* d_0 *be a fixed number such that* $\iota_0/16\le d_0\le\iota_0/2$*, and* η *be a smooth cutoff function on* $[0,\infty)$ *satisfying* $0\le\eta(s)\le1$*,* $\eta(s)=1$ *when* $s\le d_0/4$*,* $\eta(s)=0$ *when* $s\ge d_0/2$*, and* $|\eta'(s)|\le 8/d_0$*. Define*

$$\varrho^{ij}(t,x) = \delta^{ij} - \eta^2(d(t,x))\mathcal{N}^i(t,x)\mathcal{N}^j(t,x) \quad \text{in} \quad \mathscr{D}_t,$$

where

$$\mathcal{N}^j(t,x)=\delta^{ij}\mathcal{N}_i(t,x), \quad \mathcal{N}_i(t,x)=\partial_i d(t,x) \quad \text{and} \quad d(t,x)=\mathrm{dist}(x,\partial\mathscr{D}_t).$$

In particular, ϱ *gives the the induced metric on the tangential space to the boundary:*

$$\varrho^{ij}=\delta^{ij}-\mathcal{N}^i\mathcal{N}^j, \quad \varrho_{ij}=\delta_{ij}-\mathcal{N}_i\mathcal{N}_j \quad \text{on} \quad \partial\mathscr{D}_t.$$

With this setting, the above mentioned quadratic form $\mathcal{Q}(\alpha,\beta)$ for $(0,r)$ tensors is defined by

$$\mathcal{Q}(\alpha,\beta) = \varrho^{i_1j_1}\cdots\varrho^{i_rj_r}\alpha_{i_1\cdots i_r}\beta_{j_1\cdots j_r}.$$

We are concerned with the problem for the fixed κ in this paper, so we set $\kappa=1$ from now on for the simplicity of the presentation. The energy functionals of each order are then defined by

$$E_0(t) = \int_{\mathscr{D}_t}\mathcal{T}^{-1}|v|^2dx, \tag{1.12a}$$

$$E_r(t) = \int_{\mathscr{D}_t}\mathcal{T}^{-1}\delta^{mn}\mathcal{Q}(\partial^r v_m,\partial^r v_n)dx + \int_{\mathscr{D}_t}|\partial^{r-1}\mathrm{curl}v|^2dx + \int_{\mathscr{D}_t}|\partial D_t^{r-1}\mathrm{div}v|^2dx$$
$$+\int_{\partial\mathscr{D}_t}\mathcal{Q}(\partial^r p,\partial^r p)(-\partial_{\mathcal{N}}p)^{-1}dS, \qquad r\ge1, \tag{1.12b}$$

where $D_t=\partial_t+v^k\partial_k$. The higher order energy functional is defined by $\sum_{r=0}^{n+2}E_r(t)$.

In order to state the main result of the paper, we set

$$\mathrm{Vol}\mathscr{D}_0 = \int_{\mathscr{D}_0}dx, \quad K_0=\max_{x\in\partial\mathscr{D}_0}\{|\theta(0,x)|+|\iota_0^{-1}(0,x)|\}, \quad \underline{\epsilon}_0=\min_{x\in\partial\mathscr{D}_0}(-\partial_{\mathcal{N}}p)(0,x), \tag{1.13a}$$

$$\underline{\mathcal{T}}_0=\min_{x\in\mathscr{D}_0}\mathcal{T}(0,x), \quad \overline{\mathcal{T}}_0=\max_{x\in\mathscr{D}_0}\mathcal{T}(0,x), \tag{1.13b}$$

$$M_0=\max_{x\in\mathscr{D}_0}\{|\partial p(0,x)|+|\partial v(0,x)|+|\partial\mathcal{T}(0,x)|\}. \tag{1.13c}$$

The initial pressure $p_0(x)=p(0,x)$ is determined by the Dirichlet problem (1.7) in which $D_t\mathrm{div}v|_{t=0}$ is given in terms of initial data v_0 and $\mathcal{T}_0$ via (1.10a) and (1.11). With these notations, the main results of the present work are stated as follows:

Theorem 1.4. *Let $n = 2, 3$. Suppose that $0 < \mathrm{Vol}\mathscr{D}_0, \underline{\epsilon_0}, \underline{\mathcal{T}_0} < \infty$, $K_0, \overline{\mathcal{T}}_0, M_0 < \infty$. Then there exist positive continuous functions $\mathscr{T}_n\left(\mathrm{Vol}\mathscr{D}_0, K_0, \underline{\epsilon_0}^{-1}, \underline{\mathcal{T}_0}^{-1}, \overline{\mathcal{T}}_0, M_0, E_0(0), \cdots, E_{n+2}(0)\right)$ such that any smooth solution $(\mathscr{D}_t, v, p, \mathcal{T})$ to the free surface problem (1.2)-(1.4) for $0 \le t \le T$ with $T \le \mathscr{T}_n$ satisfies the following estimates*

$$\sum_{s=0}^{n+2} E_s(t) \le 2 \sum_{s=0}^{n+2} E_s(0), \qquad 0 \le t \le T, \tag{1.14a}$$

$$2^{-1}\mathrm{Vol}\mathscr{D}_0 \le \mathrm{Vol}\mathscr{D}_t \le 2\mathrm{Vol}\mathscr{D}_0, \qquad 0 \le t \le T, \tag{1.14b}$$

$$\underline{\mathcal{T}_0} \le \mathcal{T} \le \overline{\mathcal{T}}_0 \quad \text{in} \quad \mathscr{D}_t, \qquad 0 \le t \le T, \tag{1.14c}$$

$$|\theta| + |\iota_0^{-1}| \le CK_0 \quad \text{on} \quad \partial\mathscr{D}_t, \qquad 0 \le t \le T, \tag{1.14d}$$

$$-\partial_{\mathcal{N}} p \ge 2^{-1}\underline{\epsilon_0} \quad \text{on} \quad \partial\mathscr{D}_t, \qquad 0 \le t \le T, \tag{1.14e}$$

for a certain constant C, where $\mathrm{Vol}\mathscr{D}_t = \int_{\mathscr{D}_t} dx$.

As a corollary of Theorem 1.4, we have

Corollary 1.5. *For any smooth solution $(\mathscr{D}_t, v, p, \mathcal{T})$ to the problem (1.2)-(1.4) for $0 \le t \le T$ with $T \le \mathscr{T}_n$ as stated in Theorem 1.4, we have*

$$\begin{aligned}
&\|(v, p, \mathcal{T}, \mathrm{div}v, D_t p)\|^2_{H^{n+2}(\mathscr{D}_t)} + \|(D_t\mathrm{div}v, D_t^2 p, D_t^2\mathrm{div}v)\|^2_{H^{n+1}(\mathscr{D}_t)} + \|\theta\|_{H^n(\partial\mathscr{D}_t)} \\
&\le C\left(\mathrm{Vol}\mathscr{D}_0, K_0, \underline{\epsilon_0}^{-1}, \underline{\mathcal{T}_0}^{-1}, \overline{\mathcal{T}}_0, M_0, \sum_{s=0}^{n+2} E_s(0)\right), \quad n = 2, 3, \\
&\|(D_t^n\mathrm{div}v, D_t^n p)\|^2_{H^2(\mathscr{D}_t)} \le C\left(\mathrm{Vol}\mathscr{D}_0, K_0, \underline{\epsilon_0}^{-1}, \underline{\mathcal{T}_0}^{-1}, \overline{\mathcal{T}}_0, M_0, \sum_{s=0}^{n+2} E_s(0)\right), \quad n = 3.
\end{aligned}$$

The bound for $\|\partial(v, p)\|_{L^\infty(\mathscr{D}_0)}$ was not needed in [11] for incompressible Euler equations, because it could be controlled by initial values of their higher order energy functional, $\mathrm{Vol}\mathscr{D}_0$ and a boundary geometric quantity $\iota_1(0, x)$ for which the definition is given below, via Sobolev lemmas and elliptic estimates. We need this bound for the problem studied in this work, an issue which will be addressed later.

Definition 1.6. *Let $0 < \epsilon_1 \le 1/2$ be a fixed number, and let $\iota_1 = \iota_1(\epsilon_1)$ be the largest number such that*

$$|\mathcal{N}(\bar{x}_1) - \mathcal{N}(\bar{x}_2)| \le \epsilon_1 \quad \text{whenever} \quad |\bar{x}_1 - \bar{x}_2| \le \iota_1, \quad \bar{x}_1, \bar{x}_2 \in \partial\mathscr{D}_t.$$

We give some remarks on the choice of the higher order energy functional, and explain briefly the reason why we need $n+2$ derivatives in this functional, while only $n+1$ derivatives were needed in [11] when $n = 2, 3$. Let

$$\begin{aligned}
E_r^a(t) = &\int_{\mathscr{D}_t} \mathcal{T}^{-1}\delta^{mn}\mathcal{Q}(\partial^r v_m, \partial^r v_n)dx + \int_{\mathscr{D}_t} |\partial^{r-1}\mathrm{curl}v|^2 dx \\
&+ \int_{\partial\mathscr{D}_t} \mathcal{Q}(\partial^r p, \partial^r p)(-\partial_{\mathcal{N}} p)^{-1} dS, \qquad r \ge 1.
\end{aligned} \tag{1.15}$$

Note that E_r^a ($r \ge 1$) correspond to the energy functionals employed in [11] for the study of an incompressible flow when $\mathcal{T}$ is constant. In order to control the L^2-norm of $\partial^r v$ for the problem (1.2)-(1.4), one may attempt to use the following:

$$\widetilde{E}_r(t) = E_r^a(t) + \int_{\mathscr{D}_t} |\partial^{r-1}\mathrm{div}v|^2 dx, \quad r \ge 1,$$

due to the Hodge type decomposition. However, there is no control for the time evolution of $\int_{\mathscr{D}_t} |\partial^{r-1}\mathrm{div}v|^2 dx$ for our problem. Instead, by a closer look at the problem (1.10a) for the evolution of div$v$, we identify that $\int_{\mathfrak{D}_t} |\partial D_t^{r-1}\mathrm{div} v|^2 dx$ is the correct quantity to control its time evolution. We can recover the control of $\int_{\mathfrak{D}_t} |\partial^{r-1}\mathrm{div} v|^2 dx$ after then.

The choice of the higher order energy functional $\sum_{r=0}^{n+2} E_r$ enables us to prove that the temporal derivative of it can be controlled by itself under the following *a priori* assumptions:

$$\underline{V} \le \mathrm{Vol}\mathscr{D}_t(t) \le \overline{V} \quad \text{on } [0,T], \tag{1.16a}$$

$$|\theta| + 1/\iota_0 \le K, \quad -\partial_{\mathcal{N}} p \ge \epsilon_b > 0 \quad \text{on } \partial\mathscr{D}_t, \tag{1.16b}$$

$$\sum_{i=1}^{n-1} (|\partial_{\mathcal{N}} D_t^i p| + |\partial_{\mathcal{N}} D_t^i \mathrm{div} v|) + |\partial^2 p| \le L \quad \text{on } \partial\mathscr{D}_t, \tag{1.16c}$$

$$|\partial p| + |\partial v| + |\partial \mathcal{T}| \le M \quad \text{in } \mathscr{D}_t, \tag{1.16d}$$

$$|\partial \mathrm{div} v| + |D_t \mathrm{div} v| + |\partial^2 \mathcal{T}| \le \widetilde{M} \quad \text{in } \mathscr{D}_t, \tag{1.16e}$$

for positive constants $\underline{V}, \overline{V}, K, \epsilon_b, L, M, \widetilde{M}$. In closing the argument, the *a priori* assumptions, for example, on the L^∞-bounds for $\partial(v,p)$ in $\mathscr{D}_t$ and θ on $\partial\mathscr{D}_t$, need to be verified both in [11] and this article. In fact, these L^∞-bounds can be controlled in [11] by their higher order energy functional, $\mathrm{Vol}\mathscr{D}_0$, $\min_{\partial\mathscr{D}_t}(-\partial_{\mathcal{N}} p)$ and $\max_{\partial\mathscr{D}_t}(\iota_1^{-1})$ via Sobolev lemmas, elliptic estimates and projection formulae. But this is not the case for the problem studied in this paper, we do not have such simple and neat control of $\partial(v,p)$ and θ. This is a critical distinction of the problem studied here. Instead of using the method adopted in [11], we employ the evolution equations for $\partial(v,p)$ and θ, which causes the loss of derivatives. For example, in order to control $\|\theta\|_{L^\infty(\partial\mathscr{D}_t)}$, we will need the control of $\|\partial^2 v\|_{L^\infty(\partial\mathscr{D}_t)}$, while a projection formula was used in [11] to control $\|\theta\|_{L^\infty(\partial\mathscr{D}_t)}$ for which there is no need to control $\|\partial^2 v\|_{L^\infty(\partial\mathscr{D}_t)}$. This loss of derivative in the control of the L^∞-bound for θ forces us to use $n+2$ derivatives in the higher order functional, while only $n+1$ derivatives were needed in [11] for $n = 2, 3$. We will address these issues with more details in the next subsection.

1.4. **Main issues and novelty in analysis.** We highlight the main issues in extending the analysis in [11] to problem (1.2)-(1.4). The sharp estimates in [11, 36] use all the symmetries of the incompressible or isentropic Euler equations, which are missing for (1.2) we consider here. Technically, the loss of symmetries of the equations we study is reflected by the following facts: for the problems of incompressible or isentropic Euler equations studied in [11, 36], the zero-th order energy functional is conserved in time, and the temporal derivative of the r-th ($r \ge 1$) order energy functional can be controlled by lower order functionals under suitable a priori assumptions. However, in our case, the temporal derivatives of the zero-th and the first order energy functionals E_0 and E_1 depend on the higher order ones.

Another issue in our analysis is to deal with the problem of loss of derivatives when we work on evolution equations for some quantities in the *a priori* assumptions to obtain their bounds to close the argument. The first one is on the second fundamental form θ for free surfaces. The projection formula,

$$\Pi(\partial^2 p) = \theta \partial_{\mathcal{N}} p \quad \text{on } \partial\mathscr{D}_t, \tag{1.17}$$

was used to estimate the L^∞-bound for θ in [11]. The reason why this can work in [11] is because one may obtain the L^∞-bound for $\partial^2 p$ on $\partial\mathscr{D}_t$ independent of that for θ, which, together with the lower bound for $-\partial_{\mathcal{N}} p$ due to the Taylor sign condition, gives the L^∞-bound for θ. Indeed, it was proved in [11] that

$$\|\partial^2 p\|_{L^\infty(\partial\mathscr{D}_t)} \le C(K_1) \sum_{r=2}^{n+1} \|\partial^r p\|_{L^2(\partial\mathscr{D}_t)} \le C(K_1, \mathcal{E}_0, \cdots, \mathcal{E}_{n+1}, \mathrm{Vol}\mathscr{D}_t), \quad n = 2, 3,$$

where $\mathcal{E}_0 = E_0$ and $\mathcal{E}_r = E_r^a$ $(r \geq 1)$ with $\mathcal{T} = 1$, K_1 is the upper bound for $1/\iota_1$ on $\partial\mathscr{D}_t$ with ι_1 defined in Definition 1.6. In the same spirit, the L^∞-bound for θ was obtained in [36] for isentropic Euler equations by replacing the pressure $p(\rho)$ in (1.17) by the enthalpy $h(\rho) = \int^\rho s^{-1}p'(s)ds$. However, we can only obtain, for problem (1.2)-(1.4) that

$$\begin{aligned}\|\partial^2 p\|_{L^\infty(\partial\mathscr{D}_t)} \leq & C(K_1, \mathrm{Vol}\mathscr{D}_t)\|\mathcal{T}^{-1}\|_{L^\infty(\mathscr{D}_t)}\|\partial p\|_{L^\infty(\mathscr{D}_t)}\|\theta\|_{L^\infty(\partial\mathscr{D}_t)}\|\partial^n\mathcal{T}\|_{L^2(\partial\mathscr{D}_t)} \\ & + \text{other terms}, \quad n = 2, 3,\end{aligned} \tag{1.18}$$

from which it is clear that the projection formula used in [11] to give the L^∞-bound for θ cannot work directly for our problem. Here and thereafter, "other terms" means terms that do not affect those we single out to discuss. Indeed, (1.18) follows from Sobolev lemmas and the following estimates:

$$\|\partial^{n+1}p\|_{L^2(\partial\mathscr{D}_t)} \leq C\|\Pi\partial^{n+1}p\|_{L^2(\partial\mathscr{D}_t)} + C(K_1, \mathrm{Vol}\mathscr{D}_t)\sum_{r=0}^{n}\|\partial^r\Delta p\|_{L^2(\mathscr{D}_t)}, \tag{1.19a}$$

$$\|\partial^n\Delta p\|_{L^2(\mathscr{D}_t)} \leq \|\mathcal{T}^{-1}\|_{L^\infty(\mathscr{D}_t)}\|\partial p\|_{L^\infty(\mathscr{D}_t)}\|\partial^{n+1}\mathcal{T}\|_{L^2(\mathscr{D}_t)} + \text{other terms}, \tag{1.19b}$$

$$\begin{aligned}\|\partial^{n+1}\mathcal{T}\|_{L^2(\mathscr{D}_t)} &\leq C\|\Pi\partial^{n+1}\mathcal{T}\|_{L^2(\partial\mathscr{D}_t)} + \text{other terms} \\ &\leq C\|\partial_{\mathcal{N}}\mathcal{T}\|_{L^\infty(\partial\mathscr{D}_t)}\|\bar\partial^{n-1}\theta\|_{L^2(\partial\mathscr{D}_t)} + C\|\theta\|_{L^\infty(\partial\mathscr{D}_t)}\|\partial^n\mathcal{T}\|_{L^2(\partial\mathscr{D}_t)} + \text{other terms}.\end{aligned} \tag{1.19c}$$

Here (1.19a), (1.19b) and (1.19c) follow from elliptic estimates, the equation $\mathcal{T}\Delta p = -(\partial\mathcal{T})\cdot\partial p$+other terms, and the projection formula, respectively.

Instead of using the projection formula, we use the evolution equations for θ. By doing so, we are led to the following estimate:

$$|D_t\theta| \leq |\partial^2 v| + C|\theta||\partial v|,$$

from which it is apparent that we need to get the L^∞-bounds for both ∂v and $\partial^2 v$ on $\partial\mathscr{D}_t$, while only the L^∞-bound for ∂v was sufficient in [11]. Thus, the L^∞-bound for one more derivative than that in [11] of the velocity field v is needed. This is attributed to the loss of one more derivative than that in [11]. Hence, we need to estimate $n+2$ derivatives in the energy functionals to close the argument, while only $n+1$ derivatives were needed in [11] for $n = 2, 3$. It should be noted that only ∂v enters equations (1.2), but not $\partial^2 v$, and thus one may think that the L^∞-estimate of ∂v may be sufficient to close the argument as done in [11]. However, this is not the case for the problem (1.2)-(1.4) as suggested by the above arguments, reflecting the subtlety of this problem. It is extremely involved to estimate L^∞-bound for $\partial^2 v$ priori to obtaining that for θ in our case.

In fact, even for the L^∞-bound of ∂v in $\mathscr{D}_t$, we will have to use the evolution equation of ∂v, too, while it was obtained by the Sobolev lemma in [11]:

$$\|\partial v\|^2_{L^\infty(\mathscr{D}_t)} \leq C(K_1)\sum_{r=1}^{3}\|\partial^r v\|^2_{L^2(\mathscr{D}_t)} \leq C(K_1)\sum_{r=1}^{3}\mathcal{E}_r \leq C(K_1)\sum_{r=1}^{n+1}\mathcal{E}_r, \quad n = 2, 3. \tag{1.20}$$

For the problem considered in this paper, such a simple and neat estimate does not exist. Indeed, if we try to use the Sobolev lemma as in [11], we can only get a bound depending on the L^∞-bound for θ that cannot be controlled by $n+1$ derivatives, as shown in the following:

$$\begin{aligned}\|\partial v\|^2_{L^\infty(\mathscr{D}_t)} \leq & C(K_1, \mathrm{Vol}\mathscr{D}_t)\|\mathcal{T}^{-1}\|^2_{L^\infty(\mathscr{D}_t)}\|\partial\mathcal{T}\|_{L^\infty(\mathscr{D}_t)}\|v\|_{L^\infty(\mathscr{D}_t)} \\ & \times\|\theta\|_{L^\infty(\partial\mathscr{D}_t)}\|\partial^2\mathcal{T}\|_{L^2(\partial\mathscr{D}_t)}\|\partial^2 v\|_{L^2(\mathscr{D}_t)} + \text{other terms},\end{aligned} \tag{1.21}$$

which follows from Sobolev lemmas and the following estimates:

$$\|\partial^3 v\|^2_{L^\infty(\mathscr{D}_t)} \leq C(E_3^a + \|\partial^2\mathrm{div}v\|^2_{L^2(\mathscr{D}_t)}) \leq CE_3^a + C(K_1, \mathrm{Vol}\mathscr{D}_t)\|\Delta\mathrm{div}v\|^2_{L^2(\mathscr{D}_t)}, \tag{1.22a}$$

$$\|\Delta\mathrm{div}v\|^2_{L^2(\mathscr{D}_t)} \leq \|\mathcal{T}^{-1}\|^2_{L^\infty(\mathscr{D}_t)}\|\partial\mathcal{T}\|_{L^\infty(\mathscr{D}_t)}\|v\|_{L^\infty(\mathscr{D}_t)}\sum_{k=1}^{3}\|\partial^k\mathcal{T}\|_{L^2(\mathscr{D}_t)}\sum_{j=0}^{2}\|\partial^j v\|_{L^2(\mathscr{D}_t)}, \tag{1.22b}$$

$$\|\partial^3\mathcal{T}\|_{L^2(\mathscr{D}_t)} \le C\|\theta\|_{L^\infty(\partial\mathscr{D}_t)}\|\partial^2\mathcal{T}\|_{L^2(\partial\mathscr{D}_t)} + \text{other terms}. \tag{1.22c}$$

Here (1.22a), (1.22b), and (1.22c) follow from the Hodge type decomposition, the equation $\mathcal{T}\Delta\text{div}v = \partial^2\mathcal{T}\cdot\partial v$+other terms, and (1.19c), respectively. The evolution equation $D_t\partial v = -\mathcal{T}\partial^2 p$+other terms and the Sobolev lemma lead to

$$\begin{aligned}\|D_t\partial v\|_{L^\infty(\mathscr{D}_t)} \le& \|\mathcal{T}\|_{L^\infty(\mathscr{D}_t)}\|\partial^2 p\|_{L^\infty(\mathscr{D}_t)} + \text{other terms}\\ \le& C(K_1)\|\mathcal{T}\|_{L^\infty(\mathscr{D}_t)}\sum_{r=2}^{4}\|\partial^r p\|_{L^2(\mathscr{D}_t)} + \text{other terms}, \quad n=2,3,\end{aligned} \tag{1.23}$$

from which it is clear again that $n+2$ derivatives are needed to obtain the L^∞-bound for ∂v in the case of $n=2$.

We prove our theorem by using Lagrangian coordinates to fix the domain, as in [11], where the Riemannian metrics and the covariant differentiations are used to explore the intrinsic properties of equations. This approach is adopted here to deal with the problem in a coordinates free way to make full use of intrinsic properties of the studied problem independent of choice of coordinates, and all the defined energy functionals are invariant in different coordinates systems. It should be remarked that the Riemannian geometry tools including parallel transports, vector fields and covariant differentiations were intensively used in [34,36,45–47] to study fluid free surface problems, besides in [11]. The geometric approach used in [11] was also adopted to study free surface problems of incompressible MHD flows in [22] and incompressible Neo-Hookean elastodynamics in [23].

2. Preliminaries

In this section, we introduce Lagrangian transformation, the metric and covariant differentiation associated with it, the induced metric on the boundary, the geometry and regularity of the boundary, Sobolev lemmas, interpolation inequalities and estimates for the boundary. Basic materials are from [11]. We list them here for the convenience of readers and the easier reference.

