A STRUCTURE-PRESERVING ALGORITHM FOR SEMI-STABILIZING SOLUTIONS OF GENERALIZED ALGEBRAIC RICCATI EQUATIONS [†]

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Abstract. In this paper, a structure-preserving algorithm is developed for computing a semi-stabilizing solution of a Generalized Algebraic Riccati Equation (GARE). The semi-stabilizing solution of such a GARE has been used to characterize the solvability of the (J, J')-spectral factorization problem in control for general rational matrices which may have poles and zeros on the extended imaginary axis. The main difficulty for solving such a GARE is that its associated Hamiltonian/skew-Hamiltonian pencil has eigenvalues on the extended imaginary axis. Consequently, it is not clear which eigenspace of the associated Hamiltonian/skew-Hamiltonian pencil can characterize the desired semi-stabilizing solution. That is, it is not clear which eigenvectors and principal vectors corresponding to the eigenvalues on the extended imaginary axis should be contained in the eigenspace that we wish to compute. Hence, the well-known generalized eigenspace approach for the classical algebraic Riccati equations cannot be directly employed. The proposed algorithm consists of a Structure-Preserving Doubling Algorithm (SDA) and a post-process procedure for the determination of the desired eigenvectors and principal vectors corresponding to the purely imaginary and infinite eigenvalues. Under mild assumptions, the linear convergence of rate 1/2 for the SDA is proved. Numerical experiments illustrate that the proposed algorithm performs efficiently and reliably.

Key words. generalized algebraic Riccati equation; structure-preserving doubling algorithm; semi-stabilizing solution

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1. Introduction.

Throughout this paper, the sets of $m \times n$ complex and real matrices are denoted by $\mathbb{C}^{m\times n}$ and $\mathbb{R}^{m\times n}$, respectively. For convenience, we denote $\mathbb{C}^n = \mathbb{C}^{n\times 1}$, $\mathbb{C} = \mathbb{C}^1$, $\mathbb{R}^n = \mathbb{R}^{n\times 1}$ and $\mathbb{R} = \mathbb{R}^1$. The open left-half complex plane and the imaginary axis are denoted by $\mathbb{C}_-$ and $\mathbb{C}_0$, respectively. The open unit disk and the unit circle are denoted by $\mathbb{D}_-$ and $\mathbb{D}_1$, respectively. $0_{m\times n}(0_m)$ and I_m are the $m \times n(m \times m)$ zero matrix and the $m \times m$ identity matrix, respectively. The spectrums of the matrix A and the matrix pair (A, B) are denoted by $\sigma(A)$ and $\sigma(A, B)$, respectively.

In this paper, we consider the semi-stabilizing solution of the Generalized Algebraic Riccati Equation (GARE) of the form

$$\begin{cases} A_a^\top X_a + X_a^\top A_a + (C_a^\top J C_a - B_a J' B_a^\top) - X_a^\top B_a {J'}^{-1} B_a^\top X_a = 0, \\ E_a^\top X_a = X_a^\top E_a, \end{cases} \tag{1.1a}$$

where

$$E_a = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix},\ A_a = \begin{bmatrix} A & B \\ 0 & I_m \end{bmatrix},\ C_a = \begin{bmatrix} C & D \end{bmatrix},\ B_a = \begin{bmatrix} 0 \\ -I_m \end{bmatrix}, \tag{1.1b}$$

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in which $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$, $J \in \mathbb{R}^{p\times p}$, $J' \in \mathbb{R}^{m\times m}$, $p \geq m$, furthermore, the pencil $-\lambda E + A$ is regular with E being singular, J and J' are symmetric and nonsingular. A semi-stabilizing solution of the GARE (1.1) is defined as follows.

Definition 1.1 ([12, 13]). *A solution $X_a \in \mathbb{R}^{(n+m)\times(n+m)}$ of the GARE* (1.1) *is called a semi-stabilizing solution if (i) the pencil $A_a - B_a {J'}^{-1} B_a^\top X_a - \lambda E_a$ is regular and its eigenvalues lie in $\mathbb{C}_- \cup \mathbb{C}_0 \cup \{\infty\}$; (ii) the matrix pair $(C_a,\ A_a - B_a {J'}^{-1} B_a^\top X_a - \lambda E_a)$ has neither observable finite poles on $\mathbb{C}_0$ nor observable impulsive poles.*

The GARE (1.1) plays an important role on the (J, J')-spectral factorization problem in control which has found many important applications in optimal Hankel norm model reductions [1], and H_∞-optimizations [9], transport theory [10], and stochastic filterings [15].

Definition 1.2 ([12, 13]). *Let all finite generalized eigenvalues of the pencil $-\lambda E + A$ be in $\mathbb{C}_- \cup \mathbb{C}_0$. The (J, J')-spectral factorization problem for the descriptor system*

$$\begin{cases} E\dot{x} = Ax + Bu, \\ y = Cx + Du, \end{cases} \tag{1.2}$$

is solvable if $G(\lambda) = D + C(\lambda E - A)^{-1}B$ has a (J, J')-spectral factorization, i.e., there exists an invertible $\Xi(\lambda) \in \mathbb{R}^{m\times m}(\lambda)$ such that: (i) $G^T(-\lambda)JG(\lambda) = \Xi^T(-\lambda)J'\ \Xi(\lambda)$; (ii) All poles and zeros of $\Xi(\lambda)$ lie in $\mathbb{C}_- \cup \mathbb{C}_0 \cup \{\infty\}$; (iii) $G(s)\Xi^{-1}(\lambda) \in \mathbb{RL}_\infty^{p\times m}(\lambda)$, where $\mathbb{RL}_\infty^{p\times m}(\lambda)$ denotes the set of $p \times m$ proper rational matrices without poles on $\mathbb{C}_0$.

Theorem 1.3 ([12, 13]). *Assume that all the finite generalized eigenvalues of the pencil $-\lambda E + A$ lie in $\mathbb{C}_- \cup \mathbb{C}_0$, and (i) (E, A, B) is finite dynamic stabilizable and impulse controllable, i.e., rank$[-\lambda E + A \quad B] = n, \quad \forall \lambda \in \mathbb{C}\backslash\mathbb{C}_-$ and rank$\begin{bmatrix} E & A & B \\ 0 & E & 0 \end{bmatrix} = n +$ rank(E); (ii) $\max\limits_{\lambda\in\mathbb{C}} \left\{ \text{rank} \begin{bmatrix} -\lambda E + A & B \\ C & D \end{bmatrix} \right\} = n + m$. Then the (J, J')-spectral factorization problem for the descriptor system (1.2) is solvable if and only if the GARE* (1.1) *has a semi-stabilizing solution X_a, where*

$$X_a = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}, \quad X_{11} \in \mathbb{R}^{n\times n},\ X_{22} \in \mathbb{R}^{m\times m}.$$

Furthermore, in this case, a (J, J')-spectral factor $\Xi(\lambda)$ is given by

$$\Xi(\lambda) = (I - {J'}^{-1}X_{22}) - {J'}^{-1}X_{21}(\lambda E - A)^{-1}B.$$

A numerical method with a key step by seeking a nonsingular solution of a nonsymmetric ARE was proposed by [12, 13]. Indeed, there is a few numerically reliable methods for solving such a nonsymmetric ARE. Recently, numerically verifiable necessary and sufficient conditions for the existence of the semi-stabilizing solution of the GARE (1.1) and a numerically reliable method for computing such a semi-stabilizing solution were proposed by [5]. The main idea in [5] for solving the GARE (1.1) is to find a suitable semi-stable eigenspace corresponding to all eigenvalues in $\mathbb{C}_-$ and some part of eigenvalues on $\mathbb{C}_0 \cup \{\infty\}$ of the augmented matrix pencil associated with (1.1)

$$\mathcal{H}_a - \lambda\mathcal{E}_a \equiv \begin{bmatrix} A_a & -G_a \\ -H_a & -A_a^\top \end{bmatrix} - \lambda \begin{bmatrix} E_a & 0 \\ 0 & E_a^\top \end{bmatrix}, \tag{1.3}$$

where

$$H_a = C_a^\top J C_a - B_a J' B_a^\top, \quad G_a = B_a {J'}^{-1} B_a^\top.$$

It is easily seen that $(\mathcal{H}_a\mathcal{J})^T = \mathcal{H}_a\mathcal{J}$ and $(\mathcal{E}_a\mathcal{J})^T = -\mathcal{E}_a\mathcal{J}$ with $\mathcal{J} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$. Consequently, $\mathcal{H}_a - \lambda\mathcal{E}_a$ forms a Hamiltonian/skew-Hamiltonian pencil and its eigenvalues occur in quadruples $\lambda, \bar{\lambda}, -\lambda, -\bar{\lambda}$ (including $\pm\infty$). Note that the eigenstructure of the pencil $\mathcal{H}_a - \lambda\mathcal{E}_a$ corresponding to the eigenvalues on $\mathbb{C}_0 \cup \{\infty\}$ is much more complicated than the structure of its stable eigenspace. Hence, we must check whether such an eigenspace characterizes the existence of the semi-stabilizing solution of the GARE (1.1). The following connection between the semi-stabilizing solution of (1.1) and the eigenspace of the matrix pair $\mathcal{H}_a - \lambda\mathcal{E}_a$ corresponding to eigenvalues on $\mathbb{C}_-$ and $\mathbb{C}_0 \cup \{\infty\}$ can be obtained easily.

THEOREM 1.4 ([5]). *(i) X_a is a solution of the GARE* (1.1) *if and only if*

$$(\mathcal{H}_a - \lambda\mathcal{E}_a) \begin{bmatrix} I \\ X_a \end{bmatrix} = \begin{bmatrix} I \\ X_a^\top \end{bmatrix} (A_a - G_a X_a - \lambda E_a);$$

(ii) The GARE (1.1) *has a solution X_a such that the pencil $(A_a - G_a X_a) - \lambda E_a$ is regular and all its eigenvalues are on $\mathbb{C}_- \cup \mathbb{C}_0 \cup \{\infty\}$ if and only if there exist matrices $[\Phi_1^\top, \Phi_2^\top]^\top$ and $[\Psi_1^\top, \Psi_2^\top]^\top$ with $\Phi_i, \Psi_i \in \mathbb{R}^{(n+m)\times(n+m)}$ $(i = 1, 2)$ and rank$(\Phi_1) =$ rank$(\Psi_1) = n + m$ such that*

$$(\mathcal{H}_a - \lambda\mathcal{E}_a) \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} = \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix} (S_a - \lambda T_a), \;\; \Phi_1^\top \Psi_2 = \Phi_2^\top \Psi_1, \tag{1.4}$$

where $S_a - \lambda T_a \in \mathbb{R}^{(n+m)\times(n+m)}$ is regular and all its eigenvalues are on $\mathbb{C}_- \cup \mathbb{C}_0 \cup \{\infty\}$. In this case, $X_a = \Phi_2 \Phi_1^{-1}$.

Furthermore, from Weierstrass Theorem (Chap.12 of [7]) there exists a regular pair $(\widehat{S}_a, \widehat{T}_a)$ which is equivalent to (S_a, T_a) such that (1.4) can be expressed by

$$\mathcal{H}_a \begin{bmatrix} I \\ X_a \end{bmatrix} \widehat{T}_a = \mathcal{E}_a \begin{bmatrix} I \\ X_a \end{bmatrix} \widehat{S}_a. \tag{1.5}$$

However, the relation in (1.5) is only a necessary condition for (1.4).

Theorem 1.4 reveals the relationship between the semi-stabilizing solution of the GARE (1.1) and the eigenspace of $\mathcal{H}_a - \lambda\mathcal{E}_a$ corresponding to eigenvalues on $\mathbb{C}_0 \cup \{\infty\}$. As mentioned above, the eigenstructure of $\mathcal{H}_a - \lambda\mathcal{E}_a$ corresponding to eigenvalues on $\mathbb{C}_0 \cup \{\infty\}$ is much more complicated than the stable eigenstructure. This issue can be understood as follows: let τ_1 and τ_2 denote the dimensions of the eigenspaces of the pencil $\mathcal{H}_a - \lambda\mathcal{E}_a$ corresponding to the eigenvalues on $\mathbb{C}_-$ and $\mathbb{C}_0 \cup \{\infty\}$, respectively. Since E is singular, we have

$$\tau_1 < n, \quad \tau_1 + \frac{1}{2}\tau_2 = n + m,$$

provided $\mathcal{H}_a - \lambda\mathcal{E}_a$ is regular. So, there are many different eigenspaces with dimension $n+m$ corresponding to the relevant part of the eigenvalues on $\mathbb{C}_- \cup \mathbb{C}_0 \cup \{\infty\}$. Hence, it is not possible to check whether one of such eigenspaces characterizes the existence of the semi-stabilizing solution of the GARE (1.1) without having some extra insight. Consequently, it is not clear which eigenvectors and principal vectors corresponding to the eigenvalues on $\mathbb{C}_0 \cup \{\infty\}$ should be contained in the eigenspace that we wish to compute. Therefore, it is a challenge to develop a structure-preserving algorithm for the computation a semi-stabilizing solution of the GARE (1.1).

