ON THE CENTER OF QUIVER HECKE ALGEBRAS

P. SHAN, M. VARAGNOLO, and E. VASSEROT

Abstract

We compute the equivariant cohomology ring of the moduli space of framed instantons over the affine plane. It is a Rees algebra associated with the center of cyclotomic degenerate affine Hecke algebras of type A. We also give some related results on the center of quiver Hecke algebras and the cohomology of quiver varieties.

Contents

1. Introduction

A few years ago, the cohomology ring of the Hilbert scheme of points on $\mathbb{C}^2$ was computed in [27] and [38], motivated by some conjectures of Chen and Ruan on orbifold cohomology rings, which were later proved in [14]. One of the main motivations of the present work is to compute a larger class of cohomology ring of quiver varieties. More precisely, for each pair of positive integers r, n, one can consider the moduli space $\mathfrak{M}(r,n)$ of framed instantons with second Chern class n on $\mathbb{P}^2$. This is a smooth quasiprojective variety (over $\mathbb{C}$) which can be viewed as a quiver variety attached to the Jordan quiver. An $(r+2)$-dimension torus acts naturally on $\mathfrak{M}(r,n)$ and it contains an $(r+1)$-dimensional subtorus whose action preserves the symplectic form. One of our goals is to compute the equivariant cohomology ring of $\mathfrak{M}(r,n)$

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with respect to this subtorus. Since $\mathfrak{M}(r,n)$ is equivariantly formal, one can easily recover the usual cohomology ring from the equivariant one. For $r = 1$, that is, the case of the Hilbert scheme, it can be easily deduced from [38] that the equivariant cohomology ring we are interested in is the Rees algebra associated with the center of the group algebra of the symmetric group with respect to the age filtration. Here we obtain a similar description for arbitrary r with the group algebra of the symmetric group replaced by a level r cyclotomic quotient of the degenerate affine Hecke algebra of type A.

This result was conjectured soon after [38] was written. However, its proof requires two new ingredients which were introduced only very recently. An important tool used in [38] is Nakajima's action of a Heisenberg algebra on the cohomology spaces of the Hilbert scheme. A similar action on the cohomology of $\mathfrak{M}(r) = \bigsqcup_{n\geq 0}\mathfrak{M}(r,n)$ was introduced by Baranovsky a few years ago, but it is insufficient to compute the cohomology ring. What we need in fact is the action of (a degenerate version of) a new algebra $\mathcal{W}$ which is much bigger than the Heisenberg algebra. This action was introduced recently in [37] to give a proof of the AGT conjecture of Alday, Gaiotto, and Tachikawa for pure $N = 2$ gauge theories for the group $U(n)$. Here, we define a similar action of $\mathcal{W}$ on the center (or the concenter) of cyclotomic quotients of degenerate affine Hecke algebras. Then we compare it with the representation of $\mathcal{W}$ on the cohomology of $\mathfrak{M}(r)$ to obtain the desired isomorphism. To do this, we use categorical representation theory.

Categorical representations of Kac–Moody algebras have captured a lot of interest recently since the work of Khovanov and Lauda [21]–[23] and Rouquier [36]. It was observed recently by Beliakova, Habiro, Guliyev, Lauda, and Zivkovic [3], [4] that a categorical representation gives rise to some interesting structures on the center (or cocenter) of the underlying categories and not only on their Grothendieck group. More precisely, Khovanov and Lauda define a 2-Kac–Moody algebra in [23] which is a 2-category satisfying certain axioms. Their idea is that the trace of this 2-category has a natural structure of an associative algebra $L\mathfrak{g}$ which should be some kind of loop algebra over the underlying Kac–Moody algebra $\mathfrak{g}$. The $\mathfrak{sl}_2$-case was worked out in [4], and the $\mathfrak{sl}_n$-case in [3]. Naturally, the center (or cocenter) of 2-representations of the 2-Kac–Moody algebra gives rise to representations of $L\mathfrak{g}$.

In this paper, we first use a similar idea to investigate the center and cocenter of cyclotomic quiver Hecke algebras associated with a Kac–Moody algebra $\mathfrak{g}$ of arbitrary type. The category of projective modules over these algebras provides minimal categorical representations of $\mathfrak{g}$, in the sense of Rouquier [36]. We compute the representation of $L\mathfrak{g}$ on the center (or cocenter) of these minimal categorical representations. When $\mathfrak{g}$ is symmetric of finite type, we identify these $L\mathfrak{g}$-modules with the *local and global Weyl modules*, which can be realized in the (equivariant) Borel–Moore homology spaces of quiver varieties by [33].

Then, in order to compute the cohomology ring of $\mathfrak{M}(r,n)$, we consider another situation where $\mathfrak{g}$ is replaced by a Heisenberg algebra. On the categorical level this corresponds to the *Heisenberg categorifications* which have also been studied recently. But once again, instead of considering a 2-Heisenberg algebra, we focus on the particular categorical representation given by the module category of degenerate affine Hecke algebras of type A. The analogue of $L\mathfrak{g}$ in this case is the algebra $\mathcal{W}$ mentioned above. Probably one can generalize both situations (the Kac–Moody one and the Heisenberg one) using [17], where categorifications of some generalized (Borcherds–)Kac–Moody algebras are considered. Here, we do not go further in this direction.

Another motivation for this work is to check if the center of the module category of cyclotomic quiver Hecke algebras is positively graded with a 1-dimensional degree 0 component. These two conditions are difficult to check for minimal categorical representations. Using the representation of $L\mathfrak{g}$, we prove that the first condition holds. The second one is more subtle. It is equivalent to the indecomposability of the weight subcategories of the minimal categorical representations. In other words, each of these categories should have a single block. This is well known in type A and in affine type A by the work of Brundan [8] and Lyle and Mathas [29]. We can prove it in some new cases, using the fact that quiver varieties are connected (proved by Crawley-Boevey). But the general case is still unknown.

Now, let us describe more precisely the structure and the main results of the paper. Fix a symmetrizable Kac–Moody algebra $\mathfrak{g}$ and a dominant integral weight Λ of $\mathfrak{g}$.

In Section 2, we give some generalities on the centers and cocenters of linear categories. In Section 3, we introduce the cyclotomic quiver Hecke algebra of type $\mathfrak{g}$ and level Λ over a field $\Bbbk$ (of any characteristic). It is a symmetric algebra which decomposes as a direct sum $R^\Lambda = \bigoplus_{\alpha\in Q_+} R^\Lambda(\alpha)$, where α runs over the positive part of the root lattice of $\mathfrak{g}$. To $\mathfrak{g}$ we can attach another Lie algebra, $L\mathfrak{g}$, given by generators and relations. It coincides with the loop algebra of $\mathfrak{g}$ in finite types A, D, E. The first result is the following.

THEOREM 1

Assume that $\mathfrak{g}$ is symmetric and that condition (11) is satisfied. Then we have the following:

(a) *There is a $\mathbb{Z}$-graded representation of $L\mathfrak{g}$ on* $\mathrm{tr}(R^\Lambda)$.

(b) *If $\mathfrak{g}$ is of finite type and $\Bbbk$ is of characteristic zero, then* $\mathrm{tr}(R^\Lambda)$ *is isomorphic, as a $\mathbb{Z}$-graded $L\mathfrak{g}$-module, to the Weyl module with highest weight Λ.*

Note that there are two different notions of Weyl modules for loop Lie algebras used in the literature (the local and the global ones). Both versions can indeed be recovered (see Theorem 3.37 for more details). Note also that the proof of part (b) involves the geometrical incarnation, given by Nakajima, of Weyl modules of $L\mathfrak{g}$ via the equivariant cohomology of a quiver variety $\mathfrak{M}(\Lambda)$ attached to Λ.

THEOREM 2
Assume that $\Bbbk$ *is of characteristic* 0. *For any* $\alpha \in Q_+$ *the following hold:*
(a) *The trace and the center of* $R^\Lambda(\alpha)$ *are positively graded.*
(b) *If* $\mathfrak{g}$ *is symmetric of finite type, then the dimension of the degree* 0 *subspace of* $\mathrm{Z}(R^\Lambda(\alpha))$ *is* 1*-dimensional.*

The proof uses a reduction to $\mathfrak{sl}_2$. Part (b) relies on the geometrical interpretation of Weyl modules. It also uses an identification between the center $\mathrm{Z}(R^\Lambda(\alpha))$ and the dual of the trace $\mathrm{tr}(R^\Lambda(\alpha))$ of $R^\Lambda(\alpha)$ given by the symmetrizing form.

Finally, in Section 4 we focus on the Jordan quiver. In this case, instead of the cyclotomic quiver Hecke algebra, we consider a level r cyclotomic quotient $R^r(n)$ of the degenerate affine Hecke algebra of $\mathfrak{S}_n$ defined over $\Bbbk[\hbar]$. Let $R^r(n)_1$ be its specialization at $\hbar = 1$. The center of $R^r(n)_1$ has a natural filtration defined in terms of Jucy–Murphy elements. Let $\mathrm{Rees}(\mathrm{Z}(R^r(n)_1))$ be the corresponding Rees algebra. Set $R^r = \bigoplus_{n\in\mathbb{N}} R^r(n)$, and let $\mathrm{tr}(R^r)'$ be a localization of $\mathrm{tr}(R^r)$. Consider the equivariant cohomology $H^*_G(\mathfrak{M}(r,n),\Bbbk)$ of the quiver variety $\mathfrak{M}(r,n)$ relatively to an $(r+1)$-dimensional torus G with coefficient in $\Bbbk$.

THEOREM 3
The following hold:
(a) *There is a level* r *representation of* $\mathcal{W}$ *in* $\mathrm{tr}(R^r)'$.
(b) *Assume that* $\Bbbk$ *is of characteristic* 0*; then there is a* $\mathbb{Z}$*-graded algebra isomorphism*

$$\mathrm{Rees}\big(\mathrm{Z}\big(R^r(n)_1\big)\big) \simeq H^*_G\big(\mathfrak{M}(r,n),\Bbbk\big).$$

The proof of this theorem uses the representation of $\mathcal{W}$ on a localization $H^*_G(\mathfrak{M}(r),\Bbbk)'$ introduced in [37].

After our paper appeared in arXiv, A. Lauda informed us that there is some overlap between our results and his ongoing projects with collaborators.

2. Generalities

Let $\Bbbk$ be a commutative Noetherian ring.

2.1. The center and the trace of a category

2.1.1. Categories

All categories are assumed to be small. A $\Bbbk$-*linear* category is a category enriched over the tensor category of $\Bbbk$-modules, and a $\Bbbk$-*category* is an additive $\Bbbk$-linear category. For any $\Bbbk$-linear category $\mathcal{C}$ and any $\Bbbk$-algebra $\Bbbk'$, let $\Bbbk' \otimes_{\Bbbk} \mathcal{C}$ be the $\Bbbk'$-linear category whose objects are the same as those of $\mathcal{C}$ but whose morphism spaces are given by

$$\mathrm{Hom}_{\Bbbk' \otimes_{\Bbbk} \mathcal{C}}(a,b) = \Bbbk' \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(a,b), \quad \forall a,b \in \mathcal{C}.$$

We denote the identity of an object a by 1_a or by 1 if no confusion is possible. All the functors F on $\mathcal{C}$ are assumed to be additive and/or $\Bbbk$-linear. An additive and $\Bbbk$-linear functor is called a $\Bbbk$-*functor*. Let $\mathrm{End}(F)$ be the endomorphism ring of F. We may denote the identity element in $\mathrm{End}(F)$ by $F, 1_F$, or 1, and the identity functor of $\mathcal{C}$ by $1_{\mathcal{C}}$ or 1. The center of $\mathcal{C}$ is defined as $Z(\mathcal{C}) = \mathrm{End}(1_{\mathcal{C}})$. A composition of functors E and F is written as EF, while a composition of morphisms of functors y and x is written as $y \circ x$.

