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Existence and multiplicity of nontrivial solutions for biharmonic equations with singular weight functions

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ABSTRACT

In the paper, we study a class of biharmonic equations with singular weight functions as follows:

$$\begin{cases} \Delta^2 u - \beta \Delta_p u + V_\lambda(x)u = f(x)\,|u|^{q-2}\,u \quad \text{in}\mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N), \end{cases}$$

where $N \geq 3, \Delta^2 u = \Delta(\Delta u),\ \Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u), \beta \geq 0$ is a parameter, $2 < p, q < \frac{2N}{N-2}$ and $V_\lambda(x) = \lambda a(x) - b(x)$ with $\lambda > 0$. Under some suitable assumptions on a, b and f, we obtain the existence and multiplicity of nontrivial solutions for λ large enough. An interesting phenomenon is that we do not need the condition that weight function f is integrable or bounded on whole space $\mathbb{R}^N$, which can be regarded as an improvement work of Sun et al. (2017).

1. Introduction

We are concerned with the following biharmonic equations:

$$\begin{cases} \Delta^2 u - \beta \Delta_p u + V_\lambda(x)u = f(x)\,|u|^{q-2}\,u \quad \text{in } \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N), \end{cases} \tag{E_λ}$$

where $N \geq 3, \Delta^2 u = \Delta(\Delta u), \Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $\beta \geq 0$ is a parameter, $2 < p, q < 2^* := \frac{2N}{N-2}$ and $V_\lambda(x) = \lambda a(x) - b(x)$ with $\lambda > 0$. We assume that the functions a, b and f satisfy the following conditions:

- $(V1)$ $a(x)$ is a nonnegative continuous function on $\mathbb{R}^N$ and there exists a constant $c_0 > 0$ such that the set $\{a < c_0\} := \{x \in \mathbb{R}^N \mid a(x) < c_0\}$ has finite positive Lebesgue measure;
- $(V2)$ $\Omega = \operatorname{int}\{x \in \mathbb{R}^N : a(x) = 0\}$ is nonempty and has smooth boundary with $\bar{\Omega} = \{x \in \mathbb{R}^N : a(x) = 0\}$;

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$(V3)$ $b(x)$ is a measurable function on $\mathbb{R}^N$ and there exists $0 < b_0 < \alpha_N^{-1}\left(\frac{N-2}{2}\right)^2$ such that $0 \le b(x) \le \frac{b_0}{|x|^2}$ for all $x \in \mathbb{R}^N$, where α_N is defined as in (4).

Under the hypotheses $(V1)-(V2)$, $\lambda a(x)$ is called the steep potential well whose depth is controlled by the parameter λ. Such potential is first suggested by Bartsch–Wang [1] in the study of scalar Schrödinger equations.

In the last decades, the existence and multiplicity of nontrivial solutions for biharmonic equations have begun to receive much attention. The readers may refer to [2–5] for related works in the literature. For example, very recently, Sun–Chu–Wu [3] studied a class of nonlinear biharmonic equations with p-Laplacian. When the nonlinearity satisfies some different conditions, several results of the existence and multiplicity of nontrivial solutions are obtained as follows.

(i) If $2 < q < p < \min\{2^*, N\}$ and $\beta > 0$ enough small, then at least two nontrivial solutions of Eq. (E_λ) exist when weight function $f \in L^{\frac{p^*}{p^*-q}}\left(\mathbb{R}^N\right)$ $\left(p^* := \frac{pN}{N-p}\right)$ for $\lambda > 0$ sufficiently large;

(ii) If $2 < p < q < 2_*$ $\left(2_* = \infty \text{ if } N = 3, 4;\ 2_* = \frac{2N}{N-4} \text{ if } N \ge 5\right)$ and $\beta \ge 0$, then at least one nontrivial solution of Eq. (E_λ) exists when weight function $f \in L^\infty\left(\mathbb{R}^N\right)$ for $\lambda > 0$ sufficiently large.

