Invertible Lattices

Esther Beneish[(1)] and Ming-chang Kang[(2)]

[(1)]Department of Mathematics, University of Wisconsin-Parkside, Kenosha, WI, USA, E-mail: beneish@uwp.edu

[(2)]Department of Mathematics, National Taiwan University, Taipei, Taiwan, E-mail: kang@math.ntu.edu.tw

Abstract. Theorem. Let π be a finite group of order n, R be a Dedekind domain satisfying that (i) char $R = 0$, (ii) every prime divisor of n is not invertible in R, and (iii) p is unramified in R for any prime divisor p of n. Then all the flabby (resp. coflabby) $R\pi$-lattices are invertible if and only if all the Sylow subgroups of π are cyclic. The above theorem was proved by Endo and Miyata when $R = \mathbb{Z}$ [EM, Theorem 1.5]. As applications of this theorem, we give a short proof and a partial generalization of a result of Torrecillas and Weigel [TW, Theorem A], which was proved using cohomological Mackey functors.

2010 *Mathematics Subject Classification.* 11R33, 20C10, 20J06, 14E08.

Keywords and phrases. Integral representations, algebraic tori, permutation lattices, normal domains, cohomological Mackey functors.

§1. Introduction

Let π be a finite group, R be a Dedekind domain (i.e. a commutative noetherian integral domain which is integrally closed with Krull dimension one). Denote by $R\pi$ the group ring of π over R. An $R\pi$-lattice M is a finitely generated left $R\pi$-module which is a torsion-free R-module when regarded as an R-module [CR, page 524]. $R\pi$-lattices play an important role in the modular representation theory of the group π [CR, Section 18]. They arose, when $R = \mathbb{Z}$, in the study of Noether's problem and in the birational classification of algebraic tori [Sw2; EM; Vo; CTS].

Before discussing the main results, we recall some definitions.

Definition 1.1 Let M be an $R\pi$-lattice where R is a Dedekind domain and π is a finite group. M is called a permutation lattice if it is an R-free $R\pi$-module with an R-free basis permuted by π; explicitly, $M = \bigoplus_{1\le i\le m} R\cdot x_i$ and $\sigma\cdot x_i = x_j$ for all $\sigma \in \pi$, for all $1 \le i \le m$ (note that j depends on σ and i). An $R\pi$-lattice M is called an invertible lattice if, as an $R\pi$-module, it is a direct summand of some permutation $R\pi$-lattice. An $R\pi$-lattice M is called a flabby (or flasque) lattice if $H^{-1}(\pi', M) = 0$ for all subgroups π' of π [Sw2, Section 8; CTS; Be, page 103] where $H^{-1}(\pi', M)$ denotes the Tate cohomology [Be, page 102]. Similarly, M is called a coflabby (or coflasque) lattice if $H^1(\pi', M) = 0$ for all subgroups π' of π. Clearly, "permutation" ⇒ "invertible" ⇒ "flabby" and "coflabby" [Sw2, Lemma 8.4].

Definition 1.2 Let p be a prime number and R be a Dedekind domain with char $R = 0$. We call p is unramified in R if p is not invertible in R and the principal ideal pR is an intersection of some maximal ideals of R.

In [EM; CTS], many interesting results about $\mathbb{Z}\pi$-lattices were obtained. Here is one sample of them.

Theorem 1.3 (Endo and Miyata [EM, Theorem 1.5]) *Let π be a finite group, $I_{\mathbb{Z}\pi} := \operatorname{Ker}\{\varepsilon : \mathbb{Z}\pi \to \mathbb{Z}\}$ be the augmentation ideal of $\mathbb{Z}\pi$, $I^0_{\mathbb{Z}\pi} := \operatorname{Hom}_{\mathbb{Z}}(I_{\mathbb{Z}\pi}, \mathbb{Z})$ be the dual $\mathbb{Z}\pi$-lattice of $I_{\mathbb{Z}\pi}$. Then the following statements are equivalent,*

(1) *All the flabby (resp. coflabby) $\mathbb{Z}\pi$-lattices are invertible;*

(2) *$[I^0_{\mathbb{Z}\pi}]^{fl}$ is invertible;*

(3) *All the Sylow subgroups of π are cyclic.*

(The definition of $[M]^{fl}$ for an $R\pi$-lattice M can be found in Definition 2.2.)

One of the main results of this paper is to generalize the above theorem for $\mathbb{Z}\pi$-lattices to the case of $R\pi$-lattices for some "nice" Dedekind domain R. We remark that many results for $\mathbb{Z}\pi$-lattices in [EM; CTS] may be extended readily to the category of $R\pi$-lattices where R is a Dedekind domain such that char $R = 0$ and every prime divisor of $|\pi|$ is not invertible in R. However, in some situations, more delicate conditions of R are required. It is the case for the following theorem.

Theorem 1.4 *Let π be a finite group of order n, R be a Dedekind domain satisfying that* (i) $\mathrm{char}\, R = 0$, (ii) *every prime divisor of n is not invertible in R, and* (iii) *p is unramified in R for any prime divisor p of n. Then the following statements are equivalent,*

(1) *All the flabby (resp. coflabby) $R\pi$-lattices are invertible;*

(2) *$[I^0_{R\pi}]^{fl}$ is invertible where $I_{R\pi} = \mathrm{Ker}\{\varepsilon : R\pi \to R\}$ is the augmentation ideal of $R\pi$, and $I^0_{R\pi} = \mathrm{Hom}_R(I_{R\pi}, R)$ is the dual lattice of $I_{R\pi}$;*

(3) *All the Sylow subgroups of π are cyclic.*

Besides the standard method in [EM; Sw3], the crux of the proof of the above theorem is Theorem 3.3 which provides a sufficient condition to ensure $R_1 \otimes_{R_0} R_2$ is a normal domain when R_0, R_1, R_2 are normal domains.

We thank Prof. Shizuo Endo who communicated to us with two examples, Example 4.3 and Example 4.4, which showed that, without the assumption of unramifiedness on the Dedekind domain R, all of Theorem 1.4, Theorem 3.3 and Lemma 4.1 would collapse.

Two applications of Theorem 1.4 will be given. The first application is a short proof of the following theorem.

