Journal of Computational and Applied Mathematics 233 (2010) 1733–1745

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

A structure-preserving doubling algorithm for quadratic eigenvalue problems arising from time-delay systems

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ARTICLE INFO

Article history:
Received 10 September 2008

Keywords:
Doubling
Quadratic eigenvalue problem
Palindromic eigenvalue problem
Structure-preserving
Time-delay system
Unimodular eigenvalue

ABSTRACT

We propose a structure-preserving doubling algorithm for a quadratic eigenvalue problem arising from the stability analysis of time-delay systems. We are particularly interested in the eigenvalues on the unit circle, which are difficult to estimate. The convergence and backward error of the algorithm are analyzed and three numerical examples are presented. Our experience shows that our algorithm is efficient in comparison to the few existing approaches for small to medium size problems.

1. Introduction

In this paper, we consider the numerical solution of the quadratic eigenvalue problem (QEP)

$$Q(\lambda)x \equiv (\lambda^2 B + \lambda C + A)x = 0, \tag{1.1a}$$

where $A, B, C \in \mathbb{C}^{n\times n}$ satisfy

$$P\bar{B}P = \varepsilon A, \qquad P\bar{A}P = \varepsilon B, \qquad P\bar{C}P = \varepsilon C \tag{1.1b}$$

with $\varepsilon = \pm 1$, $P \in \mathbb{C}^{n\times n}$ being an idempotent matrix (i.e., $P^2 = I_n$), and $\bar{B}$ denoting the complex conjugate of B. In our application for the time-delay system (1.3), the matrix P is a real involuntary matrix (more details are given later). The scalar $\lambda \in \mathbb{C}$ and the vector $x \in \mathbb{C}^n \setminus \{0\}$ are the eigenvalue and the associated eigenvector of the quadratic pencil $Q(\lambda)$, and the pair (λ, x) is called an *eigenpair* of $Q(\lambda)$.

We shall propose a structure-preserving doubling algorithm (SDA; [1–5]) for the solution of (1.1). We define the `conjugate` and the `reverse` of a quadratic pencil, respectively, by

$$\overline{Q}(\lambda) \equiv \lambda^2\bar{B} + \lambda\bar{C} + \bar{A}, \qquad \mathtt{rev}(Q(\lambda)) \equiv \lambda^2 A + \lambda C + B.$$

It is easy to see that QEP in (1.1) satisfies

$$Q(\lambda) = \varepsilon P\mathtt{rev}(\overline{Q}(\lambda))P. \tag{1.2}$$

From (1.2), we see that (λ, x) is an eigenpair of $Q(\lambda)$ if and only if $(1/\bar{\lambda}, P\bar{x})$ is also an eigenpair of $Q(\lambda)$ [6].

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doi:10.1016/j.cam.2009.09.010

As in [7], a quadratic pencil $Q(\lambda)$ in (1.1a) is said to be palindromic if $Q(\lambda) = \mathtt{rev}(\overline{Q}(\lambda))$. Similar to the terminology of $(\star, \varepsilon)$-palindromic QEPs ($\star = \mathrm{H}$ or $\top$) in [7], the QEP (1.1) is referred to as

($\varepsilon = +1$) P-Conjugate-P Palindromic QEP (PCP_PQEP), or

($\varepsilon = -1$) anti-P-Conjugate-P Palindromic QEP ($-$PCP_PQEP).

PCP_PQEPs as in (1.1) were first proposed in [8] from the stability analysis of *retarded* time-delay systems (TDS), and generalized in [6] the more general *neutral* TDS (see [9–16] and the references therein). Consider a neutral linear time-delay system with m constant delays $h_0 = 0 < h_1 < \cdots < h_m$:

$$\sum_{k=0}^{m} D_k \dot{x}(t - h_k) = \sum_{k=0}^{m} A_k x(t - h_k) \quad (t > 0); \qquad x(t) = \varphi(t) \quad (t \in [-h_m, 0]) \tag{1.3}$$

with $x : [-h_m, \infty) \to \mathbb{R}^N$, $\varphi \in \mathbb{C}^1([-h_m, 0])$, and $A_k, D_k \in \mathbb{R}^{N\times N}$ $(k = 1, \ldots, m)$. When $D_0 = I_N$ and $D_k = 0$ $(k > 0)$, we have the retarded time-delay systems.

The stability of the TDS (1.3) can be determined by its characteristic equation

$$\left(s \sum_{k=0}^{m} D_k \mathrm{e}^{-h_k s} - \sum_{k=0}^{m} A_k \mathrm{e}^{-h_k s} \right) v = 0 \quad (v \neq 0) \tag{1.4}$$

with eigenpairs (s, v) from the nonlinear eigenvalue problem.

A TDS is said to be *critical* if and only if some eigenvalues s is purely imaginary. The set of all points $(h_1, \ldots, h_m)$ in the delay-parameter space for which the TDS (1.3) is critical are called critical curves ($m = 2$) or surfaces ($m > 2$). Under certain continuity assumptions, the boundary of the stability domain of a TDS is a subset of the critical curves/surfaces. Consequently, purely imaginary eigenvalues of (1.4) are of great interest. See [8] and the references therein for approaches to compute critical surfaces. In [6], the following parameterization of critical surfaces gives rise to an associated PCP_PQEP. Detailed discussion on palindromic linearizations, a Schur-like canonical form and other useful results can also be found in [6].

For a given eigenpair (s, v) of (1.4) with $\|v\|_2 = 1$, a point $(h_1, \ldots, h_m)$ in the delay-parameter space is critical if and only if there exist $\varphi_k \in [-\pi, \pi]$ $(k = 1, \ldots, m-1)$ and $\omega \in \mathbb{R}$ such that

$$h_k = \frac{\varphi_k + 2p_k\pi}{\omega}, \quad p_k \in \mathbb{Z}\ (k = 1, \ldots, m-1); \qquad h_m = \frac{-\operatorname{Arg} z + 2p_m\pi}{\omega}, \quad p_m \in \mathbb{Z}; \tag{1.5}$$

giving rise to the QEP

$$(z^2 E + zF + G)u = 0, \tag{1.6}$$

where the unimodular eigenvalue $z = \mathrm{e}^{-\mathrm{i}\omega h_m}$, and the corresponding eigenvector $u = \operatorname{vec} vv^* = v \otimes \bar{v}$,

$$\omega = \left(\frac{-\mathrm{i}}{w^* w} \right) w^* \left(A_m z + \sum_{k=0}^{m-1} A_k \mathrm{e}^{-\mathrm{i}\varphi_k} \right) v, \qquad w = \left(D_m z + \sum_{k=0}^{m-1} D_k \mathrm{e}^{-\mathrm{i}\varphi_k} \right) v, \tag{1.7}$$

and

$$E = \left(\sum_{k=0}^{m-1} D_k \mathrm{e}^{\mathrm{i}\varphi_k} \right) \otimes A_m + \left(\sum_{k=0}^{m-1} A_k \mathrm{e}^{\mathrm{i}\varphi_k} \right) \otimes D_m \in \mathbb{C}^{N^2\times N^2},$$

$$F = \left(\sum_{k=0}^{m-1} D_k \mathrm{e}^{\mathrm{i}\varphi_k} \right) \otimes \left(\sum_{k=0}^{m-1} A_k \mathrm{e}^{-\mathrm{i}\varphi_k} \right) + \left(\sum_{k=0}^{m-1} A_k \mathrm{e}^{\mathrm{i}\varphi_k} \right) \otimes \left(\sum_{k=0}^{m-1} D_k \mathrm{e}^{-\mathrm{i}\varphi_k} \right) + D_m \otimes A_m + A_m \otimes D_m \in \mathbb{C}^{N^2\times N^2},$$

$$G = D_m \otimes \left(\sum_{k=0}^{m-1} A_k \mathrm{e}^{-\mathrm{i}\varphi_k} \right) + A_m \otimes \left(\sum_{k=0}^{m-1} D_k \mathrm{e}^{-\mathrm{i}\varphi_k} \right) \in \mathbb{C}^{N^2\times N^2}.$$

Here $\otimes$ denotes the usual Kronecker product. As $M_1 \otimes M_2$ and $M_2 \otimes M_1$ both contain products of elements of M_1 and M_2 at different positions, we can find an involuntary matrix $P \in \mathbb{R}^{N^2\times N^2}$ $(P^{-1} = P^\top = P)$ such that $M_1 \otimes M_2 = P(M_2 \otimes M_1)P$ [17, Corollary 4.3.10]. Here, $P = \sum_{i,j=1}^{N} E_{ij} \otimes E_{ij}^\top = [E_{ij}^\top]_{i,j=1}^{N}$, $E_{ij} = e_i \otimes e_j^\top \in \mathbb{R}^{N\times N}$ and e_j is the jth column of I_N. Consequently from the structures in E, F and G, we can easily show that (1.6) is a PCP_PQEP because $E = P\overline{G}P$ and $F = P\overline{F}P$.