Motivated by the fact mentioned above, it is very natural for us to pose a question as follows:

- If $\beta \ge 0$ and the weight function f is neither integrable nor bounded in whole space $\mathbb{R}^N$, then we would much like to know whether Eq. (E_λ) admits nontrivial solutions for $\lambda > 0$ sufficiently large.

In the present paper, we focus our attention on the existence and multiplicity of nontrivial solutions for Eq. (E_λ) with $\beta \ge 0$ and the weight function f is neither integrable nor bounded in whole space $\mathbb{R}^N$, and try to search for definite answer to the above question. Our proof is mainly based on mountain pass theorem as well as Gagliardo–Nirenberg and Hardy–Sobolev inequalities. Furthermore, we will establish some new inequality estimates, helping solve our question.

We now summarize our main results as follows.

Theorem 1.1. *Suppose that $2 < q < p < \min\{2^*, N\}$ and conditions $(V1)-(V3)$ hold. In addition, the following conditions hold:*
$(D1)$ f is sign-changing weight function in $\mathbb{R}^N$ and there exists $R_0 > 0$ such that $|x|^{p(q-2)} f^{p-2}(x) \le a^{p-q}(x)$ for $|x| > R_0$ and $f \in L^{\frac{p^*}{p^*-q}}(B_{R_0}(0))$ where $B_{R_0}(0) = \left\{x \in \mathbb{R}^N : |x| \le R_0\right\}$;
$(D2)$ $\{f > 0\} \cap \Omega$ has finite positive Lebesgue measure.
Then we have the following results.
(i) For $\beta = 0$, Eq. (E_λ) admits at least one nontrivial solution for $\lambda > 0$ sufficiently large;
(ii) There exists $\beta_0 > 0$ such that for every $0 < \beta < \beta_0$, Eq. (E_λ) admits at least two nontrivial solutions for $\lambda > 0$ sufficiently large.

Theorem 1.2. *Suppose that $2 < p < q < 2^*$ and conditions $(V1)-(V3)$ and $(D2)$ hold. In addition, the following conditions hold:*
$(D1')$ f is sign-changing weight function in $\mathbb{R}^N$ which satisfies $f^{2^*-2}(x) \le a^{2^*-q}(x)$ for all $x \in \{a \ge c_0\}$ and

$$0 < \overline{f}_{\max} := \sup_{x \in \{a < c_0\}} f(x) < +\infty.$$

Then for each $\beta \ge 0$, Eq. (E_λ) admits at least one nontrivial solution for $\lambda > 0$ sufficiently large.

The remainder of this paper is organized as follows. After presenting some preliminary results in Section 2, we give the proofs of our main results in Section 3.

2. Preliminaries

Let

$$X = \left\{ u \in H^2(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} a(x) u^2 dx < \infty \right\}$$

be equipped with the inner product and norm

$$\langle u, v \rangle = \int_{\mathbb{R}^N} (\Delta u \Delta v + a(x) uv)\, dx,\ \ \|u\| = \langle u, u \rangle^{1/2}.$$

By Gagliardo–Nirenberg inequality, there exists a sharp constant $A_0 > 0$ such that

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \le A_0^2 \left(\int_{\mathbb{R}^N} |\Delta u|^2 dx \right)^{1/2} \left(\int_{\mathbb{R}^N} u^2 dx \right)^{1/2}, \tag{1}$$

which indicates that

$$\int_{\mathbb{R}^N} \left(|\Delta u|^2 + u^2 \right) dx \le \|u\|_{H^2}^2 \le \left(1 + \frac{A_0^2}{2} \right) \int_{\mathbb{R}^N} \left(|\Delta u|^2 + u^2 \right) dx. \tag{2}$$