2.1. Lagrangian coordinates, the metric, and covariant differentiation in the interior.

Let $x = x(t,y)$ be the change of variables given by

$$\partial_t x(t,y) = v\left(t, x(t,y)\right) \quad \text{and} \quad x(t,y) = x_0(y), \quad y\in\Omega. \tag{2.1}$$

Initially, when $t=0$, we can start with either the Euclidean coordinates in $\Omega = \mathscr{D}_0$ or some other coordinates $x_0 : \Omega \to \mathscr{D}_0$ where x_0 is a diffeomorphism in which the domain Ω becomes simple. For each t we will then have a change of coordinates $x : \Omega \to \mathscr{D}_t$, taking $y \to x(t,y)$. The Euclidean metric δ_{ij} in $\mathscr{D}_t$ then induces a metric

$$g_{ab}(t,y) = \delta_{ij}\frac{\partial x^i}{\partial y^a}\frac{\partial x^j}{\partial y^b} \tag{2.2}$$

on Ω for each fixed t.

The *covariant differentiation* of a $(0,r)$ tensor $w(t,y)$, is the $(0,r+1)$ tensor:

$$\nabla_a w_{a_1\cdots a_r} = \frac{\partial w_{a_1\cdots a_r}}{\partial y^a} - \Gamma^e_{aa_1}w_{ea_2\cdots a_r} - \cdots - \Gamma^e_{aa_r}w_{a_1\cdots a_{r-1}e},$$

where Γ^c_{ab} are the Christoffel symbols:

$$\Gamma^c_{ab} = \frac{g^{cd}}{2}\left(\frac{\partial g_{bd}}{\partial y^a} + \frac{\partial g_{ad}}{\partial y^b} - \frac{\partial g_{ab}}{\partial y^d}\right) = \frac{\partial y^c}{\partial x^i}\frac{\partial^2 x^i}{\partial y^a\partial y^b}$$

with g^{ab} being the inverse of g_{ab}. For a $(0,r)$ tensor $\omega(t,x)$ expressed in the x-coordinates, the same tensor $w(t,y)$ expressed in the y-coordinates is given by

$$w_{a_1\cdots a_r}(t,y) = \frac{\partial x^{i_1}}{\partial y^{a_1}}\cdots\frac{\partial x^{i_r}}{\partial y^{a_r}}\omega_{i_1\cdots i_r}(t,x), \quad x = x(t,y),$$

and by the transformation properties for tensors,

$$\nabla_a w_{a_1\cdots a_r}(t,y) = \frac{\partial x^i}{\partial y^a}\frac{\partial x^{i_1}}{\partial y^{a_1}}\cdots\frac{\partial x^{i_r}}{\partial y^{a_r}}\frac{\partial \omega_{i_1\cdots i_r}(t,x)}{\partial x^i}.$$

In this way, the norms of tensors are invariant under change of coordinates:

$$g^{a_1b_1}\cdots g^{a_rb_r}w_{a_1\cdots a_r}w_{b_1\cdots b_r} = \delta^{i_1j_1}\cdots\delta^{i_rj_r}\omega_{i_1\cdots i_r}\omega_{j_1\cdots j_r}. \tag{2.3}$$

Since the curvature vanishes in the x-coordinates, it must do so in the y-coordinates, and hence

$$[\nabla_a, \nabla_b] = 0.$$

For a tensor $w(t,y)$, set

$$w_{a\cdots}{}^{b}{}_{\cdots c} = g^{bd}w_{a\cdots d\cdots c}.$$

The material derivative is defined as

$$D_t = \partial_t|_{x=const.} + v^k\partial_k = \partial_t|_{y=const.}, \quad \partial_k = \frac{\partial}{\partial x^k} = \frac{\partial y^a}{\partial x^k}\frac{\partial}{\partial y^a}.$$

Let α be a $(0,s)$ tensor and β be a $(0,r)$ tensor. Then $\alpha\widetilde{\otimes}\beta$ is used to denote some partial symmetrization of the tensor product $\alpha\otimes\beta$, that is, a sum over some subset of the permutations of the indices divided by the number of permutations in that subset. Moreover $\alpha\tilde{\cdot}\beta$ is used to denote a partial symmetrization of the dot product $\alpha\cdot\beta$, which in turn is defined to be a contraction of the last index of α with the first index of β: $(\alpha\cdot\beta)_{i_1\cdots i_{r+s-2}} = g^{ij}\alpha_{i_1\cdots i_{s-1}i}\beta_{ji_s\cdots i_{r+s-2}}$.

The following lemmas are for temporal derivatives of the change of coordinates and commutators between temporal derivative and spatial derivatives, which are Lemmas 2.1 and 2.4 in [11], and will be used to calculate the higher order equations in Lagrangian coordinates.

Lemma 2.1. *Let $x = x(t,y)$ be the change of variables given by (2.1), and let g_{ab} be the metric given by (2.2). Let $v_i = \delta_{ij}v^j = v^i$ and $d\mu_g = \sqrt{\det g}dy$. Set*

$$u_a(t,y) = \frac{\partial x^j}{\partial y^a}v_j(t,x), \quad h_{ab} = \frac{1}{2}(\nabla_a u_b + \nabla_b u_a), \quad h^{ab} = g^{ac}g^{bd}h_{cd}, \quad \mathrm{div}u = g^{ab}\nabla_a u_b.$$

Then,

$$D_t\frac{\partial x^i}{\partial y^a} = \frac{\partial x^k}{\partial y^a}\frac{\partial v_i}{\partial x^k}, \quad D_t\frac{\partial y^a}{\partial x^i} = -\frac{\partial y^a}{\partial x^k}\frac{\partial v_k}{\partial x^i}, \tag{2.4a}$$

$$D_t g_{ab} = 2h_{ab}, \quad D_t g^{ab} = -2h^{ab}, \quad D_t d\mu_g = g^{ab}h_{ab}d\mu_g = (\mathrm{div}u)d\mu_g. \tag{2.4b}$$

Lemma 2.2. *Let $w_{a_1\cdots a_r}$ be a (0,r) tensor, q be a function, and $\Delta = g^{cd}\nabla_c\nabla_d$. Then,*

$$[D_t, \nabla_a]w_{a_1\cdots a_r} = -(\nabla_{a_1}\nabla_a u^e)w_{ea_2\cdots a_r} - \cdots - (\nabla_{a_r}\nabla_a u^e)w_{a_1\cdots a_{r-1}e}, \tag{2.5a}$$

$$[D_t, \Delta]q = -2h^{ab}\nabla_a\nabla_b q - (\Delta u^e)\nabla_e q, \tag{2.5b}$$

Furthermore,

$$[D_t, \nabla^r]q = -\sum_{s=1}^{r-1}\binom{r}{s+1}(\nabla^{s+1}u)\cdot\nabla^{r-s}q. \tag{2.6}$$

2.2. **The boundary geometry and regularity.** As in [11], we extend the normal to the boundary to the interior by a geodesic extension, which enables us to define a pseudo-Riemann metric in the whole domain whose restriction on the boundary is then the induced metric on the tangential space to the boundary. Using this induced metric, we can define the orthogonal projection of a tensor to the boundary, the covariant differentiation on the boundary, and the second fundamental form of the boundary as follows:

Definition 2.3. *Let $d(t,y) = \mathrm{dist}_g(y, \partial\Omega)$ be the geodesic distance to the boundary, which is the same as the Euclidean distance in the x-variables, and η be the smooth cut-off function given by Definition 1.3. Set $N_a(t,y) = \nabla_a d(t,y)$ and $N^a(t,y) = g^{ab}(t,y)N_b(t,y)$. Define*

$$\zeta^{ab}(t,y) = g^{ab}(t,y) - \widetilde{N}^a(t,y)\widetilde{N}^b(t,y), \quad \text{where} \quad \widetilde{N}^a(t,y) = \eta(d(t,y))N^a(t,y).$$

In particular, ζ gives the induced metric on the tangent space to the boundary:

$$\zeta_{ab} = g_{ab} - N_a N_b \quad \text{and} \quad \zeta^{ab} = g^{ab} - N^a N^b \quad \text{on} \quad \partial\Omega.$$

The orthogonal projection of a $(0,r)$ tensor $w(t,y)$ to the boundary is given by

$$(\Pi w)_{a_1\cdots a_r} = \zeta_{a_1}^{c_1}\cdots\zeta_{a_r}^{c_r} w_{c_1\cdots c_r}, \quad \text{where} \quad \zeta_a^c = \delta_a^c - N_a N^c.$$

The covariant differentiation on the boundary $\overline{\nabla}$ is given by $\overline{\nabla}_a = \zeta_a^c \nabla_c$. The second fundamental form of the boundary is given by $\theta_{ab}(t,y) = \overline{\nabla}_a N_b$.

It follows from Definitions 1.1, 1.3 and 2.3 that

$$N_a(t,y) = \frac{\partial x^j}{\partial y^a}\mathcal{N}_j(t,x) \quad \text{and} \quad \theta_{ab}(t,y) = \frac{\partial x^i}{\partial y^a}\frac{\partial x^j}{\partial y^b}\theta_{ij}(t,x).$$

Definition 2.4. *For the multi-indices $I = (i_1, \cdots, i_r)$ and $J = (j_1, \cdots, j_r)$, set $g^{IJ} = g^{i_1 j_1}\cdots g^{i_r j_r}$ and $\zeta^{IJ} = \zeta^{i_1 j_1}\cdots\zeta^{i_r j_r}$. If α and β are $(0,r)$ tensors, define*

$$< \alpha, \beta >= g^{IJ}\alpha_I \beta_J \quad \text{and} \quad |\alpha|^2 =< \alpha, \alpha >= g^{IJ}\alpha_I\alpha_J.$$

Then for the projection $(\Pi\beta)_I = \zeta_I^J \beta_J$,

$$< \Pi\alpha, \Pi\beta >= \zeta^{IJ}\alpha_I\beta_J \quad \text{and} \quad |\Pi\alpha|^2 = \zeta^{IJ}\alpha_I\alpha_J \quad \text{on} \quad \partial\Omega.$$

The L^p-norms of a $(0,r)$-tensor α on Ω and $\partial\Omega$ are denoted, respectively, by $\|\alpha\|_{L^p}$ and $|\alpha|_{L^p}$:

$$\|\alpha\|_{L^p} = \left(\int_\Omega |\alpha|^p d\mu_g\right)^{1/p} \quad \text{for} \quad 1 \le p < \infty, \quad \|\alpha\|_{L^\infty} = \text{ess sup}_\Omega |\alpha|$$

and

$$|\alpha|_{L^p} = \left(\int_{\partial\Omega} |\alpha|^p d\mu_\zeta\right)^{1/p} \quad \text{for} \quad 1 \le p < \infty, \quad |\alpha|_{L^\infty} = \text{ess sup}_{\partial\Omega}|\alpha|.$$

The following Lemma shows that ι_1 given in Definition 1.6 is equivalent to ι_0 in conjunction with a bound of the second fundamental form.

Lemma 2.5. (Lemma 3.6 in [11]) *Suppose that $|\theta| \le K$, and let ι_0 and ι_1 be as in Definitions 1.2 and 1.6. Then*

$$\iota_0 \ge \min\{\iota_1/2,\ 1/K\} \quad \text{and} \quad \iota_1 \ge \min\{2\iota_0,\ \epsilon_1/K\}. \tag{2.7}$$

The advantage to using ι_1, instead of ι_0, is that it is easier to control the evolution off. The estimates for first-order derivatives of the extension of the normal to the interior are as follows.

Lemma 2.6. (Lemma 3.9 in [11]) *Let N be the unit normal to $\partial\Omega$ and $d\mu_\zeta = \sqrt{(\sum N_a^2)^{-1}\mathrm{det} g}dS$ with dS being the Euclidean surface measure. On $[0,T] \times \partial\Omega$ we have*

$$D_t N_a = h_{NN}N_a, \quad D_t N^c = -2h_d^c N^d + h_{NN}N^c, \quad \text{where} \quad h_{NN} = N^a N^b h_{ab}, \tag{2.8a}$$

$$D_t \zeta^{ab} = -2\zeta^{ac}\zeta^{bd}h_{cd}, \tag{2.8b}$$

$$D_t d\mu_\zeta = \left(g^{ab}h_{ab} - h_{NN}\right)d\mu_\zeta = (\mathrm{div} u - h_{NN})\, d\mu_\zeta. \tag{2.8c}$$

Lemma 2.7. (Lemma 3.11 in [11]) *With the notations in Definitions 1.2, 2.3 and 2.4, we have*

$$\|\nabla\zeta\|_{L^\infty} \le 512\left(|\theta|_{L^\infty} + 1/\iota_0\right) \quad \text{and} \quad \|D_t\zeta\|_{L^\infty} \le 128\|h\|_{L^\infty}. \tag{2.9}$$

2.3. **Some boundary estimates.**

Lemma 2.8. *Let $q = q_b$ on $\partial\Omega$ with q_b being a constant, then for $r = 2, 3, 4$,*

$$\begin{aligned} |\Pi\nabla^r q|_{L^2} \leq & 2|\nabla_N q|_{L^\infty}|\overline{\nabla}^{r-2}\theta|_{L^2} + C\sum_{k=1}^{r-1}|\theta|^k_{L^\infty}|\nabla^{r-k}q|_{L^2} \\ & + C\sum_{k=1}^{r-3}|\theta|_{L^\infty}|\nabla_N q|_{L^\infty}|\overline{\nabla}^k\theta|_{L^2}, \end{aligned} \tag{2.10}$$

where C is a positive number. If, in addition, $|\nabla_N q| \geq \epsilon$ and $|\nabla_N q| \geq 2\epsilon|\nabla_N q|_{L^\infty}$ on $\partial\Omega$ for a certain positive constant ϵ, then there exists a positive number C such that

$$\begin{aligned} |\overline{\nabla}^{r-2}\theta|_{L^2} \leq & \epsilon^{-2}|\Pi\nabla^r q|_{L^2} + C\epsilon^{-3}\sum_{k=1}^{r-1}|\theta|^k_{L^\infty}|\nabla^{r-k}q|_{L^2} \\ & + C\epsilon^{-2}\sum_{k=1}^{r-3}|\theta|_{L^\infty}|\nabla_N q|_{L^\infty}|\overline{\nabla}^k\theta|_{L^2}, \quad r = 2, 3, 4. \end{aligned} \tag{2.11}$$

Proof. Simple calculations give that on $\partial\Omega$,

$$\begin{aligned} \Pi\nabla^2 q = & (\nabla_N q)\theta, \quad \Pi\nabla^3 q = (\nabla_N q)\overline{\nabla}\theta + 3(\overline{\nabla}\nabla_N q)\widetilde{\otimes}\theta + 2(\theta\,\tilde{\cdot}\,\theta)\widetilde{\otimes}\nabla q, \\ \Pi\nabla^4 q = & (\nabla_N q)\overline{\nabla}^2\theta + 4(\overline{\nabla}\theta)\widetilde{\otimes}\overline{\nabla}\nabla_N q + 6\theta\widetilde{\otimes}\overline{\nabla}^2\nabla_N q + 3(\theta\widetilde{\otimes}\theta)\nabla_N^2 q \\ & + 7((\overline{\nabla}\theta)\,\tilde{\cdot}\,\theta)\widetilde{\otimes}\nabla q - \theta\widetilde{\otimes}(\theta\,\tilde{\cdot}\,\theta)\nabla_N q + 3((\theta\,\tilde{\cdot}\,\theta)\,\tilde{\cdot}\,\theta)\widetilde{\otimes}N\widetilde{\otimes}\nabla q \\ & + 8(\theta\,\tilde{\cdot}\,\theta)\widetilde{\otimes}(\overline{\nabla}\nabla_N q)\widetilde{\otimes}N + 3\theta\widetilde{\otimes}(\theta\,\tilde{\cdot}\,\overline{\nabla}\nabla_N q)\widetilde{\otimes}N, \end{aligned} \tag{2.12}$$

where we have used the facts that $\overline{\nabla}q = 0$, $\theta \cdot N = 0$, $(\overline{\nabla}\theta)\cdot N = -\theta\cdot\theta$ and $(\overline{\nabla}^2\theta)\cdot N = -3(\overline{\nabla}\theta)\,\tilde{\cdot}\,\theta$. Clearly, (2.10) and (2.11) hold for $r = 2, 3$, because of $\overline{\nabla}\nabla_N q = N^e\overline{\nabla}\nabla_e q$ and

$$|\Pi\nabla^3 q - (\nabla_N q)\overline{\nabla}\theta|_{L^2} \leq 3|\theta|_{L^\infty}|\nabla^2 q|_{L^2} + 2|\theta|^2_{L^\infty}|\nabla q|_{L^2}. \tag{2.13}$$

For $r = 4$, we first derive from (A-1a), Hölder's inequality and Young's inequality that for any positive constant δ,

$$\begin{aligned} ||\overline{\nabla}\theta||\overline{\nabla}\nabla_N q||_{L^2} \leq & |\overline{\nabla}\theta|_{L^4}|\overline{\nabla}\nabla_N q|_{L^4} \leq C|\theta|^{1/2}_{L^\infty}|\overline{\nabla}^2\theta|^{1/2}_{L^2}|\nabla_N q|^{1/2}_{L^\infty}|\overline{\nabla}^2\nabla_N q|^{1/2}_{L^2} \\ \leq & (\delta/4)|\nabla_N q|_{L^\infty}|\overline{\nabla}^2\theta|_{L^2} + C\delta^{-1}|\theta|_{L^\infty}|\overline{\nabla}^2\nabla_N q|_{L^2}; \end{aligned}$$

which, together with $\nabla_N^2 q = N^e\nabla_N\nabla_e q$ and $|\overline{\nabla}^2\nabla_N q| \leq |\nabla^3 q| + 3|\theta||\nabla^2 q|$, gives that for any positive constant δ,

$$\begin{aligned} |\Pi\nabla^4 q - & (\nabla_N q)\overline{\nabla}^2\theta|_{L^2} \leq \delta|\nabla_N q|_{L^\infty}|\overline{\nabla}^2\theta|_{L^2} \\ & + 7|\theta|_{L^\infty}|\nabla_N q|_{L^\infty}|\overline{\nabla}\theta|_{L^2} + C\delta^{-1}\sum_{k=1}^{3}|\theta|^k_{L^\infty}|\nabla^{4-k}q|_{L^2}. \end{aligned} \tag{2.14}$$

Here C is a positive number. Clearly, choose $\delta = 1$ in (2.14) to prove (2.10). Note that

$$2\epsilon|\nabla_N q|_{L^\infty}|\overline{\nabla}^2\theta|_{L^2} \leq |(\nabla_N q)\overline{\nabla}^2\theta|_{L^2} \leq |\Pi\nabla^4 q|_{L^2} + |\Pi\nabla^4 q - (\nabla_N q)\overline{\nabla}^2\theta|_{L^2},$$

we then prove (2.11) by choosing $\delta = \epsilon$ in (2.14).

□

Lemma 2.9. *Let $q = q_b$ on $\partial\Omega$ with q_b being a constant, then*

$$\begin{aligned}|\Pi\nabla^5 q|_{L^2} \le & 2|\nabla_N q|_{L^\infty}|\overline{\nabla}^3\theta|_{L^2} + C\sum_{k=1}^{4}|\theta|_{L^\infty}^k|\nabla^{5-k}q|_{L^2} \\ & + C(K_1,|\theta|_{L^\infty})(|\overline{\nabla}^2\theta|_{L^2}+|\overline{\nabla}\theta|_{L^2})\sum_{k=1}^{4}|\nabla^k q|_{L^2}.\end{aligned} \tag{2.15}$$

If, in addition, $|\nabla_N q| \ge \epsilon$ and $|\nabla_N q| \ge 2\epsilon|\nabla_N q|_{L^\infty}$ on $\partial\Omega$ for a certain positive constant ϵ,

$$\begin{aligned}|\overline{\nabla}^3\theta|_{L^2} \le & \epsilon^{-2}|\Pi\nabla^5 q|_{L^2} + \epsilon^{-3}C\sum_{k=1}^{4}|\theta|_{L^\infty}^k|\nabla^{5-k}q|_{L^2} \\ & + \epsilon^{-3}C(K_1,|\theta|_{L^\infty})(|\overline{\nabla}^2\theta|_{L^2}+|\overline{\nabla}\theta|_{L^2})\sum_{k=1}^{4}|\nabla^k q|_{L^2}.\end{aligned} \tag{2.16}$$

Proof. This lemma can be shown in a similar way to proving Lemma 2.8 by noticing the following fact:

$$\begin{aligned}& ||\overline{\nabla}\theta||\overline{\nabla}^2\nabla_N q||_{L^2} + ||\overline{\nabla}^2\theta||\overline{\nabla}\nabla_N q||_{L^2} \le |\overline{\nabla}\theta|_{L^6}|\overline{\nabla}^2\nabla_N q|_{L^3} + |\overline{\nabla}^2\theta|_{L^3}|\overline{\nabla}\nabla_N q|_{L^6} \\ \le & C|\theta|_{L^\infty}^{2/3}|\overline{\nabla}^3\theta|_{L^2}^{1/3}|\nabla_N q|_{L^\infty}^{1/3}|\overline{\nabla}^3\nabla_N q|_{L^2}^{2/3} + C|\theta|_{L^\infty}^{1/3}|\overline{\nabla}^3\theta|_{L^2}^{2/3}|\nabla_N q|_{L^\infty}^{2/3}|\overline{\nabla}^3\nabla_N q|_{L^2}^{1/3} \\ \le & \delta|\nabla_N q|_{L^\infty}|\overline{\nabla}^3\theta|_{L^2} + C\delta^{-1}|\theta|_{L^\infty}|\overline{\nabla}^3\nabla_N q|_{L^2}\end{aligned} \tag{2.17}$$

for any positive constant δ, and

$$||\nabla^2 q||\overline{\nabla}\theta||_{L^2} \le |\nabla^2 q|_{L^\infty}|\overline{\nabla}\theta|_{L^2} \le C(K_1)|\overline{\nabla}\theta|_{L^2}\sum_{k=2}^{4}|\nabla^k q|_{L^2}. \tag{2.18}$$

Here (2.17) follows from (A-1a), Hölder's inequality and Young's inequality, and (2.18) follows from (A-2b).