Remark 1.1. In [18], it has been proved that unimodular eigenvalues occur quite often for PCP_PQEPs (and other palindromic eigenvalue problems with eigenvalue pairs $(\lambda, \varepsilon/\overline{\lambda})$). These eigenvalues can stay unimodular under perturbation, thus are numerically stable to compute. Furthermore, the probability of having too many of them is low and multiple unimodular eigenvalues are rare. This makes our problem of computing the unimodular eigenvalues z of (1.6) well-posed, unlike for complex-T palindromic eigenvalue problems.

Before proceeding further, we would like to point out some recent developments in the numerical solution of palindromic eigenvalue problems. The QEP (1.1a) is said to be $\star$-palindromic if $B^\star = A$ and $C^\star = C$, with $\star = \top$ or H. The train vibration problem and the associated $\top$-palindromic QEP were discussed in [19,20] and structure-preserving palindromic linearizations for (1.1a) were suggested in [7]. An SDA algorithm [3] and a backward stable generalized (Arnoldi) Patel algorithm [21] were proposed for the $\top$-palindromic QEP, and can be extended to H-palindromic QEPs. For other approaches for $\star$-palindromic QEPs, see [22–24]. In [2], the SDA algorithm was generalized for the g-palindromic QEPs, which do not include the PCP_PQEP. For further information on the numerical solution of palindromic EVPs, see [4]. For general survey of matrix polynomials, the associated eigenvalue problems and their applications, see [25,26].

Throughout this paper, $\mathbb{C}^{n\times m}$ is the set of all $n \times m$ complex matrices, $\mathbb{C}^n = \mathbb{C}^{n\times 1}$, and $\mathbb{C} = \mathbb{C}^1$; I_n (or I if there is no confusion) is the $n \times n$ identity matrix; and $X^H = \overline{X}^\top$ denote the Hermitian (conjugate transpose) of X. We shall also adopt MATLAB-like convention to access the entries of vectors and matrices—$X(i, j)$ is the (i, j)th entry, $X(k : \ell, i : j)$ the sub-matrix from rows $k : \ell$ and columns $i : j$, $X(:, i : j)$ from columns $i : j$ and $X(k : \ell, :)$ from rows $k : \ell$.

In Section 2, we develop an SDA algorithm for solving the εPCP_PQEP. Convergence of the SDA is proved in Section 3. For the PCP_PQEP arisen from the stability analysis of TDSs, we develop a deflation technique for finding all unimodular eigenvalues in Section 4. A structured backward error analysis for PCP_PQEPs is presented in Section 5. Numerical examples of PCP_QEPs arising from TDS are given in Section 6. Concluding remarks are given in Section 7.

2. SDA algorithm for ε PCP_PQEP

For a given PCP_PQEP (1.1), we define

$$\mathcal{M} = \begin{bmatrix} A & 0 \\ -C & -I \end{bmatrix}, \qquad \mathcal{L} = \begin{bmatrix} D & I \\ B & 0 \end{bmatrix}. \tag{2.1}$$

With $D = 0$ in (2.1), we have

$$\begin{bmatrix} A & 0 \\ -C & -I \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} 0 & I \\ B & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

leading to

$$Ax_1 = \lambda x_2, \tag{2.2a}$$

$$-Cx_1 - x_2 = \lambda B x_1. \tag{2.2b}$$

Multiplying (2.2b) by λ and substituting (2.2a) into it, we obtain

$$(\lambda^2 B + \lambda C + A)x_1 = 0.$$

We have shown that the pencil $\mathcal{M} - \lambda\mathcal{L}$ is a linearization of PCP_PQEP (1.1) with $D = 0$. Based on the SDA algorithm proposed in [5], we develop a new SDA for solving the PCP_PQEP.

For the pencil $\mathcal{M} - \lambda\mathcal{L}$ defined in (2.1), we compute

$$\mathcal{M}_* = \begin{bmatrix} -AK^{-1} & 0 \\ BK^{-1} & I \end{bmatrix}, \qquad \mathcal{L}_* = \begin{bmatrix} I & AK^{-1} \\ 0 & -BK^{-1} \end{bmatrix},$$

where $K \equiv C - D$ is assumed to be invertible. It is easy to check that $\mathcal{M}_*\mathcal{L} = \mathcal{L}_*\mathcal{M}$. Direct calculation gives rise to

$$\widehat{\mathcal{M}} = \mathcal{M}_*\mathcal{M} = \begin{bmatrix} \widehat{A} & 0 \\ -\widehat{C} & I \end{bmatrix}, \qquad \widehat{\mathcal{L}} = \mathcal{L}_*\mathcal{L} = \begin{bmatrix} \widehat{D} & I \\ \widehat{B} & 0 \end{bmatrix}, \tag{2.3}$$

where

$$\widehat{A} \equiv -AK^{-1}A, \qquad \widehat{B} \equiv -BK^{-1}B, \tag{2.4a}$$

$$\widehat{C} \equiv C - BK^{-1}A, \qquad \widehat{K} \equiv K - (BK^{-1}A + AK^{-1}B), \tag{2.4b}$$

and

$$\widehat{D} = \widehat{C} - \widehat{K}. \tag{2.4c}$$

Theorem 2.1. *The pencil $\widehat{\mathcal{M}} - \lambda\widehat{\mathcal{L}}$ has the doubling property, i.e., if*

$$\mathcal{M} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \mathcal{L} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} S,$$

where $X_1, X_2 \in \mathbb{C}^{n\times m}$ and $S \in \mathbb{C}^{m\times m}$, then

$$\widehat{\mathcal{M}} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \widehat{\mathcal{L}} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} S^2.$$

Proof. From (2.3) and the relation $\mathcal{M}_*\mathcal{L} = \mathcal{L}_*\mathcal{M}$, we have

$$\widehat{\mathcal{M}}\begin{bmatrix}X_1\\X_2\end{bmatrix} = \mathcal{M}_*\mathcal{M}\begin{bmatrix}X_1\\X_2\end{bmatrix} = \mathcal{M}_*\mathcal{L}\begin{bmatrix}X_1\\X_2\end{bmatrix}S = \mathcal{L}_*\mathcal{M}\begin{bmatrix}X_1\\X_2\end{bmatrix}S = \mathcal{L}_*\mathcal{L}\begin{bmatrix}X_1\\X_2\end{bmatrix}S^2 = \widehat{\mathcal{L}}\begin{bmatrix}X_1\\X_2\end{bmatrix}S^2. \quad \square$$

The iteration in (2.4) is structure-preserving for ε PCP_PQEP, as shown in the following theorem:

Theorem 2.2. *For a pencil $\mathcal{M} - \lambda\mathcal{L}$ given in (2.1), suppose that*

$$P\overline{A}P = \varepsilon B, \qquad P\overline{B}P = \varepsilon A, \qquad P\overline{K}P = \varepsilon K, \tag{2.5}$$

where $K = C - D$. Then it holds that

$$P\overline{\widehat{A}}P = \varepsilon\widehat{B}, \qquad P\overline{\widehat{B}}P = \varepsilon\widehat{A}, \qquad P\overline{\widehat{K}}P = \varepsilon\widehat{K},$$

where $\widehat{A}, \widehat{B}, \widehat{K}$ are defined in (2.4).

Proof. From (2.4b) and (2.5), we have

$$\begin{aligned}P\overline{\widehat{K}}P &= P\overline{K}P - \left[P\overline{B}P(P\overline{K}P)^{-1}P\overline{A}P + P\overline{A}P(P\overline{K}P)^{-1}P\overline{B}P\right]\\ &= \varepsilon K - \varepsilon^3(AK^{-1}B + BK^{-1}A) = \varepsilon\widehat{K}.\end{aligned}$$

Similarly, from (2.4a) and (2.5), we have

$$P\overline{\widehat{A}}P = -P\overline{A}P(P\overline{K}P)^{-1}P\overline{A}P = -\varepsilon^3 BK^{-1}B = \varepsilon\widehat{B}$$

and then

$$P\overline{\widehat{B}}P = \varepsilon\widehat{A}. \quad \square$$

From Theorems 2.1 and 2.2, we restate the SDA for finding a basis for the stable invariant subspace of $(\mathcal{M}, \mathcal{L})$.