An additive category $\mathcal{C}$ will be always equipped with its *trivial* exact structure; that is, the admissible exact sequences are the split short exact sequences. Therefore, a Serre subcategory $\mathcal{I} \subset \mathcal{C}$ is a full additive subcategory which is stable under taking direct summands, and the quotient additive category $\mathcal{B} = \mathcal{C}/\mathcal{I}$ is such that

$$\mathrm{Hom}_{\mathcal{B}}(a,b) = \mathrm{Hom}_{\mathcal{C}}(a,b) / \sum_{c \in \mathcal{I}} \mathrm{Hom}_{\mathcal{C}}(c,b) \circ \mathrm{Hom}_{\mathcal{C}}(a,c), \quad \forall a,b \in \mathcal{C}.$$

A *short exact sequence* of additive categories is a sequence of functors which is equivalent to a sequence $0 \to \mathcal{I} \to \mathcal{C} \to \mathcal{B} \to 0$ as above.

Fix an integer ℓ. By an $(\ell\mathbb{Z})$-*graded* $\Bbbk$-category we will mean a $\Bbbk$-category $\mathcal{C}$ equipped with a strict $\Bbbk$-automorphism $[\ell]$, which we call a *shift of the grading*. Unless specified otherwise, a functor F of $(\ell\mathbb{Z})$-graded $\Bbbk$-categories is always assumed to be graded; that is, it is a $\Bbbk$-functor F with an isomorphism $F \circ [\ell] \simeq [\ell] \circ F$. For each integer $k \in \mathbb{N} \cap (\ell\mathbb{Z})$ we will abbreviate $[k] = [\ell] \circ [\ell] \circ \cdots \circ [\ell]$ ($|k/\ell|$ times) and $[-k] = [k]^{-1}$.

Let $\mathcal{C}/\ell\mathbb{Z}$ be the category enriched over the tensor category of $(\ell\mathbb{Z})$-graded $\Bbbk$-modules whose objects are the same as those of $\mathcal{C}$ but whose morphism spaces are given by

$$\mathrm{Hom}_{\mathcal{C}/\ell\mathbb{Z}}(a,b) = \bigoplus_{k \in \ell\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}\big(a, b[k]\big).$$

Note that the center $Z(\mathcal{C}/\ell\mathbb{Z})$ is a graded ring whose degree k component is equal to $\mathrm{Hom}(1,[k])$.

Given a $\mathbb{Z}$-graded $\Bbbk$-module M, let $M^d = \{x \in M; \deg(x) = d\}$ for each $d \in \mathbb{Z}$. For any integer ℓ, the *ℓ-twist* of M is the $(\ell\mathbb{Z})$-graded $\Bbbk$-module $M^{[\ell]}$ such that $(M^{[\ell]})^d = M^{d/\ell}$ if $\ell \mid d$ and 0 otherwise. Then, for each $\mathbb{Z}$-graded $\Bbbk$-category $\mathcal{C}$ there is a canonical $(\ell\mathbb{Z})$-graded $\Bbbk$-category $\mathcal{C}^{[\ell]}$ called the *ℓ-twist* of $\mathcal{C}$ such that $\mathcal{C}^{[\ell]} = \mathcal{C}$ as a $\Bbbk$-category and the shift of the grading $[\ell]$ in $\mathcal{C}^{[\ell]}$ is the same as the shift of the grading [1] in $\mathcal{C}$. We have

$$\mathrm{Hom}_{\mathcal{C}^{[\ell]}/\ell\mathbb{Z}}(a,b) = \mathrm{Hom}_{\mathcal{C}/\mathbb{Z}}(a,b)^{[\ell]}, \quad \forall a, b.$$

Finally, for any category $\mathcal{C}$ we denote by $\mathcal{C}^c$ the idempotent completion.

2.1.2. Trace and center

Let $\mathcal{C}$ be a $\Bbbk$-linear category, and let $HH_*(\mathcal{C})$ be the Hochschild homology of $\mathcal{C}$ (see [20, Section 3.1]). It is a $\mathbb{Z}$-graded $\Bbbk$-module. We set $\mathrm{tr}(\mathcal{C}) = HH_0(\mathcal{C})$ and $\mathrm{CF}(\mathcal{C}) = \mathrm{Hom}_{\Bbbk}(\mathrm{tr}(\mathcal{C}), \Bbbk)$. We call $\mathrm{tr}(\mathcal{C})$ the *cocenter* or the *trace* of $\mathcal{C}$ and $\mathrm{CF}(\mathcal{C})$ the set of *central forms on* $\mathcal{C}$. Recall that

$$\mathrm{tr}(\mathcal{C}) = \Big(\bigoplus_{a \in \mathrm{Ob}(\mathcal{C})} \mathrm{End}_{\mathcal{C}}(a)\Big) / \sum_{f,g} \Bbbk[f,g] \quad \text{for any } f : a \to b, g : b \to a.$$

For any morphism f in $\mathcal{C}$, let $\mathrm{tr}(f)$ denote its image in $\mathrm{tr}(\mathcal{C})$.

Now, let A be any $\Bbbk$-algebra. Unless specified otherwise, all algebras are assumed to be unital. Let $Z(A)$ be the center of A, and let $HH_*(A)$ be its Hochschild homology. Define $\mathrm{tr}(A)$ and $\mathrm{CF}(A)$ as above; that is, $\mathrm{tr}(A) = A/[A,A]$, where $[A,A] \subset A$ is the $\Bbbk$-submodule spanned by the commutators of elements of A. For any element $a \in A$, let $\mathrm{tr}(a)$ denote its image $a + [A,A]$ in $\mathrm{tr}(A)$. Let A-mod and A-proj be the categories of finitely generated modules and finitely generated projective modules. For any commutative $\Bbbk$-algebra R and any $\Bbbk$-module M we abbreviate $RM = R \otimes_{\Bbbk} M$. The following is well known.

PROPOSITION 2.1

Let A, B be $\Bbbk$-algebras, and let $\mathcal{B}, \mathcal{C}$ be $\Bbbk$-linear categories.

(a) *If $\mathcal{B} \subset \mathcal{C}$ is full and any object of $\mathcal{C}$ is isomorphic to a direct summand of a direct sum of objects of $\mathcal{B}$, then the embedding $\mathcal{B} \subset \mathcal{C}$ yields an isomorphism* $\mathrm{tr}(\mathcal{B}) \to \mathrm{tr}(\mathcal{C})$.

(b) *If* $\mathcal{C} = A$-mod *or* A-proj, *then* $Z(A) = Z(\mathcal{C})$. *If* $\mathcal{C} = A$-proj, *then* $\mathrm{tr}(A) = \mathrm{tr}(\mathcal{C})$.

(c) *For any commutative $\Bbbk$-algebra R, we have* $\mathrm{tr}(RA) = R\,\mathrm{tr}(A)$.

(d) *We have* $\mathrm{tr}(A \otimes_{\Bbbk} B) = \mathrm{tr}(A) \otimes_{\Bbbk} \mathrm{tr}(B)$ *and* $Z(A \otimes_{\Bbbk} B) = Z(A) \otimes_{\Bbbk} Z(B)$.

(e) $Z(\mathcal{C})$ *acts on* $\mathrm{tr}(\mathcal{C})$ *via the map* $Z(\mathcal{C}) \to \mathrm{End}_{\Bbbk}(\mathrm{tr}(\mathcal{C}))$, $a \mapsto (\mathrm{tr}(a') \mapsto \mathrm{tr}(aa'))$.

(f) *A short exact sequence of $\Bbbk$-categories $0 \to \mathcal{I} \to \mathcal{C} \to \mathcal{B} \to 0$ yields an exact sequence of $\Bbbk$-linear maps* $\mathrm{tr}(\mathcal{I}) \to \mathrm{tr}(\mathcal{C}) \to \mathrm{tr}(\mathcal{B}) \to 0$.

For a future use, let us give some details on part (f). Assume that $\mathcal{C} = \mathcal{C}^c$. For any object X, let $\mathrm{add}(X) \subset \mathcal{C}$ be the smallest $\Bbbk$-subcategory containing X which is closed under taking direct summands. Then, the functor $\mathrm{Hom}_{\mathcal{C}}(X, \bullet)$ yields an equivalence $\mathrm{add}(X) \to \mathrm{End}_{\mathcal{C}}(X)^{\mathrm{op}}$-proj. In particular, if $\mathcal{C}$ has a finite number of indecomposable objects $X_1, X_2, \ldots, X_n$ (up to isomorphisms) and $X = \bigoplus_{i=0}^{d} X_i$, then we have an equivalence $\mathcal{C} \simeq \mathrm{End}_{\mathcal{C}}(X)^{\mathrm{op}}$-proj.

Now, assume that $\mathcal{C} = A$-proj, where A is a finitely generated $\Bbbk$-algebra. Given a Serre $\Bbbk$-subcategory $\mathcal{I} \subset \mathcal{C}$, there is an idempotent $e \in A$ such that $\mathcal{I} = eAe$-proj and the functor $\mathcal{I} \to \mathcal{C}$ is given by $M \mapsto Ae \otimes_{eAe} M$. Set $\mathcal{B} = \mathcal{C}/\mathcal{I}$. Then, we have $\mathcal{B}^c = B$-proj, where $B = A/AeA$ and the composed functor $\mathcal{C} \to \mathcal{B} \to \mathcal{B}^c$ is given by $M \mapsto B \otimes_A M$. We must prove that taking the trace we get an exact sequence of $\Bbbk$-modules $\mathrm{tr}(\mathcal{I}) \to \mathrm{tr}(\mathcal{C}) \to \mathrm{tr}(\mathcal{B}) \to 0$. Equivalently, we must check that the following complex is exact:

$$eAe/[eAe, eAe] \xrightarrow{\ i\ } A/[A, A] \xrightarrow{\ j\ } B/[B, B] \longrightarrow 0.$$

Note that $\ker j = (AeA + [A, A])/[A, A]$ and $\mathrm{im}\, i = (eAe + [A, A])/[A, A]$. Since $aeb = ebae + [ae, eb]$ for all $a, b \in A$, we deduce that $\ker j = \mathrm{im}\, i$, proving the claim.

2.1.3. Operators on the trace

Definition 2.2
Given a functor $F : \mathcal{C} \to \mathcal{C}'$ between two $\Bbbk$-categories and a morphism of functors $x \in \mathrm{End}(F)$, the *trace* of F on x is the linear map

$$\mathrm{tr}_F(x) : \mathrm{tr}(\mathcal{C}) \to \mathrm{tr}(\mathcal{C}'), \qquad \mathrm{tr}(f) \mapsto \mathrm{tr}\big(x(a) \circ F(f)\big),$$

where $f \in \mathrm{End}(a)$ and $x(a) \circ F(f) \in \mathrm{End}(F(a))$.

Note that $x(a) \circ F(f) = F(f) \circ x(a)$ by functoriality. Below are some basic properties of the trace map, whose proofs are standard and are left to the reader.

