Invent. math.
DOI 10.1007/s00222-015-0623-7

Categorifications and cyclotomic rational double affine Hecke algebras

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Received: 29 May 2013 / Accepted: 17 August 2015

Abstract Varagnolo and Vasserot conjectured an equivalence between the category $\mathcal{O}$ for CRDAHA's and a subcategory of an affine parabolic category O of type A. We prove this conjecture. As applications, we prove a conjecture of Rouquier on the dimension of simple modules of CRDAHA's and a conjecture of Chuang–Miyachi on the Koszul duality for the category $\mathcal{O}$ of CRDAHA's.

This research was partially supported by the ANR Grant number ANR-10-BLAN-0110, ANR-13-BS01-0001-01 and ANR-12-JS01-0003. R. Rouquier was partially supported by the NSF Grant DMS-1161999.

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Published online: 15 September 2015

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1 Introduction

Rational Double affine Hecke algebras (RDAHA for short) have been introduced by Etingof and Ginzburg in 2002. They are associative algebras associated with a complex reflection group W and a parameter c. Their representation theory is similar to the representation theory of semi-simple Lie algebras. In particular, they admit a category $\mathcal{O}$ which is analogous to the BGG category O. This category is highest weight with the standard modules labeled by irreducible representations of W. Representations in $\mathcal{O}$ are infinite dimensional in general, but they admit a character. An important question is to determine the characters of simple modules.

One of the most important family of RDAHA's is the cyclotomic one (CRDAHA for short), where $W = G(\ell, 1, n)$ is the wreath product of S_n and $\mathbb{Z}/\ell\mathbb{Z}$. One reason is that the representation theory of CRDAHA's is closely related to the representation theory of Ariki–Koike algebras, and that the latter are important in group theory. Another reason is that the category $\mathcal{O}$ of CRDAHA's is closely related to the representation theory of affine Kac–Moody algebras, see e.g. [17,43,46]. A third reason, is that this category has a very rich structure called a categorical action of an affine Kac–Moody algebra. This action on $\mathcal{O}$ was constructed previously in [41]. Such structures have been introduced recently in representation theory and have already had remarkable applications, see e.g. [9,29,40].

The structure of $\mathcal{O}$ depends heavily on the parameter c. For generic values of c the category is semi-simple. The most non semi-simple case (which is also the most complicated one) occurs when c takes a particular form of rational numbers, see (6.2). For these parameters Rouquier made a conjecture to determine the characters of simple modules in $\mathcal{O}$ [39]. Roughly speaking, this conjecture says that the Jordan–Hölder multiplicities of the standard modules in $\mathcal{O}$ are given by some parabolic Kazhdan–Lusztig polynomials. This conjecture was known to be true in the particular case $\ell = 1$ [39]. Motivated by this conjecture, Varagnolo–Vasserot introduced in [46] a new category **A** which is a subcategory of an affine parabolic category **O** at a negative level and should be viewed as an affine and higher level analogue of the category of polynomial representations of GL_N. They conjectured that there should be an equivalence of highest weight categories between $\mathcal{O}$ and **A**.

In this paper we prove Varagnolo–Vasserot's conjecture (Theorem 6.9). A first consequence is a proof of Rouquier's conjecture (Theorem 7.3). A second remarkable application is a proof that the category $\mathcal{O}$ is Koszul (Theorem 7.4), yielding a proof of a conjecture of Chuang–Miyachi [8], because the affine parabolic category **O** is Koszul by [42].

Our proof is based on an extension of Rouquier's theory of highest weight covers developed in [39]. Basically, [39] says that two highest weight covers of the same algebra are equivalent as highest weight categories if they satisfy a so called 1-faithful condition and if the highest weight orders on both covers are compatible. Here, given a situation where the highest weight covers are not necessarily 1-faithful, we construct bigger functors to which we can apply Rouquier's theory (see Proposition 2.20).

The category $\mathcal{O}$ is a highest weight cover over the module category $\mathcal{H}$ of the Ariki–Koike algebra via the KZ functor introduced in [22]. It is a 0-faithful cover and if the parameters of the RDAHA satisfy some technical condition, then it is even 1-faithful. A similar functor $\Phi : \mathbf{A} \to \mathcal{H}$ was introduced in [46] using the Kazhdan–Lusztig fusion product on the affine category **O** at a negative level. A previous work of Dunkl and Griffeth [16] allows to show without much difficulty that there is a highest weight order on $\mathcal{O}$ which refines the linkage order on **A**. A difficult part of the proof consists of showing that the functor Φ is indeed a cover, meaning that it is an exact quotient functor, and that it has the same faithfulness properties as the KZ functor. Once this is done, the equivalence between $\mathcal{O}$ and **A** follows directly from the unicity of 1-faithful covers if the technical condition on parameters mentioned above is satisfied. To prove the equivalence without this condition, we need to replace KZ and Φ by some other covers, see the end of the introduction for more details on this.

A key ingredient in our proof is a deformation argument. More precisely, the highest weight categories **A**, **O** admit deformed versions over a regular local ring R of dimension 2. Some technical results prove that the Kazhdan–Lusztig tensor product can also be deformed properly, which allows us to define the deformed version of Φ. Next, a theorem of Fiebig asserts that the structure of the category **O** of a Kac–Moody algebra only depends on the associated Coxeter system [20]. In particular, the localization of **A** at a height one prime ideal $\mathfrak{p} \subset R$ can be described in simpler terms. Two cases appear, either $\mathfrak{p}$ is subgeneric or generic. In the first case, considered in Sect. 5.7.2, the category **A** reduces to an analog subcategory A inside the parabolic category $\mathscr{O}$ of $\mathfrak{gl}_N$ associated with a Levi subalgebra of $\mathfrak{gl}_N$ with 2 blocks. The latter is closely related to the higher level Schur–Weyl duality studied by Brundan and Kleshchev in [5]. In the second case, considered in Sect. 5.7.3, the category $\mathbf{A}$ reduces to the corresponding category for $\ell = 1$, which is precisely the Kazhdan–Lusztig category associated with affine Lie algebras at negative levels. Finally, we show that to prove the desired properties of the functor Φ it is enough to check them for the localization of Φ at each height one prime ideal $\mathfrak{p}$ and this proves the main result.

Now, let us say a few words concerning the organization of this paper.

Section 2 contains some basic facts on highest weight categories and some developments on the theory of highest weight covers in [39].

Section 3 is a reminder on Hecke algebras, q-Schur algebras and categorifications.

Section 4 contains basic facts on the parabolic category $\mathscr{O}$ of $\mathfrak{gl}_N$ and the subcategory $A \subset \mathscr{O}$ introduced in [5]. The results in [5] are not enough for us since we need to consider a deformed category A with *integral* deformation parameters. The new material is gathered in Sect. 4.7.

In Sect. 5 we consider the affine parabolic category $\mathbf{O}$ (at a negative level). The monoidal structure on $\mathbf{O}$ is defined later in Sect. 8. Using this monoidal structure we construct a categorical action on $\mathbf{O}$ in Sect. 5.4. Then, we define the subcategory $\mathbf{A} \subset \mathbf{O}$ in Sect. 5.5. The rest of the section is devoted to the deformation argument and the proof that $\mathbf{A}$ is a highest weight cover of the module category of a cyclotomic Hecke algebra satisfying some faithfulness conditions.

In Sect. 6 we first give a reminder on the category $\mathcal{O}$ of CRDAHA's, following [22,39]. Then, we prove our main theorems in Sects. 6.3.2, 6.3.3 using the results from Sect. 5.8. This yields a proof of Varagnolo–Vasserot's conjecture [46]. For the clarity of the exposition we separate the cases of rational and irrational levels, although both proofs are very similar.

In Sect. 7 we give some applications of our main theorem, including proofs for Rouquier's conjecture and Chuang–Miyachi's conjecture.

Section 8 is a reminder on the Kazhdan–Lusztig tensor product on the affine category $\mathbf{O}$ at a negative level. We generalize their construction in order to get a monoidal structure on arbitrary parabolic categories, deformed over an analytic two-dimensional regular local ring. Several technical results concerning the Kazhdan–Lusztig tensor product are postponed to the appendix.

To finish, let us explain the relation of this work with other recent works.

The case of irrational level (proved in Theorem 6.11) was conjectured in [46, rem. 8.10(b)], as a degenerate analogue of the main conjecture [46, conj. 8.8]. There, it was mentioned that it should follow from [5, thm. C]. In the dominant case, this has been proved recently [24, thm. 6.9.1].

While we were writing this paper I. Losev made public several papers with some overlaps with ours. In [31,32] he developed a general formalism of categorical actions on highest weight categories. Then, he used this formalism in [33] to prove that the category **A** is equipped with a categorical action, induced by the categorical action on **O** introduced in [46] (using the Kazhdan–Lusztig fusion product). The categorical action on **A** gives an independent proof of Theorem 5.37(a), (b). Finally, he proposed a combinatorial approach to prove that **A** is a 1-faithful highest weight cover of the cyclotomic Hecke algebra under some technical condition on the parameters of the CRDAHA.

A first version of our paper was announced in July 2012 and has been presented at several occasions since then. There, we proved this 1-faithfulness for **A** (and the Varagnolo–Vasserot's conjecture) under the same condition on the parameters by a deformation argument similar, but weaker, to the one used in the present paper.

The proof which we give in this article avoids this technical condition on the parameters. It uses an idea introduced later, in [33]. There, I. Losev replaces the highest weight cover **A** of the cyclotomic Hecke algebra **H** by a highest weight cover, by **A**, of a bigger algebra than **H**, which has better properties.

After this paper was written, B. Webster sent us a copy of a preliminary version of his recent preprint [47] proposing another proof of Rouquier's conjecture which does not use the affine parabolic category **O**.

Note that our construction does not use any categorical action on **A**. It only uses representation theoretic arguments. However, since Theorem 6.9 yields an equivalence between **A** and $\mathcal{O}$, we can recover a categorical action on **A** from our theorem and the main result of [41]. This is explained in Sect. 7.4.

2 Highest weight categories

In the paper the symbol R will always denote a noetherian commutative domain (with 1). We denote by K its fraction field. When R is a local ring, we denote by k its residue field and by $\mathfrak{m}$ its maximal ideal.

2.1 Rings and modules

For any R-module M, let $M^* = \mathrm{Hom}_R(M, R)$ denote the dual module. An *S-point* of R is a morphism $\chi : R \to S$ of commutative rings with 1. If χ is a morphism of local rings, we say that it is a *local S-point*. We write $SM = M(\chi) = M \otimes_R S$. If ϕ is a R-module homomorphism, we abbreviate also $S\phi = \phi \otimes_R S$.

Let $\mathfrak{P}$, $\mathfrak{M}$ be the spectrum and the maximal spectrum of R. Let $\mathfrak{P}_1 \subset \mathfrak{P}$ be the subset of height 1 prime ideals. For each $\mathfrak{p} \in \mathfrak{P}$, let $R_\mathfrak{p}$ denote the localization of R at $\mathfrak{p}$. The maximal ideal of $R_\mathfrak{p}$ is $\mathfrak{m}_\mathfrak{p} = R_\mathfrak{p}\,\mathfrak{p}$ and its residue field is $\mathrm{k}_\mathfrak{p} = \mathrm{Frac}(R/\mathfrak{p})$.

A *closed* k*-point* of R is a quotient $R \to R/\mathfrak{m} = \mathrm{k}$ where $\mathfrak{m} \in \mathfrak{M}$. To unburden the notation we may write $\mathrm{k} \in \mathfrak{M}$.

A *finite projective* R-algebra is an R-algebra which is finitely generated and projective as an R-module.

We will mainly be interested in the case where R is a local ring. In this case, any projective module is free by Kaplansky's theorem. Therefore, we'll use indifferently the words free or projective.

2.2 Categories

Given A a ring, we denote by A^{op} the opposite ring in which the order of multiplication is reversed. Given $\mathscr{C}$ is a category, let $\mathscr{C}^{\mathrm{op}}$ be the opposite category.

An *R-category* $\mathscr{C}$ is an additive category enriched over the tensor category of R-modules. All the functors F on $\mathscr{C}$ are assumed to be R-linear. We denote the identity element in the endomorphism ring $\mathrm{End}(F)$ again by F. We denote the identity functor on $\mathscr{C}$ by $1_{\mathscr{C}}$. We say that $\mathscr{C}$ is *Hom-finite* if the Hom spaces are finitely generated over R. If the category $\mathscr{C}$ is abelian or exact, let $K_0(\mathscr{C})$ be the Grothendieck group and write $[\mathscr{C}] = K_0(\mathscr{C}) \otimes_{\mathbb{Z}} \mathbb{C}$. If $\mathscr{C}$ is additive, it is an exact category with split exact sequences and $[\mathscr{C}]$ is the complexified split Grothendieck group. Let $[M]$ denote the class of an object M of $\mathscr{C}$.

Assume now that $\mathscr{C}$ is abelian and has enough projectives. We say that an object $M \in \mathscr{C}$ is projective over R if $\mathrm{Hom}_{\mathscr{C}}(P, M)$ is a projective R-module for all projective objects P of $\mathscr{C}$. The full subcategory $\mathscr{C} \cap R\text{-proj}$ of objects of $\mathscr{C}$ projective over R is an exact subcategory and the canonical functor $D^b(\mathscr{C} \cap R\text{-proj}) \to D^b(\mathscr{C})$ is fully faithful. An object $X \in \mathscr{C}$ which is projective over R is *relatively R-injective* if $\mathrm{Ext}^1_{\mathscr{C}}(Y, X) = 0$ for all objects Y of $\mathscr{C}$ that are projective over R.

If $\mathscr{C}$ is the category A-mod of finitely generated (left) modules over a finite projective R-algebra A, then an object $X \in \mathscr{C}$ is projective over R if and only if it is projective as an R-module. It is relatively R-injective if in addition the dual $X^* = \mathrm{Hom}_R(X, R)$ is a projective right A-module. If there is no risk of confusion we will say injective instead of relatively R-injective. We put $\mathscr{C}^* = A^{\mathrm{op}}\text{-mod}$. The functor $\mathrm{Hom}_R(\bullet, R) : \mathscr{C}^{\mathrm{op}} \to \mathscr{C}^*$ restricts to an equivalence of exact categories $\mathscr{C}^{\mathrm{op}} \cap R\text{-proj} \xrightarrow{\sim} \mathscr{C}^* \cap R\text{-proj}$.

We denote by $\mathrm{Irr}(\mathscr{C})$ the sets of isomorphism classes of simple objects of $\mathscr{C}$. Let $\mathscr{C}^{\mathrm{proj}}, \mathscr{C}^{\mathrm{inj}} \subset \mathscr{C}$ be the full subcategories of projective and of relatively R-injective objects. If $\mathscr{C} = A\text{-mod}$, we abbreviate $\mathrm{Irr}(A) = \mathrm{Irr}(\mathscr{C})$, $A\text{-proj} = \mathscr{C}^{\mathrm{proj}}$ and $A\text{-inj} = \mathscr{C}^{\mathrm{inj}}$.

Given an S-point $R \to S$ and $\mathscr{C} = A\text{-mod}$, we can form the S-category $S\mathscr{C} = SA\text{-mod}$. Given another R-category $\mathscr{C}'$ as above and an exact (R-linear) functor $F : \mathscr{C} \to \mathscr{C}'$, then F is represented by a projective object $P \in \mathscr{C}$. We set $SF = \mathrm{Hom}_{S\mathscr{C}}(SP, \bullet) : S\mathscr{C} \to S\mathscr{C}'$.

Let $\mathscr{A}$ be a *Serre* subcategory of $\mathscr{C}$. The canonical embedding functor $h : \mathscr{A} \to \mathscr{C}$ has a left adjoint h^* which takes an object M in $\mathscr{C}$ to its maximal quotient in $\mathscr{C}$ which belongs to $\mathscr{A}$. It admits also a right adjoint $h^!$ which takes an object M in $\mathscr{C}$ to its maximal subobject in $\mathscr{C}$ which belongs to $\mathscr{A}$. The functor h^* is right exact, while $h^!$ is left exact. The functor h is fully faithful. Hence the adjunction morphisms $h^*h \to 1_{\mathscr{A}}$ and $1_{\mathscr{A}} \to h^!h$ are isomorphisms. By definition, the adjunction morphisms $1_{\mathscr{C}} \to hh^*$ and $hh^! \to 1_{\mathscr{C}}$ are respectively an epimorphism and a monomorphism.

Here, and in the rest of the paper, we use the following notation: a composition of functors E and F is written as EF while a composition of morphisms of functors ψ and ϕ is written as $\psi \circ \phi$.

2.3 Highest weight categories over local rings

Let R be a commutative local ring. We recall and complete some basic facts about highest weight categories over R (cf [39, §4.1] and [11], [15, §2]).

Let $\mathscr{C}$ be an abelian R-category which is equivalent to the category A-mod of finitely generated modules over a finite projective R-algebra A.

The category $\mathscr{C}$ is a *highest weight R-category* if it is equipped with a poset of isomorphism classes of objects $(\Delta(\mathscr{C}), \leqslant)$ called the *standard objects* satisfying the following conditions:

- the objects of $\Delta(\mathscr{C})$ are projective over R
- given $M \in \mathscr{C}$ such that $\mathrm{Hom}_{\mathscr{C}}(D, M) = 0$ for all $D \in \Delta(\mathscr{C})$, we have $M = 0$
- given $D \in \Delta(\mathscr{C})$, there is $P \in \mathscr{C}^{\mathrm{proj}}$ and a surjection $f : P \twoheadrightarrow D$ such that $\ker f$ has a (finite) filtration whose successive quotients are objects $D' \in \Delta$ with $D' > D$
- given $D \in \Delta$, we have $\mathrm{End}_{\mathscr{C}}(D) = R$
- given $D_1, D_2 \in \Delta$ with $\mathrm{Hom}_{\mathscr{C}}(D_1, D_2) \neq 0$, we have $D_1 \leq D_2$.

The partial order $\leqslant$ is called the *highest weight order* of $\mathscr{C}$. We write $\Delta(\mathscr{C}) = \{\Delta(\lambda)\}_{\lambda \in \Lambda}$, for Λ an indexing poset. Note that if $\leq'$ is an order coarser than $\leq$ (i.e., $\lambda \leq \mu$ implies $\lambda \leq' \mu$), then $\mathscr{C}$ is also a highest weight category relative to the order $\leq'$.

An *equivalence of highest weight categories* $\mathscr{C}' \xrightarrow{\sim} \mathscr{C}$ is an equivalence which induces a bijection $\Delta(\mathscr{C}') \xrightarrow{\sim} \Delta(\mathscr{C})$. A *highest weight subcategory* is a full Serre subcategory $\mathscr{C}' \subset \mathscr{C}$ that is a highest weight category with poset $\Delta(\mathscr{C}')$ an ideal of $\Delta(\mathscr{C})$ (i.e., if $D' \in \Delta(\mathscr{C}')$, $D \in \Delta(\mathscr{C})$ and $D' < D$, then $D' \in \Delta(\mathscr{C}')$).

Highest weight categories come with associated projective, injective, tilting and costandard objects, as described in the next proposition.

Proposition 2.1 *Let $\mathscr{C}$ be a highest weight R-category. Given $\lambda \in \Lambda$, there are indecomposable objects $P(\lambda) \in \mathscr{C}^{\mathrm{proj}}$, $I(\lambda) \in \mathscr{C}^{\mathrm{inj}}$, $T(\lambda) \in \mathscr{C}$ and $\nabla(\lambda) \in \mathscr{C}$ (the projective, injective, tilting and costandard objects associated with λ), unique up to isomorphism such that*

(∇) $\mathrm{Hom}_{\mathscr{C}}(\Delta(\mu), \nabla(\lambda)) \simeq \delta_{\lambda\mu} R$ *and* $\mathrm{Ext}^1_{\mathscr{C}}(\Delta(\mu), \nabla(\lambda)) = 0$ *for all* $\mu \in \Lambda$,
(P) *there is a surjection* $f : P(\lambda) \twoheadrightarrow \Delta(\lambda)$ *such that* $\ker f$ *has a filtration whose successive quotients are* $\Delta(\mu)$*'s with* $\mu > \lambda$,
(I) *there is an injection* $f : \nabla(\lambda) \hookrightarrow I(\lambda)$ *such that* $\mathrm{coker}\, f$ *has a filtration whose successive quotients are* $\nabla(\mu)$*'s with* $\mu > \lambda$,
(T) *there is an injection* $f : \Delta(\lambda) \hookrightarrow T(\lambda)$ *and a surjection* $g : T(\lambda) \twoheadrightarrow \nabla(\lambda)$ *such that* $\mathrm{coker}\, f$ *(resp.* $\ker g$*) has a filtration whose successive quotients are* $\Delta(\mu)$*'s (resp.* $\nabla(\mu)$*'s) with* $\mu < \lambda$.

We have the following properties of those objects.

- $\nabla(\lambda)$, $\Delta(\lambda)$, $P(\lambda)$, $I(\lambda)$ *and* $T(\lambda)$ *are projective over* R.
- *Given a commutative local R-algebra S, then $S\mathscr{C}$ is a highest weight S-category on the poset Λ with standard objects $S\Delta(\lambda)$ and costandard objects $S\nabla(\lambda)$. If $R \to S$ is a local S-point, then the projective, injective and tilting objects associated with λ are $SP(\lambda)$, $SI(\lambda)$ and $ST(\lambda)$.*
- *$\mathscr{C}^*$ is a highest weight R-category on the poset Λ with standard objects $\Delta^*(\lambda) = \nabla(\lambda)^*$ and with $P^*(\lambda) = I(\lambda)^*$, $I^*(\lambda) = P(\lambda)^*$, $\nabla^*(\lambda) = \Delta(\lambda)^*$ and $T^*(\lambda) = T(\lambda)^*$.*

Proof Note that the statements of the proposition are classical when R is a field.

The existence of the objects $\nabla(\lambda)$ giving $\mathscr{C}^{\mathrm{op}}$ the structure of a highest weight category and satisfying the Hom and Ext conditions is given by [39, Proposition 4.19]. The unicity follows from Lemma 2.7 below. The description of the projective, tilting and injective objects of $\mathscr{C}^*$ is clear.

It is shown in [39, Proposition 4.14] that $S\mathscr{C}$ is a highest weight category with $\Delta(S\mathscr{C}) = S\Delta(\mathscr{C})$. We denote by $P_S(\lambda)$, $I_S(\lambda)$, etc. the projective, injective, etc. of $S\mathscr{C}$ associated with λ.

The existence of $P(\lambda)$ is granted in the definition of highest weight categories. We show by descending induction on λ that $\mathrm{k}P(\lambda) \simeq P_{\mathrm{k}}(\lambda)$. This is clear if λ is maximal, for then $P(\lambda) = \Delta(\lambda)$. We have $\mathrm{k}P(\lambda) = P_{\mathrm{k}}(\lambda) \oplus Q$, where Q is a direct sum of $P_{\mathrm{k}}(\mu)$'s with $\mu > \lambda$. By induction, $P_{\mathrm{k}}(\mu) = \mathrm{k}P(\mu)$, hence Q lifts to $\tilde{Q} \in \mathscr{C}^{\mathrm{proj}}$, and there are maps $f : \tilde{Q} \to P(\lambda)$ and $g : P(\lambda) \to \tilde{Q}$ such that $\mathrm{k}(gf) = \mathrm{id}_Q$. Since R is local and $\tilde{Q}$ is a finitely generated projective R-module, we deduce that gf is an automorphism of $\tilde{Q}$, hence $\tilde{Q}$ is a direct summand of $P(\lambda)$, so $\tilde{Q} = 0$ and $\mathrm{k}P(\lambda) = P_{\mathrm{k}}(\lambda)$. The unicity of $P(\lambda)$ is then clear, since given $M, N \in \mathscr{C}^{\mathrm{proj}}$, we have $\mathrm{k}\,\mathrm{Hom}_{\mathscr{C}}(M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{k}\mathscr{C}}(\mathrm{k}M, \mathrm{k}N)$.

Given $R \to S$ a local point, the residue field k' of S is a field extension of k. Since $\mathrm{k}A$ is a split k-algebra, it follows that given P a projective indecomposable $\mathrm{k}A$-module, then $\mathrm{k}'P$ is a projective indecomposable $\mathrm{k}'A$-module. We deduce that $P_{\mathrm{k}'}(\lambda) \simeq \mathrm{k}' \otimes_{\mathrm{k}} \mathrm{k}P(\lambda)$, hence $P_S(\lambda) \simeq SP(\lambda)$.

The statements about $I(\lambda)$ follow from those about $P(\lambda)$ by duality.

The statements about $T(\lambda)$ are proven in the same way as those for $P(\lambda)$, using Proposition 2.4(b) below. □

Note that $(\mathscr{C}, \Delta(\mathscr{C}))$ is a highest weight R-category if and only if $(\mathrm{k}\mathscr{C}, \mathrm{k}\Delta(\mathscr{C}))$ is a highest weight k-category and the objects of $\Delta(\mathscr{C})$ are projective over R, see [39, thm. 4.15]. Note also that $\Delta(\lambda)$ has a unique simple quotient $L(\lambda)$, and $\mathrm{Irr}(\mathscr{C}) = \{L(\lambda)\}_{\lambda\in\Lambda}$.

Let $\mathscr{C}^{\Delta}$ and $\mathscr{C}^{\nabla}$ be the full subcategories of $\mathscr{C}$ consisting of the Δ-filtered and ∇-filtered objects, i.e., objects having a finite filtration whose successive quotients are standard, costandard respectively. These are exact subcategories of $\mathscr{C}$. Note that every object of $\mathscr{C}^{\Delta}$ has a finite projective resolution, where the kernels of the differentials are in $\mathscr{C}^{\Delta}$. As a consequence, the canonical functor $D^b(\mathscr{C}^{\Delta}) \to D^b(\mathscr{C})$ is fully faithful. Similarly, the canonical functor $D^b(\mathscr{C}^{\nabla}) \to D^b(\mathscr{C})$ is fully faithful, as every object of $\mathscr{C}^{\nabla}$ has a finite relatively R-injective resolution.

Lemma 2.2 *Let $\mathscr{C}, \mathscr{C}'$ be highest weight R-categories. An exact functor $\Phi : \mathscr{C} \to \mathscr{C}'$ which restricts to an equivalence $\Phi : \mathscr{C}^{\Delta} \xrightarrow{\sim} \mathscr{C}'^{\Delta}$ is an equivalence of highest weight categories $\mathscr{C} \xrightarrow{\sim} \mathscr{C}'$.*

Proof Since Φ identifies the projective objects in $\mathscr{C}$ and $\mathscr{C}'$, it induces an equivalence of their bounded homotopy categories, hence an equivalence $D^b(\mathscr{C}) \to D^b(\mathscr{C}')$. Since Φ is exact, we are done. □

Let $\mathscr{C}^{\mathrm{tilt}} = \mathscr{C}^{\Delta} \cap \mathscr{C}^{\nabla}$ be the full subcategory of $\mathscr{C}$ consisting of the *tilting objects*, i.e., the objects which are both Δ-filtered and ∇-filtered.

Let $T = \bigoplus_{\lambda\in\Lambda} T(\lambda)$. The *Ringel dual* of $\mathscr{C}$ is the category $\mathscr{C}^{\diamond} = \mathrm{End}_{\mathscr{C}}(T)^{\mathrm{op}}$-mod. It is a highest weight category on the poset Λ^{op}. The functor $\mathrm{Hom}(T, \bullet) : \mathscr{C} \to \mathscr{C}^{\diamond}$ restricts to an equivalence $\mathscr{R} : \mathscr{C}^{\nabla} \xrightarrow{\sim} (\mathscr{C}^{\diamond})^{\Delta}$, called the *Ringel equivalence*. We have $\mathscr{R}(\nabla(\lambda)) = \Delta^{\diamond}(\lambda)$, $\mathscr{R}(T(\lambda)) \simeq P^{\diamond}(\lambda)$ and $\mathscr{R}(I(\lambda)) \simeq T^{\diamond}(\lambda)$ for $\lambda \in \Lambda$, see [39, Proposition 4.26]. The highest weight category $\mathscr{C}$ is determined, up to equivalence, by $\mathscr{C}^{\diamond}$ and we put $(\mathscr{C}^{\diamond})^{\blacklozenge} = \mathscr{C}$. There is an equivalence of highest weight categories $\mathscr{C} \xrightarrow{\sim} \mathscr{C}^{\diamond\diamond}$ such that the composition

$$\mathscr{C}^{\mathrm{proj}} \xrightarrow{\sim} (\mathscr{C}^{\diamond\diamond})^{\mathrm{proj}} \xrightarrow[\sim]{\mathscr{R}^{-1}} (\mathscr{C}^{\diamond})^{\mathrm{tilt}} \xrightarrow[\sim]{\mathscr{R}^{-1}} \mathscr{C}^{\mathrm{inj}}$$

is isomorphic to the Nakayama duality $\mathrm{Hom}_A(\bullet, A)^*$. This provides also an equivalence of highest weight categories $\mathscr{C}^{\blacklozenge} \xrightarrow{\sim} \mathscr{C}^{\diamond}$.

Now, for $M \in \mathscr{C}$ we set

$$\begin{aligned} \mathrm{lcd}_{\mathscr{C}}(M) &= \min\{i;\ \exists \mu \in \Lambda, \mathrm{Ext}^i(M, T(\mu)) \neq 0\}, \\ \mathrm{rcd}_{\mathscr{C}}(M) &= \min\{i;\ \exists \mu \in \Lambda, \mathrm{Ext}^i(T(\mu), M) \neq 0\}. \end{aligned} \tag{2.1}$$

Lemma 2.3 *Assume R is a field. Let $\lambda \in \Lambda$. Then*

$$\begin{aligned} &\min\{i;\ \exists \mu \in \Lambda, \mathrm{Ext}^i(L(\lambda), T(\mu)) \neq 0\} \\ &\quad = \min\{i;\ \exists \mu \in \Lambda, \mathrm{Ext}^i(L(\lambda), \Delta(\mu)) \neq 0\} \\ &\quad = \min\{i;\ \exists M \in \mathscr{C}^{\Delta}, \mathrm{Ext}^i(L(\lambda), M) \neq 0\}. \end{aligned}$$

Proof Let c_1, c_2 and c_3 be the quantities defined by the terms involving respectively $T(\mu)$'s, $\Delta(\mu)$'s and $M \in \mathscr{C}^{\Delta}$ in the first two equalities. It is clear that $c_1 \geq c_2 = c_3$.

Take μ minimal such that $\mathrm{Ext}^{c_2}(L(\lambda), \Delta(\mu)) \neq 0$. There is an exact sequence $0 \to \Delta(\mu) \to T(\mu) \to M \to 0$ where M has a filtration with subquotients $\Delta(\nu)$'s where $\nu < \mu$. We deduce that $\mathrm{Ext}^{c_2}(L(\lambda), T(\mu)) \neq 0$, hence $c_1 \leq c_2$. □

Let us recall a few facts on base change for highest weight categories.

Proposition 2.4 *Let $\mathscr{C}$ be a highest weight R-category, and let $R \to S$ be a local S-point. For any $M, N \in \mathscr{C}$ the following holds*:

(a) *if S is R-flat then $S\,\mathrm{Ext}^d_{\mathscr{C}}(M, N) = \mathrm{Ext}^d_{S\mathscr{C}}(SM, SN)$ for all $d \in \mathbb{N}$,*
(b) *if either $M \in \mathscr{C}^{\mathrm{proj}}$ or ($M \in \mathscr{C}^{\Delta}$ and $N \in \mathscr{C}^{\nabla}$), then we have $S\,\mathrm{Hom}_{\mathscr{C}}(M, N) = \mathrm{Hom}_{S\mathscr{C}}(SM, SN)$,*
(c) *if M is R-projective then $M \in \mathscr{C}^{\mathrm{proj}}$ (resp. $M \in \mathscr{C}^{\mathrm{tilt}}, \mathscr{C}^{\Delta}, \mathscr{C}^{\mathrm{inj}}$) if and only if $\mathrm{k}M \in \mathrm{k}\mathscr{C}^{\mathrm{proj}}$ (resp. $\mathrm{k}M \in \mathrm{k}\mathscr{C}^{\mathrm{tilt}}, \mathrm{k}\mathscr{C}^{\Delta}, \mathrm{k}\mathscr{C}^{\mathrm{inj}}$),*
(d) *if either ($M \in \mathscr{C}^{\mathrm{proj}}$ and N is R-projective) or ($M \in \mathscr{C}^{\Delta}$ and $N \in \mathscr{C}^{\nabla}$) then $\mathrm{Hom}_{\mathscr{C}}(M, N)$ is R-projective.*

Proof Part (a) is [Bourbaki, *Algèbre*, ch. X, §6, prop. 7.c].

The statements in (b), (d) are clear if M is a free A-module, and are preserved under taking direct summands, so they hold for $M \in \mathscr{C}^{\mathrm{proj}}$.

Let $M \in \mathscr{C}^{\Delta}$ and $N \in \mathscr{C}^{\nabla}$. We have $\mathrm{Ext}^1_{\mathscr{C}}(M, N) = \mathrm{Ext}^1_{S\mathscr{C}}(SM, SN) = 0$. As a consequence, if M is an extension of $M_1, M_2 \in \mathscr{C}^{\Delta}$ and the statements (b), (d) hold for M_i, N, then they hold for M, N. We proceed now by descending induction on λ to prove that the statement for $M = \Delta(\lambda)$. There is an exact sequence $0 \to M' \to P(\lambda) \to \Delta(\lambda) \to 0$, where M' is an extension of $\Delta(\lambda')$'s with $\lambda' > \lambda$. The statements (b), (d) hold for $P(\lambda)$ and, by induction, for M'. Hence, they hold for M.

Part (c) is [39, prop. 4.30]. □

Proposition 2.5 *The indecomposable projective (resp. relatively R-injective, tilting) objects of $\mathscr{C}$ are the $P(\lambda)$ (resp. $I(\lambda)$, $T(\lambda)$), for $\lambda \in \Lambda$.*

Proof The statements are classical for $k\mathscr{C}$, and Proposition 2.4(b), (c) reduce to that case. □

Let us quote the following easy result for a later use.

Proposition 2.6 (a) *Let $\mathscr{C}_1, \mathscr{C}_2$ be highest weight* k*-categories. An equivalence of abelian* k*-categories $F : \mathscr{C}_1 \to \mathscr{C}_2$ which induces a morphism of posets* $\mathrm{Irr}(\mathscr{C}_1) \to \mathrm{Irr}(\mathscr{C}_2)$ *is an equivalence of highest weight categories.*

(b) *Let $\mathscr{C}_1, \mathscr{C}_2$ be highest weight R-categories. An equivalence of abelian R-categories $F : \mathscr{C}_1 \to \mathscr{C}_2$ which induces an equivalence of highest weight* k*-categories $kF : k\mathscr{C}_1 \to k\mathscr{C}_2$ is an equivalence of highest weight R-categories.*

Proof For part (a) we need to prove that F maps $\Delta(\mathscr{C}_1)$ to $\Delta(\mathscr{C}_2)$. An equivalence of abelian categories F takes indecomposable projective objects to indecomposable projective objects. So it preserves the standard modules, as $\Delta(\lambda)$ is the largest quotient of $P(\lambda)$ all of whose composition factors are $L(\mu)$'s with $\mu < \lambda$. Part (b) follows from Proposition 2.4(c). □

Next, we state some basic facts on ∇ and Δ-filtered modules. The situation over a base ring that is not a field is slightly more complicated.

Lemma 2.7 *Let $\mathscr{C}$ be a highest weight category over R and let $M \in \mathscr{C}$. The following conditions are equivalent:*

(a) $\mathrm{Ext}^1_{\mathscr{C}}(\Delta(\lambda), M) = 0$ *for all $\lambda \in \Lambda$*
(b) *there is a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$ and there are elements $\lambda_i \in \Lambda$ such that $M_i/M_{i-1} \simeq \nabla(\lambda_i) \otimes_R \mathrm{Hom}_{\mathscr{C}}(\Delta(\lambda_i), M)$ with $\lambda_i \neq \lambda_j$ for $i \neq j$ and $\lambda_i < \lambda_j$ implies $i < j$*
(c) *there is a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$, there are elements $\lambda_i \in \Lambda$ and there are R-modules U_i such that $M_i/M_{i-1} \simeq \nabla(\lambda_i) \otimes_R U_i$.*

If the conditions above hold and M is projective over R, then $M \in \mathscr{C}^{\nabla}$.

Proof Assume (b). Let $\lambda, \mu \in \Lambda$ and $U \in R$-mod. We have $\mathrm{Ext}^{>0}_{\mathscr{C}}(\Delta(\lambda), \nabla(\mu)) = 0$ and $\mathrm{Hom}_{\mathscr{C}}(\Delta(\lambda), \nabla(\mu)) \in R$-proj. We deduce that

$$\begin{aligned}\mathrm{Ext}^{>0}_{\mathscr{C}}(\Delta(\lambda), \nabla(\mu) \otimes_R U) &= H^{>0}(R\,\mathrm{Hom}_{\mathscr{C}}(\Delta(\lambda), \nabla(\mu) \otimes^L_R U)) \\ &\simeq H^{>0}(R\,\mathrm{Hom}_{\mathscr{C}}(\Delta(\lambda), \nabla(\mu)) \otimes^L_R U) = \mathrm{Ext}^{>0}_{\mathscr{C}}(\Delta(\lambda), \nabla(\mu)) \otimes_R U = 0.\end{aligned}$$

This shows (b) $\Rightarrow$ (a).

Now, assume (a). Let $\lambda \in \Lambda$ be minimal such that $\mathrm{Hom}_{\mathscr{C}}(\nabla(\lambda), M) \neq 0$. Fix an element $\mu \leqslant \lambda$ (no assumption on μ if $\mathrm{Hom}_{\mathscr{C}}(\nabla(\lambda), M) = 0$ for all $\lambda \in \Lambda$). There is an exact sequence $0 \to \Delta(\mu) \to T(\mu) \to M' \to 0$, where M' is filtered by $\Delta(\nu)$'s with $\nu < \mu$. So, we have $\mathrm{Ext}^1_{\mathscr{C}}(M', M) = 0$. Hence the canonical map $\mathrm{Hom}_{\mathscr{C}}(T(\mu), M) \to \mathrm{Hom}_{\mathscr{C}}(\Delta(\mu), M)$ is surjective. There is an exact sequence $0 \to M'' \to T(\mu) \to \nabla(\mu) \to 0$, where M'' is filtered by $\nabla(\nu)$'s with $\nu < \mu$. Since $\mathrm{Hom}_{\mathscr{C}}(M'', M) = 0$, the canonical map $\mathrm{Hom}_{\mathscr{C}}(\nabla(\mu), M) \to \mathrm{Hom}_{\mathscr{C}}(T(\mu), M)$ is an isomorphism. Consequently, the composition $\Delta(\mu) \to T(\mu) \to \nabla(\mu)$ induces a surjective map $\mathrm{Hom}_{\mathscr{C}}(\nabla(\mu), M) \to \mathrm{Hom}_{\mathscr{C}}(\Delta(\mu), M)$.

If $\mu \neq \lambda$, we have $\mathrm{Hom}_{\mathscr{C}}(\nabla(\mu), M) = 0$, hence $\mathrm{Hom}_{\mathscr{C}}(\Delta(\mu), M) = 0$. This shows that the canonical map $\mathrm{Hom}_{\mathscr{C}}(T(\lambda), M) \to \mathrm{Hom}_{\mathscr{C}}(\Delta(\lambda), M)$ is an isomorphism. Hence, we have canonical isomorphisms

$$\mathrm{Hom}_{\mathscr{C}}(\nabla(\lambda), M) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{C}}(T(\lambda), M) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{C}}(\Delta(\lambda), M).$$

Now, set $U = \mathrm{Hom}_{\mathscr{C}}(\Delta(\lambda), M)$. We have

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(\nabla(\lambda) \otimes_R U, M) &\simeq \mathrm{Hom}_R(U, \mathrm{Hom}_{\mathscr{C}}(\nabla(\lambda), M)) \\ &\simeq \mathrm{Hom}_R(U, \mathrm{Hom}_{\mathscr{C}}(\Delta(\lambda), M)) \\ &\simeq \mathrm{Hom}_{\mathscr{C}}(\Delta(\lambda) \otimes_R U, M). \end{aligned}$$

So, the canonical map $\Delta(\lambda) \otimes_R U \to M$ factors through a map $f : \nabla(\lambda) \otimes_R U \to M$.

If $\mu \neq \lambda$, we have $\mathrm{Hom}_{\mathscr{C}}(\Delta(\mu), \nabla(\lambda)) = 0$. Further, we have an isomorphism

$$\mathrm{Hom}_{\mathscr{C}}(\Delta(\lambda), f) : \mathrm{Hom}_{\mathscr{C}}(\Delta(\lambda), \nabla(\lambda)) \otimes_R U \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{C}}(\Delta(\lambda), M).$$

Consequently, the map $f = \mathrm{Hom}_{\mathscr{C}}(A, f)$ is injective. Hence, since

$$\mathrm{Ext}^2_{\mathscr{C}}(\Delta(\mu), \nabla(\lambda) \otimes_R U) = 0$$

for all μ, the long exact sequence gives a surjective map

$$\mathrm{Ext}^1_{\mathscr{C}}(\Delta(\mu), M) \to \mathrm{Ext}^1_{\mathscr{C}}(\Delta(\mu), \mathrm{Coker}(f)).$$

The left hand side is 0 by assumption, we deduce that $\mathrm{Ext}^1_{\mathscr{C}}(\Delta(\mu), \mathrm{Coker}(f)) = 0$. We have

$$\{\mu \in \Lambda;\ \mathrm{Hom}_{\mathscr{C}}(\Delta(\mu), \mathrm{Coker}(f)) \neq 0\} \subset \{\mu \in \Lambda;\ \mathrm{Hom}_{\mathscr{C}}(\Delta(\mu), M) \neq 0\} \setminus \{\lambda\}.$$

Therefore, by induction on the set $\{\mu \in \Lambda;\ \mathrm{Hom}_{\mathscr{C}}(\Delta(\mu), M) \neq 0\}$, we get that $\mathrm{Coker}(f)$ has a filtration as required. Since we have an exact sequence

$$0 \to \nabla(\lambda) \otimes_R U \to M \to \mathrm{Coker}(f) \to 0,$$

we deduce that M has also a filtration as required.

Assume now M is projective over R and consider a filtration as in (b). We show that $\mathrm{Hom}_{\mathscr{C}}(\Delta(\lambda), M)$ is projective over R for all λ by induction on r. There is an exact sequence $0 \to L \to P(\lambda_r) \to \Delta(\lambda_r) \to 0$ where L is filtered by $\Delta(\mu)$'s with $\mu > \lambda_r$, so we have $\mathrm{Hom}(\Delta(\lambda_r), M) \simeq \mathrm{Hom}(P(\lambda_r), M)$. We deduce that $\mathrm{Hom}(\Delta(\lambda_r), M)$ is projective over R. By induction, given $i \leq r-1$, then $\mathrm{Hom}(\Delta(\lambda_i), M_{r-1}) \simeq \mathrm{Hom}(\Delta(\lambda_i), M)$ is projective over R and the result follows. □

2.4 Highest weight covers

2.4.1 Definition and characterizations

Let $\mathscr{C}$ be a highest weight R-category and let B be a finite projective R-algebra. Consider a quotient functor $F : \mathscr{C} \to B$-mod in the general sense of [23, sec. III.I], i.e., there is $P \in \mathscr{C}^{\mathrm{proj}}$ and there are isomorphisms $B \xrightarrow{\sim} \mathrm{End}_{\mathscr{C}}(P)^{\mathrm{op}}$ and $F \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{C}}(P, \bullet)$. We denote by G a right adjoint of F and by $\eta : 1 \to GF$ the unit.

We say that F is

- a *highest weight cover* if it is fully faithful on $\mathscr{C}^{\mathrm{proj}}$
- *d-faithful* for some $d \in \mathbb{Z}_{\geqslant -1}$ if $\mathrm{Ext}^i_{\mathscr{C}}(M, N) = 0$ for all $M \in \mathscr{C}$ with $F(M) = 0$, $N \in \mathscr{C}^{\Delta}$ and $i \leqslant d+1$.

As Lemma 2.8 below shows, if F is d-faithful for some $d \geq 0$, then it is a highest weight cover.

We denote by $(B\text{-mod})^{F\Delta}$ the full exact subcategory of B-mod of objects with a filtration whose successive quotients are in $F(\Delta)$. Let $F^{\Delta} : \mathscr{C}^{\Delta} \to (B\text{-mod})^{F\Delta}$ be the restriction of F.

We provide some characterizations of d-faithfulness.

Lemma 2.8 *Let F be a quotient functor. Let $d \in \mathbb{Z}_{\geq 0}$ and let $\mathscr{E} = \mathscr{C}^{\Delta}$, $\mathscr{E} = \Delta(\mathscr{C})$ or $\mathscr{E} = \mathscr{C}^{\mathrm{tilt}}$. The following conditions are equivalent*

(i) *F is d-faithful*
(ii) *given $M \in \mathscr{C}$ with $F(M) = 0$ and $N \in \mathscr{E}$, we have $\mathrm{Ext}^{\leq d+1}_{\mathscr{C}}(M, N) = 0$*
(iii) *given $N \in \mathscr{E}$, we have $H^{\leqslant d}(\mathrm{cone}(N \xrightarrow{\eta} RGF(N))) = 0$*
(iv) *given $M \in \mathscr{C}$, $N \in \mathscr{E}$ and $i \leqslant d$, then F induces an isomorphism $\mathrm{Ext}^i_{\mathscr{C}}(M, N) \xrightarrow{\sim} \mathrm{Ext}^i_B(FM, FN)$*

(v) *given* $M \in \mathscr{C}^{\mathrm{proj}}$, $N \in \mathscr{E}$ *and* $i \leqslant d$, *then* F *induces an isomorphism* $\mathrm{Ext}^i_{\mathscr{C}}(M, N) \xrightarrow{\sim} \mathrm{Ext}^i_B(FM, FN)$.

If R *is a field, these conditions are equivalent to*

(vi) *given* $\lambda \in \Lambda$ *with* $FL(\lambda) = 0$, *then* $\mathrm{lcd}_{\mathscr{C}}(L(\lambda)) > d + 1$.

Proof Note that (ii) in the case $\mathscr{E} = \mathscr{C}^{\Delta}$ is the statement (i). It is clear that (ii) for $\mathscr{E} = \Delta(\mathscr{C})$ is equivalent to (ii) for $\mathscr{E} = \mathscr{C}^{\Delta}$, and these imply (ii) for $\mathscr{E} = \mathscr{C}^{\mathrm{tilt}}$. Assume (ii) holds in the case $\mathscr{E} = \mathscr{C}^{\mathrm{tilt}}$. Let $M \in \mathscr{C}$ with $F(M) = 0$. We prove by induction on λ that $\mathrm{Ext}^{\leq d+1}_{\mathscr{C}}(M, \Delta(\lambda)) = 0$.

There is an exact sequence $0 \to \Delta(\lambda) \to T(\lambda) \to L \to 0$, where L has a filtration by $\Delta(\mu)$'s with $\mu < \lambda$. We have $\mathrm{Ext}^{\leq d+1}_{\mathscr{C}}(M, T(\lambda)) = 0$ and, by induction, we have $\mathrm{Ext}^{\leq d+1}_{\mathscr{C}}(M, L) = 0$. We deduce that $\mathrm{Ext}^{\leq d+1}_{\mathscr{C}}(M, \Delta(\lambda)) = 0$. So, (ii) holds for $\mathscr{E} = \mathscr{C}^{\Delta}$.

Let $X = \mathrm{cone}(N \xrightarrow{\eta} RGF(N))$. We have $F(H^i(X)) = 0$ for all i. Given $Y \in D^b(\mathscr{C})$ such that $F(Y) = 0$, we have

$$\mathrm{Hom}_{D^b(\mathscr{C})}(Y, RGF(N)) \simeq \mathrm{Hom}_{D^b(B)}(F(Y), F(N)) = 0,$$

hence $\mathrm{Hom}_{D^b(\mathscr{C})}(Y, X[i]) \simeq \mathrm{Hom}_{D^b(\mathscr{C})}(Y, N[i+1])$ for all i.

Assume (ii). As usual, let $\tau_{\leqslant m}$ denote the *canonical truncation* which takes a complex $C = (C^n, d^n)$ to the subcomplex

$$\tau_{\leqslant m}(C) = \{\cdots \to C^{m-1} \to \mathrm{Ker}(d^m) \to 0 \to \cdots\}.$$

Taking $Y = \tau_{\leq d}(X)$ above, we obtain $\mathrm{Hom}_{D^b(\mathscr{C})}(\tau_{\leq d}(X), X) = 0$, hence $\tau_{\leq d}(X) = 0$. So, (iii) holds.

Note that the canonical map $\mathrm{Ext}^i_{\mathscr{C}}(M, N) \to \mathrm{Ext}^i_B(FM, FN)$ is an isomorphism if and only if the canonical map $\mathrm{Ext}^i_{\mathscr{C}}(M, N) \to \mathrm{Hom}_{D^b(\mathscr{C})}(M, RGFN[i])$ is an isomorphism.

Assume (iii). Applying $\mathrm{Hom}(M, -)$ to the distinguished triangle $N \to RGF(N) \to X \rightsquigarrow$, we deduce that (iv) holds.

It is clear that (iv) $\Rightarrow$ (v). Assume (v). It follows from Lemma 2.10 that the canonical map $\mathrm{Ext}^{d+1}_{\mathscr{C}}(M, N) \to \mathrm{Ext}^{d+1}_B(F(M), F(N))$ is injective for all $M \in \mathscr{C}$, and (ii) follows.

Assume finally R is a field. The assertion (ii), in the case $\mathscr{E} = \mathscr{C}^{\mathrm{tilt}}$, follows from the case M simple: that is assertion (vi). □

Remark 2.9 We leave it to the reader to check that the first three equivalences in Lemma 2.8 hold when $d = -1$.

Lemma 2.10 *Let* F *be an exact functor, let* $d \geq -1$ *and let* $N \in \mathscr{C}$. *Assume* F *induces*

- *an isomorphism* $\operatorname{Ext}^i_{\mathscr{C}}(P,N) \xrightarrow{\sim} \operatorname{Ext}^i_B(F(P),F(N))$ *for* $P \in \mathscr{C}^{\mathrm{proj}}$ *and* $i \le d$
- *an injection* $\operatorname{Ext}^{d+1}_{\mathscr{C}}(P,N) \hookrightarrow \operatorname{Ext}^{d+1}_B(F(P),F(N))$ *for* $P \in \mathscr{C}^{\mathrm{proj}}$.

Then, F *induces*

- *an isomorphism* $\operatorname{Ext}^i_{\mathscr{C}}(M,N) \xrightarrow{\sim} \operatorname{Ext}^i_B(F(M),F(N))$ *for* $M \in \mathscr{C}$ *and* $i \le d$
- *an injection* $\operatorname{Ext}^{d+1}_{\mathscr{C}}(M,N) \hookrightarrow \operatorname{Ext}^{d+1}_B(F(M),F(N))$ *for* $M \in \mathscr{C}$.

Proof We prove by induction on i the first statement of the lemma. Consider an exact sequence $0 \to M' \to P \to M \to 0$ with $P \in \mathscr{C}^{\mathrm{proj}}$. We have a commutative diagram with exact horizontal sequences

$$\begin{tikzcd}
\operatorname{Ext}^i_{\mathscr{C}}(P,N) \arrow[r] \arrow[d, "\sim"'] & \operatorname{Ext}^i_{\mathscr{C}}(M',N) \arrow[r] \arrow[d] & \operatorname{Ext}^{i+1}_{\mathscr{C}}(M,N) \arrow[r] \arrow[d] & \operatorname{Ext}^{i+1}_{\mathscr{C}}(P,N) \arrow[r] \arrow[d] & \operatorname{Ext}^{i+1}_{\mathscr{C}}(M',N) \arrow[d] \\
\operatorname{Ext}^i_B(FP,FN) \arrow[r] & \operatorname{Ext}^i_B(FM',FN) \arrow[r] & \operatorname{Ext}^{i+1}_B(FM,FN) \arrow[r] & \operatorname{Ext}^{i+1}_B(FP,FN) \arrow[r] & \operatorname{Ext}^{i+1}_B(FM',FN)
\end{tikzcd}$$

where the fourth vertical map is an isomorphism for $i+1 \le d$ and is injective for $i = d$. By induction, the second vertical map is an isomorphism, hence the third vertical map is injective. So, we have shown that the canonical map $\operatorname{Ext}^{i+1}_{\mathscr{C}}(L,N) \to \operatorname{Ext}^{i+1}_B(F(L),F(N))$ is injective for all $L \in \mathscr{C}$, in particular for $L = M'$. If $i+1 \le d$, we deduce that the third vertical map is an isomorphism. $\square$

Let us summarize some of the results above.

Corollary 2.11 *Let* $F : \mathscr{C} \to B\text{-mod}$ *be a quotient functor.*

- F *is* (-1)*-faithful if and only if* F^Δ *is faithful*
- F *is a highest weight cover if and only if* $\eta(M) : M \to GF(M)$ *is an isomorphism for all* $M \in \mathscr{C}^{\mathrm{proj}}$
- F *is* 0*-faithful if and only if* F^Δ *is fully faithful if and only if* $\eta(M) : M \to GF(M)$ *is an isomorphism for all* $M \in \mathscr{C}^\Delta$
- F *is* 1*-faithful if and only if* F^Δ *is an equivalence.*

The next two lemmas relate highest weight covers of $\mathscr{C}$, $\mathscr{C}^*$ and $\mathscr{C}^\diamond$.

Lemma 2.12 *Consider a highest weight cover* $F = \operatorname{Hom}_{\mathscr{C}}(P,\bullet) : \mathscr{C} \to B\text{-mod}$. *Then* $F^* = \operatorname{Hom}_{\mathscr{C}^*}(\operatorname{Hom}_A(P,A),\bullet) : \mathscr{C}^* \to B^{\mathrm{op}}\text{-mod}$ *is a highest weight cover.*

Let $d \ge 0$. *Then*, F *is* d*-faithful if and only if* F^* *induces isomorphisms* $\operatorname{Ext}^i_{\mathscr{C}^*}(M,N) \xrightarrow{\sim} \operatorname{Ext}^i_{B^{\mathrm{op}}}(F^*M,F^*N)$ *for all* $M, N \in (\mathscr{C}^*)^\nabla$ *and* $i \le d$.

Proof There is a commutative diagram

$$\begin{tikzcd}
(\mathscr{C}^{\Delta})^{\mathrm{op}} \arrow[r, "\mathrm{Hom}_R(\bullet,R)", "\sim"'] \arrow[d, "F"'] & (\mathscr{C}^*)^{\nabla} \arrow[d, "F^*"] \\
(B\text{-mod})^{\mathrm{op}} \cap R^{\mathrm{proj}} \arrow[r, "\sim", "\mathrm{Hom}_R(\bullet,R)"'] & B^{\mathrm{op}}\text{-mod} \cap R^{\mathrm{proj}}
\end{tikzcd}$$

since

$$\begin{aligned}\mathrm{Hom}_{A^{\mathrm{op}}}(\mathrm{Hom}_A(P,A),\mathrm{Hom}_R(\bullet,R)) &\simeq \mathrm{Hom}_R(\mathrm{Hom}_A(P,A)\otimes_A \bullet, R)\\ &\simeq \mathrm{Hom}_R(\mathrm{Hom}_A(P,\bullet),R).\end{aligned}$$

The lemma follows, since (higher) extensions can be computed in the exact subcategories appearing in the diagram. □

The next lemma is clear.

Lemma 2.13 *Let $T \in \mathscr{C}^{\nabla}$ and consider a finite projective R-algebra B with a morphism of algebras $\phi : B \to \mathrm{End}_{\mathscr{C}}(T)^{\mathrm{op}}$. Let $F = \mathrm{Hom}_{\mathscr{C}}(T,\bullet)$, $P = \mathscr{R}(T)$ and $F^{\diamond} = \mathrm{Hom}_{\mathscr{C}^{\diamond}}(P,\bullet) : \mathscr{C}^{\diamond} \to B\text{-mod}$.*

The functor $F^{\diamond}$ is a highest weight cover if and only if T is tilting, F is fully faithful on $\mathscr{C}^{\mathrm{tilt}}$ and ϕ is an isomorphism.

The functor $F^{\diamond}$ is d-faithful if and only if T is tilting, ϕ is an isomorphism and F induces isomorphisms $\mathrm{Ext}^i_{\mathscr{C}}(M,N) \xrightarrow{\sim} \mathrm{Ext}^i_B(FM,FN)$ for all $M, N \in \mathscr{C}^{\nabla}$ and $i \le d$.

We say that an R-algebra B is *self-injective* if B is relatively R-injective.

Lemma 2.14 *Let $F = \mathrm{Hom}_{\mathscr{C}}(P,\bullet) : \mathscr{C} \to B\text{-mod}$ be a 0-faithful functor. If B is self-injective, then P is tilting.*

Proof Let $\lambda \in \Lambda$. By Lemma 2.10, we have an injection

$$\mathrm{Ext}^1_{\mathscr{C}}(\Delta(\lambda),P) \hookrightarrow \mathrm{Ext}^1_B(F\Delta(\lambda),F(P)).$$

Since $F(P) = B$ is relatively R-injective and $F\Delta(\lambda)$ is projective over R, we deduce that $\mathrm{Ext}^1_B(F\Delta(\lambda),F(P)) = 0$, hence $\mathrm{Ext}^1_{\mathscr{C}}(\Delta(\lambda),P) = 0$. It follows from Lemma 2.7 that P is tilting. □

Lemma 2.15 *Let $\mathscr{C}$ be a highest weight category, $T \in \mathscr{C}^{\mathrm{tilt}}$ and $B = \mathrm{End}_{\mathscr{C}}(T)^{\mathrm{op}}$. Assume the restriction of $\mathrm{Hom}_{\mathscr{C}}(T,\bullet)$ to $\mathscr{C}^{\nabla}$ is fully faithful and B is self-injective. Then T is projective.*

Proof This follows from Lemma 2.14 applied to $\mathscr{C}^{\diamond}$, cf Lemma 2.13. □

2.4.2 Base change

Let S be a local commutative flat R-algebra. If F is d-faithful, then SF is d-faithful, and the converse holds if S is faithfully flat over R (for example, if it is a local S-point).

Lemma 2.16 *Let F be a quotient functor.*

If KF is (-1)-faithful, then F is (-1)-faithful.

Assume R is a regular local ring. If $R_{\mathfrak{p}}F$ is 0-faithful (resp. is a highest weight cover) for all $\mathfrak{p} \in \mathfrak{P}_1$, then F is 0-faithful (resp. is a highest weight cover).

Proof The first statement is obvious, since objects of $\mathscr{C}^{\Delta}$ are projective over R.

Assume now F is (-1)-faithful. Let $M \in \mathscr{C}^{\Delta}$. Consider the exact sequence

$$0 \to M \xrightarrow{\eta(M)} GFM \to \operatorname{coker} \eta(M) \to 0.$$

Assume $R_{\mathfrak{p}} \operatorname{coker} \eta(M) = 0$ for all $\mathfrak{p} \in \mathfrak{P}_1$. Then, the support of $\operatorname{coker} \eta(M)$ has codimension ≥ 2, hence $\operatorname{Ext}^1_R(\operatorname{coker} \eta(M), M) = 0$, since M is projective over R. It follows that $\operatorname{coker} \eta(M)$ is a direct summand of the torsion-free module $GF(M)$, hence $\operatorname{coker} \eta(M) = 0$. The lemma follows. □

The corollary below is immediate.

Corollary 2.17 *Let $\mathscr{C}$ be a highest weight category, $T \in \mathscr{C}^{\mathrm{tilt}}$ and $B = \mathrm{End}_{\mathscr{C}}(T)^{\mathrm{op}}$. Let $F = \mathrm{Hom}_{\mathscr{C}}(T, \bullet)$. Assume R is a regular local ring. Then, the restriction of F to $\mathscr{C}^{\nabla}$ is fully faithful if the restriction of $R_{\mathfrak{p}}F$ to $R_{\mathfrak{p}}\mathscr{C}^{\nabla}$ is fully faithful for all $\mathfrak{p} \in \mathfrak{P}_1$.*

Proof Let $P = \mathscr{R}(T)$ and $F^{\diamond} = \mathrm{Hom}_{\mathscr{C}^{\diamond}}(P, \bullet) : \mathscr{C}^{\diamond} \to B$-mod. The restriction of F to $\mathscr{C}^{\nabla}$ is fully faithful if and only if the restriction of $F^{\diamond}$ to $(\mathscr{C}^{\diamond})^{\nabla}$ is fully faithful. Now, $F^{\diamond}$ is a quotient functor because T is tilting. Thus, by Lemma 2.16, if $R_{\mathfrak{p}}F^{\diamond}$ is 0-faithful for all $\mathfrak{p} \in \mathfrak{P}_1$, then $F^{\diamond}$ is 0-faithful. Finally, by Lemma 2.13, $R_{\mathfrak{p}}F^{\diamond}$ is 0-faithful if the restriction of $R_{\mathfrak{p}}F$ to $R_{\mathfrak{p}}\mathscr{C}^{\nabla}$ is fully faithful. □

The following key result generalizes [39, prop. 4.42].

Proposition 2.18 *Assume R is regular. If $\mathrm{k}F$ is d-faithful, then F is d-faithful. If in addition KF is $(d+1)$-faithful, then F is $(d+1)$-faithful.*

Proof Assume $\mathrm{k}F$ is d-faithful. Let $M \in \mathscr{C}$ with $F(M) = 0$ and let $N \in \mathscr{C}^{\Delta}$. We have $R\,\mathrm{Hom}_{\mathrm{k}\mathscr{C}}(kM, kN) \simeq \mathrm{k} \otimes^{\mathbb{L}}_R R\,\mathrm{Hom}_{\mathscr{C}}(M, N)$. Let C be a bounded complex of finitely generated projective R-modules quasi-isomorphic to $R\operatorname{Hom}_{\mathscr{C}}(M,N)$ and with $C^{<r}=0$. We assume r is maximal with this property. Then, $\operatorname{Ext}^r_{k\mathscr{C}}(kM,kN)\simeq H^r(kC)\neq 0$, hence $r>d+1$, so $\operatorname{Ext}^{\le d+1}_{\mathscr{C}}(M,N)=0$. It follows that F is d-faithful.

Assume now KF is $(d+1)$-faithful. Then $H^{d+2}(C)$ is a torsion R-module. If it is non-zero, then $C^{d+1}\neq 0$ a contradiction. So, $H^{d+2}(C)=0$ and F is $(d+1)$-faithful. □

2.4.3 Uniqueness results

We assume in this section that R is normal.

Let B' be an R-algebra, finitely generated and projective over R, and such that KB' is split semi-simple.

Fix a poset structure on $\operatorname{Irr}(KB')$. Given $E\in\operatorname{Irr}(KB')$, let $(KB')_{\le E}$ (resp. $(KB')_{<E}$) be the sum of the simple KB'-submodules of KB' isomorphic to some $F\le E$ (resp. $F<E$).

We say that a family $\{S(E)\}_{E\in\operatorname{Irr}(KB')}$ of B'-modules, finitely generated and projective over R, are *Specht modules* for B' if

$$(B'\cap(KB')_{\le E})/(B'\cap(KB')_{<E})\simeq S(E)^{\dim_K E}\quad\text{for } E\in\operatorname{Irr}(K'B).$$

Note that $KS(E)\simeq E$ and $\operatorname{End}_{B'}(S(E))=R$. So, if $\{S'(E)\}_{E\in\operatorname{Irr}(KB')}$ is another family of Specht modules, then $S'(E)\simeq S(E)$ for all E: the Specht modules are unique, up to isomorphism (if they exist).

The same construction with the opposite order on $\operatorname{Irr}(KB')$ leads to the *dual Specht modules* $S'(E)\in B$-mod with $KS'(E)\simeq E$.

Assume that the K-algebra KB is semi-simple and that F is a highest weight cover. Then the K-category $K\mathscr{C}$ is split semi-simple and we have an equivalence $KF: K\mathscr{C}\xrightarrow{\sim}KB$-mod. So, the functor KF induces a bijection $\operatorname{Irr}(K\mathscr{C})\xrightarrow{\sim}\operatorname{Irr}(KB)$ and we put $S(\lambda)_K=KF(\Delta(\lambda))\in\operatorname{Irr}(KB)$. The highest weight order on $\operatorname{Irr}(K\mathscr{C})$ yields a partial order on $\operatorname{Irr}(KB)$.

We will say that F is a *highest weight cover of* B *for the order on* $\operatorname{Irr}(KB)$ *coming from the one on* $\operatorname{Irr}(K\mathscr{C})$.

The next lemma follows from [39, Lemma 4.48].

Lemma 2.19 *Let F be a highest weight cover and assume KB is semi-simple. Then B has Specht modules $S(\lambda)=F(\Delta(\lambda))$ and dual Specht modules $S'(\lambda)=F(\nabla(\lambda))$.*

Proposition 2.20 *Let* $F:\mathscr{C}\to B$-mod *and* $F':\mathscr{C}'\to B$-mod *be highest weight covers. Assume R is regular, B is self-injective, and KB is semi-simple. Assume that*

- *the order on* $\operatorname{Irr}(KB)$ *induced by* $(\mathscr{C},F)$ *refines, or is refined by, the order induced by* $(\mathscr{C}',F')$

- *F is fully faithful on $\mathscr{C}^\Delta$ and on $\mathscr{C}^\nabla$*
- *F' is fully faithful on $\mathscr{C}'^\Delta$ and on $\mathscr{C}'^\nabla$*
- (a) *$F(P(\lambda)) \in F'(\mathscr{C}'^{\mathrm{proj}})$ for all $\lambda \in \Lambda$ such that $\mathrm{lcd}_{k\mathscr{C}}(L(\lambda)) \leq 1$ and $F(I(\lambda)) \in F'(\mathscr{C}'^{\mathrm{inj}})$ for all $\lambda \in \Lambda$ such that $\mathrm{rcd}_{k\mathscr{C}}(L(\lambda)) \leq 1$ or*
  (b) *$F(T(\lambda)) \in F'(\mathscr{C}'^{\mathrm{tilt}})$ for all $\lambda \in \Lambda$ such that $\mathrm{lcd}_{k\mathscr{C}^\diamond}(L^\diamond(\lambda)) \leq 1$ or $\mathrm{rcd}_{k\mathscr{C}^\diamond}(L^\diamond(\lambda)) \leq 1$.*

Then, there is an equivalence of highest weight categories $\Phi : \mathscr{C} \xrightarrow{\sim} \mathscr{C}'$ such that $F'\Phi \simeq F$.

Proof Lemma 2.19 shows there is a bijection $p : \Lambda \xrightarrow{\sim} \Lambda'$ such that $F(\Delta(\lambda)) \simeq F'(\Delta'(p(\lambda)))$. Thus, both categories are highest weight for whichever of the orders on $\mathrm{Irr}(KB)$ is coarser, and we may assume that the partial orders coincide.