Algorithm SDA

Input: $A, B, C, P \in \mathbb{C}^{n\times n}$ with $P\overline{A}P = \varepsilon B$, $P\overline{B}P = \varepsilon A$, $P\overline{C}P = \varepsilon C$, $P^2 = P$, τ(a small tolerance);

Output: a basis $X_s = [X_{s1}^\top, X_{s2}^\top]^\top \in \mathbb{C}^{2n\times m}$ with $X_{s1}^H X_{s1} = I_m$ for the stable invariant subspace of $(\mathcal{M}, \mathcal{L})$ in (2.1);

Set $k = 0, A_k = A, B_k = B, C_k = C, K_k = C, D_k = 0$;

Do until convergence:

Compute

$$\begin{aligned}&A_{k+1} = -A_kK_k^{-1}A_k\\ &B_{k+1} = P\overline{A}_{k+1}P\\ &W_k = B_kK_k^{-1}A_k\\ &K_{k+1} = K_k - (W_k + P\overline{W}_kP)\\ &C_{k+1} = C_k - W_k\\ &(D_{k+1} = C_{k+1} - K_{k+1})\\ &k = k+1\\ &\text{If dist}(\texttt{Null}(A_k), \texttt{Null}(A_{k+1})) < \tau, \texttt{Stop}\end{aligned}$$

End

Compute an orthonormal basis X_{s1} such that $\|A_{k+1}X_{s1}\| \le \tau\|A_{k+1}\|$;

set $X_{s2} = -C_{k+1}X_{s1}$.

3. Convergence of SDA

Consider the matrix pair $(\mathcal{M}, \mathcal{L})$ as in (2.1). In order to ensure that the SDA converges to a basis of the stable invariant subspace of $(\mathcal{M}, \mathcal{L})$, we suppose that all eigenvalues of $(\mathcal{M}, \mathcal{L})$ on the unit circle are semisimple (generically, multiple unimodular eigenvalues are rare; see [18]).

From the Kronecker Canonical form [6,27], there exist nonsingular matrices Q and Z such that

$$Q\mathcal{M}Z = \begin{bmatrix}J_1 & 0\\0 & I_n\end{bmatrix} \equiv J_{\mathcal{M}}, \tag{3.1a}$$

$$Q\mathcal{L}Z = \begin{bmatrix}I & 0\\0 & J_2\end{bmatrix} \equiv J_{\mathcal{L}}, \tag{3.1b}$$

where

$$J_1 = \Omega_1 \oplus J_s, \qquad J_2 = \Omega_2 \oplus \varepsilon \bar{J}_s,$$
$$\Omega_1 = \mathrm{diag}\{e^{i\omega_1}, \ldots, e^{i\omega_\ell}\}, \qquad \Omega_2 = \mathrm{diag}\{e^{i\omega_{\ell+1}}, \ldots, e^{i\omega_{2\ell}}\},$$

and J_s is the stable Jordan block of size m (i.e., $\rho(J_s) < 1$) with $m = n - \ell$. Here ρ denotes the spectral radius and $\oplus$ the direct sum of matrices. Since $J_{\mathcal{M}}$ and $J_{\mathcal{L}}$ commute with each other, it follows from (3.1) that

$$\mathcal{M}ZJ_{\mathcal{L}} = Q^{-1}J_{\mathcal{L}}J_{\mathcal{M}} = \mathcal{L}ZJ_{\mathcal{M}}. \tag{3.2}$$

Let $\{(\mathcal{M}_k, \mathcal{L}_k)\}_{k=0}^{\infty}$ be the sequence with

$$\mathcal{M}_k = \begin{bmatrix} A_k & 0 \\ -C_k & -I_n \end{bmatrix}, \qquad \mathcal{L}_k = \begin{bmatrix} D_k & I_n \\ B_k & 0 \end{bmatrix}, \tag{3.3}$$

where A_k, B_k, C_k and D_k are generated by the SDA. With $\mathcal{M}_0 = \mathcal{M}$ and $\mathcal{L}_0 = \mathcal{L}$, it follows from (3.2) and Theorem 2.1 that

$$\mathcal{M}_k Z J_{\mathcal{L}}^{2^k} = \mathcal{L}_k Z J_{\mathcal{M}}^{2^k}. \tag{3.4}$$

Theorem 3.1. *Let $(\mathcal{M}, \mathcal{L})$ be given in (2.1) with all its unimodular eigenvalues being semisimple. Write Z in (3.1) in the form*

$$Z = \begin{bmatrix} Z_1 & Z_3 \\ Z_2 & Z_4 \end{bmatrix}, \quad Z_i \in \mathbb{C}^{n\times n},\ i = 1, \ldots, 4. \tag{3.5}$$

Suppose that the sequence $\{A_k, B_k, C_k, D_k\}_{k=1}^{\infty}$ generated by the SDA is well defined and $\{A_k\}_{k=1}^{\infty}$ is uniformed bounded on k. If Z_1 and Z_3 are invertible, we have

(i) $A_k Z_{12} = O(\rho(J_s^{2^k})) \to 0$ *as* $k \to \infty$;
(ii) $C_k Z_{12} = -Z_{22} + O(\rho(J_s^{2^k})) \to -Z_{22}$ *as* $k \to \infty$;
(iii) $D_k Z_{12} = -Z_{42} + O(\rho(\bar{J}_s^{2^k})) \to -Z_{42}$ *as* $k \to \infty$,

where $Z_{12} = Z_1(:, \ell+1, \ldots, n)$, $Z_{22} = Z_2(:, \ell+1, \ldots, n)$, $Z_{32} = Z_3(:, \ell+1, \ldots, n)$, $Z_{42} = Z_4(:, \ell+1, \ldots, n)$ and $\rho(\cdot)$ is the spectrum radius.

Proof. Substituting $(\mathcal{M}_k, \mathcal{L}_k)$ from (3.3), $J_{\mathcal{M}}$ and $J_{\mathcal{L}}$ in (3.1) as well as Z in (3.5) into (3.4), we have

$$A_k Z_1 = (D_k Z_1 + Z_2)(\Omega_1^{2^k} \oplus J_s^{2^k}), \tag{3.6a}$$
$$A_k Z_3 (\Omega_2^{2^k} \oplus \bar{J}_s^{2^k}) = D_k Z_3 + Z_4, \tag{3.6b}$$
$$-(C_k Z_1 + Z_2) = B_k Z_1 (\Omega_1^{2^k} \oplus J_s^{2^k}), \tag{3.6c}$$
$$-(C_k Z_3 + Z_4)(\Omega_2^{2^k} \oplus \bar{J}_s^{2^k}) = B_k Z_3. \tag{3.6d}$$

From (3.6b), it follows that

$$D_k = -Z_4 Z_3^{-1} + A_k Z_3 (\Omega_2^{2^k} \oplus \bar{J}_s^{2^k}) Z_3^{-1}. \tag{3.7}$$

Substituting (3.7) into (3.6a), we get

$$A_k \left[I - Z_3 \left(\Omega_2^{2^k} \oplus \bar{J}_s^{2^k} \right) Z_3^{-1} Z_1 \left(\Omega_1^{2^k} \oplus J_s^{2^k} \right) Z_1^{-1} \right] = \left(-Z_4 Z_3^{-1} Z_1 + Z_2 \right) \left(\Omega_1^{2^k} \oplus J_s^{2^k} \right) Z_1^{-1}. \tag{3.8}$$

Since $\Omega_1^{2^k}$ and $\Omega_2^{2^k}$ are uniformly bounded (independent of k), and $\rho(J_s) < 1$, (3.8) can be rewritten as

$$A_k \left[I - \left(z_3 \Omega_2^{2^k} \omega_3^{\top} \right) \left(z_1 \Omega_1^{2^k} \omega_1^{\top} \right) + O\left(\rho(J_s^{2^k}) \right) \right] = \left(-Z_4 Z_3^{-1} z_1 + z_2 \right) \Omega_1^{2^k} \omega_1^{\top} + O\left(\rho(J_s^{2^k}) \right),$$

where

$$z_1 = Z_1(:, 1:\ell), \qquad z_2 = Z_2(:, 1:\ell), \qquad z_3 = Z_3(:, 1:\ell), \qquad \omega_1^{\top} = Z_1^{-1}(1:\ell, :), \qquad \omega_3^{\top} = Z_3^{-1}(1:\ell, :).$$

By the assumption that A_k is uniformly bounded on k, we have

$$A_k = a_k \omega_1^{\top} + O(\rho(J_s^{2^k}))$$

for some suitable $a_k \in \mathbb{C}^{n\times\ell}$ with $\|a_k\|$ being uniformly bounded on k. Thus we have

$$A_k Z_{12} = O(\rho(J_s^{2^k})) \to 0, \quad \text{as } k \to \infty,$$

where $Z_{12} = Z_1(:, \ell+1, \ldots, n)$ being orthogonal to ω_1. This proves (i).