LEMMA 2.3
(a) *For each F_1, $F_2 : \mathcal{C} \to \mathcal{C}'$, $x \in \mathrm{End}(F_1 \oplus F_2)$, we have $\mathrm{tr}_{F_1 \oplus F_2}(x) = \mathrm{tr}_{F_1}(x_{11}) + \mathrm{tr}_{F_2}(x_{22})$, where $x_{11} \in \mathrm{End}(F_1)$, $x_{22} \in \mathrm{End}(F_2)$ are the diagonal coordinates of x.*
(b) *For two morphisms $\rho : F_1 \to F_2$, $\psi : F_2 \to F_1$, we have $\mathrm{tr}_{F_1}(\psi \circ \rho) = \mathrm{tr}_{F_2}(\rho \circ \psi)$. In particular, if $\rho : F_1 \to F_2$ is an isomorphism of functors, then for any $x \in \mathrm{End}(F_1)$ we have $\mathrm{tr}_{F_2}(\rho \circ x \circ \rho^{-1}) = \mathrm{tr}_{F_1}(x)$.*

(c) *For each* $F : \mathcal{C} \to \mathcal{C}'$, $G : \mathcal{C}' \to \mathcal{C}''$, $x \in \mathrm{End}(F)$ *and* $y \in \mathrm{End}(G)$, *we have* $\mathrm{tr}_{GF}(yx) = \mathrm{tr}_G(y) \circ \mathrm{tr}_F(x)$.

2.1.4. *Adjunction*

Given two $\Bbbk$-categories $\mathcal{C}_1$, $\mathcal{C}_2$, a *pair of adjoint functors* (E, F) from $\mathcal{C}_1$ to $\mathcal{C}_2$ is the datum $(E, F, \eta_E, \varepsilon_E)$ of functors $E : \mathcal{C}_1 \to \mathcal{C}_2$, $F : \mathcal{C}_2 \to \mathcal{C}_1$ and morphisms of functors $\eta_E : 1_{\mathcal{C}_1} \to FE$ and $\varepsilon_E : EF \to 1_{\mathcal{C}_2}$, called *unit* and *counit*, such that $(\varepsilon_E E) \circ (E\eta_E) = E$ and $(F\varepsilon_E) \circ (\eta_E F) = F$, where we abbreviate $E = 1_E$ and $F = 1_F$.

A *pair of biadjoint functors* $\mathcal{C}_1 \to \mathcal{C}_2$ is the datum $(E, F, \eta_E, \varepsilon_E, \eta_F, \varepsilon_F)$ of functors $E : \mathcal{C}_1 \to \mathcal{C}_2$, $F : \mathcal{C}_2 \to \mathcal{C}_1$ and morphisms of functors $\eta_E : 1_{\mathcal{C}_1} \to FE$, $\varepsilon_E : EF \to 1_{\mathcal{C}_2}$ such that $(E, F, \eta_E, \varepsilon_E)$ and $(F, E, \eta_F, \varepsilon_F)$ are adjoint pairs.

Example 2.4

Given two pairs of adjoint functors (E, F), (E', F') from $\mathcal{C}_1$ to $\mathcal{C}_2$, the direct sum $(E \oplus E', F \oplus F')$ is an adjoint pair such that

$$\eta_{E\oplus E'} = (\eta_E, 0, 0, \eta_{E'}) : 1_{\mathcal{C}_1} \to FE \oplus FE' \oplus F'E \oplus F'E',$$
$$\varepsilon_{E\oplus E'} = \varepsilon_E + \varepsilon_{E'} : EF \oplus EF' \oplus E'F \oplus E'F' \to 1_{\mathcal{C}_2}.$$

If $E : \mathcal{C}_1 \to \mathcal{C}_2$ and $E' : \mathcal{C}_2 \to \mathcal{C}_3$, then $(E'E, FF')$ is an adjoint pair such that $\eta_{E'E} = (F\eta_{E'}E) \circ \eta_E$ and $\varepsilon_{E'E} = \varepsilon_{E'} \circ (E'\varepsilon_E F')$.

Suppose that (E, F), (E', F') are two pairs of adjoint functors from $\mathcal{C}_1$ to $\mathcal{C}_2$. For any morphism $x : E \to E'$, the *left transpose* ${}^\vee x : F' \to F$ is the composition of the chain of morphisms

$$F' \xrightarrow{\eta_E F'} FEF' \xrightarrow{FxF'} FE'F' \xrightarrow{F\varepsilon_{E'}} F.$$

For any morphism $y : F' \to F$, the *right transpose* $y^\vee : E \to E'$ is the composition

$$E \xrightarrow{E\eta_{E'}} EF'E' \xrightarrow{EyE'} EFE' \xrightarrow{\varepsilon_E E'} E'.$$

2.1.5. *Operators on the center*

Let $\mathcal{C}_1$, $\mathcal{C}_2$ be two $\Bbbk$-categories, and let $(E, F, \eta_E, \varepsilon_E, \eta_F, \varepsilon_F)$ be a pair of biadjoint functors $\mathcal{C}_1 \to \mathcal{C}_2$. The isomorphisms $1_{\mathcal{C}_2}E = E = E1_{\mathcal{C}_1}$ yield a canonical $(\mathrm{Z}(\mathcal{C}_1), \mathrm{Z}(\mathcal{C}_2))$-bimodule structure on $\mathrm{End}(E)$. Let $\mathrm{Z}(\mathcal{C}_2) \to \mathrm{End}(E)$, $z \mapsto zE$ and $\mathrm{Z}(\mathcal{C}_1) \to \mathrm{End}(E)$, $z \mapsto Ez$ denote the corresponding $\Bbbk$-algebra homomorphisms.

Definition 2.5 (see [5])
For each $x \in \mathrm{End}(E)$ we define a map

$$Z_E(x) : \mathrm{Z}(\mathcal{C}_2) \to Z(\mathcal{C}_1)$$

by sending an element $z \in \mathrm{Z}(\mathcal{C}_2)$ to the composed morphism

$$1_{\mathcal{C}_1} \xrightarrow{\eta_E} F 1_{\mathcal{C}_2} E \xrightarrow{Fzx} F 1_{\mathcal{C}_2} E \xrightarrow{\varepsilon_F} 1_{\mathcal{C}_1}.$$

We define $Z_F(x) : Z(\mathcal{C}_1) \to Z(\mathcal{C}_2)$ for each $x \in \mathrm{End}(F)$ in the same manner but with the roles of E and F exchanged.

The proof of the following proposition is standard and is left to the reader.

PROPOSITION 2.6
Let $(E, F, \eta_E, \varepsilon_E, \eta_F, \varepsilon_F)$, $(E', F', \eta_{E'}, \varepsilon_{E'}, \eta_{F'}, \varepsilon_{F'})$ be two pairs of biadjoint functors. Let $x \in \mathrm{End}(E)$, $x' \in \mathrm{End}(E')$. Then, we have the following:
(a) $Z_E(x) : \mathrm{Z}(\mathcal{C}_2) \to \mathrm{Z}(\mathcal{C}_1)$ *is $\Bbbk$-linear.*
(b) $Z_{E'E}(x \circ \phi) = Z_{E'}(x') \circ Z_E(x)$ *and* $Z_{E \oplus E'}(x \oplus x) = Z_E(x) + Z_{E'}(x')$.
(c) *The map* $Z_E : \mathrm{End}(E) \to \mathrm{Hom}_{\Bbbk}(\mathrm{Z}(\mathcal{C}_2), \mathrm{Z}(\mathcal{C}_1))$ *is* $(\mathrm{Z}(\mathcal{C}_1), \mathrm{Z}(\mathcal{C}_2))$*-bilinear.*
(d) *Let $\rho : E \to E'$ be an isomorphism with $\rho^\vee = {}^\vee\rho$; then* $Z_{E'}(\rho \circ x \circ \rho^{-1}) = Z_E(x)$.

2.2. Symmetric algebras
Let A, B, C be $\Bbbk$-algebras.

2.2.1. Kernels
There is an equivalence of categories between the category of (A, B)-bimodules and the categories of functors from B-Mod to A-Mod. It associates an (A, B)-bimodule K with the functor $\Phi_K : B\text{-Mod} \to A\text{-Mod}$ given by $N \mapsto K \otimes_B N$. We say that K is the *kernel* of Φ_K. Since $\Phi_K(B) = K$, the kernel is uniquely determined by the functor Φ_K. For two (A, B)-bimodules K, K' we have $\mathrm{Hom}_{A,B}(K, K') \simeq \mathrm{Hom}(\Phi_K, \Phi_{K'})$ given by $f \mapsto f \otimes_B \mathrm{id}$.

2.2.2. Induction and restriction
We call a *B-algebra* a $\Bbbk$-algebra A with a $\Bbbk$-algebra homomorphism $i : B \to A$. We consider the restriction and induction functors

$$\mathrm{Res}_B^A : A\text{-Mod} \to B\text{-Mod}, \qquad \mathrm{Ind}_B^A = A \otimes_B - : B\text{-Mod} \to A\text{-Mod}.$$

The pair $(\mathrm{Ind}_B^A, \mathrm{Res}_B^A)$ is adjoint with the counit $\varepsilon : \mathrm{Ind}_B^A \mathrm{Res}_B^A \to 1$ represented by the (A, A)-bimodule homomorphism $\mu : A \otimes_B A \to A$ given by the multiplication, and the unit $\eta : 1 \to \mathrm{Res}_B^A \mathrm{Ind}_B^A$ is represented by the morphism i, which is (B, B)-bilinear. Let A^B be the centralizer of B in A. For any $f \in A^B$ we set

$$\mu_f : A \otimes_B A \to A, \qquad a \otimes a' = afa'. \tag{1}$$

2.2.3. Frobenius and symmetrizing forms
We refer to [36] for more details on this section.

Let A be a B-algebra that is projective and finite as a B-module. A morphism of (B, B)-bimodules $t : A \to B$ is called a *Frobenius form* if the morphism of (A, B)-bimodules $\hat{t} : A \to \mathrm{Hom}_B(A, B)$, $a \mapsto (a' \mapsto t(a'a))$ is an isomorphism. If such a form exists, then we say that A is a *Frobenius B-algebra*. If we have $t(aa') = t(a'a)$ for each $a \in A$, $a' \in A^B$, then t is called a *symmetrizing form* and A is a *symmetric B-algebra*.

Given $t : A \to B$ a Frobenius form, the composition of the isomorphism $A \otimes_B A \xrightarrow{\sim} \mathrm{Hom}_B(A, B) \otimes A$ given by $a \otimes a' \mapsto \hat{t}(a) \otimes a'$ and the canonical isomorphism $\mathrm{Hom}_B(A, B) \otimes_B A \xrightarrow{\sim} \mathrm{End}_B(A)$ yields an isomorphism $A \otimes_B A \xrightarrow{\sim} \mathrm{End}_B(A)$. The preimage of the identity under this map is the *Casimir element* $\pi \in (A \otimes_B A)^A$. We have $(t \otimes 1)(\pi) = (1 \otimes t)(\pi) = 1$.

There is a bijection between the set of Frobenius forms and the set of adjunctions $(\mathrm{Res}_B^A, \mathrm{Ind}_B^A)$ given as follows. Given a Frobenius form $t : A \to B$, the counit $\hat{\varepsilon} : \mathrm{Res}_B^A \mathrm{Ind}_B^A \to 1_B$ is represented by the (B, B)-linear map $t : A \to B$, and the unit $\hat{\eta} : 1_A \to \mathrm{Ind}_B^A \mathrm{Res}_B^A$ is represented by the unique (A, A)-linear map $\hat{\eta} : A \to A \otimes_B A$ such that $\hat{\eta}(1_A) = \pi$. This yields an adjunction for $(\mathrm{Res}_B^A, \mathrm{Ind}_B^A)$. Conversely, if $\hat{\varepsilon}$ and $\hat{\eta}$ are counit and unit, respectively, for $(\mathrm{Res}_B^A, \mathrm{Ind}_B^A)$, then the (B, B)-linear map $t : A \to B$ which represents $\hat{\varepsilon}$ is a Frobenius form.