Let $\mathscr{O} = \mathscr{C}^\diamond$ and $\mathscr{O}' = \mathscr{C}'^\diamond$. Lemma 2.14 shows that P is tilting. So, $\mathscr{R}(P)$ is tilting and projective and, identifying $\mathscr{C}^\blacklozenge$ with $\mathscr{C}^\diamond$, we have $\mathscr{R}^{-1}(P) \simeq \mathscr{R}(P)$. Since F is fully faithful on $\mathscr{C}^\nabla$, it follows from Lemma 2.13 that $F^\diamond = \mathrm{Hom}_{\mathscr{C}^\diamond}(\mathscr{R}(P), -)$ is 0-faithful. Similarly, we deduce that $F^\diamond$ is fully faithful on $(\mathscr{C}^\diamond)^\nabla$, since F is 0-faithful. We prove in the same way that $F'^\diamond = \mathrm{Hom}_{\mathscr{C}'^\diamond}(\mathscr{R}(P'), \bullet)$ is fully faithful on $(\mathscr{C}'^\diamond)^\Delta$ and on $(\mathscr{C}'^\diamond)^\nabla$.

We have $F(P(\lambda)) \in F'(\mathscr{C}'^{\mathrm{proj}})$ if and only if $F^\blacklozenge(T^\blacklozenge(\lambda)) \in F'^\blacklozenge((\mathscr{C}'^\blacklozenge)^{\mathrm{tilt}})$. Similarly, we have $F(I(\lambda)) \in F'(\mathscr{C}'^{\mathrm{inj}})$ if and only if $F^\diamond(T^\diamond(\lambda)) \in F'^\diamond((\mathscr{C}'^\diamond)^{\mathrm{tilt}})$.

Since $\mathscr{C}^\diamond \simeq \mathscr{C}^\blacklozenge$ as highest weight categories, we deduce that the case (a) of the proposition for $(\mathscr{C}, \mathscr{C}', F, F')$ is equivalent to the case (b) of the proposition for $(\mathscr{C}^\diamond, \mathscr{C}'^\diamond, F^\diamond, F'^\diamond)$. We assume from now on that we are in case (a).

Let $\tilde{P} = P \oplus \bigoplus_{\mathrm{lcd}_{k\mathscr{C}}(L(\lambda)) \leq 1} P(\lambda)$, let $\tilde{B} = \mathrm{End}_{\mathscr{C}}(\tilde{P})^{\mathrm{op}}$ and let $\tilde{F} = \mathrm{Hom}_{\mathscr{C}}(\tilde{P}, \bullet) : \mathscr{C} \to \tilde{B}\text{-mod}$. This is a 1-faithful cover by Lemma 2.8 and Proposition 2.18. So the functor $\tilde{F}$ restricts to an equivalence $\tilde{F}^\Delta : \mathscr{C}^\Delta \xrightarrow{\sim} (\tilde{B}\text{-mod})^{\tilde{F}\Delta}$, with inverse $\mathrm{Hom}_{\tilde{B}}(\tilde{F}(A), \bullet)$.

Consider $\tilde{P}' \in \mathscr{C}'^{\mathrm{proj}}$ such that $F'(\tilde{P}') \simeq F(\tilde{P})$. Fixing such an isomorphism, we obtain an isomorphism $\tilde{B} \xrightarrow{\sim} \mathrm{End}_{\mathscr{C}'}(\tilde{P}')^{\mathrm{op}}$. Note that P' is a direct summand of $\tilde{P}'$, since $F'(P') \simeq B \simeq F(P)$. Let $\tilde{F}' = \mathrm{Hom}_{\mathscr{C}'}(\tilde{P}', \bullet) : \mathscr{C}' \to \tilde{B}\text{-mod}$, a highest weight cover. Lemma 2.19 shows that $\tilde{F}'(\Delta'(\lambda)) \simeq \tilde{F}(\Delta(\lambda))$ for all $\lambda \in \Lambda$.

Let i be the idempotent of $\tilde{B}$ such that $\tilde{P}i = P$. The right action of B on P provides an isomorphism $B \xrightarrow{\sim} i\tilde{B}i$. This equips $\tilde{B}i$ with a structure of $(\tilde{B}, B)$-bimodule. Let $\hat{F} = \mathrm{Hom}_{\tilde{B}}(\tilde{B}i, \bullet) : \tilde{B}\text{-mod} \to B\text{-mod}$.

We have an isomorphism $\hat{F} \circ \tilde{F} \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{C}}(\tilde{P} \otimes_{\tilde{B}} \tilde{B}i, \bullet)$, hence $\hat{F} \circ \tilde{F} \xrightarrow{\sim} F$. Similarly, we have an isomorphism $\hat{F} \circ \tilde{F}' \xrightarrow{\sim} F'$. Consider the exact functor

$$\Phi = \mathrm{Hom}_{\mathscr{C}'}(\tilde{P}' \otimes_{\tilde{B}} \tilde{F}(A), \bullet) \simeq \mathrm{Hom}_{\tilde{B}}(\tilde{F}(A), \bullet) \circ \tilde{F}'^{\Delta} : (\mathscr{C}')^{\Delta} \to \mathscr{C}^{\Delta}.$$

We have an isomorphism $\tilde{F}^{\Delta} \circ \Phi \xrightarrow{\sim} \tilde{F}'^{\Delta}$ and there is a commutative diagram

$$\begin{tikzcd}
\mathscr{C}'^{\Delta} \arrow[rr, "\Phi"] \arrow[dr, "\tilde{F}'^{\Delta}"] \arrow[ddr, hook, "F'^{\Delta}"'] & & \mathscr{C}^{\Delta} \arrow[dl, "\tilde{F}^{\Delta}"', "\sim"] \arrow[ddl, hook', "F^{\Delta}"] \\
& (\tilde{B}\text{-mod})^{\tilde{F}\Delta} \arrow[d, "\hat{F}^{\Delta}"] & \\
& (B\text{-mod})^{F\Delta} &
\end{tikzcd}$$

Since F^{Δ} is fully faithful and $\tilde{F}^{\Delta}$ is an equivalence, we deduce that $\hat{F}^{\Delta}$ is fully faithful. Since F'^{Δ} is fully faithful, we deduce that $\tilde{F}'^{\Delta}$ is fully faithful. It follows that Φ is fully faithful. Note that $\Phi(\Delta'(\lambda)) \simeq \Delta(\lambda)$ for all $\lambda \in \Lambda$. Since $\tilde{F}(P) = \tilde{B}i \simeq \tilde{F}'(P')$, we have $\Phi(P') \simeq P$.

Define

$$\tilde{\Psi} = \mathrm{Hom}_{\mathscr{C}}(\Phi(A'), \bullet) : \mathscr{C} \to \mathscr{C}'$$

Since $\Phi(A') \in \mathscr{C}^{\Delta}$, it follows that $\tilde{\Psi}$ is exact on $\mathscr{C}^{\nabla}$. We have

$$\tilde{\Psi}(P) \simeq \mathrm{Hom}_{\mathscr{C}}(\Phi(A'), \Phi(P')) \simeq \mathrm{Hom}_{\mathscr{C}'}(A', P') \simeq P'.$$

Let us fix an isomorphism $\tilde{\Psi}(P) \xrightarrow{\sim} P'$. Let $I \subset \Lambda$ be an ideal. Define $(KP)_I$ as the sum of the simple submodules of KP isomorphic to $K\nabla(\mu)$ for some $\mu \in I$. Let $P_I = P \cap (KP)_I$. Given $\lambda \in \Lambda$, we have $P_{\leq\lambda}/P_{<\lambda} \simeq \nabla(\lambda)^n$ for some $n > 0$, since P is tilting (Lemma 2.14) and KP is a progenerator of $K\mathscr{C}$. We have $K\tilde{\Psi}((KP)_I) = (KP')_I$ for all ideals $I \subset \Lambda$, hence $\tilde{\Psi}(\nabla(\lambda)) \simeq \nabla'(\lambda)$ for all $\lambda \in \Lambda$. We deduce that $\tilde{\Psi}$ restricts to an exact functor $\Psi : \mathscr{C}^{\nabla} \to \mathscr{C}'^{\nabla}$. We have

$$\begin{aligned}
\Phi(A') \otimes_{\mathscr{C}'} P' &\simeq \mathrm{Hom}_{\mathscr{C}'}(\tilde{P}' \otimes_{\tilde{B}} \tilde{F}(A), P') \simeq \mathrm{Hom}_{\tilde{B}}(\tilde{F}(A), \tilde{F}'(P')) \\
&\simeq \mathrm{Hom}_{\tilde{B}}(\tilde{F}(A), \tilde{F}(P)) \simeq \mathrm{Hom}_{\mathscr{C}}(A, P) \simeq P,
\end{aligned}$$

hence

$$\begin{aligned}
F' \circ \tilde{\Psi} &= \mathrm{Hom}_{\mathscr{C}'}(P', \mathrm{Hom}_{\mathscr{C}}(\Phi(A'), \bullet)) \\
&\simeq \mathrm{Hom}_{\mathscr{C}}(\Phi(A') \otimes_{\mathscr{C}'} P', \bullet) \simeq \mathrm{Hom}_{\mathscr{C}}(P, \bullet) = F.
\end{aligned}$$

Since F^{∇} and F'^{∇} are fully faithful, we deduce that Ψ is fully faithful.

We now apply what we have proven to $\mathscr{C}^*$ and $\mathscr{C}'^*$ (cf Lemma 2.12). We obtain a full faithful exact functor $\Psi_* : \mathscr{C}^{*\nabla} \to \mathscr{C}'^{*\nabla}$, hence a fully faithful exact functor $\Upsilon = \Psi(\bullet^*)^* : \mathscr{C}^\Delta \to \mathscr{C}'^\Delta$ such that $\Upsilon(\Delta(\lambda)) \simeq \Delta(\lambda')$ for all $\lambda \in \Lambda$. The composition $\Phi\Upsilon$ is a fully faithful exact endofunctor of $\mathscr{C}^\Delta$ and $F\Phi\Upsilon \simeq F$. It follows that $\Phi\Upsilon$ fixes isomorphism classes of objects, hence it is an equivalence. Similarly, $\Upsilon\Phi$ is an equivalence, hence Φ is an equivalence $(\mathscr{C}')^\Delta \to \mathscr{C}^\Delta$. The proposition follows from Lemma 2.2. □

2.4.4 *Covers of truncated polynomial rings in one variable*

Let I be a non-empty finite poset and $\{q_i\}_{i\in I}$ a family of elements of R. We denote by $\bar{q}_i$ the image of q_i in k. We assume that given $i, j \in I$, then $\bar{q}_i = \bar{q}_j$ if and only $i \leqslant j$ or $j \leqslant i$.

Let $B = R[T]/\left(\prod_{i\in I}(T - q_i)\right)$. This is a free R-algebra, with basis $(1, T, \dots, T^{d-1})$. Given $j \in I$, let $S_j = R[T]/(T - q_j)$ and $Y_j = R[T]/\left(\prod_{i\geqslant j}(T - q_i)\right)$. We put $Y = \bigoplus_{j\in I} Y_j$, $A = \mathrm{End}_B(Y)^{\mathrm{op}}$, $G = \mathrm{Hom}_B(Y, \bullet) : B\text{-mod} \to A\text{-mod}$, $P = G(B)$ and $F = \mathrm{Hom}_A(P, \bullet) : A\text{-mod} \to B\text{-mod}$. Let $\Delta(j)$ be the quotient of $G(Y_j)$ by the subspace of maps $Y \to Y_j$ that factor through $Y_{j'}$ for some $j' > j$.

Proposition 2.21 (a) $\mathscr{C} = A$-mod *is a highest weight R-category on the poset I with standard objects the $\Delta(j)$'s. The functor F is a (-1)-faithful highest cover of B and we have $F(\Delta(j)) \simeq S_j$, $F(P(j)) \simeq Y_j$ and $P(j) = G(Y_j)$. If $q_i \neq q_j$ for $i \neq j$, then F is a 0-faithful cover of B.*

(b) *Assume $\mathscr{C}'$ is a highest weight R-category with poset I and $F' : \mathscr{C}' \to B$-mod is a highest weight cover. If R is a field or $KF'(\Delta(j)) \simeq KS_j$ for all j, then there is an equivalence of highest weight categories $\Phi : \mathscr{C} \xrightarrow{\sim} \mathscr{C}'$ such that $F'\Phi \simeq F$.*

Proof Let $\bar{I}$ be the quotient of I by the relation $i \sim j$ if $\bar{q}_i = \bar{q}_j$. We have a block decomposition $B \simeq \bigoplus_{J\in\bar{I}} R[T]/\left(\prod_{i\in J}(T - q_i)\right)$, and if the proposition holds for the individual blocks, then it holds for B. As a consequence, it is enough to prove the proposition when $\bar{q}_i = \bar{q}_j$ for all $i, j \in I$. Choosing $i \in I$ and replacing T by $T - q_i$, we can assume further that $\bar{q}_i = 0$ for all $i \in I$. Since the poset structure on I is now a total order, we can assume $I = \{0, \dots, d-1\}$ with the usual order, for some $d \geq 1$.

Assume first R is a field with $B = R[T]/T^d$. Note that $Y_j = R[T]/T^{d-j}$ and that $\{Y_j\}_{j\in I}$ is a complete set of representatives of isomorphism classes of indecomposable B-modules. Denote by e_j the idempotent of A corresponding to the projection onto Y_j. Then, the projective indecomposable A-modules are the $P(j) = Ae_j$, $j \in I$. Note that $\mathrm{End}(P(d-1)) = R$. Let $L = Ae_{d-1}A$. We have $L^2 = L$, $L \simeq P(d-1)^d$ as left A-modules and $A/L \simeq \mathrm{End}_{R[T]/(T^{d-1})}\left(\bigoplus_{0\leq i\leq d-2} R[T]/(T^{d-i-1})\right)^{\mathrm{op}}$. It follows that A-mod is a highest weight category on the poset I, with $\Delta(j) = Ae_j/Ae_{j+1}Ae_j$, see [10, lem. 3.4]. Let us state some properties of $\mathscr{C}$, that can be easily checked. The module $\Delta(j)$ is uniserial, with composition series $L(j), L(j-1), \dots, L(0)$, starting from the head. We have $[P(j) : \Delta(i)] = 1$ if $i \geq j$, and $[P(j) : \Delta(i)] = 0$ otherwise. The module $P = P(0)$ is projective and injective, while $P(d-1) = \Delta(d-1)$. Note that F is exact and its restriction to A-proj is fully faithful. Since every $\Delta(j)$ embeds in P, it follows that F is (-1)-faithful. Note that $F(\Delta(j)) \simeq R$.

Consider now $\mathscr{C}'$ and F' as in the proposition. Since $\mathscr{C}'$ has d non-isomorphic projective indecomposable modules, it follows that $\{F'(P'(j))\}_{j\in I} = \{Y_j\}_{j\in I}$. As a consequence, there is a permutation σ of I and an equivalence $\Phi : \mathscr{C}\text{-proj} \xrightarrow{\sim} \mathscr{C}'\text{-proj}$ such that $\Phi(P(\sigma(j))) \simeq P'(j)$ and $F'\Phi \simeq F$. Such an equivalence extends to an equivalence $\Phi : \mathscr{C} \xrightarrow{\sim} \mathscr{C}'$, and $F'\Phi \simeq F$. So, $\mathscr{C}$ is a highest weight category with the order given by $i \leqslant' j$ if $\sigma(i) \leqslant \sigma(j)$. Note that $\mathrm{End}(P(j)) = R$ if and only if $j = d-1$. It follows that $d-1$ must be maximal for the order $\leqslant'$, and considering the quotient algebra A/L as above, one sees by induction that $\leqslant' = \leqslant$, i.e., $\sigma = 1$, hence Φ is an equivalence of highest weight categories. This shows the proposition when R is a field.

Assume now R is a general local ring. The R-modules $\Delta(j)$ are free and $\mathrm{k}A \simeq \mathrm{End}_{\mathrm{k}B}(\mathrm{k}Y)$. We deduce that $\mathscr{C}$ is a highest weight category and F is a (-1)-faithful highest weight cover. If KB is semi-simple, it follows from Proposition 2.18 that F is 0-faithful (the regularity of R is not necessary here).

We consider finally $\mathscr{C}'$ and F' as in the proposition. Since the canonical map $\mathrm{k}\,\mathrm{Hom}_B(Y_i, Y_j) \to \mathrm{Hom}_{\mathrm{k}B}(\mathrm{k}Y_i, \mathrm{k}Y_j)$ is an isomorphism for all i, j, we deduce that $\mathrm{k}F'$ is a highest weight cover, hence equivalent to $\mathrm{k}F$. As a consequence, F' is 0-faithful and $\mathrm{k}F'(P'(j)) \simeq \mathrm{k}Y_j$ for all j. We deduce that $[P'(j) : \Delta(i)] = \delta_{i\geq j}$, and it follows that $[KF'(P'(j))] = [KS_j] + \cdots + [KS_{d-1}]$ in $K_0(KB\text{-mod})$. There is a surjective morphism of B-modules $B \to \mathrm{k}F'(P'(j))$. It lifts to a surjective morphism of B-modules $B \to F'(P'(j))$. Since $F'(P'(j))$ is free over R, there is a subset J of I of cardinality j with $F'(P'(j)) \simeq B/\big(\prod_{i\in J}(T-q_i)\big)$. It follows that $[KF'(P'(j))] = \sum_{i\notin J}[KS'_i]$, hence $F'(P'(j)) \simeq Y_j$, as $\{q_i\}_{i\in J} = \{q_i\}_{i\geq j}$. The proposition follows. □

Similarly, set $Z_j = R[T]/\prod_{i\leqslant j}(T-q_i)$. Then, we can prove the following.

Corollary 2.22 *Assume further that $\mathscr{C}'$ is a highest weight R-category with poset I and $F' : \mathscr{C}' \to B$-mod is a 0-faithful cover. If $KF'(\Delta'(j)) \simeq KS_j$ for all j, then we have $F'(P'(j)) \simeq Y_j$ and $F'(T'(j)) \simeq Z_j$.*

Proof The isomorphism $F'(P'(j)) \simeq Y_j$ has been proved above. Let us prove that $F'(T'(j)) \simeq Z_j$. As above, we can assume that $I = \{0, \dots, d-1\}$ with the usual order. Let $(\mathscr{C}')^{\leqslant i} \subset \mathscr{C}'$ be the highest weight subcategory associated with the ideal $\{j \leqslant i;\ j \in I\} \subset I$. By [39, prop. 4.26], under the embedding $(\mathscr{C}')^{\leqslant i} \subset \mathscr{C}'$ we have $T'(j) = P'(0)^{\leqslant j}$. The restriction of F' to $(\mathscr{C}')^{\leqslant j}$ is 0-faithful. Hence, the proof above implies that $F'(P'(0)^{\leqslant j}) = R[T]/\prod_{i \leqslant j}(T - q_i) = Z_j$. □

2.5 Complement on symmetric algebras

Let R be a commutative noetherian ring. Let B be an R-algebra. We say that B is *symmetric* if it is a finitely generated projective R-module and B is isomorphic to B^* as a (B, B)-bimodule.

Proposition 2.23 *Let B be a symmetric R-algebra. Assume R is a domain with field of fractions K and KB is a split semi-simple algebra. Let ψ be an R-algebra endomorphism of B.*

If $K\psi$ is an automorphism of KB that induces the identity map on $K_0(KB)$, then ψ is an automorphism.

Proof Let $t \in \mathrm{Hom}_R(B, R)$ be a symmetrizing form for B, the image of 1 through an isomorphism of (B, B)-bimodules $B \xrightarrow{\sim} B^*$. Note that $t([B, B]) = 0$.

Since KB is split semi-simple, the character map is an isomorphism $K \otimes_{\mathbb{Z}} K_0(KB) \to \mathrm{Hom}_K(KB/[KB, KB], K)$. We deduce that ψ induces the identity on $KB/[KB, KB]$, hence $t \circ \psi = t$.

Consider a maximal ideal $\mathfrak{m}$ of R, and let $\mathrm{k} = R/\mathfrak{m}$. The k-algebra $\mathrm{k}B$ is symmetric, with symmetrizing form $\mathrm{k}t$ and $(\mathrm{k}t) \circ (\mathrm{k}\psi) = \mathrm{k}t$. It follows that $\mathrm{k}t(\ker(\mathrm{k}\psi)) = 0$, hence $\ker(\mathrm{k}\psi) = 0$, since the kernel of a symmetrizing form contains no nonzero ideal. We deduce that $\mathrm{k}\psi$ is an isomorphism.

We have shown that $(R/\mathfrak{m})\psi$ is onto for every maximal ideal $\mathfrak{m}$ of R. It follows that ψ is onto, hence it is an isomorphism, since B is a finitely generated projective R-module. □

3 Hecke algebras, q-Schur algebras and categorifications

Let R be a $\mathbb{C}[q, q^{-1}]$-algebra. Let q_R be the image of q in R. If no confusion is possible, we may abbreviate $q = q_R$.

3.1 Quivers

Assume that $q_R \neq 1$. For any subset $\mathscr{I} \subset R^\times$ we associate a quiver $\mathscr{I}(q)$ with set of vertices $\mathscr{I}$ and with an arrow $i \to i\,q_R$ whenever $i, i\,q_R \in \mathscr{I}$. We may abbreviate $\mathscr{I} = \mathscr{I}(q)$ when there is no risk of confusion. Note that we do not assume $\mathscr{I}(q)$ to be connected or $\mathscr{I}$ to be finite. We will assume that $(q^{\mathbb{Z}}\mathscr{I}(q))/q^{\mathbb{Z}}$ is finite.

Let $Q_{R,1}, \ldots, Q_{R,\ell} \in \mathscr{I}$ such that $\mathscr{I} = \bigcup_{p=1}^{\ell} \mathscr{I}_p$, where $\mathscr{I}_p = \mathscr{I} \cap q_R^{\mathbb{Z}} Q_{R,p}$. We write $i \equiv j$ if $i \in q^{\mathbb{Z}} j$. Each equivalence class has a representative (possibly more than one) in the set $\{Q_{R,1}, Q_{R,2}, \ldots, Q_{R,\ell}\}$.

If $\mathscr{I}(q)$ is stable under multiplication by $q_R^{\mathbb{Z}}$, and q_R is not a root of 1, then each $\mathscr{I}_p$ is isomorphic to the quiver A_∞. If $\mathscr{I}(q)$ is stable under multiplication by $q_R^{\mathbb{Z}}$, and q_R is a primitive e-th of 1, then each $\mathscr{I}_p$ is isomorphic to the quiver $A_{e-1}^{(1)}$.

For any subset $I \subset R$ we consider also the quiver I_1 with the set of vertices I and with an arrow $i \to i+1$ whenever $i, i+1 \in I$. We may abbreviate $I = I_1$.

3.2 Kac–Moody algebras associated with a quiver

Let (a_{ij}) be the generalized Cartan matrix associated with the quiver $\mathscr{I}$ and let $\mathfrak{sl}_{\mathscr{I}}$ be the (derived) Kac–Moody algebra over $\mathbb{C}$ associated with (a_{ij}). The Lie algebra $\mathfrak{sl}_{\mathscr{I}}$ is generated by E_i, F_i with $i \in \mathscr{I}$, subject to the usual relations. Fix a subset $\Omega \subset [1, \ell]$ such that $\mathscr{I}$ is the disjoint union $\mathscr{I} = \bigsqcup_{p \in \Omega} \mathscr{I}_p$. We have a Lie algebra decomposition $\mathfrak{sl}_{\mathscr{I}} = \bigoplus_{p \in \Omega} \mathfrak{sl}_{\mathscr{I}_p}$.

For each $i \in \mathscr{I}$, let α_i, $\check{\alpha}_i$ be the simple root and coroot corresponding to E_i and let Λ_i be the i-th fundamental weight. Set $Q = \bigoplus_{i \in \mathscr{I}} \mathbb{Z}\alpha_i$ and $Q^+ = \bigoplus_{i \in \mathscr{I}} \mathbb{N}\alpha_i$. Set $P = \bigoplus_{i \in \mathscr{I}} \mathbb{Z}\Lambda_i$ and $P^+ = \bigoplus_{i \in \mathscr{I}} \mathbb{N}\Lambda_i$.

Let X be the free abelian group with basis $\{\varepsilon_i;\ i \in \mathscr{I}\}$. The assignment $\alpha_i \mapsto \varepsilon_i - \varepsilon_{iq}$ yields additive maps $Q, Q^+ \to X$. If $\mathscr{I}$ is bounded below then we may identify Λ_i with the (finite) sum $\sum_{d \in \mathbb{N}} \varepsilon_{iq^{-d}}$. So, we may consider P, P^+ as subsets of X.

We will write $P = P_{\mathscr{I}}$, $Q = Q_{\mathscr{I}}$, $Q^+ = Q^+_{\mathscr{I}}$ and $X = X_{\mathscr{I}}$ if necessary. For $\alpha \in Q^+$ of height d we write $\mathscr{I}^\alpha = \{\mathbf{i} = (i_1, \ldots, i_d) \in \mathscr{I}^d;\ \alpha_{i_1} + \cdots + \alpha_{i_d} = \alpha\}$. The set $\mathscr{I}^\alpha$ is an orbit for the action of the symmetric group $\mathfrak{S}_d$ on $\mathscr{I}^d$ by permutation. Each $\mathfrak{S}_d$-orbit in $\mathscr{I}^d$ is of this form.

For any subset $I \subset R$ we consider also the quiver I_1 which yields in the same way as above a Cartan datum and a Lie algebra $\mathfrak{sl}_I$.

3.3 Partitions

Set $\mathbb{Z}^\ell(n) = \{(\nu_1, \ldots, \nu_\ell) \in \mathbb{Z}^\ell;\ \nu_1 + \cdots + \nu_\ell = n\}$, $\mathscr{C}_n^\ell = \{\nu \in \mathbb{Z}^\ell(n);\ \nu_p \geqslant 0,\ \forall p\}$, and $\mathscr{C}_{n,+}^\ell = \{\nu \in \mathbb{Z}^\ell(n);\ \nu_p > 0,\ \forall p\}$. An element of $\mathscr{C}_n^\ell$ is a *composition* of n into ℓ parts. We will say that the composition ν is *dominant* if it satisfies the inequalities $\nu_1 \geqslant \nu_2 \geqslant \cdots \geqslant \nu_\ell$, and that it is *anti-dominant* if we have $\nu_1 \leqslant \nu_2 \leqslant \cdots \leqslant \nu_\ell$.

Let $\mathscr{P}_n$ be the set of *partitions* of n, i.e., the set of non-increasing sequences of positive integers with sum n. For $\lambda \in \mathscr{P}_n$, let $|\lambda| = n$ be the weight of λ, let $l(\lambda)$ be the number of parts in λ and let ${}^t\lambda$ be the transposed partition.

We associate to λ the *Young diagram* $Y(\lambda)$ with λ_i boxes in the i-th row. Let $\mathscr{P}^\ell_n$ be the set of *ℓ-partitions* of n, i.e., the set of ℓ-tuples of partitions $\lambda = (\lambda^1, \dots, \lambda^\ell)$ with $\sum_p |\lambda^p| = n$. Let $\mathscr{P} = \bigsqcup_n \mathscr{P}_n$ and $\mathscr{P}^\ell = \bigsqcup_n \mathscr{P}^\ell_n$. For each $\nu \in \mathscr{C}^\ell_n$ and $d \in [1, n]$ we set $\mathscr{P}^\nu = \{\lambda \in \mathscr{P}^\ell;\ l(\lambda^p) \leqslant \nu_p\}$ with $\mathscr{P}^\nu_d = \mathscr{P}^\nu \cap \mathscr{P}^\ell_d$.

Let $A \in Y(\lambda)$ be the box which lies in the i-th row and j-th column of the diagram of λ^p. Consider the element $p(A) = p$ in $[1, \ell]$. Given $Q_{R,1}, Q_{R,2}, \dots, Q_{R,\ell} \in \mathscr{I}$, we set $q\text{-res}^Q(A) = q_R^{j-i} Q_{R,p}$. For $\lambda, \mu \in \mathscr{P}^\ell$ we write $q\text{-res}^Q(\mu - \lambda) = a$ if μ is obtained by adding a box of residue a to the Young diagram associated with λ.

We may write $q\text{-res}^s(A) = q\text{-res}^Q(A)$ and $\mathrm{cont}^s(A) = s_p + j - i$, where s_p is a formal symbol such that $q_R^{s_p} = Q_{R,p}$. We call $q\text{-res}^s(A)$ the *shifted residue* of A and $\mathrm{cont}^s(A)$ its *shifted content*. We may also abbreviate $Q_p = Q_{R,p}$.

Let Γ be the group of ℓ-th roots of 1 in $\mathbb{C}^\times$. Let $\mathfrak{S}_d$ be the symmetric group on d letters and Γ_d be the semi-direct product $\mathfrak{S}_d \ltimes \Gamma^d$, where Γ^d is the Cartesian product of d copies of Γ. The group Γ_d is a complex reflection group. The set $\mathrm{Irr}(\mathbb{C}\Gamma_d)$ is identified with $\mathscr{P}^\ell_d$ in such a way that λ is associated with the module $\mathscr{X}(\lambda)_\mathbb{C}$ induced from the $\Gamma_{|\lambda^1|} \times \dots \times \Gamma_{|\lambda^\ell|}$-module $\phi_{\lambda^1}\chi^\ell \otimes \phi_{\lambda^2}\chi \otimes \dots \otimes \phi_{\lambda^\ell}\chi^{\ell-1}$. Here ϕ_{λ^p} is the irreducible $\mathbb{C}\mathfrak{S}_{|\lambda^p|}$-module associated with the partition λ^p and χ^p is the one dimensional $\Gamma^{|\lambda^p|}$-module given by the p-th power of the determinant.

Note that this labeling agrees with [39, sec. 6], [46, sec. 1.5] but it differs from that of [24, sec. 2.3.4].

3.4 Hecke algebras

3.4.1 Cyclotomic Hecke algebras

Write $\mathbf{H}_{R,0} = R$. For $d \geqslant 1$, the *affine Hecke algebra* $\mathbf{H}_{R,d}$ is the R-algebra generated by $T_1, \dots, T_{d-1}, X_1^{\pm 1}, \dots, X_d^{\pm 1}$ subject to the relations

$$\begin{aligned}
(T_i + 1)(T_i - q_R) &= 0,\\
T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1},\\
T_i T_j &= T_j T_i \quad \text{if } |i - j| > 1,\\
X_i X_j &= X_j X_i,\\
X_i X_i^{-1} &= X_i^{-1} X_i = 1,\\
T_i X_i T_i &= q_R X_{i+1},\\
X_i T_j &= T_j X_i \quad \text{if } i - j \neq 0, 1.
\end{aligned}$$

The *cyclotomic Hecke algebra* is the quotient $\mathbf{H}^Q_{R,d}$ of $\mathbf{H}_{R,d}$ by the two-sided ideal generated by $\prod_{p=1}^{\ell}(X_1 - Q_{R,p})$.

If $\ell = 1$, then the R-algebra $\mathbf{H}^Q_{R,d}$ is generated by T_i with $i \in [1, d)$. It does not depend on the choice of the unit $Q_{R,1}$. In this case we write $\mathbf{H}^+_{R,d} = \mathbf{H}^Q_{R,d}$.

Given $s = (s_1, \ldots, s_\ell)$ as above, we write $\mathbf{H}^s_{R,d} = \mathbf{H}^Q_{R,d}$. For any $d < d'$, the R-algebra embedding $\mathbf{H}_{R,d} \to \mathbf{H}_{R,d'}$ given by $T_i \mapsto T_i$, $X_j \mapsto X_j$ for $i \in [1, d)$, $j \in [1, d]$, induces an embedding $\mathbf{H}^s_{R,d} \to \mathbf{H}^s_{R,d'}$. The R-algebra $\mathbf{H}^s_{R,d'}$ is free as a left and as a right $\mathbf{H}^s_{R,d}$-module. This yields a pair of exact adjoint functors $(\mathrm{Ind}_d^{d'}, \mathrm{Res}_d^{d'})$ between $\mathbf{H}^s_{R,d'}$-mod and $\mathbf{H}^s_{R,d}$-mod . For $d \leqslant d'$ there is also an algebra embedding $\mathbf{H}^+_{R,d} \to \mathbf{H}^s_{R,d'}$ given by $T_i \mapsto T_i$ for $i \in [1, d)$. It yields a pair of exact adjoint functors $(\mathrm{Ind}_{d,+}^{d',s}, \mathrm{Res}_{d,+}^{d',s})$ between $\mathbf{H}^+_{R,d}$-mod and $\mathbf{H}^s_{R,d'}$-mod.

Now, assume that $R = K$ is a field. Any finite dimensional $\mathbf{H}^s_{K,d}$-module M can be decomposed into (generalized) weight spaces $M = \bigoplus_{\mathbf{i} \in \mathscr{I}^d} M_{\mathbf{i}}$, with $M_{\mathbf{i}} = \{v \in M;\ (X_r - i_r)^n v = 0,\ r \in [1, d],\ n \gg 0\}$. See [6, sec. 4.1] and the references there for details. Decomposing the regular module, we get a system of orthogonal idempotents $\{1_{\mathbf{i}};\ \mathbf{i} \in K^d\}$ in $\mathbf{H}^s_{K,d}$ such that $1_{\mathbf{i}} M = M_{\mathbf{i}}$ for each finite dimensional module M of $\mathbf{H}^s_{K,d}$.

Given $\alpha \in Q^+$ of height d, we set $1_\alpha = \sum_{\mathbf{i} \in K^\alpha} 1_{\mathbf{i}}$. The nonzero 1_α's are the primitive central idempotents in $\mathbf{H}^s_{K,d}$, i.e., the algebra $\mathbf{H}^s_{K,\alpha} = 1_\alpha \mathbf{H}^s_{K,d}$ is either zero or a single block of $\mathbf{H}^s_{K,d}$ [4,30].

3.4.2 Degenerate cyclotomic Hecke algebras

In the same way we can consider the *degenerate Hecke algebra* $H_{R,d}$ and the *degenerate cyclotomic Hecke algebra* $H^s_{R,d}$ introduced in [5]. We assume here $s \in R^\ell$. The algebra $H_{R,d}$ is generated by elements $t_1, \ldots, t_{d-1}, x_1, \ldots, x_d$ subject to the relations

$$\begin{aligned} t_i^2 &= 1, \\ t_i t_{i+1} t_i &= t_{i+1} t_i t_{i+1}, \\ t_i t_j &= t_j t_i \quad \text{if } |i - j| > 1, \\ x_i x_j &= x_j x_i, \\ t_i x_{i+1} &= x_i t_i + 1, \\ x_i t_j &= t_j x_i \quad \text{if } i - j \neq 0, 1. \end{aligned}$$

The degenerate cyclotomic Hecke algebra $H^s_{R,d}$ is the quotient of $H_{R,d}$ by the two-sided ideal generated by the element $\prod_{p=1}^{\ell}(x_1 - s_{R,p})$.

The representation theory of $H^s_{R,d}$ is very similar to that of $\mathbf{H}^s_{R,d}$. For instance, if $R = K$ is a field then the primitive central idempotents in $H^s_{K,d}$ are again labeled by the elements $\alpha \in Q^+$ of height d, which permits us to define $H^s_{K,\alpha} = 1_\alpha H^s_{K,d}$ as above. For any subset $I \subset K$ we set $H^s_I = \bigoplus_{\alpha \in Q^+_I} H^s_{K,\alpha}$, $H^s_{I,d} = H^s_I \cap H^s_{K,d}$. See e.g. [6, sec. 3] for more details.

3.4.3 Representations

We will use the following properties of $H^s_{R,d}$ and $\mathbf{H}^s_{R,d}$:

- the R-algebras $\mathbf{H}^s_{R,d}$ and $H^s_{R,d}$ are both symmetric by [34], [5, app. A],
- the K-algebra $\mathbf{H}^s_{K,d}$ is split semi-simple if and only if

$$\prod_{i=1}^{d}(1 + q_K + \cdots + q_K^{i-1}) \prod_{u<v} \prod_{-d<r<d} (q_K^r Q_{K,u} - Q_{K,v}) \neq 0. \quad (3.1)$$

Now, set $\zeta = \exp(2\sqrt{-1}\pi/\ell)$. If $q_K = 1$ and $Q_{K,p} = \zeta^{p-1}$, then $\mathbf{H}^s_{K,d}$ is the algebra $K\Gamma_d$ of the group Γ_d. Therefore, if $\mathbf{H}^s_{K,d}$ is semi-simple, then the set $\mathrm{Irr}(\mathbf{H}^s_{K,d})$ is canonically identified with $\mathrm{Irr}(K\Gamma_d)$ by Tits' deformation Theorem. For each $\lambda \in \mathscr{P}^\ell_d$, one can define a *Specht module* $S(\lambda)^{s,q}_R$ of $\mathbf{H}^s_{R,d}$ as in Sect. 2.4.3, using the dominance order $\trianglelefteq$ on $\mathscr{P}^\ell_d$, cf Sect. 3.5 below. It is free over R, and specializes to $\mathscr{X}(\lambda)_\mathbb{C}$ as $q_R \mapsto 1$ and $Q_{R,p} \mapsto \zeta^{p-1}$. The Specht modules $S(\lambda)^s_R$ of $H^s_{R,d}$ with $\lambda \in \mathscr{P}^\ell_d$ are defined similarly.

Now, assume that R is an analytic deformation ring in the sense of Sect. 5.1 below. Set $\mathscr{I} = \bigcup_{p=1}^\ell q_R^{s_p+\mathbb{Z}}$ and $I = \bigcup_{p=1}^\ell (s_p + \mathbb{Z})$. The multiplication by q_R and the shift by 1 equips the sets $\mathscr{I}$, I with structures of quivers $\mathscr{I}(q)$, I_1 as explained in Sect. 3.1.

Proposition 3.1 *Assume that R is a local ring.*

(a) *The blocks $\mathbf{H}^s_{R,\alpha}$ of $\mathbf{H}^s_{R,d}$ (resp. the blocks $H^s_{R,\alpha}$ of $H^s_{R,d}$) are labeled by the elements $\alpha \in Q^+_\mathscr{I}$ (resp. $\alpha \in Q^+_I$) of height d. We have $\mathrm{k}\mathbf{H}^s_{R,\alpha} = \mathbf{H}^s_{\mathrm{k},\alpha}$ and $\mathrm{k}H^s_{R,\alpha} = H^s_{\mathrm{k},\alpha}$ for each α.*

(b) *Assume that the map $\exp(-2\pi\sqrt{-1}\,\bullet/\kappa)$ yields an isomorphism of quivers $\beta : I_1 \to \mathscr{I}(q)$. Given an element $\alpha \in Q^+_I$, let α denote also its image in $Q^+_\mathscr{I}$. Then, we have an R-algebra isomorphism $\alpha_R : \mathbf{H}^s_{R,\alpha} \xrightarrow{\sim} H^s_{R,\alpha}$ such that $\alpha_R(S(\lambda)^s_R) \simeq S(\lambda)^{s,q}_R$ for each λ.*

Proof Part (a) is obvious, because the primitive central idempotents of $\mathbf{H}^s_{\mathrm{k},d}$, $H^s_{\mathrm{k},d}$ lift to $\mathbf{H}^s_{R,d}$, $H^s_{R,d}$ since R is henselian.

More precisely, given α in $Q^+_{\mathscr{I}_\mathrm{k}}$ or in $Q^+_{I_\mathrm{k}}$, to lift the idempotent 1_α in $\mathbf{H}^s_{\mathrm{k},d}$, $H^s_{\mathrm{k},d}$ into an idempotent in $\mathbf{H}^s_{R,d}$, $H^s_{R,d}$, we first consider the idempotent in

$\mathbf{H}^s_{K,d}$, $H^s_{K,d}$ given by the sum of all $1_{\mathbf{i}}$'s, with $\mathbf{i}$ in $\mathscr{I}^d = \mathscr{I}^d_R$ or in $I^d = I^d_R$, such that the residue class of $\mathbf{i}$ in k^d is a summand of α. Note that, although there may be an infinite number of such tuples $\mathbf{i}$, this sum contains only a finite number of non zero terms. A standard computation in linear algebra implies that it belongs indeed to $\mathbf{H}^s_{R,d}$, $H^s_{R,d}$, yielding an idempotent which specializes to 1_α.

Now, we concentrate on part (b). Note that [6, sec. 3.5, 4.5], [40, §3.2.6] yield a K-algebra isomorphism $\alpha_K : \mathbf{H}^s_{K,\alpha} \xrightarrow{\sim} H^s_{K,\alpha}$. We will prove that the isomorphism α_K in [40] (which differs from the one in [6]) restricts to an isomorphism $\alpha_R : \mathbf{H}^s_{R,\alpha} \xrightarrow{\sim} H^s_{R,\alpha}$.

We have the following formulae

$$\begin{aligned}
&\alpha_K^{-1}(1_{\mathbf{i}}) = 1_{\mathbf{j}} \quad \text{where } \mathbf{j} = \beta(\mathbf{i}),\\
&\alpha_K^{-1}(x_r 1_{\mathbf{i}}) = (j_r^{-1} X_r - 1 + i_r) 1_{\mathbf{j}},\\
&\alpha_K^{-1}((t_r+1)1_{\mathbf{i}}) = (T_r+1)\frac{X_r - X_{r+1} - j_r}{X_r - qX_{r+1}} 1_{\mathbf{j}} \quad \text{if } i_r = i_{r+1},\\
&\alpha_K^{-1}((t_r+1)1_{\mathbf{i}}) = (T_r+1)\frac{X_r - X_{r+1}}{X_r - qX_{r+1} + j_r} 1_{\mathbf{j}} \quad \text{if } i_r = i_{r+1}+1,\\
&\alpha_K^{-1}((t_r+1)1_{\mathbf{i}}) = (T_r+1)\frac{\alpha_K^{-1}(x_r) - \alpha_K^{-1}(x_{r+1}) - 1}{X_r - qX_{r+1}}\,\frac{X_r - X_{r+1}}{\alpha_K^{-1}(x_r) - \alpha_K^{-1}(x_{r+1})} 1_{\mathbf{j}} \quad \text{else.}
\end{aligned}$$

Let $\mathbf{P} \subset \mathbf{H}^s_{R,d}$ and $P \subset H^s_{R,d}$ be the R-subalgebras generated by the X_r's and the x_r's respectively. We may assume that R is in general position. Then, the K-algebras $\mathbf{H}^s_{K,d}$, $H^s_{K,d}$ are semi-simple, and the same is true for $K\mathbf{P}$ and KP. Therefore, we have $x_r 1_{\mathbf{i}} = i_r 1_{\mathbf{i}}$ and $X_r 1_{\mathbf{j}} = j_r 1_{\mathbf{j}} = \beta(i_r) 1_{\mathbf{j}} = \exp(-2\pi\sqrt{-1}\,\alpha_K^{-1}(x_r)) 1_{\mathbf{j}}$. We deduce that $\alpha_K^{-1}(P) = \mathbf{P}$.

Now, we have

$$\begin{aligned}
&\frac{\alpha_K^{-1}(x_r) - \alpha_K^{-1}(x_{r+1}) - 1}{X_r - qX_{r+1}}\\
&\quad = q^{-1} X_{r+1}^{-1} \frac{\alpha_K^{-1}(x_r) - \alpha_K^{-1}(x_{r+1}) - 1}{\exp(-2\pi\sqrt{-1}\,(\alpha_K^{-1}(x_r) - \alpha_K^{-1}(x_{r+1}) - 1)/\kappa) - 1},\\
&\frac{X_r - X_{r+1}}{\alpha_K^{-1}(x_r) - \alpha_K^{-1}(x_{r+1})}\\
&\quad = X_{r+1} \frac{\exp(-2\pi\sqrt{-1}\,(\alpha_K^{-1}(x_r) - \alpha_K^{-1}(x_{r+1}))/\kappa) - 1}{\alpha_K^{-1}(x_r) - \alpha_K^{-1}(x_{r+1})}.
\end{aligned}$$

Therefore, both expressions are units in $\mathbf{P}$. Hence α_K restricts to an isomorphism $\alpha_R : \mathbf{H}^s_{R,\alpha} \xrightarrow{\sim} H^s_{R,\alpha}$.

The isomorphism $\alpha_R(S(\lambda)^s_R) \simeq S(\lambda)^{s,q}_R$ follows from the unicity of Specht modules. □

3.5 Cyclotomic q-Schur algebras

For each $\lambda \in \mathscr{P}^\ell_d$, we consider the elements $w_\lambda = \sum_{w \in \mathfrak{S}_\lambda} T_w$ and $x_\lambda = \prod_{p=1}^{\ell} \prod_{i=1}^{a_p} (X_i - Q_{R,p})$ where $a_p = |\lambda^1| + \cdots + |\lambda^{p-1}|$ and $\mathfrak{S}_\lambda$ is the parabolic subgroup of $\mathfrak{S}_d$ associated with λ. The R-algebra $\mathbf{S}^s_{R,d} = \mathrm{End}_{\mathbf{H}^s_{R,d}}\big(\bigoplus_\lambda w_\lambda x_\lambda \mathbf{H}^s_{R,d}\big)$ is called the *cyclotomic q-Schur* algebra [13].

The category $\mathbf{S}^s_{R,d}$-mod is a highest weight category whose standard objects are the *Weyl modules* $W(\lambda)^{s,q}_R$ labeled by multipartitions $\lambda \in \mathscr{P}^\ell_d$. The highest weight order is given by the *dominance order* $\trianglelefteq$ on $\mathscr{P}^\ell_d$. The algebra $\mathbf{S}^s_{R,d}$ is Ringel self-dual, see [37, prop. 4.3, cor. 7.3].

There is a double centralizer property for $\mathbf{S}^s_{R,d}$ and $\mathbf{H}^s_{R,d}$ which produces a highest weight cover $\Xi^s_{R,d} : \mathbf{S}^s_{R,d}\text{-mod} \to \mathbf{H}^s_{R,d}\text{-mod}$, called the *cyclotomic q-Schur functor* [36, sec. 5], [39]. The Specht module $S(\lambda)^{s,q}_R$ is the image of $W(\lambda)^{s,q}_R$ under this functor. If $R = K$ is a field, then the K-algebra $\mathbf{S}^s_{K,d}$ is semi-simple if and only if condition (3.1) holds.

Using $H^s_{R,d}$ instead of $\mathbf{H}^s_{R,d}$, we define the *degenerate cyclotomic q-Schur algebra* $S^s_{R,d}$ and the cyclotomic q-Schur functor $\Xi^s_{R,d} : S^s_{R,d}\text{-mod} \to H^s_{R,d}\text{-mod}$ in a similar way. See [2,5] for details. All the results on $\mathbf{S}^s_{R,d}$ recalled above have direct analogues for $S^s_{R,d}$, see e.g., [24, sec. 6.6]. In particular, the Specht module $S(\lambda)^s_R$ is the image of the Weyl module $W(\lambda)^s_R$ by the q-Schur functor.

3.6 Categorical actions on abelian categories

Let $\mathscr{C}$ be an abelian R-category.

Definition 3.2 A *pre-categorification* (or *pre-categorical action*) on $\mathscr{C}$ is a tuple (E, F, X, T) where (E, F) is an adjoint pair of exact functors $\mathscr{C} \to \mathscr{C}$ and $X \in \mathrm{End}(E)$, $T \in \mathrm{End}(E^2)$ are endomorphisms of functors such that

- for each $d \in \mathbb{N}$, there is an R-algebra homomorphism $\phi_{E^d} : \mathbf{H}_{R,d} \to \mathrm{End}(E^d)$ given by $X_k \mapsto E^{d-k} X E^{k-1}$, $T_l \mapsto E^{d-l-1} T E^{l-1}$ for $k \in [1, d]$, $l \in [1, d)$,
- the functor E is isomorphic to a right adjoint of F.

Remark 3.3 Given a pair of adjoint functors (E, F), the adjunction yields a canonical R-algebra isomorphism $\mathrm{End}(F^d) = \mathrm{End}(E^d)^{\mathrm{op}}$ for each $d \in \mathbb{N}$, see e.g., [9, sec. 4.1.2]. Under this isomorphism, the morphisms X, T yield morphisms $X \in \mathrm{End}(F)$, $T \in \mathrm{End}(F^2)$ which induces an R-algebra homomorphism $\phi_{F^d} : \mathbf{H}_{R,d} \to \mathrm{End}(F^d)^{\mathrm{op}}$.

Now, assume that $R = K$ is a field and that $\mathscr{C}$ is Hom-finite. Let $\mathscr{I} = \mathscr{I}(q)$.

Definition 3.4 [9,40] An $\mathfrak{sl}_{\mathscr{I}}$*-categorification* (or *categorical action*) on $\mathscr{C}$ is the datum of a pre-categorification (E, F, X, T) and a decomposition $\mathscr{C} = \bigoplus_{\lambda\in X} \mathscr{C}_\lambda$. For $i \in \mathscr{I}$ let F_i, E_i be the generalized i-eigenspaces of X acting on F, E respectively. We assume in addition that

- we have $F = \bigoplus_{i\in\mathscr{I}} F_i$ and $E = \bigoplus_{i\in\mathscr{I}} E_i$,
- the action of E_i, F_i, $i \in \mathscr{I}$ on $[\mathscr{C}]$ gives an integrable representation of $\mathfrak{sl}_{\mathscr{I}}$,
- we have $E_i(\mathscr{C}_\lambda) \subset \mathscr{C}_{\lambda+\alpha_i}$ and $F_i(\mathscr{C}_\lambda) \subset \mathscr{C}_{\lambda-\alpha_i}$.

Remark 3.5 The constructions above have a degenerate analogue. Given $I \subset R$ and $\mathfrak{sl}_I$ as above, the definition of a pre-categorification and of an $\mathfrak{sl}_I$-categorification is the same, with $\mathbf{H}_{R,d}$ replaced by $H_{R,d}$ and $\mathfrak{sl}_{\mathscr{I}}$ by $\mathfrak{sl}_I$. In particular, for each $d \in \mathbb{N}$ there is an R-algebra homomorphism $\phi_{E^d} : H_{R,d} \to \mathrm{End}(E^d)$ given by $X_k \mapsto E^{d-k}XE^{k-1}$, $T_l \mapsto E^{d-l-1}TE^{l-1}$.

Example 3.6 Let $R = K$ be a field which is an analytic algebra, see Sect. 5.1. Let s be as in Sect. 3.3, and $\Lambda = \Lambda^s = \sum_{p=1}^{\ell} \Lambda_{Q_p}$. Let $\mathbf{H}^s_{\mathscr{I},d} = \bigoplus_\alpha \mathbf{H}^s_{K,\alpha}$, where α runs over elements of $Q^+_{\mathscr{I}}$ of height d.

The abelian K-category $\mathscr{L}(\Lambda)_{\mathscr{I}} = \bigoplus_{d\in\mathbb{N}} \mathbf{H}^s_{\mathscr{I},d}$-mod decomposes as $\mathscr{L}(\Lambda)_{\mathscr{I}} = \bigoplus_{\alpha\in Q^+_{\mathscr{I}}} \mathscr{L}(\Lambda)_{\mathscr{I},\Lambda-\alpha}$ with $\mathscr{L}(\Lambda)_{\mathscr{I},\Lambda-\alpha} = \mathbf{H}^s_{K,\alpha}$-mod.

The endofunctors $E = \bigoplus_{d\in\mathbb{N}} \mathrm{Res}_d^{d+1}$ and $F = \bigoplus_{d\in\mathbb{N}} \mathrm{Ind}_d^{d+1}$ of $\mathscr{L}(\Lambda)_{\mathscr{I}}$ are exact and biadjoint. The right multiplication on $\mathbf{H}^s_{\mathscr{I},d+1}$ by X_{d+1} yields an endomorphism of the functor Ind_d^{d+1}, denoted again by X_{d+1}. The right multiplication by T_{d+1} yields an endomorphism of Ind_d^{d+2}. We define $X \in \mathrm{End}(F)$ and $T \in \mathrm{End}(F^2)$ by $X = \bigoplus_{d\in\mathbb{N}} X_{d+1}$ and $T = \bigoplus_{d\in\mathbb{N}} T_{d+1}$.

The tuple (E, F, X, T) and the decomposition above give an $\mathfrak{sl}_{\mathscr{I}}$-categorification of $\mathbf{L}(\Lambda)$ (the simple $\mathfrak{sl}_{\mathscr{I}}$-module with highest weight Λ) on $\mathscr{L}(\Lambda)_{\mathscr{I}}$, called the *minimal* $\mathfrak{sl}_{\mathscr{I}}$*-categorification* of highest weight Λ.

In the degenerate case, the induction and restriction functors give an abelian $\mathfrak{sl}_I$-categorification of $\mathbf{L}(\Lambda)$ on $\mathscr{L}(\Lambda)_I = \bigoplus_{d\in\mathbb{N}} H^s_{I,d}$-mod, called again the minimal $\mathfrak{sl}_I$-categorification of highest weight Λ.

4 The category $\mathscr{O}$

Fix integers $\ell, N \geqslant 1$ and fix a composition $\nu \in \mathscr{C}^\ell_{N,+}$.

4.1 Deformation rings

A *deformation ring* is a regular commutative noetherian $\mathbb{C}$-algebra R with 1 equipped with a $\mathbb{C}$-algebra homomorphism $\mathbb{C}[\mathbb{C}^\times \times \mathbb{C}^\ell] \to R$. Let κ_R, $\tau_{R,p}$

be the images in R of the standard coordinates $z, z_1, \ldots, z_\ell$ on $\mathbb{C}^\times$ and $\mathbb{C}^\ell$. Set $\tau_R = (\tau_{R,1}, \ldots, \tau_{R,\ell})$. Define $s_{R,1}, \ldots, s_{R,\ell} \in R$ by $s_{R,p} = \nu_p + \tau_{R,p}$. We may abbreviate $s_p = s_{R,p}, \kappa = \kappa_R$ and $\tau_p = \tau_{R,p}$. For any S-point $\chi : R \to S$ we write $\kappa_S = \chi(\kappa_R)$ and $\tau_{S,p} = \chi(\tau_{R,p})$.

A *local deformation ring* is a deformation ring R which is a local ring such that the residue class $\tau_{\mathrm{k},p}$ of $\tau_{R,p}$ is 0 for each p. We will denote by $-e$ the residue class κ_{k} of κ_R. *We will always assume that e is a positive integer.*

Remark 4.1 Let R be a deformation ring. Then, for each $\mathfrak{p} \in \mathfrak{P}$, the local ring $R_\mathfrak{p}$ is regarded as a deformation ring with deformation parameters $\kappa_{R_\mathfrak{p}}, \tau_{R_\mathfrak{p}}$. It may not be a local deformation ring, since we may have $\tau_{R,p} \notin \mathfrak{p}$.

We will say that the deformation ring R is in *general position* if the elements in $\{\tau_{R,u} - \tau_{R,v} + a\kappa_R + b,\ \kappa_R - c;\ a, b \in \mathbb{Z},\ c \in \mathbb{Q},\ u \neq v\}$ are pairwise coprime.

Example 4.2 Given a tuple $\zeta = (\zeta_1, \ldots, \zeta_\ell)$ in $\mathbb{C}^\ell$, we have the deformation ring $\mathbb{C}[\mathbb{C}^\times \times \mathbb{C}^\ell] \to R = \mathbb{C}[\tau, \kappa, \kappa^{-1}]$ such that $z \mapsto \kappa$ and $z_p \mapsto \zeta_p \tau$. It is in general position if ζ is generic.

4.2 Lie algebras

Let R be a deformation ring.

Set $\mathfrak{g}_R = \mathfrak{gl}_{R,N}$. Let $U(\mathfrak{g}_R)$ be the enveloping algebra (over R) of $\mathfrak{g}_R$. Let $\mathfrak{t}_R \subset \mathfrak{b}_R \subset \mathfrak{g}_R$ be the diagonal torus and the Borel Lie subalgebra of upper triangular matrices. Let $\mathfrak{p}_{R,\nu} \subset \mathfrak{g}_R$ be the parabolic subalgebra spanned by $\mathfrak{b}_R$ and the Levi subalgebra $\mathfrak{m}_{R,\nu} = \mathfrak{gl}_{R,\nu_1} \oplus \cdots \oplus \mathfrak{gl}_{R,\nu_\ell}$.

Let $e_{i,j} \in \mathfrak{g}_R$ be the (i, j)-matrix unit, and set $e_i = e_{i,i}$. Let (ϵ_i) be the basis of $\mathfrak{t}_R^*$ dual to (e_i). It identifies $\mathfrak{t}_R^*$ with R^N. In a similar way we identify $\mathfrak{t}_R = R^N$.

Let Π, Π^+ be the sets of roots of $\mathfrak{g}_R$ and $\mathfrak{b}_R$. We say that ν is *regular* if $\mathfrak{m}_{R,\nu} = \mathfrak{t}_R$. Let Π_ν be the set of roots of $\mathfrak{m}_{R,\nu}$. Set $\Pi_\nu^+ = \Pi^+ \cap \Pi_\nu$.

The dot action of the Weyl group W on $\mathfrak{t}_R^*$ is given by $w \bullet \lambda = w(\lambda + \rho) - \rho$, where $\rho = (0, -1, \ldots, 1 - N)$. Two weights are *linked* if they belong to the same orbit of the $\bullet$-action.

Consider the partition $[1, N] = J_1^\nu \sqcup J_2^\nu \sqcup \cdots \sqcup J_\ell^\nu$ given by $i_p = 1 + \nu_1 + \cdots + \nu_{p-1}$, $j_p = i_{p+1} - 1$ and $J_p^\nu = [i_p, j_p]$ For each $k \in J_p^\nu$ we define $p_k = p$. Set $\det_p = \sum_{i \in J_p^\nu} \epsilon_i$ and $\det = \sum_{p=1}^\ell \det_p$.

The weights in the subset $P = \mathbb{Z}^N$ of $P_R = R^N$ are called *integral weights*. Given a subset $S \subset R$, we write $S^\nu = \{\lambda \in S^N;\ \lambda_i - \lambda_{i+1} \in \mathbb{N},\ \forall i \neq j_1, j_2, \ldots, j_\ell\}$. We call $P_R^\nu = R^\nu$ the set of the ν-*dominant* weights in P_R.

An ℓ-partition $\lambda \in \mathscr{P}^\nu$ can be viewed as an element in $\mathbb{N}^\nu$ by adding zeroes to the right of each partition λ^p such that $l(\lambda^p) \leqslant \nu_p$, i.e., we identify the ℓ-partition $\lambda = (\lambda^1, \lambda^2, \dots, \lambda^\ell)$ with the N-tuple $(\lambda^1 0^{\nu_1 - l(\lambda^1)} \cdots \lambda^\ell 0^{\nu_\ell - l(\lambda^\ell)})$.

Similarly, we can view the tuple $\tau_R \in R^\ell$ as a weight in P_R by identifying it with $\tau_R = \sum_p \tau_{R,p} \det_p$. To simplify we may abbreviate $\tau = \tau_R$.

Set $\rho_\nu = (\nu_1, \nu_1 - 1, \dots, 1, \dots, \nu_\ell, \nu_\ell - 1, \dots, 1)$. So, we have $\rho_\nu + \tau = (s_1, s_1 - 1, \dots, \tau_{R,1} + 1, s_2, s_2 - 1, \dots, \tau_{R,2} + 1, \dots, \tau_{R,\ell} + 1)$. We identify the set of ℓ-partitions $\mathscr{P}^\nu$ with a subset of P_R^ν via the injective map

$$\varpi : \mathscr{P}^\nu \to P^\nu + \tau, \quad \lambda \mapsto \lambda + \rho_\nu + \tau - \rho. \tag{4.1}$$

The Casimir elements are $\omega = \sum_{i,j=1}^N e_{ij} \otimes e_{ji}$ and $\mathrm{cas} = \sum_{i,j=1}^N e_{ij} e_{ji}$. We may write $\omega_N = \omega$, $\mathrm{cas}_N = \mathrm{cas}$ to avoid confusions.

4.3 Definition of the category $\mathscr{O}$

A $\mathfrak{t}_R$-module M is called a *weight* $\mathfrak{t}_R$*-module* if it is a direct sum of its *weight submodules* $M_\lambda = \{m \in M;\ xm = \lambda(x)m,\ x \in \mathfrak{t}_R\}$ as λ runs over P_R. Let $\mathscr{O}_R^\nu$ be the R-category of finitely generated $U(\mathfrak{g}_R)$-modules which are weight $\mathfrak{t}_R$-modules and such that the action of $U(\mathfrak{p}_{R,\nu})$ is locally finite over R.

For $\lambda \in P_R^\nu$, we consider the $U(\mathfrak{m}_{R,\nu})$-module $V(\lambda)_{R,\nu} = V(\lambda')_{\mathbb{C},\nu} \otimes R_{\lambda-\lambda'}$, where $\lambda' \in P^\nu$ is such that $\lambda - \lambda'$ is a character of $\mathfrak{m}_{R,\nu}$, $R_{\lambda-\lambda'}$ is R, equipped with the representation of $\mathfrak{m}_{R,\nu}$ given by this character, and $V(\lambda')_{\mathbb{C},\nu}$ is the finite-dimensional simple $\mathfrak{m}_\nu$-module with highest weight λ'. We view $V(\lambda)_{R,\nu}$ as a $\mathfrak{p}_{R,\nu}$-module and define the parabolic Verma module $M(\lambda)_{R,\nu} = U(\mathfrak{g}_R) \otimes_{U(\mathfrak{p}_{R,\nu})} V(\lambda)_{R,\nu}$. If ν is regular, we abbreviate $M(\lambda)_R = M(\lambda)_{R,\nu}$.

For $\lambda \in P_K^\nu$, let $L(\lambda)_K$ be the unique simple quotient of $M(\lambda)_{K,\nu}$.

Let $\mathscr{O}_{R,\tau}^\nu$ be the full subcategory of $\mathscr{O}_R^\nu$ consisting of the modules whose weights belong to $P + \tau$. Note that $M(\lambda)_{R,\nu} \in \mathscr{O}_{R,\tau}^\nu$ if and only if $\lambda \in P^\nu + \tau$, and that $\mathscr{O}_{K,\tau}^\nu$ is the Serre subcategory of $\mathscr{O}_K^\nu$ generated by all the simple modules $L(\lambda)_K$ with $\lambda \in P^\nu + \tau$. For $\lambda \in \mathscr{P}^\nu$ we set $\Delta(\lambda)_{R,\tau} = M(\varpi(\lambda))_{R,\nu}$.

If $R = \mathbb{C}$ or if $\tau = 0$ we drop the subscripts R or τ from the notation.

4.4 Definition of the category A

Let R be a deformation ring. Assume that R is either a field or a local ring.

The category $\mathscr{O}_{R,\tau}^\nu$ is a highest weight R-category with $\Delta(\mathscr{O}_{R,\tau}^\nu) = \{M(\lambda)_{R,\nu};\ \lambda \in P^\nu + \tau\}$. If R is a local ring with residue field k, the specialization at k identifies the poset $\Delta(\mathscr{O}_{R,\tau}^\nu)$ with $\Delta(\mathscr{O}_{\mathrm{k},\tau}^\nu)$.

The partial order is given by the *BGG-ordering* on P_R^ν, which is the smallest partial order such that $\lambda \leqslant \lambda'$ if $[M(\lambda')_{\mathrm{k},\nu} : L(\lambda)_{\mathrm{k}}] \neq 0$. It is equivalent to the *linkage ordering* on P^ν_R, which is the transitive and reflexive closure of the relation such that λ is smaller than λ' if and only if there are $\beta \in \Pi(\lambda')$, $w \in W_\nu$ such that $\beta \notin \Pi_\nu$ and $\lambda = ws_\beta \bullet \lambda' \in \lambda' - \mathbb{N}\Pi^+$ modulo $\mathfrak{m}\,\widehat{P}_R$. We will use the orderings interchangeably in the rest of the text.

Definition-Proposition 4.3 *Assume that* $\tau_{\mathrm{k},u} - \tau_{\mathrm{k},v} \notin \mathbb{N}^\times$ *for each* $u < v$. *There are unique highest weight R-subcategories* $A^\nu_{R,\tau}$, $A^\nu_{R,\tau}\{d\}$ *of* $\mathscr{O}^\nu_{R,\tau}$ *with* $\Delta(A^\nu_{R,\tau}) = \{\Delta(\lambda)_{R,\tau};\ \lambda \in \mathscr{P}^\nu\}$ *and* $\Delta(A^\nu_{R,\tau}\{d\}) = \{\Delta(\lambda)_{R,\tau};\ \lambda \in \mathscr{P}^\nu_d\}$.

Proof It is enough to assume that $R = K$ is a field and to prove that $\Delta(A^\nu_{K,\tau})$ is an ideal of the poset $\Delta(\mathscr{O}^\nu_{K,\tau})$. To do so, we must check that if $\lambda \in \mathscr{P}^\nu$, $\mu \in P^\nu + \tau$ and $\beta \in \Pi \setminus \Pi_\nu$, $w \in W_\nu$ are such that $\mu = ws_\beta \bullet \varpi(\lambda)$ and $\varpi(\lambda) - \mu \in \mathbb{N}\Pi^+$, then we have $\mu \in \varpi(\mathscr{P}^\nu)$. Write $\beta = \alpha_{k,l}$ with $k < l$ and $k = i_u + x \leqslant j_u$, $l = i_v + y \leqslant j_v$. For each $a, b \in K$ we write $a > b$ if and only if $a - b \in \mathbb{N}^\times$. Then, we have $u < v$ and

$$\lambda_k + s_{K,u} - x > \lambda_l + s_{K,v} - y, \tag{4.2}$$

where λ is viewed as a N-tuple $(\lambda_1, \lambda_2, \ldots, \lambda_N)$. We have

$$\begin{aligned}
\{(\mu+\rho)_a;\ i_u \leqslant a \leqslant j_u\} &= \{\lambda_a + s_{K,u} - (a - i_u);\ i_u \leqslant a \leqslant j_u,\ a \neq k\}\\
&\quad \cup\{\lambda_l + s_{K,v} - y\},\\
\{(\mu+\rho)_b;\ i_v \leqslant b \leqslant j_v\} &= \{\lambda_b + s_{K,v} - (b - i_v);\ i_v \leqslant b \leqslant j_v,\ b \neq l\}\\
&\quad \cup\{\lambda_k + s_{K,u} - x\}.
\end{aligned}$$

To prove that $\mu \in \varpi(\mathscr{P}^\nu)$, we must check that

$$\begin{aligned}
&\min\{(\mu+\rho)_a;\ i_u \leqslant a \leqslant j_u\} \geqslant \tau_{K,u} + 1,\\
&\min\{(\mu+\rho)_b;\ i_v \leqslant b \leqslant j_v\} \geqslant \tau_{K,v} + 1.
\end{aligned}$$

By (4.2) and the assumption in the lemma, we have $\tau_{K,v} - \tau_{K,u} \in \mathbb{N}$. Hence, the first inequality is true, because for any $\lambda \in \mathscr{P}^\nu$, $i_u \leqslant a \leqslant j_u$, we have $\lambda_a + s_{K,u} - (a - i_u) \geqslant \tau_{K,u} + 1$, and $\lambda_l + s_{K,v} - y \geqslant \tau_{K,v} + 1 \geqslant \tau_{K,u} + 1$. Now, to prove the second one, observe that by (4.2) we have

$$\min\{(\mu+\rho)_b;\ i_v \leqslant b \leqslant j_v\} \geqslant \min\{\lambda_b + s_{K,v} - (b - i_v);\ i_v \leqslant b \leqslant j_v\} \geqslant \tau_{K,v} + 1.$$

$\square$

4.5 The categorical action on $\mathscr{O}$

Let V_R be the natural representation of $\mathfrak{g}_R$ on R^N. Let $V^*_R = \mathrm{Hom}_R(V_R, R)$ be the dual representation. We have a pre-categorical action (e, f, X, T) on $\mathscr{O}^\nu_{R,\tau}$ such that

$$e(M) = M \otimes_R V_R^*, \quad f(M) = M \otimes_R V_R,$$

$X_M \in \mathrm{End}(f(M))$ is the left multiplication by the Casimir element ω, and $T_M \in \mathrm{End}(f^2(M))$ is the left multiplication by $1 \otimes \omega$, see e.g. [5, sec. 3.4].

