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Existence of homoclinic solutions for a fourth order differential equation with a parameter

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ABSTRACT

In this paper, we study the existence of homoclinic solutions for a class of fourth order differential equations. By using variational methods, the existence and the non-existence of nontrivial homoclinic solutions are obtained, depending on a parameter.

1. Introduction

In this paper, we consider a class of fourth-order differential equations with a parameter:

$$u^{(4)} + wu'' + \lambda a(x)u - f(x,u) = 0, \quad x \in \mathbb{R}, \tag{1.1}$$

where w is a constant, $\lambda > 0$ is a parameter, $f \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ and the function a satisfies the following conditions:

(V1) $a \in C(\mathbb{R}, \mathbb{R})$ and $a \geqslant 0$ on $\mathbb{R}$;
(V2) there exists $c > 0$ such that the set $\{a < c\} := \{x \in \mathbb{R} | a(x) < c\}$ is nonempty and has finite measure;
(V3) $\Omega = int\ a^{-1}(0)$ is nonempty and $\bar{\Omega} = a^{-1}(0)$ such that Ω is a finite interval;
(V4) $|\{a < c\}| < \frac{c_0}{S_\infty^2}$, where $|\cdot|$ is the Lebesgue measure, c_0 is defined by (2.1) in Section 2 and S_∞ is the best constant for the embedding of $H^2(\mathbb{R})$ in $L^\infty(\mathbb{R})$.

As usual, we say that a solution $u(x)$ of Eq. (1.1) is homoclinic (to 0) if $u(x) \to 0$ as $x \to \pm\infty$. In addition, if $u(x) \not\equiv 0$, then $u(x)$ is called a nontrivial homoclinic solution.

The above Eq. (1.1) has been put forward as mathematical model for the study of pattern formation in physics and mechanics. For example, the well-known Extended Fisher–Kolmogorov (EFK) equation proposed by Coullet et al. in 1987 [5] in study of phase transitions, and also by Dee and Van Saarlos in 1988 [6], as well as the Swift–Hohenberg (SH) equation which is general model for pattern-forming process derived in [13] to describe random thermal fluctuations in the Boussinesque equation and in the propagation of lasers in [8]. With appropriate changes of variables, stationary solutions of these equations lead to the following fourth order equation:

$$u^{(4)} + wu'' - u + u^3 = 0,$$

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where $w > 0$ corresponds to EFK equation and $w < 0$ to SH equation.

The study of homoclinic and heteroclinic solutions for the fourth order differential equations has attracted a lot of attention by many researchers, see [1,3,4,9–12,14,15]. These works are mainly concerned on the autonomous case, such as the following equation:

$$u^{(4)} + wu'' + \alpha u - \beta u^2 - \gamma u^3 = 0,$$

where $\alpha,\ \beta,\ \gamma$ are nonnegative constants.

In 2001, Tersian and Chaparova [14] first considered a class of non-autonomous fourth order problems

$$u^{(4)} + wu'' + a(x)u - b(x)u^2 - c(x)u^3 = 0. \tag{1.2}$$

Applying the mountain pass theorem, they showed that Eq. (1.2) possesses one nontrivial homoclinic solution $u \in H^2(\mathbb{R})$ when $a(x),\ c(x)$ and $d(x)$ are continuous periodic functions and satisfy some other assumptions. If there is no periodicity assumption of $a(x),\ c(x)$ and $d(x)$, then the case will be more difficult. In 2009, Li [9] studied the nonperiodic case of Eq. (1.2) and obtained the existence of nontrivial homoclinic solution by establishing a compactness lemma and using the mountain pass theorem. Furthermore, the author also studied a class of the nonhomogeneous fourth order equations with the general nonlinear term f:

$$u^{(4)} + wu'' + a(x)u = f(x,u) + h(x),$$

and obtained the existence of homoclinic solution when f satisfied the well-known (AR) condition. Very recently, Sun and Wu [12] considered a class of fourth order differential equations with a perturbation:

$$u^{(4)} + wu'' + a(x)u = f(x,u) + \lambda h(x)|u|^{p-2}u, \tag{1.3}$$

where $\lambda > 0$ is a parameter, $1 \leqslant p < 2$ and $h \in L^{\frac{2}{2-p}}(\mathbb{R})$. By using variational methods, the existence result of two homoclinic solutions for Eq. (2.6) is obtained if the parameter λ is small enough. In all these papers, in order to obtain an important inequality, the following condition

($\mathcal{A}$) there exists a constant $a_1 > 0$ such that

$$0 < a_1 \leqslant a(x) \to +\infty, \quad \text{as } |x| \to +\infty, \tag{1.4}$$

and $w \leq 2\sqrt{a_1}$;

is required. However, if there exists the function a satisfying $a = 0$ in some finite interval T of $\mathbb{R}$, then the condition ($\mathcal{A}$) does not hold.