Recall that $\mathrm{tr}(A)$ is a $\mathrm{Z}(A)$-module. We equip $\mathrm{CF}(A) = \mathrm{tr}(A)^*$ with the dual $\mathrm{Z}(A)$-action. Let us recall a few basic facts.

PROPOSITION 2.7
Let A, B, C be $\Bbbk$-algebras which are projective and finite as $\Bbbk$-modules.

(a) *If $t : A \to B$ and $t' : B \to C$ are symmetrizing forms, then $t' \circ t : A \to C$ is again a symmetrizing form.*

(b) *A symmetrizing form $t : A \to \Bbbk$ induces a $\mathrm{Z}(A)$-bilinear form*

$$t : \mathrm{Z}(A) \times \mathrm{tr}(A) \to \mathbf{k}, \qquad (a,b) \mapsto t(ab).$$

It is perfect on $\mathrm{Z}(A)$; that is, it induces an isomorphism of $\mathrm{Z}(A)$-modules $\hat{t} : \mathrm{Z}(A) \xrightarrow{\sim} \mathrm{CF}(A)$ which sends z to $t(z\bullet)$.

(c) *If $t : A \to \Bbbk$ is a symmetrizing form, then $\mathrm{tr}(A)$ is a faithful $\mathrm{Z}(A)$-module.*

Proof
Part (a) is proved in [36, Lemma 2.10]. To prove part (b), note that the bilinear form is well defined, since multiplication by an element in $Z(A)$ sends $[A, A]$ to itself. The pairing is perfect on $Z(A)$, because the induced map $\hat{t} : Z(A) \to \mathrm{tr}(A)^* = \mathrm{CF}(A)$ is given by taking (A, A)-invariants for the (A, A)-linear isomorphism $\hat{t} : A \to A^*$; hence the result is again an isomorphism. The compatibility with the $Z(A)$-module structure follows from the definition. To prove part (c), we must show that the map $Z(A) \to \mathrm{End}_{\Bbbk}(\mathrm{tr}(A))$ is injective. Indeed, if there exists $z \in Z(A)$ such that $za \in [A, A]$ for all $a \in A$, then $t(zA) = 0$, and hence $z = 0$. □

3. The center of quiver Hecke algebras

3.1. Quiver Hecke algebras

Assume that $\mathbf{k} = \bigoplus_{n\in\mathbb{N}} \mathbf{k}^n$ is Noetherian and $\mathbb{N}$-graded and that $\mathbf{k}^0$ is a field. We may abbreviate $\Bbbk = \mathbf{k}^0$, and we will identify $\Bbbk$ with the quotient $\mathbf{k}/\mathbf{k}^{>0}$ without mentioning it explicitly.

3.1.1. Cartan datum

A *Cartan datum* consists of a finite-rank free abelian group P called the *weight lattice* whose dual lattice, called the *coweight lattice*, is denoted $P^\vee$; a finite set of vectors $\Phi = \{\alpha_1, \dots, \alpha_n\} \subset P$ called *simple roots*; and a finite set of vectors $\Phi^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\} \subset P^\vee$ called *simple coroots*. Let $Q_+ = \mathbb{N}\Phi \subset P$ be the semigroup generated by the simple roots, and let $P_+ \subset P$ be the subset of dominant weights, that is, the set of weights Λ such that $\Lambda_i = \langle \alpha_i^\vee, \Lambda \rangle \geq 0$ for all $i \in I$. We will call a *Bruhat order* the partial order on P such that $\lambda \leq \mu$ whenever $\mu - \lambda \in Q_+$.

Set $I = \{1, \dots, n\}$, and let $\langle \bullet, \bullet \rangle$ be the canonical pairing on $P^\vee \times P$. The $(I \times I)$-matrix A with entries $a_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle$ is assumed to be a *generalized Cartan matrix*. We will assume that the Cartan datum is nondegenerate (i.e., the simple roots are linearly independent) and symmetrizable (i.e., there exist nonzero integers d_i such that $d_i a_{ij} = d_j a_{ji}$ for all i, j). The integers d_i are unique up to an overall common factor. They can be assumed positive and mutually prime.

Let $(\bullet|\bullet)$ be the symmetric bilinear form on $\mathfrak{h}^* = \mathbb{Q} \otimes_{\mathbb{Z}} P$ given by $(\alpha_i|\alpha_j) = d_i a_{ij}$. Let $\mathfrak{g}$ be the symmetrizable Kac–Moody algebra over $\Bbbk$ associated with the generalized Cartan matrix A and the lattice of integral weights P. Let $\mathfrak{h}, \mathfrak{n}^+ \subset \mathfrak{g}$ be the Cartan subalgebra and the maximal nilpotent subalgebra spanned by the positive root vectors of $\mathfrak{g}$. For any dominant weight $\Lambda \in P_+$, let $V(\Lambda)$ be the corresponding integrable simple $\mathfrak{g}$-module. For each $\lambda \in P$, let $V(\Lambda)_\lambda \subseteq V(\Lambda)$ be the weight subspace of weight λ.

3.1.2. Quiver Hecke algebras

Fix an element $c_{i,j,p,q} \in \mathbf{k}$ for each $i, j \in I$, $p, q \in \mathbb{N}$ such that $\deg(c_{i,j,p,q}) = -2d_i(a_{ij} + p) - 2d_j q$ and $c_{i,j,-a_{ij},0}$ is invertible. Fix a matrix $Q = (Q_{ij})_{i,j \in I}$ with entries in $\mathbf{k}[u, v]$ such that

$$Q_{ij}(u,v) = Q_{ji}(v,u),$$
$$Q_{ii}(u,v) = 0,$$
$$Q_{ij}(u,v) = \sum_{p,q \geq 0} c_{i,j,p,q} u^p v^q \quad \text{if } i \neq j.$$

Definition 3.1

The *quiver Hecke algebra* (or QHA) of rank $n \geq 0$ associated with A and Q is the $\mathbf{k}$-algebra $R(n; Q, \mathbf{k})$ generated by $e(\nu)$, x_k, τ_l with $\nu \in I^n$, $k, l \in [1, n]$, $l \neq n$, satisfying the following defining relations:

(a) $e(\nu)e(\nu') = \delta_{\nu,\nu'} e(\nu)$, $\sum_\nu e(\nu) = 1$,

(b) $x_k x_l = x_l x_k$, $x_k e(\nu) = e(\nu) x_k$,

(c) $\tau_l e(\nu) = e(s_l(\nu)) \tau_l$, $\tau_k \tau_l = \tau_l \tau_k$ if $|k - l| > 1$,

(d) $\tau_l^2 e(\nu) = Q_{\nu_l, \nu_{l+1}}(x_l, x_{l+1}) e(\nu)$,

(e) $(\tau_k x_l - x_{s_k(l)} \tau_k) e(\nu) = \delta_{\nu_k, \nu_{k+1}} (\delta_{l,k+1} - \delta_{l,k}) e(\nu)$,

(f) $(\tau_{k+1} \tau_k \tau_{k+1} - \tau_k \tau_{k+1} \tau_k) e(\nu) = \delta_{\nu_k, \nu_{k+2}} \partial_{k,k+2} Q_{\nu_k, \nu_{k+1}}(x_k, x_{k+1}) e(\nu)$,

where $\partial_{k,l}$ is the Demazure operator on $\mathbf{k}[x_1, x_2, \ldots, x_n]$ which is defined by

$$\partial_{k,l}(f) = (f - (k,l)(f))/(x_k - x_l).$$

The algebra $R(n; Q, \mathbf{k})$ is free as a $\mathbf{k}$-module. It admits a $\mathbb{Z}$-grading given by

$$\deg\big(e(\nu)\big) = 0, \qquad \deg\big(x_k e(\nu)\big) = 2d_{\nu_k}, \qquad \deg\big(\tau_k e(\nu)\big) = -d_{\nu_k} a_{\nu_k, \nu_{k+1}}.$$

For $\alpha \in Q_+$ such that $\operatorname{ht}(\alpha) = n$, we set

$$I^\alpha = \big\{\nu = (\nu_1, \ldots, \nu_n) \in I^n; \alpha_{\nu_1} + \cdots + \alpha_{\nu_n} = \alpha\big\}.$$

The idempotent $e(\alpha) = \sum_{\nu \in I^\alpha} e(\nu)$ is central in $R(n; Q, \mathbf{k})$. Given $\nu \in I^n$, $\nu' \in I^m$ we write $\nu\nu' \in I^{n+m}$ for their concatenation. Set $e(\alpha, \nu') = \sum_{\nu \in I^\alpha} e(\nu\nu')$ and $e(n, \nu') = \sum_{\nu \in I^n} e(\nu\nu')$. The quiver Hecke algebra of rank α is the algebra

$$R(\alpha; Q, \mathbf{k}) = e(\alpha) R(n; Q, \mathbf{k}) e(\alpha).$$

3.1.3. Cyclotomic quiver Hecke algebras

Given a dominant weight $\Lambda \in P_+$ we set

$$I_\Lambda = \big\{(i, p); i \in I, p = 1, \ldots, \Lambda_i\big\}.$$

For a future use, let

$$I_\Lambda \to I, \qquad t \mapsto i_t \tag{2}$$

denote the canonical map such that $(i, p) \mapsto i$. Fix a family of commuting formal variables $\{c_t; t \in I_\Lambda\}$. Let $\underline{\mathbf{k}}^\Lambda$ be the $\mathbb{N}$-graded ring given by

$$\underline{\mathbf{k}}^\Lambda = \mathbb{k}[c_t; t \in I_\Lambda], \qquad \deg(c_{ip}) = 2pd_i.$$

We will abbreviate $\underline{\mathbf{k}} = \underline{\mathbf{k}}^\Lambda$, and we will write $c_{i0} = 1$.

Now, fix a $\mathbb{N}$-graded $\underline{\mathbf{k}}$-algebra $\mathbf{k}$. Let c_t denote both the element in $\underline{\mathbf{k}}$ above and its image in $\mathbf{k}$ by the canonical map $\underline{\mathbf{k}} \to \mathbf{k}$ (which is homogeneous of degree 0). Then, set

$$a_i^\Lambda(u) = \sum_{p=0}^{\Lambda_i} c_{ip} u^{\Lambda_i - p} \in \mathbf{k}[u]. \tag{3}$$

The monic polynomial $a_i^\Lambda(u)$ is called the ith *cyclotomic polynomial* associated with $\mathbf{k}$.

For each $\alpha \in Q_+$ and $1 \leq k \leq \mathrm{ht}(\alpha)$, we set

$$a_\alpha^\Lambda(x_k) = \sum_{\nu \in I^\alpha} a_{\nu_k}^\Lambda(x_k) e(\nu). \tag{4}$$

Note that $a_\alpha^\Lambda(x_k)e(\nu)$ is a homogeneous element of $R(\alpha; Q, \mathbf{k})$ with degree $2d_{\nu_k}\Lambda_{\nu_k}$.

Definition 3.2
The *cyclotomic quiver Hecke algebra* of rank α and level Λ is the quotient $R^\Lambda(\alpha; Q, \mathbf{k})$ of $R(\alpha; Q, \mathbf{k})$ by the two-sided ideal generated by $a_\alpha^\Lambda(x_1)$.

To simplify notation, we write $R(\alpha) = R(\alpha; \mathbf{k}) = R(\alpha; Q, \mathbf{k})$ and $R^\Lambda(\alpha) = R^\Lambda(\alpha; \mathbf{k}) = R^\Lambda(\alpha; Q, \mathbf{k})$. We may also write $R = \bigoplus_\alpha R(\alpha)$, $R(\mathbf{k}) = \bigoplus_\alpha R(\alpha; \mathbf{k})$, $R^\Lambda = \bigoplus_\alpha R^\Lambda(\alpha)$, and so on. The following is proved in [16, Corollary 4.4, Theorem 4.5].