Since $B_k = P\bar{A}_k P$ by Theorem 2.2, from (3.6c) we obtain

$$C_k = -Z_2 Z_1^{-1} + c_k \omega_1^\top + O(\rho(J_s^{2^k}))$$

for some suitable $c_k \in \mathbb{C}^{n\times\ell}$ with $\|c_k\|$ being uniformly bounded on k.
Thus, we have

$$\begin{aligned} C_k Z_{12} &= -Z_2 Z_1^{-1} Z_{12} + c_k \omega_1^\top Z_{12} + O(\rho(J_s^{2^k})) \\ &= -Z_2 \begin{bmatrix} 0 \\ I_{n-\ell} \end{bmatrix} + O(\rho(J_s^{2^k})) \to -Z_{22}, \quad \text{as } k \to \infty, \end{aligned}$$

where $Z_{22} = Z_2(:, \ell+1, \ldots, n)$. This proves (ii).

Similarly from (3.7), we see that

$$\begin{aligned} D_k Z_{32} &= -Z_4 Z_3^{-1} Z_{32} + d_k \omega_3^\top Z_{32} + O(\rho(J_s^{2^k})) \\ &\to -Z_{42} \to 0, \quad \text{as } k \to \infty, \end{aligned}$$

where $Z_{32} \equiv Z_3(:, \ell+1, \ldots, n)$, $Z_{42} \equiv Z_4(:, \ell+1, \ldots, n)$ and $d_k \in \mathbb{C}^{n\times\ell}$ is some suitable matrix which is uniformly bounded on k. □

4. Application to TDS

We are ready to solve PCP_PQEPs from TDSs. For a given PCP_PQEP or $Q(\lambda)$ as in (1.1), we are interested in finding all eigenvalues on the unit circle and the associated eigenvectors. We first run the SDA until

$$\texttt{dist}(\texttt{Null}(A_{k-1}), \texttt{Null}(A_k)) < \texttt{Tolerance}.$$

Then we compute the SVD decomposition of A_k, such that

$$U_k^H A_k V_k = \begin{bmatrix} \Sigma_k & 0 \\ 0 & \Sigma_k' \end{bmatrix}, \quad \Sigma_k \gg \Sigma_k' \approx O(\tau), \tag{4.1}$$

where τ is a small tolerance.
Consequently, with ℓ denoting the number of unimodular eigenvalues, we have

$$\begin{bmatrix} A_k & 0 \\ -C_k & -I \end{bmatrix} \begin{bmatrix} V_{k2} \\ -C_k V_{k2} \end{bmatrix} \approx O(\tau), \tag{4.2}$$

where $V_{k2} = V_k(:, \ell+1 : \ldots, n)$. From Theorems 2.1 and 3.1, it follows that

$$\texttt{span}\left\{ \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \equiv \begin{bmatrix} V_{k2} \\ -C_k V_{k2} \end{bmatrix} \right\} \simeq \texttt{span}\left\{ \begin{bmatrix} Z_{12} \\ Z_{22} \end{bmatrix} \right\}. \tag{4.3}$$

That is, $\left[X_1^\top, X_2^\top\right]^\top$ approximates a basis of the stable invariant subspace of $(\mathcal{M}, \mathcal{L})$. From (4.2) and (3.1) we compute an approximate stable eigenmatrix associated with $[X_1^\top, X_2^\top]^\top$ for $(\mathcal{M}, \mathcal{L})$ by

$$\mathcal{M} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \mathcal{L} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} S.$$

This implies that

$$S = (X_2^\top X_2)^{-1} X_2^\top A X_1. \tag{4.4}$$

Let $S\xi_j = \lambda_j \xi_j$, $j = \ell+1, \ldots, n$. Then $\{(\lambda_j, X_1\xi_j)\}_{j=\ell+1}^n$ are the computed stable eigenpairs of $Q(\lambda)$. Furthermore, $\{(1/\bar{\lambda}_j, P\bar{X}_1\bar{\xi}_j)\}_{j=\ell+1}^n$ are the computed unstable eigenpairs of $Q(\lambda)$. Again from (3.1), (4.1)–(4.2) and Theorem 2.2, we have

$$\begin{bmatrix} D_k & I \\ B_k & 0 \end{bmatrix} \begin{bmatrix} P\bar{V}_{k2} \\ -D_k P\bar{V}_{k2} \end{bmatrix} \approx O(\tau).$$

Thus, from (4.3), $\left[(P\bar{X}_1)^\top, -(D_k P\bar{X}_1)^\top\right]^\top$ forms a basis for the unstable invariant subspace.

Table 1
Flop counts for the SDA+Newton algorithm.

Task		≈Flops
(a) SDA	$\frac{80}{3}n^3 \times 8=$	$213n^3$
(b) SVD in (4.1) and (4.4)	$\frac{160}{3}n^3 + 32n^3=$	$85n^3$
(c) `eig(S)`		$100n^3$
(d) Compute (R_0, T_0) in (4.5)		$75n^3$
(e) Newton's method in (4.6)		$5.3\ell n^3$
Compute all eigenvalues	(a)–(e)	$(473 + 5.3\ell)n^3$
ℓ unimodular eigenvalues only	(a), (b), (d) & (e)	$(373 + 5.3\ell)n^3$

Let Φ_1 be an orthonormal basis of $\begin{bmatrix} X_1 & P\overline{X}_1 \\ -C_kX_1 & -D_kP\overline{X}_1 \end{bmatrix}$. Then, applying deflation, there are unitary matrices $\Phi = [\Phi_1, \Phi_0]$ and $\Psi = [\Psi_1, \Psi_0]$ such that

$$\begin{bmatrix} \Psi_1^H \\ \Psi_0^H \end{bmatrix} \mathcal{M}[\Phi_1, \Phi_0] = \begin{bmatrix} R_1 & R_2 \\ 0 & R_0 \end{bmatrix}, \tag{4.5a}$$

$$\begin{bmatrix} \Psi_1^H \\ \Psi_0^H \end{bmatrix} \mathcal{L}[\Phi_1, \Phi_0] = \begin{bmatrix} T_1 & T_2 \\ 0 & T_0 \end{bmatrix}. \tag{4.5b}$$

Approximations to the unimodular eigenvalues can be obtained by solving the matrix pair $(R_0, T_0) \equiv (\Psi_0^H \mathcal{M}\Phi_0, \Psi_0^H \mathcal{L}\Phi_0)$ (e.g. using `eig` in MATLAB). To refine an approximate unimodular eigenvalue λ_0 of $Q(\lambda)$ from (R_0, T_0), we can apply Newton's method proposed by [28]:

$$\lambda_{k+1} = \lambda_k - 1/x_n^{(k)}, \quad k = 0, 1, \ldots, \tag{4.6}$$

where

$$x_n^{(k)} = \left[L_k^{-1} \Theta_k \left(\frac{\mathrm{d}Q}{\mathrm{d}\lambda}\bigg|_{\lambda=\lambda_k} \right) Q_k \right]_{nn}$$

and $\Theta_k Q(\lambda_k) = L_k Q_k$ is a QL-factorization with row pivoting Θ_k.

If the Newton process in (4.6) converges, i.e.,

$$|\lambda_{k+1} - \lambda_k| < \mathrm{Tol} \quad \text{or} \quad |[L_k]_{nn}| < \mathrm{Tol},$$

then we stop (4.6) and set $\widehat{x} = Q_k e_n$ (the last column of Q_k) to be the associated eigenvector of $Q(\lambda)$ corresponding to $\widehat{\lambda} \equiv \lambda_{k+1}$.

We now study the total flop counts of the SDA+Newton algorithm as described in (4.1)–(4.6) for computing all eigenvalues of a PCP_PQEP. Since all computations are in complex arithmetic, the addition and the multiplication of two complex numbers require 2 flops and 6 flops (4 multiplications and 2 additions), respectively. The flop counts of the SDA+Newton algorithm are listed in Table 1. From experience, at most eight iterations are required for the SDA to converge and only one iteration is required for the Newton refinement in (4.6).