Theorem 1.5 (Torrecillas and Weigel [TW, Theorem A and Corollary 6.7]) *Let π be a cyclic p-group and R be a DVR such that* $\mathrm{char}\, R = 0$ *and pR is the maximal ideal of R. Let M be an $R\pi$-lattice. Then the following statements are equivalent,*

(1) *M is a permutation $R\pi$-lattice,*

(2) *M is a coflabby $R\pi$-lattice,*

(3) *M is a flabby $R\pi$-lattice.*

The second application of Theorem 1.4 is to determine $F_{R\pi}$ when π is a cyclic group (for $F_{R\pi}$, see Definition 2.1). Consequently, a partial generalization of Theorem 1.5 is obtained if π is a cyclic group and R is some semilocal "nice" Dedekind domain (see Theorem 5.4).

We indicate briefly how to deduce Theorem 1.5 from Theorem 1.4. First rewrite Theorem 1.5 as follows.

Theorem 1.6 *Let π, R be the same as in Theorem 1.5. Then* (1), (2), (3) *and* (4) *are equivalent where* (4) *is*

(4) *M is an invertible $R\pi$-lattice.*

In fact, (2) $\Leftrightarrow$ (4) (resp. (3) $\Leftrightarrow$ (4)) follows from Theorem 1.4. As to (1) $\Leftrightarrow$ (4), it follows from the following theorem in [Be].

Theorem 1.7 (Beneish [Be, Theorem 2.1]) *Let π be a p-group and R be a DVR such that* $\operatorname{char} R = 0$ *and p is not invertible in R. If M is an invertible $R\pi$-lattice, then it is a permutation $R\pi$-lattice.*

Note that Theorem 1.7 is implicit in the proof of [EM, Theorem 3.2]. We also note that Part (3) of Theorem 1.5 is different from Part (iii) of [TW, Theorem A], but they are equivalent: From the definition of the Tate cohomology groups, we have an exact sequence $0 \to H^{-1}(\pi, M) \to H_0(\pi, M) \to H^0(\pi, M) \to \hat{H}^0(\pi, M) \to 0$. Since the module M_π in [TW, Theorem A] is nothing but $H_0(\pi, M)$, and $nH^{-1}(\pi, M) = 0$ (due to the restriction and corestriction composition), it follows that M is a flabby $R\pi$-lattice if and only if $M_{\pi'}$ is R-torsion free for any subgroup π' of π.

We remark that [TW, Theorem C] follows also from Theorem 1.4, because we may take a flabby resolution $0 \to M^0 \to P \to E \to 0$ (see Definition 2.2) and apply Theorem 1.4 to E. Then take the dual of this exact sequence.

This paper is organized as follows. In Section 2 we recall the definition of flabby resolutions and flabby class monoids. In Section 3 we prove that $R[X]/\langle \Phi_n(X) \rangle$ is a Dedekind domain when R is a "nice" Dedekind domain. The proof of Theorem 1.4 is provided in Section 4 following that of [EM; Sw3]. Section 5 contains a computation of the flabby class group $F_{R\pi}$ when π is a cyclic group, which generalizes some part of a theorem of Endo and Miyata [EM, Theorem 3.3; Sw2, Theorem 2.10]. Then a partial generalization of Theorem 1.5 is given; see Theorem 5.4.

Terminology and notations. All the groups in this paper are finite groups. We will denote by C_n the cyclic group of order n. A commutative noetherian integral domain R is called a DVR if it is a discrete rank-one valuation ring. We denote by $R[X]$ the polynomial ring of one variable over R. $\Phi_m(X)$ denotes the m-th cyclotomic polynomial, and ζ_n denotes a primitive n-th root of unity. We denote by $R\pi$ the group ring of the finite group π over the ring R. If M is an $R\pi$-lattice, then M^0 denotes its dual lattice, i.e. $M^0 = Hom_R(M, R)$; note that there is a natural action of π on M^0 from the left [Sw2, page 31]. For emphasis, we remind the reader that the definition that p is unramified in a Dedekind domain R is given in Definition 1.2.

§2. Preliminaries

From now on till the end of this paper, when we talk about the group ring $R\pi$, we always assume that π is a finite group of order n.

Let M be an $R\pi$-module. The cohomology groups $H^q(G, M)$ and the homology groups $H_q(G, M)$ can be defined via the derived functors $\operatorname{Ext}^q_{R\pi}(R, M)$ and $\operatorname{Tor}_q^{R\pi}(R, M)$; the Tate cohomology groups may be defined by the usual way [Be, page 102]. When $q \geq 1$, $H^q(G, M)$ may be defined also by the bar resolution [Se, Chapters 7 and 8;Ev].

Consider the category of $R\pi$-lattices. Most results in [EM] and [CTS, Section 1] remain valid when we replace $\mathbb{Z}\pi$ by $R\pi$ where R is a Dedekind domain such that $\operatorname{char} R = 0$ and every prime divisor of $|\pi|$ is not invertible in R. In particular, the definitions of flabby class monoids $F_{R\pi}$ and flabby resolutions may be adapted to the case of $R\pi$-lattices as follows.

Definition 2.1 ([Sw3, Definition 2.6]) Let π be a finite group of order n, R be a Dedekind domain such that $\operatorname{char} R = 0$ and every prime divisor of n is not invertible in R. In the category of flabby $R\pi$-lattices, we define an equivalence relation "$\sim$": Two flabby $R\pi$-lattices M_1 and M_2 are equivalent, denoted by $M_1 \sim M_2$, if and only if $M_1 \oplus P_1 \simeq M_2 \oplus P_2$ for some permutation lattices P_1 and P_2. Let $F_{R\pi}$ be the set of all such equivalence classes. It is a monoid under direct sum. $F_{R\pi}$ is called the flabby class monoid of π. The equivalence class containing a flabby lattice M is denoted by $[M]$.

We will say that $[M]$ is invertible (resp. permutation) if there is a lattice E such that $M \sim E$ and E is invertible (resp. permutation).

Definition 2.2 Let R and π be the same as in Definition 2.1. For any $R\pi$-lattice M, there is an exact sequence of $R\pi$-lattices $0 \to M \to P \to E \to 0$ where P is a permutation $R\pi$-lattice and E is a flabby $R\pi$-lattice. Such an exact sequence is called a flabby resolution of M [EM, Lemma 1.1; Sw1, Lemma 8.5]. If $0 \to M \to P' \to E' \to 0$ is another flabby resolution of M, it can be shown that $[E] = [E']$ in $F_{R\pi}$. We define $[M]^{fl} = [E] \in F_{R\pi}$ (see [Sw2, Lemma 8.7]).

The following results were proved for $\mathbb{Z}\pi$-lattices in [Sw3]. It is not difficult to see that they remain valid for $R\pi$-lattices.