Now, assume that $R = K$ is a field. Set $I = \{\tau_{K,1}, \ldots, \tau_{K,\ell}\} + \mathbb{Z}$.

For each $\mu \in P^\nu + \tau$, we write $\mathrm{wt}(\mu) = \sum_{k=1}^N \varepsilon_{\langle \mu+\rho, \epsilon_k \rangle}$. We have $\mathrm{wt}(\mu) \in X_I$ if and only if $\langle \mu, \epsilon_k \rangle \in I$ for all k. Note that $\mathrm{wt}(\mu) = \sum_{i \in I}(m_i(\mu) - m_{i+1}(\mu))\, \Lambda_i$, where $m_i(\mu) = \sharp\{k \in [1, N];\ \langle \mu + \rho;\ \epsilon_k \rangle = i\}$.

For each $\lambda \in X_I$, let $\mathscr{O}^\nu_{K,\tau,\lambda} \subset \mathscr{O}^\nu_{K,\tau}$ be the Serre subcategory generated by the modules $L(\mu)_K$ such that $\mu \in P^\nu + \tau$ and $\mathrm{wt}(\mu) = \lambda$. The linkage principle yields the decomposition $\mathscr{O}^\nu_{K,\tau} = \bigoplus_{\lambda \in X_I} \mathscr{O}^\nu_{K,\tau,\lambda}$. This decomposition yields an $\mathfrak{sl}_I$-categorical action on $\mathscr{O}^\nu_{K,\tau}$.

Let V_I be the natural representation of $\mathfrak{sl}_I$. It is a representation with the basis $\{v_i;\ i \in I\}$. We have the following formulas, see, e.g. [5, lem. 4.3].

Proposition 4.4 *For $\lambda, \mu \in P^\nu_K$ we write $\lambda \xrightarrow{i} \mu$ if $\mu + \rho$ is obtained from $\lambda + \rho$ by replacing an entry equal to i by $i + 1$.*

(a) *$f_i(M(\lambda)_{K,\nu})$ has a Δ-filtration with sections of the form $M(\mu)_{K,\nu}$, one for each μ such that $\lambda \xrightarrow{i} \mu$,*
(b) *$e_i(M(\lambda)_{K,\nu})$ has a Δ-filtration with sections of the form $M(\mu)_{K,\nu}$, one for each μ such that $\mu \xrightarrow{i} \lambda$,*
(c) *the elements $[L(\mu)_K], [M(\mu)_{K,\nu}]$ in $[\mathscr{O}^\nu_{K,\tau}]$ are homogeneous of weight* $\mathrm{wt}(\mu)$,
(d) *as an $\mathfrak{sl}_I$-module, we have $[\mathscr{O}^\nu_{K,\tau}] = \bigotimes_{p=1}^\ell \bigwedge^\nu (V_I)$.*

4.6 Definition of the functor Φ

Recall that R is a deformation ring which is either a field or a local ring.

Let $h : A^\nu_{R,\tau} \to \mathscr{O}^\nu_{R,\tau}$ be the canonical embedding. Its left adjoint is h^*. Consider the endofunctors E, F of $A^\nu_{R,\tau}$ given by $E = h^* e h$ and $F = h^* f h$. Since f preserves the subcategory $A^\nu_{R,\tau}$, we have $F = f|_{A^\nu_{R,\tau}}$. So F is exact and (E, F) is an adjoint pair. Further, the endomorphisms X, T of f, f^2 yield endomorphisms of F, F^2.

Next, consider the module $T_{R,d} = T^\nu_{R,\tau}\{d\} = f^d(\Delta(\emptyset)_{R,\tau})$ in $A^\nu_{R,\tau}\{d\}$. The algebra homomorphism ϕ_{f^d} factors through an R-algebra homomorphism [5, lem. 3.4]

$$\varphi^s_{R,d} : H^s_{R,d} \to \mathrm{End}_{A^\nu_{R,\tau}}(T_{R,d})^{\mathrm{op}} = \mathrm{End}_{\mathscr{O}^\nu_{R,\tau}}(T_{R,d})^{\mathrm{op}}.$$

Composing $\mathrm{Hom}_{A^\nu_{R,\tau}}(T_{R,d}, \bullet)$ with the pullback by $\varphi^s_{R,d}$ we get a functor

$$\Phi^s_{R,d} : A^\nu_{R,\tau} \to H^s_{R,d}\text{-mod}\,.$$

Remark 4.5 To avoid confusions we may write $A^{\nu}_{R,\tau}(N) = A^{\nu}_{R,\tau}$, $T_{R,d}(N) = T_{R,d}$.

Remark 4.6 For each $\mathfrak{p} \in \mathfrak{P}$, the pre-categorification (e, f, X, T) on $\mathscr{O}^{\nu}_{R,\tau}$ yields a pre-categorification on $\mathscr{O}^{\nu}_{R_{\mathfrak{p}},\tau}$ and $\mathscr{O}^{\nu}_{k_{\mathfrak{p}},\tau}$ by base-change. It yields also a tuple (E, F, X, T) on $A^{\nu}_{R_{\mathfrak{p}},\tau}$ and $A^{\nu}_{k_{\mathfrak{p}},\tau}$ as above. In particular, this yields a module $T_{R_{\mathfrak{p}},d}$ in $A^{\nu}_{R_{\mathfrak{p}},\tau}$, an $R_{\mathfrak{p}}$-algebra homomorphism $\varphi^s_{R_{\mathfrak{p}},d} : H^s_{R_{\mathfrak{p}},d} \to \mathrm{End}_{A^{\nu}_{R_{\mathfrak{p}},\tau}}(T_{R_{\mathfrak{p}},d})^{\mathrm{op}}$, and a functor $\Phi^s_{R_{\mathfrak{p}},d} : A^{\nu}_{R_{\mathfrak{p}},\tau}\{d\} \to H^s_{R_{\mathfrak{p}},d}$-mod.

Now, assume that $R = K$ is a field and recall the following.

Proposition 4.7 [5] *Let $\tau_{K,u} - \tau_{K,v} \notin \mathbb{Z}^{\times}$ all u, v.*

(a) *Assume that $\nu_p \geqslant d$ for all p. Then, the map $\varphi^s_{K,d}$ is a K-algebra isomorphism $H^s_{K,d} \to \mathrm{End}_{A^{\nu}_{K,\tau}}(T_{K,d})^{\mathrm{op}}$.*
(b) *Assume that ν is either dominant or anti-dominant. Then, the category $A^{\nu}_{K,\tau}$ is a sum of blocks of $\mathscr{O}^{\nu}_{K,\tau}$, the functors E, F are biadjoint, the module $T_{K,d}$ is projective in $A^{\nu}_{K,\tau}$ and a simple module of $A^{\nu}_{K,\tau}$ is a submodule of a parabolic Verma module if and only if it lies in the top of $T_{K,d}$.*
(c) *Assume that $\tau_{K,u} - \tau_{K,v} \neq 0$ for all $u \neq v$ and that $\nu_p \geqslant d$ for all p. Then, the category $A^{\nu}_{K,\tau}$ is split semi-simple. Assume further that ν is either dominant or anti-dominant. Then $\Phi^s_{K,d}$ is an equivalence of K-categories which maps $\Delta(\lambda)_{K,\tau}$ to $S(\lambda)^s_K$.*

Proof For ν dominant, part (a) is proved in [5, thm. 5.13, cor. 6.7]. For non-dominant ν, a proof is given in [4, lem. 5.5] using [5]. It can also be proved using [40, lem. 5.4].

Part (b) is proved in [5]. For instance, the bi-adjointness of E, F is obvious because $A^{\nu}_{K,\tau}$ is a sum of blocks of $\mathscr{O}^{\nu}_{K,\tau}$, and to prove the third claim one checks first that $T_{K,0}$ is projective and then that the functor F preserves projective modules. The last claim of (b) is proved in [5, thm. 4.8].

The first statement of (c) follows from the linkage principle. By [5, lem. 4.2], the module $T_{K,d}$ is a projective generator in this case. Therefore, the functor $\Phi^s_{K,d}$ is an equivalence of K-categories. It maps $\Delta(\lambda)_{K,\tau}$ to $S(\lambda)^s_K$ by [5, thm. 6.12]. □

Remark 4.8 Assume that $\nu_p \geqslant d$ and $\tau_{K,u} - \tau_{K,v} \notin \mathbb{Z}^{\times}$ for each p, u, v. Then, the tuple (E, F, X, T) define a pre-categorical action on $A^{\nu}_{K,\tau}$.

4.7 The category A with $\ell = 2$

If $\tau_{K,u} - \tau_{K,v} \in \mathbb{Z}_{<0}$ for some $u < v$, then the category $A^{\nu}_{K,\tau}$ is well defined but it may not be a sum of blocks of $\mathscr{O}^{\nu}_{K,\tau}$. In this section we generalize

Proposition 4.7 in order to allow *integral* deformation parameters. To simplify, we'll assume that $\ell = 2$. This is enough for our purpose. Similar results can be obtain for arbitrary ℓ. Note that, for $\ell = 2$, the composition ν is always either dominant or anti-dominant.

The aim of this section is to prove the following.

Proposition 4.9 *Assume that* $\ell = 2$, $\nu_1, \nu_2 \geqslant d$ *and* $\tau_{K,1} - \tau_{K,2} \notin \mathbb{N}^\times$. *Put* $s = \nu + \tau$. *Then, the following hold*

(a) $\varphi^s_{K,d}$ *is an isomorphism* $H^s_{K,d} \to \mathrm{End}_{A^\nu_{K,\tau}}(T_{K,d})^{\mathrm{op}}$,
(b) $T_{K,d}$ *is projective in* $A^\nu_{K,\tau}$,
(c) *a simple module of* $A^\nu_{K,\tau}$ *is a submodule of a parabolic Verma module if and only if it lies in the top of* $T_{K,d}$.

In order to prove this, we first prove the following.

Proposition 4.10 *Assume that* $\ell = 2$, $\nu_1, \nu_2 \geqslant d$ *and* $\tau_{K,1} - \tau_{K,2} \in \mathbb{Z}_{<0}$. *Set* $\nu' = (\nu'_1, \nu'_2)$ *and* $\tau'_K = (\tau'_{K,1}, \tau'_{K,2})$ *with* $\nu' = \nu + (0, 1)$, $\tau'_K = \tau_K - (0, 1)$. *Put* $s = \nu + \tau$ *and* $s' = \nu' + \tau'$. *Then, we have* $s = s'$ *and there is an equivalence of highest weight categories* $A^\nu_{K,\tau}\{d\} \simeq A^{\nu'}_{K,\tau'}\{d\}$ *which intertwines the morphisms* $\varphi^s_{K,d}, \varphi^{s'}_{K,d}$ *and the functors* $\Phi^s_{K,d}, \Phi^{s'}_{K,d}$.

Proof The proof is rather long and consists of several steps.

Write $\mathfrak{g} = \mathfrak{gl}_{K,N}$, $\mathfrak{g}' = \mathfrak{gl}_{K,N+1}$ and $e_{N+1} = \mathrm{diag}(0, \ldots, 0, 1)$. Set also $\mathfrak{n} = \bigoplus_{i=1}^{N} K e_{N+1,i}$ and $\mathfrak{u} = \bigoplus_{i=1}^{N+1} K e_{i,N+1}$.

Fix $\varkappa \in K$. Let $\mathfrak{g}$-Mod be the category of all $\mathfrak{g}$-modules. We define the functors

$$\mathcal{R} : \mathfrak{g}'\text{-Mod} \to \mathfrak{g}\text{-Mod}, \quad M \mapsto \mathrm{Ker}_M(e_{N+1} - \varkappa)$$
$$\mathcal{I} : \mathfrak{g}\text{-Mod} \to \mathfrak{g}'\text{-Mod}, \quad M \mapsto U(\mathfrak{g}') \otimes_{U(\mathfrak{p})} (M \otimes_K K_\varkappa)$$

where $\mathfrak{p} = \mathfrak{p}_{K,N,1}$ is the standard parabolic of type $(N, 1)$ and $K_\varkappa$ is the obvious $\mathfrak{gl}_{K,1}$-module. Let $\mathfrak{m} = \mathfrak{m}_{K,N,1}$ be the Levi subalgebra of $\mathfrak{p}$.

Let $\mathcal{C}_{\geqslant\varkappa} \subset \mathfrak{g}'$-Mod be the full subcategory of modules for which e_{N+1} is semi-simple with weights in $\varkappa + \mathbb{N}$. The functor $\mathcal{R}$ restricts to an exact functor $\mathcal{C}_{\geqslant\varkappa} \to \mathfrak{g}$-Mod, and since $U(\mathfrak{g}') = K[\mathfrak{n}] \otimes_K U(\mathfrak{p})$, the functor $\mathcal{I}$ takes values in $\mathcal{C}_{\geqslant\varkappa}$.

Lemma 4.11 *The functor* $\mathcal{I} : \mathfrak{g}$*-Mod* $\to \mathcal{C}_{\geqslant\varkappa}$ *is exact, fully faithful, and is left adjoint to* $\mathcal{R} : \mathcal{C}_{\geqslant\varkappa} \to \mathfrak{g}$*-Mod.*

Proof Let us first prove the adjointness. Given $M \in \mathfrak{g}$-Mod, $L \in \mathfrak{g}'$-Mod, we have $\mathrm{Hom}_{\mathfrak{g}'}(\mathcal{I}(M), L) \simeq \mathrm{Hom}_{\mathfrak{g}}(M, \mathrm{Hom}_{\bar{\mathfrak{n}}}(K_\varkappa, L))$. If $L \in \mathcal{C}_{\geqslant\varkappa}$, then we have $\mathrm{Hom}_{\bar{\mathfrak{n}}}(K_\varkappa, L) = \mathrm{Hom}_{Ke_{N+1}}(K_\varkappa, L)$. We deduce that there is an isomorphism $\mathrm{Hom}_{\mathfrak{g}'}(\mathcal{I}(M), L) \simeq \mathrm{Hom}_{\mathfrak{g}}(M, \mathcal{R}(L))$. So $\mathcal{I}$ is left adjoint to $\mathcal{R}$.

Now, let us prove the fully faithfulness of $\mathcal{I}$. We have $U(\mathfrak{g}') \otimes_{U(\mathfrak{u})} K = K[\mathfrak{n}] \otimes_K U(\mathfrak{g})$ as $(\mathfrak{m}, \mathfrak{g})$-bimodules. The left $\mathfrak{m}$-action comes from the adjoint action of $K e_{N+1}$ on $\mathfrak{n}$ and the diagonal adjoint action of $\mathfrak{g}$. The right $\mathfrak{g}$-action is the opposite of the adjoint action of $\mathfrak{g}$ on itself. We have $\mathcal{I}(M) \simeq K[\mathfrak{n}] \otimes_K (M \otimes_K K_\varkappa)$ as an $\mathfrak{m}$-module. We deduce that the unit $1 \to \mathcal{R}\mathcal{I}$ is invertible. $\square$

Lemma 4.12 *Let $\mathcal{A}$, $\mathcal{A}'$ be two abelian artinian categories, and $\mathcal{I} : \mathcal{A} \to \mathcal{A}'$ a fully faithful functor with an exact right adjoint $\mathcal{R}$. Then, the following hold*

(a) *the full subcategory* $\mathrm{Im}(\mathcal{I})$ *of $\mathcal{A}'$ is extension closed,*
(b) *if $\mathcal{R}$ induces an isomorphism $[\mathcal{A}] \to [\mathcal{A}']$ then $\mathcal{R}, \mathcal{I}$ are inverse equivalences of categories.*

Proof The functor $\mathcal{I}$ is a right exact, hence $\mathcal{I}\mathcal{R}$ is also right exact. Given an exact sequence $0 \to \mathcal{I}(M) \to L \to \mathcal{I}(M') \to 0$ in $\mathcal{A}'$ with $M, M' \in \mathcal{A}$, we obtain a commutative diagram whose rows are exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I}(M) \arrow[r] & L \arrow[r] & \mathcal{I}(M') \arrow[r] & 0 \\
& \mathcal{I}\mathcal{R}\mathcal{I}(M) \arrow[r] \arrow[u] & \mathcal{I}\mathcal{R}(L) \arrow[r] \arrow[u] & \mathcal{I}\mathcal{R}\mathcal{I}(M') \arrow[r] \arrow[u] & 0.
\end{tikzcd}$$

The vertical maps are given by the counit $\mathcal{I}\mathcal{R} \to 1$. Since $\mathcal{I}$ is fully faithful, the unit $1 \to \mathcal{R}\mathcal{I}$ is an isomorphism. Thus, the left and right vertical maps are invertible. It follows that the two sequences are actually isomorphic, hence $\mathrm{Im}(\mathcal{I})$ is extension-closed. This proves part (a).

To prove (b), since $1 \simeq \mathcal{R}\mathcal{I}$, it is enough to check that the counit is an isomorphism $\mathcal{I}\mathcal{R} \to 1$. Since $\mathcal{R}$ is exact and since $\mathcal{R}\mathcal{I}\mathcal{R} \xrightarrow{\sim} \mathcal{R}$ by adjunction, for each $M \in \mathcal{A}$ the kernel and the cokernel of $\mathcal{I}\mathcal{R}(M) \xrightarrow{\sim} M$ are killed by $\mathcal{R}$. Hence their classes in the Grothendieck groups are 0. Hence they are both 0. $\square$

Corollary 4.13 *The full subcategory* $\mathrm{Im}(\mathcal{I})$ *of $\mathcal{C}_{\geqslant\varkappa}$ is extension-closed and $\mathcal{I}, \mathcal{R}$ induce inverse equivalences* $\mathfrak{g}$-Mod $\simeq \mathrm{Im}(\mathcal{I})$.

Let $\mathfrak{t}$, $\mathfrak{t}'$ be the Cartan subalgebras of $\mathfrak{g}$, $\mathfrak{g}'$. Set $P_K = \mathfrak{t}^*$, $P'_K = (\mathfrak{t}')^*$. We abbreviate $\mathcal{O} = \mathcal{O}_K(N)$ and $\mathcal{O}' = \mathcal{O}_K(N+1)$. Given $\lambda \in P_K$, let $M(\lambda) = M(\lambda)_K$ be the corresponding Verma module in $\mathcal{O}$. For $\lambda' \in P'_K$, we define $M(\lambda') \in \mathcal{O}'$ similarly.

We have $\mathcal{I}(M(\lambda)) \simeq M(\lambda')$, where $\lambda' = \lambda + \varkappa\epsilon_{N+1}$. Thus, we have $\mathcal{R}M(\lambda') \simeq \mathcal{R}\mathcal{I}(M(\lambda)) \simeq M(\lambda)$. We deduce that $\mathcal{I}$, $\mathcal{R}$ are inverse equivalences between the category of Δ-filtered $\mathfrak{g}$-modules in $\mathcal{O}$ and the category of $\mathfrak{g}'$-modules which are extensions of objects $M(\lambda')$ with $\lambda' \in P_K + \varkappa\epsilon_{N+1}$.

Now, fix $d, \nu, \nu', \tau_K, \tau'_K$ as in Proposition 4.10. Put $\varkappa = \tau_{2,K} + N$. We abbreviate $\mathscr{O}^\nu = \mathscr{O}^\nu_K(N)$ and $\mathscr{O}^{\nu'} = \mathscr{O}^{\nu'}_K(N+1)$. Write also $A = A^\nu_{K,\tau}(N)$ and $A' = A^{\nu'}_{K,\tau'}(N+1)$. Let ϖ, ϖ' be the maps (4.1) associated with the parabolic categories $\mathscr{O}^\nu$, $\mathscr{O}^{\nu'}$. For $\lambda \in P^\nu_K$, $\lambda' \in P^{\nu'}_K$ let $M(\lambda)_\nu$, $M(\lambda')_{\nu'}$ be the parabolic Verma modules $M(\lambda)_{K,\nu}$, $M(\lambda')_{K,\nu'}$ in $\mathscr{O}^\nu$, $\mathscr{O}^{\nu'}$.

Consider the sets of weights $\mathcal{E}(d) = \{\varpi(\lambda);\ \lambda \in \mathscr{P}^2_d\}$ and $\mathcal{E}'(d) = \{\varpi'(\lambda);\ \lambda \in \mathscr{P}^2_d\}$ in P^ν_K, $P^{\nu'}_K$ respectively. Since $\nu_1, \nu_2 \geqslant d$, we have an isomorphism of posets $\mathcal{E}(d) \to \mathcal{E}'(d)$ such that $\lambda \mapsto \lambda' = \lambda + \varkappa\epsilon_{N+1}$.

Let $\mathcal{Q} : \mathscr{O} \to \mathscr{O}^\nu$ be the functor sending a module to its largest quotient in $\mathscr{O}^\nu$. This is the left adjoint to the inclusion functor $\mathscr{O}^\nu \to \mathscr{O}$. We define $\mathcal{Q}' : \mathscr{O}' \to \mathscr{O}^{\nu'}$ in the same way.

Lemma 4.14 *The functors $\mathcal{Q}'\mathcal{I}, \mathcal{R}$ induce inverse equivalences of highest categories $A\{d\} \simeq A'\{d\}$.*

Proof Let $\lambda \in P^\nu_K$ and $\lambda' = \lambda + \varkappa\epsilon_{N+1}$. Assume $\lambda' \in P^{\nu'}_K$. Let $\{\alpha_i;\ i \in I_\nu\}$ be the set of simple roots in Π^+_ν. There is an exact sequence

$$\bigoplus_{i \in I_{\nu'}} M(s_i \bullet \lambda') \to M(\lambda') \to M(\lambda')_{\nu'} \to 0.$$

We have $s_i \bullet \lambda \notin P^\nu_K$ for $i \in I_\nu$, hence $\mathcal{Q}M(s_i \bullet \lambda) = 0$. So, for $i \neq n$ we have

$$\mathcal{QR}M(s_i \bullet \lambda') \simeq \mathcal{QR}M(s_i \bullet \lambda + \varkappa\epsilon_{N+1}) \simeq \mathcal{QRI}M(s_i \bullet \lambda) \simeq \mathcal{Q}M(s_i \bullet \lambda) = 0.$$

On the other hand, we have $\mathcal{R}M(s_N \bullet \lambda') = 0$ because $M(s_N \bullet \lambda') \in \mathcal{C}_{>\varkappa}$. Since $\mathcal{QR}$ is right exact, this yields an isomorphism $\mathcal{QR}M(\lambda')_{\nu'} \simeq \mathcal{QR}M(\lambda')$. Note that $\mathcal{R}$ restricts to a functor $\mathscr{O}^{\nu'} \cap \mathcal{C}_{\geqslant \varkappa} \to \mathscr{O}^\nu$. We deduce that

$$\mathcal{R}M(\lambda')_{\nu'} \simeq \mathcal{QR}M(\lambda')_{\nu'} \simeq \mathcal{QR}M(\lambda') \simeq \mathcal{Q}M(\lambda) \simeq M(\lambda)_\nu.$$

Thus, $\mathcal{R}$ restricts to an exact functor $A'\{d\}^\Delta \to A\{d\}^\Delta$. Since $A'\{d\}^\Delta$ contains a progenerator for $A'\{d\}$, $\mathcal{R}$ is right exact and $A\{d\}$ is preserved under taking quotients, we deduce that $\mathcal{R}$ restricts to an exact functor $A'\{d\} \to A\{d\}$. For a future use, note also that $\mathcal{R}$ yields an isomorphism $[A'\{d\}] \to [A\{d\}]$.

Let $\mathcal{S}$ be the endofunctor of $\mathscr{O}'$ sending a module to the quotient by its largest submodule on which e_{N+1} doesn't have the eigenvalue $\varkappa$. Let us consider the functor $\mathcal{SI}$ on $\mathscr{O}$. It is right exact and takes values in $\mathcal{C}_{\geqslant \varkappa}$. For $N \in \mathscr{O}$, the module $\mathcal{SI}(N)$ is the quotient of $\mathcal{I}(N)$ by its largest submodule contained in $\mathcal{C}_{>\varkappa}$. Since $\mathcal{R}$ is exact and vanishes on $\mathcal{C}_{>\varkappa}$, we deduce that $1 \simeq \mathcal{RI} \simeq \mathcal{RSI}$ on $\mathscr{O}$.

Next, for $\lambda \in \mathcal{E}(d)$ the counit $\mathcal{IR} \to 1$ yields a map $\mathcal{I}M(\lambda)_\nu \to M(\lambda')_{\nu'}$ which is obviously surjective. Let M be its kernel. Applying the exact functor $\mathcal{R}$ to the exact sequence $0 \to M \to \mathcal{I}M(\lambda)_\nu \to M(\lambda')_{\nu'} \to 0$ yields the exact sequence $0 \to \mathcal{R}(M) \to M(\lambda)_\nu \to M(\lambda)_\nu \to 0$. We deduce that $\mathcal{R}(M) = 0$. Since $M \in \mathcal{C}_{\geqslant\varkappa}$, this implies that $M \in \mathcal{C}_{>\varkappa}$. Thus, applying the right exact functor $\mathcal{S}$ to the exact sequence above yields the isomorphism $\mathcal{SI}M(\lambda)_\nu \simeq \mathcal{S}M(\lambda')_{\nu'}$. Now, the constituents of $M(\lambda')_{\nu'}$ have a highest weight of the form μ' for some $\mu \in \mathfrak{t}^*$, because $M(\lambda')_{\nu'} \in A'\{d\}$. Hence, the only submodule of $M(\lambda')_{\nu'}$ contained in $\mathcal{C}_{>\varkappa}$ is 0. So $\mathcal{S}M(\lambda')_{\nu'} \simeq M(\lambda')_{\nu'}$, hence $\mathcal{SI}M(\lambda)_\nu \simeq M(\lambda')_{\nu'}$.

Now, consider an exact sequence $0 \to M_1 \to M \to M_2 \to 0$ in $A\{d\}^\Delta$. Since $\mathcal{SI}$ is right exact, we have an exact sequence $\mathcal{SI}(M_1) \to \mathcal{SI}(M) \to \mathcal{SI}(M_2) \to 0$. By induction on the length of a Δ-filtration, we have $\mathcal{SI}(M_1), \mathcal{SI}(M_2) \in A'\{d\}$. Thus, the image of the map $\mathcal{SI}(M_1) \to \mathcal{SI}(M)$ lies in $A'\{d\}$, hence $\mathcal{SI}(M) \in A'\{d\}$. We deduce that $\mathcal{SI}(A\{d\}^\Delta) \subset A'\{d\}$. Since $A\{d\}^\Delta$ contains a progenerator for $A\{d\}$, $\mathcal{SI}$ is right exact and $A'\{d\}$ is preserved under taking quotients, we deduce that $\mathcal{SI}$ restricts to a functor $A\{d\} \to A'\{d\}$.

Finally, let us consider the functor $\mathcal{Q}'\mathcal{I}$. Since $\mathcal{R}$ takes $\mathscr{O}^{\nu'} \cap \mathcal{C}_{\geqslant\varkappa}$ to $\mathscr{O}^\nu$, the functor $\mathcal{Q}'\mathcal{I} : \mathscr{O}^\nu \to \mathscr{O}^{\nu'} \cap \mathcal{C}_{\geqslant\varkappa}$ is left adjoint to $\mathcal{R}$. So $\mathcal{Q}'\mathcal{I}$ is right exact and we have an exact sequence

$$\bigoplus_{i \in I_\nu} M(s_i \bullet \lambda) \to M(\lambda) \to M(\lambda)_\nu \to 0.$$

Since $s_i \bullet \lambda' \notin P_K^{\nu'}$ for $i \in I_\nu$, we have $\mathcal{Q}'M(s_i \bullet \lambda') = 0$, hence $\mathcal{Q}'\mathcal{I}M(s_i \bullet \lambda) \simeq \mathcal{Q}'M(s_i \bullet \lambda') = 0$. We deduce that

$$\mathcal{Q}'\mathcal{I}M(\lambda)_\nu \simeq \mathcal{Q}'\mathcal{I}M(\lambda) \simeq \mathcal{Q}'M(\lambda') \simeq M(\lambda')_{\nu'}.$$

Therefore, since $\mathcal{Q}'\mathcal{I}$ is right exact and $\mathcal{Q}'\mathcal{I}M(\lambda)_\nu \simeq M(\lambda')_{\nu'}$, the same argument as for $\mathcal{SI}$, see above, implies that $\mathcal{Q}'\mathcal{I}$ restricts to a functor $A\{d\} \to A'\{d\}$ which is left adjoint to $\mathcal{R}$.

Next, we compare the functors $\mathcal{Q}'\mathcal{I}$, $\mathcal{SI}$ on $A\{d\}$. For each $N \in A\{d\}$ we write $\mathcal{SI}(N) = \mathcal{I}(N)/L$ and $\mathcal{Q}'\mathcal{I}(N) = \mathcal{I}(N)/M$. Since $d < \nu'_2 = \nu_2 + 1$ and $\mathcal{Q}'\mathcal{I}(N) \in A'\{d\}$, the constituents of $\mathcal{Q}'\mathcal{I}(N)$ are in $\mathcal{C}_{\geqslant\varkappa} \setminus \mathcal{C}_{>\varkappa}$. Hence, the constituents of $I(N)$ which are in $\mathcal{C}_{>\varkappa}$ are contained in M. Since $L \in \mathcal{C}_{>\varkappa}$, we deduce that $L \subset M$. Thus we have an epimorphism $\mathcal{SI} \to \mathcal{Q}'\mathcal{I}$ on $A\{d\}$. Hence, since $\mathcal{R}$ is exact, the isomorphism $1 \to \mathcal{RSI}$ and the unit $1 \to \mathcal{RQ}'\mathcal{I}$ yield a commutative triangle

$$\begin{tikzcd} 1 \arrow[r, "\sim"] \arrow[dr] & \mathcal{RSI} \arrow[d] \\ & \mathcal{RQ}'\mathcal{I}, \end{tikzcd}$$

from which we deduce that the unit is surjective. Now, by adjunction, composing the unit and counit gives the identity $\mathcal{R} \to \mathcal{R}\mathcal{Q}'\mathcal{I}\mathcal{R} \to \mathcal{R}$. Hence the unit is injective, hence is an isomorphism, on $\mathrm{Im}(\mathcal{R})$. But, since $1 \simeq \mathcal{R}\mathcal{S}\mathcal{I}$, the functor $\mathcal{R} : A'\{d\} \to A\{d\}$ is essentially surjective. We deduce that $1 \simeq \mathcal{R}\mathcal{Q}'\mathcal{I}$ on $A\{d\}$.

Therefore, the functor $\mathcal{R} : A'\{d\} \to A\{d\}$ is exact and yields an isomorphism $[A'\{d\}] \to [A\{d\}]$, while $\mathcal{Q}'\mathcal{I} : A\{d\} \to A'\{d\}$ is a fully faithful left adjoint. Hence, Lemma 4.12 shows that $\mathcal{Q}'\mathcal{I}$, $\mathcal{R}$ are inverse equivalences of categories. $\square$

Recall the set $I = \{\tau_{K,1}, \ldots, \tau_{K,\ell}\} + \mathbb{Z}$.

Lemma 4.15 *The functors $\mathcal{Q}'\mathcal{I}, \mathcal{R}$ between $A\{d\}, A'\{d\}$ commute with E_i, F_i, X, T (whenever $E_i, F_i, i \in I$, make sense).*

Proof Since $\mathcal{Q}'\mathcal{I}, \mathcal{R}$ are inverse equivalences, it is enough to consider the case of $\mathcal{R}$. Next, since (E, F) is an adjoint pair, by unicity of the left adjoint, it is enough to consider the case of the functor F. Let $V_N = \bigoplus_{i=1}^N K v_i$. Let $M \in \mathfrak{g}'$-Mod.

If $M \in \mathcal{C}_{\geqslant \varkappa}$, then $V_{N+1} \otimes_K M \in \mathcal{C}_{\geqslant \varkappa}$ and the decomposition $V_{N+1} = V_N \oplus K e_{N+1}$ yields an isomorphism $\mathcal{R}(V_{N+1} \otimes_K M) = V_N \otimes_K \mathcal{R}(M)$, because $\mathrm{Ker}_M(e_{N+1} - \varkappa + 1) = 0$. So, we have an isomorphism of functors $\mathcal{R} \circ f \simeq f \circ \mathcal{R} : \mathcal{C}_{\geqslant \varkappa} \to \mathfrak{g}$-Mod. Since $\mathcal{R}$ takes $A'\{d\}$ to $A\{d\}$, and since f preserves the categories A, A', this yields an isomorphism of functors $\mathcal{R} \circ f \simeq f \circ \mathcal{R} : A'\{d\} \to A\{d+1\}$. We deduce that the functors $F_i\{d\} : A'\{d\} \to A'\{d+1\}$ and $F_i\{d\} : A\{d\} \to A\{d+1\}$ are intertwined by $\mathcal{R}$ whenever they are defined (i.e., if $i \in I \setminus \{\varkappa - N + 1\}$).

Let $i : V_N \otimes M \to V_{N+1} \otimes M$ be the canonical inclusion and $p : V_{N+1} \otimes M \to V_N \otimes M$ be the canonical projection. We have $p \circ \omega_{N+1} \circ i = \omega_N$. It follows that the action of X commutes with the isomorphism $\mathcal{R} \circ f \xrightarrow{\sim} f \circ \mathcal{R}$. It is clear that the induced isomorphism $\mathcal{R} \circ f^2 \xrightarrow{\sim} f^2 \circ R$ commutes with the action of T. $\square$

This finishes the proof of Proposition 4.10. $\square$

Now, we can prove Proposition 4.9.

Proof of Proposition 4.9 We may assume that $\tau_{K,1} - \tau_{K,2} \in \mathbb{Z}_{<0}$. Set $\nu'_1 = \nu_1$, $\tau'_{K,1} = \tau_{K,1}$, $\nu'_2 = \nu_2 + \tau_{K,2} - \tau_{K,1}$ and $\tau'_{K,2} = \tau'_{K,1}$. Recall that $s = \nu + \tau$ and $s' = \nu' + \tau'$. By Proposition 4.10, there is an equivalence of highest weight categories $\Upsilon : A^{\nu}_{K,\tau}\{d\} \to A^{\nu'}_{K,\tau'}\{d\}$ which intertwines the morphisms $\varphi^s_{K,d}$, $\varphi^{s'}_{K,d}$ and the functors $\Phi^s_{K,d}$, $\Phi^{s'}_{K,d}$. In particular, we have $\Upsilon(T_{K,d}) = T_{K,d}$, see the proof of Proposition 4.10. Now, we can apply Propositions 4.7 to $A^{\nu'}_{K,\tau'}\{d\}$, because $\tau'_{K,1} = \tau'_{K,2}$. This proves the proposition. $\square$

Remark 4.16 Under the hypothesis in Proposition 4.9, the tuple (E, F, X, T) is a pre-categorical action on $A^{\nu}_{K,\tau}$.

4.8 The categories A and $\mathscr{O}$ of a Levi subalgebra

Fix a pair of distinct elements $u, v \in [1, \ell]$. We will represent an $(\ell-1)$-tuple a as a collection of elements $a_\bullet$, a_p with $p \in [1, \ell] \setminus \{u, v\}$. If a is an ℓ-tuple of elements of a ring we write $a_\circ = (a_u, a_v)$ and $a_\bullet = a_u + a_v$. Finally, we consider the positive root system $\Pi^+_{\nu,u,v} = \Pi^+ \cap \Pi_{\nu,u,v}$ with $\Pi_{\nu,u,v} = \big\{\alpha_{k,l};\ p_k = p_l \text{ or } (p_k, p_l) = (u, v), (v, u)\big\}$.

We will be interested by two types of Levi subalgebras of $\mathfrak{g}_R$:

- first, we have the Lie subalgebra $\mathfrak{m}_{R,\nu}$ associated with Π_ν,
- next, we have the Lie subalgebra $\mathfrak{m}_{R,\nu,u,v}$ associated with $\Pi_{\nu,u,v}$.

Note that the Levi subalgebra $\mathfrak{m}_{R,\nu,u,v}$ may not be standard. To each of these Lie algebras we associate a module category. To do so, fix a composition γ_p of ν_p for each p.

First, for each tuple $a = (a_p) \in \mathbb{N}^\ell$ we write $P\{a\} = \{\lambda \in P;\ \langle \lambda, \det_p \rangle = a_p,\ \forall p\}$ and $P^\nu\{a\} = P^\nu \cap P\{a\}$. Consider the categories of $\mathfrak{m}_{R,\nu}$-modules given by (the tensor product is over R)

$$\mathscr{O}^{\gamma}_{R,\tau}(\nu) = \bigotimes_{p=1}^{\ell} \mathscr{O}^{\gamma_p}_{R,\tau_p}(\nu_p), \quad \mathscr{O}^{\gamma}_{R,\tau}(\nu)\{a\} = \bigotimes_{p=1}^{\ell} \mathscr{O}^{\gamma_p}_{R,\tau_p}(\nu_p)\{a_p\}. \tag{4.3}$$

Next, for each tuple $a = (a_\bullet, a_p) \in \mathbb{N}^{\ell-1}$, we set $P\{a\} = \{\lambda \in P;\ \langle \lambda, \det_\bullet \rangle = a_\bullet,\ \langle \lambda, \det_p \rangle = a_p\}$ and $P^\nu\{a\} = P^\nu \cap P\{a\}$. Consider the categories of $\mathfrak{m}_{R,\nu,u,v}$-modules given by

$$\mathscr{O}^{\gamma}_{R,\tau}(\nu, u, v) = \mathscr{O}^{\gamma_\circ}_{R,\tau_\circ}(\nu_\bullet) \otimes_R \bigotimes_{p \neq u,v} \mathscr{O}^{\gamma_p}_{R,\tau_p}(\nu_p), \tag{4.4}$$

$$\mathscr{O}^{\gamma}_{R,\tau}(\nu, u, v)\{a\} = \mathscr{O}^{\gamma_\circ}_{R,\tau_\circ}(\nu_\bullet)\{a_\bullet\} \otimes_R \bigotimes_{p \neq u,v} \mathscr{O}^{\gamma_p}_{R,\tau_p}(\nu_p)\{a_p\}. \tag{4.5}$$

We will be mainly interested by the two extreme cases where $\gamma_p = (\nu_p)$ for each p, or where $\gamma_p = (1^{\nu_p})$ for each p. In the first case, we get the categories $\mathscr{O}^{\nu}_{R,\tau}(\nu)$, $\mathscr{O}^{\nu}_{R,\tau}(\nu, u, v)$, in the second one we get the categories $\mathscr{O}_{R,\tau}(\nu)$, $\mathscr{O}_{R,\tau}(\nu, u, v)$.

We will also use highest weight subcategories $A^{\nu}_{R,\tau}(\nu) \subset \mathscr{O}^{\nu}_{R,\tau}(\nu)$ and $A^{\nu}_{R,\tau}(\nu, u, v) \subset \mathscr{O}^{\nu}_{R,\tau}(\nu, u, v)$ which are defined as in Definition 4.3. They decompose in a similar way as in (4.3)–(4.5). We will write $\Delta(A^{\nu}_{R,\tau}(\nu)) = \{\Delta(\lambda)_{R,\tau};\ \lambda \in \mathscr{P}^\nu\}$ and $\Delta(A^{\nu}_{R,\tau}(\nu, u, v)) = \{\Delta(\lambda)_{R,\tau};\ \lambda \in \mathscr{P}^\nu\}$, hoping it will not create any confusion.

Using (4.3), (4.4) and the pre-categorification (e, f, X, T) on $\mathscr{O}^{\nu}_{R,\tau}$ introduced in Sect. 4.5, we define a pre-categorification (e, f, X, T) on $\mathscr{O}^{\nu}_{R,\tau}(\nu)$, $\mathscr{O}^{\nu}_{R,\tau}(\nu, u, v)$ such that, in both cases, the functors e, f are the direct sums of the functors e, f of each of the factors.

Next, using the canonical embeddings we define tuples (E, F, X, T) on $A^{\nu}_{R,\tau}(\nu)$ and $A^{\nu}_{R,\tau}(\nu, u, v)$ as in Sect. 4.6.

5 The category O

Fix integers ℓ, $N \geqslant 1$ and fix a composition $\nu \in \mathscr{C}^{\ell}_{N,+}$. Recall that $\mathfrak{g}_R = \mathfrak{gl}_{R,N}$.

Let R be a deformation ring. Thus, we have elements $\kappa_R \in R^{\times}$ and $\tau_{R,p} \in R$ for $p \in [1, \ell]$. For each p, we define $s_{R,p} \in R$ by $s_{R,p} = \nu_p + \tau_{R,p}$.

We may abbreviate $\kappa = \kappa_R$, $s_p = s_{R,p}$ and $\tau_p = \tau_{R,p}$.

5.1 Analytic algebras

Fix an integer $d \geqslant 1$.

Fix a compact polydisc $D \subset \mathbb{C}^d$. Here, we view $\mathbb{C}^d$ as a Stein analytic space. By an *analytic algebra* we'll mean the localization R of the ring of germs of holomorphic functions on D with respect to some multiplicative subset. See [1,25] for more details on analytic algebras. The following properties hold

- R is a noetherian regular ring of dimension d,
- R is a UFD, hence every height 1 prime ideal is principal,
- for any maximal ideal $\mathfrak{m} \in \mathfrak{M}$, the localization $R_{\mathfrak{m}}$ of R is a henselian local $\mathbb{C}$-algebra.

Since R is an analytic algebra, for any entire function $f = \sum_{n\in\mathbb{N}} a_n z^n$ on $\mathbb{C}$ and for any $x \in R$, the series $\sum_{n\in\mathbb{N}} a_n x^n$ is convergent and defines an element $f(x)$ in R. In particular, we have a well-defined element $\exp(x) \in R$. Analogously, for any analytic function $f : [0, 1] \to M_n(R)$ and for any $v \in R^n$, there is a unique analytic function $v(t)$ on $[0, 1]$ with values in R^n such that $v(0) = v$ and $dv(t)/dt = f(t)v(t)$.

An *analytic deformation ring* is an analytic algebra R which is also a deformation ring. Then, we may view κ_R, $\tau_{R,p}$ as germs of holomorphic functions on D. We will *always* assume that $\kappa_R(D) \subset \mathbb{C} \setminus \mathbb{R}_{\geqslant 0}$. Thus, for any closed point $R \to \mathbb{C}$ the element $\kappa_{\mathbb{C}}$ belongs to $\mathbb{C} \setminus \mathbb{R}_{\geqslant 0}$.

Note that if R is an analytic algebra of dimension $\geqslant 2$, then we can always choose some deformation parameters κ_R, $\tau_{R,p}$ such that R is in general position.

For an analytic deformation ring R we write $q_R = \exp(-2\pi\sqrt{-1}/\kappa_R)$ and $Q_{R,p} = q_R^{s_p} = \exp(-2\pi\sqrt{-1}s_{R,p}/\kappa_R)$. We may abbreviate $q = q_R$, $Q_p = Q_{R,p}$ and $\kappa = \kappa_R$.

5.2 Affine Lie algebras

5.2.1 Notations

Let $L\mathfrak{g}_R = \mathfrak{g} \otimes R[t, t^{-1}]$ and let $\mathbf{g}'_R$ be the Kac–Moody central extension of $L\mathfrak{g}_R$ by R. Let $\mathbf{1}$ be the canonical central element and let ∂ be the derivation of $\mathbf{g}'_R$ acting as $t\partial_t$ on $L\mathfrak{g}_R$ and acting trivially on $\mathbf{1}$.

Put $\mathbf{g}_R = R\partial \oplus \mathbf{g}'_R$ and $\mathbf{t}_R = R\partial \oplus R\mathbf{1} \oplus \mathfrak{t}_R$. Let $\mathbf{b}_R, \mathbf{p}_{R,\nu} \subset \mathbf{g}_R$ be the preimages of $\mathfrak{b}_R$ and $\mathfrak{p}_{R,\nu}$ under the projection $R\partial \oplus R\mathbf{1} \oplus (\mathfrak{g} \otimes R[t]) \to \mathfrak{g}_R$. The element $c = \kappa_R - N$ of R is called the *level*. Consider the R-algebras $\mathbf{g}_{R,\kappa} = U(\mathbf{g}_R)/(\mathbf{1} - c)$ and $\mathbf{g}'_{R,\kappa} = U(\mathbf{g}'_R)/(\mathbf{1} - c)$. For $d \in \mathbb{N}$ we set $\mathbf{g}_{R,\geqslant d} = \mathfrak{g} \otimes t^d R[t]$, $\mathbf{g}'_{R,+} = R\mathbf{1} \oplus \mathbf{g}_{R,\geqslant 0}$ and $\mathbf{g}_{R,+} = R\partial \oplus \mathbf{g}'_{R,+}$.

For a $\mathbf{g}'_{R,+}$-module M of level c we consider the induced module $\mathscr{I}nd_R(M) = \mathbf{g}'_{R,\kappa} \otimes_{U(\mathbf{g}'_{R,+})} M$. We can view a $\mathfrak{g}_R$-module as a $\mathbf{g}'_{R,+}$-module of level c where $\mathbf{g}_{R,\geqslant 1}$ acts trivially. Write again $\mathscr{I}nd_R(M)$ for the corresponding induced module.

For $d \geqslant 1$ let $Q_{R,d} \subset \mathbf{g}_{R,\kappa}$ be the R-submodule spanned by the products of d elements of $\mathbf{g}_{R,\geqslant 1}$. Set $Q_{R,0} = R$. Given a $\mathbf{g}_{R,\kappa}$-module M, let $M(d)$, $M(-d) \subset M$ be the annihilator of $Q_{R,d}$ and of $Q_{R,-d} = {}^\sharp Q_{R,d}$ respectively. Set $M(\infty) = \bigcup_{d\in\mathbb{N}} M(d)$ and $M(-\infty) = \bigcup_{d\in\mathbb{N}} M(-d)$. Note that $M(d)$ is a $\mathbf{g}_{R,+}$-submodule of M and that $M(\infty)$, $M(-\infty)$ are $\mathbf{g}_R$-submodules of M.

A $\mathbf{g}_{R,\kappa}$-module M is *smooth* if $M = M(\infty)$ and if M is flat over R. Let $\mathscr{S}_{R,\kappa}$ be the category of the smooth $\mathbf{g}_{R,\kappa}$-modules.

For each $\xi \in \mathfrak{g}$ and $r \in \mathbb{Z}$, let $\xi^{(r)}$ be the element $\xi \otimes t^r$. For each $s \in \mathbb{Z}$, the *Sugawara operator* $\mathfrak{L}_s$ is the formal sum

$$\mathfrak{L}_s = \frac{1}{2\kappa} \sum_{r\geqslant -s/2} \sum_{i,j=1}^{N} e_{i,j}^{(-r)} e_{j,i}^{(r+s)} + \frac{1}{2\kappa} \sum_{r<-s/2} \sum_{i,j=1}^{N} e_{i,j}^{(r+s)} e_{j,i}^{(-r)}$$

It lies in a completion of $\mathbf{g}_{R,\kappa}$ and it satisfies the relation $[\mathfrak{L}_s, \xi^{(r)}] = -r\xi^{(r+s)}$. The affine Casimir element is $\mathbf{cas} = \partial + \mathfrak{L}_0$.

If $R = \mathbb{C}$ we'll drop the subscript R everywhere from the notation.

5.2.2 Affine root systems

The elements of $\mathbf{t}_R$ and $\widehat{P}_R = \mathbf{t}_R^*$ are called *affine coweights* and *affine weights* respectively. Let $\widehat{\Pi}$ be the set of roots of $\mathbf{g}_R$ and let $\widehat{\Pi}^+$ be the set of roots of $\mathbf{b}_R$. We will call an element of $\widehat{\Pi}$ an *affine root*. Let $\widehat{\Pi}_{re}$ be the system of *real roots*. The set of simple roots in $\widehat{\Pi}^+$ is $\{\alpha_0, \alpha_1, \ldots, \alpha_{N-1}\}$. Let $\check{\alpha} \in \mathbf{t}_R$ be the affine coroot associated with the real affine root α.

Let $(\bullet:\bullet):\widehat{P}_R\times\mathbf{t}_R\to R$ be the canonical pairing. Let $\delta,\Lambda_0,\tilde\rho$ be the affine weights given by $(\delta:\partial)=(\Lambda_0:\mathbf{1})=1$, $(\Lambda_0:R\partial\oplus\mathfrak{t}_R)=(\delta:\mathfrak{t}_R\oplus R\mathbf{1})=0$ and $\tilde\rho=\rho+N\Lambda_0$. We will use the identification $\widehat{P}_R=R\delta\oplus P_R\oplus R\Lambda_0=R\times P_R\times R$ given by $\alpha_i\mapsto(0,\alpha_i,0)$ if $i\neq 0$, $\Lambda_0\mapsto(0,0,1)$ and $\delta\mapsto(1,0,0)$.

Let $\langle\bullet:\bullet\rangle:\widehat{P}_R\times\widehat{P}_R\to R$ be the non-degenerate symmetric bilinear form given by $(\lambda:\check\alpha_i)=2\langle\lambda:\alpha_i\rangle/\langle\alpha_i:\alpha_i\rangle$ and $(\lambda:\mathbf{1})=\langle\lambda:\delta\rangle$. It yields an isomorphism $\nu:\mathfrak{t}_R\to\mathfrak{t}_R^*$. Using it we identify $\check\alpha$ with an element of $\widehat{P}_R$ for any $\alpha\in\widehat{\Pi}_{re}$.

Let $\widehat{W}=W\ltimes\mathbb{Z}\Pi$ be the affine Weyl group and let $s_i=s_{\alpha_i}$ be the simple affine reflections relatively to α_i. The group $\widehat{W}$ acts on $\widehat{P}_R$. For $x\in\mathfrak{t}_R$ let $T_x\in\mathrm{End}(\widehat{P}_R)$ be the operator given by

$$T_x(\lambda)=\lambda+\langle\lambda:\mathbf{1}\rangle\,\nu(x)-\big(\langle\lambda,x\rangle+(\nu(x):\nu(x))\,\langle\lambda:\mathbf{1}\rangle/2\big)\,\delta.$$

The action of the reflection with respect to the affine real root $\alpha+r\delta$, with $\alpha\in\Pi$ and $r\in\mathbb{Z}$, is given by $s_{\alpha+r\delta}=s_\alpha\circ T_{r\check\alpha}$. The $\bullet$-*action* of $\widehat{W}$ is given by $w\bullet\mu=w(\mu+\tilde\rho)-\tilde\rho$ for each $\lambda\in P_R$ and $\mu\in\widehat{P}_R$. Two weights in $\widehat{P}_R^\nu$ are *linked* if they belong to the same orbit of the $\bullet$-action.

The set of *integral affine weights* is $\widehat{P}=\mathbb{Z}\delta+P+\mathbb{Z}\Lambda_0$. Replacing P by P^ν in the definitions above we get the corresponding sets of integral ν-dominant affine weights $\widehat{P}^\nu$. We define the set $\widehat{P}_R^\nu\subset\widehat{P}_R$ of ν-dominant affine weights in the obvious way. To $\lambda\in P_R^\nu$ we set $z_\lambda=-\langle\lambda:2\rho+\lambda\rangle/2\kappa$ and we associate the affine weight $\widehat{\lambda}=(z_\lambda,\lambda,c)\in\widehat{P}_R^\nu$. For $w\in W$, $x\in\mathbb{Z}\Pi$ and $\lambda\in P_R$ we have $w\bullet\widehat{\lambda}=\widehat{w\bullet\lambda}$ and $T_x\bullet\widehat{\lambda}=\widehat{\lambda+\kappa x}$.

5.3 The category O

5.3.1 Definition

A $\mathbf{t}_R$-module M is called a *weight* $\mathbf{t}_R$*-module* if it is a direct sum of the *weight submodules* $M_\lambda=\{m\in M;\ xm=\lambda(x)m,\ x\in\mathbf{t}_R\}$ with $\lambda\in\widehat{P}_R$.

Let $\mathbf{O}_R^{\nu,\kappa}$ be the R-linear abelian category of finitely generated $\mathbf{g}_{R,\kappa}$-modules M such that M is a weight $\mathbf{t}_R$-module, the $\mathbf{p}_{R,\nu}$-action on M is locally finite over R, and the highest weight of any subquotient of M is of the form $\widehat{\lambda}$ with $\lambda\in P_R^\nu$.

For each $\mu\in\widehat{P}_R^\nu$, let $M(\mu)_{R,\nu}$ be the parabolic Verma module with the highest weight μ. For $\lambda\in P_R^\nu$ we have $M(\widehat{\lambda})_{R,\nu}=\mathscr{I}nd(M(\lambda)_{R,\nu})$. Here $\partial,\mathbf{1}$ act on $M(\lambda)_{R,\nu}$ by multiplication by z,c respectively. If $R=K$ is a field, let $L(\mu)_K$ denote the top of $M(\mu)_{K,\nu}$. For $\lambda\in P_R^\nu$ we abbreviate $\mathbf{M}(\lambda)_{R,\nu}=M(\widehat{\lambda})_{R,\nu}$ and $\mathbf{L}(\lambda)_K=L(\widehat{\lambda})_K$.

If ν is regular, we write $\mathbf{O}_R = \mathbf{O}_R^{\nu,\kappa}$ and $\mathbf{M}(\lambda)_R = \mathbf{M}(\lambda)_{R,\nu}$. If $\mathfrak{p}_\nu = \mathfrak{g}$ we write $\mathbf{O}_R^{+,\kappa} = \mathbf{O}_R^{\nu,\kappa}$ and $\mathbf{M}(\lambda)_{R,+} = \mathbf{M}(\lambda)_{R,\nu}$. If $R = \mathbb{C}$ we omit the subscript $\mathbb{C}$ from the notation.

Let $\mathbf{O}_R^{\nu,\kappa,f} \subset \mathbf{O}_R^{\nu,\kappa}$ be the full subcategory consisting of the modules whose weight spaces are free of finite rank over R. Let $\mathbf{O}_R^{\nu,\kappa,\Delta} \subset \mathbf{O}_R^{\nu,\kappa,f}$ be the full extension closed additive subcategory generated by the parabolic Verma modules. The category $\mathbf{O}_R^{\nu,\kappa,\Delta}$ consists of the modules $M \in \mathbf{O}_R^{\nu,\kappa,f}$ such that $\mathrm{k}M \in \mathbf{O}_\mathrm{k}^{\nu,\kappa,\Delta}$ for each $\mathrm{k} \in \mathfrak{M}$.

Given $\tau \in P_R$ as in Sect. 4.2, let $\mathbf{O}_{R,\tau}^{\nu,\kappa} \subset \mathbf{O}_R^{\nu,\kappa}$ be the full subcategory consisting of the modules M such that the highest weight of any subquotient of M is of the form $\widehat{\lambda + \tau}$ with $\lambda \in P^\nu$. We set $\mathbf{O}_{R,\tau}^{\nu,\kappa,\Delta} = \mathbf{O}_{R,\tau}^{\nu,\kappa} \cap \mathbf{O}_R^{\nu,\kappa,\Delta}$. If $R = \mathbb{C}$ or $\tau = 0$ we drop the subscripts R or τ from the notation.

Remark 5.1 The operator **cas** acts locally nilpotently on any module of $\mathbf{O}^{\nu,\kappa}$. Replacing this condition by **cas** *is locally finite* yields a bigger category which decomposes as the direct sum $\bigoplus_{a\in\mathbb{Z}} \mathbf{O}^{\nu,\kappa}[a]$, where $\mathbf{O}^{\nu,\kappa}[a]$ consists of the modules such that $\mathbf{cas} - a$ is locally nilpotent.

More generally, for each $d \in \mathbb{Z}$, we may consider the category $\mathbf{O}_{R,\tau}^{\nu,\kappa}[a]\{d\}$ which consists of the modules whose subquotients have highest weights of the form $(z_{\lambda+\tau} + a, \lambda + \tau, c)$ with $\lambda \in P^\nu\{d\}$. Here, we set $P\{d\} = \{\lambda \in P;\ \langle\lambda, \det\rangle = d\}$ and $P^\nu\{d\} = P^\nu \cap P\{d\}$. To insist on the rank of $\mathfrak{gl}_N$ we may write $\mathbf{O}_{R,\tau}^\nu(N) = \mathbf{O}_{R,\tau}^{\nu,\kappa}$. We will use similar notation for all related categories, e.g., we may write $\mathbf{O}_{R,\tau}^{\nu,\kappa}(N)[a]\{d\} = \mathbf{O}_{R,\tau}^{\nu,\kappa}[a]\{d\}$.

Remark 5.2 In [28] the authors set $R = \mathbb{C}$ and consider a category $\mathbf{O}'$ of $\mathbf{g}'$-modules, rather than $\mathbf{g}_\kappa$-modules as above. Forgetting the ∂-action gives an equivalence $\mathbf{O}^{+,\kappa} \to \mathbf{O}'$. A quasi-inverse takes a $\mathbf{g}'$-module M to itself, with the action of ∂ equal to the semi-simplification of $-\mathfrak{L}_0$. See [44, prop. 8.1] for details.

More generally, forgetting the ∂-action gives again an equivalence from $\mathbf{O}_R^{\nu,\kappa}$ to a category of $\mathbf{g}'_R$-modules, and we may identify both categories. In particular, for $M \in \mathscr{O}_R^\nu$ we can view the $\mathbf{g}'_{R,\kappa}$-module $\mathscr{I}nd_R(M)$ as an object of $\mathbf{O}_R^{\nu,\kappa}$.

We will use this identification without further comments whenever it is necessary.

5.3.2 Basic properties

Let R be either a field or a local ring.

Let $e = -\kappa_\mathrm{k}$, where κ_k is the residue class of κ_R. *We will always assume that e is a positive integer.*

For a $\mathbf{g}_{R,\kappa}$-module M we set

- ${}^\sharp M = M$ with the $\mathbf{g}_R$-action twisted by the automorphism $\sharp$ such that $\xi^{(r)} \mapsto (-1)^r \xi^{(-r)}$ and $\mathbf{1} \mapsto -\mathbf{1}$,
- ${}^\dagger M = M$ with the $\mathbf{g}_R$-action twisted by the automorphism $\dagger$ such that $\xi^{(r)} \mapsto -{}^t\xi^{(r)}$ and $\mathbf{1} \mapsto \mathbf{1}$,
- M^* is the R-dual of M with the $\mathbf{g}_R$-action given by $(\xi^{(r)}\varphi, m) = -(\varphi, \xi^{(r)}m)$ and $(\mathbf{1}\varphi, m) = -(\varphi, \mathbf{1}m)$.

We define the $\mathbf{g}_{R,\kappa}$-modules DM, $\mathscr{D}M$ by $DM = ({}^\sharp M^*)(\infty)$ and $\mathscr{D}M = {}^\dagger DM$.

Lemma 5.3 *The functor D is a duality on $\mathbf{O}_R^{+,f}$ and $\mathscr{D}$ is a duality on $\mathbf{O}_R^{\nu,\kappa,f}$. Both commute with base change.*

Proof For any $M \in \mathbf{O}_R^{\nu,\kappa}$, the R-module DM consists of those linear forms in M^* which vanish on $Q_{R,-d}M$ for some $d \geqslant 1$. Hence, we have $\mathscr{D}M = {}^{\dagger\sharp}M^{\circledast}$, where $M^{\circledast}$ is the set of $\mathfrak{g}_\nu$-finite elements of M^*. Since the automorphism $\dagger\sharp$ takes the Borel subalgebra $\mathbf{b}_R \subset \mathbf{g}_R$ to its opposite, the functor $\mathscr{D}$ preserves $\mathbf{O}_R^{\nu,\kappa,f}$. It is the usual BGG duality, which fixes the simple objects when R is a field.

For any $M \in \mathbf{O}_R^{+}$, the R-module DM consists of those linear forms in M^* which vanish on $Q_{R,-d}M$ for some $d \geqslant 1$, we have $DM = {}^\sharp M^{\circledast}$, where $M^{\circledast}$ is the set of $\mathfrak{g}$-finite elements of M^*. The functor D preserves $\mathbf{O}_R^{+,f}$. It is the duality introduced in [28], which does not fix the simple objects when R is a field.

For the second claim we must prove that for any S-point $R \to S$ we have $D(SM) = SD(M)$ and $S\mathscr{D}(N) = \mathscr{D}(SN)$ for each $M \in \mathbf{O}_R^{+,f}$, $N \in \mathbf{O}_R^{\nu,\kappa,f}$. The proof is the same as in lemma [28, lem. 8.16]. □

A *generalized Weyl module* is a module in $\mathbf{O}_R^{\nu,\kappa,f}$ of the form $\mathscr{I}nd_R(M)$, where M is a $\mathbf{g}_{R,+}$-module with a finite filtration by $\mathbf{g}_{R,+}$-submodules such that the subquotients are annihilated by $Q_{R,1}$ and lie in $\mathscr{O}_R^\nu$ as $\mathfrak{g}_R$-modules.

Lemma 5.4 *A $\mathbf{g}_{R,\kappa}$-module which is free over R belongs to $\mathbf{O}_R^{\nu,\kappa,f}$ if and only if it is a quotient of a generalized Weyl module of $\mathbf{O}_R^{\nu,\kappa,f}$.* □

Remark 5.5 The functors $M \mapsto {}^\dagger M, {}^\sharp M, M^*$ commute with each other and we have a canonical isomorphism of $\mathbf{g}_R$-modules $({}^\sharp M)(\infty) = {}^\sharp(M(-\infty))$.

Remark 5.6 We define the involution $\dagger$ on $\mathfrak{g}_R$-modules and the dualities $\mathscr{D}$ on $\mathscr{O}^\nu$ and D on $\mathscr{O}^+$ in a similar way as above. We have a canonical $\mathbf{g}_{R,\kappa}$-module isomorphism ${}^\dagger\mathscr{I}nd_R(M) = \mathscr{I}nd_R({}^\dagger M)$.

For each $\beta \in \widehat{P}_R^\nu$, the *truncated category* ${}^\beta\mathbf{O}_R^{\nu,\kappa}$ is the Serre subcategory of $\mathbf{O}_R^{\nu,\kappa}$ consisting of the modules whose simple subquotients have a highest weight in $\beta - \mathbb{N}\widehat{\Pi}^+$. The following hold, see e.g. [19,20], [44, sec. 3, 7] for more details.

Proposition 5.7 (a) $\mathbf{O}_R^{\nu,\kappa}$ *is the direct limit of the subcategories* ${}^\beta\mathbf{O}_R^{\nu,\kappa}$,

(b) ${}^\beta\mathbf{O}_R^{\nu,\kappa}$ *is a highest weight* R*-category with* $\Delta({}^\beta\mathbf{O}_R^{\nu,\kappa,\Delta}) = {}^\beta\mathbf{O}_R^{\nu,\kappa} \cap \Delta(\mathbf{O}_R^{\nu,\kappa})$,

(c) *for* $\beta \leqslant \gamma$ *the obvious inclusion* ${}^\beta\mathbf{O}_R^{\nu,\kappa} \subset {}^\gamma\mathbf{O}_R^{\nu,\kappa}$ *preserves the tilting modules and commutes with taking extensions.* □

In particular, we'll regard the tilting modules as objects of $\mathbf{O}_R^{\nu,\kappa,\Delta}$, although $\mathbf{O}_R^{\nu,\kappa}$ is not a highest weight R-category.

Next, from Proposition 2.4 we deduce that the R-category $\mathbf{O}_R^{\nu,\kappa}$ is Hom-finite and that for any local S-point $R \to S$ the base change preserves the tilting modules. Further, if M, N are tilting, then $\mathrm{Hom}_{\mathbf{g}_R}(M, N)$ is free over R and the canonical map $S\,\mathrm{Hom}_{\mathbf{g}_R}(M, N) \to \mathrm{Hom}_{\mathbf{g}_S}(SM, SN)$ is invertible.

We call $\mathbf{O}_R^{+,\kappa}$ the *Kazhdan–Lusztig category* of $\mathfrak{g}_R$, i.e., the affine parabolic category O associated with the standard maximal parabolic in $\mathbf{g}_R$, see [28].

5.3.3 The linkage principle and the highest weight order on $\mathbf{O}$

Assume that R is a local ring. Let us recall the partial order on $\widehat{P}_R^\nu$ given in [46].

First, to each $\widehat{\lambda} = (z, \lambda, c)$ in $\widehat{P}_R$, we associate its *integral affine root system* which is given by $\widehat{\Pi}(\widehat{\lambda}) = \{\alpha \in \widehat{\Pi};\ \langle \widehat{\lambda} : \alpha\rangle_{\mathrm{k}} \in \mathbb{Z}\}$. Since $\widehat{\Pi}(\widehat{\lambda}) = \widehat{\Pi}(0, \lambda, c)$, we may write $\widehat{\Pi}(\lambda, c)$ for $\widehat{\Pi}(\widehat{\lambda})$.

Now, given $\widehat{\lambda}, \widehat{\lambda}' \in \widehat{P}_R^\nu$, we write $\widehat{\lambda} \Uparrow \widehat{\lambda}'$ if and only if there are $\beta \in \widehat{\Pi}(\widehat{\lambda}')$, $w \in W_\nu$ such that $\beta \notin \Pi_\nu$ and $\widehat{\lambda} = ws_\beta \bullet \widehat{\lambda}' \in \widehat{\lambda}' - \widehat{\Pi}^+$ modulo $\mathfrak{m}\,\widehat{P}_R$.

Definition 5.8 (a) The *linkage ordering* is the partial order $\leqslant_\ell$ on $\widehat{P}_R^\nu$ is the transitive and reflexive closure of the relation $\Uparrow$. For $\lambda, \lambda' \in P_R^\nu$ we abbreviate $\lambda \leqslant_\ell \lambda'$ if and only if $\widehat{\lambda} \leqslant_\ell \widehat{\lambda}'$. So, we may view $\leqslant_\ell$ as a partial order on P_R^ν.

(b) The *BGG ordering* $\leqslant_{\mathrm{b}}$ on P_R^ν is the smallest partial order such that $\lambda \leqslant_{\mathrm{b}} \lambda'$ if $[\mathbf{M}(\lambda')_{\mathrm{k},\nu} : \mathbf{L}(\lambda)_{\mathrm{k}}] \neq 0$.

Remark 5.9 The definition of $\leqslant_\ell$ is motivated by the following remark: the parabolic version of the Jantzen formula in [26] for the determinant of the Shapovalov form of a parabolic Verma module in $\mathbf{O}_{\mathrm{k}}^{\nu,\kappa}$ implies that $\leqslant_\ell$ refines $\leqslant_{\mathrm{b}}$. The BGG order induces an highest weight order on ${}^\beta\mathbf{O}_R^{\nu,\kappa}$ for each β. Hence $\leqslant_\ell$ induces also an highest weight order on ${}^\beta\mathbf{O}_R^{\nu,\kappa}$ for each β.

Remark 5.10 The partial orders $\leqslant_\ell, \leqslant_{\mathrm{b}}$ on P_R^ν can be viewed as partial orders on $\mathscr{P}^\nu$ under the inclusion ϖ. They depend on k. To avoid any confusion we may say that these partial orders are *relative to the field* k.