The main contribution of this paper is to propose a structure-preserving algorithm for the computation of a semi-stabilizing solution of the GARE (1.1). The main ingredients of our method include (i) computing the stable eigenspace of the Hamiltonian/skew-Hamiltonian pencil $\mathcal{H}_a - \lambda\mathcal{E}_a$ by a structure-preserving doubling algorithm, and (ii) computing a suitable semi-stable eigenspace of $\mathcal{H}_a - \lambda\mathcal{E}_a$ corresponding to the relevant part of eigenvalues on $\mathbb{C}_0 \cup \{\infty\}$ by the eigenstructure decomposition.

2. Cayley Transform of $(\mathcal{H}_a, \mathcal{E}_a)$.

Let $(\mathcal{H}_a, \mathcal{E}_a)$ be the Hamiltonian/skew-Hamiltonian pair defined in (1.3). By the Cayley transform with an appropriate parameter $\gamma > 0$, the pair $(\mathcal{H}_a, \mathcal{E}_a)$ can be transformed to a new pair $(\mathcal{H}_a + \gamma\mathcal{E}_a, \mathcal{H}_a - \gamma\mathcal{E}_a)$. The eigenpairs of $(\mathcal{H}_a, \mathcal{E}_a)$ and $(\mathcal{H}_a + \gamma\mathcal{E}_a, \mathcal{H}_a - \gamma\mathcal{E}_a)$ satisfy the relation

$$\mathcal{H}_a x = \lambda\mathcal{E}_a x \Leftrightarrow (\mathcal{H}_a + \gamma\mathcal{E}_a)x = \mu(\mathcal{H}_a - \gamma\mathcal{E}_a)x, \tag{2.1}$$

where $\mu = (\lambda + \gamma)/(\lambda - \gamma)$ and $\lambda = \gamma(\mu + 1)/(\mu - 1)$. The relation (2.1) implies the following results immediately.

Proposition 2.1. *Let λ and μ be eigenvalues of $(\mathcal{H}_a, \mathcal{E}_a)$ and $(\mathcal{H}_a + \gamma\mathcal{E}_a, \mathcal{H}_a - \gamma\mathcal{E}_a)$, respectively, satisfying* (2.1)*. Then (i) $|\lambda| = \infty$ if and only if $\mu = 1$; (ii) $\lambda = 0$ if and only if $\mu = -1$; (iii) $\lambda = i\beta$ with $\beta \in \mathbb{R}$ if and only if $|\mu| = 1$; (iv) $\lambda = \alpha + i\beta$ with $\alpha, \beta \in \mathbb{R}$ and $\alpha < 0$ ($\alpha > 0$) if and only if $|\mu| < 1$ ($|\mu| > 1$).*

Since $\mathcal{H}_a - \lambda\mathcal{E}_a$ is regular, there is a $\gamma > 0$ such that $\mathcal{H}_a - \gamma\mathcal{E}_a$ is invertible. We choose a suitable $\gamma > 0$ so that

$$A_\gamma \equiv A_a - \gamma E_a, \qquad W_\gamma \equiv A_\gamma^\top + H_a A_\gamma^{-1} G_a \tag{2.2}$$

are invertible. Let

$$\mathcal{T}_1 = \begin{bmatrix} A_\gamma^{-1} & 0 \\ 0 & I \end{bmatrix}, \; \mathcal{T}_2 = \begin{bmatrix} I & 0 \\ H_a & I \end{bmatrix}, \; \mathcal{T}_3 = \begin{bmatrix} I & 0 \\ 0 & -W_\gamma^{-1} \end{bmatrix}, \; \mathcal{T}_4 = \begin{bmatrix} I & A_\gamma^{-1} G_a \\ 0 & I \end{bmatrix}.$$

Then, the matrix pair $(\mathcal{H}_a + \gamma\mathcal{E}_a, \mathcal{H}_a - \gamma\mathcal{E}_a)$ can be transformed to the matrix pair

$(\mathcal{M}, \mathcal{L})$ with

$$\begin{aligned} \mathcal{M} &\equiv \mathcal{T}_4\mathcal{T}_3\mathcal{T}_2\mathcal{T}_1(\mathcal{H}_a + \gamma\mathcal{E}_a) = \begin{bmatrix} I + 2\gamma A_\gamma^{-1}E_a - 2\gamma A_\gamma^{-1}G_aW_\gamma^{-1}H_aA_\gamma^{-1}E_a & 0 \\ -2\gamma W_\gamma^{-1}H_aA_\gamma^{-1}E_a & I \end{bmatrix}, \\ \mathcal{L} &\equiv \mathcal{T}_4\mathcal{T}_3\mathcal{T}_2\mathcal{T}_1(\mathcal{H}_a - \gamma\mathcal{E}_a) = \begin{bmatrix} I_{n+m} & 2\gamma A_\gamma^{-1}G_aW_\gamma^{-1}E_a^\top \\ 0_{n+m} & I + 2\gamma W_\gamma^{-1}E_a^\top \end{bmatrix}. \end{aligned} \tag{2.3}$$

The Sherman-Morrison-Woodbury Formula (SMWF) gives

$$\begin{aligned} &I + (2\gamma A_\gamma^{-1} - 2\gamma A_\gamma^{-1}G_aW_\gamma^{-1}H_aA_\gamma^{-1})E_a \\ &= I + 2\gamma\left[I - A_\gamma^{-1}G_a\left(A_\gamma^\top + H_aA_\gamma^{-1}G_a\right)^{-1}H_a\right]A_\gamma^{-1}E_a \\ &= I + 2\gamma\left(A_\gamma + G_aA_\gamma^{-\top}H_a\right)^{-1}E_a \\ &= I + 2\gamma W_\gamma^{-\top}E_a \equiv I + \begin{bmatrix} A_1 & A_3 \\ A_2 & A_4 \end{bmatrix} E_a. \end{aligned}$$

Moreover, from (2.2) and the SMWF again, it follows that $G_\gamma := 2\gamma A_\gamma^{-1}G_aW_\gamma^{-1}$ and $H_\gamma := 2\gamma W_\gamma^{-1}H_aA_\gamma^{-1}$ are symmetric. Partition G_γ and H_γ by

$$G_\gamma \equiv \begin{bmatrix} G_1 & G_2^\top \\ G_2 & G_4 \end{bmatrix}, \quad H_\gamma \equiv \begin{bmatrix} H_1 & H_2^\top \\ H_2 & H_4 \end{bmatrix},$$

where $G_1 = G_1^\top$, $H_1 = H_1^\top \in \mathbb{R}^{n\times n}$ and $G_4 = G_4^\top$, $H_4 = H_4^\top \in \mathbb{R}^{m\times m}$. Then, $\mathcal{M}$ and $\mathcal{L}$ in (2.3) can be rewritten as

$$\mathcal{M} = \left[\begin{array}{cc|cc} I_n + A_1E & 0 & 0 & 0 \\ A_2E & I_m & 0 & 0 \\ \hline -H_1E & 0 & I_n & 0 \\ -H_2E & 0 & 0 & I_m \end{array}\right], \quad \mathcal{L} = \left[\begin{array}{cc|cc} I_n & 0 & G_1E^\top & 0 \\ 0 & I_m & G_2E^\top & 0 \\ \hline 0 & 0 & I_n + A_1^\top E^\top & 0 \\ 0 & 0 & A_3^\top E^\top & I_m \end{array}\right]. \tag{2.4}$$

Note that from Proposition 2.1 it is easily seen that the eigenvalues of $(\mathcal{M}, \mathcal{L})$ occur in quadruple $\{\mu, \bar{\mu}, \frac{1}{\mu}, \frac{1}{\bar{\mu}}\}$.

By (2.2) $\mathcal{H}_a - \gamma\mathcal{E}_a$ is invertible, so is $S_a - \gamma T_a$, where S_a and T_a are given in (1.4). Thus, the relation in (1.4) is equivalent to

$$(\mathcal{M} - \lambda\mathcal{L})\begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} = \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}(R_a - \lambda I), \ \Phi_1^\top\Psi_2 = \Phi_2^\top\Psi_1,$$

where R_a is similar to $(S_a + \gamma T_a)(S_a - \gamma T_a)^{-1}$ with $\sigma(R_a) \subseteq \mathbb{D}_- \cup \mathbb{D}_1$. That is, $(\mathcal{M}, \mathcal{L})$ and $(\mathcal{H}_a, \mathcal{E}_a)$ have the same invariant subspace corresponding to (R_a, I) and (S_a, T_a), respectively.

3. Structured-Preserving Algorithm for GARE (1.1).

In this Section, we want to develop a structure-preserving algorithm for solving the GARE (1.1) efficiently. We first compute a basis for an auxiliary semi-stable subspace of $(\mathcal{M}, \mathcal{L})$ in (2.4) of the form

$$\left[\begin{array}{cc|cc} I_n + A_1 E & 0 & 0 & 0 \\ A_2 E & I_m & 0 & 0 \\ \hline -H_1 E & 0 & I_n & 0 \\ -H_2 E & 0 & 0 & I_m \end{array}\right] \left[\begin{array}{cc} I_n & 0 \\ 0 & I_m \\ \hline X_1 & 0 \\ X_2 & X_4 \end{array}\right] = \left[\begin{array}{cc|cc} I_n & 0 & G_1 E^\top & 0 \\ 0 & I_m & G_2 E^\top & 0 \\ \hline 0 & 0 & I_n + A_1^\top E^\top & 0 \\ 0 & 0 & A_3^\top E^\top & I_m \end{array}\right] \left[\begin{array}{cc} I_n & 0 \\ 0 & I_m \\ \hline X_1 & 0 \\ X_2 & X_4 \end{array}\right] \begin{bmatrix} R_1 & 0 \\ R_2 & I_m \end{bmatrix}, \tag{3.1}$$

where $\sigma(R_1) \subseteq \mathbb{D}_- \cup \mathbb{D}_1$. The special basis in (3.1) spans a semi-stable subspace of $(\mathcal{M}, \mathcal{L})$ with the second block columns composing of m eigenvectors corresponding to the m trivial infinite eigenvalues of $(\mathcal{H}_a, \mathcal{E}_a)$. Using this special form of basis we will construct the basis $\left[I_{n+m}, X_a^\top\right]^\top$ for the desired semi-stable subspace of $(\mathcal{M}, \mathcal{L})$.

3.1. Structure-preserving Doubling Algorithm (SDA) for X_1.

From (2.4) we denote the submatrices of $\mathcal{M}$ and $\mathcal{L}$ by taking the 1st and 3rd block-rows and block-columns, respectively, by

$$\mathcal{M}_1 = \begin{bmatrix} I_n + A_1 E & 0_n \\ -H_1 E & I_n \end{bmatrix}, \quad \mathcal{L}_1 = \begin{bmatrix} I_n & G_1 E^\top \\ 0_n & I_n + A_1^\top E^\top \end{bmatrix}. \tag{3.2}$$

It is easy to see from (3.1) and (3.2) that X_1 satisfies

$$\mathcal{M}_1 \begin{bmatrix} I \\ X_1 \end{bmatrix} = \mathcal{L}_1 \begin{bmatrix} I \\ X_1 \end{bmatrix} R_1. \tag{3.3}$$

The matrix disk function method in [2, 3] is developed for computing X_1 by using a swapping technique built on the QR-factorization. As derived in [2, 3], for a given matrix pair $(\mathcal{M}_1, \mathcal{L}_1)$, we compute the QR-factorization of $[\mathcal{L}_1^\top, \mathcal{M}_1^\top]^\top$ by

$$\mathcal{Q} \begin{bmatrix} \mathcal{L}_1 \\ -\mathcal{M}_1 \end{bmatrix} = \begin{bmatrix} \mathcal{Q}_{11} & \mathcal{Q}_{12} \\ \mathcal{Q}_{21} & \mathcal{Q}_{22} \end{bmatrix} \begin{bmatrix} \mathcal{L}_1 \\ -\mathcal{M}_1 \end{bmatrix} = \begin{bmatrix} \mathcal{R} \\ 0 \end{bmatrix}, \tag{3.4}$$

where $\mathcal{Q}$ is orthogonal and $\mathcal{R}$ is upper triangular. Define

$$\widehat{\mathcal{M}}_1 \equiv \mathcal{Q}_{21} \mathcal{M}_1, \quad \widehat{\mathcal{L}}_1 \equiv \mathcal{Q}_{22} \mathcal{L}_1. \tag{3.5}$$

It is easily seen that $(\widehat{\mathcal{M}}_1, \widehat{\mathcal{L}}_1)$ satisfies the doubling property [14], i.e., if $\mathcal{M}_1 x = \mu \mathcal{L}_1 x$, then $\widehat{\mathcal{M}}_1 x = \mu^2 \widehat{\mathcal{L}}_1 x$. Using (3.4)–(3.5) we propose the Doubling Algorithm (DA) for computing X_1 in (3.3).