Lemma 2.3 *Let R and π be the same as in Definition 2.1*

(1) ([Sw3, Lemma 3.1]) *If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of $R\pi$-lattices where M'' is an invertible $R\pi$-lattice, then $[M]^{fl} = [M']^{fl} + [M'']^{fl}$.*

(2) ([Sw3, Lemma 3.3]) *If M is an $R\pi$-lattice which is an invertible lattice over each Sylow subgroup of π, then M is invertible.*

(3) ([Sw3, Corollary 2.5]) *If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of $R\pi$-lattices where M'' is invertible and M' is coflabby, then this exact sequence splits. Similarly, the exact sequence $0 \to M' \to M \to M'' \to 0$ splits if M' is an invertible $R\pi$-lattice and M'' is a flabby $R\pi$-lattice.*

§3. Tensor products of normal domains

The purpose of this section is to find some sufficient conditions to ensure that $R[\zeta_n]$ is a Dedekind domain when R is a Dedekind domain. The problem is reduced to the following: If R_0, R_1, R_2 are normal domains (i.e. commutative noetherian integral domains which are integrally closed), and $R_0 \subset R_1$, $R_0 \subset R_2$, when is the tensor product $R_1 \otimes_{R_0} R_2$ a normal domain?

We recall two fundamental lemmas.

Lemma 3.1 ([Na, page 172, (42.9)]) *Let R be normal domain containing a field k and K be an extension field of k. Suppose that K is separably generated over k and $K \otimes_k R$ is an integral domain. Then $K \otimes_k R$ is a normal domain.*

Lemma 3.2 ([Na, page 173, (42.12)]) *Let R_1 and R_2 be normal domains containing a DVR which is designated as R_0. Denote by u a prime element of R_0. Assume that* (i) *$R_1 \otimes_{R_0} R_2$ is a noetherian integral domain,* (ii) *both R_1 and R_2 are separably generated over R_0, and* (iii) *for any prime divisor Q of uR_1, $u \cdot (R_1)_Q = Q \cdot (R_1)_Q$ and R_1/Q is separably generated over R_0/uR_0. Then $R_1 \otimes_{R_0} R_2$ is a normal domain.*

Remark. According to [Na, page 146], if R_0 is a subring of a commutative integral domain R with k, K being the quotient fields of R_0, R respectively, we say that R is separably generated over R_0, if (i) $\operatorname{char} R = 0$, or (ii) $\operatorname{char} R = p > 0$ and $K \otimes_k k^{1/p}$ is an integral domain. Consequently, if $\operatorname{char} k = 0$ or $k = \mathbb{F}_q$ is a finite field, then R is separably generated over R_0. Note that Lemma 3.1 and Lemma 3.2 are due to Nakai and Nagata respectively; see [Na, page 220].

Theorem 3.3 *Let n be a positive integer and R be a Dedekind domain. Denote by $R[X]$ the polynomial ring over R. Assume that* (i) $\operatorname{char} R = 0$, (ii) *every prime divisor of n is not invertible in R, and* (iii) *p is unramified in R for any prime divisor p of n. Then $R[X]/\langle \Phi_n(X) \rangle$ is a Dedekind domain and $R[X]/\langle \Phi_n(X) \rangle \simeq R[\zeta_n]$.*

Proof. Step 1. Let Ω be an algebraically closed field containing K where K is the quotient field of R. Let ζ_n be a primitive n-th root of unity in Ω. Clearly $\mathbb{Z}[\zeta_n] \otimes_{\mathbb{Z}} R \simeq R[X]/\langle \Phi_n(X) \rangle$ is a one-dimensional noetherian ring. We will show that it is an integral domain.

First we will show that, within Ω, the subfields K and $\mathbb{Q}(\zeta_n)$ are linearly disjoint over $\mathbb{Q}$.

Let $k = K \cap \mathbb{Q}(\zeta_n)$. We will show that $k = \mathbb{Q}$.

Otherwise, $\mathbb{Q} \subsetneq k$. Then there is some prime number p such that p ramifies in k. Since $\mathbb{Q} \subset k \subset \mathbb{Q}(\zeta_n)$, it is necessary that p divides n. By assumptions, p is unramified in R. Thus p is also unramified in k because $\mathbb{Q} \subset k \subset K$. This is a contradiction.

Once we know $\mathbb{Q} = k$, it is easy to see that $\Phi_n(X)$ is irreducible in $K[X]$. Suppose not. Write $\Phi_n(X) = f_1(X) \cdot f_2(X)$ where $f_1(X), f_2(X) \in K[X]$ are monic polynomials and $\deg f_1(x) < \deg \Phi_n(X)$. Since the roots of $f_1(X)$ are primitive n-the roots of unity, it follows that the coefficients of $f_1(X)$ belong to $\mathbb{Q}(\zeta_n)$. Thus these coefficients lie in $K \cap \mathbb{Q}(\zeta_n) = \mathbb{Q}$. Hence $f_1(X) \in \mathbb{Q}[X]$, which is impossible.

We conclude that $\Phi_n(X)$ is irreducible in $K[X]$. Thus $\mathbb{Q}(\zeta_n)$ is linearly disjoint from K over $\mathbb{Q}$ [La, page 49].

It follows that the canonical map $\mathbb{Q}(\zeta_n) \otimes_{\mathbb{Q}} K \to \mathbb{Q}(\zeta_n) \cdot K = K(\zeta_n)$ is an isomorphism [La, page 49]. Thus $\mathbb{Z}[\zeta_n] \otimes_{\mathbb{Z}} R \to R[\zeta_n]$ is also an isomorphism. In particular, $R[X]/\langle \Phi_n(X) \rangle$ is an integral domain.

Step 2. It remains to show that $R[X]/\langle \Phi_n(X) \rangle$ is integrally closed. Remember that $R[X]/\langle \Phi_n(X) \rangle \simeq \mathbb{Z}[\zeta_n] \otimes_{\mathbb{Z}} R$.

For any non-zero prime ideal Q of R, let R_Q be the localization of R at Q. We will show that $R_Q[X]/\langle \Phi_n(X) \rangle$ is integrally closed for all such Q. Because $R[X]/\langle \Phi_n(X) \rangle = \bigcap_Q R_Q[X]/\langle \Phi_n(X) \rangle$, this will show that $R[X]/\langle \Phi_n(X) \rangle$ is integrally closed.