Remark 5.11 If $\mathfrak{p}_\nu = \mathfrak{b}$, then $\leqslant_\ell$ coincides with $\leqslant_{\mathrm{b}}$ by [26].

5.4 The categorical action on O

From now on, unless specified otherwise, we'll assume that R is a regular local analytic deformation ring of dimension $\leqslant 2$.

First, let us briefly recall the main properties of the Kazhdan–Lusztig tensor product $\dot{\otimes}_R$, see Sect. 8.3. Details will be given in Propositions 8.21, 8.29, 8.30 and 8.36.

Recall that V_R is the natural representation of $\mathfrak{g}_R$, and that the modules $\mathbf{V}_R, \mathbf{V}_R^* \in \mathbf{O}_R^{+,\kappa,\Delta}$ are given by $\mathbf{V}_R = \mathscr{I}nd_R(V_R)$, $\mathbf{V}_R^* = \mathscr{I}nd_R(V_R^*)$. We have exact endofunctors e, f on $\mathbf{O}_R^{\nu,\kappa,\Delta}$ given by $e(M) = M\dot{\otimes}_R\mathbf{V}_R^*$ and $f(M) = M\dot{\otimes}_R\mathbf{V}_R$. The functors e, f preserve the tilting modules. If $R = K$ is a field then e, f extend to biadjoint endofunctors of $\mathbf{O}_K^{\nu,\kappa}$.

Since R is an analytic algebra, the element $q_R = \exp(-2\pi\sqrt{-1}/\kappa_R)$ of R is well-defined and the operator $\exp(2\pi\sqrt{-1}\mathfrak{L}_0)$ acts on any module $M \in \mathbf{O}_R^{\nu,\kappa}$. Let X be the endomorphism of the functor f which acts on $f(M)$ by the operator $\exp(-2\pi\sqrt{-1}\mathfrak{L}_0)\,\big(\exp(2\pi\sqrt{-1}\mathfrak{L}_0)\dot{\otimes}_R\exp(2\pi\sqrt{-1}\mathfrak{L}_0)\big)$, see (8.2), (8.10). Let T be the endomorphism of f^2 defined in (8.10). By Remark 3.3 the endomorphisms X, T can be viewed as endomorphisms of e, e^2.

Now, let $R = K$ be a field. Let $\tau \in P_K$ be as in Sect. 4.2. Set $I = \{\tau_{K,1}, \tau_{K,2}, \ldots, \tau_{K,\ell}\} + \mathbb{Z} + \kappa_K\mathbb{Z}$. Write $i \sim j$ if $i - j \in \kappa_K\mathbb{Z}$. Put $\mathscr{I} = I/\sim$. We will identify q_K^i with the element $i/\sim$ in $\mathscr{I}$.

For each $i \in K$ let f_i, e_i be the generalized q_K^i-eigenspace and $q_K^{-(N+i)}$-eigenspace of X acting on f and e. The functors e_i, f_i are biadjoint, see [9, rem. 7.22]. The action of e_i, f_i on parabolic Verma modules can be computed explicitly. Recall that for $\lambda, \mu \in P_K^\nu$ we write $\lambda \xrightarrow{i} \mu$ if $\mu + \rho$ is obtained from $\lambda + \rho$ by replacing an entry equal to i by $i + 1$.

Lemma 5.12 (a) *For each $\lambda \in P_K^\nu$, the module $f_i(\mathbf{M}(\lambda)_{K,\nu})$ has a filtration with sections of the form $\mathbf{M}(\mu)_{K,\nu}$, one for each μ such that $\lambda \xrightarrow{j} \mu$ for some $j \in K$ with $i \sim j$,*

(b) *for each $\lambda \in P_K^\nu$, the module $e_i(\mathbf{M}(\lambda)_{K,\nu})$ has a filtration with sections of the form $\mathbf{M}(\mu)_{K,\nu}$, one for each μ such that $\mu \xrightarrow{j} \lambda$ for some $j \in K$ with $i \sim j$,*

(c) *e, f are exact endofunctors of $\mathbf{O}_{K,\tau}^{\nu,\kappa}$,*

(d) *$e = \bigoplus_{i\in\mathscr{I}} e_i$ and $f = \bigoplus_{i\in\mathscr{I}} f_i$ on $\mathbf{O}_{K,\tau}^{\nu,\kappa}$.*

Proof Propositions 8.21, 8.29 imply that $f(\mathbf{M}(\lambda)_{K,\nu})$ has a filtration (not necessarily unique) whose associated graded consists of the sum of the modules $\mathbf{M}(\mu)_{K,\nu}$ such that $\lambda \xrightarrow{i} \mu$ for some $i \in K$.

Next, the same proof as in [28, prop. 2.7], using the formula $\mathfrak{L}_0 = \mathrm{cas}/2\kappa + \sum_{r>0}\sum_{i,j=1}^N e_{ij}^{(-r)}e_{ji}^{(r)}/\kappa$, shows that the operator $\exp(2\pi\sqrt{-1}\mathfrak{L}_0)$ acts on

$\mathbf{M}(\mu)_{K,\nu}$ by the scalar $\exp(-2\pi\sqrt{-1}z'_\mu)$ for any $\mu \in P^\nu_K$, where $-z'_\mu = \langle \mu : 2\rho + (N-1)\det + \mu\rangle/2\kappa$.

Using this, a direct computation shows that any subquotient of $f(\mathbf{M}(\lambda)_{K,\nu})$ which is isomorphic to $\mathbf{M}(\mu)_{K,\nu}$, for some affine weight μ such that $\lambda \xrightarrow{i} \mu$, belongs to the generalized eigenspace of $X(\mathbf{M}(\lambda)_{K,\nu})$ with eigenvalue q^i_K. This proves (a).

The discussion above implies that $f = \bigoplus_{i\in K} f_i$, as endofunctors of $\mathbf{O}^{\nu,\kappa,\Delta}_K$. We deduce that $f = \bigoplus_{i\in K} f_i$ on $\mathbf{O}^{\nu,\kappa}_K$, because f is exact and any object in $\mathbf{O}^{\nu,\kappa}_K$ is a quotient of an object in $\mathbf{O}^{\nu,\kappa,\Delta}_K$. We prove that $e = \bigoplus_{i\in K} e_i$ in a similar way.

For $\lambda, \mu \in P^\nu_K$ such that $\lambda \xrightarrow{i} \mu$ for some $i \in K$, we have $\lambda \in P^\nu + \tau$ if and only if $\mu \in P^\nu + \tau$. By Lemma 5.12, we deduce that e, f restrict to exact endofunctors on $\mathbf{O}^{\nu,\kappa}_{K,\tau}$. Note that e_i, f_i act by zero on $\mathbf{O}^{\nu,\kappa}_{K,\tau}$ whenever $i \notin I$. This proves (d). □

Now, we define an $\mathfrak{sl}_{\mathscr{I}}$-categorical action on $\mathbf{O}^{\nu,\kappa}_{K,\tau}$. For each $\lambda \in P + \tau$ we write $m_i(\lambda) = \#\{k \in [1, N];\ q_K^{\langle\lambda+\rho,\epsilon_k\rangle} = i\}$ and $\mathrm{wt}(\lambda) = \sum_{i\in\mathscr{I}} \big(m_i(\lambda) - m_{iq}(\lambda)\big)\Lambda_i$. For $\beta \in X_{\mathscr{I}}$ let $\mathbf{O}^{\nu,\kappa}_{K,\tau,\beta} \subset \mathbf{O}^{\nu,\kappa}_{K,\tau}$ be the Serre subcategory generated by the modules $\mathbf{L}(\lambda)_K$ with $\sum_{i\in\mathscr{I}} m_i(\lambda)\,\epsilon_i = \beta$.

Claim 5.13 *For $\lambda, \mu \in P\{d\} + \tau$ we have*
$\widehat{\lambda}, \widehat{\mu}$ *are linked* $\iff$ $m_i(\lambda) = m_i(\mu)$ *for all* $i \in \mathscr{I}$ $\iff$ $\mathrm{wt}(\lambda) = \mathrm{wt}(\mu)$.

Hence, we have a decomposition $\mathbf{O}^{\nu,\kappa}_{K,\tau} = \bigoplus_{\beta\in X_{\mathscr{I}}} \mathbf{O}^{\nu,\kappa}_{K,\tau,\beta}$ by the linkage principle.

Proposition 5.14 *The tuple (e, f, X, T), together with the decomposition of $\mathbf{O}^{\nu,\kappa}_{K,\tau}$ above, is an $\mathfrak{sl}_{\mathscr{I}}$-categorification on $\mathbf{O}^{\nu,\kappa}_{K,\tau}$.*

Proof By Lemma 5.12 we have $e_i(\mathbf{O}^{\nu,\kappa}_{K,\tau,\beta}) \subset \mathbf{O}^{\nu,\kappa}_{K,\tau,\beta+\epsilon_i-\epsilon_{qi}}$ and $f_i(\mathbf{O}^{\nu,\kappa}_{K,\tau,\beta}) \subset \mathbf{O}^{\nu}_{K,\tau,\beta-\epsilon_i+\epsilon_{qi}}$. Further, a direct computation using Lemma 5.12 shows that the operators e_i, f_i with $i \in \mathscr{I}$ yield a representation of $\mathfrak{sl}_{\mathscr{I}}$ on $[\mathbf{O}^{\nu,\kappa}_{K,\tau}]$ such that $[\mathbf{M}(\lambda)_{K,\nu}]$ is a weight vector of weight $\mathrm{wt}(\lambda)$. The rest follows from Lemma 5.12 and Proposition 8.36. □

5.5 The category A and the functor Ψ

Let R be either a field or a local deformation ring. We have the following basic fact.

Lemma 5.15 [46] *The map ϖ identifies $\mathscr{P}^\nu$ with an ideal in P^ν for the partial orders $\leqslant_\ell$ or $\leqslant_{\mathrm{b}}$ relative to* k.

Proof It is enough to consider the case of the ordering $\leqslant_\ell$, because it refines $\leqslant_{\mathrm{b}}$. Since R is a local deformation ring with residue field k, we have $\tau_{\mathrm{k}} = 0$ and $\kappa_{\mathrm{k}} = -e$. Then, the claim follows from [46, prop. A.6.1]. □

For each $\lambda \in \mathscr{P}^\nu$, we abbreviate $\mathbf{\Delta}(\lambda)_{R,\tau} = \mathbf{M}(\varpi(\lambda))_{R,\nu}$ where ϖ is the application defined in (4.1). Following [5,46] we introduce the abelian R-category $\mathbf{A}^{\nu,\kappa}_{R,\tau} \subset \mathbf{O}^{\nu,\kappa}_{R,\tau}$ which is the Serre R-linear subcategory generated by $\{\mathbf{\Delta}(\lambda)_{R,\tau};\ \lambda \in \mathscr{P}^\nu\}$.

Since $\tau_{\mathrm{k}} = 0$, by Lemma 5.15, $\mathbf{A}^{\nu,\kappa}_{\mathrm{k}} = \mathbf{A}^{\nu,\kappa}_{\mathrm{k},\tau}$ is a highest weight k-category. Using [39, thm. 4.15], this implies that $\mathbf{A}^{\nu,\kappa}_{R,\tau}$ is a highest weight R-category such that $\Delta(\mathbf{A}^{\nu,\kappa}_{R,\tau}) = \{\mathbf{\Delta}(\lambda)_{R,\tau};\ \lambda \in \mathscr{P}^\nu\}$. The highest weight order on $\mathbf{A}^{\nu,\kappa}_{R,\tau}$ is given by the partial order $\leqslant_\ell$ or $\leqslant_{\mathrm{b}}$ on $\mathscr{P}^\nu$ relative to k.

We will write $\mathbf{L}(\lambda), \mathbf{P}(\lambda)_{R,\tau}, \mathbf{T}(\lambda)_{R,\tau}$ respectively for the simple, projective, tilting objects associated with $\mathbf{\Delta}(\lambda)_{R,\tau}$. Let $\mathbf{A}^{\nu,\kappa,\Delta}_{R,\tau} = (\mathbf{A}^{\nu,\kappa}_{R,\tau})^\Delta$ be the full exact subcategory of Δ-filtered objects. For each $d \in \mathbb{N}$, let $\mathbf{A}^{\nu,\kappa}_{R,\tau}\{d\} \subset \mathbf{A}^{\nu,\kappa}_{R,\tau}$ be the highest weight subcategory generated by $\Delta(\mathbf{A}^{\nu,\kappa}_{R,\tau}\{d\}) = \{\mathbf{\Delta}(\lambda)_{R,\tau};\ \lambda \in \mathscr{P}^\nu_d\}$.

Now, assume that R is analytic of dimension $\leqslant 2$. By Lemma 5.12 the endofunctor f of $\mathbf{O}^{\nu,\kappa,\Delta}_R$ maps $(\mathbf{A}^{\nu,\kappa}_{R,\tau}\{d\})^\Delta$ to $(\mathbf{A}^{\nu,\kappa}_{R,\tau}\{d+1\})^\Delta$. We define inductively an object $\mathbf{T}^{\nu,\kappa}_{R,\tau}\{d\}$ in $\mathbf{A}^{\nu,\kappa}_{R,\tau}\{d\}$ by setting $\mathbf{T}^{\nu,\kappa}_{R,\tau}\{0\} = \mathbf{\Delta}(\emptyset)_{R,\tau}$ and $\mathbf{T}^{\nu,\kappa}_{R,\tau}\{d\} = f(\mathbf{T}^{\nu,\kappa}_{R,\tau}\{d-1\})$. We will abbreviate $\mathbf{T}_{R,d} = \mathbf{T}^{\nu,\kappa}_{R,\tau}\{d\}$ to unburden the notation. To avoid any confusion we may write $\mathbf{T}^{\nu,\kappa}_{R,\tau}(N)\{d\} = \mathbf{T}^{\nu,\kappa}_{R,\tau}\{d\}$ and $\mathbf{T}_{R,d}(N) = \mathbf{T}_{R,d}$.

Lemma 5.16 (a) *We have* $\mathrm{k}\mathbf{T}_{R,d} = \mathbf{T}_{\mathrm{k},d}$.
(b) *The module* $\mathbf{T}_{R,d}$ *is tilting in* $\mathbf{A}^{\nu,\kappa}_{R,\tau}$.

Proof Part (a) follows from Lemma 8.34. To prove (b), note first that $\mathbf{T}_{R,0}$ is tilting by Proposition 2.4, because $\mathrm{k}\mathbf{T}_{R,0} = \mathbf{T}_{\mathrm{k},0}$ is Δ-filtered and simple. Since the functor f preserves the tilting modules of $\mathbf{O}^{\nu,\kappa}_{R,\tau}$ by Lemma 8.33, we deduce that $\mathbf{T}_{R,d}$ is tilting. □

By Proposition 8.36, we have an R-algebra homomorphism

$$\psi^s_{R,d} : \mathbf{H}^s_{R,d} \to \mathrm{End}_{\mathbf{A}^{\nu,\kappa}_{R,\tau}}\left(\mathbf{T}_{R,d}\right)^{\mathrm{op}} \tag{5.1}$$

and a functor

$$\Psi^s_{R,d} = \mathrm{Hom}_{\mathbf{A}^{\nu,\kappa}_{R,\tau}}(\mathbf{T}_{R,d}, \bullet) : \mathbf{A}^{\nu,\kappa}_{R,\tau}\{d\} \to \mathbf{H}^s_{R,d}\text{-mod}\,.$$

The main result of the section is Theorem 5.37. To prove it, we will study in the subsequent subsections some properties of $\psi^s_{R,d}$ and $\Psi^s_{R,d}$ when localized to codimension one.

Remark 5.17 Since $\mathbf{T}_{R,d}$ is tilting, it is uniquely determined by its specialization $k\mathbf{T}_{R,d} = \mathbf{T}_{k,d}$. If R is a regular local ring of dimension >2, then we may define $\mathbf{T}_{R,d}$ as the unique module in $\mathbf{A}^{\nu,\kappa}_{R,\tau}\{d\}$ (up to isomorphism) which specializes to $\mathbf{T}_{k,d}$. We do not know how to define either $\psi^s_{R,d}$ or $\Psi^s_{R,d}$ if $\dim R > 2$.

Remark 5.18 For each $p \in [1,\ell]$, let $\lambda_p \in \mathscr{P}^\ell_1$ be the ℓ-partition with (1) on the p-th component and $\emptyset$ elsewhere. The proof of Lemma 5.12 implies that the module $\mathbf{T}_{K,1}$ has a Δ-filtration with sections of the form $\mathbf{\Delta}(\lambda)_{K,\tau}$ with $\lambda \in \mathscr{P}^\ell_1$, and that the operator $X \in \mathrm{End}(\mathbf{T}_{K,1})$ has the eigenvalue $q_K^{s_p}$ on $\mathbf{\Delta}(\lambda_p)_{K,\tau}$.

5.6 The affine Lie algebra of a Levi subalgebra

Consider the root system $\widehat{\Pi}_\nu = \{\alpha + r\delta;\ \alpha \in \Pi_\nu,\ r \in \mathbb{Z}\} \cup \{r\delta;\ r \in \mathbb{Z}^\times\}$. Let $\mathbf{m}_{R,\nu}$ be the Lie subalgebra of $\mathbf{g}_R$ spanned by $\mathbf{t}_R$ and the root subspaces associated with $\widehat{\Pi}_\nu$. We may view $\mathbf{m}_{R,\nu}$ as the affine Kac–Moody algebra associated with the Levi subalgebra $\mathfrak{m}_{R,\nu}$ of $\mathfrak{g}_R$. We define the associative R-algebra $\mathbf{m}_{R,\nu,\kappa}$ in the same way as we defined $\mathbf{g}_{R,\kappa}$ in Sect. 5.2.1.

The Weyl group of $\widehat{\Pi}_\nu$ is the subgroup $\widehat{W}_\nu$ of $\widehat{W}$ generated by the affine reflections s_β with $\beta \in \widehat{\Pi}_\nu$. Thus, we have $\widehat{W}_\nu = \{wT_x;\ w \in W_\nu,\ x \in \mathbb{Z}\Pi_\nu\}$.

Set $\mathbf{b}_{R,\nu} = \mathbf{m}_{R,\nu} \cap \mathbf{b}_R$. The category $\mathbf{O}^\kappa_R(\nu)$ consists of the finitely generated $\mathbf{m}_{R,\nu,\kappa}$-modules which are weight $\mathbf{t}_R$-modules with a locally finite action of $\mathbf{b}_{R,\nu}$ (over R), and such that the highest weight of any constituent is of the form $\widehat{\lambda}$ with $\lambda \in P_R$. The decomposition $\mathfrak{m}_{R,\nu} = \bigoplus_{p=1}^\ell \mathfrak{gl}_{R,\nu_p}$ yields an equivalence $\mathbf{O}^\kappa_R(\nu) = \bigotimes_{p=1}^\ell \mathbf{O}^\kappa_R(\nu_p)$, here the tensor product is over R.

Given a tuple $\gamma = (\gamma_p)$ of compositions of the ν_p's, let $\mathbf{O}^{\gamma,\kappa}_R(\nu) \subset \mathbf{O}^\kappa_R(\nu)$ be the subcategory which is identified under the equivalence $\mathbf{O}^\kappa_R(\nu) = \bigotimes_{p=1}^\ell \mathbf{O}^\kappa_R(\nu_p)$ with the category $\bigotimes_{p=1}^\ell \mathbf{O}^{\gamma_p,\kappa}_R(\nu_p)$. Given a deformation parameter τ and a tuple $a \in \mathbb{N}^\ell$, we also consider the categories $\mathbf{O}^{\gamma,\kappa}_{R,\tau}(\nu) = \bigotimes_{p=1}^\ell \mathbf{O}^{\gamma_p,\kappa}_{R,\tau_p}(\nu_p)$ and $\mathbf{O}^{\gamma,\kappa}_{R,\tau}(\nu)\{a\} = \bigotimes_{p=1}^\ell \mathbf{O}^{\gamma_p,\kappa}_{R,\tau_p}(\nu_p)\{a_p\}$. Setting $\gamma_p = (\nu_p)$ for each p, we get the *Kazhdan–Lusztig category* $\mathbf{O}^{+,\kappa}_R(\nu) = \mathbf{O}^{\nu,\kappa}_R(\nu)$ of the Lie algebra $\mathbf{m}_{R,\nu}$. Let $\mathbf{O}^{+,\kappa}_R(\nu)\{a\} \subset \mathbf{O}^{+,\kappa}_R(\nu)$ be the full subcategory defined in the similar way.

To avoid confusions, we may set $\mathbf{A}^{\nu,\kappa}_{R,\tau}(N) = \mathbf{A}^{\nu,\kappa}_{R,\tau}$ if $\mathfrak{g} = \mathfrak{gl}_N$. Then, we define $\mathbf{A}^{+,\kappa}_{R,\tau}(\nu) \subset \mathbf{O}^{+,\kappa}_{R,\tau}(\nu)$ to be the subcategory isomorphic to $\bigotimes_{p=1}^\ell \mathbf{A}^{\nu_p,\kappa}_{R,\tau_p}(\nu_p)$.

As above, we drop the subscripts R, τ if $R = \mathbb{C}$ or $\tau = 0$.

5.7 Reductions to codimension one

5.7.1 Preliminaries

For each $z \in \mathbb{Z}$ and $u, v \in [1, \ell]$ we write $f_{u,v,z}(\tau_R, \kappa_R) = \tau_{R,u} - \tau_{R,v} - z\,\kappa_R$ and $f_{u,v}(\tau_R) = f_{u,v,0}(\tau_R, \kappa_R)$.

Definition 5.19 We will say that the deformation ring R is *generic* if $f_{u,v,z}(\tau_R, \kappa_R) \neq b$ for any tuple (u, v, z, b) with $u < v$ and $z, b \in \mathbb{Z}$, and that it is *subgeneric* if $\kappa_R \notin \mathbb{Q}$ and $f_{u,v,z}(\tau_R, \kappa_R) = b$ for a unique tuple (u, v, z, b) as above (with $u < v$).

Remark 5.20 If R is a local deformation ring, i.e., if $\tau_{\mathrm{k},p} = 0$ and $\kappa_{\mathrm{k}} = -e$ with $e \in \mathbb{N}^{\times}$, then for each $\mathfrak{p} \in \mathfrak{P}$ such that $f_{u,v,z}(\tau_{\mathrm{k}_\mathfrak{p}}, \kappa_{\mathrm{k}_\mathfrak{p}}) = b$ we have also $b = z\,e$.

Now, assume that R is a local deformation ring. Then, the category $\mathbf{A}^{\nu,\kappa}_{R,\tau}$ is a highest weight R-category by Sect. 5.5, either for the partial order $\leqslant_\ell$ or $\leqslant_{\mathrm{b}}$ relative to k by Lemma 5.15. In other words, the highest weight order on $\mathbf{A}^{\nu,\kappa}_{R,\tau}$ is induced from the highest weight order on $\mathbf{A}^{\nu,\kappa}_{\mathrm{k}}$ via base change, which yields a canonical bijection $\Delta(\mathbf{A}^{\nu,\kappa}_{R,\tau}) \to \Delta(\mathbf{A}^{\nu,\kappa}_{\mathrm{k}})$.

By base change again, these highest weight orders on $\mathbf{A}^{\nu,\kappa}_{R,\tau}$ induce highest weight orders on $\mathbf{A}^{\nu,\kappa}_{R_\mathfrak{p},\tau}$ and $\mathbf{A}^{\nu,\kappa}_{\mathrm{k}_\mathfrak{p},\tau}$ for each $\mathfrak{p} \in \mathfrak{P}$. Note that $R_\mathfrak{p}$ is a local ring, but may not be a local deformation ring because $\tau_{\mathrm{k}_\mathfrak{p},p}$ may be $\neq 0$. So, we have the posets isomorphisms

$$\Delta(\mathbf{A}^{\nu,\kappa}_{\mathrm{k}}) \xleftarrow{\otimes \mathrm{k}} \Delta(\mathbf{A}^{\nu,\kappa}_{R,\tau}) \xrightarrow{\otimes R_\mathfrak{p}} \Delta(\mathbf{A}^{\nu,\kappa}_{R_\mathfrak{p},\tau}) \xrightarrow{\otimes \mathrm{k}_\mathfrak{p}} \Delta(\mathbf{A}^{\nu,\kappa}_{\mathrm{k}_\mathfrak{p},\tau})\ .$$

We will reduce the study of $\mathbf{A}^{\nu,\kappa}_{R,\tau}$ to the study of $\mathbf{A}^{\nu,\kappa}_{\mathrm{k}_\mathfrak{p},\tau}$ for $\mathfrak{p} \in \mathfrak{P}_1$. We will say that $\mathfrak{p}$ is generic if $\mathrm{k}_\mathfrak{p}$ is generic and that $\mathfrak{p}$ *is subgeneric* if $\mathrm{k}_\mathfrak{p}$ is subgeneric.

Remark 5.21 Let R, $\mathscr{I}$ be as in Sect. 5.4. If R is subgeneric then each component $\mathscr{I}_p$ is a quiver of type A_∞, while if R is generic then $\Omega = [1, \ell]$ (i.e., the quiver $\mathscr{I}$ has exactly ℓ components).

In order to use the Kazhdan–Lusztig tensor product, we'll be mainly interested by the case where R is either a field or a regular local deformation ring of dimension $\leqslant 2$. Note that if R has dimension 2, then we can always choose it in such a way that it is in general position.

The following basic fact is important for the rest of the paper.

Proposition 5.22 *Assume that R is a local deformation ring in general position.*

(a) *If* $\mathfrak{p} \in \mathfrak{P}_1$ *then* $\mathfrak{p}$ *is either generic or subgeneric.*
(b) *The* K*-category* $\mathbf{A}^{\nu,\kappa}_{K,\tau}$ *is split semi-simple.*
(c) *If* R *is analytic, then the condition* (3.1) *holds in the fraction field* K.
(d) *If* $\nu_p \geqslant d$ *for all* p, *then the map* $\psi^s_{K,d} : \mathbf{H}^s_{K,d} \to \mathrm{End}_{\mathbf{A}^{\nu,\kappa}_{K,\tau}}(\mathbf{T}_{K,d})^{\mathrm{op}}$ *in* (5.1) *is an isomorphism of* K*-algebras. The functor* $\Psi^s_{K,d}$ *is an equivalence of categories and it maps* $\mathbf{\Delta}(\lambda)_{K,\tau}$ *to* $S(\lambda)^{s,q}_K$.

Proof Since R is a UFD and $\mathfrak{p}$ has height 1, we have $\mathfrak{p} = Rg$ for some irreducible element $g \in R$. Now, if $f_{u,v,z}(\tau_{k_\mathfrak{p}}, \kappa_{k_\mathfrak{p}}) = b$ and $\kappa_{k_\mathfrak{p}} = c$ for some $u \neq v$, $z, b \in \mathbb{Z}$ and $c \in \mathbb{Q}$ then g must be a unit of R because R is in general position. This is a contradiction. For the same reason, we may have $f_{u,v,z}(\tau_{k_\mathfrak{p}}, \kappa_{k_\mathfrak{p}}) = b$ for at most one tuple (u, v, z, b). Therefore, if $\mathfrak{p}$ is not generic, then we have $\kappa_{k_\mathfrak{p}} \notin \mathbb{Q}$. Part (a) is proved.

Part (b) follows from the linkage principle. More precisely, recall that for $k \in J^\nu_p$ we set $p_k = p$. Then, since R is in general position, we have

$$\begin{aligned}\widehat{\Pi}(\tau_K, \kappa_K) &= \{\beta \in \widehat{\Pi};\ \langle (0, \tau_K, \kappa_K) : \beta\rangle \in \mathbb{Z}\},\\ &= \{\alpha_{k,l} + z\delta;\ f_{p_k,p_l,a}(\tau_K, \kappa_K) \in \mathbb{Z}\},\\ &= \Pi_\nu.\end{aligned}$$

Thus, the linkage classes are reduced to points, because two ν-dominant weights which are W_ν-conjugate under the $\bullet$-action are equal. Hence $\mathbf{A}^{\nu,\kappa}_{K,\tau}$ is split semi-simple.

Part (c) is obvious, because $q_K = \exp(-2\pi\sqrt{-1}/\kappa_K)$, $Q_{K,p} = \exp(-2\pi\sqrt{-1}s_p/\kappa_K)$, $\kappa_K \notin \mathbb{Q}$ and $(s_{K,u} - s_{K,v} + \kappa_K\mathbb{Z}) \cap \mathbb{Z} = \emptyset$ for each $u \neq v$.

Let us prove part (d). As a finite dimensional split semi-simple K-algebra, the center of $\mathbf{H}^s_{K,d}$ is spanned by the primitive central idempotents. These idempotents are of the form $1_\alpha = \sum_{\mathbf{i}\in\mathscr{I}^\alpha} 1_\mathbf{i}$ where $\alpha \in Q^+$ has height d, see Sect. 3.4. For each nonzero 1_α, there is a unique ℓ-partition λ of d such that $\sum_{i\in K} n_i(\lambda)\alpha_i = \alpha$. From Lemma 5.12 we deduce that, if $\nu_p \geqslant d$ for all p, then for each $\mathbf{i} \in \mathscr{I}^\alpha$ we have

$$f_\mathbf{i}(\mathbf{T}_{K,0}) = \mathbf{\Delta}(\lambda)_{K,\tau}. \tag{5.2}$$

Since $\psi^s_{K,d}(1_\alpha)$ is the projection from $\mathbf{T}_{K,d}$ onto its direct summand $\bigoplus_{\mathbf{i}\in\mathscr{I}^\alpha} f_\mathbf{i}(\mathbf{T}_{K,0})$, the latter is nonzero whenever 1_α is nonzero. So, the map $\psi^s_{K,d}$ is injective. To prove that it is an isomorphism, we are reduced to check the following.

Claim 5.23 $\mathbf{H}^s_{K,d}$ *and* $\mathrm{End}_{\mathbf{A}^{\nu,\kappa}_{K,\tau}}(\mathbf{T}_{K,d})^{\mathrm{op}}$ *have the same dimension over* K.

To prove the claim, by Proposition 8.29, it is enough to check that the K-algebras $H^s_{K,d}$ and $\mathrm{End}_{A^\nu_{K,\tau}}(T_{K,d})^{\mathrm{op}}$ have the same dimension. This follows from Proposition 4.7.

Next, since the K-algebra $\mathbf{A}^{\nu,\kappa}_{K,\tau}$ is split semi-simple by part (b), the standard modules $\mathbf{\Delta}(\lambda)_{K,\tau}$, with $\lambda \in \mathscr{P}^\nu_d$, form a complete set of indecomposable projective modules in $\mathbf{A}^{\nu,\kappa}_{K,\tau}$. So, formula (5.2) implies that $\mathbf{T}_{K,d} = \bigoplus_{\mathbf{i}\in\mathscr{I}^d} f_{\mathbf{i}}(\mathbf{T}_{K,0})$ is a projective generator in $\mathbf{A}^{\nu,\kappa}_{K,\tau}$. So $\Psi^s_{K,d}$ is an equivalence. Since the unique simple and projective module in the block $\mathbf{H}^s_{K,\alpha}$ is the Specht module $S(\lambda)^{s,q}_K$, where λ is as in (5.2), we deduce that $\Psi^s_{K,d}(\Delta(\lambda)_{K,\tau}) = S(\lambda)^{s,q}_K$. □

5.7.2 *The reduction to the finite type with $\ell = 2$*

For each tuple $a \in \mathbb{N}^{\ell-1}$, let $\mathbf{O}^{\nu,\kappa}_{R,\tau}\{a\} \subset \mathbf{O}^{\nu,\kappa}_{R,\tau}$ be the full subcategory consisting of the modules whose simple subquotients have a highest weight of the form $\widehat{\lambda + \tau_{\mathrm{k}}}$ with $\lambda \in P^\nu\{a\}$. Set $\mathbf{A}^{\nu,\kappa}_{R,\tau}\{a\} = \mathbf{O}^{\nu,\kappa}_{R,\tau}\{a\} \cap \mathbf{A}^{\nu,\kappa}_{R,\tau}$. We define $\mathbf{O}^{\nu,\kappa}_{\mathrm{k},\tau}\{a\}$ and $\mathbf{A}^{\nu,\kappa}_{\mathrm{k},\tau}\{a\}$ in the obvious way.

Let $p \mapsto p^o$ be the permutation of $[1,\ell]$ such that $p^o = \ell + 1 - p$. Let $k \mapsto k^o$ be the unique permutation of $[1, N]$ which is blockwise increasing and which takes the block J^ν_p to $J^{\nu^o}_{p^o}$. Applying this permutation to the entries of a weight $\lambda \in P^\nu_R$ yields a weight $\lambda^o \in P^{\nu^o}_R$.

Assume that R is a local ring with a subgeneric residue field. Let $h = (u, v, z)$ be the unique triple such that $u < v$ and $f_{u,v,z}(\tau_{\mathrm{k}}, \kappa_{\mathrm{k}}) = z\,e$. Given a tuple $\varkappa = \varkappa_R \in R^\ell$, let $\varkappa_{\mathrm{k}} \in \mathrm{k}^\ell$ be its residue class. Assume that $f_{u,v}(\varkappa_{\mathrm{k}}) = ze$. We will identify $\varkappa_R$ with the weight $\sum_p \varkappa_{R,p}\det_p \in P_R$.

If $z \leqslant 0$, we abbreviate $\mathscr{O}^\nu_{R,h}\{a\} = \mathscr{O}^\nu_{R,\varkappa}(\nu, u, v)\{a\}$. If $z > 0$, we write $\mathscr{O}^\nu_{R,h}\{a\} = \mathscr{O}^{\nu^o}_{R,\varkappa^o}(\nu^o, v^o, u^o)\{a^o\}$. See Sect. 4.8 for the notation. We define $A^\nu_{R,h}\{a\}$ in the same manner. For each $d \in \mathbb{N}$, we write $A^\nu_{R,h}\{d\} = \bigoplus_a A^\nu_{R,h}\{a\}$, where a runs over the set of all $(\ell-1)$-compositions of d. Depending on the sign of z, we write $M(\lambda)_{R,h}$ for $M(\lambda + \varkappa)_{R,\nu}$ or $M(\lambda^o + \varkappa^o)_{R,\nu}$, and $\Delta(\lambda)_{R,h}$ for $\Delta(\lambda)_{R,\varkappa}$ or $\Delta(\lambda^o)_{R,\varkappa^o}$.

Proposition 5.24 (a) *We have* $\mathbf{O}^{\nu,\kappa}_{R,\tau} \simeq \bigoplus_{a\in\mathbb{N}^{\ell-1}} \mathbf{O}^{\nu,\kappa}_{R,\tau}\{a\}$, $\mathbf{O}^{\nu,\kappa}_{\mathrm{k},\tau} \simeq \bigoplus_{a\in\mathbb{N}^{\ell-1}} \mathbf{O}^{\nu,\kappa}_{\mathrm{k},\tau}\{a\}$.

(b) *There are equivalences of highest weight R-categories $\mathscr{Q}_R : \mathbf{O}^{\nu,\kappa}_{R,\tau}\{a\} \to \mathscr{O}^\nu_{R,h}\{a\}$ and of highest weight* k*-categories $\mathscr{Q}_{\mathrm{k}} : \mathbf{O}^{\nu,\kappa}_{\mathrm{k},\tau}\{a\} \to \mathscr{O}^\nu_{\mathrm{k},h}\{a\}$, such that $\mathrm{k}\mathscr{Q}_R(M) = \mathscr{Q}_{\mathrm{k}}(\mathrm{k}M)$ for each $M \in \mathbf{O}^{\nu,\kappa}_{R,\tau}\{a\}$ and $\mathscr{Q}_R(\mathbf{M}(\lambda + \tau_R)_{R,\nu}) = M(\lambda)_{R,h}$ for each $\lambda \in P^\nu$.*

(c) *The equivalences in* (b) *restrict to equivalences of highest weight categories $\mathscr{Q}_R : \mathbf{A}^{\nu,\kappa}_{R,\tau}\{a\} \to A^\nu_{R,h}\{a\}$ and $\mathscr{Q}_{\mathrm{k}} : \mathbf{A}^{\nu,\kappa}_{\mathrm{k},\tau}\{a\} \to A^\nu_{\mathrm{k},h}\{a\}$. In particular $\mathscr{Q}_R(\mathbf{\Delta}(\lambda)_{R,\tau}) = \Delta(\lambda)_{R,h}$ for all λ.*

Proof For $k \in J_p^\nu$ we set $p_k = p$. Since k is subgeneric, the integral root system $\widehat{\Pi}(\tau_{\mathrm{k}}, \kappa_{\mathrm{k}})$ is given by

$$\begin{aligned}\widehat{\Pi}(\tau_{\mathrm{k}}, \kappa_{\mathrm{k}}) &= \{\beta \in \widehat{\Pi};\ \langle (0, \tau_{\mathrm{k}}, \kappa_{\mathrm{k}}) : \beta\rangle \in \mathbb{Z}\},\\ &= \{\alpha_{k,l} - r\delta;\ f_{p_k,p_l,r}(\tau_{\mathrm{k}}, \kappa_{\mathrm{k}}) \in \mathbb{Z}\},\\ &= \Pi_\nu \cup \{\pm(\alpha_{k,l} - z\delta);\ p_k = u,\ p_l = v\}.\end{aligned}$$

Therefore, the linkage principle yields a decomposition $\mathbf{O}^{\nu,\kappa}_{\mathrm{k},\tau} = \bigoplus_{a\in\mathbb{N}^{\ell-1}} \mathbf{O}^{\nu,\kappa}_{\mathrm{k},\tau}\{a\}$. This decomposition holds over R by Proposition 2.4. This proves part (a).

The set $\widehat{\Pi}(\tau_{\mathrm{k}}, \kappa_{\mathrm{k}})$ is a Coxeter system whose set of positive roots is $\widehat{\Pi}(\tau_{\mathrm{k}}, \kappa_{\mathrm{k}})^+ = \widehat{\Pi}(\tau_{\mathrm{k}}, \kappa_{\mathrm{k}}) \cap \widehat{\Pi}^+$, see [27, sec. 2.2]. The set $\Pi_{\nu,u,v} = \Pi_\nu \cup \{\pm\alpha_{k,l};\ p_k = u,\ p_l = v\}$ is also a Coxeter system with positive roots $\Pi^+_{\nu,u,v} = \Pi_{\nu,u,v} \cap \Pi^+$.

If $z \leqslant 0$ then $\widehat{\Pi}(\tau_{\mathrm{k}}, \kappa_{\mathrm{k}})^+ = \Pi_\nu^+ \cup \{\alpha_{k,l} - z\delta;\ p_k = u,\ p_l = v\}$. Fix an integral coweight $\check{\chi}$ such that $\alpha_{k,l}(\check{\chi}) = -z$ if $p_k = u$, $p_l = v$ and $\alpha_{k,l}(\check{\chi}) = -z$ if $p_k = p_l$. The conjugation by $\check{\chi}$ yields a bijection $\varphi : \widehat{\Pi}(\tau_{\mathrm{k}}, \kappa_{\mathrm{k}}) \xrightarrow{\sim} \Pi_{\nu,u,v}$ such that $\alpha + r\delta \mapsto \alpha$ for all α, r. It maps positive roots to positive ones.

If $z > 0$ then $\widehat{\Pi}(\tau_{\mathrm{k}}, \kappa_{\mathrm{k}})^+ = \Pi_\nu^+ \cup \{-\alpha_{k,l} + z\delta;\ p_k = u,\ p_l = v\}$. The permutation $k \mapsto k^o$ of $[1, N]$ induces a bijection $\Pi_{\nu,u,v} \xrightarrow{\sim} \Pi_{\nu^o,v^o,u^o}, \alpha \mapsto \alpha^o$. The bijection $\varphi : \widehat{\Pi}(\tau_{\mathrm{k}}, \kappa_{\mathrm{k}}) \xrightarrow{\sim} \Pi_{\nu^o,v^o,u^o}$ such that $\alpha + r\delta \mapsto \alpha^o$ identifies the subsets of positive roots in both sides.

In both cases the map φ is an isomorphism of Coxeter systems. Now, for each weight $\lambda \in P^\nu + \rho$ we consider the sets of roots $\widehat{\Pi}[\lambda + \tau_{\mathrm{k}}, \kappa_{\mathrm{k}}] = \{\beta \in \widehat{\Pi};\ \langle (0, \lambda + \tau_{\mathrm{k}}, \kappa_{\mathrm{k}}) : \beta\rangle = 0\}$ and $\Pi_{\nu,u,v}[\lambda + \varkappa_{\mathrm{k}}] = \{\alpha \in \Pi_{\nu,u,v};\ \langle \lambda + \varkappa_{\mathrm{k}} : \alpha\rangle = 0\}$. Since k is subgeneric, we have

$$\begin{aligned}\widehat{\Pi}[\lambda + \tau_{\mathrm{k}}, \kappa_{\mathrm{k}}] &= \{\alpha_{k,l} - r\delta;\ f_{p_k,p_l,r}(\tau_{\mathrm{k}}, \kappa_{\mathrm{k}}) = -\langle \lambda : \alpha_{k,l}\rangle\},\\ &= \{\alpha \in \Pi_\nu;\ \langle \lambda : \alpha\rangle = 0\}\\ &\quad \cup\{\pm(\alpha_{k,l} - z\delta);\ p_k = u,\ p_l = v,\ \langle \lambda : \alpha_{k,l}\rangle = -z\,e\},\\ &= \Pi_\nu[\lambda] \cup \{\pm(\alpha_{k,l} - z\delta);\ p_k = u,\ p_l = v,\ \langle \lambda + \varkappa_{\mathrm{k}} : \alpha_{k,l}\rangle = 0\},\\ &= \Pi_\nu[\lambda + \varkappa_{\mathrm{k}}] \cup \{\pm(\alpha_{k,l} - z\delta);\ p_k = u,\ p_l = v,\ \alpha_{k,l}\rangle\\ &\quad \in \Pi_{\nu,u,v}[\lambda + \varkappa_{\mathrm{k}}]\}.\end{aligned}$$

If $z \leqslant 0$ then $\varphi\big(\widehat{\Pi}[\lambda + \tau_{\mathrm{k}}, \kappa_{\mathrm{k}}]\big) = \Pi_{\nu,u,v}[\lambda + \varkappa_{\mathrm{k}}]$. Therefore, by [20, thm. 11], there is an equivalence of k-categories $\mathscr{Q}_{\mathrm{k}} : \mathbf{O}^\kappa_{\mathrm{k},\tau}\{a\} \to \mathscr{O}_{\mathrm{k},\varkappa}(\nu, u, v)\{a\}$ such that $\mathbf{L}(\mu + \tau_{\mathrm{k}})_{\mathrm{k}} \mapsto L(\mu + \varkappa_{\mathrm{k}})_{\mathrm{k}}$ for each $\mu \in P\{a\}$. The proof of loc. cit. is given by constructing an analogue of Soergel's functor which identifies, block by block, the endomorphism rings of projective generators of $\mathbf{O}^\kappa_{R,\tau}\{a\}$ and $\mathscr{O}_{R,\varkappa}(\nu, u, v)\{a\}$ with the endomorphism ring of the same sheaf over a moment graph (modulo a base change of deformation rings, from a localization of the functions ring of the Cartan subalgebras of $\mathbf{g}$ and $\mathfrak{m}(\nu, u, v)$ to R). This construction yields indeed an equivalence of abelian R-categories $\mathscr{Q}_R : \mathbf{O}^{\kappa}_{R,\tau}\{a\} \to \mathscr{O}_{R,\varkappa}(\nu, u, v)\{a\}$ such that $\mathrm{k}\mathscr{Q}_R(M) = \mathscr{Q}_{\mathrm{k}}(\mathrm{k}M)$ for any $M \in \mathbf{O}^{\kappa}_{R,\tau}\{a\}$.

If $z \geqslant 0$ then $\varphi\big(\widehat{\Pi}[\lambda + \tau_{\mathrm{k}}, \kappa_{\mathrm{k}}]\big) = \big(\Pi_{\nu,u,v}[\lambda + \varkappa_{\mathrm{k}}]\big)^o = \Pi_{\nu^o,v^o,u^o}[\lambda^o + \varkappa^o_{\mathrm{k}}]$. For $-\alpha_{k,l} + z\delta \in \widehat{\Pi}(\tau_{\mathrm{k}}, \kappa_{\mathrm{k}})^+$, we also have

$$\begin{aligned}\langle \lambda^o + \varkappa^o_{\mathrm{k}} : h(-\alpha_{k,l} + z\delta)\rangle &= \langle \lambda^o + \varkappa^o_{\mathrm{k}} : -\alpha_{k^o,l^o}\rangle \\ &= -\langle \lambda, \alpha_{k,l}\rangle - z\, e, \\ &= -\langle \lambda : \alpha_{k,l}\rangle - \tau_{K,u} + \tau_{K,v} + z\,\kappa_{\mathrm{k}} \\ &= \langle (0, \lambda + \tau_{\mathrm{k}}, \kappa_{\mathrm{k}}) : -\alpha_{k,l} + z\delta\rangle.\end{aligned}$$

Thus, by [20, thm. 11] and the discussion above, we have equivalences of categories

$$\mathscr{Q}_R : \mathbf{O}^{\kappa}_{R,\tau}\{a\} \to \mathscr{O}_{R,\varkappa}(\nu^o, v^o, u^o)\{a^o\}, \quad \mathscr{Q}_{\mathrm{k}} : \mathbf{O}^{\kappa}_{\mathrm{k},\tau}\{a\} \to \mathscr{O}_{\mathrm{k},\varkappa}(\nu^o, v^o, u^o)\{a^o\}$$

such that $\mathrm{k}\mathscr{Q}_R(M) = \mathscr{Q}_{\mathrm{k}}(\mathrm{k}M)$ and $\mathbf{L}(\mu + \tau_{\mathrm{k}})_{\mathrm{k}} \mapsto L(\mu^o + \varkappa^o_{\mathrm{k}})_{\mathrm{k}}$ for each $\mu \in P\{a^o\}$.

Now, we can prove part (b). To simplify, we assume $z \leqslant 0$. The case $z > 0$ is proved in a similar way.

First, note that $\mathscr{Q}_{\mathrm{k}}$ restricts to an equivalence of abelian categories $\mathbf{O}^{\nu,\kappa}_{\mathrm{k},\tau}\{a\} \to \mathscr{O}^{\nu}_{\mathrm{k},\varkappa}(\nu, u, v)\{a\}$. We denote it again by $\mathscr{Q}_{\mathrm{k}}$. Since $\mathbf{O}^{\nu,\kappa}_{R,\tau}\{a\}$ and $\mathscr{O}^{\nu}_{R,\varkappa}(\nu, u, v)\{a\}$ are the full subcategories of $\mathbf{O}^{\kappa}_{R,\tau}\{a\}$ and $\mathscr{O}_{R,\varkappa}(\nu, u, v)\{a\}$, respectively, consisting of the modules whose simple subquotients have a highest weight of the form $\widehat{\lambda + \tau_{\mathrm{k}}}$ and $\lambda + \varkappa_{\mathrm{k}}$ respectively, with $\lambda \in P^{\nu}$, we deduce that $\mathscr{Q}_R$ restricts to an equivalence of abelian R-categories $\mathbf{O}^{\nu,\kappa}_{R,\tau}\{a\} \to \mathscr{O}^{\nu}_{R,\varkappa}(\nu, u, v)\{a\}$.

Next, since $\mathscr{Q}_{\mathrm{k}}(\mathbf{L}(\mu + \tau_{\mathrm{k}})_{\mathrm{k},\nu}) = L(\mu + \varkappa_{\mathrm{k}})_{\mathrm{k},\nu}$ for each $\mu \in P^{\nu}$, the functor $\mathscr{Q}_R$ is an equivalence of highest weight R-categories such that $\mathscr{Q}_R(\mathbf{M}(\mu + \tau_R)_{R,\nu}) = M(\mu + \varkappa_R)_{R,\nu}$ for each $\mu \in P^{\nu}$ by Proposition 2.6.

Parts (b) and (c) are proved. □

Remark 5.25 We do not know how to choose the equivalence of categories $\mathscr{Q}_R$ in such a way that it intertwines the endofunctors e, f of $\mathbf{O}$ and $\mathscr{O}$. We will not need this.

In the rest of this section, to unburden the notation, assume that $z \leqslant 0$. The case $z > 0$ is completely similar.

Fix a $(\ell - 1)$-composition $a = (a_\bullet, a_p)$ of the positive integer d. Then, we have the tilting module $T_{R,a_\bullet}(\nu_\bullet) \in A^{\nu_\circ}_{R,\varkappa_\circ}(\nu_\bullet)$ and, for each $p \neq u, v$, the tilting module $T_{R,a_p}(\nu_p) \in A_R^{\nu_p}(\nu_p)$. Recall that $\nu_\circ = (\nu_u, \nu_v)$, $\varkappa_\circ = (\varkappa_u, \varkappa_v)$ and $\nu_\bullet = \nu_u + \nu_v$. Note that, since $f_{u,v}(\varkappa) = ze \leqslant 0$, the category $A_{R,\varkappa_\circ}^{\nu_\circ}(\nu_\bullet)$ (with $\ell = 2$) satisfies the assumptions in Proposition 4.9. Let $T_{R,h,d} \in A_{R,h}^{\nu}\{d\}$ be the tilting module which is identified, under the equivalence (4.4), with the direct sum of the modules $T_{R,a_\bullet}(\nu_\bullet) \otimes \bigotimes_{p\neq u,v} T_{R,a_p}(\nu_p)$, where the sum runs over the set of all $(\ell - 1)$-compositions a of d. We also write $T_{\mathrm{k},h,d} = \mathrm{k}T_{R,h,d} \in A_{\mathrm{k},h}^{\nu}\{d\}$.

Now, let R be either a field or a regular local deformation ring of dimension 2. Assume further that R is analytic and in general position.

The category $\mathbf{A}_{K,\tau}^{\nu,\kappa}$ is split semi-simple. We have defined the module $\mathbf{T}_{R,d} \in \mathbf{A}_{R,\tau}^{\nu,\kappa}$, the R-algebra homomorphism $\psi_{R,d}^s : \mathbf{H}_{R,d}^s \to \mathrm{End}_{\mathbf{A}_{R,\tau}^{\nu,\kappa}}(\mathbf{T}_{R,d})^{\mathrm{op}}$, and the functor $\Psi_{R,d}^s : \mathbf{A}_{R,\tau}^{\nu,\kappa}\{d\} \to \mathbf{H}_{R,d}^s$-mod. By base-change, we get $\mathbf{T}_{R_\mathfrak{p},d}$, $\psi_{R_\mathfrak{p},d}^s$ and $\Psi_{R_\mathfrak{p},d}^s$ for each $\mathfrak{p} \in \mathfrak{P}$, see Remark 4.6.

Lemma 5.26 *Assume that $\mathfrak{p} \in \mathfrak{P}_1$ is subgeneric. Then, we have an isomorphism $\mathcal{Q}_{R_\mathfrak{p}}(\mathbf{T}_{R_\mathfrak{p},d}) = T_{R_\mathfrak{p},h,d}$.*

Proof The module $\mathcal{Q}_{R_\mathfrak{p}}(\mathbf{T}_{R_\mathfrak{p},d})$ is tilting, because $\mathcal{Q}_{R_\mathfrak{p}}$ is an equivalence of highest weight categories. Since $\mathbf{T}_{R_\mathfrak{p},0}$ and $T_{R_\mathfrak{p},h,0}$ are parabolic Verma modules, we have $\mathcal{Q}_{R_\mathfrak{p}}(\mathbf{T}_{K,0}) = T_{R_\mathfrak{p},h,0}$.

Next, the functor $\mathcal{Q}_{\mathrm{k}_\mathfrak{p}}$ induces an isomorphism of the (complexified) Grothendieck groups $[\mathbf{O}_{\mathrm{k}_\mathfrak{p},\tau}^{\nu,\kappa}\{a\}] \to [\mathcal{O}_{\mathrm{k}_\mathfrak{p},h}^{\nu}\{a\}]$ such that $\mathcal{Q}_{\mathrm{k}_\mathfrak{p}}([\mathbf{L}(\lambda + \tau_{\mathrm{k}_\mathfrak{p}})_{\mathrm{k}_\mathfrak{p}}]) = [L(\lambda + \varkappa_{\mathrm{k}_\mathfrak{p}})_{\mathrm{k}_\mathfrak{p}}]$. Since it also preserves the classes of the standard modules, the explicit formulae in Lemma 5.12 imply that $\mathcal{Q}_{\mathrm{k}_\mathfrak{p}} : [\mathcal{O}_{\mathrm{k}_\mathfrak{p},h}^{\nu}\{a\}] \to [\mathbf{O}_{\mathrm{k}_\mathfrak{p},\tau}^{\nu,\kappa}\{a\}]$ commutes with the action of the operators e, f on both sides.

Since $\mathbf{T}_{\mathrm{k}_\mathfrak{p},d} = f^d(\mathbf{T}_{\mathrm{k}_\mathfrak{p},0})$ and $T_{\mathrm{k}_\mathfrak{p},h,d} = f^d(T_{\mathrm{k}_\mathfrak{p},h,0})$, we deduce that $[\mathcal{Q}_{\mathrm{k}_\mathfrak{p}}(\mathbf{T}_{\mathrm{k}_\mathfrak{p},d})] = [T_{\mathrm{k}_\mathfrak{p},h,d}]$ in $[\mathcal{O}_{\mathrm{k}_\mathfrak{p},h}^{\nu}]$. Therefore, we have $\mathcal{Q}_{\mathrm{k}_\mathfrak{p}}(\mathbf{T}_{\mathrm{k}_\mathfrak{p},d}) = T_{\mathrm{k}_\mathfrak{p},h,d}$ because two tilting modules are isomorphic if they have the same class in the Grothendieck group. Since $\mathcal{Q}_{R_\mathfrak{p}}(\mathbf{T}_{R_\mathfrak{p},d})$ is tilting and $\mathrm{k}_\mathfrak{p}\mathcal{Q}_{R_\mathfrak{p}}(\mathbf{T}_{R_\mathfrak{p},d}) = \mathcal{Q}_{\mathrm{k}_\mathfrak{p}}(\mathbf{T}_{\mathrm{k}_\mathfrak{p},d})$, by Proposition 2.4(b) the isomorphism over $\mathrm{k}_\mathfrak{p}$ can be lift to an isomorphism $\mathcal{Q}_{R_\mathfrak{p}}(\mathbf{T}_{R_\mathfrak{p},d}) = T_{R_\mathfrak{p},h,d}$. □

Proposition 5.27 *Let $\mathfrak{p} \in \mathfrak{P}_1$ be subgeneric. Assume that $\nu_p \geqslant d$ for all p. Then,*

(a) $\mathbf{T}_{R_\mathfrak{p},d}$ *is projective in* $\mathbf{A}_{R_\mathfrak{p},\tau}^{\nu,\kappa}$,
(b) $\psi_{R_\mathfrak{p},d}^s$ *is an isomorphism* $\mathbf{H}_{R_\mathfrak{p},d}^s \to \mathrm{End}_{\mathbf{A}_{R_\mathfrak{p},\tau}^{\nu,\kappa}}(\mathbf{T}_{R_\mathfrak{p},d})^{\mathrm{op}}$,
(c) $\Psi_{R_\mathfrak{p},d}^s$ *is fully faithful on* $(\mathbf{A}_{R_\mathfrak{p},\tau}^{\nu,\kappa}\{d\})^{\Delta}$ *and* $(\mathbf{A}_{R_\mathfrak{p},\tau}^{\nu,\kappa}\{d\})^{\nabla}$.

Proof Since $\mathrm{k}_\mathfrak{p}$ is subgeneric, we may fix u, v, z as above. So, we have $f_{u,v,z}(\tau_{\mathrm{k}_\mathfrak{p}}, \kappa_{\mathrm{k}_\mathfrak{p}}) = z\,e$. Hence, by Proposition 5.24 and Lemma 5.26, there is an equivalence of highest weight $R_\mathfrak{p}$-categories $\mathcal{Q}_{R_\mathfrak{p}} : \mathbf{A}_{R_\mathfrak{p},\tau}^{\nu,\kappa}\{d\} \to A_{R_\mathfrak{p},h}^{\nu}\{d\}$

taking $\mathbf{\Delta}(\lambda)_{R_\mathfrak{p},\tau}$ to $\Delta(\lambda)_{R_\mathfrak{p},\varkappa}$ and $\mathbf{T}_{R_\mathfrak{p},d}$ to $T_{R_\mathfrak{p},h,d}$. By base change, it specializes to an equivalence of highest weight $\mathrm{k}_\mathfrak{p}$-categories $\mathscr{Q}_{\mathrm{k}_\mathfrak{p}} : \mathbf{A}^{\nu,\kappa}_{\mathrm{k}_\mathfrak{p},\tau}\{d\} \to A^{\nu}_{\mathrm{k}_\mathfrak{p},h}\{d\}$.

Recall that $\nu_\circ = (\nu_u, \nu_v)$ and $\nu_\bullet = \nu_u + \nu_v$. To unburden the notation, we may identify the highest weight $R_\mathfrak{p}$-categories $A^{\nu}_{R_\mathfrak{p},h}\{d\}$ and $A^{\nu_\circ}_{R_\mathfrak{p},\varkappa_\circ}(\nu_\bullet)\{d\}$ via the equivalence (4.4). The later is a particular case of the categories which have been studied in Sect. 4.7. Note that we have $\varkappa_{\mathrm{k}_\mathfrak{p},u} - \varkappa_{\mathrm{k}_\mathfrak{p},v} = ze \notin \mathbb{N}^\times$. Thus, Proposition 4.9(c) implies that $T_{\mathrm{k}_\mathfrak{p},h,d}$ is projective. Hence, part (a) follows from Proposition 2.4 and Lemma 5.26.

To prove (b) we use Proposition 2.23. Let us check the assumptions. First, the fraction field of $R_\mathfrak{p}$ is K. Since R is in general position, the K-algebra $\mathbf{H}^s_{K,d}$ is split semi-simple. Next, by [13, thm. 3.30], the decomposition map $K_0(\mathbf{H}^s_{K,d}) \to K_0(\mathbf{H}^s_{\mathrm{k}_\mathfrak{p},d})$ is surjective.

Now, let us construct an endomorphism $\theta_{R_\mathfrak{p}}$ of $\mathbf{H}^s_{R_\mathfrak{p},d}$. By Remark 4.6, we have a pre-categorification (E, F, X, T) on $A^{\nu}_{R_\mathfrak{p},h}$. Let $\varphi^s_{R_\mathfrak{p},d} : H^s_{R_\mathfrak{p},d} \to \mathrm{End}_{A^{\nu}_{R_\mathfrak{p},h}}(T_{R_\mathfrak{p},h,d})^{\mathrm{op}}$ be the corresponding $R_\mathfrak{p}$-algebra homomorphism. It is an isomorphism by Proposition 4.9 and the Nakayama's lemma. Next, by Proposition 3.1, we have an $R_\mathfrak{p}$-algebra isomorphism $\alpha_{R_\mathfrak{p}} : \mathbf{H}^s_{R_\mathfrak{p},d} \to H^s_{R_\mathfrak{p},d}$. Since $\mathscr{Q}_{R_\mathfrak{p}}(\mathbf{T}_{R_\mathfrak{p},d}) = T_{R_\mathfrak{p},h,d}$, by functoriality, we have an isomorphism $\beta_{R_\mathfrak{p}} : \mathrm{End}_{\mathbf{A}^{\nu,\kappa}_{R_\mathfrak{p},\tau}}(\mathbf{T}_{R_\mathfrak{p},d})^{\mathrm{op}} \to \mathrm{End}_{A^{\nu}_{R_\mathfrak{p},h}}(T_{R_\mathfrak{p},h,d})^{\mathrm{op}}$. We set $\theta_{R_\mathfrak{p}} = \alpha^{-1}_{R_\mathfrak{p}} \circ (\varphi^s_{R_\mathfrak{p},d})^{-1} \circ \beta_{R_\mathfrak{p}} \circ \psi^s_{R_\mathfrak{p},d}$ and we write $\theta_K = K\theta_{R_\mathfrak{p}}$.

To prove (b), we must check that $\theta_{R_\mathfrak{p}}$ is invertible. By Proposition 2.23, this follows from the following.

Claim 5.28 *The endomorphism θ_K of $\mathbf{H}^s_{K,d}$ is an automorphism and it yields the identity on the Grothendieck group.*

Now, we prove the claim. Since R is in general position, by Proposition 5.22, the K-algebra morphisms $\psi^s_{K,d} : \mathbf{H}^s_{K,d} \to \mathrm{End}_{\mathbf{A}^{\nu,\kappa}_{K,\tau}}(\mathbf{T}_{K,d})^{\mathrm{op}}$ is an isomorphism. Hence θ_K is an automorphism.

Consider the equivalences of categories $\Psi^s_{K,d} : \mathbf{A}^{\nu,\kappa}_{K,\tau}\{d\} \to \mathbf{H}^s_{K,d}$-mod and $\Phi^s_{K,d} : A^{\nu}_{K,h}\{d\} \to H^s_{K,d}$-mod induced by $\psi^s_{K,d}$ and $\varphi^s_{K,d}$. The corresponding maps between isomorphism classes of simple modules fit into the commutative square

$$\begin{tikzcd}
\mathrm{Irr}(\mathbf{A}^{\nu,\kappa}_{K,\tau}\{d\}) \arrow[r, "\Psi^s_{K,d}"] \arrow[d, "\mathscr{Q}_K"'] & \mathrm{Irr}(\mathbf{H}^s_{K,d}) \\
\mathrm{Irr}(A^{\nu}_{K,h}\{d\}) \arrow[r, "\Phi^s_{K,d}"] & \mathrm{Irr}(H^s_{K,d}), \arrow[u, "\alpha_K"']
\end{tikzcd}$$

because we have $\Psi^s_{K,d}(\boldsymbol{\Delta}(\lambda)_{K,\tau}) = S(\lambda)^{s,q}_K$, $\Phi^s_{K,d}(\Delta(\lambda)_{K,\varkappa}) = S(\lambda)^s_K$ by Propositions 5.22(d), 4.7(d), and we have $\alpha_K(S(\lambda)^s_K) = S(\lambda)^{s,q}_K$, $\mathscr{Q}_K(\boldsymbol{\Delta}(\lambda)_{K,\tau}) = \Delta(\lambda)_{K,\varkappa}$. This implies that θ_K is identity on the Grothendieck group. The claim is proved.

Finally, let us prove part (c). Let $\varphi^s_{R_\mathfrak{p},d}$, $\beta_{R_\mathfrak{p}}$ and $\alpha_{R_\mathfrak{p}}$ be as above. Then, we can view $\varphi^s_{R_\mathfrak{p},d}$ as an isomorphism $\mathbf{H}^s_{R_\mathfrak{p},d} \to \mathrm{End}_{\mathbf{A}^{\nu,\kappa}_{R_\mathfrak{p},\tau}}(\mathbf{T}_{R_\mathfrak{p},d})^{\mathrm{op}}$. We don't know whether $\psi^s_{R_\mathfrak{p},d} = \varphi^s_{R_\mathfrak{p},d}$. However, since they are both invertible, they differ obviously by an automorphism of $\mathbf{H}^s_{R_\mathfrak{p},d}$. Thus, the equivalence $\mathscr{Q}_{R_\mathfrak{p}}$ intertwines the functors $\Psi^s_{R_\mathfrak{p},d}$ and $\Phi^s_{R_\mathfrak{p},d}$, up to a twist by an automorphism of $\mathbf{H}^s_{R_\mathfrak{p},d}$. Therefore, it is enough to prove that $\Phi^s_{R_\mathfrak{p},d}$ is fully faithful on $(A^\nu_{R_\mathfrak{p},h}\{d\})^\Delta$ and $(A^\nu_{R_\mathfrak{p},h}\{d\})^\nabla$.

By Proposition 4.9, a simple module of $A^\nu_{\mathrm{k}_\mathfrak{p},h,d}$ is a submodule of a parabolic Verma module if and only if it lies in the top of $T_{\mathrm{k}_\mathfrak{p},h,d}$. Thus, the functor $\Phi^s_{\mathrm{k}_\mathfrak{p},d}$ is faithful on $(A^\nu_{\mathrm{k}_\mathfrak{p},h}\{d\})^\Delta$. By [7, cor. 4.18], the category $A^\nu_{\mathrm{k}_\mathfrak{p},h}\{d\}$ is Ringel self-dual, i.e., we have an equivalence $A^\nu_{\mathrm{k}_\mathfrak{p},h}\{d\} \simeq (A^\nu_{\mathrm{k}_\mathfrak{p},h}\{d\})^\diamond$. Therefore, by Lemma 2.13, the functor $\Phi^s_{\mathrm{k}_\mathfrak{p},d}$ is also faithful on $(A^\nu_{\mathrm{k}_\mathfrak{p},h}\{d\})^\nabla$. Note that [7] considers the category A^ν without any shift $\varkappa$, but our situation reduces to this one by Proposition 4.10. Now, part (c) follows from Proposition 2.18. □

Remark 5.29 If $\nu_u - \nu_v \notin \mathbb{Z}e$ for all $u \neq v$, then $\Psi^s_{R_\mathfrak{p},d}$ is a 1-faithful highest weight cover.

5.7.3 The reduction to $\ell = 1$

Assume that the deformation ring R is a local ring with a generic residue field k. We have the following lemma.

Lemma 5.30 *For $\lambda, \lambda' \in P^\nu$, if $\lambda + \tau_\mathrm{k} \leqslant_\ell \lambda' + \tau_\mathrm{k}$ then $\widehat{\lambda + \tau_\mathrm{k}} \in \widehat{W}_\nu \bullet \widehat{\lambda' + \tau_\mathrm{k}}$.*

Proof By an easy induction we may assume that there are elements $\beta \in \widehat{\Pi}(\tau_\mathrm{k}, \kappa_\mathrm{k}) \setminus \Pi_\nu$ and $w \in W_\nu$ with $\widehat{\lambda + \tau_\mathrm{k}} = ws_\beta \bullet \widehat{\lambda' + \tau_\mathrm{k}}$. We have

$$\begin{aligned} \widehat{\Pi}(\tau_\mathrm{k}, \kappa_\mathrm{k}) \subset \widehat{\Pi}_\nu &\iff \langle (0, \tau_\mathrm{k}, \kappa_\mathrm{k}) : \beta \rangle \notin \mathbb{Z}, \quad \forall \beta \in \widehat{\Pi} \setminus \widehat{\Pi}_\nu, \\ &\iff \langle \tau_\mathrm{k} : \alpha \rangle + r\kappa \notin \mathbb{Z}, \quad \forall \alpha \in \Pi \setminus \Pi_\nu, \quad \forall r \in \mathbb{Z}, \\ &\iff \mathrm{k} \text{ is generic}. \end{aligned}$$

Thus $\beta \in \widehat{\Pi}_\nu$, hence $ws_\beta \in \widehat{W}_\nu$. □

For $a \in \mathbb{N}^\ell$ let $\mathbf{O}^{\nu,\kappa}_{R,\tau}\{a\} \subset \mathbf{O}^{\nu,\kappa}_{R,\tau}$ be the full subcategory of the modules whose simple subquotients have a highest weight of the form $\widehat{\lambda + \tau_{\mathrm{k}}}$ with $\lambda \in P^\nu\{a\}$.