Inspired by the above facts, the aim of this paper is to consider this case. We shall establish the existence result of nontrivial homoclinic solutions for Eq. (1.1) when the nonlinear term f satisfies the asymptotically linear condition. Moreover, the non-existence of nontrivial homoclinic solutions will be discussed.

Before stating our result we need to introduce some notations.

Notation 1.1. Throughout this paper, we denote by $|\cdot|_r$ the L^r-norm, $2 \leqslant r \leqslant \infty$ and $h^{\pm} = \max\{\pm h, 0\}$. Also if we take a subsequence of a sequence $\{u_n\}$ we shall denote it again $\{u_n\}$.

Now we state our main result.

Theorem 1.1. *Assume that the conditions (V1)–(V4) hold and $w < 2$. In addition, we assume that the function f satisfy the following conditions:*

(D1) $f(x,s)$ is a continuous function on $\mathbb{R} \times \mathbb{R}$ such that $f(x,s) \equiv 0$ for all $s < 0$ and $x \in \mathbb{R}$. Moreover, there exists $p \in L^{\infty}(\mathbb{R}, \mathbb{R}^+)$ with $|p|_{\infty} < \frac{\Theta_0}{2}$ such that

$$\lim_{s \to 0^+} \frac{f(x,s)}{s} = p(x) \text{ uniformly in } x \in \mathbb{R}$$

and

$$\frac{f(x,s)}{s} \geqslant p(x) \text{ for all } s > 0 \text{ and } x \in \mathbb{R},$$

where $\Theta_0 := \frac{c_0\left(c_0 - S_\infty^2|\{a<c\}|\right)}{S_\infty^2|\{a<c\}|}$;

(D2) there exist $r > 1$ and $q \in L^{\infty}(\bar{\Omega}, \mathbb{R}^+)$ with $|q|_{\infty} > 0$ such that

$$\lim_{s \to \infty} \frac{f(x,s)}{s^r} = 0 \text{ uniformly in } x \in \mathbb{R} \setminus \bar{\Omega}$$

and

$$\lim_{s \to \infty} \frac{f(x,s)}{s} = q(x) \text{ uniformly in } x \in \bar{\Omega};$$

(D3) there exist two constants θ, d_0 satisfying $\theta > 2$ and $0 \leqslant d_0 < \frac{(\theta-2)\Theta_0}{2\theta}$ such that

$$F(x,s) - \frac{1}{\theta} f(x,s)s \leqslant d_0 s^2 \text{ for all } s > 0 \text{ and } x \in \mathbb{R};$$

(D4) $\mu^* := \inf\left\{\int_\Omega \left(u''(x)^2 - wu'(x)^2\right)dx | u \in H^2(\Omega) \cap H_0^1(\Omega), \int_\Omega q(x)u^2 dx = 1\right\} < 1.$

Then there exists $\Lambda_0 > 0$ such that for every $\lambda > \Lambda_0$, Eq. (1.1) has at least one homoclinic solution.

Remark 1.1. By [14, Lemma 8] and Sobolev embedded theorem, it is not difficult to claim that $\mu^* > 0$, which is achieved by some $\phi_1 \in H^2(\Omega) \cap H_0^1(\Omega)$ with $\int_\Omega q\phi_1^2 dx = 1$; see Appendix A

Now, we consider the following the minimum problem:

$$\mu_0 = \inf\left\{\int_{\mathbb{R}} \left(u''(x)^2 + u'(x)^2 + u(x)^2\right)dx | u \in H^2(\mathbb{R}), \int_{\mathbb{R}} q(x)u^2 dx = 1\right\}. \tag{1.5}$$

Then we have the following result.

Theorem 1.2. *Suppose that the conditions (V1)–(V4) and (D1)–(D2) hold. If $\mu_0 > \frac{1}{c_0 - S_\infty^2 |\{a<c\}|}$ and $s \mapsto \frac{f(x,s)}{s}$ is non-decreasing function for any fixed $x \in \mathbb{R}$, then for any $\lambda \geqslant \frac{1}{c}$, Eq. (1.1) does not admit any nontrivial homoclinic solution.*

The remainder of this paper is organized as follows. In Section 2, some preliminary results are presented. In Section 3, we give the proofs of our main results.

2. Variational setting and preliminaries

In this section, we give the variational setting for Eq. (1.1). We need the following result.

Lemma 2.1 [14, Lemma 8]. *Assume that $w < 2$. Then there exists a constant $c_0 > 0$ such that*

$$\int_{\mathbb{R}} \left[u''(x)^2 - wu'(x)^2 + u(x)^2\right]dx \geqslant c_0 \|u\|_{H^2}^2 \quad \text{for all } u \in H^2(\mathbb{R}), \tag{2.1}$$

where $\|u\|_{H^2} = \left(\int_{\mathbb{R}} \left[u''(x)^2 + u'(x)^2 + u(x)^2\right]dx\right)^{1/2}$ is the norm of Sobolev space $H^2(\mathbb{R})$.