□

Remark 2.10. *ϵ appearing on the right-hand side of (2.11) and (2.16) can be chosen as*

$$\epsilon = |(\nabla_N q)^{-1}|_{L^\infty}^{-1}\min\left\{1,\ 2^{-1}|\nabla_N q|_{L^\infty}^{-1}\right\}. \tag{2.19}$$

In particular, it follows from (2.12) and (2.13) that

$$|\theta|_{L^s} \le |(\nabla_N q)^{-1}|_{L^\infty}|\Pi\nabla^2 q|_{L^s}, \quad 2 \le s \le \infty, \tag{2.20a}$$

$$|\overline{\nabla}\theta|_{L^2} \le |(\nabla_N q)^{-1}|_{L^\infty}\left(|\Pi\nabla^3 q|_{L^2} + 3\sum_{k=1}^{2}|\theta|_{L^\infty}^k|\nabla^{3-k}q|_{L^2}\right). \tag{2.20b}$$

Lemma 2.11. *Let $q = q_b$ on $\partial\Omega$ with q_b being a constant. If $|\theta| + 1/\iota_0 \le K$ and $\iota_1 \ge 1/K_1$, then for any $\delta > 0$,*

$$|\nabla q|_{L^\infty} \le \begin{cases}\delta\|\nabla\Delta q\|_{L^2} + C(\delta^{-1},K,K_1,|\theta|_{L^2},\mathrm{Vol}\Omega)\|\Delta q\|_{L^2}, & n=2;\\ \delta\|\nabla^2\Delta q\|_{L^2} + C(\delta^{-1},K,K_1,|\overline{\nabla}\theta|_{L^2},\mathrm{Vol}\Omega)(\|\nabla\Delta q\|_{L^2}+\|\Delta q\|_{L^2}), & n=3.\end{cases} \tag{2.21}$$

Proof. When $n = 3$, it follows from (2.13) and (A-2b) that for any $\delta > 0$,

$$|\Pi\nabla^3 q|_{L^2} \le \delta|\nabla^3 q|_{L^2} + C(\delta^{-1},K,K_1,|\overline{\nabla}\theta|_{L^2})(|\nabla^2 q|_{L^2}+|\nabla q|_{L^2}). \tag{2.22}$$

In view of (A-5c), (A-3c) and (A-5a), we see that for any $\delta_1 > 0$,

$$|\nabla^2 q|_{L^2} \le \delta_1|\Pi\nabla^3 q|_{L^2} + C(\delta_1^{-1},K,\mathrm{Vol}\Omega)(\|\nabla\Delta q\|_{L^2}+\|\Delta q\|_{L^2}), \tag{2.23}$$

$$|\nabla q|_{L^2} \le C(K)(\|\nabla^2 q\|_{L^2}+\|\nabla q\|_{L^2}) \le C(K,\mathrm{Vol}\Omega)\|\Delta q\|_{L^2}. \tag{2.24}$$

Substitute (2.23) and (2.24) into (2.22) and choose suitable small δ_1 to obtain for any $\delta > 0$,

$$2^{-1}|\Pi\nabla^3 q|_{L^2} \le \delta|\nabla^3 q|_{L^2} + C(\delta^{-1}, K, K_1, |\overline{\nabla}\theta|_{L^2}, \mathrm{Vol}\Omega)(\|\nabla\Delta q\|_{L^2} + \|\Delta q\|_{L^2}).$$

This, together with (A-5b), gives

$$|\nabla^3 q|_{L^2} \le C(K, \mathrm{Vol}\Omega)\|\nabla^2\Delta q\|_{L^2} + C(K, K_1, |\overline{\nabla}\theta|_{L^2}, \mathrm{Vol}\Omega)(\|\nabla\Delta q\|_{L^2} + \|\Delta q\|_{L^2}). \tag{2.25}$$

So, (2.21) follows from (A-2b), (2.25) (2.23) and (2.24) in the case of $n = 3$. Similarly, (2.21) can be shown when $n = 2$.

□

3. Proof of Theorem 1.4

Let $u(t, y)$ be the same tensor of the velocity $v(t, x)$ expressed in the y-coordinates, that is, $u_a(t, y) = \frac{\partial x^j}{\partial y^a} v_j(t, x)$. Let w be a $(0, 1)$ tensor and define a scalar $\mathrm{div} w = g^{ab}\nabla_a w_b$ and a $(0, 2)$ tensor $\mathrm{curl} w_{ab} = \nabla_a w_b - \nabla_b w_a$. Then, system (1.2) can be rewritten in Lagrangian coordinates as

$$D_t u_a + \mathcal{T}\nabla_a p = (\nabla_a u_c)u^c, \tag{3.1a}$$

$$\mathrm{div} u = \Delta\mathcal{T}, \quad D_t\mathcal{T} = \mathcal{T}\Delta\mathcal{T}. \tag{3.1b}$$

It follows from (3.1a) and (2.5a) that

$$D_t\nabla_b u_a + \mathcal{T}\nabla_b\nabla_a p = -(\nabla_b\mathcal{T})\nabla_a p + (\nabla_a u_e)\nabla_b u^e, \tag{3.2}$$

which implies

$$D_t \mathrm{curl} u_{ab} = (\nabla_b\mathcal{T})\nabla_a p - (\nabla_a\mathcal{T})\nabla_b p, \tag{3.3a}$$

$$\mathcal{T}\Delta p + (\nabla\mathcal{T})\cdot\nabla p = -D_t \mathrm{div} u - (\nabla_e u)\cdot\nabla u^e. \tag{3.3b}$$

Moreover, it yields from (3.1b) and (2.5b) that

$$D_t \mathrm{div} u - \mathcal{T}\Delta \mathrm{div} u = (\nabla\mathcal{T})\cdot\nabla \mathrm{div} u - 2 < \nabla^2\mathcal{T}, \nabla u > + (\mathrm{div} u)^2 + (\nabla\mathcal{T})\cdot(\nabla \mathrm{div} u - \Delta u). \tag{3.4}$$

It should be noted that $\Delta u_a - \nabla_a \mathrm{div} u = g^{ce}\nabla_c \mathrm{curl} u_{ea}$ in the last term on the right hand side of (3.4) .

Let

$$E_r^a(t) = \int_\Omega \mathcal{T}^{-1} g^{ab}\zeta^{cd}\mathcal{Q}(\nabla^{r-1}\nabla_c u_a, \nabla^{r-1}\nabla_d u_b)d\mu_g + \int_\Omega |\nabla^{r-1}\mathrm{curl} u|^2 d\mu_g$$
$$+ \int_{\partial\Omega} \zeta^{cd}\mathcal{Q}(\nabla^{r-1}\nabla_c p, \nabla^{r-1}\nabla_d p)(-\nabla_N p)^{-1} d\mu_\zeta, \quad r \ge 1, \tag{3.5a}$$

$$E_r^d(t) = \int_\Omega |\nabla D_t^{r-1}\mathrm{div} u|^2 d\mu_g, \quad r \ge 1, \tag{3.5b}$$

where $\mathcal{Q}(\alpha, \beta) = \varsigma^{IJ}\alpha_I\beta_J$. Then,

$$E_0(t) = \int_\Omega \mathcal{T}^{-1}|u|^2 d\mu_g, \quad \text{and} \quad E_r(t) = E_r^a(t) + E_r^d(t) \quad \text{for} \quad r \ge 1. \tag{3.6}$$

We make the following *a priori* assumptions:

$$\underline{V} \le \mathrm{Vol}\Omega(t) \le \overline{V} \qquad \text{on} \quad [0, T], \tag{3.7a}$$

$$|\theta| + 1/\iota_0 \le K \qquad \text{on} \quad [0, T]\times\partial\Omega, \tag{3.7b}$$

$$-\nabla_N p \ge \epsilon_b > 0 \qquad \text{on} \quad [0, T]\times\partial\Omega, \tag{3.7c}$$

$$\sum_{i=1}^{n-1}(|\nabla_N D_t^i p| + |\nabla_N D_t^i \mathrm{div} u|) + |\nabla^2 p| \le L \qquad \text{on} \quad [0, T]\times\partial\Omega, \tag{3.7d}$$

$$|\nabla p| + |\nabla u| + |\nabla\mathcal{T}| \le M \qquad \text{in} \quad [0, T]\times\Omega, \tag{3.7e}$$

$$|\nabla \mathrm{div} u| + |D_t \mathrm{div} u| + |\nabla^2 \mathcal{T}| \le \widetilde{M} \quad \text{in } [0,T]\times\Omega, \tag{3.7f}$$

for some positive constants $\underline{V}$, $\overline{V}$, K, ϵ_b, L, M and $\widetilde{M}$, where $\mathrm{Vol}\Omega(t) = \int_\Omega d\mu_g = \int_{\mathscr{D}_t} dx = \mathrm{Vol}\mathscr{D}_t$. Let ι_0 and ι_1 be as in Definitions 1.2 and 1.6. Then we have, due to (2.7) and (3.7b), that $\iota_1^{-1} \le \max\{\epsilon_1^{-1}|\theta|_{L^\infty},\ (2\iota_0)^{-1}\} \le \epsilon_1^{-1}K$, which means

$$\iota_1^{-1}(t) \le K_1 \quad \text{on } [0,T], \quad \text{where } K_1 = \epsilon_1^{-1}K. \tag{3.8}$$

Here $\epsilon_1 > 0$ is the fixed number given in Definition 1.6. It follows from (1.4), (3.1b) and the maximum principle that

$$p = 0, \quad \mathcal{T} = \mathcal{T}_b, \quad D_t\mathcal{T} = 0 \quad \text{and} \quad \mathrm{div} u = 0 \quad \text{on } [0,T]\times\partial\Omega, \tag{3.9}$$

$$\underline{\mathcal{T}} \le \mathcal{T} \le \overline{\mathcal{T}} \quad \text{in } [0,T]\times\overline{\Omega}, \tag{3.10}$$

where $\underline{\mathcal{T}} = \min\{\min_{y\in\Omega}\mathcal{T}(0,y),\ \mathcal{T}_b\}$ and $\overline{\mathcal{T}} = \max\{\max_{y\in\Omega}\mathcal{T}(0,y),\ \mathcal{T}_b\}$.

Theorem 3.1. *Let $n = 2,3$ and $1 \le r \le n+2$. Then there exist continuous functions $\mathcal{F}_r$ with $\mathcal{F}_r\big|_{t=0} = 1$ such that any smooth solutions to the free surface problem (1.2)-(1.4) for $0 \le t \le T$ satisfying the a priori assumptions (3.7) also satisfy the following energy estimates*

$$E_0(t) \le E_0(0) + \left(\mathcal{F}_1\left(t, \overline{V}, K, \epsilon_b^{-1}, L, M, \widetilde{M}, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}\right) - 1\right)\sum_{s=1}^{2} E_s(0), \tag{3.11a}$$

$$\sum_{s=1}^{2} E_s(t) \le \mathcal{F}_2\left(t, \overline{V}, K, \epsilon_b^{-1}, L, M, \widetilde{M}, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}\right)\sum_{s=1}^{2} E_s(0), \tag{3.11b}$$

$$\sum_{s=1}^{r} E_s(t) \le \mathcal{F}_r\left(t, \overline{V}, K, \epsilon_b^{-1}, L, M, \widetilde{M}, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}, E_1(0), \cdots, E_{r-1}(0)\right)\sum_{s=1}^{r} E_s(0), \ r \ge 3. \tag{3.11c}$$

Theorem 3.2. *Let $\mathrm{Vol}\mathscr{D}_0, K_0, \underline{\epsilon_0}$ and M_0 be defined by (1.13), and $n = 2,3$. Then there are continuous functions $\mathscr{T}_n$ such that if $T \le \mathscr{T}_n\left(\mathrm{Vol}\mathscr{D}_0, K_0, \underline{\epsilon_0}^{-1}, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}, M_0, E_0(0), \cdots, E_{n+2}(0)\right)$, then any smooth solutions to the free surface problem (1.2)-(1.4) for $0 \le t \le T$ satisfy*

$$\sum_{s=0}^{n+2} E_s(t) \le 2\sum_{s=0}^{n+2} E_s(0), \quad 0 \le t \le T, \tag{3.12a}$$

$$2^{-1}\mathrm{Vol}\mathscr{D}_0 \le \mathrm{Vol}\Omega(t) \le 2\mathrm{Vol}\mathscr{D}_0, \quad 0 \le t \le T, \tag{3.12b}$$

$$|\theta(t,\cdot)|_{L^\infty} + \iota_0^{-1}(t) \le 18K_0, \quad 0 \le t \le T, \tag{3.12c}$$

$$-\nabla_N p(t,y) \ge 2^{-1}\underline{\epsilon_0} \quad \text{for } y \in \partial\Omega, \quad 0 \le t \le T, \tag{3.12d}$$

$$\|\nabla p(t,\cdot)\|_{L^\infty} + \|\nabla u(t,\cdot)\|_{L^\infty} + \|\nabla\mathcal{T}(t,\cdot)\|_{L^\infty} \le 2M_0, \quad 0 \le t \le T, \tag{3.12e}$$

$$\begin{aligned}
&\sum_{i=1}^{n-1}\left(|\nabla D_t^i p(t,\cdot)|_{L^\infty} + |\nabla D_t^i \mathrm{div} u(t,\cdot)|_{L^\infty}\right) + |\nabla^2 p(t,\cdot)|_{L^\infty} \\
&+ \|\nabla \mathrm{div} u(t,\cdot)\|_{L^\infty} + \|D_t \mathrm{div} u(t,\cdot)\|_{L^\infty} + \|\nabla^2\mathcal{T}(t,\cdot)\|_{L^\infty} \\
&\quad \le C\left(\mathrm{Vol}\mathscr{D}_0, K_0, \underline{\epsilon_0}^{-1}, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}, M_0, E_0(0), \cdots, E_{n+2}(0)\right), \quad 0 \le t \le T.
\end{aligned} \tag{3.12f}$$

Clearly, Theorem 1.4 is a conclusion of this theorem. (Indeed, the compatibility condition $\mathcal{T}(0,y) = \mathcal{T}_b$ on $\partial\Omega$ implies that $\underline{\mathcal{T}} = \min_{y\in\Omega}\mathcal{T}(0,y)$ and $\overline{\mathcal{T}} = \max_{y\in\Omega}\mathcal{T}(0,y)$.) Corollary 1.5 will follow from the regularity estimates (3.47), (3.48) and (3.49) shown in Subsection 3.3.

3.1. **Estimates on temporal derivatives of $E_k^d(t)$.** In this subsection, we prove

Proposition 3.3. *It holds that*

$$\frac{d}{dt}E_1^d(t) \leq C_1\left(\|\nabla u\|_{L^2}^2+\|\nabla \mathrm{div}u\|_{L^2}^2+\|\nabla \mathrm{curl}u\|_{L^2}^2\right), \tag{3.13a}$$

$$\begin{aligned}\frac{d}{dt}E_k^d(t) \leq& C_k\sum_{j=1}^{k-1}\sum_{i=0}^{k-j}\left(\|\nabla^i D_t^j \mathrm{div}u\|_{L^2}^2+\|\nabla^{i+1}D_t^{j-1}p\|_{L^2}^2\right)+C_k\sum_{i=1}^{k}\|\nabla^i u\|_{L^2}^2\\ &+C_k\|\nabla^k \mathrm{divu}\|_{L^2}^2, \quad 2\leq k\leq 5,\end{aligned} \tag{3.13b}$$

where

$$\begin{aligned}&C_1=C(\|\mathcal{T}^{-1}\|_{L^\infty},\|\nabla u\|_{L^\infty},\|\nabla\mathcal{T}\|_{L^\infty},\|\nabla^2\mathcal{T}\|_{L^\infty}), \quad C_2=C(\mathcal{A}_1),\\ &C_3=C(\mathcal{A}_1), \quad C_4=C(\mathcal{A}_1,\mathrm{Vol}\Omega,\|\nabla D_t p\|_{L^\infty}^2,E_3^d),\\ &C_5=C(\mathcal{A}_1,\mathrm{Vol}\Omega,\|\nabla D_t p\|_{L^\infty}^2,\|\nabla^2\mathrm{div}u\|_{L^\infty}^2,\|D_t^2\mathrm{div}u\|_{L^\infty}^2,\|\nabla^2 u\|_{L^\infty}^2,E_4^d),\end{aligned}$$

with $\mathcal{A}_1$ being given by

$$\begin{aligned}\mathcal{A}_1=(&K_1,\|\mathcal{T}^{-1}\|_{L^\infty},\|\mathcal{T}\|_{L^\infty},\|\nabla u\|_{L^\infty},\|\nabla\mathcal{T}\|_{L^\infty},\|\nabla p\|_{L^\infty},\|\nabla^2\mathcal{T}\|_{L^\infty},\\ &\|\nabla\mathrm{div}u\|_{L^\infty},\|D_t\mathrm{div}u\|_{L^\infty}).\end{aligned} \tag{3.14}$$

To prove Proposition 3.3, we need the fact that

$$\|D_t^{r-1}\mathrm{div}u\|_{L^2}^2 \leq C(\mathrm{Vol}\Omega)E_r^d(t), \quad r\geq 1, \tag{3.15}$$

which follows from (3.9) and (A-4), and the following two lemmas on the higher-order derived equations.

Lemma 3.4. *For $k\geq 1$, it holds that*

$$D_t^{k+1}\mathrm{div}u-\mathcal{T}\Delta D_t^k\mathrm{div}u=(\nabla\mathcal{T})\cdot\nabla D_t^k\mathrm{div}u+2<\nabla^2\mathcal{T},\mathcal{T}\nabla^2D_t^{k-1}p>+\mathcal{R}_{k+1}, \tag{3.16}$$

where

$$\begin{aligned}\mathcal{R}_2=&\{(D_t\mathcal{T})\Delta\mathrm{div}u+\mathcal{T}[D_t,\Delta]\mathrm{div}u\}+\{D_t((\nabla\mathcal{T})\cdot\nabla\mathrm{div}u)-(\nabla\mathcal{T})\cdot D_t\nabla\mathrm{div}u\}\\ &-2\{D_t<\nabla^2\mathcal{T},\nabla u>-<\nabla^2\mathcal{T},D_t\nabla u>\}-2<\nabla^2\mathcal{T},D_t\nabla u+\mathcal{T}\nabla^2p>\\ &+D_t\{(\mathrm{div}u)^2+g^{ab}(\nabla_a\mathcal{T})g^{ce}\nabla_c\mathrm{curl}u_{be}\},\end{aligned}$$

and

$$\begin{aligned}\mathcal{R}_{k+1}=&\{(D_t\mathcal{T})\Delta D_t^{k-1}\mathrm{divu}+\mathcal{T}[D_t,\Delta]D_t^{k-1}\mathrm{divu}\}+\{D_t((\nabla\mathcal{T})\cdot\nabla D_t^{k-1}\mathrm{div}u)\\ &-(\nabla\mathcal{T})\cdot D_t\nabla D_t^{k-1}\mathrm{divu}\}+2\{D_t<\nabla^2\mathcal{T},\mathcal{T}\nabla^2D_t^{k-2}p>\\ &-<\nabla^2\mathcal{T},\mathcal{T}D_t\nabla^2D_t^{k-2}p>\}+2<\nabla^2\mathcal{T},\mathcal{T}[D_t,\nabla^2]D_t^{k-2}p>+D_t\mathcal{R}_k, \quad k\geq 2.\end{aligned}$$

Proof. This lemma can be proved by induction. Clearly, (3.16) holds for $k=1$ by taking D_t of (3.4) and noting that $\Delta u_a-\nabla_a\mathrm{div}u=g^{ce}\nabla_c\mathrm{curl}u_{ea}$. Assume that (3.16) holds for $k=j-1$, that is,

$$D_t^j\mathrm{div}u-\mathcal{T}\Delta D_t^{j-1}\mathrm{div}u=(\nabla\mathcal{T})\cdot\nabla D_t^{j-1}\mathrm{div}u+2<\nabla^2\mathcal{T},\mathcal{T}\nabla^2D_t^{j-2}p>+\mathcal{R}_j, \tag{3.17}$$

then (3.16) for $k=j$ follows from taking D_t of (3.17).

□

Lemma 3.5. *For any integer $r\geq 2$, it holds that*

$$\begin{aligned}&\left|D_t\nabla^r u+\mathcal{T}\nabla^{r+1}p\right|+\left|D_t\nabla^{r-1}\mathrm{curl}u\right|+|\mathcal{T}\nabla^{r-1}\Delta p+\nabla^{r-1}D_t\mathrm{div}u|\\ &\leq C\sum_{s=0}^{r-1}\left(\left|\nabla^{1+s}u\right|\left|\nabla^{r-s}u\right|+\left|\nabla^{1+s}\mathcal{T}\right|\left|\nabla^{r-s}p\right|\right).\end{aligned} \tag{3.18}$$

Proof. This lemma can be proved similar to the proof of Lemma 6.1 in [11], so we only outline the proof and omit the details. First, we apply the following identities of commutators

$$[D_t, \partial_i] = -(\partial_i v^k)\partial_k \quad \text{and} \quad [D_t, \partial^r] = -\sum_{s=0}^{r-1} \binom{r}{s+1} (\partial^{1+s} v) \cdot \partial^{r-s}$$

to (1.2) and change coordinates to obtain

$$D_t \nabla^r u_a + \nabla^r (\mathcal{T} \nabla_a p) = (\nabla_a u_c - \nabla_c u_a) \nabla^r u^c - \sum_{s=1}^{r-2} \binom{r}{s+1} (\nabla^{1+s} u) \cdot \nabla^{r-s} u_a. \tag{3.19}$$

The estimate for curl$u$ can be shown similarly. For $r \geq 0$, it yields from taking ∇^r of (3.3b) that

$$\mathcal{T} \nabla^r \Delta p + \nabla^r D_t \text{div} u = -\left(\nabla^r (\mathcal{T} \Delta p) - \mathcal{T} \nabla^r \Delta p\right) - \nabla^r \left((\nabla \mathcal{T}) \cdot \nabla p + (\nabla_e u) \cdot \nabla u^e\right).$$

□

Proof of Proposition 3.3. In the proof we make use of a fact, which follows from (2.4b), that for a function $f = f(t, y)$,

$$\frac{d}{dt} \int_\Omega f d\mu_g = \int_\Omega (D_t f + f \text{div} u)\, d\mu_g. \tag{3.20}$$

We begin the proof by first noting that, for $k \geq 0$,

$$\begin{aligned} \frac{1}{2} D_t |\nabla D_t^k \text{div} u|^2 + \mathcal{T} |\Delta D_t^k \text{div} u|^2 &= (D_t^{k+1} \text{div} u - \mathcal{T} \Delta D_t^k \text{div} u)(-\Delta D_t^k \text{div} u) \\ &+ \text{div}\left((D_t^{k+1} \text{div} u) \nabla D_t^k \text{div} u\right) + \frac{1}{2} (D_t g^{ab}) (\nabla_a D_t^k \text{div} u) \nabla_b D_t^k \text{div} u. \end{aligned}$$

This, together with (3.20), (3.9) and (2.4b), implies that, for $k \geq 0$,

$$\begin{aligned} \frac{d}{dt} \int_\Omega |\nabla D_t^k \text{div} u|^2 d\mu_g + \int_\Omega \mathcal{T} |\Delta D_t^k \text{div} u|^2 d\mu_g &\leq 3 \|\nabla u\|_{L^\infty} \int_\Omega |\nabla D_t^k \text{div} u|^2 d\mu_g \\ &+ \|\mathcal{T}^{-1}\|_{L^\infty} \int_\Omega |D_t^{k+1} \text{div} u - \mathcal{T} \Delta D_t^k \text{div} u|^2 d\mu_g. \end{aligned} \tag{3.21}$$

Proof of (3.13a). It follows from (3.4) and $\Delta u_a - \nabla_a \text{div} u = g^{ce} \nabla_c \text{curl} u_{ea}$ that

$$\|D_t \text{div} u - \mathcal{T} \Delta \text{div} u\|_{L^2} \leq C \left(\|\nabla u\|_{L^2} + \|\nabla \text{div} u\|_{L^2} + \|\nabla \text{curl} u\|_{L^2}\right),$$

where $C = (\|\nabla u\|_{L^\infty}, \|\nabla \mathcal{T}\|_{L^\infty}, \|\nabla^2 \mathcal{T}\|_{L^\infty})$. This, together with (3.21), proves (3.13a).

Proof of (3.13b) *for* $k = 2$. It yields from (3.21), (3.16), (3.1b), (2.5b), (2.4b), (3.2) and (3.18) that

$$\frac{d}{dt} E_2^d(t) \leq C \sum_{i=1}^{2} \left(\|\nabla^i u\|_{L^2}^2 + \|\nabla^i p\|_{L^2}^2 + \|\nabla^{i-1} D_t \text{div} u\|_{L^2}^2\right) + C \|\nabla^2 \text{div} u\|_{L^2}^2, \tag{3.22}$$

where $C = C(\|\mathcal{T}^{-1}\|_{L^\infty}, \|\mathcal{T}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla p\|_{L^\infty}, \|\nabla \text{div} u\|_{L^\infty}, \|\nabla \mathcal{T}\|_{L^\infty}, \|\nabla^2 \mathcal{T}\|_{L^\infty})$.