Proposition 5.31 (a) *We have* $\mathbf{O}^{\nu,\kappa}_{R,\tau} = \bigoplus_{a\in\mathbb{N}^\ell} \mathbf{O}^{\nu,\kappa}_{R,\tau}\{a\}$ *and* $\mathbf{O}^{\nu,\kappa}_{\mathrm{k},\tau} = \bigoplus_{a\in\mathbb{N}^\ell} \mathbf{O}^{\nu,\kappa}_{\mathrm{k},\tau}\{a\}$.

(b) *There are equivalences of highest weight R-categories* $\mathscr{Q}_R : \mathbf{O}^{\nu,\kappa}_{R,\tau}\{a\} \to \mathbf{O}^{+,\kappa}_R(\nu)\{a\}$ *and of highest weight* k*-categories* $\mathscr{Q}_{\mathrm{k}} : \mathbf{O}^{\nu,\kappa}_{\mathrm{k},\tau}\{a\} \to \mathbf{O}^{+,\kappa}_{\mathrm{k}}(\nu)\{a\}$ *such that* $\mathrm{k}\mathscr{Q}_R(M) = \mathscr{Q}_{\mathrm{k}}(\mathrm{k}M)$ *and* $\mathscr{Q}_R(\mathbf{M}(\lambda + \tau)_{R,\nu}) = \mathbf{M}(\lambda)_{R,+}$.

(c) *The equivalences in* (b) *restricts to equivalences of highest weight categories* $\mathscr{Q}_R : \mathbf{A}^{\nu,\kappa}_{R,\tau} \to \mathbf{A}^{+,\kappa}_R(\nu)$ *and* $\mathscr{Q}_{\mathrm{k}} : \mathbf{A}^{\nu,\kappa}_{\mathrm{k},\tau} \to \mathbf{A}^{+,\kappa}_{\mathrm{k}}(\nu)$. *In particular, we have* $\mathscr{Q}_R(\boldsymbol{\Delta}(\lambda)_{R,\tau}) = \boldsymbol{\Delta}(\lambda)_R$ *for all* λ.

Proof Since k is generic, the linkage principle and Lemma 5.30 imply that if a parabolic Verma module in $\mathbf{O}^{\nu,\kappa}_{\mathrm{k},\tau}$ has a highest weight of the form $\widehat{\lambda + \tau_{\mathrm{k}}}$ with $\lambda \in P^\nu\{a\}$, then any constituent has also a highest weight of the same form. So we have a decomposition $\mathbf{O}^{\nu,\kappa}_{\mathrm{k},\tau} = \bigoplus_{a\in\mathbb{N}^\ell} \mathbf{O}^{\nu,\kappa}_{\mathrm{k},\tau}\{a\}$. The decomposition over R follows from Proposition 2.4. Part (a) is proved.

For the same reason as above, we have $\mathbf{O}^{\kappa}_{R,\tau} = \bigoplus_{a\in\mathbb{N}^\ell} \mathbf{O}^{\kappa}_{R,\tau}\{a\}$, where $\mathbf{O}^{\kappa}_{R,\tau}\{a\}$ is the full subcategory of the modules whose simple subquotients have a highest weight of the form $\widehat{\lambda + \tau_{\mathrm{k}}}$ with $\lambda \in P\{a\}$.

Further, by [20, thm. 11] there is an equivalence of highest weight k-categories $\mathscr{Q}_{\mathrm{k}} : \mathbf{O}^{\kappa}_{\mathrm{k},\tau}\{a\} \to \mathbf{O}^{\kappa}_{\mathrm{k}}(\nu)\{a\}$ such that $\mathbf{L}(\lambda + \tau_{\mathrm{k}})_{\mathrm{k}} \mapsto \mathbf{L}(\lambda)_{\mathrm{k}}$. For the same reason as explained in the proof of Proposition 5.24, the proof of [20, thm. 11] yields an equivalence $\mathscr{Q}_R : \mathbf{O}^{\kappa}_{R,\tau}\{a\} \to \mathbf{O}^{\kappa}_R(\nu)\{a\}$ such that $\mathrm{k}\mathscr{Q}_R(M) = \mathscr{Q}_{\mathrm{k}}(\mathrm{k}M)$ for any $M \in \mathbf{O}^{\kappa}_{R,\tau}\{a\}$.

Since $\lambda + \tau_{\mathrm{k}}$ is ν-dominant if and only if λ is ν-dominant, this equivalence restricts to an equivalence of abelian categories $\mathbf{O}^{\nu,\kappa}_{\mathrm{k},\tau}\{a\} \to \mathbf{O}^{+,\kappa}_{\mathrm{k}}(\nu)\{a\}$. We denote it again by $\mathscr{Q}_{\mathrm{k}}$. Since $\mathbf{O}^{\nu,\kappa}_{R,\tau}\{a\}$ and $\mathbf{O}^{+,\kappa}_R(\nu)\{a\}$ are full subcategories of $\mathbf{O}^{\kappa}_{R,\tau}\{a\}$ and $\mathbf{O}^{\kappa}_R(\nu)\{a\}$ consisting of the modules whose simple subquotients have a highest weight of the form $\widehat{\lambda + \tau_{\mathrm{k}}}$ and $\widehat{\lambda}$, respectively, with $\lambda \in P^\nu\{a\}$, we deduce that $\mathscr{Q}_R$ restricts to an equivalence of abelian R-categories $\mathscr{Q}_R : \mathbf{O}^{\nu,\kappa}_{R,\tau}\{a\} \to \mathbf{O}^{+,\kappa}_R(\nu)\{a\}$. Since $\mathscr{Q}_{\mathrm{k}}(\mathbf{L}(\lambda+\tau_{\mathrm{k}})_{\mathrm{k}}) = \mathbf{L}(\lambda)_{\mathrm{k}}$ for all $\lambda \in P^\nu\{a\}$, by Proposition 2.6 we deduce that $\mathscr{Q}_{\mathrm{k}}$ and $\mathscr{Q}_R$ are indeed equivalences of highest weight categories and that $\mathscr{Q}_R(\mathbf{M}(\lambda + \tau)_{R,\nu}) = \mathbf{M}(\lambda)_{R,+}$. This proves parts (b), (c). □

Now, let R be either a field or a regular local deformation ring of dimension 2. Assume further that R is analytic and in general position.

Consider the Kazhdan–Lusztig category $\mathbf{O}^{+,\kappa}_R(\nu_p)$ of $\mathfrak{gl}_{R,\nu_p}$. The equivalence of categories $\mathbf{O}^{\kappa}_R(\nu) = \bigotimes_{p=1}^{\ell} \mathbf{O}^{\kappa}_R(\nu_p)$ yields an equivalence of categories $\mathbf{O}^{+,\kappa}_R(\nu) = \bigotimes_{p=1}^{\ell} \mathbf{O}^{+,\kappa}_R(\nu_p)$.

Let $\mathbf{V}(\nu_p) \in \mathbf{O}^{+,\kappa}_R(\nu_p)$ be the module induced from the natural representation of $\mathfrak{gl}_{R,\nu_p}$. The endofunctor $f_p = \bullet \dot{\otimes}_R \mathbf{V}(\nu_p)$ of $\mathbf{O}^{+,\kappa}_R(\nu_p)$ extends to an endofunctor of $\mathbf{O}^{+,\kappa}_R(\nu)$ in the obvious way. We denote it again by f_p. Let $f = \bigoplus_{p=1}^{\ell} f_p$.

We set $\mathbf{T}_{R,0}(\nu) = \bigotimes_{p=1}^{\ell} \mathbf{T}_{R,0}(\nu_p)$. For each $d \in \mathbb{N}$, we consider the tilting module $\mathbf{T}_{R,d}(\nu) = f^d(\mathbf{T}_{R,0}(\nu))$ in $\mathbf{O}^{+,\kappa}_R(\nu)$. We have introduced a module $\mathbf{T}_{R,d}$ in $\mathbf{O}^{\nu,\kappa}_{R,\tau}$.

By base change, for each $\mathfrak{p} \in \mathfrak{P}$, we get the modules $\mathbf{T}_{R_\mathfrak{p},d} \in \mathbf{O}^{\nu,\kappa}_{R_\mathfrak{p},\tau}$ and $\mathbf{T}_{R_\mathfrak{p},d}(\nu) \in \mathbf{O}^{+,\kappa}_{R_\mathfrak{p}}(\nu)$. The same proof as in Lemma 5.26 yields the following.

Lemma 5.32 *Assume that* $\mathfrak{p} \in \mathfrak{P}_1$ *is generic. Then, we have an isomorphism* $\mathscr{Q}_{R_\mathfrak{p}}(\mathbf{T}_{R_\mathfrak{p},d}) = \mathbf{T}_{R_\mathfrak{p},d}(\nu)$.

On the other hand, for each $a \in \mathbb{N}^\ell$, we set $\mathbf{H}^{\ell}_{R,a} = \bigotimes_{p=1}^{\ell} \mathbf{H}^{+}_{R,a_p}$. By base change, it yields the $R_\mathfrak{p}$-algebra $\mathbf{H}^{\ell}_{R_\mathfrak{p},a}$.

Lemma 5.33 *Let* $\mathfrak{p} \in \mathfrak{P}_1$ *be generic. Then, we have an* $R_\mathfrak{p}$*-algebra isomorphism*

$$\mathbf{H}^{s}_{R_\mathfrak{p},d} = \bigoplus_{a \in \mathscr{C}^{\ell}_d} \mathrm{Mat}_{\mathfrak{S}_d/\mathfrak{S}_a}\big(\mathbf{H}^{\ell}_{R_\mathfrak{p},a}\big). \tag{5.3}$$

Proof Let $I = \{\tau_{R_\mathfrak{p},1}, \tau_{R_\mathfrak{p},2}, \dots, \tau_{R_\mathfrak{p},\ell}\} + \mathbb{Z} + \kappa_{R_\mathfrak{p}}\mathbb{Z}$ and $\mathscr{I} = \mathscr{I}_{R_\mathfrak{p}} = I/\sim$. Let $\mathscr{I}_{k_\mathfrak{p}}$ be the image of $\mathscr{I}$ in the residue field $k_\mathfrak{p}$. Since $\mathfrak{p}$ is generic, the quiver $\mathscr{I}_{k_\mathfrak{p}}$ has exactly ℓ components given by $\mathscr{I}_{k_\mathfrak{p},p} = (\tau_{k_\mathfrak{p},p} + \mathbb{Z} + \kappa_{k_\mathfrak{p}}\mathbb{Z})/\sim$ with $p \in [1,\ell]$. Hence, the quiver $\mathscr{I}_{R_\mathfrak{p}}$ has also ℓ components $\mathscr{I}_1 = \mathscr{I}_{R_\mathfrak{p},1}, \dots, \mathscr{I}_\ell = \mathscr{I}_{R_\mathfrak{p},\ell}$ which specialize to $\mathscr{I}_{k_\mathfrak{p},1}, \dots, \mathscr{I}_{k_\mathfrak{p},\ell}$ respectively.

For each tuple $\mathbf{p} = (p_1, p_2, \dots, p_d)$ in $[1,\ell]^d$, we consider the idempotent in $\mathbf{H}^{s}_{k_\mathfrak{p},d}$ given by $1_\mathbf{p} = \sum_\mathbf{i} 1_\mathbf{i}$, where $\mathbf{i} = (i_1, i_2, \dots, i_d)$ runs over the set $\mathscr{I}_{k_\mathfrak{p},\mathbf{p}} = \prod_{r=1}^{d} \mathscr{I}_{k_\mathfrak{p},p_r}$ and $1_\mathbf{i}$ is as in Sect. 3.4. Note that, although there may be an infinite number of such tuples $\mathbf{i}$, this sum contains only a finite number of non zero terms. Next, for each $a \in \mathscr{C}^{\ell}_d$, we define a central idempotent $1_{(a)}$ in $\mathbf{H}^{s}_{k_\mathfrak{p},d}$ by $1_{(a)} = \sum_{\mathbf{p} \in (a)} 1_\mathbf{p}$, where a is identified with the tuple $(1^{a_1} 2^{a_2} \cdots \ell^{a_\ell})$ and (a) is the set of all permutations of a in $[1,\ell]^d$. Then, we write $\mathbf{H}^{s}_{k_\mathfrak{p},(a)} = 1_{(a)}\mathbf{H}^{s}_{k_\mathfrak{p},d}$.

It is well-known that there are $k_\mathfrak{p}$-algebra isomorphisms $\mathbf{H}^{s}_{k_\mathfrak{p},d} = \bigoplus_{a \in \mathscr{C}^{\ell}_d} \mathbf{H}^{s}_{k_\mathfrak{p},(a)}$, $\mathbf{H}^{\ell}_{k_\mathfrak{p},a} = 1_a\,\mathbf{H}^{s}_{k_\mathfrak{p},d}1_a$ and $\mathbf{H}^{s}_{k_\mathfrak{p},(a)} = \mathrm{Mat}_{\mathfrak{S}_d/\mathfrak{S}_a}\big(\mathbf{H}^{\ell}_{k_\mathfrak{p},a}\big)$, where

$\mathfrak{S}_a = \mathfrak{S}_{a_1} \times \cdots \times \mathfrak{S}_{a_\ell}$, see [12]. We must prove that the isomorphism $\bigoplus_{a\in\mathscr{C}_d^\ell} \mathrm{Mat}_{\mathfrak{S}_d/\mathfrak{S}_a}\big(\mathbf{H}^\ell_{\mathrm{k}_\mathfrak{p},a}\big) \to \mathbf{H}^s_{\mathrm{k}_\mathfrak{p},d}$ lifts to an isomorphism of $R_\mathfrak{p}$-algebras. To do that, by the Nakayama's lemma, it is enough to prove that this isomorphism lifts to an $R_\mathfrak{p}$-algebra homomorphism $\bigoplus_{a\in\mathscr{C}_d^\ell} \mathrm{Mat}_{\mathfrak{S}_d/\mathfrak{S}_a}\big(\mathbf{H}^\ell_{R_\mathfrak{p},a}\big) \to \mathbf{H}^s_{R_\mathfrak{p},d}$.

First, by Proposition 3.1, for each tuple $\mathbf{i} \in (\mathscr{I}_{\mathrm{k}_\mathfrak{p}})^d$, the sum $1_\mathbf{i} = \sum_{\mathbf{i}'} 1_{\mathbf{i}'}$ over all elements $\mathbf{i}' \in \mathscr{I}^d$ whose residue class is equal to $\mathbf{i}$, is an idempotent in the $R_\mathfrak{p}$-subalgebra $\mathbf{H}^s_{R_\mathfrak{p},d}$ of $\mathbf{H}^s_{K,d}$. Therefore, for each tuple $\mathbf{p} \in [1,\ell]^d$, the idempotent $1_\mathbf{p} \in \mathbf{H}^s_{K,d}$ given by $1_\mathbf{p} = \sum_{\mathbf{i}'} 1_{\mathbf{i}'}$, where $\mathbf{i}'$ runs over the set $\mathscr{I}_\mathbf{p} = \prod_{r=1}^d \mathscr{I}_{p_r}$, belongs also to the $R_\mathfrak{p}$-subalgebra $\mathbf{H}^s_{R_\mathfrak{p},d}$ and it specializes to the idempotent $1_\mathbf{p} \in \mathbf{H}^s_{\mathrm{k}_\mathfrak{p},d}$ given above. In particular, for each $a \in \mathscr{C}_d^\ell$, the idempotent in $\mathbf{H}^s_{K,d}$ given by $1_{(a)} = \sum_{\mathbf{p}\in(a)} 1_\mathbf{p}$ belongs indeed to $\mathbf{H}^s_{R_\mathfrak{p},d}$ and it specializes to the idempotent $1_{\{a\}} \in \mathbf{H}^s_{\mathrm{k}_\mathfrak{p},d}$ given above. Further, setting $\mathbf{H}^s_{R_\mathfrak{p},(a)} = 1_{(a)}\mathbf{H}^s_{R_\mathfrak{p},d}$, we get $R_\mathfrak{p}$-algebra isomorphisms $\mathbf{H}^s_{R_\mathfrak{p},d} = \bigoplus_{a\in\mathscr{C}_d^\ell} \mathbf{H}^s_{R_\mathfrak{p},(a)}$ and $\mathbf{H}^\ell_{R_\mathfrak{p},a} = 1_a\,\mathbf{H}^s_{R_\mathfrak{p},d} 1_a$.

Now, we construct an $R_\mathfrak{p}$-algebra homomorphism $\bigoplus_{a\in\mathscr{C}_d^\ell} \mathrm{Mat}_{\mathfrak{S}_d/\mathfrak{S}_a}\big(1_a\mathbf{H}^s_{R_\mathfrak{p},d}1_a\big) \to \mathbf{H}^s_{R_\mathfrak{p},d}$ which lifts the isomorphism over the residue field $\mathrm{k}_\mathfrak{p}$ mentioned above.

To do that, it is convenient to use the formalism of *quiver Hecke algebras*. Let $\mathbf{R}^s_{K,d}$ be the *cyclotomic quiver Hecke algebra* of rank d associated with s. It is the K-algebra generated by elements $1_\mathbf{i}$, $x_{\mathbf{i},k}$, $\tau_{\mathbf{i},l}$ with $\mathbf{i} \in \mathscr{I}^d$, $k \in [1,d]$ and $l \in [1,d)$, subject to the relations in [40, sec. 3.2.1] associated with the quiver $\mathscr{I}$ and to the cyclotomic relations given by $(x_{\mathbf{i},1})^{\sharp\{p;\, q^{s_p}=i_1\}} = 0$ for all $\mathbf{i}$'s. Note that the K-algebra $\mathbf{R}^s_{K,d}$ is finite dimensional, and that we have $1_\mathbf{i} = 0$ except for a finite number of $\mathbf{i}$'s.

By [6,40] we have a K-algebra isomorphism $\mathbf{R}^s_{K,d} = \mathbf{H}^s_{K,d}$ which identifies the idempotents $1_\mathbf{i}$, $\mathbf{i} \in \mathscr{I}^d$, from both sides. In particular, for each integer $l \in [1,d)$ and each d-tuple $\mathbf{p}$ such that $p_l \neq p_{l+1}$, the element $\tau_{\mathbf{p},l} = \sum_{\mathbf{i}\in\mathscr{I}_\mathbf{p}} \tau_{\mathbf{i},l}$ in $\mathbf{R}^s_{K,d}$ can be viewed as an element of $\mathbf{H}^s_{K,d}$ which belongs to $\mathbf{H}^s_{R_\mathfrak{p},d}$ and which satisfies the relations $\tau_{s_l(\mathbf{p}),l}\,\tau_{\mathbf{p},l} = 1_\mathbf{p}$ and $\tau_{\mathbf{p},l}\,\tau_{s_l(\mathbf{p}),l} = 1_{s_l(\mathbf{p})}$.

Next, let $w \in \mathfrak{S}_d$. Assume that w is of minimal length in its right $\mathfrak{S}_a$-coset. Fix a reduced decomposition $w = s_{r_m}\cdots s_{r_2}s_{r_1}$. Consider the elements $\tau_{w,a}$ and $\bar\tau_{a,w}$ of $\mathbf{H}^s_{R_\mathfrak{p},d}$ given by $\tau_{w,a} = \tau_{s_{r_m},s_{r_{m-1}}\cdots s_{r_1}(a)}\cdots\tau_{s_{r_2},s_{r_1}(a)}\tau_{s_{r_1},a}$ and $\bar\tau_{a,w} = \tau_{s_{r_1},s_{r_2}\cdots s_{r_m}w(a)}\cdots\tau_{s_{r_{m-1}},s_{r_m}w(a)}\tau_{s_{r_m},w(a)}$. We have $\bar\tau_{a,w}\,\tau_{w,a} = 1_a$ and $\tau_{w,a}\,\bar\tau_{a,w} = 1_{w(a)}$.

The expected map $\bigoplus_{a\in\mathscr{C}_d^\ell} \mathrm{Mat}_{\mathfrak{S}_d/\mathfrak{S}_a}\big(1_a\mathbf{H}^s_{R_\mathfrak{p},d}1_a\big) \to \mathbf{H}^s_{R_\mathfrak{p},d}$ takes the square matrix $(x_{v(a),w(a)})_{v,w}$ in $\mathrm{Mat}_{\mathfrak{S}_d/\mathfrak{S}_a}\big(1_a\mathbf{H}^s_{R_\mathfrak{p},d}1_a\big)$ with $x_{v(a),w(a)} \in 1_a\mathbf{H}^s_{R_\mathfrak{p},d}1_a$ and $v, w \in \mathfrak{S}_d$ as above to the sum $\sum_{v,w} \tau_{w,a}\,x_{v(a),w(a)}\,\bar\tau_{v,a}$. $\square$

Now, given $\mathbf{p} = (p_1, p_2, \ldots, p_d)$ in $[1, \ell]^d$, we write $f_\mathbf{p} = f_{p_1} f_{p_2} \cdots f_{p_d}$ and $\mathbf{T}_{R,\mathbf{p}}(\nu) = f_\mathbf{p}(\mathbf{T}_{R,0}(\nu))$. We have an isomorphism $\mathbf{T}_{R,d}(\nu) = \bigoplus_\mathbf{p} \mathbf{T}_{R,\mathbf{p}}(\nu)$.

Recall that we identify a composition $a = (a_1, \ldots, a_\ell)$ in $\mathscr{C}_d^\ell$ with the ℓ-tuple $(1^{a_1} 2^{a_2} \cdots \ell^{a_\ell})$. Then, we have an isomorphism $\mathbf{T}_{R,a}(\nu) = \bigotimes_{p=1}^\ell \mathbf{T}_{R,a_p}(\nu_p)$.

Next, consider the action of the symmetric group $\mathfrak{S}_d$ on $[1, \ell]^d$ by permutation. Each orbit contains a unique element given by a composition $a \in \mathscr{C}_d^\ell$. Let (a) denote this orbit. We have a bijection $\mathfrak{S}_d/\mathfrak{S}_a \xrightarrow{\sim} \{a\}$ given by $w \mapsto w(a)$, where $\mathfrak{S}_a$ is the stabilizer of a. We write $\mathbf{T}_{R,(a)}(\nu) = \bigoplus_{\mathbf{p}\in\{a\}} \mathbf{T}_{R,\mathbf{p}}(\nu)$.

For each $\mathbf{p} \in (a)$ we have a canonical isomorphism $\mathbf{T}_{R,\mathbf{p}}(\nu) = \mathbf{T}_{R,a}(\nu)$. Therefore, we have $\mathbf{T}_{R,(a)}(\nu) = \bigoplus_{w\in\mathfrak{S}_d/\mathfrak{S}_a} \mathbf{T}_{R,a}(\nu)$. We deduce that

$$\mathrm{End}_{\mathbf{A}_R^{+,\kappa}(\nu)}(\mathbf{T}_{R,(a)}(\nu)) = \mathrm{Mat}_{\mathfrak{S}_d/\mathfrak{S}_a}\big(\mathrm{End}_{\mathbf{A}_R^{+,\kappa}(\nu)}(\mathbf{T}_{R,a}(\nu))\big).$$

Next, recall that $\mathbf{O}_R^{+,\kappa}(\nu) = \bigoplus_{a\in\mathscr{C}_d^\ell} \mathbf{O}_R^{+,\kappa}(\nu)(a)$ and that $\mathbf{T}_{R,\mathbf{p}}(\nu) \in \mathbf{O}_R^{+,\kappa}(\nu)\{a\}$ if and only if $\mathbf{p} \in (a)$. Therefore, we have

$$\begin{aligned}\mathrm{End}_{\mathbf{A}_R^{+,\kappa}(\nu)}(\mathbf{T}_{R,d}(\nu)) &= \bigoplus_{a\in\mathscr{C}_d^\ell} \mathrm{End}_{\mathbf{A}_R^{+,\kappa}(\nu)}(\mathbf{T}_{R,(a)}(\nu)) \\ &= \bigoplus_{a\in\mathscr{C}_d^\ell} \mathrm{Mat}_{\mathfrak{S}_d/\mathfrak{S}_a}\big(\mathrm{End}_{\mathbf{A}_R^{+,\kappa}(\nu)}(\mathbf{T}_{R,a}(\nu))\big). \end{aligned} \quad (5.4)$$

For each $p \in [1, \ell]$, the $\mathbf{g}_{R,\nu_p}$-module $\mathbf{T}_{R,a_p}(\nu_p) \in \mathbf{A}_R^{+,\kappa}(\nu_p)$ gives rise to an R-algebra homomorphism $\mathbf{H}_{R,a_p}^+ \to \mathrm{End}_{\mathbf{A}_R^{+,\kappa}(\nu_p)}(\mathbf{T}_{R,a_p}(\nu_p))^{\mathrm{op}}$ given by (5.1). Taking the tensor product , we get an R-algebra homomorphism $\mathbf{H}_{R,a}^\ell \to \mathrm{End}_{\mathbf{A}_R^{+,\kappa}(\nu)}(\mathbf{T}_{R,a}(\nu))^{\mathrm{op}}$.

Now, assume that $\mathfrak{p} \in \mathfrak{P}_1$ is generic. Combining the R-algebra homomorphism above with base change, (5.3) and (5.4), we get an $R_\mathfrak{p}$-algebra homomorphism $\psi_{R_\mathfrak{p},d}^+(\nu) : \mathbf{H}_{R_\mathfrak{p},d}^s \to \mathrm{End}_{\mathbf{A}_{R_\mathfrak{p}}^{+,\kappa}(\nu)}(\mathbf{T}_{R_\mathfrak{p},d}(\nu))^{\mathrm{op}}$. Further, the composition with $\psi_{R_\mathfrak{p},d}^+(\nu)$ yields a functor $\Psi_{R_\mathfrak{p},d}^+(\nu) = \mathrm{Hom}_{\mathbf{A}_{R_\mathfrak{p}}^{+,\kappa}(\nu)}\big(\mathbf{T}_{R_\mathfrak{p},d}(\nu), \bullet\big) : \mathbf{A}_{R_\mathfrak{p}}^{+,\kappa}(\nu)\{d\} \to \mathbf{H}_{R_\mathfrak{p},d}^s\text{-mod}$.

Lemma 5.34 *Let $\mathfrak{p} \in \mathfrak{P}_1$ be generic. The following holds*

(a) $\mathbf{T}_{R_\mathfrak{p},d}(\nu)$ *is projective in* $\mathbf{A}_{R_\mathfrak{p}}^{+,\kappa}(\nu)\{d\}$,

(b) $\psi_{R_\mathfrak{p},d}^+(\nu)$ *is an isomorphism* $\mathbf{H}_{R_\mathfrak{p},d}^s \to \mathrm{End}_{\mathbf{m}_{R_\mathfrak{p},\nu}}(\mathbf{T}_{R_\mathfrak{p},d}(\nu))^{\mathrm{op}}$,

(c) $\Psi_{R_\mathfrak{p},d}^+(\nu)$ *is fully faithful on* $(\mathbf{A}_{R_\mathfrak{p}}^{+,\kappa}(\nu)\{d\})^\Delta$ *and* $(\mathbf{A}_{R_\mathfrak{p}}^{+,\kappa}(\nu)\{d\})^\nabla$.

Proof We have an equivalence of highest weight categories $\mathbf{O}^{+,\kappa}_{\mathrm{k}_\mathfrak{p}}(\nu) = \bigotimes_{p=1}^{\ell} \mathbf{O}^{+,\kappa}_{\mathrm{k}_\mathfrak{p}}(\nu_p)$ and each factor $\mathbf{O}^{+,\kappa}_{\mathrm{k}_\mathfrak{p}}(\nu_p)$ is a copy of the Kazhdan–Lusztig category. Therefore $\mathbf{O}^{+,\kappa}_{\mathrm{k}_\mathfrak{p}}(\nu)$ is equivalent to a category of modules over a quantum group by [28]. Hence $\mathbf{A}^{+,\kappa}_{\mathrm{k}_\mathfrak{p}}(\nu)\{d\}$ is equivalent to the module category of a q-Schur algebra (with $\ell = 1$) as a highest weight category, and this equivalence takes $\Psi^{+}_{\mathrm{k}_\mathfrak{p},d}(\nu)$ to the q-Schur functor.

Hence, some standard facts on q-Schur algebras imply that $\mathbf{T}_{\mathrm{k}_\mathfrak{p},d}(\nu)$ is projective in $\mathbf{A}^{+,\kappa}_{\mathrm{k}_\mathfrak{p}}(\nu)\{d\}$, proving part (a), and that the $\mathrm{k}_\mathfrak{p}$-algebra homomorphism $\mathbf{H}^{s}_{\mathrm{k}_\mathfrak{p},d} \to \mathrm{End}_{\mathbf{m}_{\mathrm{k}_\mathfrak{p},\nu}}(\mathbf{T}_{\mathrm{k}_\mathfrak{p},d}(\nu))^{\mathrm{op}}$ is an isomorphism, proving part (b) by Nakayama's lemma and (5.4), (5.3), see e.g. [39].

Now, we concentrate on part (c). A standard argument due to Donkin implies that the q-Schur functor $\Xi^{s}_{\mathrm{k}_\mathfrak{p},d}$ is faithful on $(\mathbf{S}^{s}_{\mathrm{k}_\mathfrak{p},d}\text{-mod})^{\nabla}$ for $\ell = 1$. More precisely, recall that $\Xi^{s}_{\mathrm{k}_\mathfrak{p},d} = \mathrm{Hom}_{\mathbf{S}^{s}_{\mathrm{k}_\mathfrak{p},d}}(\mathbf{S}^{s}_{\mathrm{k}_\mathfrak{p},d}\, e, \bullet)$ for some idempotent $e \in \mathbf{S}^{s}_{\mathrm{k}_\mathfrak{p},d}$. Recall also that the $\mathbf{S}^{s}_{\mathrm{k}_\mathfrak{p},d}$-module $\mathbf{S}^{s}_{\mathrm{k}_\mathfrak{p},d}\, e$ is faithful and that any Weyl module embeds in $\mathbf{S}^{s}_{\mathrm{k}_\mathfrak{p},d}\, e$, see e.g., [35, p. 188]. Thus, the claim follows from [35, thm. 4.5.5]. So, from the equivalence above, we deduce that $\Psi^{+}_{\mathrm{k}_\mathfrak{p},d}(\nu)$ is faithful on $(\mathbf{A}^{+,\kappa}_{\mathrm{k}_\mathfrak{p}}(\nu)\{d\})^{\nabla}$. Since the q-Schur algebra is Ringel self-dual, we deduce that $\Psi^{+}_{\mathrm{k}_\mathfrak{p},d}(\nu)$ is also faithful on $(\mathbf{A}^{+,\kappa}_{\mathrm{k}_\mathfrak{p}}(\nu)\{d\})^{\Delta}$. Therefore, the part (c) of the lemma follows from Proposition 2.18. □

We can now prove the main result of this section. Recall that we have introduced a module $\mathbf{T}_{R,d}$ in $\mathbf{A}^{\nu,\kappa}_{R,\tau}$, an R-algebra homomorphism $\psi^{s}_{R,d} : \mathbf{H}^{s}_{R,d} \to \mathrm{End}_{\mathbf{A}^{\nu,\kappa}_{R,\tau}}(\mathbf{T}_{R,d})^{\mathrm{op}}$, and a functor $\Psi^{s}_{R,d} : \mathbf{A}^{\nu,\kappa}_{R,\tau}\{d\} \to \mathbf{H}^{s}_{R,d}$-mod.

By base-change, we get $\mathbf{T}_{R_\mathfrak{p},d}$, $\psi^{s}_{R_\mathfrak{p},d}$ and $\Psi^{s}_{R_\mathfrak{p},d}$ for each $\mathfrak{p} \in \mathfrak{P}$, see Remark 4.6. Recall also that, since R is in general position, the K-category $\mathbf{A}^{\nu,\kappa}_{K,\tau}$ is split semi-simple and condition (3.1) holds in K.

Proposition 5.35 *Let $\mathfrak{p} \in \mathfrak{P}_1$ be generic. Assume that $\nu_p \geqslant d$ for all p. Then*

(a) $\mathbf{T}_{R_\mathfrak{p},d}$ *is projective in* $\mathbf{A}^{\nu,\kappa}_{R_\mathfrak{p},\tau}$,

(b) $\psi^{s}_{R_\mathfrak{p},d}$ *is an isomorphism* $\mathbf{H}^{s}_{R_\mathfrak{p},d} \to \mathrm{End}_{\mathbf{A}^{\nu,\kappa}_{R_\mathfrak{p},\tau}}(\mathbf{T}_{R_\mathfrak{p},d})^{\mathrm{op}}$,

(c) $\Psi^{s}_{R_\mathfrak{p},d}$ *is fully faithful on* $(\mathbf{A}^{\nu,\kappa}_{R_\mathfrak{p},\tau}\{d\})^{\Delta}$ *and* $(\mathbf{A}^{\nu,\kappa}_{R_\mathfrak{p},\tau}\{d\})^{\nabla}$.

Proof Assuming part (b), the Proposition 5.31 and Lemma 5.34 imply parts (a) and (c). Let us prove (b).

The proof is similar to the proof of Proposition 5.27. It is based on Proposition 2.23. Recall that $\mathbf{H}^{s}_{K,d}$ is a split semi-simple K-algebra, and by [13, thm. 3.30], that the decomposition map $K_0(\mathbf{H}^{s}_{K,d}) \to K_0(\mathbf{H}^{s}_{\mathrm{k}_\mathfrak{p},d})$ is surjective. We construct an endomorphism $\theta_{R_\mathfrak{p}}$ of $\mathbf{H}^{s}_{R_\mathfrak{p},d}$ as follows. By Lemma 5.34, we have an isomorphism $\psi^{+}_{R_\mathfrak{p},d}(\nu) : \mathbf{H}^s_{R_\mathfrak{p},d} \to \mathrm{End}_{\mathbf{m}_{R_\mathfrak{p},\nu}}(\mathbf{T}_{R_\mathfrak{p},d}(\nu))^{\mathrm{op}}$. Next, by Proposition 5.31 and Lemma 5.32, we have an equivalence of categories $\mathscr{Q}_{R_\mathfrak{p}} : \mathbf{O}^{\nu,\kappa}_{R_\mathfrak{p},\tau}\{a\} \to \mathbf{O}^{+,\kappa}_{R_\mathfrak{p}}(\nu)\{a\}$ which maps $\mathbf{T}_{R_\mathfrak{p},d}$ to $\mathbf{T}_{R_\mathfrak{p},d}(\nu)$. By functoriality, it induces an isomorphism $\beta_{R_\mathfrak{p}} : \mathrm{End}_{\mathbf{A}^{\nu,\kappa}_{R_\mathfrak{p},\tau}}(\mathbf{T}_{R_\mathfrak{p},d})^{\mathrm{op}} \to \mathrm{End}_{\mathbf{A}^{+,\kappa}_{R_\mathfrak{p}}(\nu)}(\mathbf{T}_{R_\mathfrak{p},d}(\nu))^{\mathrm{op}}$. We set $\theta_{R_\mathfrak{p}} = (\varphi^s_{R_\mathfrak{p},d})^{-1} \circ \beta_{R_\mathfrak{p}} \circ \psi^s_{R_\mathfrak{p},d}$. The same proof as in Proposition 5.27 implies that the map $\theta_K = K\theta_{R_\mathfrak{p}}$ induces the identity on the Grothendieck group. So $\theta_{R_\mathfrak{p}}$ is an automorphism by Proposition 2.23. This implies that $\psi^s_{R_\mathfrak{p},d}$ is an isomorphism. □

Remark 5.36 If $q_{\mathrm{k}_\mathfrak{p}} \neq 1$ then $\Psi^s_{R_\mathfrak{p},d}$ is a 1-faithful highest weight cover.

5.8 The category A as a highest weight cover

Let R be a local analytic deformation ring of dimension 2 in general position. Let $\kappa_{\mathrm{k}} = -e$ and $s_R = \nu + \tau_R$. Recall the module $\mathbf{T}_{R,d} \in \mathbf{A}^{\nu,\kappa}_{R,\tau}$ and the functor $\Psi^s_{R,d} : \mathbf{A}^{\nu,\kappa}_{R,\tau} \to \mathbf{H}^s_{R,d}$-mod.

The first main result of this paper is the following theorem.

Theorem 5.37 *Assume that $\nu_p \geqslant d$ for all p.*

(a) *The map $\psi^s_{R,d} : \mathbf{H}^s_{R,d} \to \mathrm{End}_{\mathbf{A}^{\nu,\kappa}_{R,\tau}}\big(\mathbf{T}_{R,d}\big)^{\mathrm{op}}$ is an R-algebra isomorphism.*

(b) *The module $\mathbf{T}_{R,d}$ is projective in $\mathbf{A}^{\nu,\kappa}_{R,\tau}$.*

(c) *The functor $\Psi^s_{R,d}$ is fully faithful on $\mathbf{A}^{\nu,\kappa,\Delta}_{R,\tau}$ and $\mathbf{A}^{\nu,\kappa,\nabla}_{R,\tau}$.*

Proof First, by Proposition 5.22, the category $\mathbf{A}^{\nu,\kappa}_{K,\tau}$ is split semi-simple and condition (3.1) holds in the fraction field K.

To prove part (a), observe that since $\mathbf{T}_{R,d}$ is tilting, the R-module $\mathrm{End}_{\mathbf{A}^{\nu,\kappa}_{R,\tau}}(\mathbf{T}_{R,d})$ is projective. Since $\mathbf{H}^s_{R,d}$ is also projective over R, we have

$$\mathbf{H}^s_{R,d} = \bigcap_{\mathfrak{p}\in\mathfrak{P}_1} R_\mathfrak{p}\mathbf{H}^s_{R,d}, \quad \mathrm{End}_{\mathbf{A}^{\nu,\kappa}_{R,\tau}}(\mathbf{T}_{R,d}) = \bigcap_{\mathfrak{p}\in\mathfrak{P}_1} R_\mathfrak{p}\,\mathrm{End}_{\mathbf{A}^{\nu}_{R,\tau}}(\mathbf{T}_{R,d}),$$

see [Bourbaki, *Algèbre commutative*, ch. VII, §4, n°2]. Next, we have $R_\mathfrak{p}\mathbf{H}^s_{R,d} = \mathbf{H}^s_{R_\mathfrak{p},d}$ and $R_\mathfrak{p}\,\mathrm{End}_{\mathbf{A}^{\nu,\kappa}_{R,\tau}}(\mathbf{T}_{R,d}) = \mathrm{End}_{\mathbf{A}^{\nu,\kappa}_{R_\mathfrak{p},\tau}}(\mathbf{T}_{R_\mathfrak{p},d})$ for each $\mathfrak{p} \in \mathfrak{P}$. Thus, it is enough to prove that the map $\psi^s_{R_\mathfrak{p},d}$ is invertible for each $\mathfrak{p} \in \mathfrak{P}_1$. By Proposition 5.22, the prime $\mathfrak{p}$ is generic or subgeneric. Thus part (a) follows from Proposition 5.27 and Proposition 5.35.

Now, let us prove that $\Psi^s_{R,d}$ is fully faithful on $\mathbf{A}^{\nu,\kappa,\nabla}_{R,\tau}$. Since $\mathbf{T}_{R,d}$ is tilting, by Corollary 2.17 it is enough to check that $\Psi^s_{R_\mathfrak{p},d}$ is fully faithful on $(\mathbf{A}^{\nu,\kappa}_{R_\mathfrak{p},\tau}\{d\})^\nabla$ for $\mathfrak{p} \in \mathfrak{P}_1$. This has already been proved in Propositions 5.27 and 5.35.

As a consequence, the tilting module $\mathbf{T}_{R,d}$ is projective by Lemma 2.15, because the algebra $\mathrm{End}_{\mathbf{A}^{\nu,\kappa}_{R,\tau}}(\mathbf{T}_{R,d})$ being isomorphic to $\mathbf{H}^s_{R,d}$ is symmetric. Part (b) is proved.

We deduce that $\Psi^s_{R,d}$ is quotient functor. Therefore by Lemma 2.16 it is fully faithful over $\mathbf{A}^{\nu,\kappa,\Delta}_{R,\tau}$ if $\Psi^s_{R_\mathfrak{p},d}$ is fully faithful on $(\mathbf{A}^{\nu,\kappa}_{R_\mathfrak{p},\tau}\{d\})^\Delta$ for $\mathfrak{p} \in \mathfrak{P}_1$. Again, this has been proved in Propositions 5.27 and 5.35. The theorem is proved. □

The following corollary is a straightforward consequence of the theorem by specializing to the residue field, see also [33].

Corollary 5.38 *Assume that $\nu_p \geqslant d$ for all p.*

(a) *The map $\psi^\nu_{\mathrm{k},d} : \mathbf{H}^\nu_{\mathrm{k},d} \to \mathrm{End}_{\mathbf{A}^{\nu,-e}_{\mathrm{k}}}\left(\mathbf{T}_{\mathrm{k},d}\right)^{\mathrm{op}}$ is a* k*-algebra isomorphism.*

(b) *The module $\mathbf{T}_{\mathrm{k},d}$ is projective in $\mathbf{A}^{\nu,-e}_{\mathrm{k}}$.*

Remark 5.39 The module $\mathbf{T}_{\mathrm{k},d}$ may not be projective in $\mathbf{O}^{\nu,\kappa}_{\mathrm{k}}$.

Remark 5.40 Let R be any local deformation ring. Assume that $\nu_p \geqslant d$ for each p. From Theorem 5.37(b), Proposition 2.4 and Remark 5.17 we deduce that $\mathbf{T}_{R,d}$ is well-defined and is projective in $\mathbf{A}^{\nu,\kappa}_{R,\tau}$.

5.9 The functor F and induction

In Sect. 4.6 we defined a pre-categorical action (E, F, X, T) on $A^{\nu,\kappa}_{R,\tau}$. Now, we define a tuple (E, F, X, T) on $\mathbf{A}^{\nu,\kappa}_{R,\tau}$ in the following way. Let $h : \mathbf{A}^{\nu,\kappa}_{R,\tau} \to \mathbf{O}^{\nu,\kappa}_{K,\tau}$ be the canonical embedding. Consider the endofunctors E, F of $\mathbf{A}^{\nu,\kappa}_{R,\tau}$ given by $E = h^*eh$, $F = h^*fh$. Since f preserves the subcategory $\mathbf{A}^{\nu,\kappa}_{R,\tau}$, we have $F = h^!fh = f|_{\mathbf{A}^{\nu,\kappa}_{R,\tau}}$. In particular, the functor F is exact, (E, F) is an adjoint pair and we have $E(\mathbf{A}^{\nu,\kappa}_{R,\tau}\{d+1\}^\Delta) \subset (\mathbf{A}^{\nu,\kappa}_{R,\tau}\{d\})^\Delta$. Then, we define $X \in \mathrm{End}(F) = \mathrm{End}(f)$ and $T \in \mathrm{End}(F^2) = \mathrm{End}(f^2)$ as in Proposition 5.14.

Let d, k be positive integers such that $d + k \leqslant \nu_p$ for all p. In this section we compare the functors $F^k : \mathbf{A}^{\nu,\kappa}_{R,\tau}\{d\} \to \mathbf{A}^{\nu,\kappa}_{R,\tau}\{d+k\}$ and $\mathrm{Ind}^{d+k}_d = \mathbf{H}^s_{R,d+k} \otimes_{\mathbf{H}^s_{R,d}} \bullet : \mathbf{H}^s_{R,d}\text{-mod} \to \mathbf{H}^s_{R,d+k}\text{-mod}$.

By definition $F^k(\mathbf{T}_{R,d}) = \mathbf{T}_{R,d+k}$ and we have a commutative diagram

$$\begin{tikzcd}
\mathbf{H}^s_{R,d} \arrow[r, "\psi^s_{R,d}", "\sim"'] \arrow[d, hook] & \mathrm{End}_{\mathbf{A}^{\nu,\kappa}_{R,\tau}}(\mathbf{T}_{R,d})^{\mathrm{op}} \arrow[d, "F^k"] \\
\mathbf{H}^s_{R,d+k} \arrow[r, "\psi^s_{R,d+k}", "\sim"'] & \mathrm{End}_{\mathbf{A}^{\nu,\kappa}_{R,\tau}}(F^k\mathbf{T}_{R,d})^{\mathrm{op}}.
\end{tikzcd}$$

Therefore, we have a morphism of functors on $\mathbf{A}^{\nu,\kappa}_{R,\tau}\{d\}$

$$\begin{aligned}\vartheta_k:\quad \mathrm{Ind}_d^{d+k}\Psi^s_{R,d} &\simeq \mathrm{Hom}_{\mathbf{A}^{\nu,\kappa}_{R,\tau}}(\mathbf{T}_{R,d+k},F^k\mathbf{T}_{R,d})\\ &\quad\otimes_{\mathrm{End}_{\mathbf{A}^{\nu,\kappa}_{R,\tau}}(\mathbf{T}_{R,d})^{\mathrm{op}}}\mathrm{Hom}_{\mathbf{A}^{\nu,\kappa}_{R,\tau}}(\mathbf{T}_{R,d},\bullet)\\ &\simeq \mathrm{Hom}_{\mathbf{A}^{\nu,\kappa}_{R,\tau}}(E^k\mathbf{T}_{R,d+k},\mathbf{T}_{R,d})\\ &\quad\otimes_{\mathrm{End}_{\mathbf{A}^{\nu,\kappa}_{R,\tau}}(\mathbf{T}_{R,d})^{\mathrm{op}}}\mathrm{Hom}_{\mathbf{A}^{\nu,\kappa}_{R,\tau}}(\mathbf{T}_{R,d},\bullet)\\ &\to \mathrm{Hom}_{\mathbf{A}^{\nu,\kappa}_{R,\tau}}(E^k\mathbf{T}_{R,d+k},\bullet)\\ &\simeq \mathrm{Hom}_{\mathbf{A}^{\nu,\kappa}_{R,\tau}}(\mathbf{T}_{R,d+k},F^k\bullet)\\ &= \Psi^s_{R,d+k}F^k,\end{aligned}$$

where the map in the third line is given by composition.

Lemma 5.41 *Assume that $d+k\leqslant\nu_p$ for all p. Then $\vartheta_k:\mathrm{Ind}_d^{d+k}\Psi^s_{R,d}\to\Psi^s_{R,d+k}F^k$ is an isomorphism.*

Proof It is enough to prove that ϑ_k is an isomorphism of functors on $(\mathbf{A}^{\nu,\kappa}_{R,\tau}\{d\})^{\Delta}$. We must prove that the map

$$\begin{aligned}&\mathrm{Hom}_{\mathbf{A}^{\nu,\kappa}_{R,\tau}}(E^k\mathbf{T}_{R,d+k},\mathbf{T}_{R,d})\otimes_{\mathrm{End}_{\mathbf{A}^{\nu,\kappa}_{R,\tau}}(\mathbf{T}_{R,d})^{\mathrm{op}}}\mathrm{Hom}_{\mathbf{A}^{\nu,\kappa}_{R,\tau}}(\mathbf{T}_{R,d},M)\\ &\quad\to\mathrm{Hom}_{\mathbf{A}^{\nu,\kappa}_{R,\tau}}(E^k\mathbf{T}_{R,d+k},M)\end{aligned}$$

given by composition is an isomorphism for each $M\in(\mathbf{A}^{\nu,\kappa}_{R,\tau}\{d\})^{\Delta}$.

Since $\Psi^s_{R,d}$ is 0-faithful, $E(\mathbf{A}^{\nu,\kappa}_{R,\tau}\{d+1\}^{\Delta})\subset(\mathbf{A}^{\nu,\kappa}_{R,\tau}\{d\})^{\Delta}$ and $\Psi^s_{R,d}(\mathbf{T}_{R,d})\simeq\mathbf{H}^s_{R,d}$ as $(\mathbf{H}^s_{R,d},\mathbf{H}^s_{R,d})$-bimodules, the left hand side is isomorphic to

$$\mathrm{Hom}_{\mathbf{H}^s_{R,d}}(\Psi^s_{R,d}(E^k\mathbf{T}_{R,d+k}),\mathbf{H}^s_{R,d})\otimes_{\mathbf{H}^s_{R,d}}\mathrm{Hom}_{\mathbf{H}^s_{R,d}}(\mathbf{H}^s_{R,d},\Psi^s_{R,d}(M))$$

and the right hand side is isomorphic to $\mathrm{Hom}_{\mathbf{H}^s_{R,d}}(\Psi^s_{R,d}(E^k\mathbf{T}_{R,d+k}),\Psi^s_{R,d}(M))$. Hence, we are reduced to prove that the natural map

$$\begin{aligned}&\mathrm{Hom}_{\mathbf{H}^s_{R,d}}(\Psi^s_{R,d}(E^k\mathbf{T}_{R,d+k}),\mathbf{H}^s_{R,d})\otimes_{\mathbf{H}^s_{R,d}}\mathrm{Hom}_{\mathbf{H}^s_{R,d}}(\mathbf{H}^s_{R,d},\Psi^s_{R,d}(M))\\ &\quad\to\mathrm{Hom}_{\mathbf{H}^s_{R,d}}(\Psi^s_{R,d}(E^k\mathbf{T}_{R,d+k}),\Psi^s_{R,d}(M))\end{aligned}$$

given by composition is an isomorphism. We claim that $\Psi^s_{R,d}(E^k\mathbf{T}_{R,d+1})\simeq\mathbf{H}^s_{R,d+k}$ as $\mathbf{H}^s_{R,d}$-modules. Thus it is a projective $\mathbf{H}^s_{R,d}$-module, and the isomorphism follows.

To prove the claim, note that since $\Psi^s_{R,d} = \mathrm{Hom}_{\mathbf{A}^{\nu,\kappa}_{R,\tau}}(\mathbf{T}_{R,d}, \bullet)$ is fully faithful on $(\mathbf{A}^{\nu,\kappa}_{R,\tau}\{d\})^{\nabla}$, using the duality $\mathscr{D}$ on $\mathbf{A}^{\nu,\kappa}_{R,\tau}\{d\}$ and the fact that $\mathscr{D}(\mathbf{T}_{R,d}) \simeq \mathbf{T}_{R,d}$, we deduce that the contravariant functor $\mathrm{Hom}_{\mathbf{A}^{\nu,\kappa}_{R,\tau}}(\bullet, \mathbf{T}_{R,d}) : \mathbf{A}^{\nu,\kappa}_{R,\tau}\{d\} \to (\mathbf{H}^s_{R,d})^{\mathrm{op}}$-mod is fully faithful on $(\mathbf{A}^{\nu,\kappa}_{R,\tau}\{d\})^{\Delta}$. Therefore, we have isomorphisms

$$\begin{aligned}\Psi^s_{R,d}(E^k\mathbf{T}_{R,d+k}) &\simeq \mathrm{Hom}_{\mathbf{A}^{\nu,\kappa}_{R,\tau}}(\mathbf{T}_{R,d}, E^k\mathbf{T}_{R,d+k}) \\ &\simeq \mathrm{Hom}_{(\mathbf{H}^s_{R,d})^{\mathrm{op}}}\left(\mathrm{Hom}_{\mathbf{A}^{\nu,\kappa}_{R,\tau}}(E^k\mathbf{T}_{R,d+k}, \mathbf{T}_{R,d}), \mathbf{H}^s_{R,d}\right) \\ &\simeq \mathrm{Hom}_{(\mathbf{H}^s_{R,d})^{\mathrm{op}}}\left(\mathrm{Hom}_{\mathbf{A}^{\nu,\kappa}_{R,\tau}}(\mathbf{T}_{R,d+k}, F^k\mathbf{T}_{R,d}), \mathbf{H}^s_{R,d}\right) \\ &\simeq \mathrm{Hom}_{(\mathbf{H}^s_{R,d})^{\mathrm{op}}}\left(\mathbf{H}^s_{R,d+k}, \mathbf{H}^s_{R,d}\right).\end{aligned}$$

Finally, since $\mathbf{H}^s_{R,d+k}$ is self-injective, there is an isomorphism of $\mathbf{H}^s_{R,d}$-modules $\mathbf{H}^s_{R,d+k} \simeq \mathrm{Hom}_{(\mathbf{H}^s_{R,d})^{\mathrm{op}}}\left(\mathbf{H}^s_{R,d+k}, \mathbf{H}^s_{R,d}\right)$. The claim is proved. □

Remark 5.42 Recall that X acts on $\mathrm{Ind}_d^{d+1} = \mathbf{H}^s_{R,d+1} \otimes_{\mathbf{H}^s_{R,d}} \bullet$ by right multiplication by X_{d+1} on $\mathbf{H}^s_{R,d+1}$, and the action of X on E is the transposition of its action on F under the adjunction, see Remark 3.3. Hence, it follows from the definition of ϑ_1 that it intertwines the action of X on Ind_d^{d+1} and on F, i.e., we have

$$\vartheta_1 \circ (\Psi^s_{R,d+1}X) = (X\Psi^s_{R,d}) \circ \vartheta_1.$$

Similarly we have

$$\vartheta_2 \circ (\Psi^s_{R,d+2}T) = (T\Psi^s_{R,d}) \circ \vartheta_2,$$

for the action of T on Ind_d^{d+2} and on F^2.

6 The category A and CRDAHA's

6.1 Reminder on rational DAHA's

6.1.1 Definition of the category $\mathcal{O}$

Let R be a local ring with residue field $\mathbb{C}$. Let W be a complex reflection group, let $\mathfrak{h}$ be the reflection representation of W over R and let S be the set of pseudo-reflections in W. Let $\mathcal{A}$ be the set of reflection hyperplanes in $\mathfrak{h}$. We write $\mathfrak{h}_{reg} = \mathfrak{h} \setminus \bigcup_{H \in \mathcal{A}} H$.

Let $c : S \to R$ be a map that is constant on the W-conjugacy classes. The *RDAHA* (=*rational double affine Hecke algebra*) attached to W with parameter

c is the quotient $H_c(W,\mathfrak{h})_R$ of the smash product of RW and the tensor algebra of $\mathfrak{h}\oplus\mathfrak{h}^*$ by the relations

$$[x,x']=0,\quad [y,y']=0,\quad [y,x]=\langle x,y\rangle-\sum_{s\in S}c_s\langle\alpha_s,y\rangle\langle x,\check{\alpha}_s\rangle s,$$

for all $x,x'\in\mathfrak{h}^*$, $y,y'\in\mathfrak{h}$. Here $\langle\bullet,\bullet\rangle$ is the canonical pairing between $\mathfrak{h}^*$ and $\mathfrak{h}$, the element α_s is a generator of $\mathrm{Im}(s|_{\mathfrak{h}^*}-1)$ and $\check{\alpha}_s$ is the generator of $\mathrm{Im}(s|_{\mathfrak{h}}-1)$ such that $\langle\alpha_s,\check{\alpha}_s\rangle=2$.

Let $R[\mathfrak{h}]$, $R[\mathfrak{h}^*]$ be the subalgebras of $H_c(W,\mathfrak{h})_R$ generated by $\mathfrak{h}^*$ and $\mathfrak{h}$ respectively. The category $\mathcal{O}$ of $H_c(W,\mathfrak{h})_R$ is the full subcategory of the category of $H_c(W,\mathfrak{h})_R$-modules consisting of objects that are finitely generated as $R[\mathfrak{h}]$-modules and $\mathfrak{h}$-locally nilpotent, see [22, § 3]. We denote it by $\mathcal{O}_c(W,\mathfrak{h})_R$. It is a highest weight R-category. The standard modules are labeled by the set $\mathrm{Irr}(\mathbb{C}W)$ of isomorphism classes of irreducible W-modules. The standard module associated with $E\in\mathrm{Irr}(\mathbb{C}W)$ is the induced module $\Delta(E)_R=\mathrm{Ind}_{W\ltimes R[\mathfrak{h}^*]}^{H(W)_R}(RE)$. Here RE is regarded as a $W\ltimes R[\mathfrak{h}^*]$-module such that $\mathfrak{h}\subset R[\mathfrak{h}^*]$ acts by zero. Let $L(E)$ be the unique simple quotient of $\Delta(E)_R$, and let $P(E)_R$ be the projective cover of $\Delta(E)_R$.

By [22, § 4.2.1] there is a functor

$$(\bullet)^\vee:\mathcal{O}_c(W,\mathfrak{h})_R\to\mathcal{O}_{c^\vee}(W,\mathfrak{h}^*)_R^{\mathrm{op}},$$

which is an equivalence over the subcategories of modules in $\mathcal{O}_c(W,\mathfrak{h})_R$, $\mathcal{O}_{c^\vee}(W,\mathfrak{h}^*)_R^{\mathrm{op}}$ that are free over R. Here $c^\vee:S\to R$ is defined by $c^\vee(s)=c(s^{-1})$. For any $E\in\mathrm{Irr}(\mathbb{C}W)$ we write $E^\vee=\mathrm{Hom}_R(E,R)$. We have $\Delta(E)_R^\vee\simeq\nabla(E^\vee)_R$ and $\nabla(E)_R^\vee\simeq\Delta(E^\vee)_R$.

6.1.2 The KZ*-functor*

Let R be an analytic regular local ring. There is a quotient functor

$$\mathrm{KZ}_R:\mathcal{O}_c(W,\mathfrak{h})_R\to\mathbf{H}(W,\mathfrak{h})_R\text{-mod}$$

defined in [22, § 5.3], where $\mathbf{H}(W,\mathfrak{h})_R$ is the Hecke algebra associated with W and a parameter which depends on c. Note that loc. cit. uses regular complete local rings, but the same construction can be done for analytic ones.

Proposition 6.1 *The functor* KZ_R *is* 0*-faithful.*

Proof By Proposition 2.18 it is enough to prove that over the residue field $\mathbb{C}$ the functor KZ is (-1)-faithful. In other words, we must prove that KZ is faithful on $\mathcal{O}_c(W,\mathfrak{h})_{\mathbb{C}}^{\Delta}$.

Write $\mathcal{O} = \mathcal{O}_c(W, \mathfrak{h})_{\mathbb{C}}$, let $\mathcal{O}_{tor} \subset \mathcal{O}$ be the full subcategory consisting of the objects M such that $M \otimes_{\mathbb{C}[\mathfrak{h}]} \mathbb{C}[\mathfrak{h}_{reg}] = 0$. By [22, thm. 5.14], the functor KZ is isomorphic to the quotient functor $\mathcal{O} \to \mathcal{O}/\mathcal{O}_{tor}$. A Δ-filtered object M is free over $\mathbb{C}[\mathfrak{h}]$ by [22, prop. 2.21], so it has no torsion submodules. Therefore, the map $\mathrm{Hom}_{\mathcal{O}}(M, N) \to \mathrm{Hom}_{\mathcal{O}}(\mathrm{KZ}(M), \mathrm{KZ}(N))$ is injective for each $M, N \in \mathcal{O}^{\Delta}$. We are done. □

6.1.3 Induction and restriction functors

A *parabolic subgroup* $W' \subset W$ is the stabilizer of some point $b \in \mathfrak{h}$. It is a complex reflection group with the set of reflections $S' = S \cap W$ and with reflection representation $\mathfrak{h}/\mathfrak{h}^{W'}$, where $\mathfrak{h}^{W'}$ is the subspace of points fixed by W'. Bezrukavnikov and Etingof [3] defined parabolic induction and restriction functors

$$\begin{aligned} {}^{\mathcal{O}}\mathrm{Ind}_{W'}^{W} &: \mathcal{O}_c(W', \mathfrak{h}/\mathfrak{h}^{W'})_R \to \mathcal{O}_c(W, \mathfrak{h})_R, \\ {}^{\mathcal{O}}\mathrm{Res}_{W'}^{W} &: \mathcal{O}_c(W, \mathfrak{h})_R \to \mathcal{O}_c(W', \mathfrak{h}/\mathfrak{h}^{W'})_R. \end{aligned}$$

Here we view c as a parameter for W' by identifying it with its restriction to S'. In loc. cit. the authors work over a field. The definition is the same over a ring R. The functor ${}^{\mathcal{O}}\mathrm{Ind}_{W'}^{W}$ is left adjoint to ${}^{\mathcal{O}}\mathrm{Res}_{W'}^{W}$, and both functors are exact. In particular ${}^{\mathcal{O}}\mathrm{Ind}_{W'}^{W}$ maps projective objects to projective objects.

Let R be an analytic regular local ring. By [41, thm. 2.1] we have isomorphisms of functors

$$\mathrm{KZ}_R \circ {}^{\mathcal{O}}\mathrm{Res}_{W'}^{W} \simeq {}^{\mathbf{H}}\mathrm{Res}_{W'}^{W} \circ \mathrm{KZ}_R, \quad \mathrm{KZ}_R \circ {}^{\mathcal{O}}\mathrm{Ind}_{W'}^{W} \simeq {}^{\mathbf{H}}\mathrm{Ind}_{W'}^{W} \circ \mathrm{KZ}_R, \tag{6.1}$$

where ${}^{\mathcal{O}}\mathrm{Res}_{W'}^{W}$ and ${}^{\mathbf{H}}\mathrm{Ind}_{W'}^{W}$ refer to the restriction and induction functors for Hecke algebras $\mathbf{H}(W', \mathfrak{h}/\mathfrak{h}^{W'})_R \hookrightarrow \mathbf{H}(W, \mathfrak{h})_R$, see loc. cit. for more details. Again, in loc. cit. we work over a field, but the same proof works over R.

We will be mainly interested in the case where $W' = W_H$ is the pointwise stabilizer of a hyperplane H. We will abbreviate $\mathcal{O}(W_H)_R = \mathcal{O}_c(W_H, \mathfrak{h}/H)_R$ and ${}^{\mathcal{O}}\mathrm{Ind}_H = {}^{\mathcal{O}}\mathrm{Ind}_{W_H}^{W}$.

6.1.4 Support of modules

Let R be a local ring with residue field k. We abbreviate $\mathcal{O}_R = \mathcal{O}_c(W, \mathfrak{h})_R$. If $R = K$ is a field, let $\mathrm{Ch}(M)$ denote *the characteristic variety* of M as defined in [22, § 4.3.4]. It is a closed subvariety of $\mathfrak{h} \oplus \mathfrak{h}^*$. Recall the notation $\mathrm{lcd}_{\mathcal{O}}$ and $\mathrm{rcd}_{\mathcal{O}}$ from (2.1).

Lemma 6.2 *Assume $R = K$ is a field. For any $M \in \mathcal{O}_K$ we have*

$$\mathrm{lcd}_{\mathcal{O}_K}(M) = \mathrm{rcd}_{\mathcal{O}_K}(M) = \dim \mathfrak{h} - \dim \mathrm{Ch}(M).$$

Proof The equality $\dim \mathrm{Ch}(M) = \dim \mathfrak{h} - \mathrm{rcd}_{\mathcal{O}}(M)$ is proved in [22, cor. 4.14]. Further, the proof of [22, lem. 5.2] yields $\dim \mathrm{Ch}(M) = \dim \mathrm{Ch}(M^\vee)$. This implies that $\mathrm{rcd}_{\mathcal{O}}(M) = \mathrm{rcd}_{\mathcal{O}}(M^\vee)$. On the other hand, by [22, prop. 4.7], if T is a tilting generator of $\mathcal{O}_K$ then $T^\vee$ is a tilting generator of $\mathcal{O}_K^\vee$ and $\mathrm{Ext}^i_{\mathcal{O}_K}(T, M) \simeq \mathrm{Ext}^i_{\mathcal{O}_K^\vee}(M^\vee, T^\vee)$. We deduce that $\mathrm{lcd}_{\mathcal{O}_K}(M) = \mathrm{rcd}_{\mathcal{O}_K}(M^\vee) = \mathrm{rcd}_{\mathcal{O}_K}(M)$. □

Lemma 6.3 *For $E \in \mathrm{Irr}(\mathbb{C}W)$ we have $\mathrm{rcd}_{\mathcal{O}_\mathrm{k}}(L(E)) \leqslant 1$ if and only if there exist $H \in \mathcal{A}$ and $P \in \mathcal{O}(W_H)_R^{\mathrm{proj}}$ such that $P(E)$ is a direct summand of ${}^{\mathcal{O}}\mathrm{Ind}_H(P)$.*

Proof By [21, thm. 6.8] we have $\mathrm{Ch}(L(E)) = \mathfrak{h}^{W'} \bigoplus \{0\} \subset \mathfrak{h} \bigoplus \mathfrak{h}^*$ for some parabolic subgroup $W' \subset W$. So $\mathrm{rcd}_{\mathcal{O}_\mathrm{k}}(L(E)) \leqslant 1$ is equivalent, by Lemma 6.2, to the fact that $\mathfrak{h}^{W'}$ has codimension $\leqslant 1$ in $\mathfrak{h}$, which is equivalent to $W' \subset W_H$ for some hyperplane H in $\mathcal{A}$. By [43, prop. 2.2], the latter is true if and only if ${}^{\mathcal{O}}\mathrm{Res}^W_{W_H}(L(E)) \neq 0$, which is equivalent to

$$\mathrm{Hom}_{\mathcal{O}_R}\left({}^{\mathcal{O}}\mathrm{Ind}^W_{W_H}(P), L(E)\right) = \mathrm{Hom}_{\mathcal{O}(W_H)_R}\left(P, {}^{\mathcal{O}}\mathrm{Res}^W_{W_H}(L(E))\right) \neq 0,$$

for some $P \in \mathcal{O}(W_H)_R^{\mathrm{proj}}$. Hence $\mathrm{rcd}_{\mathcal{O}_\mathrm{k}}(L(E)) \leqslant 1$ is equivalent to $P(E)$ being a direct summand of ${}^{\mathcal{O}}\mathrm{Ind}^W_{W_H}(P)$ for some $H \in \mathcal{A}$ and $P \in \mathcal{O}(W_H)_R^{\mathrm{proj}}$. □

6.2 The category $\mathcal{O}$ of cyclotomic rational DAHA's

Let R be a local ring. Fix $\kappa_R \in R^\times$ and $s = (s_{R,1}, \ldots, s_{R,\ell}) \in R^\ell$.

6.2.1 Definition

Recall that Γ is the group of ℓ-th roots of unity in $\mathbb{C}^\times$ and that Γ_d is the semi-direct product $\mathfrak{S}_d \ltimes \Gamma^d$, where Γ^d is the Cartesian product of d copies of Γ. For $\gamma \in \Gamma$ let $\gamma_i \in \Gamma^d$ be the element with γ at the i-th place and with 1 at the other ones. Let $s_{ij} \in \mathfrak{S}_d$ be the transposition (i, j). Write $s_{ij}^\gamma = s_{ij}\gamma_i\gamma_j^{-1}$ for $\gamma \in \Gamma$ and $i \neq j$.

Fix a basis (x, y) of R^2. Let x_i, y_i denote the elements x, y respectively in the i-th summand of $(R^2)^{\oplus d}$. There is a unique action of the group Γ_d on $(R^2)^{\oplus d}$ such that for distinct i, j, k we have $\gamma_i(x_j) = \gamma^{-\delta_{ij}} x_j$, $\gamma_i(y_j) = \gamma^{\delta_{ij}} y_j$ and $s_{ij}(x_i) = x_j$, $s_{ij}(x_k) = x_k$, $s_{ij}(y_i) = y_j$ and $s_{ij}(y_k) = y_k$.