Let

$$X = \{u \in H^2(\mathbb{R}) | \int_{\mathbb{R}} \left[u''(x)^2 - wu'(x)^2 + a(x)u(x)^2\right]dx < +\infty\}$$

be equipped with the inner product and norm

$$(u, v) = \int_{\mathbb{R}} [u''(x)v''(x) - wu'(x)v'(x) + a(x)u(x)v(x)]dx$$

and corresponding norm $\|u\|^2 = (u,u)$. For $\lambda > 0$, we also need the following inner product and norm

$$(u, v)_\lambda = \int_{\mathbb{R}} [u''(x)v''(x) - wu'(x)v'(x) + \lambda a(x)u(x)v(x)]dx$$

and corresponding norm $\|u\|_\lambda^2 = (u,u)_\lambda$. It is clear that $\|u\| \leqslant \|u\|_\lambda$ for $\lambda \geqslant 1$. Set $X_\lambda = (X, \|u\|_\lambda)$. From the conditions (V1)–(V4), (2.1) and the Sobolev inequality, we have

$$\begin{aligned}
c_0 \int_{\mathbb{R}} \left[u''(x)^2 + u'(x)^2 + u(x)^2\right]dx &\leqslant \int_{\mathbb{R}} \left[u''(x)^2 - wu'(x)^2 + u(x)^2\right]dx \\
&= \int_{\mathbb{R}} \left[u''(x)^2 - wu'(x)^2\right]dx + \int_{\{a<c\}} u(x)^2 dx + \int_{\{a \geqslant c\}} u(x)^2 dx \\
&\leqslant \int_{\mathbb{R}} \left[u''(x)^2 - wu'(x)^2\right]dx + \|u\|_\infty^2 |\{a<c\}| + \frac{1}{\lambda c}\int_{\mathbb{R}} \lambda a(x)|u(x)|^2 dx \\
&\leqslant \int_{\mathbb{R}} \left[u''(x)^2 - wu'(x)^2\right]dx + \frac{1}{\lambda c}\int_{\mathbb{R}} \lambda a(x)u(x)^2 dx \\
&\quad + S_\infty^2 |\{a<c\}| \int_{\mathbb{R}} \left[u''(x)^2 + u'(x)^2 + u(x)^2\right]dx.
\end{aligned}$$

Thus,

$$\begin{aligned}\int_{\mathbb{R}}\left[u''(x)^2+u'(x)^2+u(x)^2\right]dx &\leqslant \frac{1}{c_0-S_\infty^2|\{a<c\}|}\left[\int_{\mathbb{R}}\left[u''(x)^2-wu'(x)^2\right]dx+\frac{1}{\lambda c}\int_{\mathbb{R}}\lambda a(x)u(x)^2dx\right]\\ &\leqslant \frac{1}{c_0-S_\infty^2|\{a<c\}|}\int_{\mathbb{R}}\left[u''(x)^2-wu'(x)^2+\lambda a(x)u(x)^2\right]dx\\ &= \frac{1}{c_0-S_\infty^2|\{a<c\}|}\|u\|_\lambda^2 \quad \text{for all } \lambda\geqslant\frac{1}{c},\end{aligned} \tag{2.2}$$

which implies that the imbedding $X_\lambda \hookrightarrow H^2(\mathbb{R})$ is continuous for all $\lambda \geqslant \frac{1}{c}$, here the set $\{a\geqslant c\}:=\{x\in\mathbb{R}|a(x)\geqslant c\}$. Furthermore, using Lemma 2.1 again, one has

$$\begin{aligned}\int_{\mathbb{R}}u(x)^2dx &= \int_{\{a<c\}}u(x)^2dx+\int_{\{a\geqslant c\}}u(x)^2dx \leqslant \|u\|_\infty^2|\{a<c\}|+\frac{1}{\lambda c}\int_{\mathbb{R}}\lambda a(x)u(x)^2dx\\ &\leqslant S_\infty^2|\{a<c\}|\int_{\mathbb{R}}\left[u''(x)^2+u'(x)^2+u(x)^2\right]dx+\frac{1}{\lambda c}\int_{\mathbb{R}}\lambda a(x)u(x)^2dx\\ &\leqslant \frac{S_\infty^2|\{a<c\}|}{c_0}\int_{\mathbb{R}}\left[u''(x)^2-wu'(x)^2+u(x)^2\right]dx+\frac{1}{\lambda c}\int_{\mathbb{R}}\lambda a(x)u(x)^2dx,\end{aligned}$$

this implies that

$$\begin{aligned}\int_{\mathbb{R}}u(x)^2dx &\leqslant \frac{1}{c_0-S_\infty^2|\{a<c\}|}\left[\frac{S_\infty^2|\{a<c\}|}{c_0}\int_{\mathbb{R}}\left[u''(x)^2-wu'(x)^2\right]dx+\frac{1}{\lambda c}\int_{\mathbb{R}}\lambda a(x)u(x)^2dx\right]\\ &\leqslant \frac{\max\left\{\frac{S_\infty^2|\{a<c\}|}{c_0},\frac{1}{\lambda c}\right\}}{c_0-S_\infty^2|\{a<c\}|}\int_{\mathbb{R}}\left[u''(x)^2-wu'(x)^2+\lambda a(x)u(x)^2\right]dx=\frac{S_\infty^2|\{a<c\}|}{c_0\left(c_0-S_\infty^2|\{a<c\}|\right)}\|u\|_\lambda^2\\ &= \Theta_0^{-1}\|u\|_\lambda^2, \quad \text{for all } \lambda\geqslant\frac{c_0}{cS_\infty^2|\{a<c\}|}\geqslant\frac{1}{c}.\end{aligned} \tag{2.3}$$