We shall compare the SDA+Newton algorithm with the QZ algorithm [29]. In [6], a "good" structured linearization for (1.1) has been proposed:

$$\begin{bmatrix} C - B & A \\ A & A \end{bmatrix} + \lambda \begin{bmatrix} B & B \\ B & C - A \end{bmatrix} \equiv X + \lambda \tilde{P}\overline{X}\tilde{P},$$

where $\tilde{P} = \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix}$. Thus, the eigenpairs for (1.1) can then be computed by the QZ algorithm on $(X, -\tilde{P}\overline{X}\tilde{P})$ or $(\mathcal{M}, \mathcal{L})$. In general, the QZ algorithm applied to $(\mathcal{M}, \mathcal{L})$ or $(X, -\tilde{P}\overline{X}\tilde{P})$ requires about $960n^3$ flops for the computation of all eigenvalues and about $1600n^3$ flops if, in additional, eigenvectors are needed.

Consequently, computing only unimodular eigenpairs of PCP_PQEPs, the number of flops required by the SDA+Newton algorithm and the QZ+Newton algorithm are summarized in Table 2 (with the last column containing ratios between the flop counts of the SDA and the QZ algorithms). For a fairer comparison, Newton refinement is applied at the end of both algorithms to achieve a similar accuracy. The SDA+Newton algorithm needs about one-half of the flop count of the QZ+Newton algorithm ($441n^3$:$1013n^3$) when $\ell = 10$. In general, the SDA+Newton algorithm is always more efficient, even more so for smaller values of ℓ. Note from [18] that the expected value of ℓ equals $E_n(\ell) = \frac{\sqrt{10n}}{4}$ for random palindromic eigenvalue problems. We have $E_{1000}(\ell) \approx 25$, $E_{10\,000}(\ell) \approx 79$ and $E_{100\,000}(\ell) \approx 250$, so even for very large TDSs, ℓ is manageably small and the SDA+Newton algorithm will always be substantially more efficient that the QZ+Newton algorithm.

Table 2
Relative flop counts vs. ℓ.

ℓ	SDA+Newton	QZ+Newton	SDA:QZ
10	$428n^3$	$1013n^3$	0.42
20	$481n^3$	$1067n^3$	0.45
30	$535n^3$	$1120n^3$	0.48
40	$588n^3$	$1173n^3$	0.50
50	$641n^3$	$1227n^3$	0.52
100	$908n^3$	$1493n^3$	0.61
500	$3041n^3$	$3627n^3$	0.84
1000	$5708n^3$	$6293n^3$	0.91

5. A Structured backward error analysis of PCP_PQEP

Let $(\hat{\lambda}, \hat{x})$ be an approximate eigenpair of $Q(\lambda)$ in (1.1). A natural definition of the normwise backward error of $(\hat{\lambda}, \hat{x})$ for (1.1) is

$$\eta(\hat{\lambda}, \hat{x}) = \min \left\{ \delta \middle| \left(Q(\hat{\lambda}) + \Delta Q(\hat{\lambda})\right)\hat{x} = 0, \|\Delta B\|_2 \le \delta\|B\|_2, \|\Delta C\|_2 \le \delta\|C\|_2, \|\Delta A\|_2 \le \delta\|A\|_2 \right\}, \tag{5.1a}$$

where $\|\cdot\|_2$ is the spectrum norm and

$$\Delta Q(\lambda) = \lambda^2\Delta B + \lambda\Delta C + \Delta A \tag{5.1b}$$

with $\Delta A, \Delta B, \Delta C \in \mathbb{C}^{n\times n}$ being the perturbation matrices.

An explicit expression for $\eta(\hat{\lambda}, \hat{x})$ with respect to the residual $r = Q(\hat{\lambda})\hat{x}$ is given by [30]:

Theorem 5.1 (*Tisseur*). *The normwise backward error $\eta(\hat{\lambda}, \hat{x})$ is given by*

$$\eta(\hat{\lambda}, \hat{x}) = \frac{\|r\|_2}{\hat{\alpha}\|\hat{x}\|_2}, \tag{5.2}$$

where $r = Q(\hat{\lambda})\hat{x}$ and $\hat{\alpha} = |\hat{\lambda}|^2\|B\|_2 + |\hat{\lambda}|\|C\|_2 + \|A\|_2$.

It is of interest to consider a backward error in which the perturbations $\{\Delta B, \Delta C, \Delta A\}$ preserve the PCP-structure in $\{B, C, A\}$. Therefore, we define the structured backward error of $(\hat{\lambda}, \hat{x})$ by

$$\eta_S(\hat{\lambda}, \hat{x}) = \min \left\{ \delta \middle| \left(Q(\hat{\lambda}) + \Delta Q(\hat{\lambda})\right)\hat{x} = 0, P\overline{\Delta B}P = \Delta A, P\overline{\Delta C}P = \Delta C, \|\Delta B\|_2 \le \delta\|B\|_2, \|\Delta C\|_2 \le \delta\|C\|_2, \right.$$
$$\left. \|\Delta A\|_2 \le \delta\|A\|_2 \right\}.$$

Note, for convenience, that we only consider the case of $\varepsilon = 1$ in (1.1).

It is clear that $\eta_S(\hat{\lambda}, \hat{x}) \ge \eta(\hat{\lambda}, \hat{x})$. The optimal perturbations in (5.1) do not, in general, have the PCP-Structure. We first prove the following property.

Theorem 5.2. *Let (λ, x) be an eigenpair of $Q(\lambda)$ in (1.1) with $\varepsilon = 1$. If λ is a simple unimodular eigenvalue, then $P\overline{x} = x$.*

Proof. From (1.1), we have

$$\begin{aligned}
&(\lambda^2 B + \lambda C + A)x = 0 \quad (\text{by (1.1a)})\\
&\Leftrightarrow \left[\lambda^2(PBP) + \lambda(PCP) + PAP\right]Px = 0\\
&\Leftrightarrow (\lambda^2\overline{A} + \lambda\overline{C} + \overline{B})Px = 0 \quad (\text{by (1.1b)})\\
&\Leftrightarrow (\lambda^2 B + \lambda C + A)P\overline{x} = 0 \quad (\text{by } \overline{\lambda} = \lambda^{-1}).
\end{aligned}$$

Since λ is simple, it follows that $P\overline{x} = x$. □

Based on the assertion of Theorem 5.2, the next theorem shows that requiring the perturbations to possess the PCP-structure has no effect on the backward error, provided that $\hat{\lambda}$ is on the unit circle and $\hat{x}$ satisfies $P\overline{\hat{x}} = \hat{x}$.

Theorem 5.3. *Let $(\hat{\lambda}, \hat{x})$ be an approximate eigenpair of $Q(\lambda)$ as in (1.1) with $\hat{\lambda} = e^{i\theta}$ and $P\overline{\hat{x}} = \hat{x}$. Then*

$$\eta_S(\hat{\lambda}, \hat{x}) = \eta(\hat{\lambda}, \hat{x}).$$

Proof. Let $r = Q(\hat\lambda)\hat x$ be the residual of the pair $(\hat\lambda, \hat x)$. We first show that $\mathrm{e}^{-\mathrm{i}\theta}\hat x^H r$ is real. Since $P\overline{\hat x} = \hat x$ and $\hat\lambda = \mathrm{e}^{\mathrm{i}\theta}$, we have

$$\begin{aligned}\mathrm{e}^{-\mathrm{i}\theta}\hat x^H r &= \mathrm{e}^{-\mathrm{i}\theta}\hat x^H Q(\hat\lambda)\hat x\\ &= \mathrm{e}^{-\mathrm{i}\theta}\left(\mathrm{e}^{2\mathrm{i}\theta}\hat x^H B\hat x + \mathrm{e}^{\mathrm{i}\theta}\hat x^H C\hat x + \hat x^H A\hat x\right)\\ &= \mathrm{e}^{\mathrm{i}\theta}\overline{\hat x}^H\overline{A\hat x} + \hat x^H C\hat x + \mathrm{e}^{-\mathrm{i}\theta}\hat x^H A\hat x \in \mathbb{R} \quad \text{(by (1.1b))}.\end{aligned}$$

Therefore, there is a Householder transform G of the form $G = I - 2uu^H$ such that

$$G\frac{\hat x}{\|\hat x\|_2} = -\mathrm{e}^{-\mathrm{i}\theta}\frac{r}{\|r\|_2}, \tag{5.3a}$$

where $u = \hat u/\|\hat u\|_2$ with

$$\hat u = \begin{cases} 0, & \text{if two vectors in (5.3a) are equal,}\\ \dfrac{\hat x}{\|\hat x\|_2} + \mathrm{e}^{-\mathrm{i}\theta}\dfrac{r}{\|r\|_2}, & \text{otherwise.}\end{cases} \tag{5.3b}$$