Fix $k \in R$ and $c_\gamma \in R$ for each $\gamma \in \Gamma$. Note that Γ_d is a complex reflection group with reflection representation $\mathfrak{h} = R^{\oplus d}$ and $S = \{s_{ij}^\gamma\}_{1\leq i\neq j\leq d,\gamma\in\Gamma} \coprod \{\gamma_i\}_{1\leq i\leq d}$. Let $c : S \to R$ be the map given by $c(s_{ij}^\gamma) = k$, $c(\gamma_i) = c_\gamma/2$. We consider the algebra $H_c(W, \mathfrak{h})_R$ for $W = \Gamma_d$. We will call $H_c(\Gamma_d, \mathfrak{h})_R$ the *CRDAHA*(=*cyclotomic RDAHA*). It is the quotient of the smash product of $R\Gamma_d$ and the tensor algebra of $(R^2)^{\oplus d}$ by the relations

$$
\begin{aligned}
&[y_i, x_i] = 1 - k\sum_{j\neq i}\sum_{\gamma\in\Gamma} s_{ij}^\gamma - \sum_{\gamma\in\Gamma\setminus\{1\}} c_\gamma\gamma_i,\\
&[y_i, x_j] = k\sum_{\gamma\in\Gamma}\gamma s_{ij}^\gamma \quad \text{if } i\neq j,\\
&[x_i, x_j] = [y_i, y_j] = 0.
\end{aligned}
$$

We will use a presentation of $H_c(\Gamma_d, \mathfrak{h})_R$ where the parameters are $h, h_0, h_1, \ldots, h_{\ell-1}$ with (setting $h_{-1} = h_{\ell-1}$)

$$
k = -h, \qquad -c_\gamma = \sum_{p=0}^{\ell-1}\gamma^{-p}(h_p - h_{p-1}) \quad \text{for } \gamma \neq 1.
$$

The notation $h = h_R, h_p = h_{R,p}$ here is the same as in [39, sec. 6.1.2]. Finally, we choose the elements h_R, $h_{R,p}$ in the following way:

$$
h_R = -1/\kappa_R, \quad h_{R,p} = -s_{R,p+1}/\kappa_R - p/\ell, \quad p = 0, 1, \ldots, \ell-1. \tag{6.2}
$$

In the rest of this section we assume that the residue field is $\mathrm{k} = \mathbb{C}$ *and that* $s_{\mathrm{k},p} \in \mathbb{Z}$ *for all* p.

Write $\kappa = \kappa_{\mathrm{k}}$ and $s_p = s_{\mathrm{k},p}$. We abbreviate $\mathcal{O}_R^{s,\kappa}\{d\} = \mathcal{O}_c(\Gamma_d, \mathfrak{h})_R$. If $\ell = 1$, then c only depends on κ, we abbreviate $\mathcal{O}_R^{\kappa}(\mathfrak{S}_d) = \mathcal{O}_c(\mathfrak{S}_d, \mathfrak{h})_R$. The category $\mathcal{O}_R^{s,\kappa}\{d\}$ is a highest weight R-category such that $\Delta(\mathcal{O}_R^{s,\kappa}\{d\}) = \{\Delta(\lambda)_R^{s,\kappa};\ \lambda \in \mathscr{P}_d^\ell\}$ and $\Delta(\lambda)_R^{s,\kappa} = \Delta(\mathscr{X}(\lambda)_{\mathbb{C}})_R$. We write $L(\lambda)^{s,\kappa}$, $P(\lambda)_R^{s,\kappa}$, $T(\lambda)_R^{s,\kappa}$, $I(\lambda)_R^{s,\kappa}$ for the corresponding simple, projective, tilting, injective object in $\mathcal{O}_R^{s,\kappa}\{d\}$.

6.2.2 *Comparison of partial orders*

The partial order on the set $\Delta(\mathcal{O}_R^{s,\kappa}\{d\}) \simeq \mathscr{P}_d^\ell$ is defined as follows. Let A, B be boxes of ℓ-partitions. We say $A \succ_s B$ if we have $\mathrm{cont}^s(A) < \mathrm{cont}^s(B)$ or if $\mathrm{cont}^s(A) = \mathrm{cont}^s(B)$ and $p(A) > p(B)$. We define a partial order $\geqslant_{s,\kappa}$ on $\mathscr{P}_d^\ell$ by setting $\lambda \geqslant_{s,\kappa} \mu$ if and only if there are orderings $Y(\lambda) = \{A_n\}$ and $Y(\mu) = \{B_n\}$ such that $A_n \succcurlyeq_s B_n$ for all n.

Lemma 6.4 *Assume* $\kappa < 0$. *Then* $\geqslant_{s,\kappa}$ *is a highest weight order on* $\mathcal{O}_R^{s,\kappa}\{d\}$.

Proof By the proof of [16, thm. 4.1], if $[\Delta(\lambda)^{s,\kappa}_{\mathrm{k}} : L(\mu)^{s,\kappa}_{\mathrm{k}}] \neq 0$ then there exist orderings $Y(\lambda) = \{A_n\}$ and $Y(\mu) = \{B_n\}$ and non negative integers D_n such that

$$D_n = p(A_n) - p(B_n) + \ell\left(\mathrm{cont}^s(A_n) - \mathrm{cont}^s(B_n)\right)/\kappa,$$

for all n and $(\mathrm{cont}^s(A_n) - \mathrm{cont}^s(B_n))/\kappa \in \mathbb{Z}$. Our notation matches those of loc. cit. in the following way: $r = \ell$, $c_0 = \kappa^{-1}$, $d_p = -\ell h_p$. Now, since $\kappa < 0$ and $p(A_n), p(B_n) \in [1, \ell]$, we have $D_n \geqslant 0$ if and only if $A_n \succcurlyeq_s B_n$. □

Set $s^\star = (-s_\ell, -s_{\ell-1}, \ldots, -s_1)$. For each $\lambda \in \mathscr{P}^\ell_d$ we write $\lambda^\star = ({}^t\lambda^\ell, \ldots, {}^t\lambda^2, {}^t\lambda^1)$. We have the following lemma which is similar to [33, lem. 2.2].

Lemma 6.5 *Assume that $\kappa < 0$ and that $s_p = \nu_p \geqslant d$ for all p. Then the order $\geqslant_{s^\star,\kappa}$ refines the order $\geqslant_\ell$, i.e., for any $\lambda, \mu \in \mathscr{P}^\ell_d$ such that $\mu \geqslant_\ell \lambda$ we have $\mu^\star \geqslant_{s^\star,\kappa} \lambda^\star$.*

Proof First, for any $\lambda \in \mathscr{P}^\nu_d$ and $A \in Y(\lambda)$, we have the transposed box $A^\star \in Y(\lambda^\star)$ such that $\mathrm{cont}^{s^\star}(A^\star) = -\mathrm{cont}^s(A)$ and $p(A^\star) = \ell + 1 - p(A)$. Therefore, we have $A \prec_s B$ if and only if $A^\star \succ_{s^\star} B^\star$.

Let $\lambda, \mu \in \mathscr{P}^\ell_d$ be such that $\mu \geqslant_\ell \lambda$. Assume that $w \in W_\nu$, $\beta \in \widehat{\Pi}^+$ are such that $\langle\widehat{\varpi(\mu)} + \tilde{\rho} : \beta\rangle > 0$ and $ws_\beta \bullet \widehat{\varpi(\mu)} = \widehat{\varpi(\lambda)}$. We must prove that $\mu^\star \geqslant_{s^\star,\kappa} \lambda^\star$, which is equivalent to $\mu \leqslant_{s,\kappa} \lambda$.

Write $\beta = \alpha_{k,l} + r\delta$ and $\lambda' = s_{k,l}(\mu + \rho_\nu) + er\alpha_{k,l} - \rho_\nu$. Set $n = \langle\mu + \rho_\nu : \alpha_{k,l}\rangle - er > 0$. We have $w(\lambda' + \rho_\nu) = \lambda + \rho_\nu$ and $(\lambda' + \rho_\nu)_k = (\mu + \rho_\nu)_k - n$, $(\lambda' + \rho_\nu)_l = (\mu + \rho_\nu)_l + n$.

For $k \in [1, N]$ let $k' = k - \nu_1 - \nu_2 - \cdots - \nu_{p_k - 1}$ where p_k is such that $k \in J^\nu_{p_k}$. Then the diagram $Y(\lambda')$ is obtained from the diagram $Y(\mu)$ by removing n boxes from the right end of the k'-th row of the p_k-th partition of μ and adding n boxes the right end of the l'-th row of the p_l-th partition of μ.

We number the removed boxes by $B_1, B_2, \ldots, B_n$ ordered from left to right, and the added boxes by $A_1, A_2, \ldots, A_n$ ordered from left to right. We claim that $B_j \preccurlyeq A_j$ for $1 \leqslant j \leqslant n$.

To prove this, note first that $B_j \preccurlyeq A_j$ if and only if $B_n \preccurlyeq A_n$, because we have $\mathrm{cont}^s(B_j) - \mathrm{cont}^s(A_j) = \mathrm{cont}^s(B_n) - \mathrm{cont}^s(A_n)$, $p(B_j) = p(B_n)$ and $p(A_j) = p(A_n)$.

Now let us compare B_n and A_n. Observe that

$$\mathrm{cont}^s(B_n) = (\mu + \rho_\nu)_k - 1, \quad \mathrm{cont}^s(A_n) = (\lambda' + \rho_\nu)_l - 1 = (\mu + \rho_\nu)_l + n - 1.$$

Recall that $\beta = \alpha_{k,l} + r\delta$ is a positive root. Therefore, we have either $r > 0$, and then

$$\mathrm{cont}^s(B_n) - \mathrm{cont}^s(A_n) = \langle \mu + \rho_\nu : \alpha_{k,l} \rangle - n = er > 0,$$

or we have $r = 0$ and $k \leqslant l$, and then $\mathrm{cont}^s(B_n) = \mathrm{cont}^s(A_n)$ and $p(B_n) = p_k \leqslant p_l = p(A_n)$. We deduce that $B_n \preccurlyeq A_n$. Hence we have shown $\mu \leqslant_{s,\kappa} \lambda'$.

Next, recall that $w \in W_\nu$ is such that the tuple $\lambda + \rho_\nu$ is ν-dominant. Thus, we can write $w = s_{\beta_m} s_{\beta_{m-1}} \cdots s_{\beta_1}$ such that $\beta_i = \alpha_{k,l}$ for some $k < l$ with $p_k = p_l$, that $\gamma_i = s_{\beta_{i-1}} s_{\beta_{i-2}} \cdots s_{\beta_1}(\lambda' + \rho_\nu) - \rho_\nu \in \mathbb{N}^N$ and that $n = -\langle \gamma_i + \rho_\nu, \beta_i \rangle > 0$. We set $\gamma_0 = \lambda'$. Repeating the argument of the last paragraph with $\beta = \beta_i$ yields that $Y(\gamma_{i+1})$ is obtained from $Y(\gamma_i)$ by removing n boxes in the l'-th row of $\gamma_i^{p_l}$ and adding them to the k'-th row. Order the removed boxes by $B_1, B_2, \ldots, B_n$ and the added one by $A_1, A_2, \ldots, A_n$ in the same way as above. Then the same computation as above yields that $\mathrm{cont}^s(A_j) = \mathrm{cont}^s(B_j)$ and $p(A_j) = p(B_j)$ for all $j = 1, 2, \ldots, n$. Therefore we have $\gamma_{i+1} = \gamma_i$ for the order $\leqslant_{s,\kappa}$. We deduce that $\lambda = \gamma_{m+1} = \gamma_{m+1} = \lambda'$. Therefore $\mu \leqslant_{s,\kappa} \lambda$. The lemma is proved. □

6.2.3 *The* KZ-*functor*

Now, let R be a local analytic deformation ring and set $q_R = \exp(-2\pi\sqrt{-1}/\kappa_R) \in R^\times$. Consider the KZ-functor $\mathrm{KZ}^s_{R,d} : \mathcal{O}^{s,\kappa}_R\{d\} \to \mathbf{H}^s_{R,d}\text{-mod}$.

Lemma 6.6 *Assume that* (3.1) *holds in* K. *Then* $\mathrm{Irr}(\mathbf{H}^s_{K,d}) = \{S(\lambda)^{s,q}_K;\ \lambda \in \mathscr{P}^\ell_d\}$, $\mathrm{Irr}(\mathcal{O}^{s,\kappa}_K\{d\}) = \{\Delta(\lambda)^{s,\kappa}_K;\ \lambda \in \mathscr{P}^\ell_d\}$ *and the bijection* $\mathrm{Irr}(\mathcal{O}^{s,\kappa}_K\{d\}) \xrightarrow{\sim} \mathrm{Irr}(\mathbf{H}^s_{K,d})$ *induced by* $\mathrm{KZ}^s_{K,d}$ *takes* $\Delta(\lambda)^{s,\kappa}_K$ *to* $S(\lambda)^{s,q}_K$.

Proof The first statement follows from the semi-simplicity of $\mathbf{H}^s_{K,d}$ and from [22, thm. 2.19]. The second one follows from Tits' deformation Theorem, because the modules $\mathrm{KZ}^s_{K,d}(\Delta(\lambda)^{s,\kappa}_K)$ and $S(\lambda)^{s,q}_K$ are both the generic point of a flat family of modules whose fiber at the special point is the $\mathbb{C}\Gamma_d$-module $\mathscr{X}(\lambda)_\mathbb{C}$, see [41, lem. 3.1] for details. □

6.2.4 *Ringel duality*

By [22, prop. 4.10], there is an equivalence of categories $\mathscr{R} : \mathcal{O}^{s^\star,\kappa}_R\{d\}^\Delta \xrightarrow{\sim} \big(\mathcal{O}^{s,\kappa}_R\{d\}^\Delta\big)^{\mathrm{op}}$ that restricts to an equivalence $\mathcal{O}^{s^\star,\kappa}_R\{d\}^{\mathrm{tilt}} \xrightarrow{\sim} \big(\mathcal{O}^{s,\kappa}_R\{d\}^{\mathrm{proj}}\big)^{\mathrm{op}}$. Hence, it induces an equivalence $\mathcal{O}^{s^\star,\kappa}_R\{d\}^\blacklozenge \xrightarrow{\sim} \mathcal{O}^{s,\kappa}_R\{d\}^{\mathrm{op}}$. We have $\mathscr{R}(\Delta(\lambda^\star)^{s^\star,\kappa}_R) \simeq \Delta(\lambda)^{s,\kappa}_R$. Consider the isomorphism of R-algebras

$$\iota : \mathbf{H}^s_{R,d} \xrightarrow{\sim} (\mathbf{H}^{s^\star}_{R,d})^{\mathrm{op}}, \quad T_i \mapsto -q_R T_i^{-1}, \quad X_j \mapsto X_j^{-1}.$$

It induces an equivalence

$$\mathscr{R}_{\mathbf{H}} = \iota^*(\bullet^\vee) : \mathbf{H}^{s^\star}_{R,d}\text{-mod} \cap R\text{-proj} \xrightarrow{\sim} (\mathbf{H}^{s}_{R,d}\text{-mod})^{\mathrm{op}} \cap R\text{-proj},$$

where $\bullet^\vee$ is the dual as an R-module.

By [22, §5.4.2], there is a commutative diagram

$$\begin{tikzcd} \mathcal{O}^{s^\star,\kappa}_R\{d\}^\Delta \arrow[rr, "\mathscr{R}", "\sim"'] \arrow[d, "\mathrm{KZ}^{s^\star}_{R,d}"'] & & \left(\mathcal{O}^{s,\kappa}_R\{d\}^\Delta\right)^{\mathrm{op}} \arrow[d, "\mathrm{KZ}^{s}_{R,d}"] \\ \mathbf{H}^{s^\star}_{R,d}\text{-mod} \cap R\text{-proj} \arrow[rr, "\mathscr{R}_{\mathbf{H}}", "\sim"'] & & (\mathbf{H}^{s}_{R,d}\text{-mod})^{\mathrm{op}} \cap R\text{-proj}\,. \end{tikzcd} \tag{6.3}$$

In particular, if $R = K$ is a field satisfying the condition (3.1), then Lemma 6.6 yields $K\mathscr{R}_{\mathbf{H}}(S(\lambda^\star)^{s^\star,q}_K) \simeq S(\lambda)^{s,q}_K$.

We will also consider the R-algebra isomorphisms

$$\mathrm{IM} : \mathbf{H}^{s}_{R,d} \xrightarrow{\sim} \mathbf{H}^{s^\star}_{R,d}, \quad T_i \mapsto -q_R T_i^{-1}, \quad X_j \mapsto X_j^{-1}$$

and

$$\sigma : (\mathbf{H}^{s}_{R,d})^{\mathrm{op}} \xrightarrow{\sim} \mathbf{H}^{s}_{R,d}, \quad T_i \mapsto T_i, \quad X_j \mapsto X_j.$$

Note that the composition $\mathrm{IM}^* \mathscr{R}_{\mathbf{H}}^{-1}$ is given by $\sigma^*(\bullet^\vee)$.

6.3 Proof of Varagnolo–Vasserot's conjecture

Let R be a local analytic deformation ring of dimension 2 in general position with residue field $\mathrm{k} = \mathbb{C}$. Fix $e, \ell, N \in \mathbb{N}^\times$. Fix $\kappa_R \in R^\times$ such that $\kappa_{\mathrm{k}} = -e$ and $\nu \in \mathscr{C}^\ell_{N,+}$. We set $s_{R,p} = \nu_p + \tau_{R,p}$, $q_R = \exp(-2\pi\sqrt{-1}/\kappa_R)$ and $Q_{R,p} = \exp(-2\pi\sqrt{-1}s_{R,p}/\kappa_R)$. We may abbreviate $\kappa = \kappa_{\mathrm{k}}$, $s_p = s_{\mathrm{k},p}$.

6.3.1 Small rank cases

As a preparation for the proof, we start by comparing the highest weight covers $\mathrm{KZ}^s_{R,d} : \mathcal{O}^{s,\kappa}_R\{d\} \to \mathbf{H}^s_{R,d}\text{-mod}$ and $\Psi^s_{R,d} : \mathbf{A}^{\nu,\kappa}_{R,\tau}\{d\} \to \mathbf{H}^s_{R,d}\text{-mod}$ for $d = 1$, 2.

First, assume that $d = 1$. Then $\Gamma_d = \Gamma$ is a cyclic group. The Hecke algebra associated with Γ is $\mathbf{H}^s_{R,1} = R[X_1]/\left(\prod_{p=1}^{\ell}(X_1 - Q_{R,p})\right)$.

Proposition 6.7 *We have* $\mathrm{KZ}^s_{R,1}(P(\lambda)^{s,\kappa}_R) \simeq \Psi^s_{R,1}(\mathbf{T}(\lambda)_{R,\tau})$ *for any* $\lambda \in \mathscr{P}^\ell_1$.

Proof For each $p \in [1,\ell]$ let $\lambda_p \in \mathscr{P}_1^\ell$ be the ℓ-partition with 1 on the p-th component and $\emptyset$ elsewhere. By Remark 5.18, Proposition 5.22(d) and Lemma 6.6, we have

$$\mathrm{KZ}^s_{K,1}(\Delta(\lambda_p)^{s,\kappa}_K) \simeq K[X_1]/(X_1 - Q_{K,p}) \simeq \Psi^s_{K,1}(\mathbf{\Delta}(\lambda_p)_{K,\tau}).$$

By Theorem 5.37 and Proposition 6.1 the functors $\mathrm{KZ}^s_{R,1}$ and $\Psi^s_{R,1}$ are 0-faithful cover of $\mathbf{H}^s_{R,1}$-mod with opposite orders. Therefore, Corollary 2.22 shows that $\mathrm{KZ}^s_{R,1}(P(\lambda)^{s,\kappa}_R) \simeq \Psi^s_{R,1}(\mathbf{T}(\lambda)_{R,\tau})$ for any $\lambda \in \mathscr{P}_1^\ell$. □

Now, assume that $d = 2$. Recall the Hecke algebra $\mathbf{H}^+_{R,2} = R[T_1]/(T_1 + 1)(T_1 - q_R)$ associated with the group $\mathfrak{S}_2$. Write $\lambda_+ = (2)$ and $\lambda_- = (1^2)$ in $\mathscr{P}_2^1$. The category $\mathscr{O}^\kappa_R(\mathfrak{S}_2)$ is a special case of $\mathscr{O}^{s,\kappa}_R\{1\}$ with $\ell = 2$. The proof of Proposition 6.7 yields

$$\mathrm{KZ}^+_{R,2}(P(\lambda_\sharp)^\kappa_R) \simeq \Psi^+_{R,2}(\mathbf{T}(\lambda_\sharp)_{R,\tau}), \quad \sharp = +, -. \tag{6.4}$$

Consider the induction functor $\mathrm{Ind}^{2,s}_{2,+} : \mathbf{H}^+_{R,2}\text{-mod} \to \mathbf{H}^s_{R,2}\text{-mod}$.

Proposition 6.8 *Assume $\nu_p \geqslant 2$ for all p. For $\sharp = +, -$, there exists a tilting object $\mathbf{T}_\sharp \in \mathbf{A}^{\nu,\kappa}_{R,\tau}\{2\}$ such that $\Psi^s_{R,2}(\mathbf{T}_\sharp) \simeq \mathrm{Ind}^{2,s}_{2,+}(\Psi^+_{R,2}(\mathbf{T}(\lambda_\sharp)_{R,\tau})$.*

Proof By Theorem 5.37(a), the module $\Psi^s_{R,2}(\mathbf{T}_{R,2})$ is the regular representation of $\mathbf{H}^s_{R,2}$. Write $\mathbf{T}^+_{R,2} = \mathbf{V}_R^{\dot\otimes 2}$. We have $\Psi^+_{R,2}(\mathbf{T}^+_{R,2}) \simeq \mathbf{H}^+_{R,2}$. Thus, there is an isomorphism of $\mathbf{H}^s_{R,2}$-modules

$$\Psi^s_{R,2}(\mathbf{T}_{R,2}) \simeq \mathrm{Ind}^{2,s}_{2,+}(\Psi^+_{R,2}(\mathbf{T}^+_{R,2})). \tag{6.5}$$

If $e > 2$ then $\kappa_\Bbbk \neq -2$, hence $\mathbf{H}^+_{\Bbbk,2}$ is semi-simple and $\mathbf{T}^+_{R,2} \simeq \mathbf{T}(\lambda_+)_{R,\tau} \bigoplus \mathbf{T}(\lambda_-)_{R,\tau}$. Since $\Psi^s_{R,2}$ is 0-faithful, it maps indecomposable factors of $\mathbf{T}_{R,2}$ to indecomposable $\mathbf{H}^s_{R,2}$-modules. So, the proposition follows from (6.5) and the Krull–Schmidt theorem.

Now, assume that $e = 2$, then $q_\Bbbk = -1$. The indecomposable tilting modules in $\mathbf{A}^{+,\kappa}_{R,\tau}\{2\}$ are $\mathbf{T}(\lambda_+)_R = \mathbf{T}^+_{R,2}$ and $\mathbf{T}(\lambda_-)_R = \mathbf{\Delta}(\lambda_-)_R$. We need to prove the proposition for $\mathbf{T}(\lambda_-)_R$. We have $\Psi^+_{R,2}(\mathbf{T}(\lambda_-)_R) \simeq R[T_1]/(T_1+1)$. Consider the action of $\mathbf{H}^+_{R,2}$ on $\mathbf{T}^+_{R,2}$. Then $\mathbf{T}(\lambda_-)_R$ is the image of $T_1 - q_R$ acting on $\mathbf{T}^+_{R,2}$. Since the functor $\mathbf{T}_{R,0}\dot\otimes_R \bullet$ is exact, we deduce that $\mathbf{T}_{R,0}\dot\otimes_R\mathbf{T}(\lambda_-)_R$ is the image of $T_1 - q_R$ acting on $\mathbf{T}_{R,2}$. By consequence, $\Psi^s_{R,2}\big(\mathbf{T}_{R,0}\dot\otimes_R\mathbf{T}(\lambda_-)_R\big)$ is the image of the right multiplication by $T_1 - q_R$ on $\Psi^s_{R,2}(\mathbf{T}_{R,2}) = \mathbf{H}^s_{R,2}$. Therefore, we have

$$\Psi^s_{R,2}\big(\mathbf{T}_{R,0}\,\dot\otimes_R\mathbf{T}(\lambda_-)_R\big) \simeq \mathrm{Ind}^{2,s}_{2,+}\,\big(\Psi^+_{R,2}(\mathbf{T}(\lambda_-)_R)\big). \tag{6.6}$$

We claim that $\mathbf{T}_{R,0}\dot{\otimes}_R\mathbf{T}(\lambda_-)_R$ is tilting in $\mathbf{A}^{\nu,\kappa}_{R,\tau}\{2\}$. Indeed, by Proposition 8.30, the specialization map $\mathrm{End}_{\mathbf{A}^{\nu,\kappa}_{R,\tau}}(\mathbf{T}_{R,2}) \to \mathrm{End}_{\mathbf{A}^{\nu,\kappa}_{\Bbbk,\tau}}(\mathbf{T}_{\Bbbk,2})$ takes $T_1 - q_R$ to $T_1 - q_\Bbbk$. Since $\mathbf{T}_{R,0}\dot{\otimes}_R\mathbf{T}(\lambda_-)_R$ is free over R by Lemma 8.27 and since it is the image of the operator $T_1 - q_R : \mathbf{T}_{R,2} \to \mathbf{T}_{R,2}$, the image of $T_1 - q_\Bbbk : \mathbf{T}_{\Bbbk,2} \to \mathbf{T}_{\Bbbk,2}$ is $\Bbbk(\mathbf{T}_{R,0}\dot{\otimes}_R\mathbf{T}(\lambda_-)_R)$. The same argument as above implies that $\mathbf{T}_{\Bbbk,0}\dot{\otimes}_\Bbbk\mathbf{T}(\lambda_-)_\Bbbk$ is also the image of the operator $T_1 - q_\Bbbk : \mathbf{T}_{\Bbbk,2} \to \mathbf{T}_{\Bbbk,2}$. We deduce that there is an isomorphism $\Bbbk(\mathbf{T}_{R,0}\dot{\otimes}_R\mathbf{T}(\lambda_-)_R) \simeq \mathbf{T}_{\Bbbk,0}\dot{\otimes}_\Bbbk\mathbf{T}(\lambda_-)_\Bbbk$. Since $\mathbf{T}_{\Bbbk,0}\dot{\otimes}_\Bbbk\mathbf{T}(\lambda_-)_\Bbbk$ is tilting by Proposition 8.11, the claim follows from Proposition 2.4(c). The proposition is proved. □

6.3.2 *Proof of the main theorem*

We can now prove conjecture [46, conj. 8.8].

Theorem 6.9 *Assume that $\nu_p \geqslant d$ for each p. Then, we have an equivalence of highest weight categories $\Upsilon^{\nu,-e}_d : \mathbf{A}^{\nu,-e}\{d\} \xrightarrow{\sim} \mathcal{O}^{\nu^\star,-e}\{d\}$ such that $\Upsilon^{\nu,-e}_d(\mathbf{\Delta}(\lambda)) \simeq \Delta(\lambda^\star)^{\nu^\star,-e}$ and $\Psi^\nu_d \simeq \mathrm{IM}^*\,\mathrm{KZ}^{\nu^\star}_d\,\Upsilon^{\nu,-e}_d$.*

Proof Let R be a local analytic deformation ring of dimension 2 in general position with residue field $\Bbbk = \mathbb{C}$. Assume that $\kappa_\Bbbk = -e$. Set $s_{R,p} = \nu_p + \tau_{R,p}$.

Let $\mathscr{C} = \mathcal{O}^{s^\star,\kappa}_R\{d\}$ and $\mathscr{C}' = \mathbf{A}^{\nu,\kappa}_{R,\tau}\{d\}$. We consider the highest weight covers $F = \mathrm{IM}^*\,\mathrm{KZ}^{s^\star}_{R,d} : \mathscr{C} \to \mathbf{H}^s_{R,d}$-mod and $F' = \Psi^s_{R,d} : \mathscr{C}' \to \mathbf{H}^s_{R,d}$-mod. We claim that they satisfy the conditions in Proposition 2.20, so the theorem holds. Let us check these conditions.

First, $\mathbf{H}^s_{R,d}$ is symmetric. Since R is in general position, the condition (3.1) holds in K, hence $\mathbf{H}^s_{K,d}$ is semi-simple.

We have $KF(\Delta(\lambda^\star)^{s^\star,\kappa}_K) = S(\lambda)^{s,q}_K$ by Lemma 6.6 and Sect. 6.2.4, and we have and $KF'(\mathbf{\Delta}(\lambda)_{K,\tau}) = S(\lambda)^{s,q}_K$ by Proposition 5.22. So the order on $\mathrm{Irr}(\mathbf{H}^s_{K,d})$ induced by $(\mathscr{C}, F)$ refines the order induced by $(\mathscr{C}', F')$ by Lemma 6.5.

Since IM^* is an equivalence, by Proposition 6.1 the functor F is fully faithful on $\mathscr{C}^\Delta$. Hence it is also fully faithful on $\mathscr{C}^\nabla$, by (6.3) and [22, §4.2.1]. Theorem 5.37(c) gives the fully faithfulness of F' on $\mathscr{C}'^\Delta$ and $\mathscr{C}'^\nabla$.

It remains to check that $F(T(\lambda)^{s^\star,\kappa}_R) \in F'(\mathcal{C}'^{\mathrm{tilt}})$ for all $\lambda \in \mathscr{P}^\ell_d$ such that $\mathrm{lcd}_{\Bbbk\mathscr{C}^\diamond}(L^\diamond(\lambda)) \leqslant 1$ or $\mathrm{rcd}_{\Bbbk\mathscr{C}^\diamond}(L^\diamond(\lambda)) \leqslant 1$. Recall from Sect. 6.2.4 that $\mathscr{C}^\diamond \simeq \mathscr{C}^\blacklozenge \simeq \mathcal{O}^{s,\kappa}_R\{d\}^{\mathrm{op}}$ and $L^\diamond(\lambda)$ corresponds to $L(\lambda^\star)^{s,\kappa}$. By Lemma 6.2, we have

$$\mathrm{rcd}_{\mathscr{C}^\diamond}(L^\diamond(\lambda)) = \mathrm{lcd}_{\mathcal{O}^{s,\kappa}_k\{d\}}(L(\lambda^\star)^{s,\kappa}) = \mathrm{rcd}_{\mathcal{O}^{s,\kappa}_k\{d\}}(L(\lambda^\star)^{s,\kappa}) = \mathrm{lcd}_{\mathscr{C}^\diamond}(L^\diamond(\lambda)).$$

We have

$$F(T(\lambda)_R^{s^\star,\kappa}) \simeq \mathrm{IM}^* \, \mathrm{KZ}_{R,d}^{s^\star}(\mathscr{R}^{-1}(P(\lambda)_R^{s,\kappa})) \simeq \sigma^*(\mathrm{KZ}_{R,d}^{s}(P(\lambda)_R^{s,\kappa})^\vee).$$

By Lemma 6.3 and the Krull–Schmidt theorem, it is enough to prove that for any reflection hyperplane H of Γ_d and any $P \in \mathcal{O}(W_H)_R^{\mathrm{proj}}$, we have

$$\sigma^*(\mathrm{KZ}_{R,d}^{s}({}^{\mathcal{O}}\mathrm{Ind}_{W_H}^{\Gamma_d}(P))^\vee) \in F'(\mathscr{C}'^{\mathrm{tilt}}).$$

Since $\sigma^*(\bullet^\vee)$ commutes with induction functors and fixes isomorphism classes of R-free $\mathbf{H}_{R,1}^s$-modules and $\mathbf{H}_{R,2}^+$-modules, we deduce from (6.1) that

$$\sigma^*(\mathrm{KZ}_{R,d}^{s}({}^{\mathcal{O}}\mathrm{Ind}_{W_H}^{\Gamma_d}(P))^\vee) \simeq \mathrm{KZ}_{R,d}^{s}({}^{\mathcal{O}}\mathrm{Ind}_{W_H}^{\Gamma_d}(P)).$$

There are two possibilities for H:

- either H is conjugate to $\ker(\gamma_i - 1)$ for some $i \in [1, d]$. Then $W_H \simeq \Gamma$. We identify $\mathcal{O}(W_H)_R \simeq \mathcal{O}_R^{s,\kappa}\{1\}$ and ${}^{\mathcal{O}}\mathrm{Ind}_{W_H}^{\Gamma_d} \simeq {}^{\mathcal{O}}\mathrm{Ind}_{\Gamma_1}^{\Gamma_d}$. By Proposition 6.7, for any projective $P \in \mathcal{O}_R^{s,\kappa}\{1\}$, there exists $\mathbf{T} \in \mathbf{A}_{R,\tau}^{\nu,\kappa}\{1\}^{\mathrm{tilt}}$ such that $\mathrm{KZ}_{R,1}^s(P) \simeq \Psi_{R,1}^s(\mathbf{T})$. By (6.1), we have $\mathrm{KZ}_{R,d}^s \; {}^{\mathcal{O}}\mathrm{Ind}_{W_H}^{\Gamma_d} \simeq \mathrm{Ind}_1^d \; \mathrm{KZ}_{R,1}^s$. Using Lemma 5.41, this yields $\mathrm{KZ}_{R,d}^s({}^{\mathcal{O}}\mathrm{Ind}_{W_H}^{\Gamma_d}(P)) \simeq \mathrm{Ind}_1^d(\Psi_{R,1}^s(\mathbf{T})) \simeq \Psi_{R,d}^s(F^{d-1}(\mathbf{T}))$. The module $F^{d-1}(\mathbf{T})$ is tilting by Proposition 8.29(a), so $\mathrm{KZ}_{R,d}^s({}^{\mathcal{O}}\mathrm{Ind}_{W_H}^{\Gamma_d}(P)) \in F'(\mathcal{C}'^{\mathrm{tilt}})$;
- or H is conjugate to $\ker(s_{ij}^\gamma - 1)$ for some $\gamma \in \Gamma$ and $i \neq j$. Then $W_H \simeq \mathfrak{S}_2$ and $\mathcal{O}(W_H)_R \simeq \mathcal{O}_R^\kappa(\mathfrak{S}_2)$. By (6.1), we have $\mathrm{KZ}_{R,d}^s({}^{\mathcal{O}}\mathrm{Ind}_{W_H}^{\Gamma_d}(P)) \simeq \mathrm{Ind}_{2,+}^{d,s}(\mathrm{KZ}(P))$. By (6.4) and Proposition 6.8 there exists $\mathbf{T} \in \mathbf{A}_{R,\tau}^{\nu,\kappa}\{2\}^{\mathrm{tilt}}$ such that $\Psi_{R,2}^s(\mathbf{T}) \simeq \mathrm{Ind}_{2,+}^{2,s}(\mathrm{KZ}(P))$. Using Lemma 5.41, this yields

$$\mathrm{Ind}_{2,+}^{d,s}(\mathrm{KZ}(P)) \simeq \mathrm{Ind}_2^d(\Psi_{R,2}^s(\mathbf{T})) \simeq \Psi_{R,d}^s(F^{d-2}(\mathbf{T})).$$

  Since $F^{d-2}(\mathbf{T})$ is tilting, we have $\mathrm{KZ}_{R,d}^s({}^{\mathcal{O}}\mathrm{Ind}_{W_H}^{\Gamma_d}(P)) \in F'(\mathscr{C}'^{\mathrm{tilt}})$.

We have checked that $(\mathscr{C}, F)$, $(\mathscr{C}', F')$ satisfy all the conditions in Proposition 2.20, the theorem is proved. □

Remark 6.10 In [46, (8.2)] the parameters of the CRDAHA are chosen in a different way. More precisely, the symbol h_p in [46] corresponds to our parameter $h_p - h_{p-1}$. Further, the parameters (h, h_p) are specialized to $(-1/e, s_{p+1}/e - p/\ell)$ in [46] instead of $(1/e, s_{p+1}/e - p/\ell)$ as above.

6.3.3 Proof of the main theorem for irrational levels

Let $\kappa \in \mathbb{C} \setminus \mathbb{Q}$. We will prove the following result, which was conjectured in [46, rem. 8.10(*b*)], as a degenerate analogue of [46, conj. 8.8]. If ν is dominant, a proof was given in [24, thm. 6.9.1].

Theorem 6.11 *Assume that $\kappa \in \mathbb{C} \setminus \mathbb{Q}$ and that $\nu_p \geqslant d$ for each p. Then, we have an equivalence of highest weight categories $\Upsilon_d^{\nu,\kappa} : A^\nu\{d\} \xrightarrow{\sim} \mathcal{O}^{\nu^\star,\kappa}\{d\}$ such that $\Upsilon_d^{\nu,\kappa}(\Delta(\lambda)) \simeq \Delta(\lambda^\star)^{\nu^\star,\kappa}$ and $\Phi_d^\nu \simeq \mathrm{IM}^*\ \mathrm{KZ}_d^{\nu^\star}\ \Upsilon_d^{\nu,\kappa}$.*

Let R be the completion at $(\kappa, 0, \dots, 0)$ of the ring of polynomials on $\mathbb{C}^{\ell+1}$. It is a local deformation ring such that $\kappa_R, \tau_{R,1}, \dots, \tau_{R,\ell}$ are the standard coordinates. The residue field is $\mathrm{k} = \mathbb{C}$ and we have $\kappa_\mathrm{k} = \kappa$, $\tau_{\mathrm{k},p} = 0$. Further, for each u, v and each $\mathfrak{p} \in \mathfrak{P}$, we have $\tau_{\mathrm{k}_\mathfrak{p},u} - \tau_{\mathrm{k}_\mathfrak{p},v} \notin \mathbb{Z}^\times$. We set $s_{R,p} = \nu_p + \tau_{R,p}$. Now, we consider the functor $\Phi_{R,d}^s : A_{R,\tau}^\nu\{d\} \to H_{R,d}^s$-mod given in Sect. 4.6.

Lemma 6.12 *The functor $\Phi_{R,d}^s$ is a highest weight cover. It is fully faithful on $(A_{R,\tau}^\nu\{d\})^\Delta$ and $(A_{R,\tau}^\nu\{d\})^\nabla$.*

The proof is by reduction to codimension one, and is very similar to the proof of Theorem 5.37. We will be sketchy.

We say that a prime ideal $\mathfrak{p} \in \mathfrak{P}$ is *generic* if $\tau_{\mathrm{k}_\mathfrak{p},u} \neq \tau_{\mathrm{k}_\mathfrak{p},v}$ for each $u \neq v$, and that it is *subgeneric* if there is a unique pair $u \neq v$ such that $\tau_{\mathrm{k}_\mathfrak{p},u} = \tau_{\mathrm{k}_\mathfrak{p},v}$.

Claim 6.13 *For each $\mathfrak{p} \in \mathfrak{P}_1$ the following hold.*

(a) *$\mathfrak{p}$ is either generic or subgeneric,*
(b) *if $\mathfrak{p}$ is generic, then there is an equivalence of highest weight categories $\mathcal{Q}_{\mathrm{k}_\mathfrak{p}} : \mathcal{O}_{\mathrm{k}_\mathfrak{p},\tau}^\nu\{a\} \to \mathcal{O}_{\mathrm{k}_\mathfrak{p}}^+(\nu)\{a\}$,*
(c) *if $\mathfrak{p}$ is subgeneric with $\tau_{\mathrm{k}_\mathfrak{p},u} = \tau_{\mathrm{k}_\mathfrak{p},v}$ and $u \neq v$, then there is an equivalence of highest weight categories $\mathcal{Q}_{\mathrm{k}_\mathfrak{p}} : \mathcal{O}_{\mathrm{k}_\mathfrak{p},\tau}^\nu\{a\} \to \mathcal{O}_{\mathrm{k}_\mathfrak{p}}^\nu(\nu, u, v)\{a\}$.*

Proof Part (a) is easy. Parts (b), (c) are proved as in Propositions 5.24, 5.31, using [20, thm. 11]. The details are left to the reader. □

Proof of Lemma 6.12 Now, the module $\mathcal{Q}_{\mathrm{k}_\mathfrak{p}}(T_{\mathrm{k}_\mathfrak{p},d})$ can be identified explicitly, using the same argument as in the proof of Lemmas 5.26, 5.32. Indeed, it is enough to check that $\mathcal{Q}_{\mathrm{k}_\mathfrak{p}}$ takes a parabolic Verma module to a parabolic Verma module with the same highest weight and that the induced linear map $[\mathcal{O}_{\mathrm{k}_\mathfrak{p},\tau}^\nu] \to [\mathcal{O}_{\mathrm{k}_\mathfrak{p}}^\nu(\nu, u, v)]$ commutes with the linear operators induced by the categorification functors e, f.

Using the same argument as in the proof of Theorem 5.37, we only need to prove the lemma for $\Phi_{\mathrm{k}_\mathfrak{p},d}^s$ and $\mathfrak{p} \in \mathfrak{P}_1$. Hence, by Claim 6.13, we are reduced to the case $\ell = 1$ or 2. If $\ell = 1$ everything is obvious, because the category $\mathscr{O}^+(\nu)$ is semi-simple. If $\ell = 2$, we may assume $\tau_{\mathrm{k_p}} = 0$, the result follows from Proposition 4.7 and the last paragraph of the proof of Proposition 5.27.

Proof of Theorem 6.11 Consider the highest weight cover $\mathrm{IM}^*\ \mathrm{KZ}^{s^\star}_{R,d}$: $\mathcal{O}^{s^\star,\kappa}_R\{d\} \to \mathbf{H}^s_{R,d}$-mod. Since the R-algebras $\mathbf{H}^s_{R,d}$ and $H^s_{R,d}$ are isomorphic by Proposition 3.1, we can regard $\mathrm{IM}^*\ \mathrm{KZ}^{s^\star}_{R,d}$ and $\Phi^s_{R,d}$ as highest weight covers of the category $\mathbf{H}^s_{R,d}$-mod . We claim that they satisfy the conditions in Proposition 2.20, so the theorem follows. Let us check the conditions.

First, $\mathbf{H}^s_{R,d}$ is Frobenius, and $\mathbf{H}^s_{R,d}$ is semi-simple because (3.1) holds obviously in K. The compatibilities of orders is again given by Lemma 6.5.

Since IM^* is an equivalence, $\mathrm{IM}^*\ \mathrm{KZ}^{s^\star}_{R,d}$ is fully faithful on Δ- and ∇-filtered objects by Proposition 6.1 and (6.3). The corresponding property for $\Phi^s_{R,d}$ follows from Lemma 6.12.

It remains to check that $\mathrm{IM}^*\ \mathrm{KZ}^{s^\star}_{R,d}(T(\lambda)^{s^\star,\kappa}_R) \in \Phi^s_{R,d}((A^\nu_{R,\tau}\{d\})^{\mathrm{tilt}})$ for all $\lambda \in \mathscr{P}^\ell_d$ such that $\mathrm{lcd}_{\mathrm{k}\mathcal{O}^{s^\star,\kappa}_R\{d\}^\diamond}(L^\diamond(\lambda)) \leqslant 1$ or $\mathrm{rcd}_{\mathrm{k}\mathcal{O}^{s^\star,\kappa}_R\{d\}^\diamond}(L^\diamond(\lambda)) \leqslant 1$. The proof is the same as in Theorem 6.9. Details are left to the reader. □

7 Consequences of the main theorem

7.1 Reminder on the Fock space

Let $R, q_R, \mathscr{I} = \mathscr{I}(q)$ and $Q_{R,1}, Q_{R,2}, \dots, Q_{R,\ell}$ be as in Sect. 3.1. Consider the dominant weight in $P = P_{\mathscr{I}}$ given by $\Lambda^Q = \sum_{p=1}^{\ell} \Lambda_{Q_p}$. Note that $\Lambda^Q = \sum_{p\in\Omega} \Lambda_p$, with $\Lambda_p = \sum_{u;\ Q_u \equiv Q_p} \Lambda_{Q_u}$. Let $s = (s_1, \dots, s_\ell)$ be as in Sect. 3.3. Then, we may write $\Lambda^s = \Lambda^Q$.

The *Fock space of multi-charge s* is the vector space $\mathbf{F}(\Lambda^s) = \bigoplus_{\lambda\in\mathscr{P}^\ell} \mathbb{C}\ |\lambda, s\rangle$. We will abbreviate $\Lambda = \Lambda^s$. We will call $\{|\lambda, s\rangle;\ \lambda \in \mathscr{P}^\ell\}$ the *standard monomial basis* of $\mathbf{F}(\Lambda)$.

There is an integrable representation of $\mathfrak{sl}_{\mathscr{I}}$ on $\mathbf{F}(\Lambda)$ given by

$$F_i(|\lambda, s\rangle) = \sum_{q\text{-res}^s(\mu-\lambda)=i} |\mu, s\rangle, \quad E_i(|\lambda, s\rangle) = \sum_{q\text{-res}^s(\lambda-\mu)=i} |\mu, s\rangle. \quad (7.1)$$

Let $n_i(\lambda)$ be the number of boxes of residue i in λ. To avoid any confusion we may write $n^s_i(\lambda) = n^Q_i(\lambda) = n_i(\lambda)$. Each basis vector $|\lambda, s\rangle$ is a weight vector of weight $\mathbf{wt}(|\lambda, s\rangle) = \Lambda - \sum_{i\in\mathscr{I}} n_i(\lambda)\,\alpha_i$.

The Λ-weight space of $\mathbf{F}(\Lambda)$ has dimension one and is spanned by the element $|\emptyset, s\rangle$. The $\mathfrak{sl}_{\mathscr{I}}$-submodule $\mathbf{L}(\Lambda) \subset \mathbf{F}(\Lambda)$ generated by $|\emptyset, s\rangle$ is the simple module of highest weight Λ. It decomposes as the tensor product

$\mathbf{L}(\Lambda) = \bigotimes_{p\in\Omega} \mathbf{L}(\Lambda_p)$, where $\mathbf{L}(\Lambda_p)$ is the simple $\mathfrak{sl}_{\mathscr{I}_p}$-module of highest weight Λ_p.

Remark 7.1 Assume that the quiver $\mathscr{I}(q)$ is the disjoint union of ℓ components of type A_∞. Then, we have $\mathbf{F}(\Lambda) = \mathbf{L}(\Lambda) = \bigotimes_{p=1}^{\ell} \mathbf{L}(\Lambda_p)$.

Remark 7.2 The weight $\mathrm{wt}(\lambda)$ associated with the element $\lambda \in P^\nu + \tau$ should not be confused with the weight $\mathbf{wt}(|\lambda, s\rangle)$ above, which is associated with the ℓ-partition $\lambda \in \mathscr{P}^\ell$. The former has the level 0 while the latter has the level ℓ. We have $\mathrm{wt}(\varpi(\lambda)) = \mathbf{wt}(|\lambda, s\rangle) - \sum_{p=1}^{\ell} \Lambda_{\tau_p} \bmod \mathbb{Z}\delta$.

Indeed, the equation above holds for $\lambda = \emptyset$. Thus, it is proved by induction using the following equivalences for $\lambda, \mu \in \mathscr{P}^\nu$, see Sect. 7.1,

$$\begin{aligned}\varpi(\lambda) \xrightarrow{i} \varpi(\mu) &\Leftrightarrow q_K\text{-res}^s(\mu - \lambda) = q_K^i,\\ &\Rightarrow \mathrm{wt}(\varpi(\lambda)) - \mathrm{wt}(\varpi(\mu)) = \mathbf{wt}(|\lambda, s\rangle) - \mathbf{wt}(|\mu, s\rangle) = \alpha_i.\end{aligned}$$

7.2 Rouquier's conjecture

Let $K = \mathbb{C}$. Fix integers $e, \ell \geqslant 1$ and fix $s = (s_1, \dots, s_\ell) \in \mathbb{Z}^\ell$. Set $\Lambda = \Lambda^s$. Set $I = \mathbb{Z}$ and $\mathscr{I} = I/e\mathbb{Z}$. So, we have $\mathfrak{sl}_{\mathscr{I}} = \widehat{\mathfrak{sl}}_e$ and the Fock space $\mathbf{F}(\Lambda)$ is an integrable $\widehat{\mathfrak{sl}}_e$-module. Consider the *Uglov's canonical* bases $\{\mathcal{G}^\pm(\lambda, s);\ \lambda \in \mathscr{P}^\ell\}$ of $\mathbf{F}(\Lambda)$ introduced in [45, sec. 4.4].

Set $\mathcal{O}^{s^\star,-e} = \bigoplus_{d\in\mathbb{N}} \mathcal{O}^{s^\star,-e}\{d\}$. We identify the complexified Grothendieck group $[\mathcal{O}^{s^\star,-e}]$ with $\mathbf{F}(\Lambda)$ via the linear map $\theta : [\mathcal{O}^{s^\star,-e}] \xrightarrow{\sim} \mathbf{F}(\Lambda)$ such that $[\Delta(\lambda^\star)^{s^\star,-e}] \mapsto |\lambda, s\rangle$.

Since the category $\mathcal{O}^{s^\star,-e}$ is preserved under the substitution $s \mapsto (1 + s_1, 1 + s_2, \dots, 1 + s_\ell)$ we may assume that $s_p = \nu_p \geqslant d$ for each p. Set $\mathbf{A}^{\nu,-e} = \bigoplus_{d\in\mathbb{N}} \mathbf{A}^{\nu,-e}\{d\}$.

The following result has been conjectured by Rouquier [39, sec. 6.5].

Theorem 7.3 *We have* $\theta([T(\lambda^\star)^{s^\star,-e}]) = \mathcal{G}^+(\lambda, s)$ *and* $\theta([L(\lambda^\star)^{s^\star,-e}]) = \mathcal{G}^-(\lambda, s)$.

Proof Let $c^\pm_{\lambda,\mu}(s) \in \mathbb{Z}$ be such that $\mathcal{G}^\pm(\lambda, s) = \sum_\mu c^\pm_{\lambda,\mu}(s)\, |\mu, s\rangle$.

Let $\mathbf{F}(\Lambda)\{d\} \subset \mathbf{F}(\Lambda)$ be the subspace spanned by the set $\{|\lambda, s\rangle;\ \lambda \in \mathscr{P}^\ell_d\}$. Assume that $\nu_p \geqslant d$ for each p. We identify the complexified Grothendieck group $[\mathbf{A}^{\nu,-e}\{d\}]$ with $\mathbf{F}(\Lambda)\{d\}$ via the linear map such that $[\boldsymbol{\Delta}(\lambda)] \mapsto |\lambda, s\rangle$.

Let $\mathbf{L}(\lambda)$ be the top of $\boldsymbol{\Delta}(\lambda)$ in $\mathbf{A}^{\nu,-e}\{d\}$. By [46, prop. 8.2], we have $[\mathbf{L}(\lambda)] = \sum_\mu c^-_{\lambda,\mu}(s)\, [\boldsymbol{\Delta}(\mu)]$ in $[\mathbf{A}^{\nu,-e}\{d\}]$. Therefore, the isomorphism $[\mathbf{A}^{\nu,-e}\{d\}] \xrightarrow{\sim} \mathbf{F}(\Lambda)\{d\}$ maps $[\mathbf{L}(\lambda)]$ to $\mathcal{G}^-(\lambda, s)$.

Since the equivalence of categories $\mathbf{A}^{\nu,-e}\{d\} \xrightarrow{\sim} \mathcal{O}^{s^\star,-e}\{d\}$ in Theorem 6.9 maps $\mathbf{L}(\lambda)$ to $L(\lambda^\star)^{s^\star,-e}$ and since the isomorphism $[\mathcal{O}^{s^\star,-e}\{d\}] \xrightarrow{\sim} \mathbf{F}(\Lambda)\{d\}$ is the composition of the map $[\mathcal{O}^{s^\star,-e}\{d\}] \xrightarrow{\sim} [\mathbf{A}^{\nu,-e}\{d\}]$ induced by the inverse of the equivalence with the isomorphism $[\mathbf{A}^{\nu,-e}\{d\}] \xrightarrow{\sim} \mathbf{F}(\Lambda)\{d\}$ above, we deduce that the map $[\mathcal{O}^{s^\star,-e}\{d\}] \xrightarrow{\sim} \mathbf{F}(\Lambda)\{d\}$ takes $[L(\lambda^\star)^{s^\star,-e}]$ to $\mathcal{G}^-(\lambda, s)$.

Next, let $\mathbf{P}(\lambda)$ be the projective cover of $\mathbf{\Delta}(\lambda)$ in $\mathbf{A}^{\nu,-e}\{d\}$. By the Brauer reciprocity we have $(\mathbf{P}(\lambda) : \mathbf{\Delta}(\mu)) = [\mathbf{\Delta}(\mu) : \mathbf{L}(\lambda)]$. Therefore we have $[\mathbf{P}(\lambda)] = \sum_\mu d^-_{\lambda,\mu}(s)[\mathbf{\Delta}(\mu)]$ in $[\mathbf{A}^{\nu,-e}\{d\}]$, where the matrix $\big(d^-_{\lambda,\mu}(s)\big)$ is the transpose of the inverse matrix of $\big(c^-_{\lambda,\mu}(s)\big)$.

By [45, thm. 5.15], we have $d^-_{\lambda,\mu}(s) = c^+_{\lambda^\star,\mu^\star}(s^\star)$. Using the equivalence of categories $\mathbf{A}^{\nu,-e}\{d\} \xrightarrow{\sim} \mathcal{O}^{s^\star,-e}\{d\}$, we get $[P(\lambda^\star)^{s^\star,-e}] = \sum_\mu c^+_{\lambda^\star,\mu^\star}(s^\star)[\Delta(\mu^\star)^{s^\star,-e}]$ in $[\mathcal{O}^{s^\star,-e}\{d\}]$. By removing $\star$ everywhere, we get the following equality in $[\mathcal{O}^{s,-e}\{d\}]$

$$[P(\lambda)^{s,-e}] = \sum_\mu c^+_{\lambda,\mu}(s)[\Delta(\mu)^{s,-e}]. \tag{7.2}$$

Next, by Sect. 6.2.4 we have the equivalence $\mathscr{R} : \mathcal{O}^{s^\star,-e,\Delta}\{d\} \xrightarrow{\sim} \mathcal{O}^{s,-e,\Delta}\{d\}^{\mathrm{op}}$ such that $\Delta(\lambda^\star)^{s^\star,-e} \mapsto \Delta(\lambda^{s,-e})$ and $T(\lambda^\star)^{s^\star,-e} \mapsto P(\lambda)^{s,-e}$. The inverse of $\mathscr{R}$ yields an isomorphism of Grothendieck groups $[\mathcal{O}^{s,-e}\{d\}] \xrightarrow{\sim} [\mathcal{O}^{s^\star,-e}\{d\}]$ such that $[\Delta(\lambda)^{s,-e}] \mapsto [\Delta(\lambda^\star)^{s^\star,-e}]$ and $[P(\lambda)^{s,-e}] \mapsto [T(\lambda^\star)^{s^\star,-e}]$. The image of the equality (7.2) under this isomorphism gives the identity $[T(\lambda^\star)^{s^\star,-e}] = \sum_\mu c^+_{\lambda,\mu}(s)[\Delta(\mu^\star)^{s^\star,-e}]$ in $[\mathcal{O}^{s^\star,-e}\{d\}]$. We deduce that the isomorphism $[\mathcal{O}^{s^\star,-e}\{d\}] \xrightarrow{\sim} \mathbf{F}(\Lambda)\{d\}$ maps the element $[T(\lambda^\star)^{s^\star,-e}]$ to $\sum_\mu c^+_{\lambda,\mu}(s)|\mu, s\rangle = \mathcal{G}^+(\lambda, s)$. We are done. □

7.3 The category $\mathcal{O}$ of CRDAHA's is Koszul

Recall that $\mathscr{I} \simeq [0, e)$ and that Λ_i, α_i are the fundamental weights and the simple roots of $\widehat{\mathfrak{sl}}_e$. For $t = (t_1, \ldots, t_e) \in \mathbb{Z}^e$ let $\mathcal{O}^s_t \subset \mathcal{O}^{s,-e}$ be the Serre subcategory generated by the modules $\Delta(\lambda)^{s,-e}$ such that the following condition holds

$$\Lambda^s - \sum_{i=1}^{e-1}(n^s_i(\lambda)-n^s_0(\lambda))\,\alpha_i = \sum_{i=1}^{e-1}(t_i - t_{i+1})\Lambda_i + (\ell + t_e - t_1)\Lambda_0. \tag{7.3}$$

Set $|s| = s_1 + \cdots + s_\ell$ and $|t| = t_1 + \cdots + t_e$. From (7.3) we get that $|t| = |s|$ modulo $\mathbb{Z}e$. Hence, up to translating the t_i's simultaneously by the same integer, we may assume that $t \in \mathbb{Z}^e(|s|)$. Note that the left hand side of (7.3) is equal to $\mathbf{wt}(|\lambda, s\rangle)$ modulo $\mathbb{Z}\delta$.

Since the category $\mathcal{O}^s_t$ is preserved under the substitutions $s \mapsto (1+s_1, 1+s_2, \ldots, 1+s_\ell)$ and $t \mapsto (1+t_1, 1+t_2, \ldots, 1+t_\ell)$, we may assume that

$s = \nu^\star$, $t = \mu^\star$ for some compositions $\nu \in \mathscr{C}_N^\ell$, $\mu \in \mathscr{C}_N^e$ such that $\mu_i, \nu_p \geqslant d$ for each i, p.

The following result has been conjectured by Chuang and Miyachi [8, conj. 6].

Theorem 7.4 *The category $\mathcal{O}_t^s$ is (standard) Koszul and its Koszul dual coincides with the Ringel dual of $\mathcal{O}_s^t$.*

Proof Recall that $s = \nu^\star$ and $t = \mu^\star$. Theorem 6.9 yields an equivalence $\Upsilon^{\nu,-e} : \mathbf{A}^{\nu,-e} \xrightarrow{\sim} \mathcal{O}^{s,-e}$ such that $\Upsilon^{\nu,-e}(\boldsymbol{\Delta}(\lambda))$ is isomorphic to $\Delta(\lambda^\star)^{s,-e}$. Let $\mathbf{O}_\mu^{\nu,-e} = \mathbf{O}_\beta^{\nu,-e} \subset \mathbf{O}^{\nu,-e}$ with $\beta = \sum_i \mu_i \epsilon_i$. It is the Serre subcategory generated by the simple modules with highest weight λ such that $m_i(\lambda) = \mu_i$ for all $i \in \mathbb{Z}/e\mathbb{Z}$, see Sect. 5.4. Let $\mathbf{A}_\mu^\nu = \mathbf{A}^{\nu,-e} \cap \mathbf{O}_\mu^{\nu,-e}$.

Lemma 7.5 *The functor $\Upsilon^{\nu,-e}$ gives an equivalence $\mathbf{A}_\mu^\nu \xrightarrow{\sim} \mathcal{O}_t^s$.*

Proof By (7.3) we have $\Delta(\lambda^\star)^{s,-e} \in \mathcal{O}_t^s$ if and only if

$$\Lambda^s - \sum_{i=1}^{e-1}(n_i^s(\lambda^\star) - n_0^s(\lambda^\star))\,\alpha_i = \sum_{i=1}^{e-1}(t_i - t_{i+1})\Lambda_i + (\ell + t_e - t_1)\Lambda_0. \quad (7.4)$$

Next, observe that $n_i^s(\lambda) = n_{-i}^{s^\star}(\lambda^\star)$ for all $i \in \mathbb{Z}/e\mathbb{Z}$. Indeed, let $b = (x, y, p)$ be a box in row x, column y of the Young diagram of the partition λ_p. Then, we have a bijection form the set of bowes of λ onto the set of boxes of $\lambda^\star$ such that $b = (x, y, p)$ maps to $b^\star = (y, x, \ell - p + 1)$. Thus, the claim follows from the relation $\mathrm{cont}^s(b) = y - x + s_p = -(x - y) - s^\star_{\ell-p+1} = -\mathrm{cont}^{s^\star}(b^\star)$.

It follows that $\Delta(\lambda^\star)^{s,-e} \in \mathcal{O}_t^s$ if and only if

$$\begin{aligned}
&\Lambda^s - \sum_{i=1}^{e-1}(n_{-i}^{s^\star}(\lambda) - n_0^{s^\star}(\lambda))\,\alpha_i = \sum_{i=1}^{e-1}(t_i - t_{i+1})\Lambda_i + (\ell + t_e - t_1)\Lambda_0,\\
&\iff \Lambda^{s^\star} - \sum_{i=1}^{e-1}(n_{-i}^{s^\star}(\lambda) - n_0^{s^\star}(\lambda))\,\alpha_{-i} = \sum_{i=1}^{e-1}(t_i - t_{i+1})\Lambda_{-i} + (\ell + t_e - t_1)\Lambda_0,\\
&\iff \Lambda^\nu - \sum_{i=1}^{e-1}(n_i^\nu(\lambda) - n_0^\nu(\lambda))\,\alpha_i = \sum_{i=1}^{e-1}(t_{-i} - t_{-i+1})\Lambda_i + (\ell + t_e - t_1)\Lambda_0,\\
&\iff \mathbf{wt}(|\lambda, \nu\rangle) = \sum_{i=1}^{e-1}(t_{-i} - t_{-i+1})\Lambda_i + (\ell + t_e - t_1)\Lambda_0 \mod \mathbb{Z}\delta.
\end{aligned}$$

Since $t^\star = (-t_e, \ldots, -t_2, -t_1)$, we deduce that $t_{-i} - t_{-i+1} = t_i^\star - t_{i+1}^\star$ for all i. Hence, we have $\Delta(\lambda^\star)^{s,-e} \in \mathcal{O}_t^s$ if and only if

$$\mathbf{wt}(|\lambda,\nu\rangle)=\sum_{i=1}^{e-1}(t_i^\star-t_{i+1}^\star)\Lambda_i+(\ell+t_e^\star-t_1^\star)\Lambda_0 \mod \mathbb{Z}\delta.$$

Recall that $\mathrm{wt}(\varpi(\lambda))=\mathbf{wt}(|\lambda,\nu\rangle)-\ell\Lambda_0$ modulo $\mathbb{Z}\,\delta$, see Remark 7.2. Since $t^\star=\mu$, we deduce that $\Delta(\lambda^\star)^{s,-e}\in\mathcal{O}_t^s$ if and only if

$$\mathrm{wt}(\varpi(\lambda))=\sum_{i=1}^{e}(\mu_i-\mu_{i+1})\Lambda_i \mod \mathbb{Z}\delta.$$

Since $\mathbf{A}_\mu^\nu=\mathbf{A}^{\nu,-e}\cap\mathbf{O}_\mu^{\nu,-e}$ and $\mathbf{\Delta}(\lambda)=\mathbf{M}(\varpi(\lambda))$, we deduce that $\Delta(\lambda^\star)^{s,-e}\in\mathcal{O}_t^s$ if and only if $\mathbf{\Delta}(\lambda)\in\mathbf{A}_\mu^\nu$. □

To conclude, note that, by [42, thm. B.4], the highest weight category $\mathbf{A}_\mu^\nu$ is (standard) Koszul, and its Koszul dual is equivalent to the Ringel dual of $\mathbf{A}_\nu^\mu$. The theorem follows. □

7.4 Categorical actions on A

Recall that $\mathscr{I}=\mathbb{Z}/e\mathbb{Z}$. By [41, thm. 5.1, cor. 4.5], there is an $\mathfrak{sl}_\mathscr{I}$-categorical action (E,F,X,T) on $\mathcal{O}^{s,-e}$ with $E=\bigoplus_{d\in\mathbb{N}}{}^{\mathcal{O}}\mathrm{Res}_d^{d+1}$, $F=\bigoplus_{d\in\mathbb{N}}{}^{\mathcal{O}}\mathrm{Ind}_d^{d+1}$, and such that the functor $\mathrm{KZ}^s=\bigoplus_{d\in\mathbb{N}}\mathrm{KZ}_d^s$ is a morphism of $\mathfrak{sl}_\mathscr{I}$-categorifications $\mathcal{O}^{s,-e}\to\mathscr{L}(\Lambda^s)_\mathscr{I}$. In this section we construct a similar $\mathfrak{sl}_\mathscr{I}$-categorification for the category $\mathbf{A}$.

Let $R=\mathbb{C}$, $\tau=0$, and let $\nu\in\mathscr{C}_{N,+}^\ell$. Assume that $d\leqslant\nu_p$ for all p. Recall the tuple (E,F,X,T) on $\mathbf{A}^{\nu,-e}$ from Sect. 5.9. Let $\Upsilon_d=\Upsilon_d^{\nu,-e}:\mathbf{A}^{\nu,-e}\{d\}\xrightarrow{\sim}\mathcal{O}^{\nu^\star,-e}\{d\}$ be the equivalence in Theorem 6.9. We have the following.

Lemma 7.6 *Assume that $d+1\leqslant\nu_p$ for all p. Then, the functors $F:\mathbf{A}^{\nu,-e}\{d\}\to\mathbf{A}^{\nu,-e}\{d+1\}$ and $E:\mathbf{A}^{\nu,-e}\{d+1\}\to\mathbf{A}^{\nu,-e}\{d\}$ are biadjoint. Further, there are isomorphisms of functors ${}^{\mathcal{O}}\mathrm{Ind}_d^{d+1}\,\Upsilon_d\simeq\Upsilon_{d+1}\,F$ and ${}^{\mathcal{O}}\mathrm{Res}_d^{d+1}\,\Upsilon_{d+1}\simeq\Upsilon_d\,E$, which intertwines $X\Upsilon_d$ with $\Upsilon_{d+1}\,\mathrm{IM}(X)$, and $T\Upsilon_d$ with $\Upsilon_{d+2}\,\mathrm{IM}(T)$.*

Proof We abbreviate $\mathrm{KZ}_d=\mathrm{KZ}_d^\nu$, $\mathrm{KZ}_d^\star=\mathrm{KZ}_d^{\nu^\star}$, $\Psi_d=\Psi_d^\nu$, $\mathbf{A}=\mathbf{A}^{\nu,-e}$, $\mathbf{O}=\mathbf{O}^{\nu,-e}$ and $\mathcal{O}=\mathcal{O}^{\nu^\star,-e}$. By Theorem 6.9 we have $\Psi_d\simeq\mathrm{IM}^*\,\mathrm{KZ}_d^\star\,\Upsilon_d$ on $\mathbf{A}\{d\}$.

Recall from Proposition 8.29 that e, f are biadjoint functors on $\mathbf{O}$. Let $F'=\Upsilon_{d+1}\,F\,\Upsilon_d^{-1}:\mathcal{O}\{d\}\to\mathcal{O}\{d+1\}$. We claim that there is an isomorphism of functors $F'\simeq{}^{\mathcal{O}}\mathrm{Ind}_d^{d+1}$. Let us prove it.

Since $\Psi_{d+1} F \simeq \mathrm{Ind}_d^{d+1} \Psi_d$ by Lemma 5.41, and since IM^* commutes with the induction functor, we have $\mathrm{KZ}_{d+1} F' \simeq \mathrm{Ind}_d^{d+1} \mathrm{KZ}_d$. By (6.1), we also have $\mathrm{KZ}_{d+1} {}^{\mathcal{O}}\mathrm{Ind}_d^{d+1} \simeq \mathrm{Ind}_d^{d+1} \mathrm{KZ}_d$. Hence, we get an isomorphism of functors

$$\theta : \mathrm{KZ}_{d+1} F' \xrightarrow{\sim} \mathrm{KZ}_{d+1} {}^{\mathcal{O}}\mathrm{Ind}_d^{d+1}.$$

The functor ${}^{\mathcal{O}}\mathrm{Ind}_d^{d+1}$ maps projectives to projectives. Let G_d be the right adjoint to KZ_d. Since KZ_d is a highest weight cover, the unit $\eta : 1 \to G_d \mathrm{KZ}_d$ is invertible on projective modules. Hence, the isomorphism θ yields an isomorphism of functors on projective modules $G_{d+1} \mathrm{KZ}_{d+1} F' \simeq G_{d+1} \mathrm{KZ}_{d+1} {}^{\mathcal{O}}\mathrm{Ind}_d^{d+1} \simeq {}^{\mathcal{O}}\mathrm{Ind}_d^{d+1}$. Composing it with η, we get a morphism $\theta' : F' \to {}^{\mathcal{O}}\mathrm{Ind}_d^{d+1}$ on the projectives, such that $\mathrm{KZ}_{d+1} \theta' = \theta$. Since KZ_{d+1} is (-1)-faithful, it follows from Lemma 2.8 and Remark 2.9 that θ' is injective, hence invertible because both terms coincide in the Grothendieck group by Lemma 5.12 and [41, prop. 4.4(3)]. Thus, θ' is an isomorphism on the projective modules.