PROPOSITION 3.3
The $\mathbf{k}$*-algebra* $R^\Lambda(\alpha; \mathbf{k})$ *is free of finite type as a* $\mathbf{k}$*-module.*

Remark 3.4
A morphism of $\mathbb{N}$-graded $\underline{\mathbf{k}}$-algebras $\mathbf{k} \to \mathbf{h}$ yields canonical graded $\mathbf{h}$-algebra isomorphisms $\mathbf{h} \otimes_\mathbf{k} R(\alpha; \mathbf{k}) \to R(\alpha; \mathbf{h})$ and $\mathbf{h} \otimes_\mathbf{k} R^\Lambda(\alpha; \mathbf{k}) \to R^\Lambda(\alpha; \mathbf{h})$.

Example 3.5

(a) Set $\underline{R}^\Lambda(\alpha) = R^\Lambda(\alpha; \underline{\mathbf{k}})$. We call $\underline{R}^\Lambda(\alpha)$ the *global* (or *universal*) cyclotomic quiver Hecke algebra.

(b) If $\mathbf{k} = \Bbbk$, then $a_i^\Lambda(u) = u^{\Lambda_i}$ for each i. We call $R^\Lambda(\alpha; \Bbbk)$ the *local* (or *restricted*) cyclotomic quiver Hecke algebra.

(c) For each $i \in I$ we fix an element $c_i \in \mathbf{k}$ of degree $2d_i$. We define the following:
- $\mathbf{k}' = \mathbf{k}$ as a $\Bbbk$-algebra with the new $\underline{\mathbf{k}}$-algebra structure associated with the elements c'_{ip} given by $c'_{ip} = e_p(y_{i1} - c_i, y_{i2} - c_i, \dots)$, where the y_{ip}'s are commuting formal variables such that $c_{ip} = e_p(y_{i1}, y_{i2}, \dots)$ and e_p is the pth elementary symmetric polynomial. Then, the ith cyclotomic polynomial associated with $\mathbf{k}'$ is $a_i^\Lambda(u - c_i)$ for each $i \in I$.
- $Q'_{ij}(u, v) = Q_{ij}(u - c_i, v - c_j)$. In particular, we have $Q'_{ij}(u, v) = Q'_{ji}(v, u)$.

Then, the assignment $e(\nu), x_k e(\nu), \tau_l e(\nu) \mapsto e(\nu), (x_k + c_{\nu_k})e(\nu), \tau_l e(\nu)$ extends uniquely to a $\mathbf{k}$-algebra isomorphism $R^\Lambda(\alpha; Q, \mathbf{k}) \xrightarrow{\sim} R^\Lambda(\alpha; Q', \mathbf{k}')$.

(d) Under the hypothesis above, assume that $i \in I$ and that $\Lambda = \omega_i$ is the ith fundamental weight. Assume also that the polynomial $Q_{ij}(u, v)$ satisfies condition (11) below. If $a_i^\Lambda(u) = u + c_i$, then we have a $\mathbf{k}$-algebra isomorphism

$$R^{\omega_i}(\alpha; Q, \mathbf{k}) \simeq \mathbf{k} \otimes_\Bbbk R^{\omega_i}(\alpha; Q, \Bbbk). \tag{5}$$

Definition 3.6

For each $k \in [1, n-1]$, the kth *intertwiner operator* is the element $\varphi_k \in R^\Lambda(n)$ defined by $\varphi_k e(\nu) = \tau_k e(\nu)$ if $\nu_k \neq \nu_{k+1}$ and by the following formulas if $\nu_k = \nu_{k+1}$:

$$\begin{aligned}\varphi_k e(\nu) &= (x_k \tau_k - \tau_k x_k) e(\nu) = (\tau_k x_{k+1} - x_{k+1}\tau_k) e(\nu) \\ &= \big((x_k - x_{k+1})\tau_k + 1\big) e(\nu) = \big(\tau_k(x_{k+1} - x_k) - 1\big) e(\nu).\end{aligned}$$

We have the following facts (see [18, Section 5.1] for details):
- $x_{s_k(\ell)} \varphi_k e(\nu) = \varphi_k x_\ell e(\nu)$.
- $\{\varphi_k\}$ satisfies the braid relations.
- If $w \in \mathfrak{S}_n$ satisfies $w(k+1) = w(k) + 1$, then $\varphi_w \tau_k = \tau_{w(k)} \varphi_w$.
- $\varphi_k^2 e(\nu) = e(\nu)$ if $\nu_k = \nu_{k+1}$ and $\varphi_k^2 e(\nu) = Q_{\nu_k, \nu_{k+1}}(x_k, x_{k+1}) e(\nu)$ if $\nu_k \neq \nu_{k+1}$.

3.1.4. *Induction and restriction*

Let $i \in I$ and $\alpha \in Q_+$ be of height n. Set $\lambda = \Lambda - \alpha$ and $\lambda_i = \langle \alpha_i^\vee, \lambda \rangle$.

We have a $\mathbb{Z}$-graded $\mathbf{k}$-algebra embedding $\iota_i : R(\alpha) \hookrightarrow R(\alpha + \alpha_i)$ given by $e(\nu), x_k, \tau_l \mapsto e(\nu, i), x_k, \tau_l$ for each $\nu \in I^\alpha$ with $1 \le k \le n$ and $1 \le l \le n-1$. It induces a $\mathbb{Z}$-graded $\mathbf{k}$-algebra homomorphism $\iota_i : R^\Lambda(\alpha) \to R^\Lambda(\alpha + \alpha_i)$.

The restriction and induction functors form an adjoint pair (F'_i, E'_i) with

$$\begin{aligned}
&E'_i : R^\Lambda(\alpha+\alpha_i)\text{-grmod} \to R^\Lambda(\alpha)\text{-grmod}, \qquad N \mapsto e(\alpha,i)N,\\
&F'_i : R^\Lambda(\alpha)\text{-grmod} \to R^\Lambda(\alpha+\alpha_i)\text{-grmod},\\
&\quad M \mapsto R^\Lambda(\alpha+\alpha_i)e(\alpha,i) \otimes_{R^\Lambda(\alpha)} M.
\end{aligned}$$

The counit $\varepsilon'_{i,\lambda} : F'_i E'_i 1_\lambda \to 1_\lambda$ and the unit $\eta'_{i,\lambda} : 1_\lambda \to E'_i F'_i 1_\lambda$ are represented, respectively, by the multiplication map μ and the map ι_i:

$$\begin{aligned}
&\varepsilon'_{i,\lambda} : R^\Lambda(\alpha)e(\alpha-\alpha_i,i) \otimes_{R^\Lambda(\alpha-\alpha_i)} e(\alpha-\alpha_i,i)R^\Lambda(\alpha) \to R^\Lambda(\alpha),\\
&\eta'_{i,\lambda} : R^\Lambda(\alpha) \to e(\alpha,i)R^\Lambda(\alpha+\alpha_i)e(\alpha,i).
\end{aligned}$$

Finally, let $\sigma'_{ij,\lambda} : F'_i E'_j 1_\lambda \to E'_j F'_i 1_\lambda$ be the morphism represented by the linear map

$$\begin{aligned}
&R^\Lambda(\alpha-\alpha_j+\alpha_i)e(\alpha-\alpha_j,i) \otimes_{R^\Lambda(\alpha-\alpha_j)} e(\alpha-\alpha_j,j)R^\Lambda(\alpha)\\
&\quad \to e(\alpha-\alpha_j+\alpha_i,j)R^\Lambda(\alpha+\alpha_i)e(\alpha,i),\\
&\qquad\qquad x \otimes y \mapsto x\tau_n y.
\end{aligned}$$

For $j = i$, the element $\tau_n \in R^\Lambda(\alpha+\alpha_i)$ centralizes the subalgebra $e(\alpha-\alpha_i, i^2) \times R^\Lambda(\alpha+\alpha_i)e(\alpha-\alpha_i, i^2)$, so we have $\sigma'_{ii,\lambda} = \mu_{\tau_n}$ (see (1)).

THEOREM 3.7 (see [16])

For each $\alpha \in Q_+$ of height n, we have the following.

(a) *If $\lambda_i \ge 0$, then the following morphism of endofunctors on* $R^\Lambda(\alpha)$-Mod *is an isomorphism:*

$$\rho'_{i,\lambda} = \sigma'_{ii,\lambda} + \sum_{k=0}^{\lambda_i-1} (E'_i x^k) \circ \eta'_{i,\lambda} : F'_i E'_i 1_\lambda \oplus \bigoplus_{k=0}^{\lambda_i-1} \mathbf{k}x^k \otimes 1_\lambda \to E'_i F'_i 1_\lambda.$$

(b) *If $\lambda_i \le 0$, then the following morphism of endofunctors on* $R^\Lambda(\alpha)$-Mod *is an isomorphism:*

$$\begin{aligned}
\rho'_{i,\lambda} = \big(\sigma'_{ii,\lambda}, \varepsilon'_{i,\lambda} \circ (F'_i x^0), \dots, \varepsilon'_{i,\lambda} \circ (F'_i x^{-\lambda_i-1})\big) :\\
F'_i E'_i 1_\lambda \to E'_i F'_i 1_\lambda \oplus \bigoplus_{k=0}^{-\lambda_i-1} \mathbf{k}(x^{-1})^k \otimes 1_\lambda.
\end{aligned}$$

The theorem can be rephrased as follows.

- Assume that $\lambda_i \geq 0$: for any $z \in e(\alpha,i)R^\Lambda(\alpha+\alpha_i)e(\alpha,i)$ there are unique elements $\pi(z) \in R^\Lambda(\alpha)e(\alpha-\alpha_i,i) \otimes_{R^\Lambda(\alpha-\alpha_i)} e(\alpha-\alpha_i,i)R^\Lambda(\alpha)$ and $p_k(z) \in R^\Lambda(\alpha)$ such that

$$z = \mu_{\tau_n}\big(\pi(z)\big) + \sum_{k=0}^{\lambda_i-1} p_k(z)x_{n+1}^k. \tag{6}$$

- Assume that $\lambda_i \leq 0$: for any $z \in e(\alpha,i)R^\Lambda(\alpha+\alpha_i)e(\alpha,i)$ and any $z_0,\dots,z_{-\lambda_i-1} \in R^\Lambda(\alpha)$, there is a unique element $y \in R^\Lambda(\alpha)e(\alpha-\alpha_i,i) \otimes_{R^\Lambda(\alpha-\alpha_i)} e(\alpha-\alpha_i,i)R^\Lambda(\alpha)$ such that

$$\mu_{\tau_n}(y) = z, \qquad \mu_{x_n^k}(y) = z_k, \quad \forall k \in [0,-\lambda_i-1]. \tag{7}$$

For a future use, let us introduce the following notation. Assume that $\lambda_i \leq 0$ and that $z \in e(\alpha,i)R^\Lambda(\alpha+\alpha_i)e(\alpha,i)$. For each $\ell \in [0,-\lambda_i-1]$, let

$$\tilde{z}, \tilde{\pi}_\ell \in R^\Lambda(\alpha)e(\alpha-\alpha_i,i) \otimes_{R^\Lambda(\alpha-\alpha_i)} e(\alpha-\alpha_i,i)R^\Lambda(\alpha) \tag{8}$$

be the unique elements such that

$$\mu_{\tau_n}(\tilde{z}) = z, \qquad \mu_{x_n^k}(\tilde{z}) = 0, \qquad \mu_{\tau_n}(\tilde{\pi}_\ell) = 0, \qquad \mu_{x_n^k}(\tilde{\pi}_\ell) = \delta_{k,\ell}.$$

THEOREM 3.8 (see [18])
The pair (E_i', F_i') is adjoint with the counit $\hat{\varepsilon}'_{i,\lambda} : E_i'F_i'1_\lambda \to 1_\lambda$ and the unit $\hat{\eta}'_{i,\lambda} : 1_\lambda \to F_i'E_i'1_\lambda$ represented by the morphisms

$$\hat{\varepsilon}'_{i,\lambda} : e(\alpha,i)R^\Lambda(\alpha+\alpha_i)e(\alpha,i) \to R^\Lambda(\alpha),$$
$$\hat{\eta}'_{i,\lambda} : R^\Lambda(\alpha) \to R^\Lambda(\alpha)e(\alpha-\alpha_i,i) \otimes_{R^\Lambda(\alpha-\alpha_i)} e(\alpha-\alpha_i,i)R^\Lambda(\alpha)$$

such that

- $\hat{\varepsilon}'_{i,\lambda}(z) = p_{\lambda_i-1}(z)$ *if* $\lambda_i > 0$ *and* $\mu_{x_n^{-\lambda_i}}(\tilde{z})$ *if* $\lambda_i \leq 0$,
- $\hat{\eta}'_{i,\lambda}(1) = -\pi(x_{n+1}^{\lambda_i})$ *if* $\lambda_i \geq 0$ *and* $\tilde{\pi}_{-\lambda_i-1}$ *if* $\lambda_i < 0$.

