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Motivic stable homotopy and the stable 51 and 52 stems

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ABSTRACT

We establish a differential $d_2(D_1) = h_0^2 h_3 g_2$ in the 51-stem of the Adams spectral sequence at the prime 2, which gives the first correct calculation of the stable 51 and 52 stems. This differential is remarkable since we know of no way to prove it without recourse to the motivic Adams spectral sequence. It is the last undetermined differential in the range of the first author's detailed calculations of the n-stems for $n < 60$ [6]. This note advertises the use of the motivic Adams spectral sequence to obtain information about classical stable homotopy groups.

1. Introduction

The problem of computing the stable homotopy groups of spheres is of fundamental importance in algebraic topology. Although this subject has been studied for a very long time, the Adams spectral sequence is still the best way to do stemwise computations at the prime 2.

Bruner produced a computer-generated table of d_2 differentials in the Adams spectral sequence [4]. He started with his theorem on the interaction between Adams differentials and squaring operations [3] to obtain several values of the d_2 differential. Then he methodically exploited all (primary) multiplicative relations, both forwards and backwards, to deduce more and more values of the d_2 differential. Through at least the 80-stem, this procedure gives the vast majority of the values of the d_2 differential.

In Bruner's approach, the first unknown value is $d_2(D_1)$, where D_1 is a certain element in the 52-stem. This gives a precise sense to the claim that $d_2(D_1)$ is "harder" than any previous d_2 differential.

Mark Mahowald communicated (unpublished) the following argument for the presence of the differential $d_2(D_1) = h_0^2 h_3 g_2$. First, there is an element x in the 37-stem that detects $\sigma\theta_4$ [2, Proposition 3.5.1]. Since

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$h_3^2 x = h_0^2 h_3 g_2$, we deduce that $h_0^2 h_3 g_2$ detects $\sigma^3 \theta_4$. Now $\sigma^3 = \nu\nu_4$, so $h_0^2 h_3 g_2$ detects $\nu\nu_4\theta_4$. The final step is the claim that $\nu_4\theta_4$ is zero.

This final step turns out to be false; $\nu_4\theta_4$ is detected by $h_2 h_5 d_0$ [6, Lemma 4.2.90]. The reason this argument falls apart arose from an incorrect understanding of the Toda bracket $\langle \theta_4, 2, \sigma^2 \rangle$ [6, Lemma 4.2.91] [10].

In this note, we will ratify Mahowald's vision by repairing the hole. The proof relies on motivic calculations in a fundamental way. In other words, our argument does not work if applied in the classical context. We will first establish a Massey product involving the element D_1. Then we will apply the higher Leibniz rule on the interaction of Adams differentials with Massey products [12]. The second author discovered this proof based on motivic calculations by the first author. We adopt notation from [6] without further explanation. For charts of the classical and motivic Adams spectral sequences, see [7].

The higher Leibniz rule of Moss [12] is an appealing result that has not been as useful in practice as one might expect. In applications, indeterminacies often interfere. The differential that we establish here is an unusually clear application of the higher Leibniz rule. Our work suggests that an extension of Bruner's program to methodically exploit all 3-fold Massey products is likely to lead to further new results about Adams d_2 differentials.

The origin of this note lies in the first author's analysis of the Adams spectral sequence through the 59-stem. The first author was able to give careful arguments for every Adams differential, with the sole exception of possible differentials on the element D_1. The second author was instrumental in finishing the last remaining differential, whose argument we present here.

The chief consequence of our Adams differential calculation is the following results about the stable 51 and 52 stems.

Theorem 1.1. *The 2-primary order of the stable 51-stem is 128. As a group, the 2-primary stable 51-stem is either* $\mathbb{Z}/8 \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/2$ *or* $\mathbb{Z}/8 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

Proof. This follows immediately from the Adams E_∞-page. The first $\mathbb{Z}/8$ lies in the image of J and has a generator that is detected by $P^6 h_2$. The last $\mathbb{Z}/2$ is detected by $h_2 B_2$.

The remaining elements are detected by $h_3 g_2$, $h_0 h_3 g_2$, and gn. These elements assemble into $\mathbb{Z}/8$ if there is a hidden 2 extension from $h_0 h_3 g_2$ to gn, and they assemble into $\mathbb{Z}/4 \oplus \mathbb{Z}/2$ if there is no hidden 2 extension. □

Remark 1.2. According to [8], there is no hidden 2 extension in the stable 51-stem. However, this claim is inconsistent with other claims in [8], as discussed in [6, Remark 4.1.17]. Because of this uncertainty, we leave this 2 extension unresolved.