We take $S = (\|r\|_2/\|\hat x\|_2)G$. Then we arrive at

$$S\hat x = -\mathrm{e}^{-\mathrm{i}\theta} r. \tag{5.4}$$

Again from (1.1b) and $P\overline{\hat x} = \hat x$, it follows that

$$r = Q(\hat\lambda)\hat x = Q(\hat\lambda)P\overline{\hat x} = P(PQ(\hat\lambda)P)\overline{\hat x} = P\mathrm{e}^{2\mathrm{i}\theta}\overline{Q}(\hat\lambda)\overline{\hat x} = \mathrm{e}^{2\mathrm{i}\theta}P\overline{r}. \tag{5.5}$$

From (5.5) and (5.3b), we have

$$P\hat u = \frac{\overline{\hat x}}{\|\hat x\|_2} + \mathrm{e}^{\mathrm{i}\theta}\frac{\overline{r}}{\|r\|_2} = \overline{\hat u}.$$

Thus, G and then S satisfies $P\overline{S}P = S$. We now define

$$\Delta B = \frac{1}{\hat\alpha}\mathrm{e}^{-\mathrm{i}\theta}\|B\|_2 S, \tag{5.6a}$$

$$\Delta C = \frac{1}{\hat\alpha}\|C\| S, \tag{5.6b}$$

$$\Delta A = \frac{1}{\hat\alpha}\mathrm{e}^{\mathrm{i}\theta}\|A\|_2 S, \tag{5.6c}$$

where $\hat\alpha = \|B\|_2 + \|C\|_2 + \|A\|_2$. From $P\overline{S}P = S$ it is easy to see that

$$P\overline{\Delta B}P = \Delta A, \qquad P\overline{\Delta C}P = \Delta C.$$

From (5.4), we have

$$\begin{aligned}Q(\hat\lambda)\hat x + \Delta Q(\hat\lambda)\hat x &= r + (\hat\lambda^2\Delta B + \hat\lambda\Delta C + \Delta A)\hat x\\ &= r - \frac{1}{\hat\alpha}\left(\mathrm{e}^{2\mathrm{i}\theta}\mathrm{e}^{-\mathrm{i}\theta}\|B\|_2 + \mathrm{e}^{\mathrm{i}\theta}\|C\|_2 + \mathrm{e}^{\mathrm{i}\theta}\|A\|_2\right)\mathrm{e}^{-\mathrm{i}\theta} r\\ &= 0.\end{aligned}$$

Using (5.2), we obtain

$$\|S\|_2 = \frac{\|r\|_2}{\|\hat x\|_2} = \eta(\hat\lambda, \hat x)\hat\alpha,$$

and then from (5.6), we prove that $\eta_S(\hat\lambda, \hat x) = \eta(\hat\lambda, \hat x)$. □

In what follows we shall focus on estimating or bounding $\|Q(\hat\lambda)\hat x_1\|$, where $(\hat\lambda, (\hat x_1^\top, \hat x_2^\top)^\top)$ is an approximate eigenpair of $(\mathcal{M}, \mathcal{L})$.

Theorem 5.4. *Let $(\hat\lambda, \hat z)$ be an approximate eigenpair of $(\mathcal{M}, \mathcal{L})$ as in (2.1) with $D = 0$, where $\|\hat z\|_2 = 1$. Let $r = (\mathcal{M} - \hat\lambda\mathcal{L})\hat z$ be the associated residual. Set*

$$\hat x_1 = \hat z(1:n), \qquad \hat x_2 = \hat z(n+1:2n), \qquad r_1 = r(1:n), \qquad r_2 = r(n+1:2n).$$

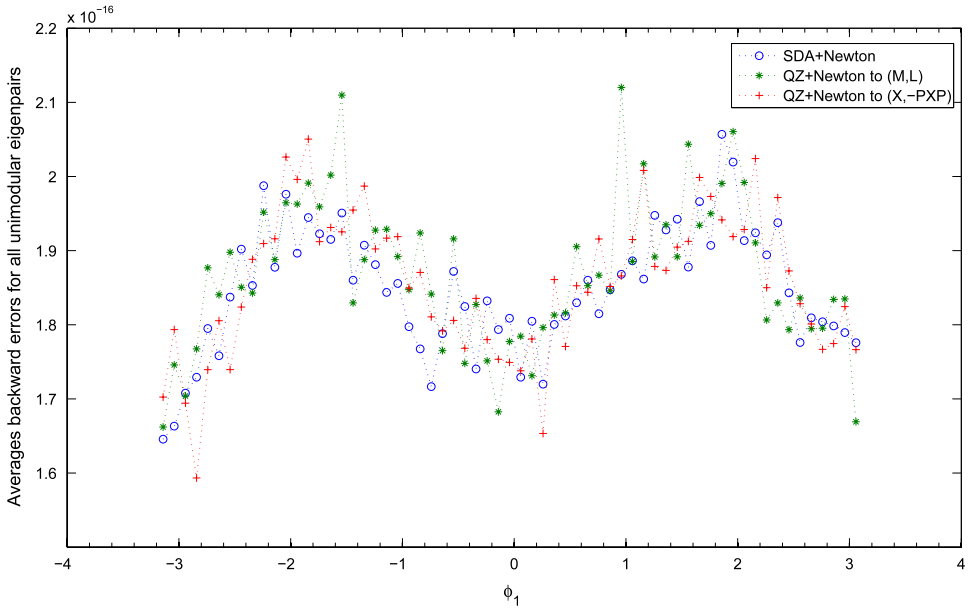

Fig. 6.1. Backward errors of approximate unimodular eigenpairs for Example 6.1.

Then

$$\|Q(\hat{\lambda})\hat{x}_1\|_2 \le \|r_1\|_2 + |\hat{\lambda}|\|r_2\|_2. \tag{5.7}$$

Proof. For $(\mathcal{M} - \hat{\lambda}\mathcal{L})\hat{z} = r$, we have

$$r_1 = A\hat{x}_1 - \hat{\lambda}\hat{x}_2, \qquad r_2 = -C\hat{x}_1 - \hat{x}_2 - \hat{\lambda}B\hat{x}_1.$$

So, $\hat{\lambda}\hat{x}_2 = A\hat{x}_1 - r_1$. Therefore,

$$\begin{aligned} -\hat{\lambda}r_2 &= \hat{\lambda}^2 B\hat{x}_1 + \hat{\lambda}C\hat{x}_1 + \hat{\lambda}\hat{x}_2 \\ &= \hat{\lambda}^2 B\hat{x}_1 + \hat{\lambda}C\hat{x}_1 + A\hat{x}_1 - r_1. \end{aligned}$$

Thus, $Q(\hat{\lambda})\hat{x}_1 = r_1 - \hat{\lambda}r_2$, which leads to (5.7). □

Remark 5.1. From Theorem 5.3, we see that the residual of $(\hat{\lambda}, \hat{x}_1)$ for $Q(\lambda)$ only depends on the associated residual $r = (\mathcal{M} - \hat{\lambda}\mathcal{L})\hat{z}$ which is independent of $\|B\|_2$, $\|C\|_2$ and $\|A\|_2$.

6. Numerical examples

We perform all computation in MATLAB R2007a with eps $\approx 2.22 \times 10^{-16}$. Note, from Remark 1.1 and [18], that computing unimodular eigenvalues for PCP_PQEPs is a well-posed problem, as unimodular eigenvalues can stay on the unit circle under perturbation. In addition, ℓ (the number of unimodular eigenvalues) is small which makes the refinement of unimodular eigenvalues inexpensive. From Table 2, the SDA+Newton algorithm is much more efficient than the QZ+Newton algorithm.

Example 6.1. We consider a neutral TDS (1.3), with corresponding characteristic equation (1.4):

$$Q(\lambda) = \lambda^2 E + \lambda F + G, \quad G = P\bar{E}P, \quad F = P\bar{F}P.$$

We are interested in the computation of all unimodular eigenpairs.

Now, we take $m = 2$, $n = 10$, A_k and D_k are randomly generated with entries being normally distributed in $[-100, 100]$ for $k = 0, 1, 2$, $\varphi_0 = 0$ and $\varphi_1 = -\pi : 0.1 : \pi$.

In Fig. 6.1, we show the average of backward errors of approximate unimodular eigenpairs for $Q(\lambda)$ computed by the SDA+Newton algorithm, as well as, by applying the QZ+Newton algorithm to $(\mathcal{M}, \mathcal{L})$ and to $(X, -\tilde{P}\bar{X}\tilde{P})$ [6], respectively. The number of iterations for the SDA is about eight. For each unimodular eigenvalue only one refinement Newton step is needed. In Fig. 6.1, we see that the SDA+Newton algorithm as well as those by the QZ+Newton algorithm for $(X, -\tilde{P}\bar{X}\tilde{P})$ and $(\mathcal{M}, \mathcal{L})$, lead to almost the same accuracy. However, the computational cost of the QZ+Newton algorithm is about twice of that of the SDA+Newton algorithm as shown in Section 4.