Suppose that $Q \cap \mathbb{Z} \neq 0$ and $Q \cap \mathbb{Z} = \langle q \rangle$ for some prime number q. If q is a divisor of n, then q is unramified in R. Let $S = \mathbb{Z} \backslash \langle q \rangle$ and $\mathbb{Z}_q = S^{-1}\mathbb{Z}$ be the localization of $\mathbb{Z}$ at $\langle q \rangle$. Then $S^{-1}R[X]/\langle \Phi_n(X) \rangle \simeq \mathbb{Z}_q[\zeta_n] \otimes_{\mathbb{Z}_q} (S^{-1}R)$. Apply Lemma 3.2. Note that the assumptions of Lemma 3.2 are fulfilled, e.g. if Q' is a prime divisor of qR, then $S^{-1}R/S^{-1}Q'$ is separably generated over $\mathbb{Z}_q/q\mathbb{Z}_q$ because $\mathbb{Z}_q/q\mathbb{Z}_q \simeq \mathbb{F}_q$ is a finite field. Hence $S^{-1}R[X]/\langle \Phi_n(X) \rangle$ is a normal domain. Since $R_Q[X]/\langle \Phi_n(X) \rangle$ is a localization of $S^{-1}R[X]/\langle \Phi_n(X) \rangle$, it follows that $R_Q[X]/\langle \Phi_n(X) \rangle$ is integrally closed.

Suppose that $Q \cap \mathbb{Z} \neq 0$ and $Q \cap \mathbb{Z} = \langle q \rangle$ for some prime number q such that q is not a divisor of n. Then q is unramified in $\mathbb{Z}[\zeta_n]$. Thus we my apply the same arguments as above and apply Lemma 3.2 to $S^{-1}\mathbb{Z}[\zeta_n] = \mathbb{Z}_q[\zeta_n]$. Hence $R_Q[X]/\langle \Phi_n(X) \rangle$ is also integrally closed.

Suppose that $Q \cap \mathbb{Z} = 0$. Then $\mathbb{Q} \subset R_Q$. Let $T = \mathbb{Z} \backslash \{0\}$. Then $T^{-1}R[X]/\langle \Phi_n(X) \rangle \simeq T^{-1}\mathbb{Z}[\zeta_n] \otimes_{\mathbb{Q}} R_Q = \mathbb{Q}(\zeta_n) \otimes_{\mathbb{Q}} R_Q$. Apply Lemma 3.1. We find that $T^{-1}R[X]/\langle \Phi_n(X) \rangle$ is a normal domain. Hence the result. ■

Remark. We thank Nick Ramsey for pointing out that Proposition 17 of [Se, page 19] provides a special case of Theorem 3.3 : If p is a prime number and R is a DVR with maximal ideal pR, then $R[X]/\langle \Phi_{p^t}(X) \rangle$ is again a DVR where t is any positive integer.

§4. Proof of Theorem 1.4

The following lemma is a generalization of [Sw3, Lemma 4.3].

Lemma 4.1 *Let π be a cyclic p-group of order n. Write $\pi = \langle \sigma \rangle$. Let R be a Dedekind domain such that* $\operatorname{char} R = 0$ *and p is unramified in R. Let M be a finitely generated module over $R\pi/\langle \Phi_n(\sigma) \rangle$ such that M is a torsion-free R-module when it is regarded as an R-module. Then $[M]^{fl}$ is an invertible $R\pi$-lattice.*

Proof. Write $n = pq$ and define $\pi'' = \pi/\langle \sigma^q \rangle$. From the factorization $X^n - 1 = (X^q - 1)\Phi_n(X)$, we get an exact sequence $0 \to R\pi/\langle \Phi_n(\sigma) \rangle \to R\pi \to R\pi'' \to 0$. Note that $R\pi/\langle \Phi_n(\sigma) \rangle \simeq R[X]/\langle \Phi_n(X) \rangle \simeq R[\zeta_n]$ is a Dedekind domain by Theorem 3.3.

This provides a flabby resolution of the $R\pi$-lattice $R\pi/\langle\Phi_n(\sigma)\rangle$. Hence $[R\pi/\langle\Phi_n(\sigma)\rangle]^{fl}$ $= [R\pi''] = 0$.

Since M is torsion-free, M is a projective module over the Dedekind domain $R\pi/\langle\Phi_n(\sigma)\rangle$. Thus we may find another module N satisfying that $M \oplus N \simeq (R\pi/\langle\Phi_n(\sigma)\rangle)^{(t)}$ for some integer t. Thus $[M]^{fl} + [N]^{fl} = t[R\pi/\langle\Phi_n(\sigma)\rangle]^{fl} = 0$. It follows that $[M]^{fl}$ is invertible. ■

Proof of Theorem 1.4.

The proof of Theorem 1.4 is almost the same as that in [EM, Theorem 1.3; Sw2, Theorem 4.4], once Lemma 4.1 is obtained. In order not to commit a blunder mistake, we choose to rewrite the proof once again.

(3) $\Rightarrow$ (1)

Step 1. Assume that all the Sylow subgroups of π are cyclic. Let M be an $R\pi$-lattice which is flabby (resp. coflabby). We will show that M is invertible.

By Lemma 2.3, it suffices to show that M is an invertible $R\pi_p$-lattice where π_p is a p-Sylow subgroup of π and p is a prime divisor of $|\pi|$. Thus we may assume that $\pi = \langle\sigma\rangle$ is a cyclic p-group of order n, without loss of generality.

Step 2. For any $R\pi$-lattice M, we claim that M is flabby if and only if it is coflabby.

Since $\pi = \langle\sigma\rangle$ is cyclic of order n, we find that $H^{-1}(\pi, M) \simeq \operatorname{Ker}\varphi/\langle\sigma v - v : v \in M\rangle$ where $\varphi : M \to M$ is defined by $\varphi(u) = u + \sigma\cdot u + \cdots + \sigma^{n-1}\cdot u$. On the other hand, $H^1(\pi, M) = \operatorname{Ker}\varphi/\langle\sigma\cdot v - v : v \in M\rangle$ by definition. Hence $H^1(\pi, M) \simeq H^{-1}(\pi, M)$. Similarly, for any subgroup $\pi' \subset \pi$, $H^1(\pi', M) \simeq H^{-1}(\pi', M)$. Hence the result.

Step 3. Let M be a flabby $R\pi$-lattice. We will show that M is invertible.

Write $n = pq$ where q is a power of p. Define $M' = \{u \in M : \Phi_n(\sigma)\cdot u = 0\}$, $M'' = M/M'$. Then we have an exact sequence of $R\pi$-lattices $0 \to M' \to M \to M'' \to 0$ where M' is a module over $R\pi/\langle\Phi_n(\sigma)\rangle$ and M'' is a lattice over $R\pi''$ with $\pi'' = \pi/\langle\sigma^q\rangle$.