Proof of (3.13b) *for* $k = 3$. It gives from (3.16) and Hölder's inequality that

$$\begin{aligned} \|D_t^3 \text{div} u - \mathcal{T} \Delta D_t^2 \text{div} u\|_{L^2} &\leq C \|(\nabla^2 p) \cdot \nabla^2 u\|_{L^2} + C \|(\nabla^2 p) \cdot \nabla^2 \text{div} u\|_{L^2} \\ &+ C \|(\nabla D_t \text{div} u) \cdot \nabla^2 u\|_{L^2} + C \sum_{i=0}^{1} \left(\|\nabla^i D_t^2 \text{div} u\|_{L^2} + \|\nabla^{i+1} D_t p\|_{L^2}\right) \\ &+ C \sum_{i=1}^{3} \left(\|\nabla^i u\|_{L^2} + \|\nabla^i p\|_{L^2} + \|\nabla^{i-1} D_t \text{div} u\|_{L^2}\right), \end{aligned} \tag{3.23}$$

where $C = C(\|\mathcal{T}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla\mathcal{T}\|_{L^\infty}, \|\nabla p\|_{L^\infty}, \|\nabla^2\mathcal{T}\|_{L^\infty}, \|D_t\mathrm{div}u\|_{L^\infty})$. Using Hölder's inequality, (A-1b) and Young's inequality, we have

$$\begin{aligned}
&\|(\nabla^2 p)\cdot\nabla^2 u\|_{L^2} \le \|\nabla^2 p\|_{L^4}\|\nabla^2 u\|_{L^4}\\
\le& C(K_1)\|\nabla p\|_{L^\infty}^{1/2}\left(\sum_{i=1}^{3}\|\nabla^i p\|_{L^2}\right)^{1/2}\|\nabla u\|_{L^\infty}^{1/2}\left(\sum_{i=1}^{3}\|\nabla^i u\|_{L^2}\right)^{1/2}\\
\le& C(K_1)\left(\|\nabla p\|_{L^\infty}+\|\nabla u\|_{L^\infty}\right)\sum_{i=1}^{3}\left(\|\nabla^i p\|_{L^2}+\|\nabla^i u\|_{L^2}\right).
\end{aligned}\tag{3.24}$$

Similarly,

$$\begin{aligned}
&\|(\nabla^2 p)\cdot\nabla^2\mathrm{div}u\|_{L^2}\\
\le& C(K_1)\left(\|\nabla p\|_{L^\infty}+\|\nabla\mathrm{div}u\|_{L^\infty}\right)\sum_{i=1}^{3}\left(\|\nabla^i p\|_{L^2}+\|\nabla^i\mathrm{div}u\|_{L^2}\right)
\end{aligned}\tag{3.25}$$

and

$$\begin{aligned}
&\|(\nabla D_t\mathrm{div}u)\cdot\nabla^2 u\|_{L^2}\\
\le& C(K_1)\left(\|D_t\mathrm{div}u\|_{L^\infty}+\|\nabla u\|_{L^\infty}\right)\sum_{i=1}^{3}\left(\|\nabla^{i-1}D_t\mathrm{div}u\|_{L^2}+\|\nabla^i u\|_{L^2}\right).
\end{aligned}\tag{3.26}$$

Thus, it follows from (3.21) and (3.23)-(3.26) that

$$\begin{aligned}
\frac{d}{dt}E_3^d(t) \le& C\sum_{i=0}^{1}\left(\|\nabla^i D_t^2\mathrm{div}u\|_{L^2}^2+\|\nabla^{i+1}D_t p\|_{L^2}^2\right)+C\|\nabla^3\mathrm{div}u\|_{L^2}^2\\
&+C\sum_{i=1}^{3}\left(\|\nabla^i u\|_{L^2}^2+\|\nabla^i p\|_{L^2}^2+\|\nabla^{i-1}D_t\mathrm{div}u\|_{L^2}^2\right),
\end{aligned}\tag{3.27}$$

where $C = C(\mathcal{A}_1)$ and $\mathcal{A}_1$ is defined by (3.14).

Proof of (3.13b) *for* $k = 4$. Due to (3.16), Hölder's inequality, (A-2d), (A-1b) and Young's inequality, one has

$$\begin{aligned}
\|D_t^4\mathrm{div}u-\mathcal{T}\Delta D_t^3\mathrm{div}u\|_{L^2} \le& C\sum_{i=0}^{1}\left(\|\nabla^i D_t^3\mathrm{div}u\|_{L^2}+\|\nabla^{i+1}D_t^2 p\|_{L^2}\right)\\
&+C\sum_{i=0}^{2}\left(\|\nabla^i D_t^2\mathrm{div}u\|_{L^2}+\|\nabla^{i+1}D_t p\|_{L^2}\right)+\|\nabla^4\mathrm{div}u\|_{L^2}\\
&+C\sum_{i=1}^{4}\left(\|\nabla^i u\|_{L^2}+\|\nabla^i p\|_{L^2}+\|\nabla^{i-1}D_t\mathrm{div}u\|_{L^2}\right),
\end{aligned}\tag{3.28}$$

where $C = C(\mathcal{A}_1, \|\nabla D_t p\|_{L^\infty}, \|\nabla D_t^2\mathrm{div}u\|_{L^2}, \|D_t^2\mathrm{div}u\|_{L^2})$ and $\mathcal{A}_1$ is defined by (3.14). In addition to the same type of estimates as that in (3.24), the following type of estimates has been used to derive (3.28).

$$\begin{aligned}
&\|(\nabla D_t^2\mathrm{div}u)\cdot\nabla^2 u\|_{L^2}+\|(\nabla^2\mathrm{div}u)D_t^2\mathrm{div}u\|_{L^2}+\|(\nabla^2 p)D_t^2\mathrm{div}u\|_{L^2}\\
\le&\|\nabla D_t^2\mathrm{div}u\|_{L^2}\|\nabla^2 u\|_{L^\infty}+\|D_t^2\mathrm{div}u\|_{L^2}\left(\|\nabla^2\mathrm{div}u\|_{L^\infty}+\|\nabla^2 p\|_{L^\infty}\right)\\
\le& C(K_1)\sum_{i=0}^{1}\|\nabla^i D_t^2\mathrm{div}u\|_{L^2}\sum_{i=2}^{4}\left(\|\nabla^i u\|_{L^2}+\|\nabla^i\mathrm{div}u\|_{L^2}+\|\nabla^i p\|_{L^2}\right)
\end{aligned}$$

and

$$
\begin{aligned}
&\|(\nabla^2 p)\cdot\nabla^3 u\|_{L^2}+\|(\nabla^2 p)\cdot\nabla^2 D_t\mathrm{div}u\|_{L^2}+\|(\nabla^3 p)\cdot\nabla D_t\mathrm{div}u\|_{L^2}\\
\le&\|\nabla^2 p\|_{L^6}\left(\|\nabla^3 u\|_{L^3}+\|\nabla^2 D_t\mathrm{div}u\|_{L^3}\right)+\|\nabla^3 p\|_{L^3}\|\nabla D_t\mathrm{div}u\|_{L^6}\\
\le&C(K_1)\|\nabla p\|_{L^\infty}^{2/3}\left(\sum_{i=1}^4\|\nabla^i p\|_{L^2}\right)^{1/3}\|\nabla u\|_{L^\infty}^{1/3}\left(\sum_{i=1}^4\|\nabla^i u\|_{L^2}\right)^{2/3}\\
&+C(K_1)\|\nabla p\|_{L^\infty}^{2/3}\left(\sum_{i=1}^4\|\nabla^i p\|_{L^2}\right)^{1/3}\|D_t\mathrm{div}u\|_{L^\infty}^{1/3}\left(\sum_{i=0}^3\|\nabla^i D_t\mathrm{div}u\|_{L^2}\right)^{2/3}\\
&+C(K_1)\|\nabla p\|_{L^\infty}^{1/3}\left(\sum_{i=1}^4\|\nabla^i p\|_{L^2}\right)^{2/3}\|D_t\mathrm{div}u\|_{L^\infty}^{2/3}\left(\sum_{i=0}^3\|\nabla^i D_t\mathrm{div}u\|_{L^2}\right)^{1/3}\\
\le&\bar{C}\sum_{i=1}^4\left(\|\nabla^i p\|_{L^2}+\|\nabla^i u\|_{L^2}+\|\nabla^{i-1}D_t\mathrm{div}u\|_{L^2}\right),
\end{aligned}
$$

where $\bar{C}=C(K_1)(\|\nabla p\|_{L^\infty}+\|\nabla u\|_{L^\infty}+\|D_t\mathrm{div}u\|_{L^\infty})$. Combining (3.21), (3.28) and (3.15) together, we arrive at

$$
\begin{aligned}
\frac{d}{dt}E_4^d(t)\le&C\sum_{i=0}^1\left(\|\nabla^i D_t^3\mathrm{div}u\|_{L^2}^2+\|\nabla^{i+1}D_t^2 p\|_{L^2}^2\right)\\
&+C\sum_{i=0}^2\left(\|\nabla^i D_t^2\mathrm{div}u\|_{L^2}^2+\|\nabla^{i+1}D_t p\|_{L^2}^2\right)+\|\nabla^4\mathrm{div}u\|_{L^2}^2\\
&+C\sum_{i=1}^4\left(\|\nabla^i u\|_{L^2}^2+\|\nabla^i p\|_{L^2}^2+\|\nabla^{i-1}D_t\mathrm{div}u\|_{L^2}^2\right),
\end{aligned}\tag{3.29}
$$

where $C=C(\mathcal{A}_1,\mathrm{Vol}\Omega,\|\nabla D_t p\|_{L^\infty},E_3^d)$ and $\mathcal{A}_1$ is defined by (3.14).

Proof of (3.13b) *for* $k=5$. Similarly, (3.13b) for $k=5$ can be proved by noting that

$$
\begin{aligned}
&\|(\nabla^3 p)\cdot\nabla^2 D_t\mathrm{div}u\|_{L^2}\le\|\nabla^3 p\|_{L^4}\|\nabla^2 D_t\mathrm{div}u\|_{L^4}\\
\le&C(K_1)\|\nabla p\|_{L^\infty}^{1/2}\left(\sum_{i=1}^5\|\nabla^i p\|_{L^2}\right)^{1/2}\|D_t\mathrm{div}u\|_{L^\infty}^{1/2}\left(\sum_{i=0}^4\|\nabla^i D_t\mathrm{div}u\|_{L^2}\right)^{1/2}\\
\le&C(K_1)\left(\|\nabla p\|_{L^\infty}+\|D_t\mathrm{div}u\|_{L^\infty}\right)\sum_{i=1}^5\left(\|\nabla^i p\|_{L^2}+\|\nabla^{i-1}D_t\mathrm{div}u\|_{L^2}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&\|(\nabla^4 p)\cdot\nabla^2 p\|_{L^2}\le\|\nabla^4 p\|_{L^{8/3}}\|\nabla^2 p\|_{L^8}\\
\le&C(K_1)\|\nabla p\|_{L^\infty}^{1/4}\left(\sum_{i=1}^5\|\nabla^i p\|_{L^2}\right)^{3/4}\|\nabla p\|_{L^\infty}^{3/4}\left(\sum_{i=1}^5\|\nabla^i p\|_{L^2}\right)^{1/4}\\
=&C(K_1)\|\nabla p\|_{L^\infty}\sum_{i=1}^5\|\nabla^i p\|_{L^2}.
\end{aligned}
$$

□

3.2. **Estimates on temporal derivatives of $E_k^a(t)$.** This subsection is devoted to proving

Proposition 3.6. *It holds that*

$$\frac{d}{dt}E_1^a(t) \leq C_1 \left(\|\nabla p\|_{L^2}^2 + \|\nabla \mathrm{div} u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\right), \tag{3.30a}$$

$$\begin{aligned}\frac{d}{dt}E_r^a(t) \leq C_r \Bigg(& \sum_{i=1}^{r} \left(\|\nabla^i u\|_{L^2}^2 + \|\nabla^i p\|_{L^2}^2 + \|\nabla^i \mathcal{T}\|_{L^2}^2\right) + \|\nabla^r \mathrm{div} u\|_{L^2}^2 + |\Pi\nabla^r p|_{L^2}^2 \\ & + |\Pi \nabla^r D_t p|_{L^2}^2 + \sum_{i=2}^{r-1} |\nabla^i u|_{L^2}^2 + \sum_{i=3}^{r} |\nabla^i p|_{L^2}^2 + \sum_{i=2}^{r-3} |\overline{\nabla}^i \theta|_{L^2}^2 \Bigg), \ 2 \leq r \leq 5, \end{aligned} \tag{3.30b}$$

where $C_1 = C(K, \|\mathcal{T}^{-1}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla \mathcal{T}\|_{L^\infty})$ *and* $C_r = C(\mathcal{A}_2)$ *with* $\mathcal{A}_2$ *being given by*

$$\begin{aligned}\mathcal{A}_2 =& (K, K_1, \|\mathcal{T}^{-1}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla p\|_{L^\infty}, \|\nabla \mathcal{T}\|_{L^\infty}, |(\nabla_N p)^{-1}|_{L^\infty}, |\nabla p|_{L^\infty}, \\ & |\nabla D_t p|_{L^\infty}, |\nabla u|_{L^\infty}, |\nabla^2 p|_{L^\infty}).\end{aligned} \tag{3.31}$$

Proof. The proof of this proposition consists of three steps. In Step 1, we prove (3.30a). Steps 2 and 3 are devoted to showing (3.30b).

Step 1. It follows from (3.1b) and (3.2) that

$$\begin{aligned}&\frac{1}{2}D_t\left(\mathcal{T}^{-1}g^{ac}\zeta^{bd}(\nabla_b u_a)\nabla_d u_c\right) + \frac{1}{2}\mathcal{T}^{-1}g^{ac}\zeta^{bd}(\nabla_b u_a)(\nabla_d u_c)\mathrm{div} u \\ =& -\mathrm{div}\left(\zeta^{bd}(\nabla_b p)\nabla_d u\right) + \frac{1}{2}\mathcal{T}^{-1}\left(D_t\left(g^{ac}\zeta^{bd}\right)\right)(\nabla_b u_a)\nabla_d u_c + (\nabla_d u)\cdot(\nabla \zeta^{bd})\nabla_b p \\ & + \zeta^{bd}(\nabla_b p)\nabla_d \mathrm{div} u + \zeta^{bd}\mathcal{T}^{-1}\left((\nabla_d u)\cdot(\nabla u_e)\nabla_b u^e - (\nabla_b \mathcal{T})(\nabla_d u)\cdot\nabla p\right).\end{aligned} \tag{3.32}$$

This, together with (3.20), (2.4b) and (2.9), implies that

$$\begin{aligned}&\frac{d}{dt}\int_\Omega \mathcal{T}^{-1}g^{ac}\zeta^{bd}(\nabla_b u_a)\nabla_d u_c d\mu_g \\ \leq & C\int_\Omega \left(|\nabla u|^2 + (|\nabla u| + |\nabla \mathrm{div} u|)|\nabla p|\right)d\mu_g \\ \leq & C\int_\Omega \left(|\nabla u|^2 + |\nabla \mathrm{div} u|^2 + |\nabla p|^2\right)d\mu_g,\end{aligned} \tag{3.33}$$

where $C = C(K, \|\mathcal{T}^{-1}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla \mathcal{T}\|_{L^\infty})$. It follows from (3.20), (2.4b) and (3.3a) that

$$\begin{aligned}\frac{d}{dt}\int_\Omega |\mathrm{curl} u|^2 d\mu_g \leq & C(\|\nabla u\|_{L^\infty})\int_\Omega \left(|\mathrm{curl} u||D_t \mathrm{curl} u| + |\mathrm{curl} u|^2\right)d\mu_g \\ \leq & C(\|\nabla u\|_{L^\infty}, \|\nabla \mathcal{T}\|_{L^\infty})\int_\Omega \left(|\nabla u|^2 + |\nabla p|^2\right)d\mu_g.\end{aligned}$$

This, together with (3.33), implies (3.30a).

Step 2. In this step, we prove that for $r \geq 2$,

$$\begin{aligned}\frac{d}{dt}E_r^a(t) \leq & C\left(\|\nabla^r u\|_{L^2}^2 + \|\nabla^r \mathrm{div} u\|_{L^2}^2 + \|\nabla^r p\|_{L^2}^2 + \|D_t \nabla^r u + \mathcal{T}\nabla^{r+1}p\|_{L^2}^2\right. \\ & \left. + \|D_t \nabla^{r-1}\mathrm{curl} u\|_{L^2}^2 + |\Pi \nabla^r p|_{L^2}^2 + |\Pi(D_t \nabla^r p + (\nabla^r u)\cdot\nabla p)|_{L^2}^2\right),\end{aligned} \tag{3.34}$$

where $C = C(K, \|\mathcal{T}^{-1}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, |(\nabla_N p)^{-1}|_{L^\infty}, |\nabla p|_{L^\infty}, |\nabla D_t p|_{L^\infty}, |\nabla u|_{L^\infty})$.

In a similar way to deriving (3.32), we have that for $r \geq 1$,

$$\begin{aligned}
&\frac{1}{2}D_t\left(\mathcal{T}^{-1}g^{ab}\zeta^{cd}\zeta^{IJ}(\nabla_I^r\nabla_c u_a)\nabla_J^r\nabla_d u_b\right)+\frac{1}{2}\mathcal{T}^{-1}g^{ab}\zeta^{cd}\zeta^{IJ}(\nabla_I^r\nabla_c u_a)(\nabla_J^r\nabla_d u_b)\mathrm{div}u\\
&=-\mathrm{div}\left(\zeta^{cd}\zeta^{IJ}(\nabla_I^r\nabla_c p)\nabla_J^r\nabla_d u\right)+\frac{1}{2}\mathcal{T}^{-1}\left(D_t\left(g^{ab}\zeta^{cd}\zeta^{IJ}\right)\right)(\nabla_I^r\nabla_c u_a)\nabla_J^r\nabla_d u_b\\
&\quad+\mathcal{T}^{-1}g^{ab}\zeta^{cd}\zeta^{IJ}(D_t\nabla_I^r\nabla_c u_a+\mathcal{T}\nabla_I^r\nabla_c\nabla_a p)\nabla_J^r\nabla_d u_b\\
&\quad+(\nabla_J^r\nabla_d u)\cdot\left(\nabla(\zeta^{cd}\zeta^{IJ})\right)(\nabla_I^r\nabla_c p)+\zeta^{cd}\zeta^{IJ}(\nabla_I^r\nabla_c p)\nabla_J^r\nabla_d\mathrm{div}u.
\end{aligned}\tag{3.35}$$

Due to (3.9), we have $\nabla_a p=\overline{\nabla}_a p+N_a\nabla_N p=N_a\nabla_N p$ on $\partial\Omega$ and

$$\begin{aligned}
&\int_\Omega \mathrm{div}\left(\zeta^{cd}\zeta^{IJ}(\nabla_I^r\nabla_c p)\nabla_J^r\nabla_d u\right)d\mu_g=\int_{\partial\Omega}N_a\zeta^{cd}\zeta^{IJ}(\nabla_I^r\nabla_c p)(\nabla_J^r\nabla_d u^a)d\mu_\zeta\\
&=-\int_{\partial\Omega}N_a(\nabla_N p)\zeta^{cd}\zeta^{IJ}(\nabla_I^r\nabla_c p)(\nabla_J^r\nabla_d u^a)(-\nabla_N p)^{-1}d\mu_\zeta\\
&=-\int_{\partial\Omega}\zeta^{cd}\zeta^{IJ}(\nabla_I^r\nabla_c p)(D_t\nabla_J^r\nabla_d p+(\nabla_J^r\nabla_d u^a)\nabla_a p)(-\nabla_N p)^{-1}d\mu_\zeta\\
&\quad+\int_{\partial\Omega}\zeta^{cd}\zeta^{IJ}(\nabla_I^r\nabla_c p)(D_t\nabla_J^r\nabla_d p)(-\nabla_N p)^{-1}d\mu_\zeta.
\end{aligned}\tag{3.36}$$

Moreover, it follows from (2.8c) that

$$\begin{aligned}
&\int_{\partial\Omega}\zeta^{cd}\zeta^{IJ}(\nabla_I^r\nabla_c p)(D_t\nabla_J^r\nabla_d p)(-\nabla_N p)^{-1}d\mu_\zeta\\
=&\frac{1}{2}\frac{d}{dt}\int_{\partial\Omega}\zeta^{cd}\zeta^{IJ}(\nabla_I^r\nabla_c p)(\nabla_J^r\nabla_d p)(-\nabla_N p)^{-1}d\mu_\zeta\\
&-\frac{1}{2}\int_{\partial\Omega}\left(D_t(\zeta^{cd}\zeta^{IJ}(-\nabla_N p)^{-1})\right)(\nabla_I^r\nabla_c p)(\nabla_J^r\nabla_d p)d\mu_\zeta\\
&-\frac{1}{2}\int_{\partial\Omega}\zeta^{cd}\zeta^{IJ}(\nabla_I^r\nabla_c p)(\nabla_J^r\nabla_d p)(-\nabla_N p)^{-1}(\mathrm{div}u-h_{NN})d\mu_\zeta.
\end{aligned}\tag{3.37}$$

Notice that on $\partial\Omega$,

$$\begin{aligned}
|D_t(-\nabla_N p)^{-1}|\leq&|(-\nabla_N p)^{-1}|^2|D_t\nabla_N p|\\
\leq&|(-\nabla_N p)^{-1}|^2(|(D_t N^c)\nabla_c p|+|\nabla_N D_t p|)\\
=&|(-\nabla_N p)^{-1}|^2(|\left(-2h_d^c N^d+h_{NN}N^c\right)\nabla_c p|+|\nabla_N D_t p|)\\
=&|(-\nabla_N p)^{-1}|^2(|h_{NN}\nabla_N p|+|\nabla_N D_t p|),
\end{aligned}\tag{3.38}$$

due to (2.8a) and (3.9). We then obtain, with the help of (3.35)-(3.37), (3.20), (2.4b), (2.8b) and (2.9), that for $r \geq 2$,

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega\mathcal{T}^{-1}g^{ab}\zeta^{cd}\zeta^{IJ}(\nabla_I^{r-1}\nabla_c u_a)\nabla_J^{r-1}\nabla_d u_b d\mu_g\\
&+\frac{d}{dt}\int_{\partial\Omega}\zeta^{cd}\zeta^{IJ}(\nabla_I^{r-1}\nabla_c p)(\nabla_J^{r-1}\nabla_d p)(-\nabla_N p)^{-1}d\mu_\zeta\\
\leq&C_1\int_\Omega\{|\nabla^r u|(|\nabla^r u|+|D_t\nabla^r u+\mathcal{T}\nabla^{r+1}p|+|\nabla^r p|)+|\nabla^r p||\nabla^r\mathrm{div}u|\}d\mu_g\\
&+C_2\int_{\partial\Omega}(|\Pi\nabla^r p||\Pi(D_t\nabla^r p+(\nabla^r u)\cdot\nabla p)|+|\Pi\nabla^r p|^2)d\mu_\zeta,
\end{aligned}\tag{3.39}$$

where $C_1=C(K,\|\mathcal{T}^{-1}\|_{L^\infty},\|\nabla u\|_{L^\infty})$ and $C_2=C(|(\nabla_N p)^{-1}|_{L^\infty},|\nabla p|_{L^\infty},|\nabla D_t p|_{L^\infty},|\nabla u|_{L^\infty})$.

It follows from (3.20) and (2.4b) that for $r \geq 2$

$$\begin{aligned}&\frac{d}{dt}\int_\Omega |\nabla^{r-1}\mathrm{curl}u|^2 d\mu_g\\ \leq& C(\|\nabla u\|_{L^\infty})\int_\Omega \left(|\nabla^{r-1}\mathrm{curl}u|^2 + |D_t\nabla^{r-1}\mathrm{curl}u||\nabla^{r-1}\mathrm{curl}u|\right) d\mu_g,\end{aligned}$$

which, together with (3.39), implies (3.34).

Step 3. To prove (3.30b), it remains to handle the terms $\|D_t\nabla^r u+\mathcal{T}\nabla^{r+1}p\|_{L^2}$, $\|D_t\nabla^{r-1}\mathrm{curl}u\|_{L^2}$ and $|\Pi(D_t\nabla^r p+(\nabla^r u)\cdot\nabla p)|_{L^2}$ on the right-hand side of (3.34), which can be done by Lemma 3.5 and the following estimate

$$|\Pi(D_t\nabla^r p + (\nabla^r u)\cdot\nabla p - \nabla^r D_t p)| \leq C\sum_{s=1}^{r-2}\left|\Pi((\nabla^{1+s}u)\cdot\nabla^{r-s}p)\right|. \tag{3.40}$$

Indeed, (3.40) is an immediate consequence of (2.6).