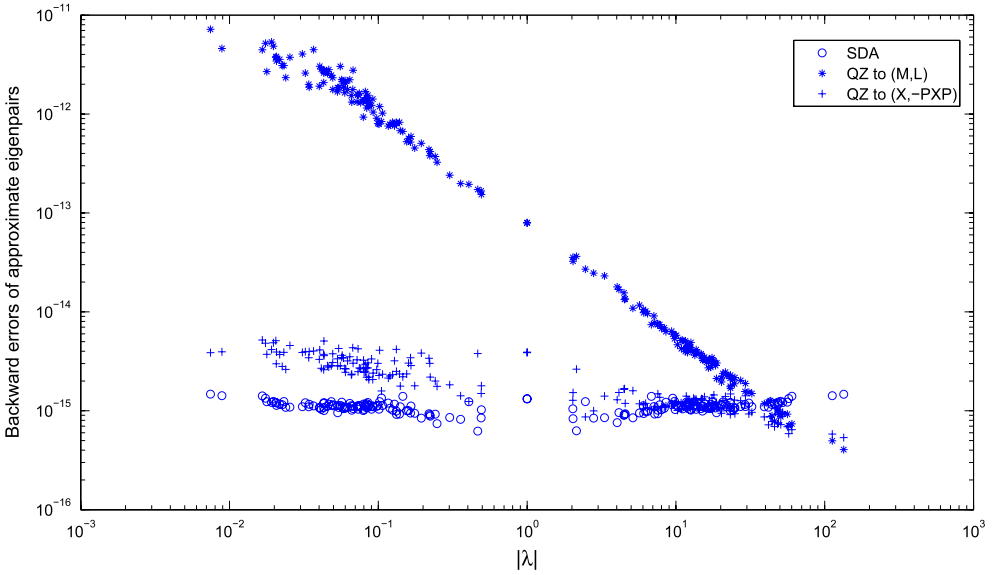

Fig. 6.2. Backward errors of approximate eigenpairs for Example 6.2.

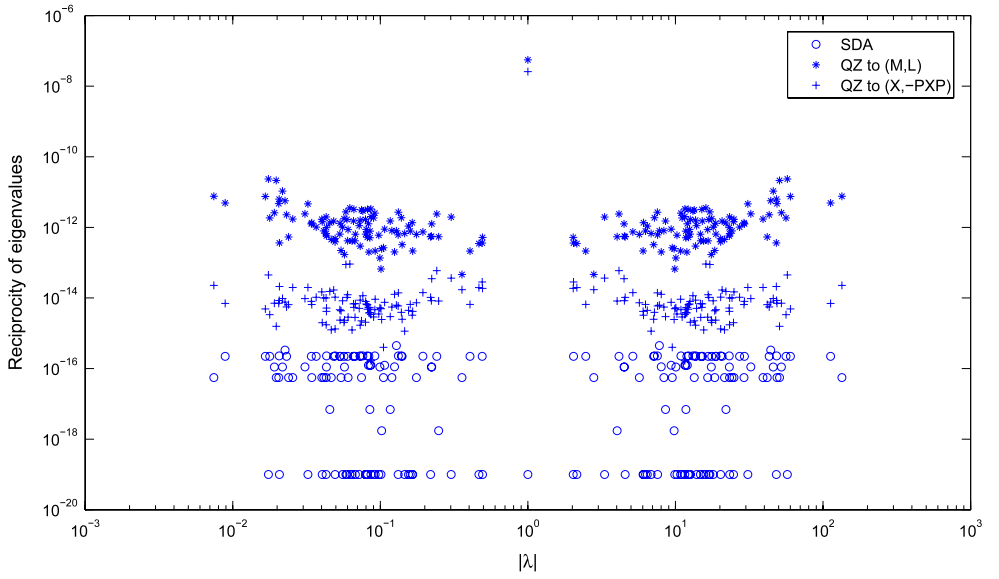

Fig. 6.3. Reciprocity of eigenvalues for Example 6.2.

Example 6.2. We generate random $A, B, C \in \mathbb{C}^{100\times 100}$ with $P\overline{A}P = B$, $P\overline{C}P = C$ and satisfying

$$\min\{|\mu| - 1 :\ \mu \in \text{spectrum of } Q(\lambda)\} = 1.18 \times 10^{-6}.$$

In Figs. 6.2 and 6.3, we show the backward errors of all eigenpairs of $Q(\lambda)$ and the reciprocity property of approximate λ and $1/\overline{\lambda}$ (i.e. $|\lambda_i\overline{\lambda}_{2n+1-i}| = 1$), respectively. We see that the SDA and QZ_$(X, -\tilde{P}\overline{X}\tilde{P})$ perform better than QZ_$(\mathcal{M}, \mathcal{L})$, and the SDA has better reciprocity property than that of QZ_$(X, -\tilde{P}\overline{X}\tilde{P})$ and QZ_$(\mathcal{M}, \mathcal{L})$.

Example 6.3. Consider the delay-free-feedback version of the example in [15],

$$\dot{x}(t) + D_1\dot{x}(t - h_1) + D_2\dot{x}(t - h_2) = A_0x(t),$$

where

$$D_1 = -\begin{bmatrix} 0 & 0.2 & -0.4 \\ -0.5 & 0.3 & 0 \\ 0.2 & 0.7 & 0 \end{bmatrix}, \qquad D_2 = -\begin{bmatrix} -0.3 & -0.1 & 0 \\ 0 & 0.2 & 0 \\ 0.1 & 0 & 0.4 \end{bmatrix},$$

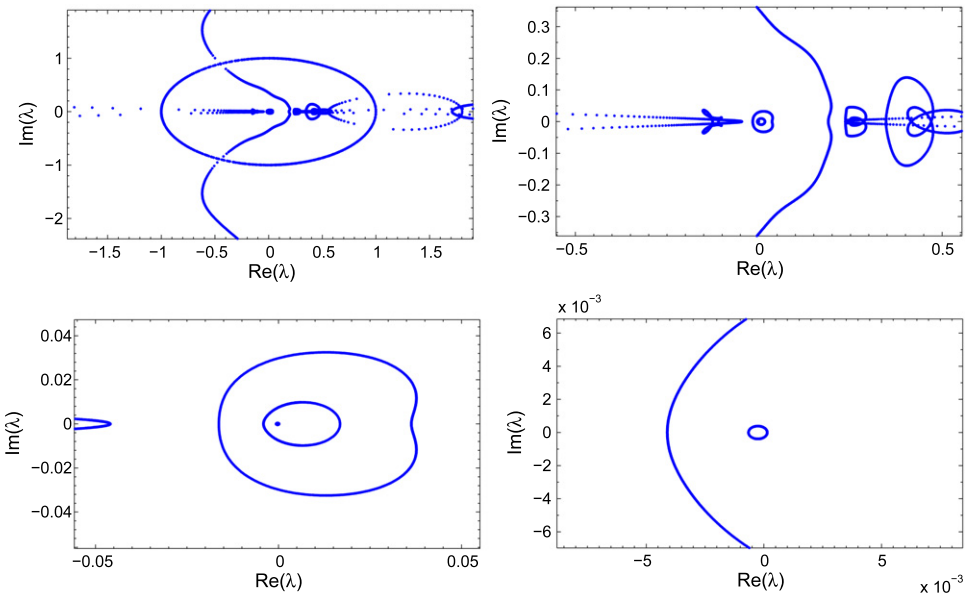

Fig. 6.4. Distribution of eigenvalues for Example 6.3 with different scales.