By Theorem 3.3, $R\pi/\langle\Phi_n(\sigma)\rangle \simeq R[\zeta_n]$ is a Dedekind domain. Thus $[M']^{fl}$ is invertible by Lemma 4.1.

We will show that M'' is a flabby $R\pi''$-lattice. This will be proved in the next step.

Assume the above claim. By induction on $|\pi|$, we find that M'' is invertible. Thus $[M'']^{fl}$ is invertible also. Apply Lemma 2.3. We find that $[M]^{fl} = [M']^{fl} + [M'']^{fl}$ is invertible.

Since $[M]^{fl}$ is invertible, we get a flabby resolution of M, $0 \to M \to P \to E \to 0$ where P is permutation and E is invertible. By Step 1, M is coflabby. Hence the exact sequence $0 \to M \to P \to E \to 0$ splits by Lemma 2.3. We get $P \simeq M \oplus E$. Thus M is invertible.

Step 4. We will show that M'' is a flabby $R\pi''$-lattice.

For any subgroup π' of π, we will show that $H^{-1}(\pi', M'') = 0$.

If $\pi' = \{1\}$, it is clear that $H^{-1}(\pi', M'') = 0$.

Now assume $\pi' \supsetneq \{1\}$. Write $\pi' = \langle \sigma^d \rangle$ with $d \mid n$ and $d \neq n$. Then $M'^{\pi'} := \{u \in M' : \lambda \cdot u = u \text{ for any } \lambda \in \pi'\} = 0$ because, for any $v \in M'^{\pi'}$, $\Phi_n(\sigma) \cdot v = 0$, $(\sigma^d - 1) \cdot v = 0$, and M' is torsion-free. From the exact sequence $0 = H^{-1}(\pi', M) \to H^{-1}(\pi', M'') \to \hat{H}^0(\pi', M') = 0$, we find $H^{-1}(\pi', M'') = 0$.

Now that M'' is flabby as an $R\pi$-lattice, it is flabby as an $R\pi''$-lattice (where $\pi'' = \pi/\langle \sigma^q \rangle$) because every subgroup of π'' may be written as $\pi_1/\langle \sigma^q \rangle$ for some subgroup $\langle \sigma^q \rangle \subset \pi_1$ and $H^1(\pi_1/\langle \sigma^q \rangle, M'') \to H^1(\pi_1, M'')$ is injective by the five-term exact sequence of the Hochschild-Serre's spectral sequence. Done.

(1) $\Rightarrow$ (2) In general, $[I^0_{R\pi}]^{fl}$ is flabby. By (1), it is invertible.

(2) $\Rightarrow$ (3) Let π be a group of order n. Let $I_{R\pi}$ be the augmentation ideal. Then we have an exact sequence $0 \to I_{R\pi} \to R\pi \to R \to 0$. Thus $H^1(\pi, I_{R\pi}) = R/nR$.

If $[I^0_{R\pi}]^{fl}$ is invertible, then we have an exact sequence $0 \to I^0_{R\pi} \to P \to E \to 0$ where P is permutation and E is invertible. Taking the dual of each lattice, we get $0 \to E^0 \to P^0 \to I_{R\pi} \to 0$. Note that P^0 is also permutation and E^0 is invertible. Moreover, we have $0 = H^1(\pi, P^0) \to H^1(\pi, I_{R\pi}) \to H^2(\pi, E^0)$. Thus there is an embedding (i.e. an injective map of R-modules) $0 \to R/nR \to H^2(\pi, E^0)$.

Write $E^0 \oplus E' = Q$ where Q is some permutative $R\pi$-lattice. It follows that there is also an embedding $0 \to R/nR \to H^2(\pi, Q)$.

Write $Q = \bigoplus_i R\pi/\pi_i$ where π_i's are subgroups of π. Then $H^2(\pi, Q) = \bigoplus_i H^2(\pi, R\pi/\pi_i) \simeq \bigoplus_i H^2(\pi_i, R)$.

Since p is unramified in R, choose a prime ideal P containing pR. Let R_P be the localization of R at P. Consider $R_P \otimes Q$. In other words, we may assume that R is a DVR with maximal ideal pR. We will show that all the Sylow subgroups of π are cyclic. Let p be a prime divisor of n $(:= |\pi|)$. Write $n = p^t n'$ with $t \geq 1$ and $p \nmid n'$.

Since $0 \to R/nR \to H^2(\pi, Q) \simeq \bigoplus_i H^2(\pi_i, R)$ and $R/nR \simeq R/p^tR \oplus R/n'R$, it follows that there is an embedding $0 \to R/p^tR \to H^2(\pi_i, R)$ for some i, because R/p^tR is an indecomposable R-module.

Thus the proof is finished by the following lemma. ■

Lemma 4.2 ([Sw3, Lemma 4.5]) *Let π be a finite group of order n, p be a prime divisor of n. Let R be a DVR such that* $\operatorname{char} R = 0$ *and the maximal ideal of R is pR. Write $n = p^t n'$ where $t \geq 1$ and $p \nmid n'$. If there is an embedding $0 \to R/p^tR \to H^2(\pi, R)$, then the p-Sylow subgroup of π is cyclic of order p^t.*

Proof. From the exact sequence $0 \to R \xrightarrow{p^t} R \to R/p^tR \to 0$ of $R\pi$-modules, we get $0 = H^1(\pi, R) \to H^1(\pi, R/p^tR) \to H^2(\pi, R) \xrightarrow{p^t} H^2(\pi, R)$, it follows that there is an embedding $0 \to R/p^tR \to H^1(\pi, R/p^tR)$. Since $H^1(\pi, R/p^tR) \simeq \operatorname{Hom}(\pi, R/p^tR)$, there is a group homomorphism $f : \pi \to R/p^tR$ such that the annihilator $\operatorname{Ann}_R(f) = p^tR$ (here $\operatorname{Hom}(\pi, R/p^tR)$ is regarded as an R-module). Hence π contains an element of order p^t. ■

The following two examples are due to Shizuo Endo.

Example 4.3 Let p be an odd prime number and $R = \mathbb{Z}[\zeta_p]$. Write $\zeta = \zeta_p$. Let $\pi = \langle\sigma\rangle \simeq C_p$ be the cyclic group of order p. Then p is ramified in R; in fact, $pR = (1-\zeta)^{p-1}$.