Algorithm 3.1 DA algorithm for X_1

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Require: $A_1, E, G_1, H_1$; $\tau$ (a small tolerance);
Ensure: An $X_1$ satisfying (3.3) with $X_1 = H_\infty E$ and $H_\infty$ being symmetric.
Initialize $k \leftarrow 1$, $\mathcal{R}_1 \leftarrow 0_{2n}$, $\mathcal{M}_1 \leftarrow \begin{bmatrix} I + A_1 E & 0 \\ -H_1 E & I \end{bmatrix}$, $\mathcal{L}_1 \leftarrow \begin{bmatrix} I & G_1 E^\top \\ 0 & I + A_1^\top E^\top \end{bmatrix}$;
repeat
    Compute the QR-factorization $\begin{bmatrix} \mathcal{Q}_{11} & \mathcal{Q}_{12} \\ \mathcal{Q}_{21} & \mathcal{Q}_{22} \end{bmatrix} \begin{bmatrix} \mathcal{L}_k \\ -\mathcal{M}_k \end{bmatrix} = \begin{bmatrix} \mathcal{R}_{k+1} \\ 0 \end{bmatrix}$;
    if $\|\mathcal{R}_{k+1} - \mathcal{R}_k\| \leq \tau \|\mathcal{R}_{k+1}\|$, then
        solve the least squares problem for
        $X_1 : -\mathcal{M}_k(:, 1:n) = \mathcal{M}(:, n+1:2n) X_1$;
    else
        set $\mathcal{M}_{k+1} \leftarrow \mathcal{Q}_{21}\mathcal{M}_k$,  $\mathcal{L}_{k+1} \leftarrow \mathcal{Q}_{22}\mathcal{L}_k$,  $k \leftarrow k+1$,
    end if
until there is a symmetric $H_\infty$ such that $X_1 = H_\infty E$
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Algorithm 3.1 has the disadvantage of destroying the special blocks structure as in (3.2). To remedy the shortcoming we shall develop a Structure-preserving Doubling Algorithm (SDA) for solving (3.3).

Note that in [4, 6, 11, 14] some SDAs are proposed for the computation of a basis for the semi-stable subspace of a symplectic matrix pair of the form $(\mathcal{M}_1, \mathcal{L}_1)$ as in (3.2) with $E = I_n$. However, in general, $(\mathcal{M}_1, \mathcal{L}_1)$ in (3.2) is no longer a symplectic pair. In this subsection, we can still develop a new SDA algorithm for the computation of X_1 satisfying (3.3) with $\sigma(R_1) \subseteq \mathbb{D}_- \cup \mathbb{D}_1$.

As derived in [11, 14], for the matrix pair $(\mathcal{M}_1, \mathcal{L}_1)$ we construct

$$\mathcal{M}_{1*} = \begin{bmatrix} T_1 & 0_n \\ -T_2 H_1 E & I_n \end{bmatrix}, \quad \mathcal{L}_{1*} = \begin{bmatrix} I_n & T_1 G_1 E^\top \\ 0_n & T_2 \end{bmatrix} \tag{3.6}$$

with $T_1 = (I + A_1 E)(I + G_1 E^\top H_1 E)^{-1}$ and $T_2 = (I + A_1^\top E^\top)(I + H_1 E G_1 E^\top)^{-1}$ provided that $(I + G_1 E^\top H_1 E)^{-1}$ exists, and deduce that $\mathcal{M}_{1*}\mathcal{L}_1 = \mathcal{L}_{1*}\mathcal{M}_1$. Note that $I + G_1 E^\top H_1 E$ is invertible if and only if $I + H_1 E G_1 E^\top$ is invertible because of $\sigma(G_1(E^\top H_1 E)) = \sigma(H_1(E G_1 E^\top))$. Define

$$\widehat{\mathcal{M}_1} \equiv \mathcal{M}_{1*}\mathcal{M}_1, \quad \widehat{\mathcal{L}_1} \equiv \mathcal{L}_{1*}\mathcal{L}_1. \tag{3.7}$$

Then $(\widehat{\mathcal{M}_1}, \widehat{\mathcal{L}}_1)$ satisfies the doubling property. By careful calculation the pair $(\widehat{\mathcal{M}_1}, \widehat{\mathcal{L}}_1)$ in (3.7) can be simplified to the special form as in (3.2) with

$$\widehat{\mathcal{M}_1} = \begin{bmatrix} I + \widehat{A}_1 E & 0 \\ -\widehat{H}_1 E & I \end{bmatrix}, \quad \widehat{\mathcal{L}}_1 = \begin{bmatrix} 0 & \widehat{G}_1 E^\top \\ I & I + \widehat{A}_1^\top E^\top \end{bmatrix},$$

where

$$
\begin{aligned}
I + \hat{A}_1 E &= (I + A_1 E)(I + G_1 E^\top H_1 E)^{-1}(I + A_1 E) \\
&= (I + A_1 E)\left\{I - G_1(I + E^\top H_1 E G_1)^{-1} E^\top H_1 E\right\}(I + A_1 E) \\
&\equiv I + \left[A_1 + (A_1 - G_1 E^\top H_1 (I + E G_1 E^\top H_1)^{-1}(I + E A_1)\right] E, \qquad (3.8a) \\
\hat{H}_1 E &= H_1 E + (I + A_1^\top E^\top)(I + H_1 E G_1 E^\top)^{-1} H_1 E (I + A_1 E) \\
&\equiv \left[H_1 + (I + A_1^\top E^\top)(I + H_1 E G_1 E^\top)^{-1} H_1 (I + E A_1)\right] E, \qquad (3.8b) \\
\hat{G}_1 E^\top &= G_1 E^\top + (I + A_1 E)(I + G_1 E^\top H_1 E)^{-1} G_1 E^\top (I + A_1^\top E^\top) \\
&\equiv \left[G_1 + (I + A_1 E)(I + G_1 E^\top H_1 E)^{-1} G_1 (I + E^\top A_1^\top)\right] E^\top, \qquad (3.8c) \\
I + \hat{A}_1^\top E^\top &= (I + A_1^\top E^\top)(I + H_1 E G_1 E^\top)^{-1}(I + A_1^\top E^\top) \\
&= (I + A_1^\top E^\top)\left\{I - H_1 (I + E G_1 E^\top H_1)^{-1} E G_1 E^\top\right\}(I + A_1^\top E^\top) \\
&\equiv I + \left[A_1^\top + (I + A_1^\top E^\top)(A_1^\top - (I + H_1 E G_1 E^\top)^{-1} H_1 E G_1)\right] E^\top.
\end{aligned}
$$

Since $H_1(I+EG_1E^\top H_1) = (I+H_1EG_1E^\top)H_1$, the matrix $\hat{H}_1$ in (3.8b) is symmetric. Similarly, $\hat{G}_1$ in (3.8c) can also be shown to be symmetric. Note that the matrix $(I + EG_1E^\top H_1)$ in (3.8) should be assumed to be invertible so that the structure-preserving doubling process can continue. Hence, for the case that $(I + E^\top H_1 E G_1)$ is singular, the doubling process should be switched back to Algorithm 3.1.

Using (3.7)–(3.8) the new SDA algorithm for computing X_1 is summarized as follows.

Algorithm 3.2 SDA algorithm for X_1

```
Require: $A_1, E, G_1, H_1$; $\tau$ (a small tolerance);
Ensure: An $X_1$ satisfying (3.3) with $X_1 = H_\infty E$ (see (4.18) of Theorem 4.4 for
    details) and $H_\infty$ being symmetric.
Initialize $k \leftarrow 1,\ A_{1,1} \leftarrow A_1, G_{1,1} \leftarrow G_1,\ H_{1,1} \leftarrow H_1$;
repeat
    if $(I + EG_{1,k}E^\top H_{1,k})$ is nearly singular or singular, then
        $A_1 \leftarrow A_{1,k}, G_1 \leftarrow G_{1,k}, H_1 \leftarrow H_{1,k}$; and call Algorithm 3.1;
    else
        $A_{1,k+1} \leftarrow A_{1,k} + (A_{1,k} - G_{1,k}E^\top H_{1,k}(I + EG_{1,k}E^\top H_{1,k})^{-1})(I + EA_{1,k})$,
        $G_{1,k+1} \leftarrow G_{1,k} + (I + A_{1,k}E)(I + G_{1,k}E^\top H_{1,k}E)^{-1}G_{1,k}(I + E^\top A_{1,k}^\top)$,
        $H_{1,k+1} \leftarrow H_{1,k} + (I + A_{1,k}^\top E^\top)H_{1,k}(I + EG_{1,k}E^\top H_{1,k})^{-1}(I + EA_{1,k})$,
    end if
    $k \leftarrow k + 1$;
until $\|H_{1,k+1}E - H_{1,k}E\| \le \tau\|H_{1,k+1}E\|$
$X_1 \leftarrow H_{1,k+1}E \equiv H_\infty E$.
```

Under Assumption 4.1 in Section 4, the convergence of DA (Algorithm 3.1) can be shown in a similar way as in Theorem 4.4 of [11] and the convergence of SDA (Algorithm 3.2) will be given in Theorem 4.4 in detail. In practice, the matrix $I + EG_{1,k}E^\top H_{1,k}$ in SDA is often invertible. Thus, it is extremely rare to switch from SDA to DA.

When Algorithm 3.2 converges, X_1 satisfies (3.3) with some suitable $R_1 \in \mathbb{R}^{n\times n}$ with $\sigma(R_1) \subseteq \mathbb{D}_- \cup \mathbb{D}_1$. That is, span$\left\{\left[I_n, X_1^\top\right]^\top\right\}$ forms a semi-stable subspace of

$(\mathcal{M}_1, \mathcal{L}_1)$. In the next subsetion, we will use this result to compute the unknown submatrices R_2, X_2 and X_4 in (3.1).

3.2. Computation of X_2 and X_4. Once X_1 is obtained by the Algorithm 3.2, from (3.2)–(3.3) the matrix $\left[R_1^\top, R_2^\top\right]^\top$ in (3.1) can be computed by

$$\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = \begin{bmatrix} (I + G_1 E^\top X_1)^{-1}(I + A_1 E) \\ A_2 E - G_2 E^\top X_1 (I + G_1 E^\top X_1)^{-1}(I + A_1 E) \end{bmatrix}.$$

Subsequently, we compare the $(4, 1)$–block of (3.1) and obtain

$$-H_2 E + X_2 = (A_3^\top E^\top X_1 + X_2) R_1 + X_4 R_2.$$

Thus, the matrix $[X_2, X_4]$ can be computed by solving the underdetermined equation

$$[X_2, X_4] \begin{bmatrix} I - R_1 \\ -R_2 \end{bmatrix} = H_2 E + A_3^\top E^\top X_1 R_1. \tag{3.9}$$

Remark 3.1: A number of methods can be applied to solve the underdetermined system (3.9), and any solution of (3.9) can be chosen as $[X_2, X_4]$.

In the following subsection, we want to use the auxiliary basis in (3.1) to construct bases V_s and V_∞ for the semi-stable subspaces corresponding to $\lambda \in \mathbb{D}_- \cup \mathbb{D}_1 \backslash \{1\}$ and $\lambda \in \{1\}$, respectively, which are essential for the computation of the desired X_a.

3.3. Computation of V_s and V_∞. From (3.1) we see that the matrix $R \equiv \begin{bmatrix} R_1 & 0 \\ R_2 & I_m \end{bmatrix}$ has the same eigenvalues with the matrix pair

$$(\mathcal{A}, \mathcal{B}) \equiv \left(\begin{bmatrix} I + A_1 E & 0 \\ A_2 E & I_m \end{bmatrix}, \begin{bmatrix} I + G_1 E^\top X_1 & 0 \\ G_2 E^\top X_1 & I_m \end{bmatrix} \right).$$

Let

$$E[V_0, V_r] = [U_0, U_r] \begin{bmatrix} 0_e & 0 \\ 0 & \Delta \end{bmatrix} \tag{3.10}$$

be the singular value decomposition of E, where $\Delta =$ diagonal > 0, and $[V_0, V_r]$ and $[U_0, U_r]$ are orthogonal with $V_0, U_0 \in \mathbb{R}^{n \times e}$. Then, it holds that

(3.11)

$$\begin{bmatrix} V_0^\top & 0 \\ 0 & I_m \\ V_r^\top & 0 \end{bmatrix} \begin{bmatrix} I + A_1 E & 0 \\ A_2 E & I_m \end{bmatrix} \begin{bmatrix} V_0 & 0 & V_r \\ 0 & I_m & 0 \end{bmatrix} = \left[\begin{array}{cc|c} I_e & 0 & V_0^\top A_1 U_r \Delta \\ 0 & I_m & A_2 U_r \Delta \\ \hline 0 & 0 & I + V_r^\top A_1 U_r \Delta \end{array}\right] \equiv \left[\begin{array}{c|c} I_{e+m} & \star \\ \hline 0 & \mathcal{C} \end{array}\right],$$

$$\begin{bmatrix} V_0^\top & 0 \\ 0 & I_m \\ V_r^\top & 0 \end{bmatrix} \begin{bmatrix} I + G_1 E^\top X_1 & 0 \\ G_2 E^\top X_1 & I_m \end{bmatrix} \begin{bmatrix} V_0 & 0 & V_r \\ 0 & I_m & 0 \end{bmatrix} = \left[\begin{array}{cc|c} I_e & 0 & V_0^\top G_1 E^\top X_1 V_r \\ 0 & I_m & G_2 E^\top X_1 V_r \\ \hline 0 & 0 & I + V_r^\top G_1 E^\top X_1 V_r \end{array}\right] \equiv \left[\begin{array}{c|c} I_{e+m} & \star \\ \hline 0 & \mathcal{D} \end{array}\right].$$