Theorem 1.3. *The 2-primary stable 52-stem is equal to* $\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

Proof. This is immediate from the Adams E_∞-page. According to [6], there are no possible hidden 2 extensions. □

2. The higher Leibniz rule

We describe the higher Leibniz rule of Moss [12] on the interaction between Massey products and Adams differentials.

Theorem 2.1. *Let a_0, a_1, and a_2 be elements in the Adams E_2-page such that $a_0 a_1 = 0$ and $a_1 a_2 = 0$. Suppose further that $d_2(a_0) a_1 = 0$ and $d_2(a_1) a_2 = 0$. Then*

$$d_2(\langle a_0, a_1, a_2 \rangle) \subset \langle d_2(a_0), a_1, a_2 \rangle + \langle a_0, d_2(a_1), a_2 \rangle + \langle a_0, d_2(a_1), a_2 \rangle.$$

The brackets in Theorem 2.1 refer to Massey products computed in the cohomology of the Steenrod algebra, which is equal to the Adams E_2-page. These Massey products may have indeterminacy, so the theorem gives an inclusion of sets. Note that we are working at the prime 2, so there are no signs involved.

For clarity, we have stated only a special case of Moss's higher Leibniz rule. The more general form allows for higher Adams differentials and can handle situations in which $d_r(a_0)a_1$ or $a_1 d_r(a_2)$ are non-zero. Lawrence further generalized Moss's higher Leibniz rule to Massey products of length greater than three [9].

Moss also proved a theorem relating Massey products in the cohomology of the Steenrod algebra to their corresponding Toda brackets in stable homotopy groups. We only briefly mention this result in Remark 3.9. See [6] for a detailed discussion of this theorem.

3. The differential

The crucial calculation is the following lemma which is due to Tangora [13].

Lemma 3.1. *In the classical Adams E_2 page, we have a Massey product*

$$G = \langle h_1, h_0, D_1 \rangle.$$

Proof. This Massey product is proven using the Lambda algebra. Since the elements D_1 and G lie beyond the range of the published Curtis tables [5], Tangora used another Massey product to deduce that the representative of D_1 in the Lambda algebra has a leading term $\lambda_4\lambda_7\lambda_{11}\lambda_{15}\lambda_{15}$. One can also use unpublished results of Mahowald and Tangora to see directly that G is represented by $\lambda_2\lambda_4\lambda_7\lambda_{11}\lambda_{15}\lambda_{15}$. Then the Massey product follows from $d(\lambda_2) = \lambda_1\lambda_0$. □

Remark 3.2. One cannot apply May's convergence theorem [11] directly to get this Massey product from the May differential $d_2(h_2 b_{22} b_{40}) = h_0 D_1$. The problem is that there is a "crossing" May differential $d_4(h_1 b_{31}^2) = h_1 h_3 g_2 + h_1 h_5 g$ that voids the hypotheses of May's convergence theorem.

Remark 3.3. Bruner has verified this Massey product using computer data.

The classical calculation of Lemma 3.1 allow us to deduce the analogous motivic calculation.

Corollary 3.4. *In the motivic Adams E_2 page, we have a Massey product*

$$\tau G = \langle h_1, h_0, D_1 \rangle.$$

Proof. This is immediate from the fact that classical calculations can be recovered from motivic calculations by inverting τ [6]. □

Theorem 3.5. *In the motivic Adams spectral sequence, $d_2(D_1) = h_0^2 h_3 g_2$.*

Proof. Corollary 3.4 gives us the Massey product $\tau G = \langle h_1, h_0, D_1 \rangle$ in the motivic Adams spectral sequence. The higher Leibniz rule of Theorem 2.1 implies that $d_2(\tau G) = \langle h_1, h_0, d_2(D_1) \rangle$ since there is no indeterminacy. Since $d_2(\tau G) = h_5 c_0 d_0$ [6, Lemma 3.3.12], we have $\langle h_1, h_0, d_2(D_1) \rangle = h_5 c_0 d_0$. In particular, this is nonzero. Then the only possibility is that $d_2(D_1) = h_0^2 h_3 g_2$. □

Remark 3.6. Note that $h_0^2 h_3 g_2 = h_2^2 h_5 d_0$, so

$$\langle h_1, h_0, h_0^2 h_3 g_2 \rangle = \langle h_1, h_0, h_2^2 h_5 d_0 \rangle = \langle h_1, h_0, h_2^2 \rangle h_5 d_0 = h_5 c_0 d_0,$$

as dictated by the higher Leibniz rule.