Let $M = R \cdot u$ be the cyclic $R\pi$-lattice defined by $\sigma \cdot u = \zeta u$. Taking a flabby resolution of M^0 and then taking the dual, we obtain an exact sequence of $R\pi$-lattices $0 \to E \to P \to M \to 0$ where P is a permutation lattice and E is a coflabby lattice (and also a flabby lattice by the periodicity of cohomolgy groups). We will show that E is not an invertible lattice.

It is easy to show that $H^{-1}(\pi, M) = R/(1-\zeta)$, and $\hat{H}^0(\pi, Q) = (R/(1-\zeta)^{p-1})^{(n)}$ if $Q = R^{(n)} \oplus (R\pi)^{(n')}$ is a permutation lattice.

Suppose that E is an invertible lattice, then $\hat{H}^0(\pi, E)$ is a direct summand of $(R/(1-\zeta)^{p-1})^{(m)}$ for some integer m. Since the module $(R/(1-\zeta)^{p-1})^{(m)}$ satisfies the ascending chain condition and the descending chain condition, the Krull-Schmidt-Azumaya Theorem may be applied to it [CR, page 128, Theorem 6.12]. Hence $\hat{H}^0(\pi, E) \simeq (R/(1-\zeta)^{p-1})^{(m')}$ for some integer m'.

From the exact sequence $0 \to E \to P \to M \to 0$, we get an exact sequence of R-modules $0 \to H^{-1}(\pi, M) \to \hat{H}^0(\pi, E) \to \hat{H}^0(\pi, P) \to 0$, i.e. an exact sequence $0 \to R/(1-\zeta) \to (R/(1-\zeta)^{p-1})^{(m')} \to (R/(1-\zeta)^{p-1})^{(n)} \to 0$. Counting the lengths of these modules, we find a contradiction.

For the case $p = 2$, let $R = \mathbb{Z}[\sqrt{-1}]$ and $\pi = \langle\sigma\rangle \simeq C_2$ be the cyclic group of order 2. Let $M = R \cdot u$ be the cyclic $R\pi$-lattice defined by $\sigma \cdot u = -u$. We can find a flabby $R\pi$-lattice E which is not invertible as before.

More generally, let π be a group of order n such that there is a quotient group π' of π with $|\pi'| = p$ where p is a prime divisor of n (for example, π is a solvable group). Suppose that R is a Dedekind domain such that (i) $\operatorname{char} R = 0$, (ii) every prime divisor of n is not invertible in R, and (iii) $\zeta_p \in R$ if p is odd (resp. $\sqrt{-1} \in R$ if $p = 2$). Then we may find a flabby, but not invertible $R\pi'$-lattice E as above. As an $R\pi$-lattice, E is flabby and is not invertible (see, for example, [CTS, page 180, Lemma 2]).

If π is a finite group such that all the Sylow subgroups of π are cyclic, it is known that π is solvable [Is, page 160, Corollary 5.15]. Thus the assumption of unramifiedness in Theorem 1.4 is indispensable.

Example 4.4 Let p be an odd prime number. Let $\pi = \langle\sigma\rangle \simeq C_p$, and $R = \mathbb{Z}[\sqrt{-p}]$ if $p \equiv 1 \pmod 4$, $R = \mathbb{Z}[\sqrt{p}]$ if $p \equiv 3 \pmod 4$. Then R is a Dedekind domain in which p ramifies.

Note that R and $\mathbb{Z}[\zeta_p]$ are linearly disjoint over $\mathbb{Z}$, because the unique quadratic subfield of $\mathbb{Q}(\zeta_p)$ is $\mathbb{Q}(\sqrt{p})$ if $p \equiv 1 \pmod 4$ (resp. $\mathbb{Q}(\sqrt{-p})$ if $p \equiv 3$). Thus $R[X]/\langle\Phi_p(X)\rangle$ is an integral domain and we will write it as $R[\zeta_p]$.

We will show that there is a flabby $R\pi$-lattice which is not invertible. This shows that Lemma 4.1 fails without the unramifiedness assumption.

To establish the above claim, it suffices to show that, there is some torsion-free $R[\zeta_p]$-module N such that E is not invertible with E defined by the exact sequence of

$R\pi$-lattices $0 \to E \to P \to N \to 0$ where E is a coflabby lattice, P is a permutation lattice.

Suppose not. We will find a contradiction. The strategy is to show that the homological dimension of N is ≤ 1 (as a torsion-free $R[\zeta_p]$-module).

Let Q be the prime ideal of R with $pR = Q^2$. Denote by R_Q the localization of R at Q; write $E_Q = R_Q \otimes_R E$. Since we assume that E is an invertible $R\pi$-lattice, E_Q becomes a permutation $R_Q\pi$-lattice by Theorem 1.7.

Tensoring $R_Q\pi/\langle\Phi_p(\sigma)\rangle$ with the exact sequence $0 \to E_Q \to P_Q \to N_Q \to 0$ over $R_Q\pi$, we get $Tor_1^{R_Q\pi}(R_Q[\zeta_p], N_Q) \to E_Q/\Phi_p(\sigma)E_Q \to P_Q/\Phi_p(\sigma)P_Q \to N_Q \to 0$. Since $Tor_1^{R_Q\pi}(R_Q[\zeta_p], N_Q)$ is a torsion group, we may apply [Sw3, Lemma 5.4] and we get an exact sequence of $R_Q[\zeta_p]$-modules $0 \to (E_Q/\Phi_p(\sigma)E_Q)_0 \to (P_Q/\Phi_p(\sigma)P_Q)_0 \to N_Q \to 0$ where, for an $R_Q[\zeta_p]$-module L, L_0 is defined as $L/\{\text{torsion elements}\}$. This exact sequence is just the same as $0 \to (R_Q[\zeta_p])^{(n')} \to (R_Q[\zeta_p])^{(n)} \to N_Q \to 0$. In summary, as an $R_Q[\zeta_p]$-module, the homological dimension of N_Q is ≤ 1.

On the other hand, if Q' is a prime ideal of R with $p \notin Q'$, then p is invertible in $R_{Q'}$ and $R_{Q'}\pi$ is a maximal order. Hence $N_{Q'}$ is a projective module over $R_{Q'}\pi$. Apply the same arguments as above. We find that, as an $R_{Q'}[\zeta_p]$-module, $N_{Q'}$ is projective.

We conclude that, as an $R[\zeta_p]$-module, the homological dimension of N is ≤ 1.