First, it follows from (3.18) and Hölder's inequality that for $r = 2, 3, 4, 5$,

$$\begin{aligned}&\|D_t\nabla^r u + \mathcal{T}\nabla^{r+1}p\|_{L^2} + \|D_t\nabla^{r-1}\mathrm{curl}u\|_{L^2}\\ \leq& C(\|\nabla u\|_{L^\infty}, \|\nabla p\|_{L^\infty}, \|\nabla\mathcal{T}\|_{L^\infty})\left(\|\nabla^r u\|_{L^2} + \|\nabla^r p\|_{L^2} + \|\nabla^r\mathcal{T}\|_{L^2} + X_r\right)\end{aligned}$$

where

$$X_r = \begin{cases} 0, & r=2,\\ \|\nabla^2 u\|_{L^4}^2 + \|\nabla^2 p\|_{L^4}\|\nabla^2\mathcal{T}\|_{L^4}, & r=3,\\ \|\nabla^2 u\|_{L^6}\|\nabla^3 u\|_{L^3} + \|\nabla^2 p\|_{L^6}\|\nabla^3\mathcal{T}\|_{L^3} + \|\nabla^3 p\|_{L^3}\|\nabla^2\mathcal{T}\|_{L^6}, & r=4,\\ \|\nabla^2 u\|_{L^8}\|\nabla^4 u\|_{L^{\frac{8}{3}}} + \|\nabla^3 u\|_{L^4}^2 + \|\nabla^2 p\|_{L^8}\|\nabla^4\mathcal{T}\|_{L^{\frac{8}{3}}} & \\ \quad +\|\nabla^3 p\|_{L^4}\|\nabla^3\mathcal{T}\|_{L^4} + \|\nabla^4 p\|_{L^{\frac{8}{3}}}\|\nabla^2\mathcal{T}\|_{L^8}, & r=5,\end{cases}$$

This, together with (A-1b) and Young's inequality, implies that for $r = 2, 3, 4, 5,$,

$$\begin{aligned}&\|D_t\nabla^r u + \mathcal{T}\nabla^{r+1}p\|_{L^2} + \|D_t\nabla^{r-1}\mathrm{curl}u\|_{L^2}\\ \leq& C(K_1, \|\nabla u\|_{L^\infty}, \|\nabla p\|_{L^\infty}, \|\nabla\mathcal{T}\|_{L^\infty})\sum_{i=1}^{r}\left(\|\nabla^i u\|_{L^2} + \|\nabla^i p\|_{L^2} + \|\nabla^i\mathcal{T}\|_{L^2}\right).\end{aligned} \tag{3.41}$$

Using (3.40) and (3.9), one has

$$|\Pi(D_t\nabla^r p + (\nabla^r u)\cdot\nabla p)|_{L^2} \leq |\Pi\nabla^r D_t p|_{L^2} + CY_r, \tag{3.42}$$

where

$$Y_r = \begin{cases} 0, & r=2,\\ |\nabla^2 u|_{L^2}|\nabla^2 p|_{L^\infty}, & r=3,\\ |\nabla^3 u|_{L^2}|\nabla^2 p|_{L^\infty} + |\Pi((\nabla^2 u)\cdot\nabla^3 p)|_{L^2}, & r=4,\\ |\nabla^4 u|_{L^2}|\nabla^2 p|_{L^\infty} + |\Pi((\nabla^3 u)\cdot\nabla^3 p)|_{L^2} + |\Pi((\nabla^2 u)\cdot\nabla^4 p)|_{L^2}, & r=5.\end{cases}$$

Notice that on $\partial\Omega$,

$$\begin{aligned}|\Pi((\nabla^2 u)\cdot\nabla^3 p)| &\leq |\overline{\nabla}\nabla u||\overline{\nabla}\nabla^2 p|,\\ |\Pi((\nabla^3 u)\cdot\nabla^3 p)| &\leq |\overline{\nabla}^2\nabla u||\overline{\nabla}\nabla^2 p| + CK|\overline{\nabla}\nabla u||\overline{\nabla}\nabla^2 p|,\\ |\Pi((\nabla^2 u)\cdot\nabla^4 p)| &\leq |\overline{\nabla}\nabla u||\overline{\nabla}^2\nabla^2 p| + CK|\overline{\nabla}\nabla u||\overline{\nabla}\nabla^2 p|.\end{aligned}$$

We then obtain, with the help of Hölder's inequality, (A-1a) and Young's inequality, that

$$
\begin{aligned}
&|\Pi((\nabla^2 u)\cdot\nabla^3 p)|_{L^2} \le |\overline{\nabla}\nabla u|_{L^4}|\overline{\nabla}\nabla^2 p|_{L^4} \\
\le& C|\nabla u|_{L^\infty}^{1/2}|\overline{\nabla}^2\nabla u|_{L^2}^{1/2}|\nabla^2 p|_{L^\infty}^{1/2}|\overline{\nabla}^2\nabla^2 p|_{L^2}^{1/2} \\
\le& C\left(|\nabla u|_{L^\infty},|\nabla^2 p|_{L^\infty}\right)\left(|\overline{\nabla}^2\nabla u|_{L^2}+|\overline{\nabla}^2\nabla^2 p|_{L^2}\right)
\end{aligned}
\tag{3.43}
$$

and

$$
\begin{aligned}
&|\Pi((\nabla^3 u)\cdot\nabla^3 p)|_{L^2}+|\Pi((\nabla^2 u)\cdot\nabla^4 p)|_{L^2} \\
\le&|\overline{\nabla}^2\nabla u|_{L^3}|\overline{\nabla}\nabla^2 p|_{L^6}+|\overline{\nabla}\nabla u|_{L^6}|\overline{\nabla}^2\nabla^2 p|_{L^3}+CK|\overline{\nabla}\nabla u|_{L^4}|\overline{\nabla}\nabla^2 p|_{L^4} \\
\le& C|\nabla u|_{L^\infty}^{1/3}|\overline{\nabla}^3\nabla u|_{L^2}^{2/3}|\nabla^2 p|_{L^\infty}^{2/3}|\overline{\nabla}^3\nabla^2 p|_{L^2}^{1/3}+CK|\overline{\nabla}\nabla u|_{L^4}|\overline{\nabla}\nabla^2 p|_{L^4} \\
&+C|\nabla u|_{L^\infty}^{2/3}|\overline{\nabla}^3\nabla u|_{L^2}^{1/3}|\nabla^2 p|_{L^\infty}^{1/3}|\overline{\nabla}^3\nabla^2 p|_{L^2}^{2/3} \\
\le& C(K,|\nabla u|_{L^\infty},|\nabla^2 p|_{L^\infty})\sum_{i=2}^{3}\left(|\overline{\nabla}^i\nabla u|_{L^2}+|\overline{\nabla}^i\nabla^2 p|_{L^2}\right).
\end{aligned}
\tag{3.44}
$$

For a $(0,2)$ tensor α, we have on $\partial\Omega$,

$$
|\overline{\nabla}^2\alpha|\le|\nabla^2\alpha|+CK|\nabla\alpha| \quad\text{and}\quad |\overline{\nabla}^3\alpha|\le|\nabla^3\alpha|+C\left(K|\nabla^2\alpha|+K^2|\nabla\alpha|+|\overline{\nabla}\theta||\overline{\nabla}\alpha|\right).
$$

This, together with (3.42), (3.43) and (3.44), implies that for $r=4,5$,

$$
|\Pi(D_t\nabla^r p+(\nabla^r u)\cdot\nabla p)|_{L^2}\le|\Pi\nabla^r D_t p|_{L^2}+Z_r, \tag{3.45}
$$

where

$$
Z_r=\begin{cases} C\left(|\nabla u|_{L^\infty},|\nabla^2 p|_{L^\infty}\right)\sum_{i=2}^{3}\left(|\nabla^i u|_{L^2}+|\nabla^{i+1}p|_{L^2}\right), & r=4,\\ C(K,|\nabla u|_{L^\infty},|\nabla^2 p|_{L^\infty})\left(\sum_{i=2}^{4}\left(|\nabla^i u|_{L^2}+|\nabla^{i+1}p|_{L^2}\right)+|\overline{\nabla}^2\theta|_{L^2}\right), & r=5.\end{cases}
$$

It implies from (3.42) and (3.45) that for $r=2,3,4,5$,

$$
\begin{aligned}
&|\Pi(D_t\nabla^r p+(\nabla^r u)\cdot\nabla p)|_{L^2}\le|\Pi\nabla^r D_t p|_{L^2} \\
&+C(K,|\nabla u|_{L^\infty},|\nabla^2 p|_{L^\infty})\left(\sum_{i=2}^{r-1}\left(|\nabla^i u|_{L^2}+|\nabla^{i+1}p|_{L^2}\right)+\sum_{i=2}^{r-3}|\overline{\nabla}^i\theta|_{L^2}\right).
\end{aligned}
\tag{3.46}
$$

So, (3.30b) follows from (3.34), (3.41) and (3.46).

□

3.3. **Regularity Estimates.** In this subsection, we derive the bounds that the higher order energy (3.6) implies. Clearly,

$$
\|u\|_{L^2}^2\le\|\mathcal{T}\|_{L^\infty}E_0, \tag{3.47a}
$$

$$
|\Pi\nabla^r p|_{L^2}^2\le|\nabla p|_{L^\infty}E_r^a, \quad r\ge 2, \tag{3.47b}
$$

$$
\|D_t^{r-1}\mathrm{div}u\|_{L^2}^2+\|\nabla D_t^{r-1}\mathrm{div}u\|_{L^2}^2\le C(\mathrm{Vol}\Omega)E_r^d(t), \quad r\ge 1. \tag{3.47c}
$$

Indeed, (3.47c) follows from (3.9) and (A-4). The following bounds are also apparent.

$$
\|\nabla u\|_{L^2}^2\le C(\mathrm{Vol}\Omega,\|\mathcal{T}\|_{L^\infty})E_1, \qquad |u|_{L^2}^2\le C(K,\mathrm{Vol}\Omega,\|\mathcal{T}\|_{L^\infty})(E_0+E_1), \tag{3.48a}
$$

$$
\|\mathcal{T}-\mathcal{T}_b\|_{L^2}^2+\|\nabla\mathcal{T}\|_{L^2}^2+\|\nabla^2\mathcal{T}\|_{L^2}^2+|\nabla\mathcal{T}|_{L^2}^2\le C(K,\mathrm{Vol}\Omega)E_1^d. \tag{3.48b}
$$

In fact, the bound for $\|\nabla u\|_{L^2}$ in (3.48a) follows from (A-3a) and (3.47c); that for $|u|_{L^2}$ follows from (A-3c), (3.47a) and the bound just obtained for $\|\nabla u\|_{L^2}$; those for the first three terms in (3.48b) are consequences of (3.9), (A-5a), (3.1b) and (3.47c); and that for $|\nabla\mathcal{T}|_{L^2}$ follows from (A-3c) and the bounds just obtained for $\|\nabla^r\mathcal{T}\|_{L^2}$ $(r=1,2)$. The main content of this subsection is the proof of the following proposition.

Proposition 3.7. *For $2 \le r \le n+2$, we have*

$$\begin{aligned}
&\|\nabla^r u\|_{L^2}^2 + \|\nabla^r p\|_{L^2}^2 + \|\nabla^r \mathcal{T}\|_{L^2}^2 + \|\nabla^r \mathrm{div} u\|_{L^2}^2 + |\nabla^r p|_{L^2}^2 + |\nabla^r \mathcal{T}|_{L^2}^2 \\
&+ |\Pi \nabla^r D_t p|_{L^2}^2 + |\nabla^{r-1} u|_{L^2}^2 + |\nabla^{r-1}\mathrm{div} u|_{L^2}^2 + |\overline{\nabla}^{r-2}\theta|_{L^2}^2 \\
&+ \sum_{i=0}^{2} \left(\|\nabla^i D_t^{r-2} p\|_{L^2}^2 + \|\nabla^i D_t^{r-2}\mathrm{div} u\|_{L^2}^2\right) + |\nabla D_t^{r-2} p|_{L^2}^2 + |\nabla D_t^{r-2}\mathrm{div} u|_{L^2}^2 \\
&\quad \le C\left(\mathcal{A}_3, \|\nabla^2 \mathcal{T}\|_{L^\infty}, |\nabla D_t p|_{L^\infty}, \sum_{j=0}^{r-3} |\nabla D_t^j \mathrm{div} u|_{L^\infty}, \sum_{i=1}^{r-1} E_i\right) \sum_{i=1}^{r} E_i, \quad r \ge 2,
\end{aligned} \tag{3.49a}$$

$$\begin{aligned}
&|\nabla^{r-1} D_t p|_{L^2}^2 + \|\nabla^r D_t p\|_{L^2}^2 + |\nabla^{r-2} D_t \mathrm{div} u|_{L^2}^2 + \|\nabla^{r-1} D_t \mathrm{div} u\|_{L^2}^2 \\
&\quad \le C\left(\mathcal{A}_3, \|\nabla^2 \mathcal{T}\|_{L^\infty}, |\nabla D_t p|_{L^\infty}, \sum_{j=0}^{r-3} |\nabla D_t^j \mathrm{div} u|_{L^\infty}, \sum_{i=1}^{r-1} E_i\right) \sum_{i=1}^{r} E_i, \quad r \ge 3,
\end{aligned} \tag{3.49b}$$

$$\begin{aligned}
&\sum_{r=3}^{4} \left(|\nabla^{r-1} D_t^2 \mathrm{div} u|_{L^2}^2 + \|\nabla^r D_t^2 \mathrm{div} u\|_{L^2}^2 + |\nabla^{r-1} D_t^2 p|_{L^2}^2 + \|\nabla^r D_t^2 p\|_{L^2}^2\right) \\
&\quad \le C\left(\mathcal{A}_3, \sum_{j=1}^{2} |\nabla D_t^j p|, \sum_{j=0}^{2} |\nabla D_t^j \mathrm{div} u|_{L^\infty}, \sum_{i=1}^{4} E_i\right) \sum_{i=1}^{5} E_i,
\end{aligned} \tag{3.49c}$$

where

$$\begin{aligned}
\mathcal{A}_3 = (&K, K_1, \mathrm{Vol}\Omega, \|\mathcal{T}^{-1}\|_{L^\infty}, \|\mathcal{T}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla \mathcal{T}\|_{L^\infty}, \|\nabla p\|_{L^\infty}, \\
&|(\nabla_N p)^{-1}|_{L^\infty}, |\nabla p|_{L^\infty}, |\nabla \mathcal{T}|_{L^\infty}).
\end{aligned} \tag{3.50}$$

The key to the proof of this proposition is to use the elliptic estimates, Corollary A-4, for which we need to estimate the terms of the forms $\|\mathrm{div}(\mathcal{T}\nabla D_t^j p)\|_{L^2} = \|\mathcal{T}\Delta D_t^j p + (\nabla \mathcal{T}) \cdot \nabla D_t^j p\|_{L^2}$, $\|\mathcal{T}\nabla^r \Delta D_t^j p\|_{L^2}$ and $\|\mathcal{T}\nabla^r \Delta D_t^j \mathrm{div} u\|_{L^2}$. To do so, we need the following lemma:

Lemma 3.8. *We have for $k \ge 0$,*

$$\mathcal{T}\Delta D_t^k p + (\nabla \mathcal{T}) \cdot \nabla D_t^k p = -D_t^{k+1}\mathrm{div} u - D_t^k((\nabla_e u) \cdot \nabla u^e) - \mathfrak{R}_{k+2}, \tag{3.51}$$

where $\mathfrak{R}_2 = 0$ and for $k \ge 1$,

$$\begin{aligned}
\mathfrak{R}_{k+2} = &\mathcal{T}[D_t, \Delta] D_t^{k-1} p + (D_t \mathcal{T})\Delta D_t^{k-1} p + D_t((\nabla \mathcal{T}) \cdot \nabla D_t^{k-1} p) \\
&- (\nabla \mathcal{T}) \cdot D_t \nabla D_t^{k-1} p + D_t \mathfrak{R}_{k+1}.
\end{aligned}$$

Proof. Take D_t of (3.3b) to get

$$\mathcal{T}\Delta D_t p + (\nabla \mathcal{T}) \cdot \nabla D_t p = -D_t^2 \mathrm{div} u - D_t((\nabla_e u) \cdot \nabla u^e) - \mathfrak{R}_3,$$

where

$$\mathfrak{R}_3 = \mathcal{T}[D_t, \Delta] p + (D_t \mathcal{T})\Delta p + D_t((\nabla \mathcal{T}) \cdot \nabla p) - (\nabla \mathcal{T}) \cdot D_t \nabla p.$$

This proves (3.51) for $k = 1$. Suppose (3.51) holds for $k = j-1$, that is,

$$\mathcal{T}\Delta D_t^{j-1} p + (\nabla \mathcal{T}) \cdot \nabla D_t^{j-1} p = -D_t^j \mathrm{div} u - D_t^{j-1}((\nabla_e u) \cdot \nabla u^e) - \mathfrak{R}_{j+1}$$

Take D_t of the equation above to get (3.51) for $k = j$. □

Proof of Proposition 3.7. The proof consists of four steps.

Step 1. In this step, we prove

$$\|\nabla^2 u\|_{L^2}^2 + |\nabla u|_{L^2}^2 + \|\nabla^2 \mathrm{div} u\|_{L^2}^2 + |\nabla \mathrm{div} u|_{L^2}^2 \le C(A_4)\left(E_1 + E_2\right), \tag{3.52a}$$

$$\sum_{i=0}^{2}\|\nabla^i p\|_{L^2}^2+\sum_{i=1}^{2}|\nabla^i p|_{L^2}^2\le C(A_4,K_1,\|\nabla p\|_{L^\infty},|\nabla p|_{L^\infty})\left(E_1+E_2\right), \tag{3.52b}$$

$$|\theta|_{L^2}^2\le|(\nabla_N p)^{-1}|_{L^\infty}^2|\nabla p|_{L^\infty}E_2, \tag{3.52c}$$

$$|\nabla^2\mathcal{T}|_{L^2}^2\le C(K,\mathrm{Vol}\Omega,|\nabla p|_{L^\infty},|(\nabla_N p)^{-1}|_{L^\infty},|\nabla\mathcal{T}|_{L^\infty})\left(E_1+E_2\right), \tag{3.52d}$$

$$|\Pi\nabla^2 D_t p|_{L^2}^2\le C(|\nabla D_t p|_{L^\infty},|\nabla p|_{L^\infty},|(\nabla_N p)^{-1}|_{L^\infty})E_2, \tag{3.52e}$$

where $A_4=(K,\mathrm{Vol}\Omega,\|\mathcal{T}^{-1}\|_{L^\infty},\|\mathcal{T}\|_{L^\infty},\|\nabla u\|_{L^\infty},\|\nabla\mathcal{T}\|_{L^\infty})$.

The bound for $\|\nabla^2 u\|_{L^2}$ in (3.52a) follows from (A-3a). It follows from (3.4) and $\Delta u_a-\nabla_a\mathrm{div}u=g^{ce}\nabla_c\mathrm{curl}u_{ea}$ that

$$\|D_t\mathrm{div}u-\mathcal{T}\Delta\mathrm{div}u\|_{L^2}\le C\sum_{i=0}^{1}\|\nabla^i\mathrm{div}u\|_{L^2}+C(\|\nabla^2\mathcal{T}\|_{L^2}+\|\nabla\mathrm{curl}u\|_{L^2}),$$

where $C=(\|\nabla u\|_{L^\infty},\|\nabla\mathcal{T}\|_{L^\infty})$. This, together with (A-5a), (3.47c) and (3.48b), gives the bound for $\|\nabla^2\mathrm{div}u\|_{L^2}$ in (3.52a). The bounds obtained for $\|\nabla^2 u\|_{L^2}$ and $\|\nabla^2\mathrm{div}u\|_{L^2}$ in turn give the bounds for $|\nabla u|_{L^2}$ and $|\nabla\mathrm{div}u|_{L^2}$, with the help of (A-3c), (3.48a).

Next, we show (3.52b). Let q be a function satisfying $q=0$ on $\partial\Omega$, we have for any $\delta>0$,

$$\begin{aligned}\int_\Omega\mathcal{T}|\nabla q|^2d\mu_g=&\int_\Omega\mathrm{div}(\mathcal{T}q\nabla q)d\mu_g-\int_\Omega(\mathcal{T}\Delta q+(\nabla\mathcal{T})\cdot\nabla q)qd\mu_g\\ \le&\frac{1}{2\delta}\int_\Omega|\mathcal{T}\Delta q+(\nabla\mathcal{T})\cdot\nabla q|^2d\mu_g+\frac{\delta}{2}\int_\Omega q^2d\mu_g.\end{aligned}$$

Due to (A-4), we can then choose a suitably small δ to obtain

$$\|\nabla q\|_{L^2}^2\le C(\mathrm{Vol}\Omega,\|\mathcal{T}^{-1}\|_{L^\infty})\|\mathcal{T}\Delta q+(\nabla\mathcal{T})\cdot\nabla q\|_{L^2}^2. \tag{3.53}$$

It follows from (3.3b) that

$$\|\mathcal{T}\Delta p+(\nabla\mathcal{T})\cdot\nabla p\|_{L^2}\le\|D_t\mathrm{div}u\|_{L^2}+\|\nabla u\|_{L^\infty}\|\nabla u\|_{L^2}, \tag{3.54}$$

which, together with (3.53), (3.47c) and (3.48a), gives the bound for $\|\nabla p\|_{L^2}$ in (3.52b). (A-4) and the bound just obtained for $\|\nabla p\|_{L^2}$ imply the bound for $\|p\|_{L^2}$. The bound for $\|\nabla^2 p\|_{L^2}$ follows from (A-5a), (3.54), (3.47c) and (3.48). The bound for $|\nabla p|_{L^2}$ follows from (A-3c) and the bounds just obtained for $\|\nabla^r p\|_{L^2}$ $(r=1,2)$. Due to (3.18), Hölder's inequality, (A-1b) and Young's inequality, we have for $r\ge 0$,

$$\|\mathcal{T}\nabla^r\Delta p\|_{L^2}\le C\|\nabla^r D_t\mathrm{div}u\|_{L^2}+\bar{C}\sum_{i=1}^{r+1}\left(\|\nabla^i u\|_{L^2}+\|\nabla^i\mathcal{T}\|_{L^2}+\|\nabla^i p\|_{L^2}\right), \tag{3.55}$$

where $\bar{C}=C(K_1,\|\nabla u\|_{L^\infty},\|\nabla\mathcal{T}\|_{L^\infty},\|\nabla p\|_{L^\infty})$. (A-5b), (3.47b), (3.55), and the bounds obtained for $\|\nabla^r u\|_{L^2}$, $\|\nabla^r\mathcal{T}\|_{L^2}$ and $\|\nabla^r p\|_{L^2}$ $(r=1,2)$ lead to the bound for $|\nabla^2 p|_{L^2}$ in (3.52b).

(3.52c) follows from (2.20a) and (3.47b). (3.52d) is a consequence of (A-5b), (2.12), (3.1b), (3.52c) and (3.47c). (3.52e) follows from (2.12) and (3.52c).