Applying conditions $(V1) - (V2)$, by the Hö lder, Young and Gagliardo–Nirenberg inequalities, following the argument of [6], there exists a sharp constant of Gagliardo–Nirenberg inequality $\overline{A}_N > 0$ such that

$$\|u\|_{H^2}^2 \le \alpha_N \int_{\mathbb{R}^N} \left(|\Delta u|^2 + \lambda a(x) u^2 \right) dx \text{ for } \lambda \ge \Lambda_N, \tag{3}$$

where

$$\alpha_N := \begin{cases} \left(1 + \frac{A_0^2}{2}\right) \overline{A}_N^{16/N} \left|\{a < c_0\}\right|^{4/N}, & \text{if } N = 3, 4, \\ \left(1 + \frac{A_0^2}{2}\right) \overline{A}_N^{2} \left|\{a < c_0\}\right|^{4/N}, & \text{if } N \ge 5, \end{cases} \tag{4}$$

and

$$\Lambda_N := \begin{cases} \frac{8}{N c_0 \overline{A}_N^{16/N} |\{a < c_0\}|^{4/N}} & \text{if } N = 3, 4, \\ \frac{1}{c_0 \overline{A}_N^{2} |\{a < c_0\}|^{4/N}} & \text{if } N \ge 5. \end{cases}$$

It follows from (3), condition $(V3)$ and the standard Hardy inequality that

$$\begin{aligned} \int_{\mathbb{R}^N} b(x) u^2 dx &\le b_0 \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} dx \le b_0 \left(\frac{2}{N-2} \right)^2 \int_{\mathbb{R}^N} |\nabla u|^2 dx \\ &\le b_0 \alpha_N \left(\frac{2}{N-2} \right)^2 \int_{\mathbb{R}^N} \left(|\Delta u|^2 + \lambda a(x) u^2 \right) dx. \end{aligned} \tag{5}$$

Furthermore, it follows from (5) that

$$\int_{\mathbb{R}^N} \left(|\Delta u|^2 + V_\lambda(x) u^2 \right) dx \ge \left[1 - b_0 \alpha_N \left(\frac{2}{N-2} \right)^2 \right] \int_{\mathbb{R}^N} \left(|\Delta u|^2 + \lambda a(x) u^2 \right) dx \tag{6}$$

and

$$\int_{\mathbb{R}^N} \left(|\Delta u|^2 + V_\lambda(x) u^2 \right) dx \le \left[1 + b_0 \alpha_N \left(\frac{2}{N-2} \right)^2 \right] \int_{\mathbb{R}^N} \left(|\Delta u|^2 + \lambda a(x) u^2 \right) dx, \tag{7}$$

for all $\lambda \ge \Lambda_N$. Thus, for $\lambda \ge \Lambda_N$, we also define the following inner product and norm

$$\langle u, v \rangle_\lambda = \int_{\mathbb{R}^N} (\Delta u \Delta v + V_\lambda(x) uv)\, dx,\ \ \|u\|_\lambda = \langle u, u \rangle_\lambda^{1/2}.$$

Set $X_\lambda = (X, \|u\|_\lambda)$. Then this implies that the imbedding $X_\lambda \hookrightarrow H^2(\mathbb{R}^N)$ is continuous for all $\lambda \geq \Lambda_N$. Moreover, by condition $(D1)$, (3), (6) and the Hölder, Sobolev and Hardy inequalities, there exists $D_0 > 0$ such that