Thus, by (2.2) and (2.3), for any $r\in(2,\infty)$ and $\lambda\geqslant\frac{c_0}{cS_\infty^2|\{a<c\}|}$, one has

$$\begin{aligned}\int_{\mathbb{R}}|u(x)|^rdx &\leqslant \|u\|_\infty^{r-2}\int_{\mathbb{R}}u(x)^2dx\leqslant S_\infty^{r-2}\left(\frac{1}{c_0-S_\infty^2|\{a<c\}|}\|u\|_\lambda^2\right)^{\frac{r-2}{2}}\frac{S_\infty^2|\{a<c\}|}{c_0-S_\infty^2|\{a<c\}|}\|u\|_\lambda^2\\ &= \frac{1}{|\{a<c\}|^{\frac{r-2}{2}}}\left(\frac{S_\infty^2|\{a<c\}|}{c_0-S_\infty^2|\{a<c\}|}\right)^{\frac{r}{2}}\|u\|_\lambda^r=\frac{1}{|\{a<c\}|^{\frac{r-2}{2}}\Theta_0^{\frac{r}{2}}}\|u\|_\lambda^r.\end{aligned} \tag{2.4}$$

Now we begin describing the variational formulation of Eq. (1.1). Consider the functional $J: X_\lambda \to \mathbb{R}$ defined by

$$J(u)=\frac{1}{2}\|u\|_\lambda^2-\int_{\mathbb{R}}F(x,u)dx, \tag{2.5}$$

where F is the primitive

$$F(x,u)=\int_0^u f(x,s)ds.$$

Since f is continuous, we deduce that J is of class C^1 and its derivative is given by

$$\langle J'(u),\varphi\rangle=\int_{\mathbb{R}}[u''(x)v''(x)-wu'(x)v'(x)+\lambda a(x)u(x)v(x)]dx-\int_{\mathbb{R}}f(x,u(x))\varphi(x)dx,$$

for all $u,\ \varphi\in X_\lambda$. Then, we can infer that $u\in X_\lambda$ is a critical point of J if and only if it is a homoclinic solution of Eq. (1.1). Furthermore, we have the following result.

Lemma 2.2. *Suppose that the conditions (D1) and (D3) hold. Let u_0 be a nontrivial homoclinic solution of Eq. (1.1), we have $J(u_0)>0$.*

Proof. Since u_0 is a nontrivial homoclinic solution of Eq. (1.1),

$$\|u_0\|_\lambda^2=\int_{\mathbb{R}}f(x,u_0)u_0dx. \tag{2.6}$$

By the conditions (D1) and (D3), the Hö lder inequality, (2.3) and (2.6), we have

$$J(u_0)=\frac{1}{2}\|u_0\|_\lambda^2-\int_{\mathbb{R}}F(x,u_0)dx\geqslant\frac{1}{2}\|u_0\|_\lambda^2-d_0\int_{\mathbb{R}}u_0(x)^2dx-\frac{1}{\theta}\int_{\mathbb{R}}f(x,u_0)u_0dx\geqslant\left(\frac{\theta-2}{2\theta}-\frac{d_0}{\Theta_0}\right)\|u_0\|_\lambda^2>0,$$

where $\Theta_0:=\frac{1-S_\infty^2|\{\beta<c\}|}{S_\infty^2|\{\beta<c\}|}>0$ as in the condition (D1). This completes the proof. □

Next, we give a useful theorem. It is the variant version of the mountain pass theorem, which allows us to find a so-called Cerami type (PS) sequence.

Theorem 2.1 ([7], Mountain Pass Theorem). *Let E be a real Banach space with its dual space E^*, and suppose that $I\in C^1(E,\mathbb{R})$ satisfies*

$$\max\{I(0),I(e)\}\leqslant\mu<\eta\leqslant\inf_{\|u\|=\rho}I(u),$$

for some $\mu<\eta,\ \rho>0$ and $e\in E$ with $\|e\|>\rho$. Let $\hat{c}\geqslant\eta$ be characterized by

$$\hat{c}=\inf_{\gamma\in\Gamma}\max_{0\leqslant\tau\leqslant1}I(\gamma(\tau)),$$

where $\Gamma=\{\gamma\in C([0,1],E):\gamma(0)=0,\ \gamma(1)=e\}$ is the set of continuous paths joining 0 and e, then there exists a sequence $\{u_n\}\subset E$ such that

$$I(u_n)\to\hat{c}\geqslant\eta\quad\text{and}\quad(1+\|u_n\|)\|I'(u_n)\|_{E^*}\to0,\quad\text{as } n\to\infty.$$

3. Proofs of Theorems 1.1 and 1.2

In what follows, we give the following two lemmas which ensure that the functional J has the mountain pass geometry, which will be used in the proof of Theorem 1.1.