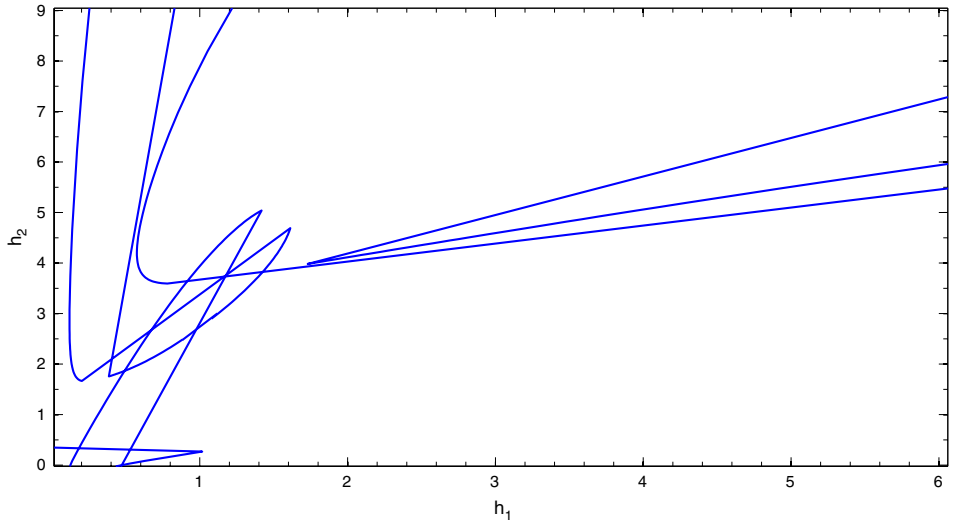

Fig. 6.5. Critical curves for Example 6.3.

$$A_0 = \begin{bmatrix} -4.8 & 4.7 & 3 \\ 0.1 & 1.4 & -0.4 \\ 0.7 & 3.1 & -1.5 \end{bmatrix} + BK^\top,$$

$$B = \begin{bmatrix} 0.3 & 0.7 & 0.1 \end{bmatrix}^\top, \qquad K = \begin{bmatrix} -2.593 & 1.284 & 1.826 \end{bmatrix}^\top.$$

With the parameterization of the critical surface ($m > 2$) and critical curve ($m = 2$) from [6] as described in the Introduction, critical points $(h_1, \ldots, h_m)$ are calculated from (1.5)–(1.7). With φ being 2π-periodic, we let φ run through $[-\pi, \pi]$.

We take $\varphi = -\pi : 0.01 : \pi$ and compute 11 322 eigenpairs by the SDA+Newton algorithm. We plot the distribution of eigenvalues inside and on the unit circle in Fig. 6.4 with different scales. In Fig. 6.5 we plot the critical curves of unimodular eigenvalues against h_1 and h_2.

7. Conclusions

We have proposed a structure-preserving doubling algorithm to compute all the eigenvalues of a PCP_PQEP. When only the unimodular eigenvalues are required, as in the stability analysis of time-delay systems, Newton refinement is applied

after doubling. The algorithm is efficient for small to medium size examples, as illustrated by numerical simulation. For large and structured problems, modifications to the SDA+Newton algorithm or new algorithms may have to be devised.

For time-delay systems, a smaller nonlinear eigenvalue problem can be solved instead [8]. It is nontrivial to compare numerical techniques for such nonlinear eigenvalue problems with our algorithm. The nonlinear eigenvalue problems are smaller but more complicated but our PCP_PQEPs are larger but more structured. Detailed comparison, especially with more properties from the actual time-delay system, will be interesting.

Acknowledgements

This work is partially supported by grants from the National Centre for Theoretical Sciences, National Tsing Hua University (Hsinchu, Taiwan), the Department of Applied Mathematics, National Chiao Tung University (Hsinchu, Taiwan), and the Faculty of Science, Monash University (Melbourne, Australia).

References

[1] E.K.-W. Chu, H.-Y. Fan, W.-W. Lin, C.-S. Wang, Structure-preserving algorithms for periodic discrete-time algebraic Riccati equations, Internat. J. Control 77 (2004) 767–788.
[2] E.K.-W. Chu, T.-M. Huang, W.-W. Lin, Structured doubling algorithms for solving g-palindromic quadratic eigenvalue problems, Technical Report, NCTS Preprints in Mathematics, National Tsing Hua University, Hsinchu, Taiwan, 2008.
[3] E.K.-W. Chu, T.-M. Huang, W.-W. Lin, C.-T. Wu, Vibration of fast trains, palindromic eigenvalue problems and structure-preserving doubling algorithms, J. Comput. Appl. Math. 219 (1) (2008) 237–252.
[4] E.K.-W. Chu, T.-M. Huang, W.-W. Lin, C.-T. Wu, Palindromic eigenvalue problem: A brief survey, Technical Report, NCTS Preprints in Mathematics, National Tsing Hua University, Hsinchu, Taiwan, 2008; Taiwanese J. Maths. (in press).
[5] W.-W. Lin, S.-F. Xu, Convergence analysis of structure-preserving doubling algorithms for Riccati-type matrix equations, SIAM J. Matrix Anal. Appl. 28 (1) (2006) 26–39.
[6] H. Fassbender, N. Mackey, D.S. Mackey, C. Schröder, A structured polynomial eigenproblem arising in the analysis of time delay systems and related polynomial eigenproblems, Technical Report, TU Braunschweig, 2007.
[7] D.S. Mackey, N. Mackey, C. Mehl, V. Mehrmann, Structured polynomial eigenvalue problems: Good vibrations from good linearizations, SIAM J. Matrix Anal. Appl. 28 (2006) 1029–1051.
[8] E. Jarlebring, The spectrum of delay-differential equations: Computational analysis, Ph.D. Thesis, Institute Computational Mathematics, TU Braunschweig, Germany, 2008.
[9] R. Bellman, K.L. Cooke, Differential–Difference Equations, Academic Press, 1963.
[10] E.K. Boukas, Z.K. Liu, Deterministic and Stochastic Time Delay Systems, Birkhäuser, Boston, 2002.
[11] O. Diekmann, S.A. Van Gils, S.M. Verduyn Lunel, H.O. Walther, Delay Equations: Functional-, Complex-, and Nonlinear Analysis, in: Applied Mathematical Sciences, vol. 110, Springer-Verlag, New York, 1995.
[12] K. Gu, V. Kharitonov, J. Chen, Stability of Time-delay Systems, in: Control Engineering, Birkhäuser, Boston, 2003.
[13] J. Hale, S.M. Verduyn Lunel, Introduction to Functional Differential Equations, Springer-Verlag, New York, 1993.
[14] W. Michiels, S.-I. Niculescu, Stability and Stabilization of Time-delay Systems: An Eigenvalue-based Approach, in: Advances in Design and Control, vol. 12, SIAM Publications, Philadelphia, 2007.
[15] W. Michiels, T. Vyhlídal, An eigenvalue based approach for the stabilization of linear time-delay systems of neutral type, Automatica 41 (6) (2005) 991–998.
[16] Q.C. Zhong, Robust Control of Time-delay Systems, Springer Verlag, New York, 2006.
[17] R.A. Horn, C.R. Johnson, Topics in Matrix Analysis, Cambridge University Press, Cambridge, 1995.
[18] C.Y. Chiang, E.K.-W. Chu, T.-M. Huang, Unimodular eigenvalues for palindromic eigenvalue problems. Technical report, NCTS Preprints in Mathematics, National Tsing Hua University, Hsinchu, Taiwan, 2008.
[19] C.F. Ipsen, Accurate eigenvalues for fast trains, SIAM News 37 (2004).
[20] D.S. Mackey, N. Mackey, C. Mehl, V. Mehrmann, Vector spaces of linearizations for matrix polynomials, SIAM J. Matrix Anal. Appl. 28 (2006) 971–1004.
[21] T.-M. Huang, W.-W. Lin, J. Qian, Structured algorithms for palindromic quadratic eigenvalue problems arising in vibration of fast trains, Technical Report, NCTS Preprints in Mathematics, National Tsing Hua University, Hsinchu, Taiwan, 2008-2-001, 2008.
[22] D.S. Mackey, N. Mackey, C. Mehl, V. Mehrmann, Numerical methods for palindromic eigenvalue problems, Technical Report, TU Berlin, MATHEON, Germany, 2007.
[23] C. Schröder, A QR-like algorithm for the palindromic eigenvalue problem, Technical Report 388, TU Berlin, MATHEON, Germany, 2007.
[24] C. Schröder, URV decomposition based structured methods for palindromic and even eigenvalue problems, Technical Report 375, TU Berlin, MATHEON, Germany, 2007.
[25] I. Gohberg, P. Lancaster, L. Rodman, Matrix Polynomials, Academic Press, New York, 1982.
[26] F. Tisseur, K. Meerbergen, A survey of the quadratic eigenvalue problem, SIAM Rev. 43 (2001) 234–286.
[27] F.R. Gantmacher, The Theory of Matrices, vol. II, Chelsea Publishing Company, New York, 1977.
[28] V.N. Kublanovskaya, On an approach to the solution of the generalzed latent value problem for λ-matrices, SIAM J. Numer. Anal 7 (1970) 532–537.
[29] G.H. Golub, C.F. Van Loan, Matrix Computations, 3rd ed., Johns Hopkins University Press, Baltimore, MD, 1996.
[30] F. Tisseur, Backward error and condition of polynomial eigenvalue problems, Linear Algebra Appl. 309 (1–3) (2000) 339–361.