Now, since Υ_{d+1}, Υ_d are equivalences, both F' and ${}^{\mathcal{O}}\mathrm{Ind}_d^{d+1}$ are exact on $\mathcal{O}\{d\}$. Thus, θ' extends to an isomorphism of functors $\theta' : F' \xrightarrow{\sim} {}^{\mathcal{O}}\mathrm{Ind}_d^{d+1}$ on $\mathcal{O}\{d\}$ such that $\mathrm{KZ}_{d+1} \theta' = \theta$ by [41, lem. 1.2]. The claim is proved.

Let $E' : \mathbf{A}\{d+1\} \to \mathbf{A}\{d\}$ be the right adjoint of F. The uniqueness of right adjoints implies that $\Upsilon_d E' \Upsilon_{d+1}^{-1} \simeq {}^{\mathcal{O}}\mathrm{Res}_d^{d+1}$. Now, since ${}^{\mathcal{O}}\mathrm{Res}_d^{d+1}$ is also left adjoint to ${}^{\mathcal{O}}\mathrm{Ind}_d^{d+1}$ by [41, prop. 2.9] and since Υ_d is an equivalence, we deduce that E' is left adjoint to F, hence $E \simeq E'$ on $\mathbf{A}\{d+1\}$.

Now, let $X_{\mathbf{H}} \in \mathrm{End}(\mathrm{Ind}_d^{d+1})$ and $T_{\mathbf{H}} \in \mathrm{End}(\mathrm{Ind}_d^{d+2})$ be as in Example 3.6. The isomorphism $\mathrm{Ind}_d^{d+1} \mathrm{KZ}_d \simeq \mathrm{KZ}_{d+1} {}^{\mathcal{O}}\mathrm{Ind}_d^{d+1}$ in (6.1) intertwines $X_{\mathbf{H}}^{-1} \mathrm{KZ}_d$ with $\mathrm{KZ}_{d+1} X^{-1}$. The isomorphism $\mathrm{Ind}_d^{d+1} \Psi_d \simeq \Psi_{d+1} F$ in Lemma 5.41 intertwines $X_{\mathbf{H}} \Psi_d$ with $\Psi_{d+1} X$ by Remark 5.42. Hence, θ intertwines $\mathrm{KZ}_{d+1} \Upsilon_{d+1} X \Upsilon_d^{-1}$ with $\mathrm{KZ}_{d+1} X^{-1}$. We deduce that θ' intertwines $\Upsilon_{d+1} X \Upsilon_d^{-1}$ with X^{-1}. The proof for T is similar. The lemma is proved. □

For each $a \in \mathbb{N}$, set $\nu + a = (\nu_1 + a, \nu_2 + a, \ldots, \nu_\ell + a)$.

Lemma 7.7 *For any $d \in \mathbb{N}$ and any $a \leqslant a' \in \mathbb{N}$ such that $d \leqslant \nu_p + a$ for all p, there is an equivalence of highest weight categories $\Sigma = \Sigma^{a,a'} : \mathbf{A}^{\nu+a,-e}\{d\} \xrightarrow{\sim} \mathbf{A}^{\nu+a',-e}\{d\}$ which maps $\mathbf{\Delta}(\lambda)$ to $\mathbf{\Delta}(\lambda)$, intertwines (E, F, X, T) on both sides and such that $\Psi_d^{\nu+a} \simeq \Psi_d^{\nu+a'} \Sigma^{a,a'}$.*

Proof The CRDAHA's associated with $(\nu + a)^\star$ and $(\nu + a')^\star$ are the same. Hence, we have $\mathcal{O}^{(\nu+a)^\star,-e}\{d\} = \mathcal{O}^{(\nu+a')^\star,-e}\{d\}$. We define $\Sigma^{a,a'} = (\Upsilon_d^{\nu+a',-e})^{-1} \Upsilon_d^{\nu+a,-e}$. By Lemma 7.6, the functor Σ intertwines (E, F) on both sides. □

For each d, we define the category $\widetilde{\mathbf{A}}^{\nu,-e}\{d\}$ as the limit of the inductive system of categories $(\mathbf{A}^{\nu+a,-e}\{d\}, \Sigma^{a,a'})_{a,a'\in\mathbb{N}}$. We have an equivalence of highest weight categories $\tilde{\Upsilon}_d^{\nu,-e} : \widetilde{\mathbf{A}}^{\nu,-e}\{d\} \xrightarrow{\sim} \mathcal{O}^{\nu^\star,-e}\{d\}$ and a highest weight cover $\tilde{\Psi}_d^{\nu} : \widetilde{\mathbf{A}}^{\nu,-e}\{d\} \to \mathbf{H}_d^{\nu}$-mod. In particular, the blocks of $\widetilde{\mathbf{A}}^{\nu,-e}\{d\}$ are in bijection with the blocks of $\mathbf{H}_d^{\nu}$-mod via $\tilde{\Psi}_d^{\nu}$. For $\mu = \Lambda^{\nu} - \alpha$, let $\widetilde{\mathbf{A}}_{\mu}^{\nu,-e}$ be the block corresponding to $\mathbf{H}_{\alpha}^{\nu}$-mod.

Now, let $\widetilde{\mathbf{A}}^{\nu,-e} = \bigoplus_{d\in\mathbb{N}} \widetilde{\mathbf{A}}^{\nu,-e}\{d\}$. The category $\widetilde{\mathbf{A}}^{\nu,-e}$ carries a pre-categorical action (E, F, X, T) given by Lemma 7.7. The following is now obvious.

Proposition 7.8 *The tuple (E, F, X, T) and the decomposition $\widetilde{\mathbf{A}}^{\nu,-e} = \bigoplus_{\mu\in X_{\mathscr{I}}} \widetilde{\mathbf{A}}_{\mu}^{\nu,-e}$ define an $\mathfrak{sl}_{\mathscr{I}}$-categorical action on $\tilde{\mathbf{A}}^{\nu,-e}$.*

Proof We have $E = \bigoplus_{i\in\mathscr{I}} E_i$ and $F = \bigoplus_{i\in\mathscr{I}} F_i$, where E_i, F_i are defined as in Sect. 5.4. By Theorem 6.9, the equivalence $\tilde{\Upsilon}^{\nu,-e} = \bigoplus_{d\in\mathbb{N}} \tilde{\Upsilon}_d^{\nu,-e} : \tilde{\mathbf{A}}^{\nu,-e} \xrightarrow{\sim} \mathcal{O}^{\nu^\star,-e}$ yields a linear isomorphism $[\tilde{\mathbf{A}}^{\nu,-e}] \xrightarrow{\sim} [\mathcal{O}^{\nu^\star,-e}]$ which maps $[\mathbf{\Delta}(\lambda)]$ to $[\Delta(\lambda^\star)^{\nu^\star,-e}]$. Hence by Lemma 5.12 and [41, prop. 4.4], it intertwines the operators E_i, F_i on the left hand side with the operators the operators E_{-i}, F_{-i} on the right hand side. Thus, the operators E_i, F_i with $i \in \mathscr{I}$ yield a representation of $\mathfrak{sl}_{\mathscr{I}}$ on $[\tilde{\mathbf{A}}^{\nu,-e}]$. $\square$

7.5 The category A and the cyclotomic q-Schur algebra

Let $\Bbbk$ be a field containing $\mathbb{C}$. Fix a positive integer d and a composition ν. We will say that ν is *d-dominant* if we have $\nu_p - \nu_{p+1} \geqslant d$ for each $p = 1, \ldots, \ell - 1$ and that it is *anti-dominant* if we have $\nu_{p+1} - \nu_p \leqslant d$ for each p as above. The following propositions generalize some of the results in [5]. They are proved as Theorem 6.9 using Proposition 2.20.

Proposition 7.9 *Let $\nu_p \geqslant d$ and $\tau_{\Bbbk,u} - \tau_{\Bbbk,v} \notin \mathbb{N}^\times$ for all $p = 1 \cdots \ell$ and all $u < v$. Set $s = \nu + \tau$. Assume that ν is either d-dominant or d-anti-dominant. Then, there is an equivalence of highest weight* $\Bbbk$*-categories*

$$\mathscr{G}_{\Bbbk,d}^{s} : A_{\Bbbk,\tau}^{\nu}\{d\} \xrightarrow{\sim} S_{\Bbbk,d}^{s^\star}\text{-mod}$$

which intertwines the functors

$$\Phi_{\Bbbk,d}^{s} : A_{\Bbbk,\tau}^{\nu}\{d\} \to H_{\Bbbk,d}^{s}\text{-mod},$$
$$\mathrm{IM}^* \ \Xi_{\Bbbk,d}^{s^\star} : S_{\Bbbk,d}^{s^\star}\text{-mod} \to H_{\Bbbk,d}^{s}\text{-mod}\,.$$

Furthermore, we have $\mathscr{G}_{\Bbbk,d}^{s}(\Delta(\lambda)_{\Bbbk,\tau}) \simeq W(\lambda^\star)_{\Bbbk}^{s^\star}$ for all λ's.

Proof If $\tau_{\mathrm{k},u} - \tau_{\mathrm{k},v} \notin \mathbb{Z}^{\times}$ for all u, v, then the proposition is proved in [5]. The general case, i.e., the case where $\tau_{\mathrm{k},u} - \tau_{\mathrm{k},v} \notin \mathbb{N}^{\times}$, is proved as Proposition 7.10 below. $\square$

Proposition 7.10 *Let $\nu_p \geqslant d$ for all $p = 1, \ldots, \ell$. Assume that ν is either d-dominant or d-anti-dominant. Set $s = \nu$. Then, there is an equivalence of highest weight* k*-categories*

$$\mathscr{G}^s_{\mathrm{k},d} : \mathbf{A}^{\nu,\kappa}_{\mathrm{k}}\{d\} \xrightarrow{\sim} \mathbf{S}^{s^\star}_{\mathrm{k},d}\text{-mod}$$

which intertwines the functors

$$\Phi^s_{\mathrm{k},d} : \mathbf{A}^{\nu,\kappa}_{\mathrm{k}}\{d\} \to \mathbf{H}^s_{\mathrm{k},d}\text{-mod},$$
$$\mathrm{IM}^* \ \Xi^{s^\star}_{\mathrm{k},d} : \mathbf{S}^{s^\star}_{\mathrm{k},d}\text{-mod} \to \mathbf{H}^s_{\mathrm{k},d}\text{-mod}.$$

Furthermore, we have $\mathscr{G}^s_{\mathrm{k},d}(\boldsymbol{\Delta}(\lambda)_{\mathrm{k}}) \simeq W(\lambda^\star)^{s^\star,q}_{\mathrm{k}}$ for all λ's.

Proof We can assume $\mathrm{k} = \mathbb{C}$. Let R be a local analytic deformation ring of dimension 2 in general position with residue field k. Assume that $\kappa_{\mathrm{k}} = -e$. Set $s_{R,p} = \nu_p + \tau_{R,p}$. Let $\mathscr{C} = \mathcal{O}^{s,\kappa}_R\{d\}$ and $\mathscr{C}' = \mathbf{S}^s_{R,d}$-mod. Since the highest weight categories $\mathcal{O}^{s^\star,\kappa}_R\{d\}$ and $\mathbf{A}^{\nu,\kappa}_{R,\tau}\{d\}$ are equivalent by Theorem 6.9, it is enough to compare $\mathscr{C}, \mathscr{C}'$.

Consider the highest weight covers

$$F = \mathrm{KZ}^s_{R,d} : \mathscr{C} \to \mathbf{H}^s_{R,d}\text{-mod},$$
$$F' = \Xi^s_{R,d} : \mathscr{C}' \to \mathbf{H}^s_{R,d}\text{-mod}.$$

We claim that they satisfy the conditions in Proposition 2.20, so the theorem holds. Let us check these conditions.

We'll assume that ν is d-dominant. Then, there is a partial order which refines both highest weight orders ${}_{s,\kappa}\!\leqslant$ on $\mathscr{C}$ and $\trianglelefteq$ on $\mathscr{C}'$, see [39, prop. 6.4].

The functor F is fully faithful on $\mathscr{C}^{\Delta}$ and $\mathscr{C}^{\nabla}$, by the proof of Theorem 6.9.

By [37, prop. 3.1, 3.5, cor. 6.11, thm. 6.18] and Proposition 4.9, that any tilting module in $\mathscr{C}'$ is isomorphic to the image of an object of $\mathbf{H}^s_{R,d}$-mod by the right adjoint to the Schur functor F'. Note that [37, thm. 6.18] is proved over a field, but it remains true over the ring R by Proposition 2.4. We deduce that F' is fully faithful on $(\mathscr{C}')^{\Delta}$, by [39, prop. 4.40]. Next, by [37, prop. 4.3, cor. 7.2], the R-category $\mathscr{C}'$ is Ringel self-dual, i.e., we have an equivalence $\mathscr{C}' \simeq (\mathscr{C}')^{\diamond}$. Therefore, by Lemma 2.13, the functor F' is also fully faithful on $(\mathscr{C}')^{\nabla}$.

Finally, we prove that $F(T(\lambda)) \in F'(\mathscr{C}'^{\mathrm{tilt}})$ for all $\lambda \in \mathscr{P}^{\ell}_d$ such that $\mathrm{lcd}_{\mathrm{k}\mathscr{C}^\diamond}(L^\diamond(\lambda)) \leq 1$ or $\mathrm{rcd}_{\mathrm{k}\mathscr{C}^\diamond}(L^\diamond(\lambda)) \leq 1$ as in Theorem 6.9, using some analogues (for the Schur algebra) of Propositions 6.7, 6.8.

Note that Proposition 7.10 gives a proof of Yvonne's conjecture in [48]. $\square$

8 The Kazhdan–Lusztig category

Fix integers ℓ, $N \geqslant 1$ and fix a composition $\nu \in \mathscr{C}^{\ell}_{N,+}$. Let R be a deformation ring. We may abbreviate $\kappa = \kappa_R$. If $R = \mathbb{C}$ we may also drop the subscript R from the notation.

8.1 Coinvariants

Fix a finite totally ordered set A. Set $R^A = \bigoplus_{a\in A} R((t_a))$, where t_a is a formal variable. Let $\mathbf{g}^A_R$ be the central extension of $\mathfrak{g} \otimes R^A$ by R associated with the cocycle $(\xi \otimes f, \zeta \otimes g) \mapsto \langle \xi : \zeta \rangle \sum_{a\in A} \mathrm{Res}_{t_a=0}(gdf)$.

Write $\mathbf{1}$ for the canonical central element of $\mathbf{g}^A_R$, and let $U(\mathbf{g}^A_R) \to \mathbf{g}^A_{R,\kappa}$ be the quotient of the enveloping algebra (over R) by the two-sided ideal generated by $\mathbf{1} - c$. By the symbol $\bigotimes_{R,a}$ we'll mean the (ordered) tensor product of R-modules with respect to the ordering of A. Given a module $M_a \in \mathscr{S}_{R,\kappa}$ for each $a \in A$, the Lie algebra $\mathbf{g}^A_{R,\kappa}$ acts naturally on the tensor product $\bigotimes_{R,a} M_a$, where a runs over the set A.

Let C be a connected projective curve isomorphic to $\mathbb{P}^1$. By a *chart* on C centered at x we mean an automorphism γ of $\mathbb{P}^1$ such that $\gamma(x) = 0$. We will say that $\gamma = \{\gamma_a;\ a \in A\}$ is an *admissible system of charts* if the conditions (a), (b) in [28, sec. 13.1] hold. Let $\eta_a = \gamma_a^{-1}$ denote the automorphism which is the inverse of γ_a and let x_a be the center of γ_a. The x_a's are distinct points of C. We write $C_\gamma = C \setminus \{x_a; a \in A\}$, $D_R = D_{R,\gamma} = R[C_\gamma]$ and $\Gamma_R = \Gamma_{R,\gamma} = \mathfrak{g} \otimes D_R$.

For any $f \in D_R$, let ${}^a f \in R((t_a))$ be the power series expansion at 0 of the rational function $f \circ \eta_a$ on $\mathbb{P}^1$. Taking f to the A-tuple ${}^A f = ({}^a f)$ gives a R-algebra homomorphism $D_R \to R^A$ and a R-Lie algebra homomorphism $\Gamma_R \to \mathbf{g}^A_R$ by the residue theorem.

We can now define the sets of *coinvariants*.

Definition 8.1 Let $A = [1, n]$. Given $N_a \in U(\mathfrak{g}_R)$-mod and $M_a \in \mathscr{S}_{R,\kappa}$ for each $a \in A$, we set

$$\langle N_1, \ldots, N_n \rangle_R = H_0(\mathfrak{g}_R, \textstyle\bigotimes_{R,a} N_a), \quad \langle\!\langle M_1, \ldots, M_n \rangle\!\rangle_R = H_0(\Gamma_R, \textstyle\bigotimes_{R,a} M_a).$$

By [28, sec. 13.3] the R-module $\langle\!\langle M_1, \ldots, M_n \rangle\!\rangle_R$ does not depend on the choice of the admissible system of charts, up to a canonical isomorphism. Further, it only depends on the cyclic ordering of A, so that we have a canonical isomorphism

$$\langle\!\langle M_1, \ldots, M_n \rangle\!\rangle_R = \langle\!\langle M_2, \ldots, M_n, M_1 \rangle\!\rangle_R.$$

Let $\mathbf{1} = \mathbf{1}_R$ denote the parabolic Verma module $\mathbf{M}(0)_{R,+}$.

Lemma 8.2 (a) *Taking coinvariants is a right exact functor. It commutes with base change. More precisely, if $A = [1, n]$, then for any morphism of deformation rings $R \to S$ the obvious map $\bigotimes_{R,a} M_a \to \bigotimes_{S,a} SM_a$ induces an S-module isomorphism*

$$S\langle\langle M_1, \dots, M_n\rangle\rangle_R = \langle\langle SM_1, \dots, SM_n\rangle\rangle_S.$$

(b) *Assume that for each $a \in A$ there is an integer $d_a \geqslant 1$ such that $M_a(d_a)$ generates M_a as a $\mathbf{g}_{R,\kappa}$-module. Then, the obvious inclusion $\bigotimes_{R,a} M_a(d_a) \to \bigotimes_{R,a} M_a$ induces a surjective R-module homomorphism*

$$\langle M_1(d_1), \dots, M_n(d_n)\rangle_R \to \langle\langle M_1, \dots, M_n\rangle\rangle_R.$$

In particular, if $A = [-m, n]$ and $M_0 \in \mathbf{O}_R$, $M_a \in \mathbf{O}_R^{+,\kappa}$ for each $a \in A \setminus \{0\}$, then the R-module $\langle\langle M_{-m}, \dots, M_n\rangle\rangle_R$ is finitely generated.

(c) *Assume that $M_a = \mathscr{I}nd_R(N_a)$ is a generalized Weyl module for each $a \in A$. Then, the obvious inclusion $\bigotimes_{R,a} N_a \to \bigotimes_{R,a} M_a$ induces an isomorphism of R-modules $\langle N_1, \dots, N_n\rangle_R \to \langle\langle M_1, \dots, M_n\rangle\rangle_R$.*

(d) *If $M_1 = \mathbf{1}$, then the canonical inclusion $\bigotimes_{R,a\neq 1} M_a \to \bigotimes_{R,a} M_a$ induces an R-module isomorphism $\langle\langle M_2, \dots, M_n\rangle\rangle_R \to \langle\langle M_1, M_2, \dots, M_n\rangle\rangle_R$.*

Proof Part (a) is obvious. See, e.g., [28, sec. 9.13]. For part (b) note that the $\mathfrak{g}_R$-action on M_a preserves the R-submodule $M_a(d_a)$ for each $a \in A$. Then, the first claim follows from [28, prop. 9.12]. The proof in loc. cit. is done under the hypothesis that $\kappa \in \mathbb{C}$. It extends easily to the case of any $\kappa = \kappa_R \in R$. The second claim follows from Lemma 5.4(b), the first claim, part (a) and from the fact that $\mathscr{O}_R$ is Hom finite (over R) and that the tensor product maps $\mathscr{O}_R \times \mathscr{O}_R^+$ into $\mathscr{O}_R$. Part (c) is proved in [28, prop. 9.15]. The proof in loc. cit. is done under some restrictive conditions on the $\mathfrak{g}_R$-modules N_a and under the hypothesis that $\kappa \in \mathbb{C}$, but it extends to our setting. Part (d) is proved in [28, prop. 9.18]. The loc. cit. the proof is given for $R = \mathbb{C}$, but it generalizes to our case. □

Remark 8.3 Assume that $R = K$ is a field and that $\kappa_K \notin \mathbb{Q}_{\geqslant 0}$. Then, we have $\mathbf{1} = \mathbf{L}(0)_K$, see e.g. [28, prop. 2.12].

8.2 The monoidal structure on O over a field

Let $R = K$ be a field which is an analytic algebra. Fix an element κ_K in $K \setminus \mathbb{Q}_{\geqslant 0}$. Let $\mathbf{1} \in \mathbf{O}_K^{+,\kappa}$ be as in Sect. 8.1. In [28], Kazhdan and Lusztig have defined a braided monoidal structure $(\mathbf{O}_K^{+,\kappa}, \dot{\otimes}_K, \mathbf{a}_K, \mathbf{c}_K)$ with unit $\mathbf{1}$ on $\mathbf{O}_K^{+,\kappa}$. In this section we'll define a *bimodule category* $(\mathbf{O}_K^{\nu,\kappa}, \dot{\otimes}_K, \mathbf{a}, \mathbf{c})$ over it. This means that $\mathbf{O}_K^{\nu,\kappa}$ is a left and right *module category* over $\mathbf{O}_K^{+,\kappa}$, see [38] for details, and that the functors $\mathbf{a}_K$, $\mathbf{c}_K$ satisfy an analogue of the hexagon axiom that expresses the commutativity of the left and right actions.

8.2.1 Definition of the bimodule category

First, we define a tuple $(\mathbf{O}_K^{\nu,\kappa}, \dot{\otimes}_K, \mathbf{a}, \mathbf{c})$ such that the following hold:

- $\dot{\otimes}_K : \mathbf{O}_K^{+,\kappa} \times \mathbf{O}_K^{\nu,\kappa} \to \mathbf{O}_K^{\nu,\kappa}$ and $\dot{\otimes}_K : \mathbf{O}_K^{\nu,\kappa} \times \mathbf{O}_K^{+,\kappa} \to \mathbf{O}_K^{\nu,\kappa}$ are bilinear functors such that $V\dot{\otimes}_K \bullet$ and $\bullet\dot{\otimes}_K V$ are exact for each $V \in \mathbf{O}_K^{+,\kappa}$,
- there are functorial (left and right) unit isomorphisms for each $M \in \mathbf{O}_K^{\nu,\kappa}$

$$\mathbf{1}\dot{\otimes}_K M \to M, \quad M\dot{\otimes}_K \mathbf{1} \to M,$$

- there are functorial associativity isomorphisms for each $V_1, V_2 \in \mathbf{O}_K^{+,\kappa}$ and $M \in \mathbf{O}_K^{\nu,\kappa}$

$$\begin{aligned}
\mathbf{a}_{V_1,V_2,M} &: (V_1\dot{\otimes}_K V_2)\dot{\otimes}_K M \to V_1\dot{\otimes}_K (V_2\dot{\otimes}_K M), \\
\mathbf{a}_{V_1,M,V_2} &: (V_1\dot{\otimes}_K M)\dot{\otimes}_K V_2 \to V_1\dot{\otimes}_K (M\dot{\otimes}_K V_2), \\
\mathbf{a}_{M,V_1,V_2} &: (M\dot{\otimes}_K V_1)\dot{\otimes}_K V_2 \to M\dot{\otimes}_K (V_1\dot{\otimes}_K V_2),
\end{aligned}$$

- there are functorial commutativity isomorphisms for each $V \in \mathbf{O}_K^{+,\kappa}$ and $M \in \mathbf{O}_K^{\nu,\kappa}$

$$\mathbf{c}_{V,M} : V\dot{\otimes}_K M \to M\dot{\otimes}_K V,$$

- $\mathbf{1}, \mathbf{a}$ satisfy the triangle axioms (left and right) for $V \in \mathbf{O}_K^{+,\kappa}$ and $M \in \mathbf{O}_K^{\nu,\kappa}$,

$$\begin{tikzcd}
(V\dot{\otimes}_K \mathbf{1})\dot{\otimes}_K M \arrow[rr] \arrow[dr] & & V\dot{\otimes}_K (\mathbf{1}\dot{\otimes}_K M) \arrow[dl] \\
& V\dot{\otimes}_K M &
\end{tikzcd}$$

- $\mathbf{a}$ satisfies the pentagon axiom (left and right) for each $V_1, V_2, V_3 \in \mathbf{O}^{+,\kappa}$ and $M \in \mathbf{O}^{\nu,\kappa}$

$$\begin{tikzcd}
V_1\dot{\otimes}_K ((V_2\dot{\otimes}_K V_3)\dot{\otimes}_K M) \arrow[rr] & & V_1\dot{\otimes}_K (V_2\dot{\otimes}_K (V_3\dot{\otimes}_K M)) \\
(V_1\dot{\otimes}_K (V_2\dot{\otimes}_K V_3))\dot{\otimes}_K M \arrow[u] & & (V_1\dot{\otimes}_K V_2)\dot{\otimes}_K (V_3\dot{\otimes}_K M) \arrow[u] \\
& ((V_1\dot{\otimes}_K V_2)\dot{\otimes}_K V_3)\dot{\otimes}_K M, \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

plus the diagrams obtained by cyclic permutation of M, V_1, V_2, V_3,

- $\mathbf{a}_K$, $\mathbf{c}_K$ satisfy the hexagon axiom for each $V_1, V_2 \in \mathbf{O}_K^{+,\kappa}$ and $M \in \mathbf{O}_K^{\nu,\kappa}$

$$\begin{tikzcd}
 & V_1\dot{\otimes}_K(M\dot{\otimes}_K V_2) \arrow[r] & (M\dot{\otimes}_K V_2)\dot{\otimes}_K V_1 \arrow[dr] & \\
(V_1\dot{\otimes}_K M)\dot{\otimes}_K V_2 \arrow[ur] \arrow[dr] & & & M\dot{\otimes}_K(V_2\dot{\otimes}_K V_1) \\
 & (M\dot{\otimes}_K V_1)\dot{\otimes}_K V_2 \arrow[r] & M\dot{\otimes}_K(V_1\dot{\otimes}_K V_2), \arrow[ur] &
\end{tikzcd}$$

plus the diagrams obtained by cyclic permutation of M, V_1, V_2.

Remark 8.4 The notion of *bimodule functors*, and, in particular, of equivalence of bimodule categories is defined in the obvious way. Generally one impose the functor $\dot{\otimes}$ to be biexact. Our choice simplifies the exposition in the rest of the paper.

Now, let us define the functor $\dot{\otimes}_K$. The bifunctor $\dot{\otimes}_K$ on $\mathbf{O}_K^{+,\kappa}$ is defined in [28]. By [49], the same definition yields functors $\mathbf{O}_K^{+,\kappa} \times \mathbf{O}_K^{\nu,\kappa} \to \mathbf{O}_K^{\nu,\kappa}$ and $\mathbf{O}_K^{\nu,\kappa} \times \mathbf{O}_K^{+,\kappa} \to \mathbf{O}_K^{\nu,\kappa}$. Note that [28,49] deal only with the field $K = \mathbb{C}$ and $\kappa \in \mathbb{C} \setminus \mathbb{Q}_{\geqslant 0}$. The same definition works equally well over any field K containing $\mathbb{C}$ and for any $\kappa \in K \setminus \mathbb{Q}_{\geqslant 0}$.

More precisely, let $\spadesuit = [-m, n]$, $A = [-m, n+1]$ and fix an admissible system of charts γ. Given a smooth $\mathbf{g}_\kappa$-module M_a for each $a \in \spadesuit$, we consider the functor $M \mapsto \langle\!\langle M_{-m}, \ldots, M_n, DM\rangle\!\rangle_K^*$.

Proposition 8.5 *Assume that $M_0 \in \mathbf{O}_K^{\nu,\kappa}$ and that $M_a \in \mathbf{O}_K^{+,\kappa}$ for $a \neq 0$.*

(a) *There is a module $\dot{\bigotimes}_{K,a} M_a \in \mathbf{O}_K^{\nu,\kappa}$ such that, for each $M \in \mathscr{S}_{K,\kappa}$, we have*

$$\mathrm{Hom}_{\mathbf{g}_K}(\dot{\bigotimes}_{K,a} M_a, M) = \langle\!\langle M_{-m}, \ldots, M_n, DM\rangle\!\rangle_K^*.$$

(b) *We have a functorial isomorphism*

$$\langle\!\langle \dot{\bigotimes}_{K,a} M_a, DM\rangle\!\rangle_K = \langle\!\langle M_{-m}, \ldots, M_n, DM\rangle\!\rangle_K.$$

Proof It is easy to prove that $\mathbf{O}_K^{\nu,\kappa}$ is the category of the finitely generated smooth $\mathbf{g}_{K,\kappa}$-modules M such that $M(d)$ belongs to $\mathscr{O}_K^\nu$ for all $d \geqslant 1$. Thus, part (a) follows from [49, def. 1.2, thm. 1.6]. Part (b) is proved as in [28, sec. 7.10, 13.4]. ◻

Remark 8.6 The $\mathbf{g}_{K,\kappa}$-module $\dot{\bigotimes}_{K,a} M_a$ does not depend on the choice of the admissible system of charts, up to a canonical isomorphism.

Now, we set $A = [-1, 0]$ and we consider the charts γ_{-1}, γ_0, γ_1 centered at $1, \infty, 0$ respectively, given in [28, sec. 13.5]. Then, Proposition 8.5 yields modules $V \dot{\otimes}_K M$ and $M \dot{\otimes}_K V$ in $\mathbf{O}_K^{\nu,\kappa}$ for each $V \in \mathbf{O}^{+,\kappa}$ and $M \in \mathbf{O}_K^{\nu,\kappa}$.

The endomorphisms of functors $\mathbf{a}_K$, $\mathbf{c}_K$ are defined in [28, sec. 14, 18]. There, they are only defined for $\mathbf{O}^{+,\kappa}$, but for $\mathbf{O}_K^{\nu,\kappa}$ one can proceed in the same way. More precisely, since the spaces of coinvariants are finite dimensional by Lemma 8.2 and since K is an analytic algebra, the proof of [28, thm. 17.29] works equally well in our case. Hence, standard facts about linear ordinary differential equations yield a canonical isomorphism

$$\langle\!\langle V_1 \dot{\otimes}_K M_1, V_2 \dot{\otimes}_K DM_2 \rangle\!\rangle_K = \langle\!\langle V_1, M_1, V_2, DM_2 \rangle\!\rangle_K$$

for all $M_1, M_2 \in \mathbf{O}_K^{\nu,\kappa}$ and all $V_1, V_2 \in \mathbf{O}_K^{+,\kappa}$. Then, we define $\mathbf{a}_{V_1,M_1,V_2}$ using this isomorphism and Proposition 8.5 as in [28, sec. 18.2]. The other associativity constraints are constructed in the same way using the cyclic invariance of coinvariants. The braiding $\mathbf{c}_K$ is also defined as in [28], since any module from $\mathbf{O}_K^{\nu,\kappa}$ admits an action of the Sugawara operators. For more details, see the proof of Proposition 8.30 below, where some analogues $\mathbf{a}_R$, $\mathbf{c}_R$ of $\mathbf{a}_K$, $\mathbf{c}_K$ are constructed over a ring R.

8.2.2 Proof of the axioms

Now, we must check that $\mathbf{a}_K$ and $\mathbf{c}_K$ satisfy the axioms of a bimodule category over K. The proof is essentially the same as in [28]. We will give a few details for the comfort of the reader. We must prove the following.

Proposition 8.7 *The functors $\dot{\otimes}_K : \mathbf{O}_K^{+,\kappa} \times \mathbf{O}_K^{\nu,\kappa} \to \mathbf{O}_K^{\nu,\kappa}$ and $\dot{\otimes}_K : \mathbf{O}_K^{\nu,\kappa} \times \mathbf{O}_K^{+,\kappa} \to \mathbf{O}_K^{\nu,\kappa}$ give a bimodule category $(\mathbf{O}_K^{\nu,\kappa}, \dot{\otimes}_K, \mathbf{a}_K, \mathbf{c}_K)$ over the braided monoidal category $(\mathbf{O}_K^{+,\kappa}, \dot{\otimes}_K, \mathbf{a}_K, \mathbf{c}_K)$. The unit of $(\mathbf{O}_K^{+,\kappa}, \dot{\otimes}_K, \mathbf{a}_K, \mathbf{c}_K)$ is the module* $\mathbf{1}$. □

By [28, sec. 31, 32], the braided monoidal category $(\mathbf{O}_K^{+,\kappa}, \dot{\otimes}_K, \mathbf{a}_K, \mathbf{c}_K)$ is rigid with the duality functor D. This means that D is exact and that for any module $M \in \mathbf{O}_K^{+,\kappa}$ there are functorial morphisms

$$i_M : \mathbf{1} \to M \dot{\otimes}_K DM, \quad e_M : DM \dot{\otimes}_K M \to \mathbf{1}$$

such that the functor $DM \dot{\otimes}_K \bullet$ is left adjoint to $M \dot{\otimes}_K \bullet$. Equivalently, the functor $\bullet \dot{\otimes}_K M$ is left adjoint to the functor $\bullet \dot{\otimes}_K DM$. Since D is an involution, the functors above are indeed biadjoint.

Lemma 8.8 *For each $M \in \mathbf{O}_K^{\nu,\kappa}$, there are functorial isomorphisms $\mathbf{1} \dot{\otimes}_K M \to M$ and $M \dot{\otimes}_K \mathbf{1} \to M$ which satisfy the triangle axioms.*

Proof We define the unit isomorphism $\boldsymbol{\ell}_M : \mathbf{1}\dot{\otimes}_K M \to M$ to represent the isomorphism of functors given, for each $M_1, M_2 \in \mathbf{O}_K^{\nu,\kappa}$, by

$$\begin{aligned}\mathrm{Hom}_{\mathbf{g}_K}(M_1, M_2) &\simeq \langle\!\langle M_1, DM_2\rangle\!\rangle_K^* \\ &\simeq \langle\!\langle \mathbf{1}, M_1, DM_2\rangle\!\rangle_K^* \\ &\simeq \mathrm{Hom}_{\mathbf{g}_K}(\mathbf{1}\dot{\otimes}_K M_1, M_2).\end{aligned}$$

In the chain of isomorphisms above, the second one is given in Lemma 8.2(b), the other ones are as in Proposition 8.5. A similar construction yields the isomorphism $\boldsymbol{r}_M : M\dot{\otimes}_K\mathbf{1} \to M$.

Now, it is enough to check the triangle axiom for $V = \mathbf{1} \in \mathbf{O}_K^{+,\kappa}$ and $M \in \mathbf{O}_K^{\nu,\kappa}$ (then, the general version follows using the pentagon axiom for the quadruple $V, \mathbf{1}, \mathbf{1}, M$). So we must check that the composition

$$(\mathbf{1}\dot{\otimes}_K\mathbf{1})\dot{\otimes}M \xrightarrow{\mathbf{a}_{\mathbf{1},\mathbf{1},M}} \mathbf{1}\dot{\otimes}_K(\mathbf{1}\dot{\otimes}_K M) \xrightarrow{1\dot{\otimes}_K\boldsymbol{\ell}_M} \mathbf{1}\dot{\otimes}_K M$$

is given by the unit $\boldsymbol{r}_{\mathbf{1}} : \mathbf{1}\dot{\otimes}_K\mathbf{1} \to \mathbf{1}$. This follows from Proposition 8.5(b) and the invariance of coinvariants under cyclic permutation as in [28, sec. 18.2]. This allows us to identify the morphism

$$\langle\!\langle(\mathbf{1}\dot{\otimes}_K\mathbf{1})\dot{\otimes}_K M, N\rangle\!\rangle_K \to \langle\!\langle\mathbf{1}\dot{\otimes}_K(\mathbf{1}\dot{\otimes}_K M), N\rangle\!\rangle_K \to \langle\!\langle\mathbf{1}\dot{\otimes}_K M, N\rangle\!\rangle_K$$

induced by $\boldsymbol{\ell}_M, \mathbf{a}_{\mathbf{1},\mathbf{1},M}$ with the morphism

$$\langle\!\langle(\mathbf{1}\dot{\otimes}_K\mathbf{1})\dot{\otimes}_K M, N\rangle\!\rangle_K \to \langle\!\langle\mathbf{1}\dot{\otimes}_K M, N\rangle\!\rangle_K$$

induced by $\boldsymbol{r}_{\mathbf{1}}$. □

Next, let us quote the following technical lemma.

Lemma 8.9 *For $M \in \mathbf{O}_K^{+,\kappa}$ the functors $\bullet\dot{\otimes}_K M$ and $\bullet\dot{\otimes}_K DM$ on $\mathbf{O}_K^{\nu,\kappa}$ are exact and biadjoint to each other. The same holds for the functors $M\dot{\otimes}_K\bullet$ and $DM\dot{\otimes}_K\bullet$.*

Proof If $\mathbf{O}_K^{\nu,\kappa} = \mathbf{O}_K^{+,\kappa}$ the lemma follows from the rigidity of $(\mathbf{O}_K^{+,\kappa}, \dot{\otimes}_K, \mathbf{a}_K, \mathbf{c}_K)$. The general case is proved in the same way, using the rigidity of M, DM in $\mathbf{O}_K^{+,\kappa}$ and Lemma 8.8 instead of the unit axiom of $(\mathbf{O}_K^{+,\kappa}, \dot{\otimes}_K, \mathbf{a}_K, \mathbf{c}_K)$. □

We can now prove Proposition 8.7.

Proof of Proposition 8.7 We must check that the isomorphisms $\mathbf{a}_K, \mathbf{c}_K$ satisfy the pentagon and the hexagon axioms. This is proved as in proposition [28, prop. 31.2], using an auxiliary module category called the *Drinfeld category*.

Set $R_\infty = K[[\varpi]]$ and $K_\infty = K((\varpi))$. Put $\kappa_{R_\infty} = \kappa_{K_\infty} = -1/\varpi$. Consider the elements $v = \exp(\sqrt{-1}\pi\varpi)$ and $q = v^{-2}$ in R_∞. Whenever this makes sense we write $v^z = \sum_{r\in\mathbb{N}} \varpi^r(\sqrt{-1}\pi z)^r/r!$.

The category $\mathscr{O}_\infty$ of *deformation representations* of $\mathfrak{g}$ consists of the representations of $\mathfrak{g}_{R_\infty}$ on topologically free R_∞-modules M such that M is a weight $\mathfrak{t}_{R_\infty}$-module and the weights of M belong to a union of finitely many cones $\lambda - Q^+$ and the weight subspaces are free of finite type over R_∞.

Following Drinfeld and [18,28] we put on $\mathscr{O}_\infty$ a structure of a braided monoidal category $(\mathscr{O}_\infty, \otimes_{R_\infty}, \mathbf{a}_\infty, \mathbf{c}_\infty)$ where $\otimes_{R_\infty}$ is the tensor product of R_∞-modules and $\mathbf{a}_\infty$ is the Knizhnik–Zamolodchikov associator, i.e.,

$$\mathbf{a}_\infty = \{\mathbf{a}_{M_1,M_2,M_3} : (M_1 \otimes_{R_\infty} M_2) \otimes_{R_\infty} M_3 \to M_1 \otimes_{R_\infty} (M_2 \otimes_{R_\infty} M_3)\}$$

is defined in [28, sec. 19.10]. Note that we do not impose M to be of finite rank over R_∞. However, since the weight subspaces of M are free of finite type over R_∞, by standard facts about linear ordinary differential equations, the series obtained by restricting $\mathbf{a}_\infty$ to a weight subspace in the tensor product of three objects of $\mathscr{O}_\infty$ is well-defined. The braiding is given by the following formula, see [28, sec. 19.12],

$$\mathbf{c}_\infty = \{\mathbf{c}_{M_1,M_2} = v^\omega\sigma : M_1 \otimes_{R_\infty} M_2 \to M_2 \otimes_{R_\infty} M_1\}$$

where σ flips the factors and ω is the Casimir element. Recall that

- the functor $\otimes_{R_\infty}$ is R_∞-bilinear and biexact,
- there is a unit object $\mathbf{1}$, which is simple (equal to R_∞ with the trivial action), with functorial unit isomorphisms $\mathbf{1} \otimes_{R_\infty} M \to M$, $M \otimes_{R_\infty} \mathbf{1} \to M$,
- the unit $\mathbf{1}$ and the functor $\mathbf{a}_\infty$ satisfy the triangle axiom,
- the functor $\mathbf{a}_\infty$ satisfies the pentagon axiom,
- the functors $\mathbf{a}_\infty$ and $\mathbf{c}_\infty$ satisfy the hexagon axiom.

Restricting the braided monoidal structure on $\mathscr{O}_\infty$ to some parabolic subcategories, we define in the obvious way

- a braided monoidal category $(\mathscr{O}_\infty^+, \otimes_{R_\infty}, \mathbf{a}_\infty, \mathbf{c}_\infty)$ called the *Drinfeld category,* which consists of the modules which are free of finite rank over R_∞,
- a bimodule category $(\mathscr{O}_\infty^\nu, \otimes_{R_\infty}, \mathbf{a}_\infty, \mathbf{c}_\infty)$ over $(\mathscr{O}_\infty^+, \otimes_\infty, \mathbf{a}_\infty, \mathbf{c}_\infty)$.

Now, we may assume that there is a local analytic deformation ring $R \subset R_\infty$ of dimension 1 with residue field K such that the inclusion $R \subset R_\infty$ is given by the expansion at $\varpi = \infty$. Assume also that $\kappa_R = \kappa_{R_\infty} = -1/\varpi$ is the germ of an holomorphic function over some polydisc such that the specialization map $R \to K$ takes κ_R to κ_K. Since R_∞ is flat over R, the base change yields an exact functor $\mathbf{O}_R^{\kappa,\Delta} \to \mathbf{O}_{R_\infty}^{\kappa,\Delta}$.

Lemma 8.10 (a) *There is a faithful braided functor and a faithful bimodule functor*

$$(\mathbf{O}_R^{+,\kappa,\Delta}, \dot{\otimes}_R, \mathbf{a}_R, \mathbf{c}_R) \to (\mathbf{O}_{R_\infty}^{+,\kappa,\Delta}, \dot{\otimes}_{R_\infty}, \mathbf{a}_{R_\infty}, \mathbf{c}_{R_\infty}),$$
$$(\mathbf{O}_R^{\nu,\kappa,\Delta}, \dot{\otimes}_R, \mathbf{a}_R, \mathbf{c}_R) \to (\mathbf{O}_{R_\infty}^{\nu,\kappa,\Delta}, \dot{\otimes}_{R_\infty}, \mathbf{a}_{R_\infty}, \mathbf{c}_{R_\infty}).$$

(b) *There is a braided equivalence and a bimodule equivalence*

$$(\mathbf{O}_{R_\infty}^{+,\kappa,\Delta}, \dot{\otimes}_{R_\infty}, \mathbf{a}_{R_\infty}, \mathbf{c}_{R_\infty}) \to (\mathscr{O}_\infty^{+,\Delta}, \otimes_{R_\infty}, \mathbf{a}_\infty, \mathbf{c}_\infty),$$
$$(\mathbf{O}_{R_\infty}^{\nu,\kappa,\Delta}, \dot{\otimes}_{R_\infty}, \mathbf{a}_{R_\infty}, \mathbf{c}_{R_\infty}) \to (\mathscr{O}_\infty^{\nu,\Delta}, \otimes_{R_\infty}, \mathbf{a}_\infty, \mathbf{c}_\infty).$$

(c) *The specialization gives a braided functor and a bimodule functor*

$$(\mathbf{O}_R^{+,\kappa,\Delta}, \dot{\otimes}_R, \mathbf{a}_R, \mathbf{c}_R) \to (\mathbf{O}_K^{+,\kappa,\Delta}, \dot{\otimes}_K, \mathbf{a}_K, \mathbf{c}_K),$$
$$(\mathbf{O}_R^{\nu,\kappa,\Delta}, \dot{\otimes}_R, \mathbf{a}_R, \mathbf{c}_R) \to (\mathbf{O}_K^{\nu,\kappa,\Delta}, \dot{\otimes}_K, \mathbf{a}_K, \mathbf{c}_K).$$

Proof Since R is a regular local ring of dimension 1, we define the functor $\dot{\otimes}_R$ and the morphisms of functors $\mathbf{a}_R$, $\mathbf{c}_R$ as in [28, sec. 29, 31]. We may as well define them as in Sect. 8.3 below. Part (a) follows from Lemma 8.22. Part (b) is proved as in [28, sec. 31]. Part (c) is proved as in [28, thm. 29.1]. □

We can now finish the proof of Proposition 8.7. Composing the functors in (a), (b), we get faithful functors $(\mathbf{O}_R^{+,\kappa,\Delta}, \dot{\otimes}_R, \mathbf{a}_R, \mathbf{c}_R) \to (\mathscr{O}_\infty^{+,\Delta}, \otimes_{R_\infty}, \mathbf{a}_\infty, \mathbf{c}_\infty)$ and $(\mathbf{O}_R^{\nu,\kappa,\Delta}, \dot{\otimes}_R, \mathbf{a}_R, \mathbf{c}_R) \to (\mathscr{O}_\infty^{\nu,\Delta}, \otimes_{R_\infty}, \mathbf{a}_\infty, \mathbf{c}_\infty)$. This implies that $\mathbf{a}_R$, $\mathbf{c}_R$ satisfy the pentagon and the hexagon axioms. Hence, from (c), we deduce that $\mathbf{a}_K$, $\mathbf{c}_K$ also satisfy the pentagon and the hexagon axioms. The details are left to the reader.

8.2.3 Properties of the functor $\dot{\otimes}_K$

We have $\mathscr{O}_K^{+,\Delta} = \mathscr{O}_K^{+}$, because the category $\mathscr{O}_K^{+}$ is semi-simple. The tensor product equips the $\mathbb{C}$-vector space $[\mathscr{O}_K^{+,\Delta}]$ with a commutative $\mathbb{C}$-algebra structure and the $\mathbb{C}$-vector space $[\mathscr{O}_K^{\nu,\Delta}]$ with a bimodule structure over $[\mathscr{O}_K^{+,\Delta}]$. We'll need the following properties of the functor $\dot{\otimes}_K$.

Proposition 8.11 (a) *The functor $\dot{\otimes}_K$ preserves the Δ-filtered modules.*
(b) *The functor $\dot{\otimes}_K$ preserves the tilting modules.*
(c) *The functor $\dot{\otimes}_K$ is biexact on $\mathbf{O}_K^{+,\kappa,\Delta} \times \mathbf{O}_K^{\nu,\kappa,\Delta}$ and on $\mathbf{O}_K^{\nu,\kappa,\Delta} \times \mathbf{O}_K^{+,\kappa,\Delta}$. It equips $[\mathbf{O}_K^{+,\kappa,\Delta}]$ with a commutative $\mathbb{C}$-algebra structure and $[\mathbf{O}_K^{\nu,\kappa,\Delta}]$ with a bimodule structure over $[\mathbf{O}_K^{+,\kappa,\Delta}]$.*
(d) *The functor $\mathscr{I}nd$ gives a $\mathbb{C}$-algebra isomorphism $[\mathscr{O}_K^{+,\Delta}] \to [\mathbf{O}_K^{+,\kappa,\Delta}]$ and a module isomorphism $[\mathscr{O}_K^{\nu,\Delta}] \to [\mathbf{O}_K^{\nu,\kappa,\Delta}]$.*

Proof First, we prove part (a). Fix $M_1 \in \mathbf{O}_K^{+,\kappa,\Delta}$ and $M_2 \in \mathbf{O}_K^{\nu,\kappa,\Delta}$. The module $M_1 \dot{\otimes}_K M_2$ belongs to the category $\mathbf{O}_K^{\nu,\kappa}$ by Proposition 8.5. We must prove that it lies in $\mathbf{O}_K^{\nu,\kappa,\Delta}$.

Fix β such that M_2 and $M_1 \dot{\otimes}_K M_2$ belong to the Serre subcategory ${}^\beta\mathbf{O}_K^{\nu,\kappa}$ of $\mathbf{O}_K^{\nu,\kappa}$. Since ${}^\beta\mathbf{O}_K^{\nu,\kappa}$ is a highest weight category with a duality functor $\mathscr{D}$, it is enough to check that for $M_3 \in \Delta({}^\beta\mathbf{O}_K^{\nu,\kappa})$ we have the equality $\mathrm{Ext}^1_{{}^\beta\mathbf{O}_K^{\nu,\kappa}}(M_1 \dot{\otimes}_K M_2, \mathscr{D}M_3) = 0$.

Fix an exact sequence $0 \to Q \to P \to M_3 \to 0$ with P a projective module in ${}^\beta\mathbf{O}_K^{\nu,\kappa}$. Since P, M_3 have Δ-filtrations, the module Q is again a Δ-filtered object of ${}^\beta\mathbf{O}_K^{\nu,\kappa}$. Since P is projective, we have $\mathrm{Ext}^1_{{}^\beta\mathbf{O}_K^{\nu,\kappa}}(M_1 \dot{\otimes}_K M_2, \mathscr{D}P) = 0$. Therefore, since $\mathscr{D}$ is exact and contravariant, the long exact sequence of the Ext-group and Proposition 8.5 yield a vector space exact sequence

$$\begin{aligned}&0 \to \langle\langle M_1, M_2, {}^\dagger M_3\rangle\rangle_K^* \to \langle\langle M_1, M_2, {}^\dagger P\rangle\rangle_K^* \to \langle\langle M_1, M_2, {}^\dagger Q\rangle\rangle_K^* \\ &\quad \to \mathrm{Ext}^1_{{}^\beta\mathbf{O}_K^{\nu,\kappa}}(M_1 \dot{\otimes}_K M_2, \mathscr{D}M_3) \to 0.\end{aligned}$$

Thus, we get the equality of dimensions

$$\begin{aligned}\dim \mathrm{Ext}^1_{{}^\beta\mathbf{O}_K^{\nu,\kappa}}(M_1 \dot{\otimes}_K M_2, \mathscr{D}M_3) = \dim \langle\langle M_1, M_2, {}^\dagger P\rangle\rangle_K \\ - \dim \langle\langle M_1, M_2, {}^\dagger Q\rangle\rangle_K \\ - \dim \langle\langle M_1, M_2, {}^\dagger M_3\rangle\rangle_K.\end{aligned}$$

The right hand side is zero by the following lemma.

Lemma 8.12 *For $M_2, M_3 \in \mathbf{O}_K^{\nu,\kappa,\Delta}$ and $M_1 \in \mathbf{O}_K^{+,\kappa,\Delta}$ we have*

$$\begin{aligned}\dim \langle\langle M_1, M_2, {}^\dagger M_3\rangle\rangle_K = \sum (M_1 : \mathbf{M}(\lambda_1)_+)\,(M_2 : \mathbf{M}(\lambda_2)_\nu) \\ \times (M_3 : \mathbf{M}(\lambda_3)_\nu)\,(L(\lambda_1) \otimes M(\lambda_2)_\nu : M(\lambda_3)_\nu),\end{aligned}$$

where the sum is over all $\lambda_1 \in P_K^+$ and $\lambda_2, \lambda_3 \in P_K^\nu$.

Proof Let $d(M_1, M_2, {}^\dagger M_3)$ denote the right hand side in the equality of the lemma.

First, assume that $M_3 = \mathbf{M}(\lambda_3)_\nu$, $M_2 = \mathbf{M}(\lambda_2)_\nu$ and $M_1 = \mathbf{M}(\lambda_1)_+$. We have ${}^\dagger M_3 = \mathscr{I}nd({}^\dagger M(\lambda_3)_\nu)$. Thus ${}^\dagger M_3$ is again a generalized Weyl module and Lemma 8.2(a) yields

$$\begin{aligned}\langle\langle M_1, M_2, {}^\dagger M_3\rangle\rangle_K &= \langle L(\lambda_1), M(\lambda_2)_\nu, {}^\dagger M(\lambda_3)_\nu\rangle_K, \\ &= \mathrm{Hom}_{\mathfrak{g}_K}(L(\lambda_1) \otimes_K M(\lambda_2)_\nu, \mathscr{D}M(\lambda_3)_\nu)^*.\end{aligned}$$

Since $L(\lambda_1)\otimes_K M(\lambda_2)_\nu \in \mathscr{O}_K^{\nu,\Delta}$, by [14, prop. A.2.2$(ii)$] we get dim $\langle\langle M_1, M_2, {}^\dagger M_3\rangle\rangle_K = (L(\lambda_1)\otimes_K M(\lambda_2)_\nu : M(\lambda_3)_\nu)$.

The same argument implies that dim $\langle\langle M_1, M_2, {}^\dagger M_3\rangle\rangle_K = d(M_1, M_2, {}^\dagger M_3)$ if M_1, M_2, M_3 are generalized Weyl modules.

Now, we concentrate on the general case. First, observe that using the *third* construction of $\dot\otimes_K$ in [28, sec. 6] it is easy to check that $\dot\otimes_K$ is right biexact. Further, by Proposition 8.5, we have $\langle\langle M_1, M_2, {}^\dagger M_3\rangle\rangle_K^* = \mathrm{Hom}_{\mathbf{g}_K}(M_1\dot\otimes_K M_2, \mathscr{D}M_3)$. Thus the left hand side is left exact in each of its variables. So, given exact sequences $M_a^{(2)} \to M_a^{(1)} \to M_a^{(3)} \to 0$ of Δ-filtered modules with $a = 1, 2, 3$, we have

$$\dim \langle\langle M_1^{(1)}, M_2^{(1)}, {}^\dagger M_3^{(1)}\rangle\rangle_K \leqslant \sum_{\alpha,\beta,\gamma=2,3} \dim \langle\langle M_1^{(\alpha)}, M_2^{(\beta)}, {}^\dagger M_3^{(\gamma)}\rangle\rangle_K. \quad (8.1)$$

Using the first part of the proof (i.e., the case of generalized Weyl modules) and (8.1), we get that for any $M_2, M_3 \in \mathbf{O}_K^{\nu,\kappa,\Delta}$ and $M_1 \in \mathbf{O}_K^{+,\kappa,\Delta}$ we have

$$\dim \langle\langle M_1, M_2, {}^\dagger M_3\rangle\rangle_K \leqslant d(M_1, M_2, {}^\dagger M_3).$$

To prove the equality, for each a we fix an exact sequence $0 \to M_a^{(2)} \to M_a^{(1)} \to M_a^{(3)} \to 0$ of Δ-filtered modules such that $M_a^{(1)}$ is a generalized Weyl module and $M_a^{(3)} = M_a$. Clearly, such exact sequences always exist. Then, we have

$$\begin{aligned}
&\dim \langle\langle M_1^{(1)}, M_2^{(1)}, {}^\dagger M_3^{(1)}\rangle\rangle_K = d(M_1^{(1)}, M_2^{(1)}, {}^\dagger M_3^{(1)}),\\
&\dim \langle\langle M_1^{(\alpha)}, M_2^{(\beta)}, {}^\dagger M_3^{(\gamma)}\rangle\rangle_K \leqslant d(M_1^{(\alpha)}, M_2^{(\beta)}, {}^\dagger M_3^{(\gamma)}), \quad \forall \alpha, \beta, \gamma.
\end{aligned}$$

Thus the equality follows from (8.1). □

Next, we prove part (b). Assume that $M_1 \in \mathbf{O}_K^{+,\kappa}$ and $M_2 \in \mathbf{O}_K^{\nu,\kappa}$ are tilting. We must check that $M_1\dot\otimes_K M_2$ is still tilting. For $N \in \mathbf{O}_K^{\nu,\kappa,\Delta}$ we must prove that $\mathrm{Ext}^1_{\mathbf{O}_K^{\nu,\kappa}}(N, M_1\dot\otimes_K M_2) = 0$. Since the functors $\bullet\dot\otimes_K M_2$ and $\bullet\dot\otimes_K DM_2$ are exact and biadjoint by Lemma 8.9, we have

$$\mathrm{Ext}^1_{\mathbf{O}_K^{\nu,\kappa}}(N, M_1\dot\otimes_K M_2) = \mathrm{Ext}^1_{\mathbf{O}_K^{\nu,\kappa}}(N\dot\otimes_K DM_2, M_1).$$

Since DM_2 is Δ-filtered, part (a) yields $\mathrm{Ext}^1_{\mathbf{O}_K^{\nu,\kappa}}(N\dot\otimes_K DM_2, M_1) = 0$.

Finally, we prove parts (c), (d). The functor $\dot\otimes_K$ is right biexact. The same argument as above using Proposition 8.5 and Lemma 8.12 implies that it is biexact on Δ-filtered modules. More precisely, for each $M \in \mathbf{O}_K^{\nu,\kappa,\Delta}$ the functor $\mathrm{Hom}_{\mathbf{g}_K}(\bullet\dot\otimes\bullet, \mathscr{D}M)$ on $\mathbf{O}_K^{+,\kappa,\Delta}\times\mathbf{O}_K^{\nu,\kappa,\Delta}$ is exact and $\bullet\dot\otimes_K\bullet$ takes values in $\mathbf{O}_K^{\nu,\kappa,\Delta}$. Thus, if $0 \to N_1 \to N_2 \to N_3 \to 0$ is exact in $\mathbf{O}_K^{+,\kappa,\Delta}$ and if $N \in \mathbf{O}_K^{\nu,\kappa,\Delta}$, then we have an exact sequence $N_1\dot{\otimes}_K N \to N_2\dot{\otimes}_K N \to N_3\dot{\otimes}_K N \to 0$ and $(N_2\dot{\otimes} N : M_3) = (N_1\dot{\otimes}_K N : M_3) + (N_3\dot{\otimes}_K N : M_3)$ if M_3 is a parabolic Verma module. Thus, also we have an exact sequence $0 \to N_1\dot{\otimes}_K N \to N_2\dot{\otimes}_K N \to N_3\dot{\otimes}_K N \to 0$.

Since it is exact, the tensor product $\dot{\otimes}_K$ factors to the Grothendieck groups $[\mathbf{O}_K^{+,\kappa,\Delta}]$ and $[\mathbf{O}_K^{\nu,\kappa,\Delta}]$. The exact functor $\mathscr{I}nd(\bullet)$ gives $\mathbb{C}$-linear isomorphisms $[\mathscr{O}_K^{+,\Delta}] \to [\mathbf{O}_K^{+,\kappa,\Delta}]$ and $[\mathscr{O}_K^{\nu,\Delta}] \to [\mathbf{O}_K^{\nu,\kappa,\Delta}]$, because the parabolic Verma modules form bases of the Grothendieck groups of Δ-filtered objects. The compatibility with the monoidal structures follows from Proposition 8.5 and Lemma 8.12. □

Remark 8.13 By [28, prop. 31.2] the braided monoidal category $(\mathbf{O}_K^{+,\kappa}, \dot{\otimes}_K, \mathbf{a}, \mathbf{c})$ admits a *balancing*. More precisely, we have

$$\mathbf{c}^2 = \exp(-2\pi\sqrt{-1}\mathfrak{L}_0)\big(\exp(2\pi\sqrt{-1}\mathfrak{L}_0)\dot{\otimes}_K \exp(2\pi\sqrt{-1}\mathfrak{L}_0)\big). \quad (8.2)$$

The proof in loc. cit. implies that (8.2) holds also for tensor products of modules from $\mathbf{O}_K^{\nu,\kappa}$ and $\mathbf{O}_K^{+,\kappa}$.

8.3 The monoidal structure on O over a ring

Let R be either a field or a regular local ring of dimension $\leqslant 2$ with residue field k. Assume that $\kappa_{\mathrm{k}} = -e$ where e is a positive integer. In this section we'll construct a version $\dot{\otimes}_R$ of the functor $\dot{\otimes}_K$ above, which is defined over the ring R.

8.3.1 Definition of the functor $\dot{\otimes}_R$

Let $\spadesuit = [-m, n]$. Fix a module $M_0 \in \mathbf{O}_R^{\nu,\kappa,\Delta}$ and a module $M_a \in \mathbf{O}_R^{+,\kappa,\Delta}$ for each $a \neq 0$. The goal of this section is to construct a module $\dot{\bigotimes}_{R,a} M_a$ in $\mathbf{O}_R^{\nu,\kappa,\Delta}$, where a runs over $\spadesuit$, which is functorial in the M_a's. The construction of the $\mathbf{g}_{R,\kappa}$-module $\dot{\bigotimes}_{R,a} M_a$ is essentially the same as in [28, sec. 29]. However, our setting differs from that of [28] from several aspects

- the category $\mathbf{O}_R^{\nu,\kappa}$ is defined over a regular local ring R of dimension $\leqslant 2$,
- the modules M_a do not all belong to the category $\mathbf{O}_R^{+,\kappa}$,
- the modules M_a may not have integral weights,
- we work with $\mathbf{g}_R$-modules, rather than $\mathbf{g}'_R$-modules.

The last point is easy to deal with: we'll switch from $\mathbf{g}_R$-modules to $\mathbf{g}'_R$-modules as in Remark 5.2 without mentioning it explicitly.

First, assume first that $R = K$ is a field and $M_a \in \mathscr{S}_{K,\kappa}$ for each a. Then, the module $\dot{\bigotimes}_{K,a} M_a$ is defined in Proposition 8.5. More precisely, if $K = \mathbb{C}$, then the smooth $\mathbf{g}'_\kappa$-module $\dot{\bigotimes}_{K,a} M_a$ is constructed in [28]. It is proved there that, if M_a belongs to $\mathbf{O}^{+,\kappa}$ for each a, then $\dot{\bigotimes}_{K,a} M_a$ belongs also to $\mathbf{O}^{+,\kappa}$. Next, it is proved in [49] that, if $M_a \in \mathbf{O}^{+,\kappa}$ for $a \neq 0$ and if $M_0 \in \mathbf{O}^{\nu,\kappa}$, then $\dot{\bigotimes}_{K,a} M_a$ belongs to $\mathbf{O}^{\nu,\kappa}$. If $K \neq \mathbb{C}$, then we define $\dot{\bigotimes}_{K,a} M_a$ as in the case $K = \mathbb{C}$.

Now, let R be any commutative noetherian $\mathbb{C}$-algebra with 1. Set $A = [-m, n+1]$ and $\heartsuit = \{n+1\}$. To simplify the notation we'll also write $\heartsuit = n+1$. Recall that γ_a is a chart on C centered at x_a for each $a \in A$, and that $D_R = R[C \backslash \{x_a; a \in A\}]$. Let $\widehat{\Gamma}_R$ be the central extension of $\Gamma_R = \mathfrak{g} \otimes D_R$ by R associated with the cocycle $(\xi_1 \otimes f_1, \xi_2 \otimes f_2) \mapsto \mathrm{Res}_{\gamma_\heartsuit=0}(f_2 df_1)$. Set $\kappa = c + N$ and $\kappa' = -c + N$. The quotient by the ideal $(\mathbf{1} - c)$ yields an algebra homomorphism $U(\widehat{\Gamma}_R) \to \Gamma_{R,\kappa}$.

Lemma 8.14 (a) *There is an R-algebra homomorphism $\Gamma_{R,\kappa'} \to \mathbf{g}^{\spadesuit}_{R,\kappa}$ such that $\xi \otimes f \mapsto \xi \otimes {}^{\spadesuit}f$.*

(b) *There is an R-algebra homomorphism $\mathbf{g}'_{R,\kappa'} \to \Gamma_{R,\kappa'}$ such that $\xi \otimes f(t) \mapsto \xi \otimes f(\gamma_\heartsuit)$, and an R-algebra homomorphism $\Gamma_{R,\kappa'} \to \mathbf{g}^{\heartsuit}_{R,\kappa'}$ such that $\xi \otimes f \mapsto \xi \otimes {}^{\heartsuit}f$.*

(c) *Composing the maps in* (b) *we get an R-algebra embedding $\mathbf{g}'_{R,\kappa'} \to \mathbf{g}^{\heartsuit}_{R,\kappa'}$ such that $\xi \otimes f(t) \mapsto \xi \otimes f(t_\heartsuit)$.*

Proof Part (a) is standard. To prove (b), observe that the chart $\gamma_\heartsuit$ can be regarded as an element in the subalgebra $\{f \in D_R;\ f(x_\heartsuit) = 0\}$. Thus, we have an R-algebra homomorphism $R[t, t^{-1}] \to D_R$ such that $f(t) \mapsto f(\gamma_\heartsuit)$ and an R-algebra homomorphism $D_R \to R((t_\heartsuit))$ such that $f \mapsto {}^{\heartsuit}f$. $\square$

Now, for each $a \in \spadesuit$ we fix a smooth module $M_a \in \mathscr{S}_{R,\kappa}$ which is a weight $\mathfrak{t}_R$-module. Set $W_R = \bigotimes_{R,a\in\spadesuit} M_a$. Since the M_a's are smooth, the R-module W_R has a natural structure of $\mathbf{g}^{\spadesuit}_{R,\kappa}$-module. We view W_R as a $\Gamma_{R,\kappa'}$-module via the map in Lemma 8.14(a). Note that W_R is a weight $\mathfrak{t}_R$-module.

For $d \geqslant 1$ let $G_{R,d}$ be the R-submodule of $\Gamma_{R,\kappa'}$ spanned by the products of d elements in $\mathfrak{g} \otimes D^1_R$ with $D^1_R = \{f \in D_R;\ f(x_\heartsuit) = 0\}$. Note that $G_{R,d}$ is a weight $\mathfrak{t}_R$-module for the adjoint action. We have the following natural decreasing filtration of weight $\mathfrak{t}_R$-modules $W_R \supset G_{R,1}W_R \supset G_{R,2}W_R \supset \cdots$ Consider the weight $\mathfrak{t}_R$-module $W_{R,d}$ given by $W_{R,d} = W_R / G_{R,d} W_R$. Let

$$W_{R,d} = \bigoplus_{\lambda \in P_R} W_{R,d,\lambda}$$

be the decomposition of $W_{R,d}$ into the sum of its weight R-submodules.

The modules $W_{R,d}$ with $d \geqslant 1$ form a projective system. The limit $\widehat{W}_R = \lim\limits_{\leftarrow} W_{R,d}$ in the category of weight $\mathfrak{t}_R$-modules decomposes as the direct sum of weight R-submodules $\widehat{W}_R = \bigoplus_{\lambda \in P_R} \widehat{W}_{R,\lambda}$, where $\widehat{W}_{R,\lambda}$ is the projective limit of R-modules $\lim\limits_{\leftarrow} W_{R,d,\lambda}$.