Now, we want to separate the eigenvalue "1" from the other semi-stable eigenvalues of $(\mathcal{A}, \mathcal{B})$. Using the backward stable numerical algorithm [17] to compute the Kronecker structure of the eigenvalue "1" of $(\mathcal{C}, \mathcal{D})$, there are orthogonal matrices Q and $Y \equiv [Y_1, Y_2]$ such that

$$Q^\top (\mathcal{C}, \mathcal{D}) Y = \left(\begin{bmatrix} \hat{C}_1 & \star \\ 0 & C_2 \end{bmatrix}, \begin{bmatrix} \hat{D}_1 & \star \\ 0 & D_2 \end{bmatrix} \right),$$

where $Y_1 \in \mathbb{R}^{(n-e)\times f}$, $\hat{C}_1$ and $\hat{D}_1 \in \mathbb{R}^{f\times f}$ are upper triangular with diagonal elements being one, and $1 \notin \sigma(C_2, D_2)$.
Let

$$V = \begin{bmatrix} V_0 & 0 & V_r \\ 0 & I_m & 0 \end{bmatrix}, \quad \widetilde{Y} = \begin{bmatrix} I_{e+m} & 0 \\ 0 & Y \end{bmatrix}, \quad \widetilde{Q}^\top = \begin{bmatrix} I_{e+m} & 0 \\ 0 & Q^\top \end{bmatrix}. \tag{3.12}$$

Then from (3.11)–(3.12) we have

$$\widetilde{Q} V^\top (\mathcal{A}, \mathcal{B}) V \widetilde{Y} = \left(\begin{bmatrix} C_1 & C_3 \\ 0 & C_2 \end{bmatrix}, \begin{bmatrix} D_1 & D_3 \\ 0 & D_2 \end{bmatrix} \right) \tag{3.13}$$

with

$$(C_1, D_1) = \left(\begin{bmatrix} I_{e+m} & \star \\ 0 & \hat{C}_1 \end{bmatrix}, \begin{bmatrix} I_{e+m} & \star \\ 0 & \hat{D}_1 \end{bmatrix} \right).$$

Since $\sigma(C_1, D_1) \cap \sigma(C_2, D_2) = \emptyset$, with $m' = e + m + f$ and $n' = n - e - f$, there are matrices W_1 and W_2 such that

$$\begin{bmatrix} I_{m'} & W_2 \\ 0 & I_{n'} \end{bmatrix} \begin{bmatrix} C_1 & C_3 \\ 0 & C_2 \end{bmatrix} \begin{bmatrix} I_{m'} & W_1 \\ 0 & I_{n'} \end{bmatrix} = C_1 \oplus C_2,$$

$$\begin{bmatrix} I_{m'} & W_2 \\ 0 & I_{n'} \end{bmatrix} \begin{bmatrix} D_1 & D_3 \\ 0 & D_2 \end{bmatrix} \begin{bmatrix} I_{m'} & W_1 \\ 0 & I_{n'} \end{bmatrix} = D_1 \oplus D_2,$$

where W_1 and W_2 solve the generalized sylvester equations $C_1 W_1 + W_2 C_2 = -C_3$ and $D_1 W_1 + W_2 D_2 = -D_3$. Here and hereafter "$\oplus$" denotes the direct sum of two matrices. Let

$$\mathcal{V} = V \widetilde{Y} \begin{bmatrix} I_{m'} & W_1 \\ 0 & I_{n'} \end{bmatrix}. \tag{3.14}$$

Then from (3.1) and (3.13)–(3.14) we have

$$V_s = \left[\begin{array}{c} V_{s,1} \\ \hline V_{s,2} \end{array} \right] \equiv \left[\begin{array}{cc} I_n & 0 \\ 0 & I_m \\ \hline X_1 & 0 \\ X_2 & X_4 \end{array} \right] \mathcal{V}(:, m'+1 : n+m) \in \mathbb{R}^{2(n+m)\times(n-(e+f))}, \tag{3.15}$$

whose columns span the semi-stable subspace of $(\mathcal{M}, \mathcal{L})$ corresponding to (C_2, D_2). Note that $1 \notin \sigma(C_2, D_2)$.

On the other hand, using (3.12),(3.14) and $Y = [Y_1, Y_2]$ we get the generalized eigenvectors

$$W_\infty \equiv \begin{bmatrix} I_n & 0 \\ 0 & I_m \\ X_1 & 0 \\ X_2 & X_4 \end{bmatrix} \mathcal{V}(:, e+m+1 : m') = \begin{bmatrix} V_r \\ 0 \\ X_1 V_r \\ X_2 V_r \end{bmatrix} Y_1 \tag{3.16}$$

corresponding to $(\hat{C}_1, \hat{D}_1)$. Then from (3.1) and (3.10) we have

$$V_\infty = \left[\begin{array}{c} V_{\infty,1} \\ \hline V_{\infty,2} \end{array}\right] \equiv \left[\begin{array}{cccc|c} V_0 & 0 & 0 & 0 & \\ 0 & I_m & 0 & 0 & \\ 0 & 0 & U_0 & 0 & W_\infty \\ 0 & 0 & 0 & I_m & \end{array}\right] \in \mathbb{R}^{2(n+m)\times\nu} \tag{3.17}$$

spanning the semi-stable subspace of $(\mathcal{M}, \mathcal{L})$ corresponding to $(I_{e+m} \oplus C_1, I_{e+m} \oplus D_1)$ with $\nu = 2(e+m) + f$. Note that $\sigma(C_1, D_1) = \{1\}$. Moreover, we have the following lemma.

Lemma 3.1. *V_∞ in (3.17) satisfies $V_\infty^\top \mathcal{J}\mathcal{E}_a V_\infty = 0$.*

Proof. From (3.17) we have $V_\infty^\top \mathcal{J}\mathcal{E}_a V_\infty = 0_{2(e+m)} \oplus (W_\infty^\top \mathcal{J}\mathcal{E}_a W_\infty)$. It suffices to show that $W_\infty^\top \mathcal{J}\mathcal{E}_a W_\infty = 0$. Since $X_1 = H_\infty E$ and $H_\infty^\top = H_\infty$, from (3.16) we have

$$\begin{aligned} W_\infty^\top \mathcal{J}\mathcal{E}_a W_\infty &= Y_1^\top \left[-V_r^\top X_1^\top E, 0, V_r^\top E^\top, 0\right] \begin{bmatrix} V_r \\ 0 \\ X_1 V_r \\ X_2 V_r \end{bmatrix} Y_1 \\ &= Y_1^\top(-V_r^\top X_1^\top E V_r + V_r^\top E^\top X_1 V_r) Y_1 = 0. \end{aligned}$$

□

Furthermore, from (2.1), (3.15) and (3.17), there exist $R_s \in \mathbb{R}^{n'\times n'}$ and $N_\infty \in \mathbb{R}^{\nu\times\nu}$ such that

$$\mathcal{H}_a \begin{bmatrix} V_{s,1} \\ V_{s,2} \end{bmatrix} = \begin{bmatrix} U_{s,1} \\ U_{s,2} \end{bmatrix} R_s, \quad \mathcal{E}_a \begin{bmatrix} V_{s,1} \\ V_{s,2} \end{bmatrix} = \begin{bmatrix} U_{s,1} \\ U_{s,2} \end{bmatrix}, \tag{3.18}$$

$$\mathcal{H}_a \begin{bmatrix} V_{\infty,1} \\ V_{\infty,2} \end{bmatrix} = \begin{bmatrix} U_{\infty,1} \\ U_{\infty,2} \end{bmatrix}, \quad \mathcal{E}_a \begin{bmatrix} V_{\infty,1} \\ V_{\infty,2} \end{bmatrix} = \begin{bmatrix} U_{\infty,1} \\ U_{\infty,2} \end{bmatrix} N_\infty, \tag{3.19}$$

where R_s is equivalent to $\gamma(C_2 + D_2)(C_2 - D_2)^{-1}$ with $\sigma(R_s) \subseteq \mathbb{C}_- \cup \mathbb{C}_0$ and N_∞ is equivalent to $\left(0_{2(e+m)-1} \oplus K_{0,f+1}\right)$ with $K_{0,f+1}$ being the nilpotent matrix of size $f+1$. (This coincides with Assumption 4.1(*ii*) in Section 4.)

3.4. Computation of X_a. From (3.15) and (3.17) we see that $\dim(\mathrm{span}\{[V_s, V_\infty]\}) = (n+m)+(e+m) > n+m$. According to the second condition of (1.4) we shall find a compression matrix $Z_\infty \in \mathbb{R}^{\nu \times m'}$ for V_∞ and U_∞ such that

$$\begin{bmatrix} V_{s,1}^\top \\ Z_\infty^\top V_{\infty,1}^\top \end{bmatrix} [U_{s,2}, U_{\infty,2} Z_\infty] = \begin{bmatrix} V_{s,2}^\top \\ Z_\infty^\top V_{\infty,2}^\top \end{bmatrix} [U_{s,1}, U_{\infty,1} Z_\infty], \tag{3.20}$$

and $\mathcal{E}_a V_\infty Z_\infty = U_\infty Z_\infty \hat{N}_\infty$ for some appropriate nilpotent matrix $\hat{N}_\infty$. The latter statement will be proved in Theorem 3.3 later.

From (3.14) and (3.15) we have

$$V_s \equiv \begin{bmatrix} V_{s,1} \\ V_{s,2} \end{bmatrix} = \left[\begin{array}{ccc} V_0 & 0 & V_r \\ 0 & I_m & 0 \\ \hline 0 & 0 & X_1 V_r \\ X_2 V_0 & X_4 & X_1 V_r \end{array}\right] \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix} \tag{3.21}$$

and

$$U_s \equiv \begin{bmatrix} U_{s,1} \\ U_{s,2} \end{bmatrix} = \mathcal{E}_a V_s = \left[\begin{array}{ccc} 0 & 0 & E V_r \\ 0 & 0 & 0 \\ \hline 0 & 0 & E^\top X_1 V_r \\ 0 & 0 & 0 \end{array}\right] \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix},$$

where

$$\begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix} \equiv \begin{bmatrix} I_{e+m} & 0 \\ 0 & Y_1 \end{bmatrix} W_1 + \begin{bmatrix} 0 \\ Y_2 \end{bmatrix}$$

with $\Gamma_1 \in \mathbb{R}^{(e+m)\times n'}$ and $\Gamma_2 \in \mathbb{R}^{f \times n'}$. Since $X_1 = H_\infty E$ and $H_\infty^\top = H_\infty$, it holds that

$$V_{s,1}^\top U_{s,2} = \left[\Gamma_1^\top, \Gamma_2^\top\right] \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & V_r^\top E^\top X_1 V_r \end{bmatrix} \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix} = V_{s,2}^\top U_{s,1}.$$

From (3.18) and (3.19), it follows that

(3.22)

$$N_\infty^\top (V_\infty^\top \mathcal{J} U_s) R_s = N_\infty^\top V_\infty^\top \mathcal{J} \mathcal{H}_a V_s = -N_\infty^\top V_\infty^\top \mathcal{H}_a^\top \mathcal{J} V_s = -V_\infty^\top \mathcal{E}_a^\top \mathcal{J} V_s = -V_\infty^\top \mathcal{J} \mathcal{E}_a V_s = -V_\infty^\top \mathcal{J} U_s$$

and

$$N_\infty^\top (U_\infty^\top \mathcal{J} V_s) R_s = V_\infty^\top \mathcal{J} \mathcal{E}_a V_s R_s = (V_\infty^\top \mathcal{J} U_s) R_s = -V_\infty^\top \mathcal{H}_a^\top \mathcal{J} V_s = -U_\infty^\top \mathcal{J} V_s. \tag{3.23}$$

Since $\sigma(N_\infty) = \{0\}$, the Stein equations (3.22) and (3.23) (after ignoring all intermediate terms) have only trivial solutions, i.e., $V_\infty^\top \mathcal{J} U_s = 0$ and $U_\infty^\top \mathcal{J} V_s = 0$.

To show (3.20), it remains to construct a Z_∞ of full rank such that $Z_\infty^\top V_{\infty,1}^\top U_{\infty,2} Z_\infty = Z_\infty^\top V_{\infty,2}^\top U_{\infty,1} Z_\infty$. Let

$$\Upsilon \equiv V_{\infty,1}^\top U_{\infty,2} - V_{\infty,2}^\top U_{\infty,1} = V_\infty^\top \mathcal{J}\mathcal{H}_a V_\infty = V_\infty^\top \mathcal{J} U_\infty. \tag{3.24}$$

Since Υ is symmetric, we compute the spectrum decomposition of Υ by

$$\Upsilon = Q^\top \Sigma Q, \tag{3.25}$$

where $\Sigma = \text{diagonal} \equiv \Sigma_1 \oplus (-\Sigma_2) \oplus 0_{\eta_0}$ with $\Sigma_1 > 0$ and $\Sigma_2 > 0$ of dimension η_1 and η_2, respectively, and $\eta_0 = \nu - (\eta_1 + \eta_2)$.

Theorem 3.2. *With $m' = e + m + f$, there is a full rank matrix $Z_\infty \in \mathbb{R}^{\nu \times m'}$ such that*

$$Z_\infty^\top \Upsilon Z_\infty = 0 \tag{3.26}$$

if and only if $\eta_0 + \min\{\eta_1, \eta_2\} \geq m'$.