Since N is an arbitrary torsion-free $R[\zeta_p]$-module, it follows that the homological dimension of any finitely generated $R[\zeta_p]$-module is ≤ 2. Hence $R[\zeta_p]$ is a regular domain by Serre's Theorem [Na, page 101, Exercise 2]; thus it is a Dedekind domain.

We will see that, in fact, $R[\zeta_p]$ is not a Dedekind domain. By this, we get a contradiction. This also shows that, without the unramifiedness assumption, Theorem 3.3 would break down.

Let K be the quotient field of R. Since $K(\zeta_p)$ contains $\mathbb{Q}(\zeta_p)$ and R, it follows that $\sqrt{-1} \in K(\zeta_p)$. It is not difficult to show that $K(\zeta_p) = \mathbb{Q}(\zeta_{4p})$ and the integral closure of $R[\zeta_p]$ is $\mathbb{Z}[\zeta_{4p}]$. On the other hand, $\sqrt{-1} \in \mathbb{Z}[\zeta_{4p}]$, but $\sqrt{-1} \notin R[\zeta_p] = R \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta_p]$. Thus $R[\zeta_p]$ is not a Dedekind domain.

For the convenience of the reader, we reproduce a proof of Theorem 1.7.

Proof of Theorem 1.7.

Let M be an invertible $R\pi$-lattice. We will show that M is permutation.

Write $M \oplus M' = P$ for some permutation $R\pi$-lattice and some M'.

Denote by $\widehat{R}$ the completion of R at its maximal ideal. In the category of $\widehat{R}\pi$-lattices, $\widehat{R}\pi/\pi'$ is indecomposable for any subgroup π' of π by [CR, page 678, Theorem 32.14] (note that the assumptions of [CR, Theorem 32.11] are satisfied).

Since the Krull-Schmidt-Azumaya Theorem is valid in the category of $\widehat{R}\pi$-lattices [CR, page 128, Theorem 6.12], from $\widehat{R}M \oplus \widehat{R}M' = \widehat{R}P$, we find a permutation $R\pi$-lattice Q such that $\widehat{R}M \simeq \widehat{R}Q$. By [CR, page 627, Proposition (30.17)] we find that $M \simeq Q$. ∎

§5. The flabby class group

Let R be a Dedekind domain. Recall that the class group of R, denoted by $C(R)$, is defined as $C(R) = I(R)/P(R)$ where $I(R)$ is the group of fractional ideals of R and $P(R)$ is the group of principal ideals of R. If J is a fractional ideal of R, $[J]$ denotes the image of J in $C(R)$. The group operation in $C(R)$ is written multiplicatively.

Let R be a Dedekind domain and M be a finitely generated torsion-free R-module. Then M is isomorphic to a direct sum of a free module and a non-zero ideal of R; write $M \simeq R^{(m-1)} \oplus I$ where I is a non-zero ideal of R. Define the Steinitz class of M, denoted by $\mathrm{cl}(M)$, by $\mathrm{cl}(M) = [I]$ (see [CR, page 85]). If M_1 and M_2 are finitely generated torsion-free R-modules, it is not difficult to verify that $\mathrm{cl}(M \oplus N) = \mathrm{cl}(M) \cdot \mathrm{cl}(N)$.

Definition 5.1 Let R be a Dedekind domain, M be a finitely generated R-module. Define $M_0 = M/\{\text{torsion elements in } M\}$.

Theorem 5.2 *Let $\pi = \langle \sigma \rangle$ be a cyclic group of order n and R be a Dedekind domain satisfying that* (i) $\mathrm{char}\, R = 0$, (ii) *every prime divisor of n is not invertible in R, and* (iii) *p is unramified in R for any prime divisor p of n. Define a group homomorphism $c : F_{R\pi} \to \bigoplus_{d|n} C(R\pi/\langle \Phi_n(\sigma) \rangle)$ by $c([M]) = (\ldots, \mathrm{cl}((M/\Phi_d(\sigma)M)_0), \ldots)$ where M is a flabby $R\pi$-lattice. Then c is an isomorphism.*

Remark. By Theorem 1.4, $F_{R\pi}$ is a group. By Theorem 3.3, $R\pi/\langle \Phi_d(\sigma) \rangle \simeq R[X]/\langle \Phi_d(X) \rangle \simeq R[\zeta_d]$ is a Dedekind domain; thus $C(R\pi/\langle \Phi_d(\sigma) \rangle)$ is well-defined.

The proof of Theorem 5.2 follows by the same way as in [Sw3, Sections 5 and 6]. Before giving the proof of it, we recall a key lemma in [Sw3].

Theorem 5.3 *Let π and R be the same as in Theorem 5.2. If M is an invertible $R\pi$-lattice, then*

$$[M]^{fl} = \sum_{d|n} [(M/\Phi_d(\sigma)M)_0]^{fl},$$

$$[(M/\Phi_n(\sigma)M)_0]^{fl} = \sum_{d|n} \mu\left(\frac{n}{d}\right) [M/(\sigma^d - 1)M]^{fl}.$$

Proof. The proof of Theorem 5.1 and Corollary 5.2 in [Sw3, Section 5] works as well in the present situation. The details are omitted. ■

Proof of Theorem 5.2.

First of all, we will show that c is injective. Let M be a flabby $R\pi$-lattice. It is invertible by Theorem 1.4. If $c([M]) = 0$ in $\bigoplus_{d|n} C(R\pi/\langle \Phi_d(\sigma) \rangle)$, then $(M/\Phi_d(\sigma)M)_0$ is a free module over $R\pi/\langle \Phi_d(\sigma) \rangle$ for all $d \mid n$. Since $[R\pi/\langle \Phi_d(\sigma) \rangle]^{fl} = 0$ (by applying Theorem 5.3 with $M = R\pi$), we find $[(M/\Phi_d(\sigma)M)_0]^{fl} = 0$. By Theorem 5.3, we find

$[M]^{fl} = 0$. Thus we have a flabby resolution of M, $0 \to M \to P_1 \to P_2 \to 0$ where P_1 and P_2 are permutation $R\pi$-lattices. Since M is invertible, we may apply Lemma 2.3 to conclude that $P_1 \simeq M \oplus P_2$. Thus $M \sim P_1$ and $[M] = 0$ in $F_{R\pi}$.

It remains to show that c is surjective. We also follow the proof of [Sw3, page 247–248].