Step 2. In this step, we prove

$$\begin{aligned}&\|\nabla^3 u\|_{L^2}^2+|\nabla^2 u|_{L^2}^2+\|\nabla^3 p\|_{L^2}^2+|\nabla^3 p|_{L^2}^2+|\overline{\nabla}\theta|_{L^2}^2+\|\nabla^3\mathcal{T}\|_{L^2}^2+|\nabla^3\mathcal{T}|_{L^2}^2\\ &+\|\nabla^2 D_t\mathrm{div}u\|_{L^2}^2+|\nabla D_t\mathrm{div}u|_{L^2}^2+\sum_{i=0}^{2}\|\nabla^i D_t p\|_{L^2}^2+|\nabla D_t p|_{L^2}^2\le C(\mathcal{A}_3)\sum_{i=1}^{3}E_i,\end{aligned} \tag{3.56a}$$

$$|\nabla^2\mathrm{div}u|_{L^2}^2+|\Pi\nabla^3\mathrm{div}u|_{L^2}^2+\|\nabla^3\mathrm{div}u\|_{L^2}^2\le C(\mathcal{A}_3,|\nabla\mathrm{div}u|_{L^\infty})\sum_{i=1}^{3}E_i, \tag{3.56b}$$

$$|\nabla^2 D_t p|_{L^2}^2 + |\Pi\nabla^3 D_t p|_{L^2}^2 + \|\nabla^3 D_t p\|_{L^2}^2 \le \widetilde{C}\sum_{i=1}^{3} E_i, \tag{3.56c}$$

where $\widetilde{C} = \min\{C(\mathcal{A}_3, |\nabla D_t p|_{L^\infty}, \|\nabla^2 \mathcal{T}\|_{L^\infty}),\ \ C(\mathcal{A}_3, |\nabla D_t p|_{L^\infty}, \|D_t p\|_{L^\infty}^2)\}$, and $\mathcal{A}_3$ is defined by (3.50).

First, we show (3.56a). The bounds for $\|\nabla^3 u\|_{L^2}$ and $|\nabla^2 u|_{L^2}$ can be obtained easily. Using (A-5c), (3.47b), (3.55), (3.48) and (3.52), one has the bound for $\|\nabla^3 p\|_{L^2}$. The bound for $|\overline{\nabla}\theta|_{L^2}$ follows from (2.20b), (3.47b) and (3.52). This, together with (A-5b), (3.1b), (3.47c), (3.52), (2.13) and (3.48), gives the bounds for $\|\nabla^3\mathcal{T}\|_{L^2}$ and $|\nabla^3\mathcal{T}|_{L^2}$. It follows from (3.16), Hölder's inequality, (A-1b), and Young's inequality that

$$\begin{aligned}&\|D_t^2 \mathrm{div} u - \mathcal{T}\Delta D_t \mathrm{div} u\|_{L^2}\\ \le& C\sum_{i=0}^{1}\|\nabla^i D_t \mathrm{div} u\|_{L^2} + C\sum_{i=1}^{3}\left(\|\nabla^i u\|_{L^2} + \|\nabla^i \mathcal{T}\|_{L^2} + \|\nabla^i p\|_{L^2}\right),\end{aligned} \tag{3.57}$$

where $C = C(K_1, \|\mathcal{T}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla\mathcal{T}\|_{L^\infty}, \|\nabla p\|_{L^\infty})$. The bounds obtained for $\|\nabla^r u\|_{L^2}$, $\|\nabla^r\mathcal{T}\|_{L^2}$ and $\|\nabla^r p\|_{L^2}$ $(r = 1, 2, 3)$ imply that for $\|\nabla^2 D_t \mathrm{div} u\|_{L^2}$ in (3.56a), by use of (A-5a), (3.57) and (3.47c). The bound for $|\nabla D_t \mathrm{div} u|_{L^2}$ follows from (A-3c) and the bound just obtained for $\|\nabla^2 D_t \mathrm{div} u\|_{L^2}$. The bound for $|\nabla^3 p|_{L^2}$ is a consequence of (A-5b), (3.47b), (3.55), and the bound-s obtained for $\|\nabla^2 D_t \mathrm{div} u\|_{L^2}$, $\|\nabla^r u\|_{L^2}$, $\|\nabla^r\mathcal{T}\|_{L^2}$ and $\|\nabla^r p\|_{L^2}$ $(r = 1, 2, 3)$. The bounds for $\sum_{i=0}^{2}\|\nabla^i D_t p\|_{L^2}^2$ and $|\nabla D_t p|_{L^2}^2$ can be obtained in the same way to the derivation of (3.52b), by noting that

$$\|\mathcal{T}\Delta D_t p + (\nabla\mathcal{T})\cdot\nabla D_t p\|_{L^2} \le \|D_t^2 \mathrm{div} u\|_{L^2} + C\|\nabla^2 p\|_{L^2} + C\sum_{i=1}^{2}\|\nabla^i u\|_{L^2}, \tag{3.58}$$

where $C = C(\|\mathcal{T}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla\mathcal{T}\|_{L^\infty}, \|\nabla p\|_{L^\infty})$. Indeed, (3.58) follows from (3.51), (3.2) and (2.5b).

Next, we show (3.56b). It follows from (3.4), Hölder's inequality, (A-1b) and Young's inequality that for $r = 1, 2, 3$,

$$\|\mathcal{T}\nabla^r \Delta \mathrm{div} u\|_{L^2} \le C\|\nabla^r D_t \mathrm{div} u\|_{L^2} + \bar{C}\sum_{i=1}^{r+2}\left(\|\nabla^i u\|_{L^2} + \|\nabla^i\mathcal{T}\|_{L^2}\right), \tag{3.59}$$

where $\bar{C} = C(K_1, \|\nabla u\|_{L^\infty}, \|\nabla\mathcal{T}\|_{L^\infty})$. The bound for $|\nabla^2 \mathrm{div} u|_{L^2}$ follows from (A-5b), (2.12), (3.52c), (3.59), and the bounds obtained for $\|\nabla^r u\|_{L^2}$ and $\|\nabla^r\mathcal{T}\|_{L^2}$ $(r = 1, 2, 3)$. The bound obtained for $|\nabla^2 \mathrm{div} u|_{L^2}$ in turn gives that for $|\Pi\nabla^3 \mathrm{div} u|_{L^2}$ by using (2.13), (3.56a) and (3.52a). The bound for $\|\nabla^3 \mathrm{div} u\|_{L^2}$ can be derived from (A-5c), (3.59), and the bounds obtained for $|\Pi\nabla^3 \mathrm{div} u|_{L^2}$, $\|\nabla^r u\|_{L^2}$ and $\|\nabla^r\mathcal{T}\|_{L^2}$ $(r = 1, 2, 3)$.

It follows from (3.51), Hölder's inequality, (A-1b) and Young's inequality that for $r = 1, 2, 3$,

$$\begin{aligned}\|\mathcal{T}\nabla^r \Delta D_t p\|_{L^2} \le& C\|\nabla^r D_t^2 \mathrm{div} u\|_{L^2} + \bar{C}\sum_{i=0}^{r+1}\|\nabla^i D_t p\|_{L^2}\\ &+ \bar{C}\sum_{i=1}^{r+2}\left(\|\nabla^i u\|_{L^2} + \|\nabla^i\mathcal{T}\|_{L^2} + \|\nabla^i p\|_{L^2}\right),\end{aligned} \tag{3.60}$$

where $\bar{C} = C(K_1, \|\mathcal{T}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla\mathcal{T}\|_{L^\infty}, \|\nabla p\|_{L^\infty}, \|D_t p\|_{L^\infty})$; and

$$\|\mathcal{T}\nabla\Delta D_t p\|_{L^2} \le C\|\nabla D_t^2 \mathrm{div} u\|_{L^2} + \bar{C}\sum_{i=1}^{2}\|\nabla^i D_t p\|_{L^2} + \bar{C}\sum_{i=1}^{3}\left(\|\nabla^i u\|_{L^2} + \|\nabla^i p\|_{L^2}\right), \tag{3.61}$$

where $\bar{C} = C(K_1, \|\mathcal{T}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla \mathcal{T}\|_{L^\infty}, \|\nabla p\|_{L^\infty}, \|\nabla^2 \mathcal{T}\|_{L^\infty})$. With (3.60) and (3.61), we obtain (3.56c) in a similar way to the derivation of (3.56b).

Step 3. In this step, we prove

$$\|\nabla^4 p\|_{L^2}^2 + |\overline{\nabla}^2 \theta|_{L^2}^2 + \|\nabla^4 \mathcal{T}\|_{L^2}^2 \le C(\mathcal{A}_3) \sum_{i=1}^4 E_i, \tag{3.62a}$$

$$\begin{aligned} &\|\nabla^4 u\|_{L^2}^2 + |\nabla^3 u|_{L^2}^2 + |\nabla^4 \mathcal{T}|_{L^2}^2 + |\nabla^3 \mathrm{div} u|_{L^2}^2 + |\Pi \nabla^4 \mathrm{div} u|_{L^2}^2 + \|\nabla^4 \mathrm{div} u\|_{L^2}^2 \\ &\quad \le C(\mathcal{A}_3, |\nabla \mathrm{div} u|_{L^\infty}) \sum_{i=1}^4 E_i, \end{aligned} \tag{3.62b}$$

$$\begin{aligned} &|\nabla^2 D_t \mathrm{div} u|_{L^2}^2 + |\Pi \nabla^3 D_t \mathrm{div} u|_{L^2}^2 + \|\nabla^3 D_t \mathrm{div} u\|_{L^2}^2 + |\nabla^4 p|_{L^2}^2 \\ &\quad \le C\left(\mathcal{A}_3, |\nabla \mathrm{div} u|_{L^\infty}, |\nabla D_t \mathrm{div} u|_{L^\infty}, \sum_{i=1}^3 E_i\right) \sum_{i=1}^4 E_i, \end{aligned} \tag{3.62c}$$

$$\begin{aligned} &\|\nabla^2 D_t^2 \mathrm{div} u\|_{L^2}^2 + |\nabla D_t^2 \mathrm{div} u|_{L^2}^2 + |\nabla^3 D_t p|_{L^2}^2 + |\Pi \nabla^4 D_t p|_{L^2}^2 + \|\nabla^4 D_t p\|_{L^2}^2 \\ &\quad \le C\left(\mathcal{A}_3, |\nabla \mathrm{div} u|_{L^\infty}, |\nabla D_t p|_{L^\infty}, \sum_{i=1}^3 E_i\right) \sum_{i=1}^4 E_i, \end{aligned} \tag{3.62d}$$

$$\begin{aligned} &|\nabla^2 D_t^2 \mathrm{div} u|_{L^2}^2 + |\Pi \nabla^3 D_t^2 \mathrm{div} u|_{L^2}^2 + \|\nabla^3 D_t^2 \mathrm{div} u\|_{L^2}^2 \\ &\quad \le C\left(\mathcal{A}_3, |\nabla D_t p|_{L^\infty}, \sum_{j=0}^2 |\nabla D_t^j \mathrm{div} u|_{L^\infty}, \sum_{i=1}^3 E_i\right) \sum_{i=1}^4 E_i, \end{aligned} \tag{3.62e}$$

$$\sum_{i=0}^2 \|\nabla^i D_t^2 p\|_{L^2}^2 + |\nabla D_t^2 p|_{L^2}^2 \le C\left(\mathcal{A}_3, \sum_{i=1}^3 E_i\right) \sum_{i=1}^4 E_i, \tag{3.62f}$$

$$\begin{aligned} &|\nabla^2 D_t^2 p|_{L^2}^2 + |\Pi \nabla^3 D_t^2 p|_{L^2}^2 + \|\nabla^3 D_t^2 p\|_{L^2}^2 \\ &\quad \le C\left(\mathcal{A}_3, \|D_t^2 p\|_{L^\infty}^2, |\nabla \mathrm{div} u|_{L^\infty}, \sum_{j=1}^2 |\nabla D_t^j p|_{L^\infty}, \sum_{i=1}^3 E_i\right) \sum_{i=1}^4 E_i, \end{aligned} \tag{3.62g}$$

where $\mathcal{A}_3$ is defined by (3.50).

(3.62a) and (3.62b) can be obtained easily. Indeed, (2.11) and (2.19) have been used to derive the bound for $|\overline{\nabla}^2 \theta|_{L^2}$ in (3.62a).

It follows from (3.16), Hölder's inequality, (A-1b) and Young's inequality that for $r = 1, 2$,

$$\begin{aligned} &\|\mathcal{T} \nabla^r \Delta D_t \mathrm{div} u\|_{L^2} \le C \|\nabla^r D_t^2 \mathrm{div} u\|_{L^2} \\ &\quad + \bar{C} \sum_{i=0}^{r+1} \|\nabla^i D_t \mathrm{div} u\|_{L^2} + \bar{C} \sum_{i=1}^{r+3} \left(\|\nabla^i u\|_{L^2} + \|\nabla^i \mathcal{T}\|_{L^2} + \|\nabla^i p\|_{L^2} \right), \end{aligned} \tag{3.63}$$

where $\bar{C} = C(K_1, \|\mathcal{T}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla \mathcal{T}\|_{L^\infty}, \|\nabla p\|_{L^\infty}, \|D_t \mathrm{div} u\|_{L^\infty})$. With (3.63), we can obtain the bounds for $|\nabla^2 D_t \mathrm{div} u|_{L^2}$, $|\Pi \nabla^3 D_t \mathrm{div} u|_{L^2}^2$ and $\|\nabla^3 D_t \mathrm{div} u\|_{L^2}$ in (3.62c) in a similar way to the derivation of (3.56b), because

$$\|D_t \mathrm{div} u\|_{L^\infty}^2 \le C(\mathcal{A}_3) \sum_{i=1}^3 E_i,$$

which follows from (A-2d), (3.47c) and (3.56a). The bound for $|\nabla^4 p|_{L^2}$ in (3.62c) can be derived easily by use of (A-5b), (3.55) and the bound just obtained for $\|\nabla^3 D_t \mathrm{div} u\|_{L^2}$.

It follows from (3.16), Hölder's inequality, (A-1b) and Young's inequality that

$$\|D_t^3\mathrm{div}u - \mathcal{T}\Delta D_t^2\mathrm{div}u\|_{L^2} \leq C(A_5)\sum_{i=0}^{1}\|\nabla^i D_t^2\mathrm{div}u\|_{L^2} + C(A_5)\sum_{i=0}^{2}\|\nabla^i D_t\mathrm{div}u\|_{L^2},$$
$$+ C(A_5)\sum_{i=1}^{3}\left(\|\nabla^i u\|_{L^2} + \|\nabla^i\mathcal{T}\|_{L^2} + \|\nabla^i p\|_{L^2} + \|\nabla^i\mathrm{div}u\|_{L^2} + \|\nabla^i D_t p\|_{L^2}\right), \tag{3.64}$$

where $A_5 = (K_1, \|\mathcal{T}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla\mathcal{T}\|_{L^\infty}, \|\nabla p\|_{L^\infty}, \|\nabla\mathrm{div}u\|_{L^\infty}, \|\nabla D_t p\|_{L^\infty}, \|D_t\mathrm{div}u\|_{L^\infty})$. With (3.64), we can first obtain the bounds for $\|\nabla^2 D_t^2\mathrm{div}u\|_{L^2}$ and $|\nabla D_t^2\mathrm{div}u|_{L^2}$, and then the bounds for $|\nabla^3 D_t p|_{L^2}$, $|\Pi\nabla^4 D_t p|_{L^2}$ and $\|\nabla^4 D_t p\|_{L^2}$ in (3.62d). Here we have used that

$$\|\nabla\mathrm{div}u\|_{L^\infty}^2 + \|\nabla D_t p\|_{L^\infty}^2 + \|D_t\mathrm{div}u\|_{L^\infty}^2 \leq C\left(\mathcal{A}_3, |\nabla\mathrm{div}u|_{L^\infty}, |\nabla D_t p|_{L^\infty}, \sum_{i=1}^{3}E_i\right), \tag{3.65}$$

which follows from (A-2d), (3.47c), (3.52) and (3.56).

It follows from (3.16), Hölder's inequality, (A-1b) and Young's inequality that for $r = 1, 2$,

$$\begin{aligned}\|\mathcal{T}\nabla\Delta D_t^2\mathrm{div}u\| \leq & C\|\nabla D_t^3\mathrm{div}u\| + \bar{C}(\|\nabla^2 u\|_{L^\infty} + \|\nabla^2\mathcal{T}\|_{L^\infty} + \|\nabla^4\mathrm{div}u\|_{L^2}) \\ & + \bar{C}\sum_{i=1}^{4}\left(\|\nabla^i u\|_{L^2} + \|\nabla^i\mathcal{T}\|_{L^2} + \|\nabla^i p\|_{L^2} + \|\nabla^i D_t p\|_{L^2}\right) \\ & + \bar{C}\sum_{i=1}^{2}\|\nabla^i D_t^2\mathrm{div}u\|_{L^2} + \bar{C}\sum_{i=0}^{3}\|\nabla^i D_t\mathrm{div}u\|_{L^2},\end{aligned} \tag{3.66}$$

where $\bar{C} = C(A_5, \sum_{i=0}^{1}\|\nabla^i D_t^2\mathrm{div}u\|_{L^2})$ and A_5 is given in (3.64). This, together with (3.47c) and (3.65), gives (3.62e) easily, by noting that

$$\|\nabla^2 u\|_{L^\infty}^2 + \|\nabla^2\mathcal{T}\|_{L^\infty}^2 \leq C(\mathcal{A}_3, |\nabla\mathrm{div}u|_{L^\infty})\sum_{i=1}^{4}E_i.$$

It follows from (3.51), Hölder's inequality, (A-1b) and Young's inequality that

$$\|\mathcal{T}\Delta D_t^2 p + (\nabla\mathcal{T})\cdot\nabla D_t^2 p\|_{L^2} \leq C\|D_t^3\mathrm{div}u\|_{L^2} + \bar{C}\left(\|\nabla D_t\mathrm{div}u\|_{L^2} + \|\nabla^2\mathcal{T}\|_{L^2}\right)$$
$$+ \bar{C}\sum_{i=0}^{2}\|\nabla^i D_t p\|_{L^2} + \bar{C}\sum_{i=1}^{3}\left(\|\nabla^i u\|_{L^2} + \|\nabla^i p\|_{L^2}\right), \tag{3.67}$$

where $\bar{C} = C(K_1, \|\mathcal{T}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla\mathcal{T}\|_{L^\infty}, \|\nabla p\|_{L^\infty}, \|D_t p\|_{L^\infty}, \|D_t\mathrm{div}u\|_{L^\infty})$; and

$$\|\mathcal{T}\nabla^r\Delta D_t^2 p\|_{L^2} \leq C\|\nabla^r D_t^3\mathrm{div}u\|_{L^2} + \bar{C}\sum_{i=0}^{r+1}\left(\|\nabla^i D_t^2 p\|_{L^2} + \|\nabla^i D_t\mathrm{div}u\|_{L^2}\right)$$
$$+ \bar{C}\sum_{i=1}^{r+3}\left(\|\nabla^i u\|_{L^2} + \|\nabla^i\mathcal{T}\|_{L^2} + \|\nabla^i p\|_{L^2} + \|\nabla^{i-1}D_t p\|_{L^2}\right), \quad r = 1, 2, \tag{3.68}$$

where $\bar{C} = C(K_1, \|\mathcal{T}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla\mathcal{T}\|_{L^\infty}, \|\nabla p\|_{L^\infty}, \|D_t\mathrm{div}u\|_{L^\infty}, \|D_t p\|_{L^\infty}, \|D_t^2 p\|_{L^\infty})$. With (3.67) and (3.68), we can obtain (3.62f) and (3.62g) in a similar way to the derivation of (3.52b) and (3.56b), respectively.

Step 4. Clearly, we can obtain

$$\|\nabla^5 u\|_{L^2}^2 + |\nabla^4 u|_{L^2}^2 \leq C(\mathcal{A}_3, |\nabla\mathrm{div}u|_{L^\infty})\sum_{i=1}^{5}E_i, \tag{3.69a}$$

$$\|\nabla^5 p\|_{L^2}^2 + |\overline{\nabla}^3\theta|_{L^2}^2 + \|\nabla^5\mathcal{T}\|_{L^2}^2 + |\nabla^5\mathcal{T}|_{L^2}^2 + |\nabla^4\mathrm{div}u|_{L^2}^2 + |\Pi\nabla^5\mathrm{div}u|_{L^2}^2 + \|\nabla^5\mathrm{div}u\|_{L^2}^2$$
$$\le C\left(\mathcal{A}_3, \sum_{j=0}^{1}|\nabla D_t^j\mathrm{div}u|_{L^\infty}, \sum_{i=1}^{4}E_i\right)\sum_{i=1}^{5}E_i, \tag{3.69b}$$
$$|\nabla^3 D_t\mathrm{div}u|_{L^2}^2 + |\Pi\nabla^4 D_t\mathrm{div}u|_{L^2}^2 + \|\nabla^4 D_t\mathrm{div}u\|_{L^2}^2 + |\nabla^5 p|_{L^2}^2$$
$$\le C\left(\mathcal{A}_3, |\nabla D_t p|_{L^\infty}, \sum_{j=0}^{1}|\nabla D_t^j\mathrm{div}u|_{L^\infty}, \sum_{i=1}^{4}E_i\right)\sum_{i=1}^{5}E_i, \tag{3.69c}$$
$$|\nabla^4 D_t p|_{L^2}^2 + |\Pi\nabla^5 D_t p|_{L^2}^2 + \|\nabla^5 D_t p\|_{L^2}^2$$
$$\le C\left(\mathcal{A}_3, |\nabla D_t p|_{L^\infty}, \sum_{j=0}^{2}|\nabla D_t^j\mathrm{div}u|_{L^\infty}, \sum_{i=1}^{4}E_i\right)\sum_{i=1}^{5}E_i, \tag{3.69d}$$
$$\|\nabla^2 D_t^3\mathrm{div}u\|_{L^2}^2 \le C\left(\mathcal{A}_3, |\nabla D_t p|_{L^\infty}, \sum_{j=0}^{1}|\nabla D_t^j\mathrm{div}u|_{L^\infty}, \sum_{i=1}^{4}E_i\right)\sum_{i=1}^{5}E_i, \tag{3.69e}$$
$$\sum_{i=0}^{2}\|\nabla^i D_t^3 p\|_{L^2}^2 + |\nabla D_t^3 p|_{L^2}^2 \le C\left(\mathcal{A}_3, |\nabla D_t p|_{L^\infty}, |\nabla\mathrm{div}u|_{L^\infty}, \sum_{i=1}^{4}E_i\right)\sum_{i=1}^{5}E_i, \tag{3.69f}$$
$$|\nabla^3 D_t^2\mathrm{div}u|_{L^2}^2 + |\Pi\nabla^4 D_t^2\mathrm{div}u|_{L^2}^2 + \|\nabla^4 D_t^2\mathrm{div}u\|_{L^2}^2$$
$$\le C\left(\mathcal{A}_3, |\nabla D_t p|_{L^\infty}, \sum_{j=0}^{2}|\nabla D_t^j\mathrm{div}u|_{L^\infty}, \sum_{i=1}^{4}E_i\right)\sum_{i=1}^{5}E_i, \tag{3.69g}$$
$$|\nabla^3 D_t^2 p|_{L^2}^2 + |\Pi\nabla^4 D_t^2 p|_{L^2}^2 + \|\nabla^4 D_t^2 p\|_{L^2}^2$$
$$\le C\left(\mathcal{A}_3, \sum_{j=1}^{2}|\nabla D_t^j p|, \sum_{j=0}^{1}|\nabla D_t^j\mathrm{div}u|, \sum_{i=1}^{4}E_i\right)\sum_{i=1}^{5}E_i, \tag{3.69h}$$

where $\mathcal{A}_3$ is given by (3.50).