$$\begin{aligned}\int_{\mathbb{R}^N} f|u|^q dx &= \int_{|x|>R_0} f|u|^q dx + \int_{|x|\leq R_0} f|u|^q dx \\ &\leq \int_{|x|>R_0}\left(a(x)|u|^2\right)^{\frac{p-q}{p-2}}\left(\frac{|u|^p}{|x|^p}\right)^{\frac{q-2}{p-2}} dx + \|f\|_{L^{\frac{p^*}{p^*-q}}(B_{R_0}(0))}\left(\int_{|x|\leq R_0}|u|^{p^*}dx\right)^{q/p*} \\ &\leq \left(\int_{\mathbb{R}^N} a(x)|u|^2 dx\right)^{\frac{p-q}{p-2}}\left(\int_{\mathbb{R}^N}\frac{|u|^p}{|x|^p}dx\right)^{\frac{q-2}{p-2}} + D_0\left(\int_{\mathbb{R}^N}|\nabla u|^p dx\right)^{q/p} \\ &\leq \left(\frac{p}{N-p}\right)^{p\frac{q-2}{p-2}}\left(\frac{1}{\lambda d_N}\|u\|_\lambda^2\right)^{\frac{p-q}{p-2}}\left(\int_{\mathbb{R}^N}|\nabla u|^p dx\right)^{\frac{q-2}{p-2}} + D_0\left(\int_{\mathbb{R}^N}|\nabla u|^p dx\right)^{q/p} \\ &\leq \left(\frac{1}{\lambda d_N}\right)^{\frac{p-q}{p-2}}\left(\frac{p}{N-p}\right)^{p\frac{q-2}{p-2}}\left[\frac{p-q}{p-2}\|u\|_\lambda^2 + \frac{q-2}{p-2}\int_{\mathbb{R}^N}|\nabla u|^p dx\right] \\ &\quad + D_0\left(\int_{\mathbb{R}^N}|\nabla u|^p dx\right)^{q/p} \end{aligned} \tag{8}$$

and

$$\int_{\mathbb{R}^N}|\nabla u|^p dx \leq \left(\frac{\alpha_N}{1-b_0\alpha_N}\right)^{p/2}\left[\frac{\widehat{S}_p(N-2)}{2}\right]^p \|u\|_\lambda^p, \tag{9}$$

where $d_N = 1 - b_0\alpha_N\left(\frac{2}{N-2}\right)^2$ and $\widehat{S}_p > 0$ is the best Sobolev constant for the embedding of $H^2(\mathbb{R}^N)$ in $W^{1,p}(\mathbb{R}^N)$. Similarly, by condition $(D1')$, (3), (6) and the Hölder and Sobolev inequalities, we have

$$\begin{aligned}\int_{\mathbb{R}^N} f(x)|u|^q dx &\leq \int_{\{a\geq c_0\}}\left(a(x)u^2\right)^{\frac{2^*-q}{2^*-2}}|u|^{\frac{2^*(q-2)}{2^*-2}}dx + \overline{f}_{\max}\int_{\{a<c_0\}}|u|^q dx \\ &\leq \left(\int_{\mathbb{R}^N} a(x)u^2 dx\right)^{\frac{2^*-q}{2^*-2}}\left(\int_{\mathbb{R}^N}|u|^{2^*}dx\right)^{\frac{q-2}{2^*-2}} \\ &\quad + \overline{f}_{\max}|\{a<c_0\}|^{\frac{2^*-q}{2^*}}\left(\int_{\{a<c_0\}}|u|^{2^*}dx\right)^{\frac{q}{2^*}} \\ &\leq \left(\frac{1}{\lambda}\int_{\mathbb{R}^N}\left(|\Delta u|^2 + \lambda a(x)u^2\right)dx\right)^{\frac{2^*-q}{2^*-2}}\left(\frac{\|u\|_{H^2}^{2^*}}{S^{2^*}}\right)^{\frac{q-2}{2^*-2}} \\ &\quad + \frac{\overline{f}_{\max}|\{a<c_0\}|^{\frac{2^*-q}{2^*}}}{S^q}\|u\|_{H^2}^q \\ &\leq \left[\left(\frac{1}{\lambda}\right)^{\frac{2^*-q}{2^*-2}}\left(\frac{\alpha_N^{\frac{2^*}{2}}}{S^{2^*}}\right)^{\frac{q-2}{2^*-2}} + \frac{\overline{f}_{\max}|\{a<c_0\}|^{\frac{2^*-q}{2^*}}\alpha_N^{\frac{q}{2}}}{S^q}\right]\left(\frac{1}{d_N}\right)^{\frac{q}{2}}\|u\|_\lambda^q. \end{aligned} \tag{10}$$