Proof. From (3.24) and (3.25), it follows that $Z_\infty^\top \Upsilon Z_\infty = 0$ if and only if $Z_\infty^\top Q^\top \Sigma Q Z_\infty = 0$. Let $\zeta = Q Z_\infty \equiv [\zeta_1^\top, \zeta_2^\top, \zeta_3^\top]^\top$ have the same partitions in Σ. Then $Z_\infty^\top \Upsilon Z_\infty = 0$ is equivalent to

$$\zeta_1^\top \Sigma_1 \zeta_1 - \zeta_2^\top \Sigma_2 \zeta_2 = 0. \tag{3.27}$$

(Necessity) Without loss of generalization, we assume that $\eta_1 = \min\{\eta_1, \eta_2\}$. Since $\eta_0 + \eta_1 \geq m'$ and $\eta_0 + \eta_1 + \eta_2 = \nu$, it implies that $\eta_1 \leq \eta_2 \leq \nu - m' = e + m$. We choose

$$\zeta_1 = (\Sigma_1)^{-\frac{1}{2}} \widehat{\zeta}, \quad \zeta_2 = (\Sigma_2)^{-\frac{1}{2}} \begin{bmatrix} \widehat{\zeta} \\ 0 \end{bmatrix} \tag{3.28}$$

with $\widehat{\zeta} \in \mathbb{R}^{\eta_1 \times m'}$ being any full row rank matrix, and choose $\zeta_3 \in \mathbb{R}^{\eta_0 \times m'}$ such that $\begin{bmatrix} \widehat{\zeta}^\top & \zeta_3^\top \end{bmatrix}^\top$ is of full column rank. It is easily seen that ζ_1 and ζ_2 satisfy (3.27). Thus, we have a full column rank matrix

$$Z_\infty = Q^\top [\zeta_1^\top, \zeta_2^\top, \zeta_3^\top]^\top \tag{3.29}$$

satisfying $Z_\infty^\top \Upsilon Z_\infty = 0$.

(Sufficiency) If $\eta_0 + \eta_1 < m'$, then from (3.27) we see that

$$\text{rank} \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} = \text{rank}(\zeta_1) \leq \eta_1 \text{ and } \text{rank}(\zeta) = \text{rank}\left([\zeta_1^\top, \zeta_2^\top, \zeta_3^\top]\right) \leq \eta_0 + \eta_1 < m'.$$

Thus, we have $\text{rank}(Z_\infty) = \text{rank}(Q^\top \zeta) < m'$. Therefore, there is no full rank Z_∞ satisfying (3.26). □

Remark 3.2: Note that $Z_\infty^\top \Upsilon Z_\infty = Z_\infty^\top (V_{\infty,1}^\top U_{\infty,2} - V_{\infty,2}^\top U_{\infty,1}) Z_\infty$, and such a matrix pair $(V_\infty Z_\infty, U_\infty Z_\infty)$ is called bi-isotropic. In Theorem 3.2 we give a sufficient and necessary condition for the bi-isotropicity of $V_\infty Z_\infty$ and $U_\infty Z_\infty$. In the following theorem, we will show that the matrix pair $(V_\infty Z_\infty, U_\infty Z_\infty)$ span a deflating subspace pair of $(\mathcal{H}_a, \mathcal{E}_a)$ corresponding to $(I_{m'}, \hat{N}_\infty)$ with some suitable nilpotent matrix $\hat{N}_\infty$.

Theorem 3.3. *(i) If $\eta_0 + \min\{\eta_1, \eta_2\} = m'$, then there is a nilpotent matrix $\hat{N}_\infty \in \mathbb{R}^{m' \times m'}$ such that*

$$\mathcal{E}_a V_\infty Z_\infty = U_\infty Z_\infty \hat{N}_\infty, \tag{3.30}$$

where Z_∞ *is given by* (3.29).
(ii) If $\eta_0 + \min\{\eta_1, \eta_2\} > m'$, *then,* generically, *there is a nilpotent matrix* $\hat{N}_\infty \in \mathbb{R}^{m' \times m'}$ *such that* (3.30) *holds, where* Z_∞ *is given by* (3.40) *below.*

Proof. Without loss of generalization, we assume that $\eta_1 = \min\{\eta_1, \eta_2\}$ and adopt the notations used in Theorem 3.2.

From (3.17) and (3.19) there is a matrix $B_\infty \in \mathbb{R}^{\nu \times f}$ of full column rank such that $\mathcal{E}_a W_\infty = U_\infty B_\infty$. Let $N_\infty = [0_{\nu, 2(e+m)} \mid B_\infty]$. We then have

$$\mathcal{E}_a V_\infty = [0 \mid \mathcal{E}_a W_\infty] = [0 \mid U_\infty B_\infty] = U_\infty N_\infty, \tag{3.31}$$

where $0 = 0_{2(n+m), 2(e+m)}$. Partition Q and Z_∞ in (3.29) by

$$Q = [\underbrace{Q_1}_{e+m}, \underbrace{Q_2}_{e+m}, \underbrace{Q_3}_{f}] = \begin{bmatrix} Q' \\ Q'' \\ Q''' \end{bmatrix} \begin{matrix} \}\,\eta_1 \\ \}\,\eta_2 \\ \}\,\eta_0 \end{matrix}, \quad Z_\infty = \begin{bmatrix} Z_{\infty,1} \\ Z_{\infty,2} \\ Z_{\infty,3} \end{bmatrix} \begin{matrix} \}\,e+m \\ \}\,e+m \\ \}\,f \end{matrix}. \tag{3.32}$$

From Lemma 3.1 and (3.24), it follows that $V_\infty^\top \mathcal{J} \mathcal{E}_a V_\infty = (V_\infty^\top \mathcal{J} U_\infty) N_\infty = \Upsilon N_\infty = 0$. Therefore, $\Upsilon B_\infty = (Q^\top \Sigma Q) B_\infty = Q^\top [\Sigma_1 \oplus (-\Sigma_2) \oplus 0_{\eta_0}] Q B_\infty = 0$. From (3.32) we have

$$\begin{bmatrix} Q' \\ Q'' \end{bmatrix} B_\infty = 0. \tag{3.33}$$

By (3.28), ζ can be expressed by

$$\zeta = \begin{bmatrix} \Sigma_1' & 0 \\ \Sigma_2' & 0 \\ 0 & 0 \\ 0 & I_{\eta_0} \end{bmatrix} \begin{bmatrix} \hat{\zeta} \\ \zeta_3 \end{bmatrix}, \tag{3.34}$$

where $\Sigma_1' = \Sigma_1^{-\frac{1}{2}}$ and $\Sigma_2' = \Sigma_2^{-\frac{1}{2}}(1 : \eta_1, 1 : \eta_1)$. Since $\mathcal{E}_a V_\infty Z_\infty = U_\infty N_\infty Z_\infty$, it holds that $\mathcal{E}_a V_\infty Z_\infty \subseteq \mathcal{R}(U_\infty Z_\infty), U_\infty B_\infty Z_{\infty,3} \subseteq \mathcal{R}(U_\infty Z_\infty)$ (by (3.31) and (3.32)), $B_\infty Z_{\infty,3} \subseteq \mathcal{R}(Z_\infty), B_\infty Q_3^\top \zeta \subseteq \mathcal{R}(Q^\top \zeta)$ (by (3.29) and (3.32)), or $Q B_\infty Q_3^\top \zeta \subseteq \mathcal{R}(\zeta)$. Equivalently, by (3.33) and (3.34), there is a $\Phi \in \mathbb{R}^{m' \times m'}$ such that

$$\begin{bmatrix} 0_{(\eta_1+\eta_2) \times m'} \\ F \begin{bmatrix} \hat{\zeta} \\ \zeta_3 \end{bmatrix} \end{bmatrix} = \zeta \Phi \Leftrightarrow \begin{bmatrix} \hat{\zeta} \\ \zeta_3 \end{bmatrix} \Phi = \begin{bmatrix} 0_{\eta_1 \times m'} \\ F \begin{bmatrix} \hat{\zeta} \\ \zeta_3 \end{bmatrix} \end{bmatrix}, \tag{3.35}$$

where $F := Q''' B_\infty Q_3^\top \begin{bmatrix} \Sigma_1' & 0 \\ \Sigma_2' & 0 \\ 0 & 0 \\ 0 & I_{\eta_0} \end{bmatrix}$. Thus, to show $\mathcal{E}_a V_\infty Z_\infty \subseteq \mathcal{R}(U_\infty Z_\infty)$ is equivalent to showing that (3.35) holds.

Case (i): For $\eta_0+\eta_1 = m'$, $\left[\hat{\zeta}^\top, \zeta_3^\top\right]^\top$ is an $m' \times m'$-matrix. In this case $\left[\hat{\zeta}^\top, \zeta_3^\top\right]^\top$ can be chosen to be invertible. So, Φ in (3.35) is always solvable. Hence, there is a nilpotent $\hat{N}_\infty$ such that (3.30) holds, where Z_∞ is given by (3.29).

Case (ii): For $\eta_0 + \eta_1 > m'$, we partition ζ_3 in (3.34) and F in (3.35) by

$$\zeta_3 = \begin{bmatrix} \zeta_{3,0} \\ \zeta_{3,1} \end{bmatrix} \begin{matrix} \}\, l' \equiv m' - \eta_1 \\ \}\, d' \equiv \eta_0 - l' \end{matrix} \quad , \quad F = \begin{bmatrix} \underbrace{\begin{matrix} F_{11} \\ F_{21} \end{matrix}}_{m'} & \underbrace{\begin{matrix} F_{12} \\ F_{22} \end{matrix}}_{d'} \end{bmatrix} \begin{matrix} \}\, l' \\ \}\, d' \end{matrix} \quad .$$

Rewrite

$$\begin{bmatrix} \hat{\zeta} \\ \zeta_3 \end{bmatrix} = \begin{bmatrix} \hat{\zeta}_0 \\ \hat{\zeta}_3 \end{bmatrix} \begin{matrix} \}\, m' \\ \}\, d' \end{matrix} \quad , \tag{3.36}$$

where $\hat{\zeta}_0 := \begin{bmatrix} \hat{\zeta} \\ \zeta_{3,0} \end{bmatrix}$ and $\hat{\zeta}_3 := \zeta_{3,1}$. Equation (3.35) becomes

$$\begin{bmatrix} 0 \\ F_{11}\hat{\zeta}_0 + F_{12}\hat{\zeta}_3 \end{bmatrix} = \hat{\zeta}_0 \Phi, \quad F_{21}\hat{\zeta}_0 + F_{22}\hat{\zeta}_3 = \hat{\zeta}_3 \Phi. \tag{3.37}$$

Since $\hat{\zeta}_0$ can be chosen invertible, we partition $\hat{\zeta}_3\hat{\zeta}_0^{-1}$, F_{11} and F_{21}, respectively, by

$$\hat{\zeta}_3\hat{\zeta}_0^{-1} = [\underbrace{\Omega_1}_{\eta_1} \mid \underbrace{\Omega_2}_{l'}]\}d', \quad F_{11} = [\underbrace{F_{11}^a}_{\eta_1} \mid \underbrace{F_{11}^b}_{l'}]\}l', \quad F_{21} = [\underbrace{F_{21}^a}_{\eta_1} \mid \underbrace{F_{21}^b}_{l'}]\}d'. \tag{3.38}$$

With (3.38), (3.37) can be written as a Riccati equation in Ω_2 and the linear equation in Ω_1:

$$\Omega_2 F_{12} \Omega_2 + \Omega_2 F_{11}^b - F_{22}\Omega_2 - F_{21}^b = 0, \tag{3.39a}$$

$$(F_{22} - \Omega_2 F_{12})\Omega_1 = \Omega_2 F_{11}^a - F_{21}^a. \tag{3.39b}$$

Equation (3.39a) for Ω_2 is, generically, solvable by the Schur method, and (3.39b) is generically solvable, and so is Φ in (3.35). By (3.38), (3.36) and (3.34), Z_∞ can be chosen by

$$Z_\infty = Q^\top \begin{bmatrix} \Sigma_1' & 0 \\ \Sigma_2' & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I_{m'} \\ [\Omega_1, \Omega_2] \end{bmatrix} \hat{\zeta}_0. \tag{3.40}$$

Hence, there is a nilpotent $\hat{N}_\infty$ such that (3.30) holds. □

Remark 3.3: In our test example 5.1 [12, 13] and example 5.2 [16] in Section 5, we will check that $\eta_0 + \min\{\eta_1, \eta_2\} = m'$ holds which coincides with the condition of case (i) in Theorem 3.3.