Lemma 3.1. *Suppose that the conditions (V1)–(V4) and (D1)–(D2) hold. Then for every $\lambda\geqslant\frac{c_0}{cS_\infty^2|\{a<c\}|}$ there exist two positive constants ρ,η such that $J(u)|_{\|u\|_\lambda=\rho}\geqslant\eta>0$.*

Proof. For any $\epsilon>0$, it follows from the conditions (D1) and (D2) that there exist $C_\epsilon>0$ and $r>2$ such that

$$F(x,s)\leqslant\frac{|p|_\infty+\epsilon}{2}s^2+\frac{C_\epsilon}{r}|s|^r,\quad\text{for all } s\in\mathbb{R}.\tag{3.1}$$

So that, from Lemma 2.1, (3.1) and the Sobolev inequality, we have for all $u\in X_\lambda$,

$$\int_{\mathbb{R}}F(x,u)dx\leqslant\frac{|p|_\infty+\epsilon}{2}\int_{\mathbb{R}}u^2dx+\frac{C_\epsilon}{r}\int_{\mathbb{R}}|u|^rdx\leqslant\frac{|p|_\infty+\epsilon}{2\Theta_0}\|u\|_\lambda^2+\frac{C_\epsilon}{r}\frac{c_0^{\frac{r-2}{2}}}{|\{a<c\}|^{\frac{r-2}{2}}\Theta_0^{\frac{r}{2}}}\|u\|_\lambda^r,$$

which implies that

$$\begin{aligned}J(u)&=\frac{1}{2}\|u\|_\lambda^2-\int_{\mathbb{R}}F(x,u)dx\geqslant\frac{1}{2}\|u\|_\lambda^2-\frac{|p|_\infty+\epsilon}{2\Theta_0}\|u\|_\lambda^2-\frac{C_\epsilon}{r}\frac{c_0^{\frac{r-2}{2}}}{|\{a<c\}|^{\frac{r-2}{2}}\Theta_0^{\frac{r}{2}}}\|u\|_\lambda^r\\&=\|u\|_\lambda^2\left[\frac{1}{2}\left(1-\frac{|p|_\infty+\epsilon}{\Theta_0}\right)-\frac{C_\epsilon c_0^{\frac{r-2}{2}}}{r|\{a<c\}|^{\frac{r-2}{2}}\Theta_0^{\frac{r}{2}}}\|u\|_\lambda^{r-2}\right].\end{aligned}\tag{3.2}$$

Take $\epsilon=\frac{\Theta_0}{2}-|p|_\infty$. It follows from (3.2) that there exist $\rho,\ \eta>0$ such that $J(u)|_{\|u\|_\lambda=\rho}\geqslant\eta$. □

Lemma 3.2. *Suppose that the conditions (V1)–(V4), (D2) and (D4) hold. Let $\rho>0$ be as in Lemma 3.1. Then there exists $e\in X_\lambda$ with $\|e\|_\lambda>\rho$ such that $J(e)<0$ for all $\lambda\geqslant\frac{c_0}{cS_\infty^2|\{a<c\}|}$.*

Proof. By the condition $(D4)$ and Remark 1.1, we can choose a nonnegative function $\phi_1\in H^2(\Omega)\cap H_0^1(\Omega)$ with

$$\int_\Omega q(x)\phi_1^2(x)dx=1\quad\text{such that}\quad\int_\Omega\left[\phi_1''(x)^2-w\phi_1'(x)^2\right]dx=\mu^*<1.$$

Therefore, by the condition (D2) and Fatou's lemma, we have

$$\begin{aligned}\lim_{t\to+\infty}\frac{J(t\phi_1)}{t^2}&=\frac{1}{2}\|\phi_1\|_\lambda^2-\lim_{t\to+\infty}\int_{\mathbb{R}}\frac{F(x,t\phi_1)}{t^2\phi_1^2}\phi_1^2dx=\frac{1}{2}\int_\Omega\left[\phi_1''(x)^2-w\phi_1'(x)^2\right]dx-\lim_{t\to+\infty}\int_\Omega\frac{F(x,t\phi_1)}{t^2\phi_1^2}\phi_1^2dx\\&\leqslant\frac{1}{2}\mu^*-\frac{1}{2}\int_{\mathbb{R}}q(x)\phi_1^2(x)dx=\frac{1}{2}(\mu^*-1)<0.\end{aligned}$$

So, if $J(t\phi_1) \to -\infty$ as $t \to +\infty$, then there exists $e \in X_\lambda$ with $\|e\|_\lambda > \rho$ such that $J(e) < 0$. □