For the sake of convenience, throughout in this paper, we always assume the parameter $\lambda \geq \Lambda_N$. It is easily seen that Eq. (E_λ) is variational and its solutions are critical points of the functional defined in X_λ by

$$J_{\lambda,\beta}(u) = \frac{1}{2}\|u\|_\lambda^2 + \frac{\beta}{p}\int_{\mathbb{R}^N}|\nabla u|^p dx - \frac{1}{q}\int_{\mathbb{R}^N} f|u|^q dx. \tag{11}$$

It is not difficult to prove that the functional J_λ is of class C^1 in X_λ, and that

$$\langle J'_{\lambda,\beta}(u), v\rangle = \int_{\mathbb{R}^N}[\Delta u\cdot\Delta v + V_\lambda(x)uv]\,dx + \beta\int_{\mathbb{R}^N}|\nabla u|^{p-2}\nabla u\nabla v dx - \int_{\mathbb{R}^N} f|u|^{q-2}uv dx.$$

Lemma 2.1. *Suppose that $\beta > 0$, $2 < q < p < \min\left\{N, \frac{2N}{N-2}\right\}$ and conditions $(V1)-(V3)$ and $(D1)$ hold. Then $J_{\lambda,\beta}$ is coercive and bounded below on X_λ for $\lambda > 0$ sufficiently large. Specifically, there exists $\widetilde{K} < 0$ such that*

$$J_{\lambda,\beta}(u) \geq \widetilde{K} \text{ for all } u \in X_\lambda.$$

Proof. By (8),

$$\begin{aligned} J_{\lambda,\beta}(u) &= \frac{1}{2}\|u\|_\lambda^2 + \frac{\beta}{p}\int_{\mathbb{R}^N} |\nabla u|^p\,dx - \frac{1}{q}\int_{\mathbb{R}^N} f\,|u|^q\,dx \\ &\geq \frac{1}{4}\|u\|_\lambda^2 + \frac{\beta}{2p}\int_{\mathbb{R}^N} |\nabla u|^p\,dx - D_0\left(\int_{\mathbb{R}^N} |\nabla u|^p\,dx\right)^{q/p} \\ &\geq \frac{1}{4}\|u\|_\lambda^2 - \left(\frac{p-q}{p}\right)\left(\frac{2q}{\beta}\right)^{\frac{q}{p-q}} D_0^{\frac{p}{p-q}} \text{ for all } u \in X_\lambda. \end{aligned}$$

This shows that $J_{\lambda,\beta}$ is coercive and bounded below on X_λ for $\lambda > 0$ sufficiently large. □

Next, we give a useful theorem, which is the variant version of the mountain pass theorem. It can help us to find a so-called (PS) sequence.

Theorem 2.2 (*[7], Mountain Pass Theorem*)**.** *Let E be a real Banach space with its dual space E^*, and suppose that $I \in C^1(E,\mathbb{R})$ satisfies*

$$\max\{I(0), I(e)\} \leq \mu < \eta \leq \inf_{\|u\|=\rho} I(u),$$

for some $\mu < \eta, \rho > 0$ and $e \in E$ with $\|e\| > \rho$. Let $c \geq \eta$ be characterized by

$$c = \inf_{\gamma\in\Gamma}\max_{0\leq\tau\leq1} I(\gamma(\tau)),$$

where $\Gamma = \{\gamma \in C([0,1],E) : \gamma(0) = 0, \gamma(1) = e\}$ is the set of continuous paths joining 0 and e, then there exists a sequence $\{u_n\} \subset E$ such that

$$I(u_n) \to c \geq \eta \quad \text{and} \quad \|I'(u_n)\|_{E^*} \to 0 \quad \text{as } n \to \infty.$$