For each $\lambda \in P_R$ and each $d \geqslant 1$, we define the R-module

$$Z_{R,d,\lambda} = (W_{R,d,\lambda})^*.$$

The R-modules $Z_{R,d,\lambda}$ with $d \geqslant 1$ form an inductive system of R-submodules $Z_{R,1,\lambda} \subset Z_{R,2,\lambda} \subset \cdots$ Consider the weight $\mathfrak{t}_R$-module $Z_{R,\infty}$ given by $Z_{R,\infty} = \bigoplus_{\lambda \in P_R} Z_{R,\infty,\lambda}$, where $Z_{R,\infty,\lambda} = \lim\limits_{\rightarrow} Z_{R,d,\lambda}$.

From now on, we'll assume that R is a regular ring of dimension $\leqslant 2$ and that the modules M_0, M_a belong to $\mathbf{O}_R^{\nu,\kappa,f}$, $\mathbf{O}_R^{+,\kappa,f}$ respectively, for each $a \in \spadesuit$ with $a \neq 0$.

Lemma 8.15 (a) *The R-module $W_{R,d,\lambda}$ is finitely generated.*
(b) *The R-module $Z_{R,d,\lambda}$ is finitely generated and projective.*

Proof Since M_a belongs to $\mathbf{O}_R^{\kappa}$, there is an integer $d_a \geqslant 0$ such that M_a is generated by the R-submodule $M_a(d_a)$ as a $\mathbf{g}_{R,\kappa}$-module. Then, the same proof as in [28, prop. 7.4] implies that

$$W_R = X_{R,d} W_R^1 + G_{R,d} W_R, \quad W_R^1 = \bigotimes_{R,a} M_a(d_a), \tag{8.3}$$

where $X_{R,d}$ is the R-submodule of $\Gamma_{R,\kappa'}$ spanned by the product of $< d$ elements in $\mathfrak{g} \otimes \gamma_{\heartsuit}$. The right hand side of the first equality in (8.3) is defined using the $\Gamma_{R,\kappa'}$-module structure on W_R.

Now, since $M_a \in \mathbf{O}_R^{\kappa}$ and R is noetherian, the weight $\mathfrak{t}_R$-submodules of the $\mathfrak{t}_R$-submodule $M_a(d_a) \subset M_a$ are finitely generated over R. Indeed, the weight $\mathbf{t}_R$-submodules of M_a are finitely generated because $M_a \in \mathbf{O}_R^{\kappa}$, and each weight $\mathfrak{t}_R$-submodule of $M_a(d_a)$ is contained in the sum of a finite number of weight $\mathbf{t}_R$-submodules of M_a (because M_a is flat over R and the result is well-known over the fraction field K of R). Therefore, part (a) of the lemma is an easy consequence of (8.3).

Since R is a regular ring of dimension $\leqslant 2$, any finitely generated reflexive R-module is projective. Since it is the dual of a finitely generated R-module, the R-module $Z_{R,d,\lambda}$ is finitely generated and reflexive. Hence it is projective as an R-module for each d, λ. $\square$

Under the previous hypothesis, we can now prove the following.

Lemma 8.16 (a) *There is a natural representation of $\mathbf{g}'_{R,\kappa'}$ on $\widehat{W}_R$.*
(b) *There is a natural smooth representation of $\mathbf{g}'_{R,\kappa}$ on $Z_{R,\infty}$.*

Proof The proof is adapted from [28]. We will be sketchy. Recall that W_R is a $\Gamma_{R,\kappa'}$-module. The $\Gamma_{R,\kappa'}$-action does not induce a $\Gamma_{R,\kappa'}$-action on $\widehat{W}_R$ in a natural way. However, under the second map in Lemma 8.14(b), it descends to a representation of $\mathbf{g}^{\heartsuit}_{R,\kappa'}$ on $\widehat{W}_R$ as in [28, sec. 4.9]. More precisely, given $f(t_\heartsuit)$ in $t_\heartsuit^{-n}R[[t_\heartsuit]]$ for some $n \in \mathbb{N}$, we fix a sequence of elements $g_1, g_2, \dots$ in D_R such that ${}^\heartsuit g_d - f(t_\heartsuit) \in t_\heartsuit^d R[[t_\heartsuit]]$ for each d, and we define the action of $\xi \otimes f(t_\heartsuit)$ on the element $(w_d) \in \widehat{W}_R$, with $w_d \in W_{R,d}$ and $d \geqslant 1$, by setting $\xi \otimes f(t_\heartsuit) \cdot (w_d) = (\xi \otimes g_d \cdot w_{n+d})$.

Twisting this representation by the map $\mathbf{g}'_{R,\kappa'} \to \mathbf{g}^{\heartsuit}_{R,\kappa'}$ in Lemma 8.14(c), we get a representation of $\mathbf{g}'_{R,\kappa'}$ on $\widehat{W}_R$. Taking its dual, we get a representation of $\mathbf{g}'_{R,\kappa}$ on $Z_{R,\infty}$. See [28, sec. 6.3] for details.

The R-module $Z_{R,\infty}$ is flat, because the direct summand $Z_{R,\infty,\lambda}$ is the limit of the inductive system of flat submodules $(Z_{R,d,\lambda})$. To prove that it is smooth, it is enough to check that $Z_{R,\infty} = Z_{R,\infty}(\infty)$. This is obvious, because we have $Z_{R,d} \subset Z_{R,\infty}(d)$, where $Z_{R,d} = \bigoplus_\lambda Z_{R,d,\lambda}$. □

Now, we consider the behavior of $Z_{R,\infty}$ under flat base changes.

Lemma 8.17 *Let S be a commutative noetherian R-algebra with* 1 *which is flat as an R-module. Then, we have a canonical $\mathbf{g}'_{S,\kappa}$-module isomorphism $SZ_{R,\infty} = Z_{S,\infty}$.*

Proof Since taking tensor products is right exact, we have a canonical S-module isomorphism $SW_{R,d,\lambda} = W_{S,d,\lambda}$. Since S is flat over R, for any R-modules X, Y such that X is finitely presented over R, the canonical homomorphism $S\,\mathrm{Hom}_R(X, Y) \to \mathrm{Hom}_S(SX, SY)$ is an isomorphism. By Lemma 8.15, the R-module $W_{R,d,\lambda}$ is finitely generated. Therefore, since direct limits commute with tensor products, we have

$$SZ_{R,\infty} = \bigoplus_\lambda \lim_{\to} S\,\mathrm{Hom}_R(W_{R,d,\lambda}, R) = \bigoplus_\lambda \lim_{\to} \mathrm{Hom}_S(W_{S,d,\lambda}, S) = Z_{S,\infty}.$$

□

We can now prove the following

Lemma 8.18 *Assume that $R = K$ is a field.*

(a) *The $\mathbf{g}'_{K,\kappa}$-module $Z_{K,\infty}$ belongs to $\mathbf{O}^{\nu,\kappa}_K$.*
(b) *The Sugawara operator $\mathfrak{L}_0$ preserves the finite dimensional K-subspace $Z_{K,d,\lambda}$ of $Z_{K,\infty}$ for each d, λ.*
(c) *The characteristic polynomial of $\mathfrak{L}_0$ on $Z_{K,d,\lambda}$ is a product of linear factors with coefficients in R.*

Proof For any smooth modules $M_a \in \mathscr{S}_{K,\kappa}$, a construction of the $\mathbf{g}'_\kappa$-module $\dot{\bigotimes}_{K,a} M_a$ is given in [28, sec. 4]. It is called there the *first construction*. The smooth $\mathbf{g}'_\kappa$-module $DZ_{K,\infty}$ is precisely the one given by the *third construction* in [28, sec. 6]. If M_a belongs to $\mathbf{O}_K^{+,\kappa}$ for all a, then the first and third constructions coincide by [28, thm. 7.9]. If $M_0 \in \mathbf{O}_K^{\nu,\kappa}$ and $M_a \in \mathbf{O}_K^{+,\kappa}$ for each $a \neq 0$, then both constructions coincide by [49, prop. 5.8], and the first construction yields a module in $\mathbf{O}_K^{\nu,\kappa}$ by [49, thm. 1.6]. This proves part (a).

Part (b) is a standard computation using the relation $[\mathfrak{L}_0, \xi^{(r)}] = -r\xi^{(r)}$ for each $\xi \in \mathfrak{g}$ and $r \in \mathbb{Z}$.

Since $Z_{K,\infty} \in \mathbf{O}_K^{\nu,\kappa}$, part (c) follows from elementary properties of the action of the Sugawara operator on objects of $\mathbf{O}_K^{\nu,\kappa}$. □

Now, we come back to the case where R is a regular local ring of dimension $\leqslant 2$.

Lemma 8.19 *There is a natural smooth representation of* $\mathbf{g}_{R,\kappa}$ *on* $Z_{R,\infty}$.

Proof Since the $\mathbf{g}'_{R,\kappa}$-module $Z_{R,\infty}$ is smooth, it is equipped with a canonical action of the Sugawara operator $\mathfrak{L}_0$. For each $r \in R$ we set

$$^rZ_{R,\infty} = \{v \in Z_{R,\infty};\ (\mathfrak{L}_0 - r)^n v = 0,\ n \gg 0\}.$$

Replacing R by K everywhere in the construction above, we get the $\mathbf{g}'_{K,\kappa}$-module $Z_{K,\infty}$. Since the $\mathbf{g}'_{R,\kappa}$-module $Z_{R,\infty}$ is smooth, it is flat over R. Thus, we have an obvious inclusion $Z_{R,\infty} \subset KZ_{R,\infty} = Z_{K,\infty}$. Hence, by Lemma 8.18, we have a direct sum decomposition $Z_{R,\infty} = \bigoplus_r {}^rZ_{R,\infty}$.

Therefore, we can consider the R-linear operator ∂ on $Z_{R,\infty}$ which acts by multiplication with $(-r)$ on the R-submodule ${}^rZ_{R,\infty}$. It equips $Z_{R,\infty}$ with the structure of a smooth $\mathbf{g}_{R,\kappa}$-module. □

Definition 8.20 Assume that R is a regular local ring of dimension $\leqslant 2$. Let $\spadesuit = [-m, n]$. Assume that $M_0 \in \mathbf{O}_R^{\nu,\kappa,f}$ and $M_a \in \mathbf{O}_R^{+,\kappa,f}$ for $a \in \spadesuit$ with $a \neq 0$. Then, we define the $\mathbf{g}_{R,\kappa}$-module $\dot{\bigotimes}_{R,a} M_a$, where a runs over the set $\spadesuit$, to be equal to $DZ_{R,\infty}$. It is a smooth module by Lemma 8.19 and by the definition of D.

8.3.2 Properties of the functor $\dot{\otimes}_R$

Set $\spadesuit = [-m, n]$. Our next goal is to prove the following.

Proposition 8.21 (a) *Assume that* $M_0 \in \mathbf{O}_R^{\nu,\kappa,\Delta}$ *and* $M_a \in \mathbf{O}_R^{+,\kappa,\Delta}$ *for each* $a \in \spadesuit$ *with* $a \neq 0$. *Then, there is a module* $\dot{\bigotimes}_{R,a} M_a$ *in* $\mathbf{O}_R^{\nu,\kappa,\Delta}$ *which* *is functorial in the M_a's and such that for each $M \in \mathbf{O}_R^{\nu,\kappa,f}$ we have a functorial isomorphism*

$$\mathrm{Hom}_{\mathbf{g}_R}(\dot{\bigotimes}_{R,a} M_a, M) = \langle\langle M_{-m}, \ldots, M_n, DM\rangle\rangle_R^*.$$

(b) *The functor $\dot{\otimes}_R$ commutes with flat base change (of the ring R).* □

First, assume that $M_0 \in \mathbf{O}_R^{\nu,\kappa,f}$ and $M_a \in \mathbf{O}_R^{+,\kappa,f}$ for each $a \in \spadesuit$ with $a \neq 0$.

Lemma 8.22 *Let S be a commutative noetherian R-algebra with 1 which is regular of dimension $\leqslant 2$ and which is flat as an R-module. We have canonical $\mathbf{g}_{S,\kappa}$-module isomorphism $S(\dot{\bigotimes}_{R,a} M_a) = \dot{\bigotimes}_{S,a} SM_a$.*

Proof By Lemma 8.17 we have $SZ_{R,\infty} = Z_{S,\infty}$. Thus, the lemma follows from the proof of Lemma 5.3, which insures that D commutes with base change. □

Next, we prove the following.

Lemma 8.23 *We have $\dot{\bigotimes}_{R,a} M_a \in \mathbf{O}_R^{\nu,\kappa,f}$. The functor $\dot{\bigotimes}_{R,a}$ on $\mathbf{O}_R^{\nu,\kappa,f}$ and $\mathbf{O}_R^{+,\kappa,f}$ is right exact.*

Proof The $\mathbf{g}'_{R,\kappa'}$-action on W_R yields a representation of $\mathbf{g}'_{R,\kappa}$ on ${}^\sharp W_R$. Consider the R-submodule $W_R^1 \subset W_R$ introduced in (8.3). We claim that ${}^\sharp W_R^1$ is a $\mathbf{g}'_{R,+}$-submodule of ${}^\sharp W_R$. Indeed, the element $\xi \otimes f(t)$ in $\mathbf{g}'_{R,\kappa}$ acts on ${}^\sharp W_R$ by the operator $\sum_{a\in\spadesuit} \xi \otimes {}^a f(-1/\gamma_\heartsuit)$. Further, for each $a \in \spadesuit$ the function $1/\gamma_\heartsuit$ is regular at x_a and, thus, since the system of charts is defined over $\mathbb{C}$, the expansion ${}^a(1/\gamma_\heartsuit)$ is a well-defined Laurent formal series in $\mathbb{C}[[t_a]]$. Therefore we have ${}^a f(-1/\gamma_\heartsuit) \in R[[t_a]]$ for each $f \in R[t]$. We deduce that there is a $\mathbf{g}'_{R,\kappa}$-homomorphism

$$\mathscr{I}nd({}^\sharp W_R^1) \to {}^\sharp W_R. \tag{8.4}$$

Next, recall that the first map in Lemma 8.14(b) yields a $\mathbf{g}'_{R,\kappa'}$-action on W_R and that $\mathbf{g}'_{R,\kappa'}$ acts on $\widehat{W}_R$ by Lemma 8.16. By definition of the actions, the canonical R-module homomorphism $W_R \to \widehat{W}_R$ is a $\mathbf{g}'_{R,\kappa'}$-module homomorphism. Taking the dual of W_R in the category of weight $\mathfrak{t}_R$-modules, we get the $\mathbf{g}'_{R,\kappa}$-module Z_R given by

$$Z_R = \bigoplus_{\lambda\in P_R} Z_{R,\lambda}, \quad Z_{R,\lambda} = (W_{R,\lambda})^*. \tag{8.5}$$

Twisting (8.4) by $\sharp$ and taking its transpose, we get a $\mathbf{g}'_{R,\kappa}$-homomorphism $Z_R \to {}^\sharp\mathscr{I}nd({}^\sharp W_R^1)^*$. Since $Z_{R,\infty} \subset Z_R(\infty)$, this map restricts to a $\mathbf{g}'_{R,\kappa}$-homomorphism

$$ {}^\dagger Z_{R,\infty} \to \mathscr{D}\mathscr{I}nd({}^\sharp W_R^1). \tag{8.6} $$

Using (8.3) it is easy to see that the map (8.6) is an inclusion.

Claim 8.24 *Let $N \in \mathbf{O}_R^{\nu,\kappa,f}$ and let $M \subset N$ be a submodule which is flat as an R-module. Then, we have $M \in \mathbf{O}_R^{\nu,\kappa,f}$.*

To prove the claim, observe first that, since N is a weight $\mathbf{t}_R$-module with finitely generated weight subspaces over R, so is also M. Thus, since M is flat and since any flat finitely generated R-module is free (because R is a noetherian local ring), the R-module M is indeed free. It is easy to check that M satisfies the other axioms of the category $\mathbf{O}_R^{\nu,\kappa,f}$, except the fact that it is finitely generated. For this last property, recall that for each β the category ${}^\beta\mathbf{O}_R^{\nu,\kappa}$ is a highest weight category over R. Since it is equivalent to the category of finitely generated modules over a finitely generated projective R-algebra, it is noetherian. Therefore M is finitely generated. The claim is proved.

Now, recall that ${}^\dagger Z_{R,\infty}$ is flat over R and that $\mathscr{I}nd({}^\sharp W_R^1)$ is a generalized Weyl module of $\mathbf{O}_R^{\nu,\kappa,f}$. Thus, the claim implies that ${}^\dagger Z_{R,\infty} \in \mathbf{O}_R^{\nu,\kappa,f}$. Hence $DZ_{R,\infty} \in \mathbf{O}_R^{\nu,\kappa,f}$. This proves the first part of the lemma.

To prove the second part, it is enough to observe that the functor $(M_a) \mapsto {}^\dagger Z_{R,\infty}$ is left exact, because it is the composition of a tensor product over R of free R-modules, of a dual over R of free R-modules, and of the functor of taking smooth vectors (which is left exact), and that $\mathscr{D}$ is an exact endofunctor of $\mathbf{O}_R^{\nu,\kappa,f}$. □

Now, we consider the functor represented by the module $\dot{\bigotimes}_{R,a} M_a$. The lemma below gives a functorial isomorphism for each module M in $\mathbf{O}_R^{\nu,\kappa,f}$

$$ \mathrm{Hom}_{\mathbf{g}_R}(\dot{\bigotimes}_{R,a} M_a, M) = \langle\!\langle M_{-m}, \ldots, M_n, DM\rangle\!\rangle_R^*. \tag{8.7} $$

Lemma 8.25 *For each $M, N \in \mathbf{O}_R^{\nu,\kappa,f}$, we have functorial R-module isomorphisms*

$$ \mathrm{Hom}_{\mathbf{g}_R}(N, M) = \langle\!\langle N, DM\rangle\!\rangle_R^*, \quad \langle\!\langle \dot{\bigotimes}_{R,a} M_a, DM\rangle\!\rangle_R^* = \langle\!\langle M_{-m}, \ldots, M_n, DM\rangle\!\rangle_R^*. $$

Proof There is a natural R-module inclusion $\mathrm{Hom}_{\mathbf{g}_R}(N, M) \to \mathrm{Hom}_{\mathbf{g}_R}(N \otimes_R DM, R)$, because M, N are weight $\mathbf{t}_R$-modules whose weight subspaces are free R-modules of finite type. We must prove that this inclusion is indeed an isomorphism. The proof is the same as in [46, prop. A.2.6], see also [28, prop. 2.31].

Next, by definition of coinvariants, we also have a canonical R-module isomorphism $\mathrm{Hom}_{\mathbf{g}_R}(N \otimes_R DM, R) \to \langle\!\langle N, DM\rangle\!\rangle_R^*$. This proves the first isomorphism in the lemma.

Now, we concentrate on the second one. Consider the $\Gamma_{R,\kappa}$-module Z_R. By construction, we have $\mathrm{Hom}_R(DM, Z_R) = \mathrm{Hom}_R(W_R \otimes_R DM, R)$. Thus, we have also $\mathrm{Hom}_{\Gamma_{R,\kappa}}(DM, Z_R) = \mathrm{Hom}_{\Gamma_R}(W_R \otimes_R DM, R)$. Thus, since DM is smooth and $Z_{R,\infty} = Z_R(\infty)$, the canonical inclusion $\mathrm{Hom}_{\Gamma_{R,\kappa}}(DM, Z_{R,\infty}) \subset \mathrm{Hom}_{\Gamma_{R,\kappa}}(DM, Z_R)$ is indeed an isomorphism. So we get an isomorphism $\mathrm{Hom}_{\mathbf{g}_{R,\kappa}}(DM, Z_{R,\infty}) = \langle\!\langle M_{-m}, \dots, M_n, DM\rangle\!\rangle_R^*$. Finally, since ${}^\dagger Z_{R,\infty}$ belongs to $\mathbf{O}_R^{\nu,\kappa,f}$, we have

$$\mathrm{Hom}_{\mathbf{g}_{R,\kappa}}(\dot{\bigotimes}_{R,a} M_a, M) = \mathrm{Hom}_{\mathbf{g}_{R,\kappa}}(DZ_{R,\infty}, M) = \mathrm{Hom}_{\mathbf{g}_{R,\kappa}}(DM, Z_{R,\infty}).$$

□

Next, we consider the behavior of the tensor product $\dot{\otimes}_R$ on Δ-filtered modules. Assume that $M_0 \in \mathbf{O}_R^{\nu,\kappa,\Delta}$ and $M_a \in \mathbf{O}_R^{+,\kappa,\Delta}$ for $a \neq 0$. First, note the following.

Lemma 8.26 *For each $M \in \mathbf{O}_R^{\nu,\kappa,\Delta}$ the R-module $\langle\!\langle M_{-m}, \dots, M_n, {}^\dagger M\rangle\!\rangle_R$ is free of finite type.*

Proof Since this R-module is finitely generated by Lemma 8.2, it is enough to check that its rank is the same at the special point and at the generic point of $\mathrm{Spec}(R)$. By Lemma 8.2 we must check that

$$\dim_{\mathrm{k}} \langle\!\langle \mathrm{k}M_{-m}, \dots, \mathrm{k}M_n, {}^\dagger \mathrm{k}M\rangle\!\rangle_{\mathrm{k}} = \dim_K \langle\!\langle KM_{-m}, \dots, KM_n, {}^\dagger KM\rangle\!\rangle_K.$$

For each $M \in \mathbf{O}_R^{\nu,\kappa,\Delta}$, $N \in \Delta(\mathbf{O}_R^{\nu,\kappa})$, we have $(KM : KN) = (\mathrm{k}M : \mathrm{k}N)$. Therefore, the claim follows from Lemma 8.12. □

Now, we can prove the following.

Lemma 8.27 *We have $\dot{\bigotimes}_{R,a} M_a \in \mathbf{O}_R^{\nu,\kappa,\Delta}$.*

Proof Taking β large enough we can assume that all modules belong to the category ${}^\beta\mathbf{O}_R^{\nu,\kappa,f}$. Since ${}^\beta\mathbf{O}_R^{\nu,\kappa}$ is a highest weight category over R, to prove that $\dot{\bigotimes}_{R,a} M_a$ has a Δ-filtration, it suffices to check that $\mathrm{Ext}^1_{{}^\beta\mathbf{O}_R^{\nu,\kappa}}(\dot{\bigotimes}_{R,a} M_a, M) = 0$ for each $M \in \nabla({}^\beta\mathbf{O}_R^{\nu,\kappa})$, see [39, lem. 4.21]. Since the category ${}^\beta\mathbf{O}_R^{\nu,\kappa}$ is preserved by taking extensions in $\mathbf{O}_R^{\nu,\kappa}$, we must check that $\mathrm{Ext}^1_{\mathbf{O}_R^{\nu,\kappa}}(\dot{\bigotimes}_{R,a} M_a, \mathscr{D}M) = 0$ for each $M \in \Delta(\mathbf{O}_R^{\nu,\kappa})$. To simplify the notation, we assume that $[-m, n] = [1, 2]$. By (8.7), it is enough to check that, given an exact sequence

$$0 \to Q \to P \to M \to 0$$

with P projective, the following left exact sequence of R-modules is indeed exact

$$0 \to \langle\langle M_1, M_2, {}^{\dagger}M\rangle\rangle_R^* \to \langle\langle M_1, M_2, {}^{\dagger}P\rangle\rangle_R^* \to \langle\langle M_1, M_2, {}^{\dagger}Q\rangle\rangle_R^* \to 0.$$

Note that M_1, M_2, M_3, Q, P, M are Δ-filtered. To prove the claim we may consider the right exact sequence of free R-modules of finite type

$$0 \to \langle\langle M_1, M_2, {}^{\dagger}Q\rangle\rangle_R \to \langle\langle M_1, M_2, {}^{\dagger}P\rangle\rangle_R \to \langle\langle M_1, M_2, {}^{\dagger}M\rangle\rangle_R \to 0.$$

We must prove that it is exact. To do so, it is enough to prove that it is exact after specialization at the special point and at the generic point of $\mathrm{Spec}(R)$. Now, the sequences

$$0 \to \langle\langle \mathrm{k}M_1, \mathrm{k}M_2, {}^{\dagger}\mathrm{k}Q\rangle\rangle_{\mathrm{k}} \to \langle\langle \mathrm{k}M_1, \mathrm{k}M_2, {}^{\dagger}\mathrm{k}P\rangle\rangle_{\mathrm{k}} \to \langle\langle \mathrm{k}M_1, \mathrm{k}M_2, {}^{\dagger}\mathrm{k}M\rangle\rangle_{\mathrm{k}} \to 0,$$
$$0 \to \langle\langle KM_1, KM_2, {}^{\dagger}KQ\rangle\rangle_K \to \langle\langle KM_1, KM_2, {}^{\dagger}KP\rangle\rangle_K \to \langle\langle KM_1, KM_2, {}^{\dagger}KM\rangle\rangle_K \to 0$$

are both exact by Lemma 8.12. Thus, the lemma follows from Lemma 8.2. □

We can now prove Proposition 8.21: it is a direct consequence of Lemmas 8.22, 8.25 and 8.27.

8.3.3 The functors e and f

We consider the modules $\mathbf{V}_R, \mathbf{V}_R^*$ in $\mathbf{O}_R^{+,\kappa}$ given by

$$\mathbf{V}_R = \mathbf{M}(\epsilon_1)_{R,+}, \quad \mathbf{V}_R^* = \mathbf{M}(-\epsilon_N)_{R,+}.$$

The following hold.

Lemma 8.28 (a) *If $R = K$ is a field then $\mathbf{V}_K, \mathbf{V}_K^*$ are simple.*
(b) *The modules $\mathbf{V}_R, \mathbf{V}_R^*$ are tilting.*
(c) *We have $D\mathbf{V}_R = {}^{\dagger}\mathbf{V}_R = \mathbf{V}_R^*$, $\mathbf{V}_R = \mathscr{I}nd(V_R)$ and $\mathbf{V}_R^* = \mathscr{I}nd(V_R^*)$.*

Proof If $R = K$ is a field, then $\mathbf{V}_K, \mathbf{V}_K^*$ are simple, proving part (a). To prove (b), note that under base change we get $\mathbf{V}_{\mathrm{k}} = \mathrm{k}\mathbf{V}_R$ and $\mathbf{V}_{\mathrm{k}}^* = \mathrm{k}\mathbf{V}_R^*$. The modules $\mathbf{V}_{\mathrm{k}}$ and $\mathbf{V}_{\mathrm{k}}^*$ are simple and standard. Thus, they are both tilting. Therefore $\mathbf{V}_R, \mathbf{V}_R^*$ are also tilting modules by Proposition 2.4. Part (c) is clear, because (b) implies that $\mathscr{D}\mathbf{V}_R = \mathbf{V}_R$ and $\mathscr{D}\mathbf{V}_R^* = \mathbf{V}_R^*$. □

Next, we define the endofunctors e, f of $\mathbf{O}_R^{\nu,\kappa,\Delta}$ and $\mathscr{O}_R^{\nu,\Delta}$ respectively by

$$e = \bullet\,\dot{\otimes}_R \mathbf{V}_R^*, \quad f = \bullet\,\dot{\otimes}_R \mathbf{V}_R,$$
$$e = \bullet\,\otimes_R V_R^*, \quad f = \bullet\,\otimes_R V_R.$$

The goal of this section is to prove the following.

Proposition 8.29 (a) *The endofunctors e, f of $\mathbf{O}_R^{\nu,\kappa,\Delta}$ are exact and preserve the subcategory $\mathbf{O}_R^{\nu,\kappa,\mathrm{tilt}}$.*

(b) *We have functorial isomorphisms $\mathrm{k}e(M) \simeq e(\mathrm{k}M)$ and $\mathrm{k}f(M) \simeq f(\mathrm{k}M)$ for each module M in $\mathbf{O}_R^{\nu,\kappa,\mathrm{tilt}}$.*

(c) *If $R = K$ is a field then e, f extend to exact biadjoint endofunctors of $\mathbf{O}_K^{\nu,\kappa}$.*

(d) *The functor $\mathscr{I}nd_R$ gives a $\mathbb{C}$-vector space isomorphism $[\mathscr{O}_R^{\nu,\Delta}] \to [\mathbf{O}_R^{\nu,\kappa,\Delta}]$ which commutes with the $\mathbb{C}$-linear maps e, f.* □

Proposition 8.30 *Assume that R is a local analytic algebra.*

(a) *There is a braided monoidal category $(\mathbf{O}_R^{+,\kappa,\Delta}, \dot{\otimes}_R, \mathbf{a}_R, \mathbf{c}_R)$.*

(b) *There is a bimodule category $(\mathbf{O}_R^{\nu,\kappa,\Delta}, \dot{\otimes}_R, \mathbf{a}_R, \mathbf{c}_R)$ over $(\mathbf{O}_R^{+,\kappa,\Delta}, \dot{\otimes}_R, \mathbf{a}_R, \mathbf{c}_R)$.*

(c) *For each module $M \in \mathbf{O}_R^{\nu,\kappa,\mathrm{tilt}}$ and each integer $d \geqslant 1$, we have a k-algebra isomorphism $\mathrm{k}\,\mathrm{End}_{\mathbf{g}_R}(f^d(M)) \to \mathrm{End}_{\mathbf{g}_\mathrm{k}}(f^d(\mathrm{k}M))$ which commutes with the associativity and the commutativity constraints $\mathbf{a}_R, \mathbf{c}_R$.* □

To prove these propositions, we need more material. First, we define the associativity and the commutativity constraints $\mathbf{a}_R$, $\mathbf{c}_R$ for $\dot{\otimes}_R$. From now on we'll assume that R is a local analytic algebra.

Lemma 8.31 *Assume that $V_1, V_2 \in \mathbf{O}_R^{+,\kappa,f}$ and $M \in \mathbf{O}_R^{\nu,\kappa,f}$. Then, there are functorial isomorphisms $\mathbf{a}_{V_1,M,V_2} : (V_1\dot{\otimes}_R M)\dot{\otimes}_R V_2 \to V_1\dot{\otimes}_R(M\dot{\otimes}_R V_2)$.*

Proof We apply the same construction as in the case $R = \mathbb{C}$ in [28, sec. 17, 18]. We will be very brief. We allow the system of charts γ to vary in the set of $\mathbb{C}$-points of an affine scheme $\mathscr{V}$. Taking the coinvariants, we construct a bundle of R-modules of finite rank over $\mathscr{V}$. This bundle is equipped with an integrable R-linear connection. Since R is an analytic algebra, it admits a flat section. This section gives R-linear isomorphisms, see [28, thm. 17.29],

$$\langle\langle V_1\dot{\otimes}_R M, V_2\dot{\otimes}_R DN\rangle\rangle_R = \langle\langle V_1, M, V_2, DN\rangle\rangle_R,$$
$$\langle\langle DN\dot{\otimes}_R V_1, M\dot{\otimes}_R V_2\rangle\rangle_R = \langle\langle DN, V_1, M, V_2\rangle\rangle_R$$

for each $M, N \in \mathbf{O}_R^{\nu,\kappa,f}$ and $V_1, V_2 \in \mathbf{O}_R^{+,\kappa}$.

Using these isomorphisms and the invariance of coinvariants under cyclic permutation, we get functorial isomorphisms [28, sec. 18.2]

$$\begin{array}{ccc} \langle\!\langle V_1\dot{\otimes}_R M,\, V_2\dot{\otimes}_R DN\rangle\!\rangle_R & = & \langle\!\langle (V_1\dot{\otimes}_R M)\otimes_R V_2,\, DN\rangle\!\rangle_R, \\ \| & & \\ \langle\!\langle DN\dot{\otimes}_R V_1,\, M\dot{\otimes}_R V_2\rangle\!\rangle_R & = & \langle\!\langle V_1\dot{\otimes}_R (M\otimes_R V_2),\, DN\rangle\!\rangle_R. \end{array}$$

Hence, from (8.7) we deduce a functorial isomorphism

$$\mathrm{Hom}_{\mathbf{g}_R}((V_1\dot{\otimes}_R M)\dot{\otimes}_R V_2, DN) = \mathrm{Hom}_{\mathbf{g}_R}(V_1\dot{\otimes}_R(M\dot{\otimes}_R V_2), DN),$$

which yields a module isomorphism $\mathbf{a}_{V_1,M,V_2} : (V_1\dot{\otimes}_R M)\dot{\otimes}_R V_2 \to V_1\dot{\otimes}_R(M\dot{\otimes}_R V_2)$. □

The isomorphisms $\mathbf{a}_{M,V_1,V_2}$ and $\mathbf{a}_{M,V_1,V_2}$ are constructed in a similar way. The details are left to the reader.

Now, we consider the commutativity constraint. To do so, for each modules $V \in \mathbf{O}_R^{+,\kappa,f}$ and $M \in \mathbf{O}_R^{\nu,\kappa,f}$ we consider the morphism of functors $\langle\!\langle V, M, DN\rangle\!\rangle_R \to \langle\!\langle M, V, DN\rangle\!\rangle_R$ induced by the R-linear map

$$V\otimes_R M\otimes_R DN \to M\otimes_R V\otimes_R DN, \quad x\otimes y\otimes z \mapsto \tau y\otimes\tau x\otimes\bar{\tau}z.$$

Here, we set $\tau = \exp(\sqrt{-1}\pi\,\mathfrak{L}_0)\exp(\mathfrak{L}_1)$ and $\bar{\tau} = \exp(-\sqrt{-1}\pi\,\mathfrak{L}_0)\exp(\mathfrak{L}_1)$.

Lemma 8.32 *Assume that $V \in \mathbf{O}_R^{+,\kappa,f}$ and $M \in \mathbf{O}_R^{\nu,\kappa,f}$. Then, there is a functorial isomorphism $\mathbf{c}_{V,M} : V\dot{\otimes}_R M \to M\dot{\otimes}_R V$ which represents the morphism of functors $\langle\!\langle V, M, DN\rangle\!\rangle_R \to \langle\!\langle M, V, DN\rangle\!\rangle_R$.*

Proof The isomorphism $\mathbf{c}_{V,M}$ is defined as in [28, sec. 14]. More precisely, setting $A = [0,1]$, $M_0 = V$ and $M_1 = M$, we consider the $\Gamma_{R,\kappa}$-module Z_R in (8.5). Switching V and M we define Z'_R in a similar way. Since the Sugawara operators $\mathfrak{L}_0, \mathfrak{L}_1$ act on V, M and since R is an analytic algebra, we can define the R-module isomorphism $P : Z'_R \to Z_R$ which is the transpose of the R-linear map $V\otimes_R M \to M\otimes_R V$ such that $x\otimes y\mapsto \tau y\otimes\tau x$. Now, recall that $Z'_{R,\infty} = Z'_R(\infty)$ and $Z_{R,\infty} = Z_R(\infty)$. One check as in loc. cit. that P induces a $\mathbf{g}'_{R,\kappa}$-isomorphism $Z'_{R,\infty} \to Z_{R,\infty}$. We define the isomorphism $\mathbf{c}_{V,M}$ to be the map $DZ_{R,\infty} \to DZ'_{R,\infty}$ which is the transpose of P. The second part of the lemma is proved as in [28, sec. 14.6]. □

Next, we consider the behavior of the functors e, f on tilting modules.

Lemma 8.33 *The functors e, f on $\mathbf{O}_R^{\nu,\kappa,\Delta}$ are exact. They preserve the subcategory $\mathbf{O}_R^{\nu,\kappa,\mathrm{tilt}}$.*

Proof Let $S \subset R$ be the $\mathbb{C}$-subalgebra of R generated by κ. The modules $\mathbf{V}_R$, $\mathbf{V}_R^*$ are defined over S, i.e., we have $\mathbf{V}_R = R\mathbf{V}_S$ and $\mathbf{V}_R^* = R\mathbf{V}_S^*$ with $\mathbf{V}_S = \mathscr{I}nd_S(V_S)$, $\mathbf{V}_S^* = \mathscr{I}nd_S(V_S^*)$.

Now, the second claim is proved as Proposition 8.11(b). Since $\mathbf{V}_R$, $\mathbf{V}_R^*$ are tilting by Lemma 8.28, it is enough to check that e, f are biadjoint on $\mathbf{O}_R^{\nu,\kappa,\Delta}$ (hence exact) proving the first claim on the way. To do so, since R is a regular ring, we may assume that R is flat over S. Then, since e, f commute with flat base change by Lemma 8.22, we may assume that $R = S$ is a regular local ring of dimension 1. So, we are in the same setting as in [28, sec. 31].

Next, proving the lemma is equivalent to proving that $\mathbf{V}_R$ and $\mathbf{V}_R^*$ are *rigid*, see the appendix to part IV of [28] for details. This is proved in the proof of [28, prop. 31.3], modulo a technical assumption which is checked in [28, lem. 31.6]. □

Finally, we consider the behavior of the tensor product $\dot{\otimes}_R$ under the specialization of R to the residue field k.

Lemma 8.34 *For each module $M \in \mathbf{O}_R^{\nu,\kappa,\mathrm{tilt}}$, we have functorial isomorphisms* $\mathrm{k}e(M) = e(\mathrm{k}M)$ *and* $\mathrm{k}f(M) = f(\mathrm{k}M)$.

Proof By (8.7), for $N \in \mathbf{O}_R^{\nu,\kappa,f}$ we have functorial isomorphisms

$$\mathrm{Hom}_{\mathbf{g}_R}(\dot{\bigotimes}_{R,a} M_a, N) = \langle\!\langle M_{-m}, \ldots, M_n, DN\rangle\!\rangle_R^*,$$
$$\mathrm{Hom}_{\mathbf{g}_\mathrm{k}}(\dot{\bigotimes}_{\mathrm{k},a} \mathrm{k}M_a, \mathrm{k}N) = \langle\!\langle \mathrm{k}M_{-m}, \ldots, \mathrm{k}M_n, D\mathrm{k}N\rangle\!\rangle_\mathrm{k}^*.$$

If M_a, N are tilting, then $\langle\!\langle M_{-m}, \ldots, M_n, DN\rangle\!\rangle_R$ is free of finite type over R by Lemma 8.26. Therefore, by Lemma 8.2 we have a functorial isomorphism $\mathrm{k}\,\mathrm{Hom}_{\mathbf{g}_R}(\dot{\bigotimes}_{R,a} M_a, N) = \mathrm{Hom}_{\mathbf{g}_\mathrm{k}}(\dot{\bigotimes}_{\mathrm{k},a} \mathrm{k}M_a, \mathrm{k}N)$. So, for $M, N \in \mathbf{O}_R^{\nu,\kappa,\mathrm{tilt}}$ we have functorial isomorphisms $\mathrm{k}\,\mathrm{Hom}_{\mathbf{g}_R}(e(M), N) = \mathrm{Hom}_{\mathbf{g}_\mathrm{k}}(e(\mathrm{k}M), \mathrm{k}N)$ and similar isomorphisms for f.

On the other hand, by Lemma 8.33 the modules $e(M)$, $f(M)$ are tilting. Thus, we have functorial isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathbf{g}_\mathrm{k}}(\mathrm{k}e(M), \mathrm{k}N) &= \mathrm{Hom}_{\mathbf{g}_\mathrm{k}}(e(\mathrm{k}M), \mathrm{k}N), \\ \mathrm{Hom}_{\mathbf{g}_\mathrm{k}}(\mathrm{k}f(M), \mathrm{k}N) &= \mathrm{Hom}_{\mathbf{g}_\mathrm{k}}(f(\mathrm{k}M), \mathrm{k}N). \end{aligned} \tag{8.8}$$

This proves the lemma. □

Remark 8.35 In Lemma 8.10 we considered the specialization functor in [28, thm. 29.1], from a regular local ring of dimension 1 to its residue field. In Lemma 8.34, we consider a specialization functor from a regular local ring of dimension 2 to its residue field.

We can now finish the proof of Propositions 8.29 and 8.30.

Proof of Proposition 8.29 Parts (a), (b), (c) follow from Lemmas 8.22, 8.33, 8.34 and 8.9. Part (d) is proved as Proposition 8.11(b). □

Proof of Proposition 8.30 The isomorphisms of functors $\mathbf{a}_R$, $\mathbf{c}_R$ are constructed in Lemmas 8.31, 8.32. For parts (a), (b) we must prove that $\mathbf{a}_R$, $\mathbf{c}_R$ satisfy the hexagon and the pentagon axioms. The tensor product $\dot{\otimes}_R$ commutes with a flat base change of the ring R by Lemma 8.22. The isomorphisms of functors $\mathbf{a}_R$, $\mathbf{c}_R$ commute also with a flat base change. Therefore, embedding R in its fraction field K, we are reduced to prove that $\mathbf{a}_K$, $\mathbf{c}_K$ satisfy the hexagon and the pentagon axioms. This is proved in Proposition 8.7.

Now, let $M \in \mathbf{O}_R^{\nu,\kappa,\mathrm{tilt}}$ and $N \in \mathbf{O}_R^{\nu,\kappa,\mathrm{tilt}}$. By (8.8) and Propositions 2.4, 8.29, the specialization at k gives functorial isomorphisms

$$\begin{aligned}
\mathrm{k}\,\mathrm{Hom}_{\mathbf{g}_R}(\mathbf{V}_R^*\dot{\otimes}_R M, N) &= \mathrm{Hom}_{\mathbf{g}_\mathrm{k}}(\mathbf{V}_\mathrm{k}^*\dot{\otimes}_\mathrm{k}(\mathrm{k}M), \mathrm{k}N),\\
\mathrm{k}\,\mathrm{Hom}_{\mathbf{g}_R}(M\dot{\otimes}_R\mathbf{V}_R^*, N) &= \mathrm{Hom}_{\mathbf{g}_\mathrm{k}}((\mathrm{k}M)\dot{\otimes}_\mathrm{k}\mathbf{V}_\mathrm{k}^*, \mathrm{k}N).
\end{aligned}$$

They are induced by the base-change homomorphisms

$$\begin{aligned}
\mathrm{k}\langle\!\langle \mathbf{V}_R^*, M, DN\rangle\!\rangle_R &\to \langle\!\langle \mathbf{V}_\mathrm{k}^*, \mathrm{k}M, D(\mathrm{k}N)\rangle\!\rangle_\mathrm{k},\\
\mathrm{k}\langle\!\langle M, \mathbf{V}_R^*, DN\rangle\!\rangle_R &\to \langle\!\langle \mathbf{V}_\mathrm{k}^*, \mathrm{k}M, D(\mathrm{k}N)\rangle\!\rangle_\mathrm{k}.
\end{aligned}\tag{8.9}$$

We must check that they intertwine the isomorphisms

$$\mathbf{c}_{\mathbf{V}_R^*,M} : \mathbf{V}_R^*\dot{\otimes}_R M \to M\dot{\otimes}_R\mathbf{V}_R^*, \quad \mathbf{c}_{\mathbf{V}_\mathrm{k}^*,\mathrm{k}M} : \mathbf{V}_\mathrm{k}^*\dot{\otimes}_\mathrm{k}(\mathrm{k}M) \to (\mathrm{k}M)\dot{\otimes}_\mathrm{k}\mathbf{V}_\mathrm{k}^*.$$

To do so, recall that $\mathbf{c}_{\mathbf{V}_R^*,M}$ represents the transpose of the morphism of functors

$$P_R : \langle\!\langle \mathbf{V}_R^*, M, DN\rangle\!\rangle_R \to \langle\!\langle M, \mathbf{V}_R^*, DN\rangle\!\rangle_R, \quad x\otimes y\otimes z \mapsto \tau y\otimes\tau x\otimes\bar{\tau}z.$$

So the claim follows from the commutativity of the following square

$$\begin{tikzcd}
\mathrm{k}\langle\!\langle \mathbf{V}_R^*, M, DN\rangle\!\rangle_R \arrow[r, "\mathrm{k}P_R"] \arrow[d, "(8.9)"] & \mathrm{k}\langle\!\langle M, \mathbf{V}_R^*, DN\rangle\!\rangle_R \arrow[d, "(8.9)"] \\
\langle\!\langle \mathbf{V}_\mathrm{k}^*, \mathrm{k}M, D(\mathrm{k}N)\rangle\!\rangle_\mathrm{k} \arrow[r, "P_\mathrm{k}"] & \langle\!\langle \mathrm{k}M, \mathbf{V}_\mathrm{k}^*, D(\mathrm{k}N)\rangle\!\rangle_\mathrm{k}.
\end{tikzcd}$$

The commutation of the specialization with the associativity constraint is proved in a similar way. □

8.4 From O to the cyclotomic Hecke algebra

Let R be a local analytic deformation ring of dimension $\leqslant 2$. Set $v = v_R = \exp(-\sqrt{-1}\pi/\kappa_R)$ and $q = q_R = v_R^2$. The endomorphisms X, T of the functors

f, f^2 are given by $X = \mathbf{c}_R \circ \mathbf{c}_R$ and $T = v_R \cdot \mathbf{a}_R^{-1} \circ (1\dot{\otimes}_R \mathbf{c}_R) \circ \mathbf{a}_R$. More precisely, for each $M \in \mathbf{O}_R^{\nu,\kappa}$ we have

$$\begin{aligned} X_M &= \mathbf{c}_{\mathbf{V}_R,M} \circ \mathbf{c}_{M,\mathbf{V}_R}, \\ T_M &= v_R \cdot \mathbf{a}^{-1}_{M,\mathbf{V}_R,\mathbf{V}_R} \circ (1_M \dot{\otimes}_R \mathbf{c}_{\mathbf{V}_R,\mathbf{V}_R}) \circ \mathbf{a}_{M,\mathbf{V}_R,\mathbf{V}_R}. \end{aligned} \tag{8.10}$$

Next, fix an integer $d \geqslant 1$ and consider the endomorphisms of f^d given by $X_j = 1^{d-j} X 1^{j-1}$ and $T_i = 1^{d-i-1} T 1^{i-1}$ with $j \in [1, d]$, $i \in [1, d)$. We can now prove the following.

Proposition 8.36 (a) *X_j, T_i yield an R-algebra homomorphism*

$$\psi_{R,d} : \mathbf{H}_{R,d} \to \mathrm{End}(f^d)^{\mathrm{op}}.$$

(b) *$\psi_{R,d}$ gives an R-algebra homomorphism*

$$\psi^s_{R,d} : \mathbf{H}^s_{R,d} \to \mathrm{End}_{\mathbf{g}_R}(\mathbf{T}_{R,d})^{\mathrm{op}}.$$

Proof The braid relations $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$ and $T_i T_j = T_j T_i$ if $|i-j| > 1$ are well-known formal consequences of the axioms of a braided monoidal category.

Next, consider the relation $T_i X_i T_i = v_R^2 X_{i+1}$. The hexagon axiom yields the following relations

$$\begin{aligned} \mathbf{a}_{\mathbf{V}_R,M,\mathbf{V}_R} \circ (\mathbf{c}_{M,\mathbf{V}_R} \dot{\otimes}_R 1_{\mathbf{V}_R}) \circ T_M &= v_R \cdot \mathbf{c}_{f(M),\mathbf{V}_R}, \\ T_M \circ (\mathbf{c}_{\mathbf{V}_R,M} \dot{\otimes}_R 1_{\mathbf{V}_R}) \circ \mathbf{a}^{-1}_{\mathbf{V}_R,M,\mathbf{V}_R} &= v_R \cdot \mathbf{c}_{\mathbf{V}_R,f(M)}. \end{aligned}$$

Therefore we have $T_M \circ (X_M \dot{\otimes}_R 1_{\mathbf{V}_R}) \circ T_M = v_R^2 \cdot X_{f(M)}$. We deduce that $(T_i)_M \circ (X_i)_M \circ (T_i)_M = v_R^2 \cdot (X_{i+1})_M$.

Now, let us prove the relations $X_i X_j = X_j X_i$ and $X_i T_j = T_j X_i$ for $i \neq j, j+1$. We are reduced to check the relations $(X_1)_M \circ (X_i)_M = (X_i)_M \circ (X_1)_M$ and $(T_i)_M \circ (X_1)_M = (X_1)_M \circ (T_i)_M$ for $i \neq 1$. They follow from the functoriality of $\mathbf{c}$ and $\mathbf{a}$. Let us check the first one in details for $i = 1$, $j = 2$. The diagram

$$\begin{tikzcd} f(M)\dot{\otimes}_R \mathbf{V}_R \arrow[r, "X_M \dot{\otimes} 1"] \arrow[d, "\mathbf{c}_{f(M),\mathbf{V}_R}"'] & f(M)\dot{\otimes}_R \mathbf{V}_R \arrow[d, "\mathbf{c}_{f(M),\mathbf{V}_R}"] \\ \mathbf{V}_R \dot{\otimes}_R f(M) \arrow[r, "1 \dot{\otimes} X_M"] \arrow[d, "\mathbf{c}_{\mathbf{V}_R,f(M)}"'] & \mathbf{V}_R \dot{\otimes}_R f(M) \arrow[d, "\mathbf{c}_{\mathbf{V}_R,f(M)}"] \\ f(M)\dot{\otimes}_R \mathbf{V}_R \arrow[r, "X_M \dot{\otimes} 1"] & f(M)\dot{\otimes}_R \mathbf{V}_R \end{tikzcd}$$

is commutative because X_M is an endomorphism of $f(M)$ and $\mathbf{c}_R$ is a morphism of functors. Now the composition of both vertical maps is equal to $X_{f(M)} = (X_2)_M$, while the upper and the lower horizontal maps are both equal to $(X_1)_M$. We are done.

To prove the Hecke relation $(T_i + 1)(T_i - q_R) = 0$, observe that the action of ω on $V_R \otimes_R V_R$ is a diagonalizable operator with eigenvalues 1 and -1. Thus, from (8.10) we get that $(T_i - v_R^2)(T_i + 1) = 0$.

Finally, let us check the cyclotomic relation. By (8.2), (8.10), the endomorphism $(X_1)_{\mathbf{T}_{R,0}}$ of $\mathbf{T}_{R,d}$ is identified with the endomorphism $f^{d-1}(X_{\mathbf{T}_{R,0},\mathbf{V}_R})$ of $f^{d-1}(\mathbf{T}_{R,1})$, where $X_{\mathbf{T}_{R,0},\mathbf{V}_R}$ is an operator on $\mathbf{T}_{R,1} = \mathbf{T}_{R,0}\dot{\otimes}_R\mathbf{V}_R$. We must prove that this operator satisfies the equation $\prod_{p=1}^{\ell}(X_{\mathbf{T}_{R,0},\mathbf{V}_R} - q_R^{s_p}) = 0$. We may assume that $R = K$ is a field. Then, the claim follows from Remark 5.18. $\square$

Index of notation

2: R, K, k, $\mathfrak{m}$.

2.1: M^*, SM, $S\phi$, $\mathfrak{P}$, $\mathfrak{M}$, $\mathfrak{P}_1$, $R_\mathfrak{p}$, $\mathfrak{m}_\mathfrak{p}$, $\mathrm{k}_\mathfrak{p}$.

2.2: A^{op}, $\mathscr{C}^{\mathrm{op}}$, $1_\mathscr{C}$, $K_0(\mathscr{C})$, $[\mathscr{C}]$, $[M]$, A-mod, $\mathscr{C}^*$, $\mathrm{Irr}(\mathscr{C})$, $\mathscr{C}^{\mathrm{proj}}$, $\mathscr{C}^{\mathrm{inj}}$, $\mathrm{Irr}(A)$, A-proj, A-inj, $S\mathscr{C}$, SF, h, h^*, $h^!$.

2.3: $\Delta(\mathscr{C})$, $\leqslant$, Λ, $P(\lambda)$, $I(\lambda)$, $T(\lambda)$, $\nabla(\lambda)$, $\Delta^*(\lambda)$, $P^*(\lambda)$, $I^*(\lambda)$, $T^*(\lambda)$, $\nabla^*(\lambda)$, $L(\lambda)$, $\mathscr{C}^\Delta$, $\mathscr{C}^\nabla$, $\mathscr{C}^{\mathrm{tilt}}$, $\mathscr{C}^\diamond$, $\mathscr{R}$, $\Delta^\diamond(\lambda)$, $P^\diamond(\lambda)$, $T^\diamond(\lambda)$, $\mathscr{C}^\blacklozenge$, $\mathrm{lcd}_\mathscr{C}(M)$, $\mathrm{rcd}_\mathscr{C}(M)$.

2.4.1: $F : \mathscr{C} \to B$-mod, G, $(B\text{-mod})^{F\Delta}$, F^Δ, F^*, $F^\diamond$.

2.4.3: $(KB')_{\leq E}$, $S(\lambda)$, $S'(\lambda)$.

3: q, q_R.

3.1: $\mathscr{I}$, $\mathscr{I}(q)$, $Q_{R,p}$, $\mathscr{I}_p$, I_1.

3.2: $\mathfrak{sl}_\mathscr{I}$, Ω, α_i, $\check{\alpha}_i$, Λ_i, $Q = Q_\mathscr{I}$, $Q^+ = Q^+_\mathscr{I}$, $P = P_\mathscr{I}$, $P^+ = P^+_\mathscr{I}$, $X = X_\mathscr{I}$, ε_i, $\mathscr{I}^\alpha$, $\mathfrak{sl}_I$.

3.3: $\mathbb{Z}^\ell(n)$, $\mathscr{C}^\ell_n$, $\mathscr{C}^\ell_{n,+}$, $\mathscr{P}_n$, $|\lambda|$, $l(\lambda)$, ${}^t\lambda$, $Y(\lambda)$, $\mathscr{P}^\ell_n$, $\mathscr{P}$, $\mathscr{P}^\ell$, $\mathscr{P}^\nu$, $\mathscr{P}^\nu_d$, $p(A)$, q-res^Q, q-res^s, cont^s, $q_R^{s_p} = Q_{R,p}$, $Q_p = Q_{R,p}$, Γ, $\mathfrak{S}_d$, Γ_d, $\mathscr{X}(\lambda)_\mathbb{C}$.

3.4.1: $\mathbf{H}_{R,d}$, T_i, X_i, $\mathbf{H}^Q_{R,d}$, $\mathbf{H}^+_{R,d}$, $\mathbf{H}^s_{R,d}$, $\mathrm{Ind}_d^{d'}$, $\mathrm{Res}_d^{d'}$, $\mathrm{Ind}_{d,+}^{d',s}$, $\mathrm{Res}_{d,+}^{d',s}$, $M_\mathbf{i}$, $1_\mathbf{i}$, 1_α, $\mathbf{H}^s_{K,\alpha}$.

3.4.2: $H_{R,d}$, $H^s_{R,d}$, t_i, x_i, $H^s_{K,\alpha}$, H^s_I, $H^s_{I,d}$.

3.4.3: ζ, $S(\lambda)^{s,q}_R$, $\trianglelefteq$, $S(\lambda)^s_R$.

3.5: w_λ, x_λ, $\mathfrak{S}_\lambda$, $\mathbf{S}^s_{R,d}$, $W(\lambda)^{s,q}_R$, $\Xi^s_{R,d}$, $S^s_{R,d}$, $W(\lambda)^s_R$.

3.6: (E, F, X, T), ϕ_{E^d}, $\Lambda = \Lambda^s$, $\mathbf{H}^s_{\mathscr{I},d}$, $\mathscr{L}(\Lambda)_\mathscr{I}$, $\mathscr{L}(\Lambda)_{\mathscr{I},\Lambda-\alpha}$, $\mathbf{L}(\Lambda)$, $\mathscr{L}(\Lambda)_I$.

4: ℓ, N, ν.

4.1: $\kappa_R = \kappa$, $\tau_{R,p} = \tau_p$, τ_R, $s_{R,p} = s_p$, κ_S, $\tau_{S,p}$, e.

4.2: $\mathfrak{g}_R$, $U(\mathfrak{g}_R)$, $\mathfrak{t}_R$, $\mathfrak{b}_R$, $\mathfrak{p}_{R,\nu}$, $\mathfrak{m}_{R,\nu}$, $e_{i,j}$, e_i, ϵ_i, $\mathfrak{t}_R^*$, Π, Π^+, Π_ν, Π_ν^+, W, $w \bullet \lambda$, ρ, i_p, j_p, J_p^ν, p_k, $\det_p$, $\det$, P, P_R, S^ν, P_R^ν, ρ_ν, $\tau = \tau_R$, ϖ, $\omega = \omega_N$, $\mathrm{cas} = \mathrm{cas}_N$.
4.3: M_λ, $\mathscr{O}_R^\nu$, $V(\lambda)_{R,\nu}$, $M(\lambda)_{R,\nu}$, $L(\lambda)_K$, $\mathscr{O}_{R,\tau}^\nu$, $\Delta(\lambda)_{R,\tau}$.
4.4: $A_{R,\tau}^\nu$, $A_{R,\tau}^\nu\{d\}$.
4.5: V_R, V_R^*, e, f, I, $\mathrm{wt}(\mu)$, $m_i(\mu)$, $\mathscr{O}_{K,\tau,\lambda}^\nu$, V_I, $\lambda \xrightarrow{i} \mu$.
4.6: h, E, F, $T_{R,d} = T_{R,\tau}^\nu\{d\}$, $\varphi_{R,d}^s$, $\Phi_{R,d}^s$, $A_{R,\tau}^\nu(N) = A_{R,\tau}^\nu$, $T_{R,d}(N) = T_{R,d}$.
4.7: $a_\bullet$, a_p, $a_\circ$, $\Pi_{\nu,u,v}$, $\Pi_{\nu,u,v}^+$, $\mathfrak{m}_{R,\nu}$, $\mathfrak{m}_{R,\nu,u,v}$, $P\{a\}$, $P^\nu\{a\}$, $\det_\bullet$, $\nu_\circ$, $\nu_\bullet$, $\Pi_{\nu,u,v}$, $\Pi_{\nu,u,v}^+$, $\mathscr{O}_{R,\tau}^\gamma(\nu)$, $\mathscr{O}_{R,\tau}^\gamma(\nu)\{a\}$, $\mathscr{O}_{R,\tau}^\gamma(\nu,u,v)$, $\mathscr{O}_{R,\tau}^\gamma(\nu,u,v)\{a\}$, $A_{R,\tau}^\nu(\nu)$, $A_{R,\tau}^\nu(\nu,u,v)$.
5.1: $q_R = \exp(-2\pi\sqrt{-1}/\kappa_R)$, $Q_{R,p} = q_R^{s_p} = \exp(-2\pi\sqrt{-1}s_{R,p}/\kappa_R)$.
5.2.1: $L\mathfrak{g}_R$, $\mathbf{g}_R'$, $\mathbf{1}$, ∂, $\mathbf{g}_R$, $\mathbf{t}_R$, $\mathbf{b}_R$, $\mathbf{p}_{R,\nu}$, c, $\mathbf{g}_{R,\kappa}$, $\mathbf{g}_{R,\kappa}'$, $\mathbf{g}_{R,\geqslant d}$, $\mathbf{g}_{R,+}'$, $\mathbf{g}_{R,+}$, $\mathscr{I}nd_R(M)$, $Q_{R,d}$, $M(d)$, $M(-d)$, $M(\infty)$, $M(-\infty)$, $\mathscr{S}_{R,\kappa}$, $\xi^{(r)}$, $\mathfrak{L}_s$, **cas**.
5.2.2: $\widehat{P}_R$, $\widehat{\Pi}$, $\widehat{\Pi}^+$, $\widehat{\Pi}_{re}$, $\check{\alpha}$, $(\bullet : \bullet)$, δ, Λ_0, $\tilde{\rho}$, $\langle \bullet : \bullet \rangle$, $\widehat{W}$, s_i, T_x, $w \bullet \mu$, $\widehat{P}$, $\widehat{P}^\nu$, $\widehat{P}_R^\nu$, $\widehat{\lambda}$, z_λ.
5.3.1: $\mathbf{O}_R^{\nu,\kappa}$, $M(\mu)_{R,\nu}$, $L(\mu)_K$, $\mathbf{M}(\lambda)_{R,\nu}$, $\mathbf{L}(\lambda)_K$, $\mathbf{O}_R$, $\mathbf{M}(\lambda)_R$, $\mathbf{O}_R^{+,\kappa}$, $\mathbf{M}(\lambda)_{R,+}$, $\mathbf{O}_R^{\nu,\kappa,f}$, $\mathbf{O}_R^{\nu,\kappa,\Delta}$, $\mathbf{O}_{R,\tau}^{\nu,\kappa}$, $\mathbf{O}_{R,\tau}^{\nu,\kappa,\Delta}$, $\mathbf{O}^{\nu,\kappa}[a]$, $P\{d\}$, $P^\nu\{d\}$, $\mathbf{O}_{R,\tau}^\nu(N)$, $\mathbf{O}_{R,\tau}^{\nu,\kappa}(N)[a]\{d\}$, $\mathbf{O}'$.
5.3.2: ${}^\sharp M$, ${}^\dagger M$, M^*, DM, $\mathscr{D}M$, ${}^\beta\mathbf{O}_R^{\nu,\kappa}$.
5.3.3: $\widehat{\Pi}(\widehat{\lambda})$, $\widehat{\Pi}(\lambda, c)$, $\widehat{\lambda} \Uparrow \widehat{\lambda}'$, $\leqslant_\ell$, $\leqslant_{\mathrm{b}}$.
5.4: $\dot{\otimes}_R$, $\mathbf{V}_R$, $\mathbf{V}_R^*$, e, f, X, T, I, $i \sim j$, $\mathscr{I}$, e_i, f_i, $m_i(\lambda)$, $\mathrm{wt}(\lambda)$, $\mathbf{O}_{K,\tau,\beta}^{\nu,\kappa}$.
5.5: $\mathbf{\Delta}(\lambda)_{R,\tau}$, $\mathbf{A}_{R,\tau}^{\nu,\kappa}$, $\mathbf{L}(\lambda)$, $\mathbf{P}(\lambda)_{R,\tau}$, $\mathbf{T}(\lambda)_{R,\tau}$, $\mathbf{A}_{R,\tau}^{\nu,\kappa}\{d\}$, $\mathbf{T}_{R,\tau}^{\nu,\kappa}\{d\}$, $\mathbf{T}_{R,d}$, $\mathbf{T}_{R,d}(N)$, $\psi_{R,d}^s$, $\Psi_{R,d}^s$.
5.6: $\mathbf{m}_{R,\nu}$, $\mathbf{m}_{R,\nu,\kappa}$, $\widehat{W}_\nu$, $\mathbf{b}_{R,\nu}$, $\mathbf{O}_R^\kappa(\nu)$, $\mathbf{O}_R^{\gamma,\kappa}(\nu)$, $\mathbf{O}_{R,\tau}^{\gamma,\kappa}(\nu)$, $\mathbf{O}_{R,\tau}^{\gamma,\kappa}(\nu)\{a\}$, $\mathbf{O}_R^{+,\kappa}(\nu)$, $\mathbf{O}_R^{+,\kappa}(\nu)\{a\}$, $\mathbf{A}_{R,\tau}^{+,\kappa}(\nu)$.
5.7.1: $f_{u,v,z}(\tau_R, \kappa_R)$.
5.7.2: $\mathbf{O}_{R,\tau}^{\nu,\kappa}\{a\}$, $\mathbf{A}_{R,\tau}^{\nu,\kappa}\{a\}$, p^o, λ^o, h, $\varkappa = \varkappa_R$, $\mathscr{O}_{R,h}^\nu\{a\}$, $A_{R,h}^\nu\{a\}$, $M(\lambda)_{R,h}$, $\Delta(\lambda)_{R,h}$, $\mathscr{Q}_R$, $T_{R,a_\bullet}(\nu_\bullet)$, $T_{R,h,d}$.
5.7.3: $\mathbf{V}(\nu_p)$, f_p, $\mathbf{T}_{R,d}(\nu)$, $\mathbf{H}_{R,a}^\ell$, $f_{\mathbf{p}}$, $\mathbf{T}_{R,\mathbf{p}}(\nu)$, $\mathbf{T}_{R,(a)}(\nu)$, $\psi_{R_{\mathrm{p}},d}^+(\nu)$, $\Psi_{R_{\mathrm{p}},d}^+(\nu)$.
5.9: E, F.
6.1.1: W, $\mathfrak{h}$, S, $\mathcal{A}$, $\mathfrak{h}_{reg}$, c, $H_c(W,\mathfrak{h})_R$, α_s, $\check{\alpha}_s$, $R[\mathfrak{h}]$, $R[\mathfrak{h}^*]$, $\mathcal{O}_c(W,\mathfrak{h})_R$, $\Delta(E)_R$, $L(E)$, $P(E)_R$, $(\bullet)^\vee$, $c^\vee$.
6.1.2: KZ_R.
6.1.3: W', S', $\mathfrak{h}^{W'}$, ${}^{\mathcal{O}}\mathrm{Ind}_{W'}^W$, ${}^{\mathcal{O}}\mathrm{Res}_{W'}^W$, W_H, $\mathcal{O}(W_H)_R$, ${}^{\mathcal{O}}\mathrm{Ind}_H$.
6.1.4: $\mathrm{Ch}(M)$.
6.2.1: γ_i, s_{ij}^γ, x_i, y_i, k, c_γ, h_R, $h_{R,p}$, $\mathcal{O}_R^{s,\kappa}\{d\}$, $\mathcal{O}_R^\kappa(\mathfrak{S}_d)$, $\Delta(\lambda)_R^{s,\kappa}$, $L(\lambda)^{s,\kappa}$, $P(\lambda)_R^{s,\kappa}$, $T(\lambda)_R^{s,\kappa}$, $I(\lambda)_R^{s,\kappa}$.

6.2.2: $\succ_s, \geqslant_{s,\kappa}, s^\star, \lambda^\star$.
6.2.3: $\mathrm{KZ}^s_{R,d}$.
6.2.4: $\mathscr{R}_{\mathbf{H}}$.
6.3.1: λ_+, λ_-.
7.1: $\Lambda^Q, \mathbf{F}(\Lambda^s), |\lambda, s\rangle, n_i(\lambda) = n_i^s(\lambda) = n_i^Q(\lambda), \mathbf{wt}(|\lambda, s\rangle)$.
7.2: $\mathcal{G}^\pm(\lambda, s), \mathcal{O}^{s^\star,-e}$.
7.3: $\mathcal{O}^s_t, |s|$.
7.4: $\Upsilon_d, \Sigma = \Sigma^{a,a'}, \widetilde{\mathbf{A}}^{\nu,-e}\{d\}, \tilde{\Upsilon}_d^{\nu,-e}, \tilde{\Psi}^\nu_d, \widetilde{\mathbf{A}}^{\nu,-e}_\mu, \widetilde{\mathbf{A}}^{\nu,-e}$.
8.1: $R^A, \mathbf{g}^A_R, \mathbf{g}^A_{R,\kappa}, \bigotimes_{R,a}, C, \gamma = \{\gamma_a;\ a \in A\}, \eta_a, x_a, C_\gamma, D_R = D_{R,\gamma}$, $\Gamma_R = \Gamma_{R,\gamma}, {}^a f, {}^A f, \langle N_1, \dots, N_n\rangle_R, \langle\langle M_1, \dots, M_n\rangle\rangle_R$.
8.2: $(\mathbf{O}_K^{+,\kappa}, \dot{\otimes}_K, \mathbf{a}_K, \mathbf{c}_K), (\mathbf{O}_K^{\nu,\kappa}, \dot{\otimes}_K, \mathbf{a}, \mathbf{c}), \gamma_{-1}, \gamma_0, \gamma_1$,
8.4: $v = v_R$.

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