Finally, we let $\hat{V}_\infty = V_\infty Z_\infty$ and $V_a = \left[V_s, \widehat{V}_\infty\right] \equiv \begin{bmatrix} V_{a,1} \\ V_{a,2} \end{bmatrix}$. If $V_{a,1}$ is invertible, then the solution X_a for (1.1) is given by $X_a = V_{a,2}V_{a,1}^{-1}$. Therefore, we are required to ensure that $V_{a,1}$ is invertible. In fact, from (3.16), (3.17) and (3.21) we have

$$
\begin{aligned}
V_{a,1} = [V_{s,1},\ V_{\infty,1}Z_\infty] &= \left[\begin{bmatrix} V_0 & 0 & V_r \\ 0 & I_m & 0 \end{bmatrix}\begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix}, \begin{bmatrix} V_0 & 0 & 0 & 0 & V_rY_1 \\ 0 & I_m & 0 & 0 & 0 \end{bmatrix} Z_\infty \right] \\
&= \begin{bmatrix} V_0 & 0 & V_r \\ 0 & I_m & 0 \end{bmatrix}\begin{bmatrix} \Gamma_1 & Z_{\infty,1} \\ \Gamma_2 & Y_1Z_{\infty,3} \end{bmatrix} = \begin{bmatrix} V_0 & 0 & V_r \\ 0 & I_m & 0 \end{bmatrix}\begin{bmatrix} I_{e+m} & 0 \\ 0 & Y \end{bmatrix}\begin{bmatrix} W_1 & \begin{bmatrix} Z_{\infty,1} \\ Z_{\infty,3} \end{bmatrix} \\ I_{e+m} & 0 \end{bmatrix},
\end{aligned}
$$

where $Z_\infty = \left[Z_{\infty,1}^\top, Z_{\infty,2}^\top, Z_{\infty,3}^\top\right]^\top$ is defined in (3.32). Therefore, $V_{a,1}$ is nonsingular if and only if $\left[Z_{\infty,1}^\top, Z_{\infty,3}^\top\right]^\top$ is nonsingular.

Now, we summarize the above procedures for the computation of X_a in the following algorithm.

Algorithm 3.3 Structure-preserving Algorithm (SA) for GARE (1.1)

```
Require: $A_1,\ A_2,\ A_3,\ H_1,\ H_2,\ G_1,\ G_2,\ E$ as in (3.1);
Ensure: An $X_a$ for GARE (1.1).
 1: Compute $X_1$ by Algorithm 3.2;
 2: Compute $X_2$ and $X_4$ by (3.9);
 3: Compute $V_s$ and $V_\infty$ by (3.15) and (3.17), respectively;
 4: if the condition $\eta_0 + \min\{\eta_1, \eta_2\} \geq m'$ in Theorem 3.3 holds, then
 5:     compute $Z_\infty$ by (3.29) or (3.40)
 6: else
 7:     there is no solution;
 8: end if
 9: Compute $\begin{bmatrix} V_{a,1} \\ V_{a,2} \end{bmatrix} \equiv [V_s, V_\infty Z_\infty]$;
10: if $V_{a,1}$ is invertible, then
11:     $X_a = V_{a,2}V_{a,1}^{-1}$,
12: else
13:     fails
14: end if
```

Remark 3.4: In Algorithm 3.3, step 1 is carried out iteratively and converges quadratically under some mild assumptions, as proved in Theorem 4.4. As for step 4-8, since $\left[\hat{\zeta}^\top, \zeta_3^\top\right]^\top$ in (3.32) or $\widehat{\zeta}_0$ in (3.43) can be chosen as arbitrary nonsingular matrices, there are many degrees of freedom in obtaining invertible $\left[Z_{\infty,1}^\top, Z_{\infty,3}^\top\right]^\top$, and desirable $V_{a,1}$. Thus, Algorithm 3.3 solves GARE (1.1) efficiently and reliably, in most cases, as illustrated by the numerical experiments presented in Section 5.

4. Convergence of SDA Algorithm. We denote the Jordan block of size p corresponding to a unimodular eigenvalue $\omega \equiv e^{i\theta}$ by

$$K_{\omega,p} = \begin{bmatrix} \omega & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \omega \end{bmatrix}_{p\times p}.$$

The Jordan block $K_{\omega,p}$ to the power of 2^k can be explicitly evaluated by (see e.g. p.557 of [8])

$$K_{\omega,p}^{2^k} = \begin{bmatrix} \gamma_{1,k} & \gamma_{2,k} & \cdots & \gamma_{p,k} \\ & \ddots & \ddots & \vdots \\ & & \ddots & \gamma_{2,k} \\ 0 & & & \gamma_{1,k} \end{bmatrix}, \tag{4.1}$$

where

$$\gamma_{i,k} = \frac{2^k(2^k-1)\cdots(2^k-i+2)}{(i-1)!}\omega^{2^k-i+1} = \mathcal{O}(2^{k(i-1)}),$$

for $i = 1, \cdots, p$. If $p = 2q$, let

$$L_{\omega,k} \equiv K_{\omega,p}^{2^k}(1:q:q+1:p). \tag{4.2}$$

We quote the useful lemma from [11].

Lemma 4.1. *For $p = 2q$, the matrix in* (4.2) *is invertible and satisfies*

$$\|L_{\omega,k}^{-1}K_{\omega,q}^{2^k}\| = \mathcal{O}(2^{-k}), \quad \|K_{\omega,q}^{2^k}L_{\omega,k}^{-1}K_{\omega,q}^{2^k}\| = \mathcal{O}(2^{-k}). \tag{4.3}$$

To show the convergence of the SDA algorithm we first assume that the original matrix pencil $\mathcal{H}_a - \lambda\mathcal{E}_a$ satisfies the following assumption.

Assumption 4.1. *For the Hamiltonian/skew-Hamiltonian pair $(\mathcal{H}_a, \mathcal{E}_a)$ we assume that*

(i) eigenvalue 0 *has even partial multiplicity and 2μ semi-simple zeros with*

$$\textit{rank}\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = n + m - \mu;$$

(ii) eigenvalue ∞ has Jordan structure $\left(I_{2(e+f+m)}, 0_{2m} \oplus 0_{2e-1} \oplus K_{0,2f+1}\right)$ with

$$\textit{nullity}\begin{bmatrix} \widehat{F}_a & E_a & & 0 \\ & F_a & E_a & \\ & & \ddots & \ddots \\ 0 & & & F_a & \widehat{E}_a \end{bmatrix}_{g\times(g+e)} \geq e+1,$$

where $g = (n+m)f$ *and*

$$\widehat{F}_a = \begin{bmatrix} AV_0 & B \\ CV_0 & D \end{bmatrix},\ F_a = \begin{bmatrix} A & B \\ C & D \end{bmatrix},\ E_a = \begin{bmatrix} E & 0 \\ 0 & 0_m \end{bmatrix},\ \widehat{E}_a = \begin{bmatrix} E \\ 0 \end{bmatrix},$$

in which V_0 *is given in* (3.10)*;*
(iii) each nonzero purely imaginary eigenvalue has even partial multiplicity.
Remark 4.1: The Jordan structure of ∞ in (ii) can be considered to a more general case as $\left(I_{2(e+f+m)}, 0_{2m} \oplus 0_e \oplus 0_{e-d} \overset{d}{\underset{i=1}{\oplus}} K_{0,2f_i+1}\right)$ with $f = f_1 + \cdots + f_d$ and $f_d \geq \cdots \geq f_1 \geq 1$. Then the condition in (4.4) should be generalized to

$$\text{nullity} \begin{bmatrix} \widehat{F}_a & E_a & & \\ & F_a & \ddots & \\ & & \ddots & E_a \\ & & F_a & \widehat{E}_a \end{bmatrix}_{g_i \times (g_i+e)} \geq e + (d-i+1),$$

for $i = 1, \cdots, d$ and $f_i > f_{i-1}$ ($f_0 \equiv 0$), where $g_i = (n+m)f_i$. Since the proof for convergence of SDA in Theorem 4.4 is a straightforward extension for the case $d > 1$, we only consider the simple case with $d = 1$ as in (ii) for convenience.

LEMMA 4.2. *Let* $(\mathcal{H}_a, \mathcal{E}_a)$ *satisfy Assumption 4.1. Then (i) for* $\mu > 0$*, the null space of* $\mathcal{H}_a$ *contains* μ *linearly independent vectors of the form* $\zeta \equiv \left[\zeta_1^\top, \zeta_2^\top, 0_{n,\mu}^\top, \zeta_4^\top\right]^\top \in \mathbb{R}^{2(n+m)\times\mu}$ *with* $\zeta_1 \in \mathbb{R}^n$*,* ζ_2 *and* $\zeta_4 \in \mathbb{R}^m$*; (ii) for* $f \geq 1$*, the generalized eigenvectors of* $(\mathcal{H}_a, \mathcal{E}_a)$ *corresponding to* ∞ *of degree* j *are of the forms* $\eta_j \equiv \left[\eta_{j1}^\top, \eta_{j2}^\top, 0_{n,1}^\top, \eta_{j4}^\top\right]^\top \in \mathbb{R}^{2(n+m)}$ *with* $\eta_{j1} \neq 0 \in \mathbb{R}^n$*,* η_{j2}*,* $\eta_{j4} \in \mathbb{R}^m$*, for* $j = 1, \cdots, f$*, i.e., there is* $\eta_0 \equiv \left[(V_0\alpha)^\top, \beta^\top, (U_0\gamma)^\top, \delta^\top\right]^\top$ *with* $0 \neq \alpha, \gamma \in \mathbb{R}^e$ *and* $\beta, \delta \in \mathbb{R}^m$ *such that*

$$\mathcal{E}_a \eta_j = \mathcal{H}_a \eta_{j-1}, \quad j = 1, \cdots, f. \tag{4.5}$$

Proof. (1) Since

$$\begin{bmatrix} A & B \\ -C^\top JC & -C^\top JD \\ -D^\top JC & -D^\top JD \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & -C^\top J \\ 0 & -D^\top J \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \leq n + m - \mu,$$

from Assumption 4.1(i), (1.1b) and (1.3), it follows that

(4.6)

$$\text{nullity}\left(\mathcal{H}_a \begin{bmatrix} I_{n+m} & 0 \\ 0 & 0_{n,m} \\ 0 & I_m \end{bmatrix}\right) = \text{nullity}\left(\left[\begin{array}{cc|c} A & B & 0 \\ 0 & I_m & -(J')^{-1} \\ \hline -C^\top JC & -C^\top JD & 0 \\ -D^\top JC & -D^\top JD - J' & I_m \end{array}\right]\right) \geq \mu.$$

This proves the assertion.
(2) Denote

$$\begin{aligned}
&\widehat{\mathcal{H}}_a = \mathcal{H}_a \begin{bmatrix} I_{n+m} & 0 \\ 0 & 0_{n,m} \\ 0 & I_m \end{bmatrix}, \qquad \widehat{\mathcal{E}}_a = \mathcal{E}_a \begin{bmatrix} I_{n+m} & 0 \\ 0 & 0_{n,m} \\ 0 & I_m \end{bmatrix}, \\
&\text{and } \ \widehat{\mathcal{H}}_{a,1} = \widehat{\mathcal{H}}_a \begin{bmatrix} V_0 & 0 \\ 0 & I_{2m} \end{bmatrix}, \qquad \widehat{\mathcal{E}}_{a,1} = \widehat{\mathcal{E}}_a \begin{bmatrix} I_n \\ 0_{2m,n} \end{bmatrix}.
\end{aligned} \tag{4.7}$$

From Assumption 4.1(ii), (4.7) and the equality of matrices in (4.6) we have

$$\text{nullity}\left(\begin{bmatrix} \widehat{\mathcal{H}}_{a,1} & \widehat{\mathcal{E}}_a & & 0 \\ & \widehat{\mathcal{H}}_a & \widehat{\mathcal{E}}_a & \\ & & \ddots & \ddots & \\ 0 & & & \widehat{\mathcal{H}}_a & \widehat{\mathcal{E}}_{a,1} \end{bmatrix}_{g\times(g+e)} \right) \geq e+1. \tag{4.8}$$

Since nullity$(E) = e$, $\widehat{\mathcal{H}}_{a,1}$ is of full column rank and $\mathcal{H}_a - \lambda\mathcal{E}_a$ is regular, from (4.8) there are $\widehat{\eta}_0 = \left(\alpha^\top, \beta^\top, \delta^\top\right)^\top$ with $0 \neq \alpha \in \mathbb{R}^e,\ \beta,\ \delta \in \mathbb{R}^m,\ 0 \neq \widehat{\eta}_f \in \mathbb{R}^n$ and $\widehat{\eta}_j = \left(\eta_{j1}^\top, \eta_{j2}^\top, \eta_{j4}^\top\right)^\top$ with $0 \neq \eta_{j1} \in \mathbb{R}^n,\ \eta_{j2}, \eta_{j4} \in \mathbb{R}^m,\ j = 1, \cdots, f-1$ such that

$$\widehat{\mathcal{E}}_a \widehat{\eta}_1 = \widehat{\mathcal{H}}_{a,1}\widehat{\eta}_0,\ \widehat{\mathcal{E}}_a \widehat{\eta}_j = \widehat{\mathcal{H}}_a \widehat{\eta}_{j-1},\ \widehat{\mathcal{E}}_{a,1}\widehat{\eta}_f = \widehat{\mathcal{H}}_a \widehat{\eta}_{f-1}, \tag{4.9}$$

for $j = 2, \cdots, f-1$. By taking $\gamma = 0,\ \eta_{j2},\ \eta_{j4}$ arbitrary, from (4.9), it follows that (4.5) holds. ◻

Let

$$\mathcal{M}_1 = \begin{bmatrix} I_n + A_1 E & 0 \\ -H_1 E & I_n \end{bmatrix}, \quad \mathcal{L}_1 = \begin{bmatrix} I_n & G_1 E^\top \\ 0 & I_n + A_1^\top E^\top \end{bmatrix} \tag{4.10}$$

be the submatrices of $\mathcal{M}$ and $\mathcal{L}$ in (2.4), respectively. By (2.1) the matrix pair $(\mathcal{M}, \mathcal{L})$ is a Cayley transform of $(\mathcal{H}_a, \mathcal{E}_a)$. Therefore, $(\mathcal{M}_1, \mathcal{L}_1)$ adopts Assumption 4.1 as follows.