Next, we define

$$\alpha = \inf_{\gamma\in\Gamma_\lambda} \max_{0\leqslant t\leqslant 1} J(\gamma(t))$$

and

$$\alpha_0(\Omega) = \inf_{\gamma\in\bar{\Gamma}_\lambda(T)} \max_{0\leqslant t\leqslant 1} J|_{H^2(\Omega)\cap H_0^1(\Omega)}(\gamma(t)),$$

where $J|_{H^2(\Omega)\cap H_0^1(\Omega)}$ is a restriction of J on $H^2(\Omega)\cap H_0^1(\Omega)$,

$$\Gamma = \{\gamma \in C([0,1], X_\lambda) : \gamma(0) = 0, \gamma(1) = e\}$$

and

$$\bar{\Gamma}(T) = \{\gamma \in C([0,1], H^2(\Omega)\cap H_0^1(\Omega)) : \gamma(0) = 0, \gamma(1) = e\}.$$

Note that

$$J|_{H^2(\Omega)\cap H_0^1(\Omega)}(u) = \frac{1}{2}\int_{\mathbb{R}}\left[u''(x)^2 - wu'(x)^2\right]dx - \int_{\mathbb{R}} F(x,u)dx,$$

for $u \in H^2(\Omega)\cap H_0^1(\Omega)$ and $\alpha_0(\Omega)$ independent of λ. Moreover, if the conditions (D1)–(D4) hold, then by the proofs of Lemmas 2.1 and 3.1, we can conclude that $J|_{H^2(\Omega)\cap H_0^1(\Omega)}$ satisfies the mountain pass hypothesis as in Theorem 2.1. Since $\left(H^2(\Omega)\cap H_0^1(\Omega)\right) \subset X_\lambda$ for all $\lambda > 0$, we have $0 < \eta \leqslant \alpha \leqslant \alpha_0(\Omega)$ for all $\lambda \geqslant \frac{1}{cS_\infty^2|\{a<c\}|}$. Take $D_0 > \alpha_0(T)$. Thus,

$$0 < \eta \leqslant \alpha_\lambda \leqslant \alpha_0(\Omega) < D_0 \text{ for all } \lambda \geqslant \frac{1}{cS_\infty^2|\{a<c\}|}. \tag{3.3}$$

From Lemmas 2.1 and 3.1 and Theorem 2.1, we obtain that for each $\lambda \geqslant \frac{1}{cS_\infty^2|\{a<c\}|}$, there exists $\{u_n\} \subset X_\lambda$ such that

$$J(u_n) \to \alpha > 0 \quad \text{and} \quad (1+\|u_n\|_\lambda)\|J'(u_n)\|_{X_\lambda^{-1}} \to 0, \quad \text{as } n \to \infty, \tag{3.4}$$

where $0 < \eta \leqslant \alpha \leqslant \alpha_0(\Omega) < D_0$. Furthermore, we have the following results.

Lemma 3.3. *Suppose that the conditions (V1)–(V4) and (D1)–(D3) hold. Then $\{u_n\}$ defined by (3.4) is bounded in X_λ for all $\lambda \geqslant \frac{1}{cS_\infty^2|\{\beta<c\}|}$.*

Proof. For n large enough, by ($D3$), the Hölder inequality and Lemmas 2.1 and 3.1, one has

$$\alpha + 1 \geqslant J(u_n) - \frac{1}{\theta}\langle J'(u_n), u_n\rangle = \left(\frac{1}{2}-\frac{1}{\theta}\right)\|u_n\|_\lambda^2 - \int_{\mathbb{R}}\left[F(x,u_n) - \frac{1}{\theta}f(x,u_n)\right]dx \geqslant \frac{\theta-2}{2\theta}\|u_n\|_\lambda^2 - d_0\int_{\mathbb{R}} u_n^2 dx$$
$$\geqslant \left(\frac{\theta-2}{2\theta} - \frac{d_0}{\Theta_0}\right)\|u_n\|_\lambda^2,$$

which implies that $\{u_n\}$ is bounded in X_λ. □

Proposition 3.4. *Suppose that the conditions (V1)–(V4) and (D1)–(D4) hold. Let $D_0 > 0$ be as in (3.3). Then there exists $\Lambda_* = \Lambda(D_0) \geqslant \frac{1}{cS_\infty^2|\{\beta<c\}|}$ such that J satisfies the $(C)_\alpha$–condition in X_λ for all $\alpha < D_0$ and $\lambda > \Lambda_*$.*

Proof. Let $\{u_n\}$ be a $(C)_\alpha$–sequence with $\alpha < D_0$. By Lemma 3.3, there exist a subsequence $\{u_n\}$ and u_0 in X_λ such that

$$\begin{aligned} &u_n \rightharpoonup u_0 \text{ weakly in } X_\lambda;\\ &u_n \to u_0 \text{ strongly in } L_{loc}^r(\mathbb{R}), \quad \text{for } 2 \leqslant r < \infty. \end{aligned} \tag{3.5}$$