Step 1. Let $K_0(R\pi)$ be the Grothendieck group of the category of finitely generated projective $R\pi$-modules. Every such projective module is isomorphic to a direct sum of a free module and a projective ideal $\mathcal{A}$ [Sw1, Theorem A]. Define $C(R\pi)$ as a subgroup of $K_0(R\pi)$ by $C(R\pi) = \{[\mathcal{A}] - [R\pi] \in K_0(R\pi) : \mathcal{A}$ is a projective ideal over $R\pi\}$. The group $C(R\pi)$ is called the locally free class group of $R\pi$ [CR, page 659; EM, page 86].

Step 2. Define a map $c' : C(R\pi) \to F_{R\pi}$ by $c'([\mathcal{A}] - [R\pi]) = [\mathcal{A}] \in F_{R\pi}$. Since $\mathcal{A}$ is a projective ideal over $R\pi$, it is an invertible $R\pi$-lattice; thus c' is well-defined.

We will show that the composition map $c \circ c' : C(R\pi) \to F_{R\pi} \to \bigoplus_{d|n} C(R\pi/\langle\Phi_d(\sigma)\rangle)$ is surjective in the next step. Once it is proved, c is also surjective.

Step 3. We will show that $c \circ c' : C(R\pi) \to F_{R\pi} \to \bigoplus_{d|n} C(R\pi/\langle\Phi_d(\sigma)\rangle)$ is surjective.

Let K be the quotient field of R. Write $\Omega_{R\pi} := \prod_{d|n} R\pi/\langle\Phi_d(\sigma)\rangle$. It is not difficult to verify that $\Omega_{R\pi}$ is the maximal R-order in $K\pi$ containing $R\pi$ [CR, page 559 and page 563]. We may define the locally free class group $C(\Omega_{R\pi})$ as in the case $C(R\pi)$ (see [CR, page 659]). It follows that $C(\Omega_{R\pi}) \simeq \bigoplus_{d|n} C(R\pi/\langle\Phi_d(\sigma)\rangle)$.

The composite map $c \circ c'$ turns out to be $c \circ c'([\mathcal{A}] - [R\pi]) = [\Omega_{R\pi} \otimes_{R\pi} \mathcal{A}] - [\Omega_{R\pi}] \in C(\Omega_{R\pi})$, which is just the natural map $C(R\pi) \to C(\Omega_{R\pi})$ (the map induced by the inclusion map $R\pi \to \Omega_{R\pi}$). Thus the surjectivity of $c \circ c'$ is equivalent to the surjectivity of the map $C(R\pi) \to C(\Omega_{R\pi})$. However, the map $C(R\pi) \to C(\Omega_{R\pi})$ is surjective by [Ri, Corollary 11]; in applying Rim's Theorem, we should verify the fact that $R\pi$ has no nilpotent ideal, which may be see from the embedding $R\pi \hookrightarrow K\pi \simeq \prod_{d|n} K(\zeta_d)$ and hence $R\pi$ has no nilpotent element. This finishes the proof that $c \circ c'$ is surjective.

Alternatively, the reader may show that $C(R\pi) \to C(\Omega_{R\pi})$ is surjective by modifying the proof of [Sw3, Lemma 6.1]. ■

Now we give a partial generalization of Theorem 1.5.

Theorem 5.4 *Let π be a cyclic group of order n, R be a semilocal Dedekind domain satisfying* (i) $\operatorname{char} R = 0$, (ii) *every prime divisor of n is not invertible in R, and* (iii) *p is unramified in R for every prime divisor p of n. Then $F_{R\pi} = \{0\}$ and all the flabby $R\pi$-lattices are stably permutation, i.e. if M is a flabby $R\pi$-lattice, there are permutation $R\pi$-lattices P_1 and P_2 such that $M \oplus P_1 \simeq P_2$.*

Proof. Apply Theorem 5.2. It suffices to show that $C(R\pi/\langle\Phi_d(\sigma)\rangle) = 0$ where $\pi = \langle\sigma\rangle$ is of order n and $d \mid n$. Note that $R\pi/\langle\Phi_d(\sigma)\rangle \simeq R[\zeta_d]$ is a Dedekind domain integral over R. Since R is semilocal, $R[\zeta_d]$ is also semilocal. Thus $R[\zeta_d]$ is a principal ideal domain and $C(R[\zeta_d]) = 0$. Thus $F_{R\pi} = \{0\}$.

If M is a flabby $R\pi$-lattice, from $[M] \in F_{R\pi} = \{0\}$, we find that $[M] = 0$, i.e. $M \sim 0$ which is equivalent to that M is stably permutation. ■

Remark. In the above theorem, if R is a local ring, then this theorem is reduced to Theorem 1.5. On the other hand, if R is not a local ring, it is necessary that R is not a complete semilocal ring because we assume R is an integral domain (note that a complete semilocal ring is isomorphic to a direct product of complete local rings). In such a situation, it is impossible to strengthen the above result about stable permutation as the following example shows (also see [Dr, page 273]).

Let π be a finite group which is not a p-group. Let R be the same in Theorem 5.4 and, by abusing the notation, denote R the $R\pi$-lattice on which π acts trivially.

For each prime divisor p of $\mid \pi \mid$, choose a p-Sylow subgroup π_p. Write the coset decomposition $\pi = \cup_i \sigma_i^{(p)} \pi_p$. Define a morphism $\phi_p : R \to R\pi/\pi_p$ by $\phi_p(1) = \sum_i \sigma_i^{(p)} \in R\pi/\pi_p$. Find integers a_p such that $\sum_p a_p[\pi : \pi_p] = 1$.

Define a permutation $R\pi$-lattice $P = \oplus_p R\pi/\pi_p$. Consider the morphisms $\phi : R \to P$ and $\varepsilon : P \to R$ where ε is the augmentation map and $\phi(1) = (\ldots, a_p\phi_p(1), \ldots)$. Since $\varepsilon\phi = id$, it follows that $P = R \oplus E$ for some $R\pi$-lattice E. Thus E is stably permutation.

But E is not a permutation lattice in general. For examples, consider the special case when π is a group of order pq and p, q are distinct odd prime numbers. Suppose E is a permutation lattice. From $H^0(\pi, P) = H^0(\pi, R) \oplus H^0(\pi, E)$ (the usual cohomology groups), we find that $E = R\pi/\pi'$ for some subgroup π'. Counting the R-ranks of $P = R \oplus E$, we find that $[\pi : \pi'] = p + q - 1$, which is impossible.

In general, if π is any finite group admitting a surjective group homomorphism $\pi \to \pi_0$ where π_0 is a group of order pq and p, q are distinct odd prime numbers, the lattice E constructed through $R\pi_0$ is a stably permutation $R\pi$-lattice, but is not a permutation lattice.

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