ASSUMPTION 4.2. *For* $(\mathcal{M}_1, \mathcal{L}_1)$ *we assume that*
(i) eigenvalue -1 *has even partial multiplicity and has* 2μ *semi-simple* -1 *with rank* $\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = n + m - \mu$*;*
(ii) eigenvalue 1 *has Jordan structure* $\left(I_{2(e+f)}, I_e \oplus I_{e-1} \oplus K_{1,2f+1}\right)$ *satisfying* (4.4)*;*
(iii) each unimodular eigenvalue ω_j *with* $\omega_j \neq -1$ *and 1 has even partial multiplicity* $2m_j$*.*

LEMMA 4.3. *Let* $(\mathcal{H}_a, \mathcal{E}_a)$ *satisfy Assumption 4.2. Then*
(i) for $\mu > 0$*, the null space of* $\mathcal{M}_1 + \mathcal{L}_1$ *contains* μ *linearly independent vectors of the form* $[\zeta_1^\top, 0_{n,\mu}^\top]^\top \in \mathbb{R}^{2n\times\mu}$*;*
(ii) for $f \geq 1$*, the generalized eigenvectors of* $(\mathcal{M}_1, \mathcal{L}_1)$ *corresponding to* 1 *of degree*

j are of the form $\eta_j \equiv \left(\eta_{j1}^\top, 0_{n,1}^\top\right)^\top \neq 0 \in \mathbb{R}^{2n}$, $j = 1, \cdots, f$, *i.e., there is* $\eta_0 \equiv \left[(V_0\alpha)^\top, 0_{n,1}^\top\right]^\top$ *such that*

$$(\mathcal{M}_1 - \mathcal{L}_1)\eta_j = \mathcal{L}_1\eta_{j-1},\ j = 1, \cdots, f.$$

Proof. The assertions (i)(ii) follows from Lemma 4.2 and the Cayley transform immediately. □

From Kronecker's Theorem (Chap.12 of [7]) there are nonsingular matrices $\mathcal{Q}$ and $\mathcal{Z}$ such that

$$\mathcal{Q}\mathcal{M}_1\mathcal{Z} = \begin{bmatrix} J_s \oplus (-I_\mu) \oplus J_\omega \oplus J_1 & 0 \oplus 0_\mu \oplus \Gamma_\omega \oplus \Gamma_1 \\ 0_n & I_s \oplus (-I_\mu) \oplus J_\omega \oplus \widehat{J}_1 \end{bmatrix} \equiv J_{\mathcal{M}_1},$$

$$\mathcal{Q}\mathcal{L}_1\mathcal{Z} = \begin{bmatrix} I_n & 0_n \\ 0_n & J_s \oplus I_\mu \oplus I_r \oplus I_{e+f} \end{bmatrix} \equiv J_{\mathcal{L}_1}, \tag{4.11}$$

where $J_s \in \mathbb{R}^{s \times s}$ consists of asymptotic stable blocks with $\rho(J_s) < 1$,

$$\begin{aligned} &J_\omega = K_{\omega_1,m_1} \oplus \cdots \oplus K_{\omega_l,m_l} \in \mathbb{R}^{r \times r} \text{ with } \omega_j \neq 1, \\ &\Gamma_\omega = \Gamma_{1,m_1} \oplus \cdots \oplus \Gamma_{l,m_l} \text{ with } \Gamma_{1,m_j} = e_{m_j}e_1^\top, \\ &J_1 = K_{1,f+1} \oplus I_{e-1},\quad \widehat{J}_1 = K_{1,f} \oplus I_e,\quad \Gamma_1 = e_{f+1}e_1^\top. \end{aligned}$$

On the other hand, if we interchange the roles of $\mathcal{M}_1$ and $\mathcal{L}_1$ in (4.11), and consider the pair $(\mathcal{L}_1, \mathcal{M}_1)$, then there are nonsingular matrices $\mathcal{P}$ and $\mathcal{Y}$ such that

$$\mathcal{P}\mathcal{L}_1\mathcal{Y} = J_{\mathcal{M}_1},\quad \mathcal{P}\mathcal{M}_1\mathcal{Y} = J_{\mathcal{L}_1}. \tag{4.12}$$

Since $J_{\mathcal{M}_1}$ and $J_{\mathcal{L}_1}$ in (4.11) commute with each other and from (4.11) and (4.12), one can derive

$$\mathcal{M}_1\mathcal{Z}J_{\mathcal{L}_1} = \mathcal{L}_1\mathcal{Z}J_{\mathcal{M}_1},\quad \mathcal{L}_1\mathcal{Y}J_{\mathcal{L}_1} = \mathcal{M}_1\mathcal{Y}J_{\mathcal{M}_1}. \tag{4.13}$$

Partition $\mathcal{Z}$ and $\mathcal{Y}$ in (4.13) with

$$\mathcal{Z} = \begin{bmatrix} Z_1 & Z_3 \\ Z_2 & Z_4 \end{bmatrix},\quad \mathcal{Y} = \begin{bmatrix} Y_1 & Y_3 \\ Y_2 & Y_4 \end{bmatrix}, \tag{4.14}$$

where $Z_i,\ Y_i \in \mathbb{R}^{n \times n},\ i = 1, \cdots, 4$. From Lemma 4.3 we see that

$$Z_2 \begin{bmatrix} 0_{s,\mu} \\ I_\mu \\ 0_{n-s-\mu,\mu} \end{bmatrix} = 0 \text{ and } Z_2 \begin{bmatrix} 0_{n-e,e} \\ I_e \end{bmatrix} = 0. \tag{4.15}$$

Let $\{(\mathcal{M}_{1,k}, \mathcal{L}_{1,k})\}_{k=1}^\infty$ be the sequence generated by the SDA algorithm of the form

$$\mathcal{M}_{1,k} = \begin{bmatrix} I_n + A_{1,k}E & 0 \\ -H_{1,k}E & I_n \end{bmatrix},\quad \mathcal{L}_{1,k} = \begin{bmatrix} I_n & G_{1,k}E^\top \\ 0 & I_n + A_{1,k}^\top E^\top \end{bmatrix} \tag{4.16}$$

with $\mathcal{M}_{1,1} = \mathcal{M}_1$ and $\mathcal{L}_{1,1} = \mathcal{L}_1$. From (3.6)–(3.7) and (4.13) we have

$$\mathcal{M}_{1,k}\mathcal{Z}J_{\mathcal{L}_1}^{2^k} = \mathcal{L}_{1,k}\mathcal{Z}J_{\mathcal{M}_1}^{2^k}, \quad \mathcal{L}_{1,k}\mathcal{Y}J_{\mathcal{L}_1}^{2^k} = \mathcal{M}_{1,k}\mathcal{Y}J_{\mathcal{M}_1}^{2^k}. \tag{4.17}$$

THEOREM 4.4. *Let* $(\mathcal{M}_1, \mathcal{L}_1)$ *be given in* (4.10) *and satisfy Assumption 4.2. Suppose that the SVD of* E *in* (3.10) *holds, and* Z_1 *and* Y_2 *in* (4.14) *are invertible. If the sequence* $\{(A_{1,k}, G_{1,k},\ H_{1,k})\}$ *generated by SDA is well-defined, then we have*

$$\|H_{1,k}E - Z_2Z_1^{-1}\| \leq \mathcal{O}(\rho(J_s)^{2^k}) + \mathcal{O}(2^{-k}) \to 0, \ \text{as} \ \ k \to \infty. \tag{4.18}$$

Here and hereafter $\|\cdot\|$ *denotes any matrix norm.*

Proof. Substituting $(\mathcal{M}_{1,k}, \mathcal{L}_{1,k})$ of (4.16), $\mathcal{Z}$ in (4.14), $J_{\mathcal{M}_1}$ and $J_{\mathcal{L}_1}$ of (4.11) into the first equation of (4.17), and comparing both sides we have

(4.19a)

$$-H_{1,k}EZ_1 + Z_2 = (I_n + A_{1,k}^{\top}E^{\top})Z_2(J_s^{2^k} \oplus I_\mu \oplus J_\omega^{2^k} \oplus J_1^{2^k}),$$

$$(-H_{1,k}EZ_3 + Z_4)(J_s^{2^k} \oplus I_\mu \oplus I_r \oplus I_{e+f}) = (I_n + A_{1,k}^{\top}E^{\top})Z_2(0_s \oplus 0_\mu \oplus \Gamma_{\omega,k} \oplus \Gamma_{1,k})$$

(4.19b)

$$+ (I_n + A_{1,k}^{\top}E^{\top})Z_4(I_s \oplus I_\mu \oplus J_\omega^{2^k} \oplus \widehat{J}_1^{2^k}),$$

where

$$\Gamma_{\omega,k} = \bigoplus_{j=1}^{l} K_{\omega_j,2m_j}^{2^k}(1:m_j, m_{j+1}:2m_j), \quad \Gamma_{1,k} = K_{1,2f+1}^{2^k}(1:f+1, f+2:2f+1) \oplus 0_{e-1,e},$$

$$J_\omega^{2^k} = \bigoplus_{j=1}^{l} K_{\omega_j,m_j}^{2^k}, \quad J_1^{2^k} = K_{1,f+1}^{2^k} \oplus I_{e-1}, \widehat{J}_1^{2^k} = K_{1,f}^{2^k} \oplus I_e.$$

Define

$$\widehat{\Gamma}_{1,k} := K_{1,2f+1}^{2^k}(1:f, f+2:2f+1) \oplus 0_{e,e}$$

for $k = 0, 1, 2, \cdots$. Then from (4.1) we have $\Gamma_{1,k}\widehat{\Gamma}_{1,k}^{+} = \begin{bmatrix} I_f & 0_{f,e} \\ \zeta_k & 0_e \end{bmatrix}$ with $\zeta_k = \left[\begin{array}{c} 2^{-fk}, \cdots, 2^{-k} \\ \hline 0_{e-1,f} \end{array}\right]$. Consequently, from (4.3) of Lemma 4.1 we have

$$\left(I_n - \Gamma_{1,k}\widehat{\Gamma}_{1,k}^{+}\right) J_1^{2^k} = \begin{bmatrix} 0_f & 0_{f,e} \\ \zeta_k & I_e \end{bmatrix} \begin{bmatrix} K_{1,f}^{2^k} & \zeta_k^{+} \\ 0_{e,f} & I_e \end{bmatrix} = \begin{bmatrix} 0_f & 0_{f,e} \\ \zeta_k & I_e \end{bmatrix} \tag{4.20}$$

and

$$\begin{aligned} \left\| \widehat{J}_1^{2^k} \widehat{\Gamma}_{1,k}^{+} J_1^{2^k} \right\| &= \left\| \left(K_{1,f}^{2^k} \oplus I_e\right) \widehat{\Gamma}_{1,k}^{+} \begin{bmatrix} K_{1,f}^{2^k} & \zeta_k^{+} \\ 0_{e,f} & I_e \end{bmatrix} \right\| \\ &= \left\| \left(K_{1,f}^{2^k} \oplus I_e\right) \widehat{\Gamma}_{1,k}^{+} \left(K_{1,f}^{2^k} \oplus I_e\right) + \begin{bmatrix} 2^{-k} \\ \vdots \\ 2^{-fk} \\ 0_{e,1} \end{bmatrix} e_{f+1}^{\top} \right\| = \mathcal{O}(2^{-k}) \to 0, \end{aligned} \tag{4.21}$$

as $k \to \infty$.
Postmultiplying (4.19b) by $\left(0_s \oplus 0_\mu \oplus \Gamma_{\omega,k}^{-1} J_\omega^{2^k} \oplus \widehat{\Gamma}_{1,k}^{+} J_1^{2^k}\right)$ we have

$$(-H_{1,k}EZ_3 + Z_4)\left(0_s \oplus 0_\mu \oplus \Gamma_{\omega,k}^{-1} J_\omega^{2^k} \oplus \widehat{\Gamma}_{1,k}^{+} J_1^{2^k}\right) = (I_n + A_{1,k}^\top E^\top)Z_2\left(0_s \oplus 0_\mu \oplus J_\omega^{2^k} \oplus \Gamma_{1,k}\widehat{\Gamma}_{1,k}^{+} J_1^{2^k}\right)$$
$$+ (I_n + A_{1,k}^\top E^\top)Z_4\left(0_s \oplus 0_\mu \oplus J_\omega^{2^k}\Gamma_{\omega,k}^{-1} J_\omega^{2^k} \oplus \widehat{\Gamma}_1^{2^k}\widehat{\Gamma}_{1,k}^{+} J_1^{2^k}\right). \tag{4.22}$$