3. Proofs of Theorems 1.1, 1.2

First, we investigate the compactness condition for the functional $J_{\lambda,\beta}$. Here we call that a C^1-functional $J_{\lambda,\beta}$ satisfies Palais–Smale condition at level c ($(PS)_c$ condition for short) in X_λ, if any sequence $\{u_n\} \subset X_\lambda$ such that $J_{\lambda,\beta}(u_n) \to c$ and $\|J'_{\lambda,\beta}(u_n)\|_{X^{-1}} \to 0$ has a convergent subsequence.

Proposition 3.1. *Suppose that $\beta \geq 0$ and conditions $(V1)-(V3)$ and $(D1)$ hold. Then for each $D > 0$ independent of λ, $J_{\lambda,\beta}$ satisfies the $(PS)_c$-condition in X_λ with $c < D$, for $\lambda > 0$ sufficiently large.*

Proof. Let $\{u_n\}$ be a $(PS)_c$-sequence. By Lemma 2.1, there exists $C_0 > 0$ independent of λ such that $\|u_n\|_\lambda \leq C_0$. Without loss of generality, we can assume that there exist a subsequence $\{u_n\}$ and u_0 in X_λ such that

$$\begin{aligned} &u_n \rightharpoonup u_0 \text{ weakly in } X_\lambda, \\ &u_n \to u_0 \text{ strongly in } L^r_{loc}(\mathbb{R}^N), \text{ for } 2 \leq r < 2_*, \\ &u_n \to u_0 \text{ a.e. in } \mathbb{R}^N. \end{aligned}$$

Thus, by Brezis–Lieb lemma [8] gives

$$\int_{\mathbb{R}^N} f\,|u_n-u_0|^q\,dx=\int_{\mathbb{R}^N} f\,|u_n|^q\,dx-\int_{\mathbb{R}^N} f\,|u_0|^q\,dx+o(1). \tag{12}$$

We now prove that $u_n\to u_0$ strongly in X_λ. Let $v_n=u_n-u_0$. Then $v_n\rightharpoonup 0$ in X_λ and $\|v_n\|_\lambda\le 2C_0$. Using $u_n\rightharpoonup u_0$ weakly in X_λ and (12) leads to

$$J_{\lambda,\beta}(v_n)=J_{\lambda,\beta}(u_n)-J_{\lambda,\beta}(u_0)+o(1)\ \text{ and }\ J'_{\lambda,\beta}(v_n)=o(1). \tag{13}$$

Using (9), the condition $(D1)$ and the Hardy inequality, similar to the estimate on inequality (8), for any $\lambda>\Lambda_N$ one has

$$\begin{aligned}\int_{\mathbb{R}^N} f\,|v_n|^q\,dx &= \int_{|x|>R^*} f\,|v_n|^q\,dx+\int_{|x|\le R^*} f\,|v_n|^q\,dx\\ &\le \left(\frac{1}{\lambda d_N}\right)^{\frac{p-q}{p-2}}\left(\frac{p}{N-p}\right)^{\frac{p(q-2)}{p-2}}\\ &\quad\times\left[\left(\frac{p-q}{p-2}\right)\|v_n\|_\lambda^2+\left(\frac{q-2}{p-2}\right)\left(\frac{\alpha_N}{1-b_0\alpha_N}\right)^{\frac{p}{2}}\left(\frac{\hat{S}_N(N-2)}{2}\right)^p\|v_n\|_\lambda^p\right]+o(1).\end{aligned} \tag{14}$$