Indeed, we have used (3.28) and the following two estimate to derive (3.69e)-(3.69g).

$$\|\mathcal{T}\Delta D_t^3 p + (\nabla\mathcal{T})\cdot\nabla D_t^3 p\|_{L^2} \le C\|D_t^4\mathrm{div}u\|_{L^2} + \bar{C}\|\nabla D_t^2\mathrm{div}u\|_{L^2} + \bar{C}\sum_{i=0}^{2}(\|\nabla^i D_t\mathrm{div}u\|_{L^2}$$
$$+ \|\nabla^i D_t^2 p\|_{L^2} + \|\nabla^{i+1}\mathcal{T}\|_{L^2}) + \bar{C}\sum_{i=1}^{4}(\|\nabla^i u\|_{L^2} + \|\nabla^i p\|_{L^2} + \|\nabla^{i-1}D_t p\|_{L^2})$$

where $\bar{C} = C(K_1, \|\mathcal{T}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla\mathcal{T}\|_{L^\infty}, \|\nabla p\|_{L^\infty}, \|D_t^i p\|_{L^\infty}, \|D_t^i\mathrm{div}u\|_{L^\infty})$ $(i = 1, 2)$.

$$\begin{aligned}\|\mathcal{T}\nabla^2\Delta D_t^2\mathrm{div}u\|_{L^2} \le{}& C\|\nabla^2 D_t^3\mathrm{div}u\|_{L^2} + \bar{C}\sum_{i=0}^{3}(\|\nabla^i D_t^2\mathrm{div}u\|_{L^2} + \|\nabla^{i+1}D_t p\|_{L^2})\\ &+ \bar{C}\sum_{i=1}^{5}\left(\|\nabla^i u\|_{L^2} + \|\nabla^i\mathcal{T}\|_{L^2} + \|\nabla^i p\|_{L^2} + \|\nabla^{i-1}D_t\mathrm{div}u\|_{L^2}\right).\end{aligned} \tag{3.70}$$

where $\bar{C} = C(K_1, \|\nabla^j\mathcal{T}\|_{L^\infty}, \|\nabla^i u\|_{L^\infty}, \|\nabla^i p\|_{L^\infty}, \|D_t^i\mathrm{div}u\|_{L^\infty}, \|\nabla D_t p\|_{L^\infty})$ $(i = 1, 2, j = 0, 1, 2)$.

3.4. **Proof of Theorem 3.1.** First, we derive the estimate for the zeroth order energy E_0:

$$\frac{d}{dt}E_0(t) \le \|p\|_{L^2}^2 + \|\mathrm{div}u\|_{L^2}^2. \tag{3.71}$$

Indeed, one can see easily from (3.1a) and (3.1b) that

$$\frac{1}{2}D_t\left(\mathcal{T}^{-1}|u|^2\right) + \frac{1}{2}\mathcal{T}^{-1}|u|^2\mathrm{div}u = -\mathrm{div}(pu) + p\mathrm{div}u,$$

which, together with (3.20) and (3.9), implies

$$\frac{d}{dt}\int_\Omega \mathcal{T}^{-1}|u|^2 d\mu_g = 2\int_\Omega p\mathrm{div}u d\mu_g \le \int_\Omega (p^2 + |\mathrm{div}u|^2) d\mu_g.$$

It follows from (3.71), (3.13), (3.30), (3.47), (3.48) and (3.49) that

$$\frac{d}{dt}E_0(t) \le C(\mathfrak{B}_0)(E_1 + E_2), \tag{3.72a}$$

$$\frac{d}{dt}E_1(t) \le C(\mathfrak{B}_1)(E_1 + E_2), \tag{3.72b}$$

$$\frac{d}{dt}E_2(t) \le C(\mathfrak{B}_2)(E_1 + E_2), \tag{3.72c}$$

$$\frac{d}{dt}E_r(t) \le C\left(\mathfrak{B}_r,\ \sum_{i=1}^{r-1} E_i(t)\right)\sum_{i=1}^{r} E_i(t), \quad r \ge 3, \tag{3.72d}$$

where

$$\begin{aligned}
\mathfrak{B}_0 &= (K, K_1, \mathrm{Vol}\Omega, \|\mathcal{T}^{-1}\|_{L^\infty}, \|\mathcal{T}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla\mathcal{T}\|_{L^\infty}, \|\nabla p\|_{L^\infty}, |\nabla p|_{L^\infty}),\\
\mathfrak{B}_1 &= (\mathcal{F}_0, \|\nabla^2\mathcal{T}\|_{L^\infty}),\\
\mathfrak{B}_2 &= (\mathcal{F}_1, \|\nabla\mathrm{div}u\|_{L^\infty}, |\nabla u|_{L^\infty}, |(\nabla_N p)^{-1}|_{L^\infty}, |\nabla D_t p|_{L^\infty}, |\nabla^2 p|_{L^\infty}),\\
\mathfrak{B}_r &= \left(\mathcal{F}_2, \|D_t\mathrm{div}u\|_{L^\infty}, |\nabla\mathcal{T}|_{L^\infty}, \sum_{j=0}^{r-3}|\nabla D_t^j\mathrm{div}u|_{L^\infty}, \sum_{j=0}^{r-3}|\nabla D_t^j p|_{L^\infty}\right).
\end{aligned}$$

Indeed, (3.72a) follows from (3.71), (3.52b) and (3.47c); (3.72b) follows from (3.13a), (3.30a), (3.48a) and (3.52b); (3.72c) follows from (3.22), (3.30b), (3.47), (3.48) and (3.52); and the following estimates have been used to derive (3.72d).

$$\|\nabla D_t p\|_{L^\infty} \le C\left(\mathcal{A}_3, |\nabla D_t p|_{L^\infty}, \sum_{i=1}^{3} E_i\right),$$

$$\|\nabla^2 u\|_{L^\infty}^2 + \|\nabla^2\mathrm{div}u\|_{L^\infty}^2 + \|D_t^2\mathrm{div}u\|_{L^\infty}^2 \le C\left(\mathcal{A}_3, |\nabla\mathrm{div}u|_{L^\infty}, |\nabla D_t p|_{L^\infty}, \sum_{i=1}^{4} E_i\right),$$

which follows from (A-2d), (3.47c), (3.52a), (3.56), (3.62b) and (3.62d). Here $\mathcal{A}_3$ is defined by (3.50). It is produced by substituting (3.7), (3.8), (3.10) into (3.72) that there are continuous functions $\mathfrak{C}_r$ $(0 \le r \le n+2)$ such that

$$\begin{aligned}
&\frac{d}{dt}E_0(t) \le \mathfrak{C}_0\left(\overline{V}, K, M, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}\right)(E_1(t) + E_2(t)),\\
&\frac{d}{dt}E_1(t) \le \mathfrak{C}_1\left(\overline{V}, K, M, \widetilde{M}, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}\right)(E_1(t) + E_2(t)),\\
&\frac{d}{dt}E_2(t) \le \mathfrak{C}_2\left(\overline{V}, K, \epsilon_b^{-1}, L, M, \widetilde{M}, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}\right)(E_1(t) + E_2(t)),
\end{aligned}$$

$$\frac{d}{dt}E_r(t) \leq \mathfrak{C}_r\left(\overline{V}, K, \epsilon_b^{-1}, L, M, \widetilde{M}, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}, \sum_{i=1}^{r-1} E_i(t)\right)\sum_{i=1}^{r} E_i(t), \quad 3 \leq r \leq n+2.$$

This concludes the proof of Theorem 3.1.

3.5. **Proof of Theorem 3.2.** The proof follows from the Lemmas 3.9 and 3.10.

Lemma 3.9. *Let $n = 2, 3$. Then there are continuous functions $T_{n1} > 0$ such that*

$$\begin{aligned}
&\sum_{s=0}^{r} E_s(t) \leq 2\sum_{s=0}^{r} E_s(0), \quad 2 \leq r \leq n+2, \qquad 2^{-1}\mathrm{Vol}\mathscr{D}_0 \leq \mathrm{Vol}\Omega(t) \leq 2\mathrm{Vol}\mathscr{D}_0, \\
&|\theta(t,\cdot)|_{L^\infty} + \iota_0^{-1}(t) \leq 18K_0, \qquad -\nabla_N p(t,y) \geq 2^{-1}\underline{\epsilon_0} \ \text{ for } \ y \in \partial\Omega, \\
&\|\nabla p(t,\cdot)\|_{L^\infty} + \|\nabla u(t,\cdot)\|_{L^\infty} + \|\nabla \mathcal{T}(t,\cdot)\|_{L^\infty} \leq 2M_0,
\end{aligned} \tag{3.73}$$

for $t \leq T_{n1}(\mathrm{Vol}\mathscr{D}_0, K_0, \underline{\epsilon_0}^{-1}, L, \widetilde{M}, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}, M_0, E_0(0), \cdots, E_{n+2}(0))$.

Proof. It follows from (3.20) that

$$\left|\frac{d}{dt}\mathrm{Vol}\Omega(t)\right| = \left|\frac{d}{dt}\int_\Omega d\mu_g\right| = \left|\int_\Omega \mathrm{div} u \, d\mu_g\right| \leq M\mathrm{Vol}\Omega(t),$$

which implies $\mathrm{Vol}\Omega(0)\exp\{-Mt\} \leq \mathrm{Vol}\Omega(t) \leq \mathrm{Vol}\Omega(0)\exp\{Mt\}$. Thus we have, due to the fact $\mathrm{Vol}\Omega(0) = \mathrm{Vol}\mathscr{D}_0$, that for $t \leq M^{-1}\ln 2$,

$$2^{-1}\mathrm{Vol}\mathscr{D}_0 \leq \mathrm{Vol}\Omega(t) \leq 2\mathrm{Vol}\mathscr{D}_0. \tag{3.74}$$

It follows from (3.38), (3.7c)-(3.7e) and (2.8a) that

$$|D_t(-\nabla_N p)^{-1}| \leq |(-\nabla_N p)^{-1}|^2(|h_{NN}\nabla_N p| + |\nabla_N D_t p|) \leq \epsilon_b^{-2}(M^2 + L) \ \text{ on } \ \partial\Omega,$$

which, together with $\partial_{\mathcal{N}} p = \nabla_N p$ and (1.13a), implies that for $t \leq \epsilon_b^2(M^2+L)^{-1}\underline{\epsilon_0}^{-1}$,

$$-\nabla_N p(t,y) \geq 2^{-1}\underline{\epsilon_0} \ \text{ for } \ y \in \partial\Omega. \tag{3.75}$$

Let $\epsilon_1 \in (0, 1/2]$ be a fixed constant (for example, $\epsilon_1 = 1/4$), $\iota_1(0)$ the largest number such that

$$\begin{aligned}
&|\mathcal{N}(x(0,y_1)) - \mathcal{N}(x(0,y_2))| \leq \epsilon_1/2, \\
&\qquad \text{whenever } \ |x(0,y_1) - x(0,y_2)| \leq 2\iota_1(0), \quad y_1, y_2 \in \partial\Omega.
\end{aligned} \tag{3.76}$$

Then we have from (2.7) and (1.13a) that

$$\iota_1(0) \geq 2^{-1}K_0^{-1}\epsilon_1. \tag{3.77}$$

Due to $\partial_t x(t,y) = v(t, x(t,y))$ in Ω, $|D_t\mathcal{N}| \leq 2|\nabla u| \leq 2M$ on $\partial\Omega$, and

$$\|v(t,x(t,\cdot))\|_{L^\infty(\Omega)} \leq 2\|v(0,x(0,\cdot))\|_{L^\infty(\Omega)} \ \text{ for } \ t \leq (\overline{\mathcal{T}}M)^{-1}\|v(0,x(0,\cdot))\|_{L^\infty(\Omega)}, \tag{3.78}$$

we have

$$|x(t,y) - x(0,y)| \leq 2^{-1}\iota_1(0) \ \text{ for } \ y \in \Omega \ \text{ and } \ t \leq T_1, \tag{3.79}$$

$$|\mathcal{N}(x(t,y)) - \mathcal{N}(x(0,y))| \leq 4^{-1}\epsilon_1 \ \text{ for } \ y \in \partial\Omega \ \text{ and } \ t \leq 8^{-1}M^{-1}\epsilon_1, \tag{3.80}$$

where $T_1 = \min\{(\overline{\mathcal{T}}M)^{-1}\|v(0,x(0,\cdot))\|_{L^\infty(\Omega)}, \ 4^{-1}\|v(0,x(0,\cdot))\|_{L^\infty(\Omega)}^{-1}\iota_1(0)\}$. Indeed, the bound $|D_t v| = |\mathcal{T}\partial p| = |\mathcal{T}\nabla p| \leq \overline{\mathcal{T}}M$ in Ω, which follows from (1.2a), (3.10) and (3.7e), has been used to derive (3.78). It follows from (3.76), (3.79) and (3.80) that for $t \leq \min\{T_1, \ 8^{-1}M^{-1}\epsilon_1\}$,

$$\begin{aligned}
&|\mathcal{N}(x(t,y_1)) - \mathcal{N}(x(t,y_2))| \leq \epsilon_1, \\
&\qquad \text{whenever } \ |x(t,y_1) - x(t,y_2)| \leq \iota_1(0), \quad y_1, y_2 \in \partial\Omega.
\end{aligned} \tag{3.81}$$

This, together with (3.77), implies that for $t \leq \min\{T_1, \ 8^{-1}M^{-1}\epsilon_1\}$,

$$\iota_1(t) \geq \iota_1(0) \geq 2^{-1}K_0^{-1}\epsilon_1. \tag{3.82}$$

It follows from Theorem 3.1 that there are continuous functions $T_r > 0$ such that

$$\sum_{s=0}^{r} E_s(t) \le 2 \sum_{s=0}^{r} E_s(0), \quad 2 \le r \le n+2, \tag{3.83}$$

for $t \le T_r(\overline{V}, K, \epsilon_b^{-1}, L, M, \widetilde{M}, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}, E_1(0), \cdots, E_{r-1}(0))$. This, together with (A-2b), (A-2d), (3.49), (3.7), (3.8) and (3.10), implies that

$$\begin{aligned} &\|\nabla D_t p(t,\cdot)\|_{L^\infty}^2 + \|\nabla^2 p(t,\cdot)\|_{L^\infty}^2 + |\nabla^2 u(t,\cdot)|_{L^\infty}^2 \\ \le & C\left(\overline{V}, K, \epsilon_b^{-1}, L, M, \widetilde{M}, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}, \sum_{i=0}^{n+2} E_i(0)\right), \quad t \le T_{n+2}. \end{aligned} \tag{3.84}$$

Notice that

$$|D_t \nabla p| = |\nabla D_t p| \le \|\nabla D_t p\|_{L^\infty} \text{ in } \Omega,$$

$$|D_t \nabla u| \le \mathcal{T}|\nabla^2 p| + |\nabla \mathcal{T}||\nabla u| + |\nabla u|^2 \le \overline{\mathcal{T}}\|\nabla^2 p\|_{L^\infty} + 2M^2 \text{ in } \Omega, \tag{3.85}$$

$$|D_t \nabla \mathcal{T}| \le \mathcal{T}|\nabla \text{div} u| + |\nabla \mathcal{T}||\nabla u| \le \overline{\mathcal{T}}\widetilde{M} + M^2 \text{ in } \Omega, \tag{3.86}$$

$$|D_t \theta| \le |\nabla^2 u| + C|\theta||\nabla u| \le |\nabla^2 u|_{L^\infty} + CKM \text{ on } \partial\Omega,$$

where (3.85) (or respectively, (3.86)) follows from (3.2) (or respectively, (3.1b)), (3.10) and (3.7). Then we have from (1.13a), (1.13c), (2.4b), (2.8b), the fact that $|\partial p| = |\nabla p|$, $|\partial v| = |\nabla u|$ and $|\partial \mathcal{T}| = |\nabla \mathcal{T}|$, and (3.84) that there is a continuous function $T_6 > 0$ such that

$$\|\nabla p(t,\cdot)\|_{L^\infty} + \|\nabla u(t,\cdot)\|_{L^\infty} + \|\nabla \mathcal{T}(t,\cdot)\|_{L^\infty} \le 2M_0 \text{ and } |\theta(t,\cdot)|_{L^\infty} \le 2K_0, \tag{3.87}$$

for $t \le T_6(\overline{V}, K, \epsilon_b^{-1}, L, M, \widetilde{M}, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}, E_0(0), \cdots, E_{n+2}(0), M_0, K_0)$. Moreover, we can derive from (2.7), (3.87) and (3.82) that for $t \le \min\{T_1,\ 8^{-1}M^{-1}\epsilon_1,\ T_6\}$,

$$\iota_0^{-1}(t) \le \max\{2\iota_1^{-1}(t),\ 2K_0\} \le 4\epsilon_1^{-1}K_0. \tag{3.88}$$

Clearly, there is a continuous function $T_7 > 0$ such that (3.74), (3.75), (3.83), (3.87) and (3.88) hold for $t \le T_7(\overline{V}, K, \epsilon_b^{-1}, L, M, \widetilde{M}, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}, E_0(0), \cdots, E_{n+2}(0), \underline{\epsilon}_0, M_0, K_0, \text{Vol}\mathscr{D}_0)$, due to

$$\|v(0, x(0,\cdot))\|_{L^\infty(\Omega)}^2 \le C(\text{Vol}\mathscr{D}_0, K_0, M_0, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}) \sum_{i=0}^{2} E_i(0)$$

which follows from $|v| = |u|$, (A-2d), (3.47a), (3.48a), (3.52a) and (3.10). So, (3.73) holds for $t \le T_{n1}$ for some continuous function

$$T_{n1}(\text{Vol}\mathscr{D}_0, K_0, \underline{\epsilon}_0^{-1}, L, \widetilde{M}, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}, E_0(0), \cdots, E_{n+2}(0), M_0) > 0,$$

by choosing $\overline{V} = 4\text{Vol}\mathscr{D}_0$, $\epsilon_b = 4^{-1}\underline{\epsilon}_0$, $M = 4M_0$, and $K = 2(2 + 4\epsilon_1^{-1})K_0 = 36K_0$ with ϵ_1 being 4^{-1} in T_7.

□

Lemma 3.10. *Let $n = 2, 3$. Then there are continuous functions $T_{n2} > 0$ such that*

$$\begin{aligned} &\sum_{i=1}^{n-1} \left(|\nabla D_t^i p(t,\cdot)|_{L^\infty} + |\nabla D_t^i \text{div} u(t,\cdot)|_{L^\infty}\right) + |\nabla^2 p(t,\cdot)|_{L^\infty} \\ &+ \|\nabla \text{div} u(t,\cdot)\|_{L^\infty} + \|D_t \text{div} u(t,\cdot)\|_{L^\infty} + \|\nabla^2 \mathcal{T}(t,\cdot)\|_{L^\infty} \\ &\le C\left(\text{Vol}\mathscr{D}_0, K_0, \underline{\epsilon}_0^{-1}, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}, M_0, E_0(0), \cdots, E_{n+2}(0)\right) \end{aligned} \tag{3.89}$$

for $t \le T_{n2}(\text{Vol}\mathscr{D}_0, K_0, \underline{\epsilon}_0^{-1}, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}, M_0, E_0(0), \cdots, E_{n+2}(0))$.

Moreover, (3.73) also holds for $t \le T_{n2}$.

Proof. It follows from (A-2d), (3.47c), (3.56a), (3.48b), (3.62a), (3.8) and (3.73) that

$$\begin{aligned}&\|D_t\mathrm{div}u(t,\cdot)\|_{L^\infty}\le C_1\left(\mathrm{Vol}\mathscr{D}_0,K_0,\epsilon_0^{-1},\underline{\mathcal{T}}^{-1},\overline{\mathcal{T}},M_0,E_0(0),\cdots,E_3(0)\right),\quad t\le T_{n1},\\&\|\nabla^2\mathcal{T}(t,\cdot)\|_{L^\infty}\le C_2\left(\mathrm{Vol}\mathscr{D}_0,K_0,\epsilon_0^{-1},\underline{\mathcal{T}}^{-1},\overline{\mathcal{T}},M_0,E_0(0),\cdots,E_4(0)\right),\quad t\le T_{n1},\end{aligned}\tag{3.90}$$

where T_{n1} is defined in Lemma 3.9. The proof of the remainder terms is divided into two cases.

Case 1. Let n=2. It follows from (A-2b), (3.52b), (3.56a), (3.8) and (3.73) that for $t\le T_{21}$,

$$|\nabla^2 p|_{L^\infty}\le C_3\left(\mathrm{Vol}\mathscr{D}_0,K_0,\epsilon_0^{-1},\underline{\mathcal{T}}^{-1},\overline{\mathcal{T}},M_0,E_0(0),\cdots,E_3(0)\right).\tag{3.91}$$

It follows from (2.21) and (3.52c) that

$$|\nabla\mathrm{div}u|_{L^\infty}\le\|\nabla\Delta\mathrm{div}u\|_{L^2}+C\left(K,K_1,|(\nabla_N p)^{-1}|_{L^\infty},|\nabla p|_{L^\infty},E_2,\mathrm{Vol}\Omega\right)\|\Delta\mathrm{div}u\|_{L^2},$$

which, together with (3.59), (3.48), (3.52a), (3.56a), (3.8) and (3.73), implies that for $t\le T_{21}$,

$$|\nabla\mathrm{div}u(t,\cdot)|_{L^\infty}\le C\left(\mathrm{Vol}\mathscr{D}_0,K_0,\epsilon_0^{-1},\underline{\mathcal{T}}^{-1},\overline{\mathcal{T}},M_0,E_0(0),\cdots,E_3(0)\right).\tag{3.92}$$

With the bound just obtained for $|\nabla\mathrm{div}u|_{L^\infty}$ in (3.92), we get that for $t\le T_{21}$,

$$\|\nabla\mathrm{div}u(t,\cdot)\|_{L^\infty}\le C_4\left(\mathrm{Vol}\mathscr{D}_0,K_0,\epsilon_0^{-1},\underline{\mathcal{T}}^{-1},\overline{\mathcal{T}},M_0,E_0(0),\cdots,E_3(0)\right),\tag{3.93}$$

due to (A-2d), (3.52a), (3.56b), (3.8) and (3.73). In a similar way to deriving (3.92), we can obtain that for $t\le T_{21}$,

$$|\nabla D_t p(t,\cdot)|_{L^\infty}\le C_5\left(\mathrm{Vol}\mathscr{D}_0,K_0,\epsilon_0^{-1},\underline{\mathcal{T}}^{-1},\overline{\mathcal{T}},M_0,E_0(0),\cdots,E_3(0)\right),\tag{3.94}$$

$$|\nabla D_t\mathrm{div}u(t,\cdot)|_{L^\infty}\le C_6\left(\mathrm{Vol}\mathscr{D}_0,K_0,\epsilon_0^{-1},\underline{\mathcal{T}}^{-1},\overline{\mathcal{T}},M_0,E_0(0),\cdots,E_4(0)\right),\tag{3.95}$$

where (3.60) and (3.63) have been used to derive (3.94) and (3.95), respectively.

So, it follows from (3.90)-(3.95) that (3.89) holds for $t\le T_{22}$ for some continuous function $T_{22}(\mathrm{Vol}\mathscr{D}_0,K_0,\epsilon_0^{-1},\underline{\mathcal{T}}^{-1},\overline{\mathcal{T}},M_0,E_0(0),\cdots,E_4(0))>0$, by choosing $L=2(C_3+C_5+C_6)$ and $\widetilde{M}=2(C_1+C_2+C_4)$ in the continuous function T_{21} given by Lemma 3.9. Clearly, (3.73) holds for $t\le T_{22}$.