Now we prove that $u_n \to u_0$ strongly in X_λ. Let $v_n = u_n - u_0$. Then by the conditions (D1)–(D4) and Brezis–Lieb Lemma [2], we have

$$J(v_n) = J(u_n) - J(u_0) + o(1) \text{ and } J'(v_n) = o(1).$$

It follows from (V2) and (3.5) that

$$\int_{\mathbb{R}} v_n^2(x)dx = \int_{\{a\geqslant c\}} v_n^2(x)dx + \int_{\{a<c\}} v_n^2(x)dx \leqslant \frac{1}{\lambda c}\int_{\{a\geqslant c\}} \lambda a(x)v_n^2(x)dx + o(1) \leqslant \frac{1}{\lambda c}\|v_n\|_\lambda^2 + o(1). \tag{3.6}$$

Then, by the Sobolev inequality and (2.2), we have

$$\int_{\mathbb{R}} |v_n(x)|^r dx \leqslant |v_n|_\infty^{r-2} \int_{\mathbb{R}} v_n(x)^2 dx \leqslant \frac{S_\infty^{r-2}}{\lambda c} \|v_n\|_{H^2}^{r-2} \|v_n\|_\lambda^2 + o(1) \leqslant \frac{S_\infty^{r-2}}{\lambda c} \left(\frac{1}{c_0 - S_\infty^2 |\{a < c\}|} \right)^{\frac{r-2}{2}} \|v_n\|_\lambda^r + o(1). \quad (3.7)$$

By the conditions (D1)–(D3) and Brezis–Lieb Lemma [2], we have

$J(v_n) = J(u_n) - J(u_0) + o(1)$ and $J'(v_n) = o(1)$.

Consequently, this together with the condition (D3), Lemma 2.2 and (2.3), we obtain

$$D_0 > \alpha - J(u_0) \geqslant J(v_n) - \frac{1}{\theta} \langle J'(v_n), v_n \rangle + o(1) \geqslant \left(\frac{\theta - 2}{2\theta} - \frac{d_0}{\Theta_0} \right) \|v_n\|_\lambda^2 + o(1),$$

which implies that

$$\|v_n\|_\lambda^2 \leqslant \frac{2\theta\Theta_0 D_0}{\Theta_0(\theta - 2) - 2\theta d_0} + o(1).$$

Moreover, by (2.4), one has

$$\int_{\mathbb{R}} |v_n(x)|^r dx \leqslant \frac{c_0^{\frac{r-2}{2}}}{|\{a < c\}|^{\frac{r-2}{2}} \Theta_0^{\frac{r}{2}}} \|v_n\|_\lambda^r \leqslant \frac{1}{|\{a < c\}|^{\frac{r-2}{2}}} \left[\frac{2\theta\Theta_0 D_0}{\Theta_0(\theta - 2) - 2\theta d_0} \right]^{\frac{r}{2}} + o(1). \quad (3.8)$$

Since $\langle J'(v_n), v_n \rangle = o(1)$ and

$$\int_{\mathbb{R}} f(x, v_n) v_n dx \leqslant (|p^+|_\infty + \epsilon) \int_{\mathbb{R}} v_n(x)^2 dx + C_\epsilon \int_{\mathbb{R}} |v_n(x)|^r dx, \quad (3.9)$$

it follows from (3.6) and (3.8) and (3.9) that

$$\begin{aligned} o(1) &= \|v_n\|_\lambda^2 - \int_{\mathbb{R}} f(x, v_n) v_n dx \geqslant \|v_n\|_\lambda^2 - (|p^+|_\infty + \epsilon) \int_{\mathbb{R}} v_n^2(x) dx - C_\epsilon \int_{\mathbb{R}} |v_n(x)|^r dx \\ &\geqslant \|v_n\|_\lambda^2 - \frac{|p^+|_\infty + \epsilon}{\lambda c} \|v_n\|_\lambda^2 - C_\epsilon \left(\int_{\mathbb{R}^N} |v_n|^r dx \right)^{(r-2)/r} \left(\int_{\mathbb{R}^N} |v_n|^r dx \right)^{2/r} \\ &\geqslant \|v_n\|_\lambda^2 \left\{ 1 - \frac{(|p^+|_\infty + \epsilon)}{\lambda c} - \left(\frac{2\theta\Theta_0 D_0}{(\Theta_0(\theta - 2) - 2\theta d_0)|\{a < c\}|^{\frac{r-2}{r}}} \right)^{\frac{r-2}{2}} \left[\frac{S_\infty^{r-2}}{\lambda c} \left(\frac{1}{c_0 - S_\infty^2 |\{a < c\}|} \right)^{\frac{r-2}{2}} \right]^{2/r} \right\}. \end{aligned}$$

Thus, there exists $\Lambda_* = \Lambda(D_0) \geqslant \max\left\{\frac{1}{c}, \frac{1}{cS_\infty^2|\{a<c\}|}\right\}$ such that $v_n \to 0$ strongly in X_λ for $\lambda > \Lambda_*$. This completes the proof. □