By condition $(D1)$, it follows from $\|v_n\|_\lambda\le 2C_0$ and (14) that

$$o(1)=\left\langle J'_{\lambda,\beta}(v_n),v_n\right\rangle\ge\left[1-\left(\frac{1}{\lambda d_N}\right)^{\frac{p-q}{p-2}}\left(\frac{p}{N-p}\right)^{\frac{p(q-2)}{p-2}}\left(\frac{p-q}{p-2}\right)\right]\|v_n\|_\lambda^2-\Pi_\lambda(2C_0)^p,$$

where $\Pi_\lambda:=\left(\frac{1}{\lambda d_N}\right)^{\frac{p-q}{p-2}}\left(\frac{p}{N-p}\right)^{\frac{p(q-2)}{p-2}}\left(\frac{q-2}{p-2}\right)\left(\frac{\alpha_N}{1-b_0\alpha_N}\right)^{\frac{p}{2}}\left(\frac{\hat{S}_N(N-2)}{2}\right)^p$. This implies that $v_n\to 0$ strongly in X_λ for $\lambda>0$ sufficiently large, and so

$$J_{\lambda,\beta}(u_0)=\lim_{n\to\infty}J_{\lambda,\beta}(u_n)=c \text{ and } J'_{\lambda,\beta}(u_0)=0.$$

This completes the proof. □

Lemma 3.2. *Suppose that conditions $(V1)-(V3)$ and $(D1)$ hold. Then we have the following results.*
(i) For each $\beta\ge 0$ there exist $\rho,\eta>0$ such that

$$\inf\{J_{\lambda,\beta}(u):u\in X_\lambda \text{ with } \|u\|=\rho\}>\eta \text{ for all } \lambda>\Lambda_N.$$

(ii) There exist $\beta_0>0$ and $e\in X$ with $\|e\|_\lambda>\rho$ such that $J_{\lambda,\beta}(e)<0$ for all $0\le\beta<\beta_0$ and $\lambda>\Lambda_N$.

Proof. (i) By (8) and (9), there exists $\widehat{D}_0>0$ such that

$$\begin{aligned}J_{\lambda,\beta}(u)&\ge\frac{1}{4}\|u\|_\lambda^2+\frac{\beta}{2p}\int_{\mathbb{R}^N}|\nabla u|^p\,dx-\widehat{D}_0\,\|u\|_\lambda^q\\ &\ge\frac{1}{4}\|u\|_\lambda^2-\widehat{D}_0\,\|u\|_\lambda^q\ \text{ for all } \lambda>\Lambda_N.\end{aligned}$$

So, letting $\|u\|_\lambda=\rho>0$ small enough, it is easy to see that there exists $\eta>0$ such that part (i) holds.
(ii) By condition $(D1)$, we can choose a nonnegative function $\phi_0\in X_\lambda$ such that $\int_{\mathbb{R}^N} f(x)\,\phi_0^q dx>0$. Then for $\beta=0$, one has $J_{\lambda,0}(t\phi_0)\to-\infty$ as $t\to+\infty$. Thus, there exists $e\in X_\lambda$ with $\|e\|_\lambda>\rho$ such that $J_{\lambda,0}(e)<0$, and so we see that there exists $\beta_0>0$ such that $J_{\lambda,\beta}(e)<0$ for all $\beta\in[0,\beta_0)$. □

Now, we define

$$c_\lambda = \inf_{\gamma\in\Gamma_\lambda} \max_{0\le t\le 1} J_{\lambda,\beta}(\gamma(t))$$

and

$$c_0(\Omega) = \inf_{\gamma\in\overline{\Gamma}_\lambda(\Omega)} \max_{0\le t\le 1} J_{\lambda,\beta}|_{H_0^1(\Omega)\cap H^2(\Omega)}(\gamma(t)),$$

where $J_{\lambda,\beta}|_{H_0^1(\Omega)\cap H^2(\Omega)}$ is a restriction of $J_{\lambda,\beta}$ on $H_0^1(\Omega)\cap H^2(\Omega)$,

$$\Gamma_\lambda = \{\gamma\in C([0,1],X_\lambda) : \gamma(0)=0, \gamma(1)=e\}$$

and

$$\overline{\Gamma}_\lambda(\Omega) = \{\gamma\in C([0,1],H_0^1(\Omega)\cap H^2(\Omega)) : \gamma(0)=0, \gamma(1)=e\}.$$