We abbreviate $\varepsilon_i' = \varepsilon'_{i,\lambda}$, $\eta_i' = \eta'_{i,\lambda}$, $\hat{\varepsilon}_i' = \hat{\varepsilon}'_{i,\lambda}$, $\hat{\eta}_i' = \hat{\eta}'_{i,\lambda}$, and so on, when λ is clear from the context.

COROLLARY 3.9
The linear maps ε_i', η_i' are homogeneous of degree 0. The linear maps $\hat{\varepsilon}_i'$, $\hat{\eta}_i'$ are homogeneous of degrees $2d_i(1-\lambda_i)$, $2d_i(1+\lambda_i)$, respectively. The linear map σ'_{ij} is homogeneous of degree $-d_ia_{ij}$.

3.1.5. The symmetrizing form

For each $\alpha \in Q_+$ we set

$$d_{\Lambda,\alpha} = (\Lambda|\Lambda) - (\Lambda - \alpha \mid \Lambda - \alpha).$$

We will need the following result from [40].

PROPOSITION 3.10 ([40, Remark 3.19])
The $\mathbf{k}$*-algebra* $R^\Lambda(\alpha)$ *is symmetric and admits a symmetrizing form* $t_{\Lambda,\alpha}$ *which is homogeneous of degree* $-d_{\Lambda,\alpha}$.

The definition of $t_{\Lambda,\alpha}$ is given in Definition A.6. We will abbreviate $t_\alpha = t_{\Lambda,\alpha}$ and $t_\Lambda = \sum_\alpha t_\alpha$. Since we have not found any proof of the proposition in the literature, we have given one in Appendix A.

3.2. Categorical representations

Let $\mathbf{k}$ be an $\mathbb{N}$-graded commutative ring as in Section 3.1. Write $\mathfrak{g}_\mathbf{k} = \mathbf{k} \otimes_\Bbbk \mathfrak{g}$. Fix an integer ℓ.

3.2.1. Definition

For each $\lambda \in P$, let $\mathscr{C}_\lambda$ be an $(\ell\mathbb{Z})$-graded $\Bbbk$-category. Set $\mathscr{C} = \bigoplus_\lambda \mathscr{C}_\lambda$ and denote by 1_λ the obvious functor $1_\lambda : \mathscr{C} \to \mathscr{C}_\lambda$. For each $i, j \in I$, $\lambda \in P$, we fix

- a $\mathbb{Z}$-graded $\Bbbk$-algebra homomorphism $\mathbf{k}^{[\ell]} \to Z(\mathscr{C}/\ell\mathbb{Z})$;
- a functor $1_{\lambda-\alpha_i} F_i = F_i 1_\lambda$ with a right adjoint $1_\lambda E_i[\ell d_i(1-\lambda_i)] = E_i 1_{\lambda-\alpha_i}[\ell d_i(1-\lambda_i)]$;
- morphisms of functors $x_i 1_\lambda : F_i 1_\lambda \to F_i 1_\lambda[2\ell d_i]$ and $\tau_{ij} 1_\lambda : F_i F_j 1_\lambda \to F_j F_i 1_\lambda[-\ell d_i a_{ij}]$.

Thus $\mathscr{C}/\ell\mathbb{Z}$ is a $\mathbf{k}$-category, and the functors $F_i 1_\lambda$, $E_i 1_\lambda$ are $\mathbf{k}$-linear. Let

$$\varepsilon_i 1_\lambda : F_i E_i 1_\lambda \to 1_\lambda\big[\ell d_i(1+\lambda_i)\big], \qquad \eta_i 1_\lambda : 1_\lambda \to E_i F_i 1_\lambda\big[\ell d_i(1-\lambda_i)\big]$$

be the counit and the unit of the adjoint pair $(1_\lambda F_i, E_i 1_\lambda[-\ell d_i(1+\lambda_i)])$. We will abbreviate

$$E_i = \bigoplus_\lambda E_i 1_\lambda, \qquad F_i = \bigoplus_\lambda F_i 1_\lambda, \qquad F_\alpha = \bigoplus_{\nu \in I^\alpha} F_\nu, \quad \text{and so on,}$$

where $F_\nu = F_{\nu_1} F_{\nu_2} \cdots F_{\nu_n}$ for $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$. Next, we define the following morphisms:

- $\sigma_{ij} = (E_j F_i \varepsilon_j) \circ (E_j \tau_{ji} E_i) \circ (\eta_j F_i E_j) : F_i E_j \to E_j F_i$;
- $\rho_i 1_\lambda = \sigma_{ii} 1_\lambda + \sum_{l=0}^{-\lambda_i - 1} (\varepsilon_i 1_\lambda) \circ (x_i^l E_i 1_\lambda) : F_i E_i 1_\lambda \to E_i F_i 1_\lambda \oplus \bigoplus_{l=0}^{-\lambda_i-1} 1_\lambda[\ell d_i(1+2l+\lambda_i)]$ if $\lambda_i \leq 0$;

- $\rho_i 1_\lambda = \sigma_{ii} 1_\lambda + \sum_{l=0}^{\lambda_i - 1} (E_i x_i^l 1_\lambda) \circ (\eta_i 1_\lambda) : F_i E_i 1_\lambda \oplus \bigoplus_{l=0}^{\lambda_i-1} 1_\lambda[\ell d_i(1+2l-\lambda_i)] \to E_i F_i 1_\lambda$ if $\lambda_i \geq 0$.

Definition 3.11
A *categorical representation* of $\mathfrak{g}_{\mathbf{k}}$ of degree ℓ in $\mathcal{C}$ is a tuple $\mathcal{C}_\lambda$, E_i, F_i, ε_i, η_i, x_i, τ_{ij} as above such that the following hold:
- the assignment $e(\nu) \mapsto 1_{F_\nu}$, $x_k e(\nu) \mapsto x_{\nu_k} 1_{F_\nu}$, $\tau_l e(\nu) \mapsto \tau_{\nu_l, \nu_{l+1}} 1_{F_\nu}$ for each $\nu \in I^\alpha$ defines a $\mathbb{Z}$-graded $\mathbf{k}^{[\ell]}$-algebra homomorphism $R(\alpha; \mathbf{k})^{[\ell]} \to \mathrm{End}_{\mathcal{C}/\ell\mathbb{Z}}(F_\alpha)$;
- the morphisms $\rho_i 1_\lambda$, σ_{ij}, $i \neq j$, are isomorphisms.

Morphisms of categorical representations are defined in the obvious way.

We will call the map $R(\alpha; \mathbf{k})^{[\ell]} \to \mathrm{End}_{\mathcal{C}/\ell\mathbb{Z}}(F_\alpha)$ the *canonical homomorphism* associated with the categorical representation of $\mathfrak{g}_{\mathbf{k}}$ in $\mathcal{C}$.

Unless specified otherwise, a categorical representation will be of degree 1. Degrees $\ell \neq 1$ are used only in the nonsymmetric case and the reader interested only in symmetric ones may set $\ell = 1$ everywhere. Note that, given a categorical representation of $\mathfrak{g}_{\mathbf{k}}$ in $\mathcal{C}$, there is a canonical categorical representation of $\mathfrak{g}_{\mathbf{k}}$ of degree ℓ in $\mathcal{C}^{[\ell]}$ called its *ℓ-twist* such that the $\mathbb{Z}$-graded $\mathbf{k}^{[\ell]}$-algebra homomorphism

$$R(\alpha; \mathbf{k})^{[\ell]} \to \mathrm{End}_{\mathcal{C}^{[\ell]}/\ell\mathbb{Z}}(F_\alpha) = \mathrm{End}_{\mathcal{C}/\mathbb{Z}}(F_\alpha)^{[\ell]}$$

is equal to the homomorphism $R(\alpha; \mathbf{k}) \to \mathrm{End}_{\mathcal{C}/\mathbb{Z}}(F_\alpha)$ associated with the $\mathfrak{g}_{\mathbf{k}}$-action on $\mathcal{C}$.

We will also use the following definitions:
- $\mathcal{C}$ is *integrable* if E_i, F_i are locally nilpotent for all i.
- $\mathcal{C}$ is *bounded above* if the set of weights of $\mathcal{C}$ is contained in a finite union of cones of type $\mu - Q_+$ with $\mu \in P$.
- The *highest weight subcategory* $\mathcal{C}^{\mathrm{hw}} \subset \mathcal{C}$ is the full subcategory given by

$$\mathcal{C}^{\mathrm{hw}} = \{M \in \mathcal{C}; E_i(M) = 0, \forall i \in I\}.$$

Remark 3.12
(a) Taking the left transpose of the morphisms of functors

$$x_i 1_\lambda : F_i 1_\lambda \to F_i 1_\lambda[2\ell d_i], \qquad \tau_{ij} 1_\lambda : F_i F_j 1_\lambda \to F_j F_i 1_\lambda[-\ell d_i a_{ij}]$$

we get the morphisms of functors

$$1_\lambda{}^\vee x_i : 1_\lambda E_i \to 1_\lambda E_i[2\ell d_i], \qquad 1_\lambda{}^\vee \tau_{ij} 1_\lambda : 1_\lambda E_i E_j \to 1_\lambda E_j E_i[-\ell d_i a_{ij}].$$

We will abbreviate $x_i = {}^\vee x_i$ and $\tau_{ij} = {}^\vee \tau_{ij}$.

(b) Forgetting the grading at each place we define as above a categorical representation of $\mathfrak{g}_{\mathbf{k}}$ in a (not graded) $\Bbbk$-category $\mathcal{C}$.

(c) For each short exact sequence of $\mathbb{Z}$-graded $\Bbbk$-categories $0 \to \mathcal{I} \to \mathcal{C} \to \mathcal{B} \to 0$ such that $\mathcal{I} \to \mathcal{C}$ is a morphism of categorical representations of $\mathfrak{g}_{\mathbf{k}}$, there is a unique categorical representation of $\mathfrak{g}_{\mathbf{k}}$ on $\mathcal{B}$ such that $\mathcal{C} \to \mathcal{B}$ is a morphism of categorical representations.