From (4.19a), it follows that

$$\begin{aligned}(I_n + A_{1,k}^\top E^\top)Z_2&\left(0_s \oplus 0_\mu \oplus J_\omega^{2^k} \oplus \Gamma_{1,k}\widehat{\Gamma}_{1,k}^{+} J_1^{2^k}\right)\\ &= -H_{1,k}EZ_1 + Z_2 - (I_n + A_{1,k}^\top E^\top)\left(J_s^{2^k} \oplus I_\mu \oplus 0_r \oplus (I - \Gamma_{1,k}\widehat{\Gamma}_{1,k}^{+})J_1^{2^k}\right).\end{aligned} \tag{4.23}$$

Substituting (4.23) into (4.22) we get

(4.24)

$$\begin{aligned}-H_{1,k}E\left[Z_1 - Z_3\left(0_s \oplus 0_\mu \oplus \Gamma_{\omega,k}^{-1} J_\omega^{2^k} \oplus \widehat{\Gamma}_{1,k}^{+} J_1^{2^k}\right)\right] + Z_2 &= Z_4\left(0_s \oplus 0_\mu \oplus \Gamma_{\omega,k}^{-1} J_\omega^{2^k} \oplus \widehat{\Gamma}_{1,k}^{+} J_1^{2^k}\right)\\ &- (I_n + A_{1,k}^\top E^\top)Z_4\left(0_s \oplus 0_\mu \oplus J_\omega^{2^k}\Gamma_{\omega,k}^{-1} J_\omega^{2^k} \oplus \widehat{J_1}^{2^k}\widehat{\Gamma}_{1,k}^{+} J_1^{2^k}\right)\\ &+ (I_n + A_{1,k}^\top E^\top)Z_2\left(J_s^{2^k} \oplus I_\mu \oplus 0_r \oplus \begin{bmatrix} 0 & 0 \\ \zeta_k & I_e \end{bmatrix}\right).\end{aligned}$$

On the other hand, substituting $(\mathcal{L}_{1,k}, \mathcal{M}_{1,k})$ of (4.16) and $\mathcal{Y}$ of (4.14) into the second equation of (4.17), we have

(4.25a)

$$\left(I_n + A_{1,k}^\top E^\top\right)Y_2 = (-H_{1,k}EY_1 + Y_2)\left(J_s^{2^k} \oplus I_\mu \oplus J_\omega^{2^k} \oplus J_1^{2^k}\right),$$

(4.25b)

$$\begin{aligned}\left(I_n + A_{1,k}^\top E^\top\right)Y_4\left(J_s^{2^k} \oplus I_\mu \oplus I_r \oplus I_{e+f}\right) &= (-H_{1,k}EY_1 + Y_2)\left(0_s \oplus 0_\mu \oplus \Gamma_{\omega,k} \oplus \Gamma_{1,k}\right)\\ &+ (-H_{1,k}EY_3 + Y_4)\left(I_s \oplus I_\mu \oplus J_\omega^{2^k} \oplus \widehat{J_1}^{2^k}\right).\end{aligned}$$

As above, postmultiplying (4.25b) by $\left(0_s \oplus 0_\mu \oplus \Gamma_{\omega,k}^{-1} J_\omega^{2^k} \oplus \widehat{\Gamma}_{1,k}^{+} J_1^{2^k}\right)$ and using (4.25a), we get

$$\begin{aligned}(I_n + A_{1,k}^\top E^\top)&\left[Y_2 - Y_4\left(0_s \oplus 0_\mu \oplus \Gamma_{\omega,k}^{-1} J_1^{2^k} \oplus \widehat{\Gamma}_{1,k}^{+} J_1^{2^k}\right)\right] =\\ &(-H_{1,k}EY_1 + Y_2)\left[J_s^{2^k} \oplus I_\mu \oplus 0_r \oplus \begin{bmatrix} 0 & 0 \\ \zeta_k & I_e \end{bmatrix}\right]\\ &+ (-H_{1,k}EY_3 + Y_4)\left(0_s \oplus 0_\mu \oplus J_\omega^{2^k}\Gamma_{\omega,k}^{-1} J_\omega^{2^k} \oplus \widehat{J_1}^{2^k}\widehat{\Gamma}_{1,k}^{+} J_1^{2^k}\right).\end{aligned} \tag{4.26}$$

Then from (4.15), (4.20)–(4.21) and Lemma 4.1, (4.24) can be simplified by

(4.27)

$$H_{1,k}EZ_1\left(I_n + \mathcal{O}(2^{-k})\right) = -Z_2 + \mathcal{O}(2^{-k}) + (I_n + A_{1,k}^\top E^\top)\left(\mathcal{O}(\rho(J_s^{2^k})) + \mathcal{O}(2^{-k})\right),$$

as k sufficiently large. Since Z_1 is invertible, substituting $H_{1k}E$ in (4.27) into (4.26), we get

$$\left(I_n + A_{1,k}^\top E^\top\right)\left(Y_2 + \mathcal{O}(\rho(J_s^{2^k})) + \mathcal{O}(2^{-k})\right) = \mathcal{O}(1).$$

Since Y_2 is invertible, it holds that $\|I_n + A_{1,k}^\top E^\top\| \le \mathcal{O}(1)$, for all k. Again from (4.15), (4.20)–(4.21) and (4.3), it follows that

$$\left\|H_{1,k}E - Z_2 Z_1^{-1}\right\| \le \mathcal{O}(\rho(J_s^{2^k})) + \mathcal{O}(2^{-k}),$$

as $k \to \infty$. □

Remark 4.2: In Theorem 4.4, we assume that the sequence $\{A_{1,k}, G_{1,k}, H_{1,k}\}$ is well-defined (or the SDA does not break down). How to guarantee the existence of the sequence is still an open problem and is under investigation.

5. Numerical Results.

In this Section, we test the Structure-preserving Algorithm (SA) (Algorithm 3.3) for GARE (1.1) on two numerical examples of [12, 13] and [16] under Assumptions 4.1 or 4.2, to illustrate the convergence behavior. All computations were performed in MATLAB R2008a on a PC with IEEE double-precision floating-point arithmetic (eps$\approx 2.22 \times 10^{-16}$).

Example 5.1 [12, 13] Given

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -9 & -6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}, E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}, J = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, J' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The Kronecker structure of $(\mathcal{H}_a, \mathcal{E}_a)$ is

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} \pm 1.414i & 1 \\ 0 & \pm 1.414i \end{bmatrix} \oplus I_8,\ I_8 \oplus 0_5 \oplus \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right).$$

We choose $\gamma = 9$ to transform $(\mathcal{H}_a, \mathcal{E}_a)$ to $(\mathcal{M}, \mathcal{L})$ as in (2.4). More details in finding a γ by Fibonacci sequence, so that the condition numbers of A_γ and W_γ in (2.2) are as small as possible, can be found in [6].

The corresponding Kronecker structure of $(\mathcal{M}, \mathcal{L})$ becomes

$$\left(\begin{bmatrix} -1.25 & 0 \\ 0 & -0.8 \end{bmatrix} \oplus \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \oplus \begin{bmatrix} -0.952 \pm 0.3067i & 1 \\ 0 & -0.952 \pm 0.3067i \end{bmatrix} \oplus I_5 \oplus \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},\ I_{16}\right).$$

The related quantities in (4.11) are given by $n = 6,\ m = 2,\ e \equiv \text{nullity}(E) = 1,\ f = 1,\ s = 1,\ r = 3,\ \mu = 0$. We compute $\eta_1 = \eta_2 = 3,\ \eta_0 = 1$ and $\eta_0 + \eta_1 = e + m + f = m' = 4$ which coincides with the case (i) in Theorem 3.3. We check that Assumption 4.1 holds, as

$$\text{rank}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = 7 < n + m - \mu = 8,\ \text{nullity}\begin{bmatrix} AV_0 & B & E \\ CV_0 & D & 0 \end{bmatrix} = e + 1 = 2.$$

The SDA (Algorithm 3.2) converges to $X_1 \equiv H_{1,9}E$ in 9 iterations. Using Algorithm 3.3 we get

$$X_a = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 8.0 & 0 & 4.0 & -1.0e-7 & 0 & 0 \\ 0 & 0 & 4.0 & 0 & 2.0 & -5.0e-8 & 0 & 0 \\ 0 & 0 & -1.0e-7 & 0 & -5.0e-8 & -1.5e-7 & 0 & 0 \\ 0 & -0.43 & -1.70 & -0.64 & -0.64 & 0.11 & 0 & -1.38 \\ 0 & 0.50 & -1.74 & -1.12 & -1.12 & 0.93 & 1 & -3.12 \\ 0 & -1.12 & -0.12 & 0.50 & 0.50 & -0.80 & 0 & 0.81 \end{bmatrix}$$

satisfying

$$\begin{aligned} \|E_a^\top X_a - X_a^\top E_a\|_2 &= 1.47 \times 10^{-15}, \\ \text{Res} \equiv \|A_a^\top X_a + X_a^\top A_a + H_a - X_a^\top G_a X_a\|_2 &= 4.71 \times 10^{-14}, \\ \text{Rel\_Res} \equiv \text{Res}/\left(2\|A_a^\top X_a\|_2 + \|X_a^\top G_a X_a\|_2 + \|H_a\|_2\right) &= 9.12 \times 10^{-16}. \end{aligned}$$

Example 5.2 [16] Given

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 500 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -1 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^\top, D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$E = \begin{bmatrix} -1 & -1 & 0.005 & -0.005 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.005 & -0.005 & 0 & 0 & 0 & 0 \\ -0.001 & 0 & 0 & -0.25 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 0 & 0.1 & 0 & 0 & 0 \\ 0 & 0 & 0.25 & -0.25 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.75 & 0 & 0.1 & -0.2 & -0.2 \end{bmatrix}, J = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, J' = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The Kronecker structure of $(\mathcal{H}_a, \mathcal{E}_a)$ is

$$\left(0_2 \oplus \begin{bmatrix} \pm 1 & & & \\ & \pm 4.998 & & \\ & & \pm 3.79 & \\ & & & \pm 0.211 \end{bmatrix} \oplus I_{10},\ I_{10} \oplus 0_7 \oplus \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right).$$

We choose $\gamma = 9$ to transform $(\mathcal{H}_a, \mathcal{E}_a)$ to $(\mathcal{M}, \mathcal{L})$. The corresponding Kronecker structure of $(\mathcal{M}, \mathcal{L})$ becomes

$$\left(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \oplus \begin{bmatrix} -0.8 & & & \\ & -0.407 & & \\ & & -0.954 & \\ & & & 0.996 \end{bmatrix} \oplus \begin{bmatrix} -1.25 & & & \\ & -2.45 & & \\ & & -1.05 & \\ & & & 1.004 \end{bmatrix} \oplus I_7 \oplus \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},\ I_{20} \right).$$

The related quantities in (4.11) are given by $n = 8,\ m = 2,\ e \equiv \text{nullity}(E) = 2,\ f = 1,\ s = 4,\ r = 0,\ \mu = 1$. We compute $\eta_1 = \eta_2 = 4,\ \eta_0 = 1$ and $\eta_0 + \eta_1 = e + m + f = m' = 5$ which coincides with the case (i) in Theorem 3.3. To check Assumption 4.1, we have

$$\text{rank} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = 9 < n + m - \mu = 10,\ \text{nullity} \begin{bmatrix} AV_0 & B & E \\ CV_0 & D & 0 \end{bmatrix} = e + 1 = 3.$$

The SDA (Algorithm 3.2) converges to $X_1 \equiv H_{1,16}E$ in 16 iterations. Then using

Algorithm 3.3 we get

$$X_a = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.01 & -4.9e-4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.65 & -1.2 & 0.034 & -0.52 & -7.1e-4 & 0 & 0.037 & -0.038 & -1.24 & 0.72 \\ 0.26 & -0.17 & 0.16 & -0.58 & -3.4e-3 & 0 & 0.88 & -0.88 & -0.17 & -0.42 \\ 0 & 0 & 0.5 & 0.025 & -0.01 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.47 & 0.025 & -0.01 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.79 & 1.1 & -0.14 & 1.6 & 2.9e-3 & 0 & -0.28 & 0.27 & 0.15 & 0.51 \\ -0.37 & -1.3 & -0.14 & -2.0 & 2.9e-3 & 0 & -0.031 & 0.031 & -1.3 & 0.35 \end{bmatrix}$$

which satisfies

$$\|E_a^\top X_a - X_a^\top E_a\|_2 = 1.37 \times 10^{-16}, \ \mathrm{Res} = 5.09 \times 10^{-14}, \ \mathrm{Rel\_Res} = 2.99 \times 10^{-15}.$$

6. Conclusions. In this paper, we propose a structure-preserving algorithm (SDA+post-process procedure) for a semi-stabilizing solution for GARE (1.1). Under Assumptions 4.1 or 4.2, in Theorem 4.4 we prove that the SDA algorithm converges globally and linearly provided that it does not break down. The advantage of SDA algorithm is evident in that the E-symmetric solution $X_1 \equiv H_{1,\infty}E$ with $H_{1,\infty}$ being symmetric is obtained by a structure-preserving doubling iterative process without performing any preprocess for deflating the associated unimodular eigenvalues. The normalized residuals of desired E_a-symmetric solution X_a for the tested examples computed by the structure-preserving algorithm are accurate to the machine accuracy.

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