Now we give the proof of Theorem 1.1: By Proposition 3.4 and $0 < \eta \leqslant \alpha \leqslant \alpha_0(\Omega)$ for all $\lambda \geqslant \frac{1}{cS_\infty^2|\{a<c\}|}$, for each $D_0 > \alpha_0(\Omega)$ there exists

$$\Lambda_* \geqslant \max\left\{ \frac{1}{c}, \frac{1}{cS_\infty^2 |\{a < c\}|} \right\} > 0$$

such that for every $\lambda > \Lambda^*$ and $(C)_\alpha$–sequence $\{u_n\}$ for J on X_λ there exist a subsequence $\{u_n\}$ and $u_\lambda \in X_\lambda$ such that $u_n \to u_\lambda$ strongly in X_λ. Moreover, $J(u_\lambda) = \alpha$ and u_λ is a nontrivial homoclinic solution of Eq. (1.1).

Now we give the proof of Theorem 1.2: Let $u_0 \in H^2(\mathbb{R})$ be a nontrivial homoclinic solution of Eq. (1.1). Then by (2.2), for $\lambda \geqslant \frac{1}{c}$ we have

$$\begin{aligned} \int_{\mathbb{R}} \left[u_0''(x)^2 - wu_0'(x)^2 + \lambda a(x) u(x)_0^2 \right] dx &= \int_{\mathbb{R}} f(x, u_0) u_0 dx \leqslant \int_{\mathbb{R}} q u_0^2 dx \leqslant \frac{1}{\mu_0} \|u_0\|_{H^2}^2 \\ &\leqslant \frac{1}{\mu_0 \left(c_0 - S_\infty^2 |\{a < c\}| \right)} \int_{\mathbb{R}} \left[u_0''(x)^2 - wu_0'(x)^2 + \lambda a(x) u(x)_0^2 \right] dx \\ &< \int_{\mathbb{R}} \left[u_0''(x)^2 - wu_0'(x)^2 + \lambda a(x) u(x)_0^2 \right] dx, \end{aligned}$$

which is a contradiction.

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Appendix A

Consider the minimum problem

$$\mu^* = \inf\left\{\int_\Omega \left[u''(x)^2 - wu'(x)^2\right]dx \,|\, u \in H^2(\Omega)\cap H_0^1(\Omega),\ \int_\Omega qu^2dx = 1\right\}. \tag{4.1}$$

Then we have the following result.

Lemma 4.1. *There exist a constant $c_1 > 0$ and $\phi_1 \in H^2(\Omega)\cap H_0^1(\Omega)$ with $\int_\Omega q\phi_1^2dx = 1$ such that*

$$\mu^* = \int_\Omega \left[\phi_1''(x)^2 - w\phi_1'(x)^2\right]dx \geqslant c_1,$$

i.e., the minimum problem (4.1) is achieved.

Proof. For any $u \in H^2(\Omega)\cap H_0^1(\Omega)$ with $\int_\Omega qu^2dx = 1$, by Lemma 2.1 and Sobolev embedded theorem, we have

$$\int_\Omega \left[u''(x)^2 - wu'(x)^2\right]dx = \int_{\mathbb{R}} \left[u''(x)^2 - wu'(x)^2 + a(x)u(x)^2\right]dx \geqslant c_0\|u\|_{H^2}^2 \geqslant c_0S_2^2\int_\Omega u^2dx \geqslant \frac{c_0S_2^2}{|q|_\infty} > 0.$$

Therefore, $\mu^* \geqslant \frac{c_0S_2^2}{|q|_\infty} > 0$. Let $\{u_n\} \subset H^2(\Omega)\cap H_0^1(\Omega)$ be a minimizing sequence of (4.1). Clearly, $\int_\Omega qu_n^2dx = 1$ and $\{u_n\}$ is bounded. Then by the compact imbedding theorem, there exist a subsequence $\{u_n\}$ and $\phi_1 \in H^2(\Omega)\cap H_0^1(\Omega)$ such that $u_n \rightharpoonup \phi_1$ weakly in $H^2(\Omega)\cap H_0^1(\Omega)$ and $u_n \to \phi_1$ strongly in $L^2(\mathbb{R})$. So it is easy to verify that $\int_\Omega qu_n^2dx \to \int_\Omega q\phi_1^2dx$ as $n\to\infty$ and $\int_\Omega q\phi_1^2dx = 1$. Therefore,

$$\mu^* \leqslant \int_\Omega \left[\phi_1''(x)^2 - w\phi_1'(x)^2\right]dx \leqslant \liminf_{n\to\infty}\int_\Omega \left[u_n''(x)^2 - wu_n'(x)^2\right]dx = \mu^*,$$

which implies that $\int_\Omega \left[\phi_1''(x)^2 - w\phi_1'(x)^2\right]dx = \mu^*$. This completes the proof. □

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