(d) Given a categorical representation of $\mathfrak{g}_{\mathbf{k}}$ on $\mathcal{C}$, there is a unique categorical representation of $\mathfrak{g}_{\mathbf{k}}$ on $\mathcal{C}^c$ such that the canonical fully faithful functor $\mathcal{C} \to \mathcal{C}^c$ is a morphism of categorical representations. Recall that the objects of the idempotent completion $\mathcal{C}^c$ are the pairs (M, e), where M is an object of $\mathcal{C}$ and e is an idempotent of $\mathrm{End}_{\mathcal{C}}(M)$, and that $\mathrm{Hom}_{\mathcal{C}^c}((M,e),(N,f)) = f\,\mathrm{Hom}_{\mathcal{C}}(M,N)e$. Then, we have $F_i(M,e) = (F_i(M), F_i(e))$, $E_i(M,e) = (E_i(M), E_i(e))$, and $x_i 1_\lambda$, $\tau_{ij} 1_\lambda$ are defined in a similar way.

3.2.2. The minimal categorical representation

Fix a dominant weight $\Lambda \in P_+$ and an $\mathbb{N}$-graded $\underline{\mathbf{k}}$-algebra $\mathbf{k}$. Given $\alpha \in Q_+$ we write $\lambda = \Lambda - \alpha$. Recall that we abbreviate $R^\Lambda(\alpha) = R^\Lambda(\alpha; \mathbf{k})$. Let $\mathbf{k}\mathcal{A}^\Lambda_\lambda = R^\Lambda(\alpha)$-grmod be the $\mathbb{Z}$-graded abelian $\Bbbk$-category consisting of the finitely generated $\mathbb{Z}$-graded $R^\Lambda(\alpha)$-modules, and let $\mathbf{k}\mathcal{V}^\Lambda_\lambda = R^\Lambda(\alpha)$-grproj be the full subcategory formed by the projective $\mathbb{Z}$-graded modules. When there is no confusion, we will abbreviate $\mathcal{A}^\Lambda_\lambda = \mathbf{k}\mathcal{A}^\Lambda_\lambda$ and $\mathcal{V}^\Lambda_\lambda = \mathbf{k}\mathcal{A}^\Lambda_\lambda$. Let $\mathcal{A}^\Lambda$, $\mathcal{V}^\Lambda$ be the categories

$$\mathcal{A}^\Lambda = \bigoplus_\lambda \mathcal{A}^\Lambda_\lambda, \qquad \mathcal{V}^\Lambda = \bigoplus_\lambda \mathcal{V}^\Lambda_\lambda.$$

Fix an integer ℓ. Let $\mathcal{V}^{\Lambda,[\ell]} = (\mathcal{V}^\Lambda)^{[\ell]}$ be the ℓ-twist of $\mathcal{V}^\Lambda$, and let $R^\Lambda(\alpha)^{[\ell]}$ be the ℓ-twist of $R^\Lambda(\alpha)$. Thus, $R^\Lambda(\alpha)^{[\ell]}$ is a $(\ell\mathbb{Z})$-graded $\mathbf{k}^{[\ell]}$-algebra and $\mathcal{V}^{\Lambda,[\ell]}_\lambda$ is the category of finitely generated projective $(\ell\mathbb{Z})$-graded modules; that is,

$$\mathcal{V}^{\Lambda,[\ell]}_\lambda = R^\Lambda(\alpha)^{[\ell]}\text{-grproj}.$$

Definition 3.13

The *minimal categorical representation* of $\mathfrak{g}_{\mathbf{k}}$ of highest weight Λ and degree ℓ is the representation on $\mathcal{V}^{\Lambda,[\ell]}$ given by the following:

- $E_i 1_\lambda = E'_i[-\ell d_i(1+\lambda_i)]$;
- $F_i 1_\lambda = F'_i$;
- $\varepsilon_i 1_\lambda = \varepsilon'_{i,\lambda}$ and $\eta_i 1_\lambda = \eta'_{i,\lambda}$;
- $x_i 1_\lambda \in \mathrm{Hom}(F_i 1_\lambda, F_i 1_\lambda[2\ell d_i])$ is represented by the right multiplication by x_{n+1} on $R^\Lambda(\alpha+\alpha_i)e(\alpha, i)$;
- $\tau_{ij} 1_\lambda \in \mathrm{Hom}(F_i F_j 1_\lambda, F_j F_i 1_\lambda[-\ell d_i a_{ij}])$ is represented by the right multiplication by τ_{n+1} on $R^\Lambda(\alpha+\alpha_i+\alpha_j)e(\alpha, ji)$.

The category $\mathcal{A}_\lambda^{\Lambda,[\ell]}$ is Krull–Schmidt with a finite number of indecomposable projective objects. The category $\mathcal{V}^{\Lambda,[\ell]}/(\ell\mathbb{Z})$ is the category of $(\ell\mathbb{Z})$-graded finitely generated projective $R^{\Lambda,[\ell]}$-modules with morphisms which are not necessarily homogeneous. We will call it the category of all $(\ell\mathbb{Z})$-*gradable* projective modules.

Example 3.14
We will abbreviate $\mathfrak{e} = \mathfrak{sl}_2$. Assume that $\mathfrak{g} = \mathfrak{e}$, $\Lambda = k\omega_1$, and $\alpha = n\alpha_1$ with $k, n \in \mathbb{N}$. In this case, we write $\mathcal{V}^k = \mathcal{V}^\Lambda$ and $\mathcal{V}^k_{k-2n} = \mathcal{V}^\Lambda_\lambda$. Consider the polynomial ring $\mathrm{Z}^k = \Bbbk[c_1, \ldots, c_k]$ with $\deg(c_p) = 2p$ for all p. Let $\underline{\mathrm{H}}^k_n$ be the *global cyclotomic affine nil Hecke algebra* of rank n and level k, that is, the $\mathbb{Z}$-graded Z^k-algebra denoted by $H_{n,k}$ in [35, Section 4.3.2]. Note that we have $\underline{\mathrm{H}}^k_0 = \mathrm{Z}^k$ and $\underline{\mathbf{k}} = \mathrm{Z}^k$. Given an $\mathbb{N}$-graded Z^k-algebra $\mathbf{k}$, the cyclotomic quiver Hecke algebra $R^\Lambda(\alpha; \mathbf{k})$ is isomorphic to $\mathbf{k} \otimes_{\mathrm{Z}^k} \underline{\mathrm{H}}^k_n$ as a $\mathbb{Z}$-graded $\mathbf{k}$-algebra by [35, Lemma 4.27]. In particular, we have $\underline{R}^\Lambda(\alpha) = \underline{\mathrm{H}}^k_n$. For each integer ℓ, we abbreviate $\mathrm{Z}^{k,[\ell]} = (\mathrm{Z}^k)^{[\ell]}$ and $\underline{\mathrm{H}}^{k,[\ell]}_n = (\underline{\mathrm{H}}^k_n)^{[\ell]}$. We have

$$\mathcal{V}^{k,[\ell]} = \bigoplus_{n \geq 0} (\mathbf{k}^{[\ell]} \otimes_{\mathrm{Z}^{k,[\ell]}} \underline{\mathrm{H}}^{k,[\ell]}_n)\text{-grproj}, \tag{9}$$

and the trace $\mathrm{tr}(\mathcal{V}^{k,[\ell]})$ is given in Proposition 3.30 below. We will also identify $R^\Lambda(\alpha; \Bbbk)$ with the *local cyclotomic affine nil Hecke algebra* of rank n and level k, which is the quotient $\mathrm{H}^k_n = \Bbbk \otimes_{\mathrm{Z}^k} \underline{\mathrm{H}}^k_n$ of $\underline{\mathrm{H}}^k_n$ by the ideal $(c_1, \ldots, c_k)$.

3.2.3. Factorization
Fix a dominant weight $\Lambda \in P_+$. Recall the map $I_\Lambda \to I$, $t \mapsto i_t$ introduced in (2). We will abbreviate $\omega_t = \omega_{i_t}$ and $d_t = d_{i_t}$. Consider the $\mathbb{N}$-graded $\Bbbk$-algebra $\underline{\mathbf{h}} = \Bbbk[y_t; t \in I_\Lambda]$, where y_t is a formal variable of degree $\deg(y_t) = 2d_t$. It has a natural structure of an $\mathbb{N}$-graded $\underline{\mathbf{k}}$-algebra such that the element $c_{ip} \in \underline{\mathbf{h}}$ is given by $c_{ip} = e_p(y_{i1}, \ldots, y_{i\Lambda_i})$. The corresponding cyclotomic polynomials are

$$a_i^\Lambda(u) = \prod_{p=1}^{\Lambda_i} (u + y_{ip}), \quad \forall i \in I. \tag{10}$$

Let $\underline{\mathbf{h}}'$ be the fraction field of $\underline{\mathbf{h}}$. Then we have the algebras $R^\Lambda(n, \underline{\mathbf{h}})$ and $R^\Lambda(n, \underline{\mathbf{h}}')$ over $\underline{\mathbf{h}}$ and $\underline{\mathbf{h}}'$ such that $R^\Lambda(n, \underline{\mathbf{h}}') = \underline{\mathbf{h}}' \otimes_{\underline{\mathbf{h}}} R^\Lambda(n, \underline{\mathbf{h}})$. Next, for each $t \in I_\Lambda$, we have

$$\underline{\mathbf{h}} = \bigotimes_{t \in I_\Lambda} \Bbbk[y_t] = \bigotimes_{t \in I_\Lambda} \underline{\mathbf{k}}^{\omega_t}.$$

Therefore, we can view $\underline{\mathbf{h}}$ as a $\underline{\mathbf{k}}^{\omega_t}$-algebra, and hence the $\underline{\mathbf{h}}$-algebra $R^{\omega_t}(n, \underline{\mathbf{h}})$ associated with the cyclotomic polynomials $a_i^{\omega_t}(u) = u + y_t$ is well defined. Note that by

Remark 3.5(d), if condition (11) below is satisfied, then we may as well assume that $a_i^{\omega_t}(u) = u$. Recall that

$$R^\Lambda(n,\underline{\mathbf{h}}) = \underline{\mathbf{h}} \otimes_{\underline{\mathbf{k}}} \underline{R}^\Lambda(n),$$
$$R^{\omega_t}(n,\underline{\mathbf{h}}) = \underline{\mathbf{h}} \otimes_{\underline{\mathbf{k}}^{\omega_t}} \underline{R}^{\omega_t}(n).$$

When no confusion is possible, we will abbreviate

$$\mathcal{V}^\Lambda = \bigoplus_n \underline{R}^\Lambda(n)\text{-grproj}, \qquad \mathcal{V}^{\omega_t} = \bigoplus_n \underline{R}^{\omega_t}(n)\text{-grproj}.$$

Finally, we consider the condition

$$Q_{ij}(u,v) = r_{ij}(u-v)^{-a_{ij}} \quad \text{for some } r_{ij} \text{ in } \Bbbk^\times \text{ such that}$$
$$r_{ij} = (-1)^{a_{ij}} r_{ji} \quad \text{for all } i \neq j. \tag{11}$$

THEOREM 3.15
Let n be a positive integer, and let $\alpha \in Q_+$ be of height n. If the condition (11) is satisfied, then the following hold:

(a) *There is an $\underline{\mathbf{h}}'$-algebra isomorphism*

$$R^\Lambda(n,\underline{\mathbf{h}}') \to \bigoplus_{(n_t)} \mathrm{Mat}_{\mathfrak{S}_n/\prod_t \mathfrak{S}_{n_t}}\Big(\bigotimes_{t\in I_\Lambda} R^{\omega_t}(n_t,\underline{\mathbf{h}}')\Big),$$

where (n_t) runs over the set of I_Λ-tuples of nonnegative integers with sum n.