Note that for $u\in H_0^1(\Omega)\cap H^2(\Omega)$,

$$J_{\lambda,\beta}|_{H_0^1(\Omega)\cap H^2(\Omega)}(u) = \frac{1}{2}\|u\|_\lambda^2 + \frac{\beta}{p}\int_{\mathbb{R}^N}|\nabla u|^p\,dx - \frac{1}{q}\int_{\mathbb{R}^N} f\,|u|^q\,dx$$

and $c_0(\Omega)$ independent of λ. Moreover, if the conditions $(D1)-(D2)$ hold, then by the proof of Lemma 3.2, we can conclude that $J_{\lambda,\beta}|_{H_0^1(\Omega)\cap H^2(\Omega)}$ satisfies the mountain pass hypothesis as in Theorem 2.2.

Since $H_0^1(\Omega)\cap H^2(\Omega)\subset X_\lambda$ for all $\lambda>0$, one can see that $0<\eta\le c_\lambda\le c_0(\Omega)$ for $\lambda>0$ sufficiently large. Then we have the following result.

Theorem 3.3. *Suppose that conditions $(V1)-(V3)$ and $(D1)-(D2)$ hold. Let $\beta_0>0$ be as in Lemma 3.2. Then for every $0\le\beta<\beta_0$, Eq. (E_λ) has one nontrivial solution u_0^+ such that $J_{\lambda,\beta}(u_0^+)=c_\lambda$ for $\lambda>0$ sufficiently large.*

Proof. By Lemma 3.2 and Theorem 2.2, we obtain that for $\lambda>0$ enough large, there exists a sequence $\{u_n\}\subset X_\lambda$ such that

$$J_{\lambda,\beta}(u_n)\to c_\lambda>0 \quad\text{and}\quad \|J'_{\lambda,\beta}(u_n)\|_{X_\lambda^{-1}}\to 0 \quad\text{as } n\to\infty.$$

Since $0<\eta\le c_\lambda\le c_0(\Omega)$, by Proposition 3.1, there exists $u_0^+\in X_\lambda$ such that $J'_{\lambda,\beta}(u_0^+)=0$ and $J_{\lambda,\beta}(u_0^+)=c_\lambda$, which indicates that u_0^+ is a nontrivial solution of Eq. (E_λ). □

We are now ready to prove Theorem 1.1: (i) By Theorem 3.3.
(ii) By Lemmas 2.1, 3.2, one has

$$\widetilde{K}\le\inf_{u\in X_\lambda} J_{\lambda,\beta}(u)\le J_{\lambda,\beta}(e)<0.$$

Thus, by the Ekeland variational principle and Proposition 3.1, there exists $u_0^-\in X_\lambda$ such that $J'_{\lambda,\beta}\left(u_0^-\right)=0$ and $J_{\lambda,\beta}\left(u_0^-\right)=\inf_{u\in X_\lambda}J_{\lambda,\beta}(u)<0$, and so u_0^- is a nontrivial solution of Eq. (E_λ). Together with Theorem 3.3, we obtain the final conclusion. This completes the proof.

We are now ready to prove Theorem 1.2: According to (10), an estimation similar to (14) will be obtained, which leads to Proposition 3.1 and Lemma 3.2 still hold when $\beta\ge 0$ and the condition $(D1)$ is replaced by $(D1')$. Thus, by using the same argument as in Theorem 3.3, we can get the conclusion of Theorem 1.2.

CRediT authorship contribution statement

Han-Su Zhang: Writing - original draft, Writing - review & editing. **Tiexiang Li:** Supervision. **Tsung-fang Wu:** Resources, Project administration.

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