Case 2. Let $n=3$. We first show how to get the bound for $|\nabla\mathrm{div}u|_{L^\infty}$. It follows from (3.9), (A-5c) and (2.13) that for any $\delta\in(0,1]$,

$$\begin{aligned}&\|\nabla^3\mathrm{div}u\|_{L^2}+|\nabla^2\mathrm{div}u|_{L^2}\\\le&\delta|\Pi\nabla^3\mathrm{div}u|_{L^2}+C(\delta^{-1},K,\mathrm{Vol}\Omega)\sum_{s=0}^{1}\|\nabla^s\Delta\mathrm{div}u\|_{L^2}\\\le&\delta\left(|\nabla\mathrm{div}u|_{L^\infty}|\overline{\nabla}\theta|_{L^2}+3K|\nabla^2\mathrm{div}u|_{L^2}+2K^2|\nabla\mathrm{div}u|_{L^2}\right)\\&+C(\delta^{-1},K,\mathrm{Vol}\Omega)\sum_{s=0}^{1}\|\nabla^s\Delta\mathrm{div}u\|_{L^2},\end{aligned}$$

which implies, by choosing $\delta=\min\{(6K)^{-1},1\}$, that

$$\begin{aligned}&\|\nabla^3\mathrm{div}u\|_{L^2}+2^{-1}|\nabla^2\mathrm{div}u|_{L^2}\\\le&|\nabla\mathrm{div}u|_{L^\infty}|\overline{\nabla}\theta|_{L^2}+C(K)|\nabla\mathrm{div}u|_{L^2}+C(K,\mathrm{Vol}\Omega)\sum_{s=0}^{1}\|\nabla^s\Delta\mathrm{div}u\|_{L^2}.\end{aligned}$$

This, together with (A-3a), gives

$$\|\nabla^4u\|_{L^2}^2\le C\|\nabla^3\mathrm{div}u\|_{L^2}^2+C(\|\mathcal{T}\|_{L^\infty}+1)E_4\le C|\nabla\mathrm{div}u|_{L^\infty}^2|\overline{\nabla}\theta|_{L^2}^2+L_1,\tag{3.96}$$

where

$$L_1=C(K)|\nabla\mathrm{div}u|_{L^2}^2+C(K,\mathrm{Vol}\Omega)\sum_{s=0}^{1}\|\nabla^s\Delta\mathrm{div}u\|_{L^2}^2+C(\|\mathcal{T}\|_{L^\infty})E_4.$$

It follows from (2.21), (3.59) and (3.96) that for any $\delta \in (0,1]$,

$$\begin{aligned} |\nabla \mathrm{div} u|_{L^\infty}^2 \le & \delta \|\nabla^2 \Delta \mathrm{div} u\|_{L^2}^2 + C(\delta^{-1}, K, K_1, |\overline{\nabla}\theta|_{L^2}, \mathrm{Vol}\Omega) \sum_{s=0}^{1} \|\nabla^s \Delta \mathrm{div} u\|_{L^2}^2 \\ \le & \delta C(K_1, \|\nabla u\|_{L^\infty}, \|\nabla \mathcal{T}\|_{L^\infty}) \|\mathcal{T}^{-1}\|_{L^\infty}^2 \|\nabla^4 u\|_{L^2}^2 + L_2 \\ \le & \delta C(K_1, \|\mathcal{T}^{-1}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla \mathcal{T}\|_{L^\infty}) |\nabla \mathrm{div} u|_{L^\infty}^2 |\overline{\nabla}\theta|_{L^2}^2 + L_3, \end{aligned} \tag{3.97}$$

where

$$\begin{aligned} L_2 = & C(\delta^{-1}, K, K_1, |\overline{\nabla}\theta|_{L^2}, \mathrm{Vol}\Omega) \sum_{s=0}^{1} \|\nabla^s \Delta \mathrm{div} u\|_{L^2}^2 + \delta C(\|\mathcal{T}^{-1}\|_{L^\infty}) \|\nabla^2 D_t \mathrm{div} u\|_{L^2}^2 \\ & + \delta C(K_1, \|\mathcal{T}^{-1}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla \mathcal{T}\|_{L^\infty}) \left(\sum_{i=1}^{3} \|\nabla^i u\|_{L^2}^2 + \sum_{i=1}^{4} \|\nabla^i \mathcal{T}\|_{L^2}^2 \right) \end{aligned}$$

and

$$L_3 = \delta C(K_1, \|\mathcal{T}^{-1}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla \mathcal{T}\|_{L^\infty}) L_1 + L_2.$$

By choosing δ suitably small in (3.97), and using (3.1), (3.8) and (3.73), we have that for $t \le T_{31}$,

$$|\nabla \mathrm{div} u(t,\cdot)|_{L^\infty} \le C_7 \left(\mathrm{Vol}\mathscr{D}_0, K_0, \epsilon_0^{-1}, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}, M_0, E_0(0), \cdots, E_4(0) \right). \tag{3.98}$$

In a similar way to the derivation of (3.98), we have, using (3.60), (3.63), (3.68), (3.66) and (3.70), that for $t \le T_{31}$,

$$|\nabla D_t p(t,\cdot)|_{L^\infty} \le C_8 \left(\mathrm{Vol}\mathscr{D}_0, K_0, \epsilon_0^{-1}, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}, M_0, E_0(0), \cdots, E_4(0) \right), \tag{3.99}$$

$$\begin{aligned} & |\nabla D_t \mathrm{div} u(t,\cdot)|_{L^\infty} + |\nabla D_t^2 p(t,\cdot)|_{L^\infty} + |\nabla D_t^2 \mathrm{div} u(t,\cdot)|_{L^\infty} \\ & \quad \le C_9 \left(\mathrm{Vol}\mathscr{D}_0, K_0, \epsilon_0^{-1}, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}, M_0, E_0(0), \cdots, E_5(0) \right). \end{aligned} \tag{3.100}$$

With these bounds, we can obtain, in the same manner as the case of $n = 2$, that for $t \le T_{31}$,

$$\|\nabla \mathrm{div} u(t,\cdot)\|_{L^\infty} \le C_{10} \left(\mathrm{Vol}\mathscr{D}_0, K_0, \epsilon_0^{-1}, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}, M_0, E_0(0), \cdots, E_4(0) \right), \tag{3.101}$$

$$|\nabla^2 p(t,\cdot)|_{L^\infty} \le C_{11} \left(\mathrm{Vol}\mathscr{D}_0, K_0, \epsilon_0^{-1}, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}, M_0, E_0(0), \cdots, E_5(0) \right). \tag{3.102}$$

It yields from (3.90), (3.99)-(3.102) that (3.89) holds for $t \le T_{32}$ for some continuous function $T_{32}(\mathrm{Vol}\mathscr{D}_0, K_0, \epsilon_0^{-1}, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}, M_0, E_0(0), \cdots, E_5(0)) > 0$, by choosing $L = 2(C_8 + C_9 + C_{11})$ and $\widetilde{M} = 2(C_1 + C_2 + C_{10})$ in the continuous function T_{31} given by Lemma 3.9. Clearly, (3.73) holds for $t \le T_{32}$.

□

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Appendix: Sobolev lemmas, interpolation inequalities and elliptic estiamtes

We list here Sobolev lemmas and interpolation inequalities with the dependence of the Sobolev constants on the lower bound of $1/\iota_1$; and the elliptic estimates for the boundary problem. Before that, recall the following notations for easier reference. The L^p-norms of a $(0,r)$-tensor α on Ω and $\partial\Omega$ are denoted, respectively, by $\|\alpha\|_{L^p}$ and $|\alpha|_{L^p}$:

$$\|\alpha\|_{L^p} = \left(\int_\Omega |\alpha|^p d\mu_g \right)^{1/p} \quad \text{for} \quad 1 \le p < \infty, \quad \|\alpha\|_{L^\infty} = \text{ess sup}_\Omega |\alpha|$$

and

$$|\alpha|_{L^p} = \left(\int_{\partial\Omega} |\alpha|^p d\mu_\zeta \right)^{1/p} \quad \text{for} \quad 1 \le p < \infty, \quad |\alpha|_{L^\infty} = \text{ess sup}_{\partial\Omega} |\alpha|.$$

Lemma A-1. (Lemmas A.1-A.4 in [11]) *Let α be a $(0,r)$ tensor and $\iota_1 \ge 1/K_1$. Assume k, m are positive integers, and $p \ge 1$. We have*
(i) if $2 \le p \le s \le q \le \infty$ and $m/s = k/p + (m-k)/q$,

$$|\overline{\nabla}^k \alpha|^m_{L^s} \le C(k,m,n,s) |\alpha|^{m-k}_{L^q} |\overline{\nabla}^m \alpha|^k_{L^p}, \tag{A-1a}$$

$$\left(\sum_{i=0}^{k} \|\nabla^i \alpha\|_{L^s} \right)^m \le C(k,m,n,s) \|\alpha\|^{m-k}_{L^q} \left(\sum_{i=0}^{m} K_1^{m-i} \|\nabla^i \alpha\|_{L^p} \right)^k; \tag{A-1b}$$

(ii) for any $\delta > 0$,

$$|\alpha|_{L^{(n-1)p/(n-1-kp)}} \le C(k,n,p) \sum_{i=0}^{k} K_1^{k-i} |\nabla^i \alpha|_{L^p}, \quad 1 \le p < (n-1)/k, \tag{A-2a}$$

$$|\alpha|_{L^\infty} \le \delta |\nabla^k \alpha|_{L^p} + C(\delta^{-1}, K_1, k, n, p) \sum_{i=0}^{k-1} |\nabla^i \alpha|_{L^p}, \quad p > (n-1)/k, \tag{A-2b}$$

$$\|\alpha\|_{L^{np/(n-kp)}} \le C(k,n,p) \sum_{i=0}^{k} K_1^{k-i} \|\nabla^i \alpha\|_{L^p}, \quad 1 \le p < n/k, \tag{A-2c}$$

$$\|\alpha\|_{L^\infty} \le C(k,n,p) \sum_{i=0}^{k} K_1^{k-i} \|\nabla^i \alpha\|_{L^p}, \quad p > n/k. \tag{A-2d}$$

Lemma A-2. (Lemmas 5.5-5.6 in [11]) *Let w be a $(0,1)$ tensor and define a scalar* $\text{div} w = g^{ab} \nabla_a w_b$ *and a $(0,2)$ tensor* $\text{curl} w_{ab} = \nabla_a w_b - \nabla_b w_a$. *If $|\theta| + 1/\iota_0 \le K$, then for any nonnegative integer r,*

$$|\nabla^{r+1} w|^2 \le C \left(g^{ij} \zeta^{kl} \zeta^{IJ} (\nabla_k \nabla^r_I w_i) \nabla_l \nabla^r_J w_j + |\nabla^r \text{div} w|^2 + |\nabla^r \text{curl} w|^2 \right), \tag{A-3a}$$

$$\begin{aligned} \|\nabla^{r+1} w\|^2_{L^2} \le {} & C \int_\Omega \widetilde{N}^i \widetilde{N}^j g^{kl} \zeta^{IJ} (\nabla_k \nabla^r_I w_i) \nabla_l \nabla^r_J w_j d\mu_g \\ & + C \left(\|\nabla^r \text{div} w\|^2_{L^2} + \|\nabla^r \text{curl} w\|^2_{L^2} + K^2 \|\nabla^r w\|^2_{L^2} \right), \end{aligned} \tag{A-3b}$$

$$|\nabla^r w|^2_{L^2} \le C \left(\|\nabla^{r+1} w\|_{L^2} + K \|\nabla^r w\|_{L^2} \right) \|\nabla^r w\|_{L^2}, \tag{A-3c}$$

$$|\nabla^r w|^2_{L^2} \le C |\Pi \nabla^r w|^2_{L^2} + C \left(\|\nabla^r \text{div} w\|_{L^2} + \|\nabla^r \text{curl} w\|_{L^2} + K \|\nabla^r w\|_{L^2} \right) \|\nabla^r w\|_{L^2}, \tag{A-3d}$$

$$\|\nabla^{r+1} w\|^2_{L^2} \le C |\nabla^{r+1} w|_{L^2} |\nabla^r w|_{L^2} + C \left(\|\nabla^r \text{div} w\|^2_{L^2} + \|\nabla^r \text{curl} w\|^2_{L^2} \right), \tag{A-3e}$$

$$\begin{aligned} \|\nabla^{r+1} w\|^2_{L^2} \le {} & C |\Pi \nabla^{r+1} w|_{L^2} |\Pi (N^i \nabla^r w_i)|_{L^2} \\ & + C \left(\|\nabla^r \text{div} w\|^2_{L^2} + \|\nabla^r \text{curl} w\|^2_{L^2} + K^2 \|\nabla^r w\|^2_{L^2} \right), \end{aligned} \tag{A-3f}$$

$$\begin{aligned} \|\nabla^{r+1} w\|^2_{L^2} \le {} & C |\Pi (N^i \nabla^{r+1} w_i)|_{L^2} |\Pi \nabla^r w|_{L^2} \\ & + C \left(\|\nabla^r \text{div} w\|^2_{L^2} + \|\nabla^r \text{curl} w\|^2_{L^2} + K^2 \|\nabla^r w\|^2_{L^2} \right). \end{aligned} \tag{A-3g}$$

Indeed, the proof of (A-3a)-(A-3b) can also be found in [34]. The proof of (A-3c)-(A-3g) are based on the divergence theorem, and (A-3f)-(A-3g) are based additionally on (A-3b).

Lemma A-3. (Lemma A.5 in [11]) *Suppose that $q = 0$ on $\partial\Omega$. Then*

$$\|q\|_{L^2} \le C (\text{Vol}\Omega)^{1/n} \|\nabla q\|_{L^2} \quad \text{and} \quad \|\nabla q\|_{L^2} \le C (\text{Vol}\Omega)^{1/n} \|\Delta q\|_{L^2}. \tag{A-4}$$

As a consequence of Lemmas A-2 and A-3, we have

Corollary A-4. *Let $q = q_b$ on $\partial\Omega$ with q_b being a constant. If $|\theta| + 1/\iota_0 \leq K$, we have for any $r \geq 2$ and $\delta > 0$,*

$$\|q - q_b\|_{L^2} \leq C(\mathrm{Vol}\Omega)^{1/n}\|\nabla q\|_{L^2}, \quad \|\nabla q\|_{L^2} + \|\nabla^2 q\|_{L^2} \leq C(K, \mathrm{Vol}\Omega)\|\Delta q\|_{L^2}, \tag{A-5a}$$

$$\|\nabla^r q\|_{L^2} + |\nabla^r q|_{L^2} \leq C|\Pi\nabla^r q|_{L^2} + C(K, \mathrm{Vol}\Omega)\sum_{s=0}^{r-1}\|\nabla^s \Delta q\|_{L^2}, \tag{A-5b}$$

$$\|\nabla^r q\|_{L^2} + |\nabla^{r-1} q|_{L^2} \leq \delta|\Pi\nabla^r q|_{L^2} + C(\delta^{-1}, K, \mathrm{Vol}\Omega)\sum_{s=0}^{r-2}\|\nabla^s \Delta q\|_{L^2}. \tag{A-5c}$$

Clearly, (A-5a) is a consequence of (A-4) and (A-3g). The proof of (A-5b) and (A-5c) can be found in Proposition 5.8, [11]. (Indeed, (A-5b) follows from (A-3c)-(A-3e) and (A-4), and (A-5c) follows from (A-3c), (A-3e), (A-3f) and (A-4).)

References

[1] T. Alazard, *Low Mach number limit of the full Navier-Stokes equations*, Arch. Ration. Mech. Anal. **180** (2006), 1–73.

[2] T. Alazard, N. Burq and C. Zuily, *On the water waves equations with surface tension*, Duke Math. J. **158** (2011), 413–499.

[3] ——, *On the Cauchy problem for gravity water waves*, Invent. Math. **198** (2014), 71–163.

[4] T. Alazard and J.-M. Delort, *Global solutions and asymptotic behavior for two dimensional gravity water waves*, Ann. Sci. Ec. Norm. Super. **48** (2015), 1149–1238.

[5] D. Ambrose and N. Masmoudi, *The zero surface tension limit of two-dimensional water waves*, Comm. Pure Appl. Math. **58** (2005), 1287–1315

[6] V. I. Arnold, *Sur la geometrie differentielle des groupes de Lie de dimension infinie et ses applications aal'hydrodynamique des fluides parfaits*, Ann. Inst. Fourier (Grenoble) **16** (1966), 316–361.

[7] L. Bieri, S. Miao, S. Shahshahani and S. Wu, *On the motion of a self-gravitating incompressible fluid with free boundary*, Communications in Mathematical Physics, **355** (2017), 161–243.

[8] Y. Brenier, *Minimal geodesics on groups of volume-preserving maps and generalized solutions of the Euler equations*, Comm. Pure Appl. Math. **52** (1999), 411–452.

[9] A.Castro, D. Cordoba, C. Fefferman, F. Gancedo, and J. Gomez-Serrano, *Finite time singularities for the free boundary incompressible Euler equations*, Ann. of Math. **178** (2013), 1061–1134.

[10] D. Christodoulou, The action principle and partial differential equations. In: Griffiths, P.A., Mather, J.N., Stein, E.M. (eds.) Annals of Mathematics Studies, vol. 146. Princeton University Press, Princeton (2000).

[11] D. Christodoulou and H. Lindblad, *On the motion of the free surface of a liquid*, Comm. Pure Appl. Math. **53** (2000), 1536–1602.

[12] D. Coutand, J. Hole and S. Shkoller, *Well-posedness of the free-boundary compressible 3-D Euler equations with surface tension and the zero surface tension limit*, SIAM J. Math. Anal. **45** (2013), 3690–3767.

[13] D. Coutand and S. Shkoller, *Well-posedness of the free-surface incompressible Euler equations with or without surface tension*, J. Amer. Math. Soc. **20** (2007), 829–930.

[14] ——, *Well-Posedness in Smooth Function Spaces for the Moving-Boundary Three-Dimensional Compressible Euler Equations in Physical Vacuum*, Arch. Ration. Mech. Anal. **206** (2012), 515–616.

[15] ——, *On the finite-time splash and splat singularities for the 3-D free-surface Euler equations*, Comm. Math. Phys. **325** (2014), 143–83.

[16] Y. Deng, A. D. Ionescu, B. Pausader and F. Pusateri, *Global solutions of the gravity-capillary water-wave system in three dimensions*, Acta Math. **219** (2017), 213–402.

[17] D.G Ebin, *The equations of motion of a perfect fluid with free boundary are not well posed*, Comm. Partial Differential Equations **12** (1987), 1175–1201.

[18] D. G. Ebin and G. Marsden, *Groups of diffeomorphisms and the notion of an incompressible fluid*, Ann. of Math. **92** (1970), 102–163.

[19] P. Germain, N. Masmoudi and J. Shatah, *Global solutions for the gravity water waves equation in dimension 3*, Ann. of Math. **175** (2012), 691–754.

[20] M. Hadzic and J. Jang, *Nonlinear stability of expanding star solutions in the radially-symmetric mass-critical Euler-Poisson system*, Comm. Pure Appl. Math. **71** (2018), 827–891.

[21] ——, *Expanding large global solutions of the equations of compressible fluid mechanics*, Invent. Math. **214** (2018), 1205–1266.

[22] C. Hao and T. Luo, *Free Boundary Problem of Incompressible Inviscid Magnetohydrodynamic Flows*, Arch. Ration. Mech. Anal. **212** (2014), 805–847.
[23] C. Hao and D. Wang, *A Priori Estimates for the Free Boundary Problem of Incompressible Neo-Hookean Elastodynamics*, J. Differential Equations **261** (2016), 712–737.
[24] F. Huang, T. Wang and Y. Wang, *Diffusive wave in the low Mach limit for compressible Navier-Stokes equations.*, Adv. Math. **319** (2017), 348–395.
[25] M. Ifrim and D. Tataru, *Two dimensional water waves in holomorphic coordinates II: Global solutions*, Bull. Soc. Math. France **144** (2016), 369–394.
[26] A. D. Ionescu and F. Pusateri, *Global solutions for the gravity water waves system in 2d*, Invent. Math. **199** (2015), 653–804.
[27] J. Jang and N. Masmoudi, *Well-posedness of compressible Euler equations in a physical vacuum*, Comm. Pure Appl. Math. **68** (2015), 61–111.
[28] S. Jiang, Q. Ju, F. Li and Z. Xin, *Low Mach number limit for the full compressible magnetohydrodynamic equations with general initial data*, Adv. Math. **259** (2014), 384–20.
[29] S. Klainerman and A. Majda, *Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids*, Comm. Pure Appl. Math. **34** (1981), 481–524.
[30] ——, *Compressible and incompressible fluids*, Comm. Pure Appl. Math. **35** (1982), 629–653.
[31] D. Lannes, *Well-posedness of the water waves equations*, J. Am. Math. Soc. **18** (2005), 605–54.
[32] ——, The water waves problem: mathematical analysis and asympototics. In: Mathematical Surveys and Monographs, vol. 188. American Mathematical Society Providence, RI, (2013)
[33] N. R. Lebovitz, *On the fluid dynamics of evolving stars*, Proc. Roy. Soc. London Ser. A **375** (1981), 249–269.
[34] H. Lindblad, *Well-posedness for the motion of an incompressible liquid with free surface boundary*, Ann. of Math. **162** (2005), 109–194.
[35] ——, *Well posedness for the motion of a compressible liquid with free surface boundary*, Commun. Math. Phys. **260** (2005), 319–392.
[36] H. Lindblad and C. Luo, *A priori Estimates for the Compressible Euler Equations for a Liquid with Free Surface Boundary and the Incompressible Limit*, Comm. Pure Appl. Math. **71** (2018), 1273–1333.
[37] J.-G. Liu, R. L. Pego and D. Slepcev, *Least action principles for incompressible flows and geodesics between shapes*, arXiv:1604.03387.
[38] C. Luo, *On the Motion of a Compressible Gravity Water Wave with Vorticity,* arXiv:1701.03987
[39] T. Luo, Z. Xin and H. Zeng, *Well-Posedness for the Motion of Physical Vacuum of the Three-dimensional Compressible Euler Equations with or without Self-Gravitation*, Arch. Ration. Mech. Anal. **213** (2014), 763–831.
[40] T. Luo and H. Zeng, *Global Existence of Smooth Solutions and Convergence to Barenblatt Solutions for the Physical Vacuum Free Boundary Problem of Compressible Euler Equations with Damping*, Comm. Pure Appl. Math. **69** (2016), 1354–1396.
[41] A. Majda, *Compressible fluid flow and systems of conservation laws in several space variables.* In: Applied Mathematical Sciences **53**, Springer-Verlag, 1984
[42] G. Métivier and S. Schochet, *The incompressible limit of the non-isentropic Euler equations.*, Arch. Ration. Mech. Anal. **158** (2001), 61–90.
[43] R. G. Rehm and H.R. Baum, *The equations of motion for thermally driven, buoyant flows*, Journal of Research of the National Bureau of Standards, **83** (1978), 297–308.
[44] P. Secchi and A. Valli, *A free boundary problem for compressible viscous fluids*, J. Reine Angew. Math. **341** (1983), 1–31.
[45] J. Shatah and C. Zeng, *Geometry and a priori estimates for free boundary problems of the Euler equation*, Comm. Pure Appl. Math. **61** (2008), 698–744.
[46] ——, *A priori estimates for fluid interface problems*, Comm. Pure Appl. Math. **61** (2008), 848–876 .
[47] ——, *Local well-posedness for fluid interface problems*, Arch. Ration. Mech. Anal. **199** (2011), 653–705.
[48] A. Shnirelman, *The geometry of the group of diffeomorphisms and the dynamics of an ideal incompressible fluid*, Mat. Sb. (N.S.) **128** (1985), 82–109; translation in Math. USSR-Sb. **56** (1987), 79–105.
[49] Y. Trakhinin, *Local existence for the free boundary problem for the non-relativistic and relativistic compressible Euler equations with a vacuum boundary condition*, Commun. Pure Appl. Math. **62** (2009), 1551-1594.
[50] S. Wu, *Well-posedness in Sobolev spaces of the full water wave problem in* 2*-D*, Invent. Math. **130** (1997), 39–72.
[51] ——, *Well-posedness in Sobolev spaces of the full water wave problem in 3-D*, J. Amer. Math. Soc. **12** (1999), 445–495.
[52] ——, *Almost global wellposedness of the 2-D full water wave problem*, Invent. Math. **177** (2009), 45–135.
[53] ——-, *Global wellposedness of the 3-D full water wave problem*, Invent. Math. **184** (2011), 125–220.
[54] ——-, *On a class of self-similar 2D surface water waves*, arXiv: 1206.2208.
[55] H. Zeng, *Global Resolution of the Physical Vacuum Singularity for 3-D Isentropic Inviscid Flows with Damping in Spherically Symmetric Motions*, Arch. Ration. Mech. Anal. **226** (2017), 33–82.

[56] H. Zeng, *Almost global solutions to the three-dimensional isentropic inviscid flows with damping in physical vacuum around Barenlatt solutions* , arXiv:1910.05516

[57] P. Zhang, and Z. Zhang, *On the free boundary problem of three-dimensional incompressible Euler equations*, Comm. Pure Appl. Math. **61** (2008), 877–940.

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