Corollary 3.7. *In the classical Adams spectral sequence,* $d_2(D_1) = h_0^2 h_3 g_2$.

Proof. This is immediate from the fact that classical calculations can be recovered from motivic calculations by inverting τ [6]. □

Remark 3.8. In the classical Adams spectral sequence, $h_5 c_0 d_0$ and $d_2(G)$ are both zero. This means that our proof is strictly motivic in nature. We do not know a proof using only the classical Adams spectral sequence.

Remark 3.9. Using only classical information, we suggest a possible argument that D_1 does not survive the classical Adams spectral sequence. Start with the Massey product $G = \langle h_1, h_0, D_1 \rangle$ and the classical differential on $d_3(G) = Ph_5 d_0$. Moss's convergence theorem [12] then implies that $\langle \eta, 2, \{D_1\} \rangle$ is not a well-formed Toda bracket. The only possibility is that D_1 does not survive. However, that still leaves two possible differentials: $d_2(D_1) = h_0^2 h_3 g_2$, or $d_4(D_1) = gn$. We do not know how to rule out the second alternative using only the classical Adams spectral sequence.

Remark 3.10. Mahowald's original argument for the differential $d_2(D_1)$ used a faulty computation of the Toda bracket $\langle \theta_4, 2, \sigma^2 \rangle$ [6, Lemma 4.2.91] [10]. Partial information about this Toda bracket is used in the construction of a Kervaire invariant element θ_5 in dimension 62 [1]. Our improved understanding of the Toda bracket does not contradict any of the claims of [1]. For an alternative construction of a Kervaire invariant element θ_5 in dimension 62, see [14].

References

[1] M.G. Barratt, J.D.S. Jones, M.E. Mahowald, Relations amongst Toda brackets and the Kervaire invariant in dimension 62, J. Lond. Math. Soc. 30 (1984) 533–550.
[2] M.G. Barratt, M.E. Mahowald, M.C. Tangora, Some differentials in the Adams spectral sequence—II, Topology 9 (1970) 309–316.
[3] R.R. Bruner, J.P. May, J.E. McClure, M. Steinberger, H_∞ Ring Spectra and Their Applications, Lecture Notes in Math., vol. 1176, Springer, Berlin, 1986.
[4] R.R. Bruner, The Adams d_2 differential, Preprint, 2010.
[5] E.B. Curtis, P. Goerss, M. Mahowald, R.J. Milgram, Calculations of unstable Adams E_2 terms for spheres, in: Algebraic Topology, Seattle, Wash., 1985, in: Lecture Notes in Math., vol. 1286, Springer, Berlin, 1987, pp. 208–266.
[6] D.C. Isaksen, Stable stems, arXiv:1407.8418.
[7] D.C. Isaksen, Classical and motivic Adams charts, arXiv:1401.4983.
[8] S.O. Kochman, Stable Homotopy Groups of Spheres, a Computer-Assisted Approach, Lecture Notes in Mathematics, vol. 1423, Springer-Verlag, 1990.
[9] A. Lawrence, Higher order compositions in the Adams spectral sequence, Bull. Am. Math. Soc. 76 (1970) 874–877.
[10] W.H. Lin, A proof of the strong Kervaire invariant in dimension 62, in: First International Congress of Chinese Mathematicians, Beijing, 1998, in: AMS/IP Stud. Adv. Math., vol. 20, Amer. Math. Soc., Providence, RI, 2001, pp. 351–358.
[11] J.P. May, Matric Massey products, J. Algebra 12 (1969) 533–568.
[12] R.M.F. Moss, Secondary compositions and the Adams spectral sequence, Math. Z. 115 (1970) 283–310.
[13] M. Tangora, Some Massey products in Ext, in: Topology and Representation Theory, Evanston, IL, 1992, in: Contemp. Math., vol. 158, Amer. Math. Soc., Providence, RI, 1994, pp. 269–280.
[14] Z. Xu, The strong Kervaire invariant problem in dimension 62, arXiv:1410.6199.