(b) *There is an isomorphism $\underline{\mathbf{h}}' \otimes_{\underline{\mathbf{k}}} \mathcal{V}^\Lambda \to \underline{\mathbf{h}}' \otimes_{\underline{\mathbf{h}}} \bigotimes_{t\in I_\Lambda} \mathcal{V}^{\omega_t}$ of (nongraded) categorical $\mathfrak{g}_{\underline{\mathbf{h}}'}$-representations taking the functor F_α to $\bigoplus_{(\alpha_t)} \bigotimes_t F_{\alpha_t}$, where the sum runs over the set of I_Λ-tuples (α_t) of elements of Q_+ with sum α. More precisely, the canonical homomorphism*

$$\bigotimes_{t\in I_\Lambda} R(\alpha_t;\underline{\mathbf{h}}') \to \mathrm{End}\Big(\bigotimes_{t\in I_\Lambda} F_{\alpha_t}\Big)$$

is the composition of the inclusion $\bigotimes_{t\in I_\Lambda} R(\alpha_t;\underline{\mathbf{h}}') \subset R(\alpha;\underline{\mathbf{h}}')$ underlying (a) and of the canonical homomorphism $R(\alpha;\underline{\mathbf{h}}') \to \mathrm{End}(F_\alpha)$.

Proof
Fix $M \in R^\Lambda(n;\underline{\mathbf{h}}')$-mod and $g(u) \in \underline{\mathbf{h}}'[u]$. From [16, pp. 715–716], we get

$$g(x_a)e(\nu)M = 0$$
$$\Rightarrow Q_{\nu_a,\nu_{a+1}}(x_a,x_{a+1})g(x_{a+1})e\big(s_a(\nu)\big)M = 0, \quad \forall a \in [1,n), \forall \nu \in I^n.$$

Set $Q(u,v) = \prod_{i \neq j} Q_{ij}(u,v)$. We deduce that

$$g(x_a)e(\nu)M = 0 \Rightarrow Q(x_a, x_{a+1})g(x_{a+1})e\big(s_a(\nu)\big)M = 0, \quad \forall a \in [1,n). \tag{12}$$

Now, assume that the polynomial $Q(u,v) \in \underline{\mathbf{h}}'[u,v]$ has the following form,

$$Q(u,v) = r \prod_{\lambda \in S} (u - v - \lambda), \tag{13}$$

for some finite family S of elements of $\underline{\mathbf{h}}'$ and some element $r \in \underline{\mathbf{h}}'$. Let $\mathrm{sp}_{e(\nu)M}(x_a) \subset \underline{\mathbf{h}}'$ be the set of $\lambda \in \underline{\mathbf{h}}'$ such that the operator $x_a - \lambda\,\mathrm{id} \in \mathrm{End}_{\underline{\mathbf{h}}'}(e(\nu)M)$ is not invertible. Since x_a and x_{a+1} commute with each other, from (12) and (13) we deduce that

$$\mathrm{sp}_{e(s_a(\nu))M}(x_{a+1}) \subseteq \mathrm{sp}_{e(\nu)M}(x_a) \sqcup \big(\mathrm{sp}_{e(s_a(\nu))M}(x_a) - S\big).$$

Thus, switching ν and $s_a(\nu)$, we deduce that

$$\mathrm{sp}_{e(\nu)M}(x_{a+1}) \subseteq \mathrm{sp}_{e(s_a(\nu))M}(x_a) \sqcup \big(\mathrm{sp}_{e(\nu)M}(x_a) - S\big).$$

Next, recall that $g(x_1)(e(\nu)M) = 0$ if $g(u) = a^{\Lambda}_{\nu_1}(u)$. We deduce that

$$\mathrm{sp}_{e(\nu)M}(x_1) \subseteq \{-y_{\nu_1,p};\, p = 1, \dots, \Lambda_{\nu_1}\}.$$

Therefore, an easy induction implies that

$$\mathrm{sp}_{e(\nu)M}(x_a) \subseteq \{-y_{\nu_b,p};\, p = 1, \dots, \Lambda_{\nu_b}, b = 1, \dots, a\} - \mathbb{N}S, \quad \forall a \in [1,n].$$

Assume further that condition (11) holds. Then we have $S = \{0\}$, and hence

$$\mathrm{sp}_{e(\nu)M}(x_a) \subseteq \{-y_{\nu_b,p};\, p = 1, \dots, \Lambda_{\nu_b}, b = 1, \dots, a\}, \quad \forall a \in [1,n]. \tag{14}$$

In the rest of the proof we write $\tilde{I} = I_\Lambda$ to simplify the notation. For each n-tuple $\tilde{\nu} \in \tilde{I}^n$, set

$$M_{\tilde{\nu}} = \big\{m \in M;\, (x_k + y_{\tilde{\nu}_k})^D m = 0, \forall k \in [1,n], \forall D \gg 0\big\}.$$

Considering the decomposition of the regular module, we deduce from (14) that there is a complete collection of orthogonal idempotents $\{e(\tilde{\nu});\, \tilde{\nu} \in \tilde{I}^n\}$ in $R^\Lambda(n, \underline{\mathbf{h}}')$ such that $e(\tilde{\nu})M = M_{\tilde{\nu}}$. The group $\mathfrak{S}_n$ acts on $\tilde{I}^n$ in the obvious way. The following hold:

- $e(\tilde{\nu})e(\nu) = e(\nu)e(\tilde{\nu})$,
- $x_l e(\tilde{\nu}) = e(\tilde{\nu})x_l$,
- $\varphi_k e(\tilde{\nu}) = e(s_k(\tilde{\nu}))\varphi_k$,
- $\tau_k e(\tilde{\nu}) = e(\tilde{\nu})\tau_k$ if $\tilde{\nu}_k = \tilde{\nu}_{k+1}$,

where k, l, $\tilde{\nu}$, and ν run over the sets $[1,n)$, $[1,n]$, $\tilde{I}^n$, and I^n, respectively. The relations above are immediate, except the last one. To prove it, note that if $\tilde{\nu}_k = \tilde{\nu}_{k+1}$, then the operators $x_k + y_{\tilde{\nu}_k}$ and $x_{k+1} + y_{\tilde{\nu}_k}$ commute to each other and their sum and product commute with τ_k and act nilpotently on $\tau_k e(\tilde{\nu})M$. Therefore $x_k + y_{\tilde{\nu}_k}$ and $x_{k+1} + y_{\tilde{\nu}_k}$ also act nilpotently on $\tau_k e(\tilde{\nu})M$. Note that the first relation above implies that we may view $\{e(\tilde{\nu}); \tilde{\nu} \in \tilde{I}^n\}$ as a complete collection of orthogonal idempotents in $R^\Lambda(\alpha, \underline{\mathbf{h}}')$ for each $\alpha \in Q_+$ of height n.

LEMMA 3.16
For each $M \in R^\Lambda(n; \underline{\mathbf{h}}')$-mod, *the map* $\varphi_k e(\tilde{\nu}) : e(\tilde{\nu})M \to e(s_k(\tilde{\nu}))M$ *is invertible whenever* $\tilde{\nu}_k \neq \tilde{\nu}_{k+1}$.

Proof
The lemma is an immediate consequence of the following relations, for each $\nu \in I^n$,

$$\varphi_k^2 e(\nu) = 1 \quad \text{if } \nu_k = \nu_{k+1},$$
$$\varphi_k^2 e(\nu) = Q_{\nu_k,\nu_{k+1}}(x_k, x_{k+1}) e(\nu) \quad \text{if } \nu_k \neq \nu_{k+1},$$

because (11) implies that $Q_{\nu_k,\nu_{k+1}}(x_k, x_{k+1})$ is invertible if $\nu_k \neq \nu_{k+1}$ and $\tilde{\nu}_k \neq \tilde{\nu}_{k+1}$. □

Set $\tilde{Q}_+ = \mathbb{N}\tilde{I}$. For any element $\tilde{\alpha} = \sum_{t \in I_\Lambda} n_t \cdot t$ in $\tilde{Q}_+$ we set $\mathrm{ht}(\tilde{\alpha}) = \sum_t a_t$. Assume that $\mathrm{ht}(\tilde{\alpha}) = n$; then we consider the set $\tilde{I}^{\tilde{\alpha}} = \{\tilde{\nu} \in \tilde{I}^n; \sum_k \tilde{\nu}_k = \tilde{\alpha}\}$. We will say that two n-tuples $\tilde{\nu}, \tilde{\nu}' \in \tilde{I}^n$ are equivalent, and we write $\tilde{\nu} \sim \tilde{\nu}'$ if we have $\tilde{\nu}, \tilde{\nu}' \in \tilde{I}^{\tilde{\alpha}}$ for some $\tilde{\alpha} \in \tilde{Q}_+$. We define the idempotent $e(\tilde{\alpha})$ in $R^\Lambda(n, \underline{\mathbf{h}}')$ by $e(\tilde{\alpha}) = \sum_{\tilde{\nu} \in \tilde{I}^{\tilde{\alpha}}} e(\tilde{\nu})$.

Next, fix a total order on $\tilde{I}$, and set $\tilde{I}^n_+ = \{\tilde{\nu} \in \tilde{I}^n; i < j \Rightarrow \tilde{\nu}_i \leq \tilde{\nu}_j\}$. For any tuple $\tilde{\nu} \in \tilde{I}^n$ there is a unique element $\tilde{\nu}^+ \in \tilde{I}^n_+$ such that $\tilde{\nu} \sim \tilde{\nu}^+$. Let $\tilde{\alpha}^+$ be the unique element in $\tilde{I}^n_+ \cap \tilde{I}^{\tilde{\alpha}}$. The theorem is an easy consequence of the following lemma.

LEMMA 3.17
Let $\alpha \in Q_+$ *and* $\tilde{\alpha} \in \tilde{Q}_+$ *be of height* n, *with* $\tilde{\alpha} = \sum_{t \in \tilde{I}} n_t \cdot t$. *The following hold:*

(a) $e(\tilde{\nu}) R^\Lambda(\alpha, \underline{\mathbf{h}}') e(\tilde{\nu}') = 0$ *unless* $\tilde{\nu} \sim \tilde{\nu}'$;

(b) $e(\tilde{\alpha}) R^\Lambda(\alpha, \underline{\mathbf{h}}') e(\tilde{\alpha}) \simeq \mathrm{Mat}_{\tilde{I}^{\tilde{\alpha}}}(e(\tilde{\alpha}^+) R^\Lambda(\alpha, \underline{\mathbf{h}}') e(\tilde{\alpha}^+))$ *as* $\underline{\mathbf{h}}'$*-algebras;*

(c) $e(\tilde{\alpha}^+) R^\Lambda(\alpha, \underline{\mathbf{h}}') e(\tilde{\alpha}^+) \simeq \bigoplus_{(\alpha_t)} \bigotimes_{t \in \tilde{I}} R^{\omega_t}(\alpha_t, \underline{\mathbf{h}}')$ *as* $\underline{\mathbf{h}}'$*-algebras, where* (α_t) *runs over the set of all* $\tilde{I}$*-tuples of elements in* Q_+ *such that* $\sum_t \alpha_t = \alpha$ *and* $\mathrm{ht}(\alpha_t) = n_t$.

Proof
Part (a) of the lemma follows from the following.

CLAIM 3.18
The $\underline{\mathbf{h}}'$-algebra $R^\Lambda(n,\underline{\mathbf{h}}')$ is generated by the subset

$$\big\{e(\nu)e(\tilde\nu), \tau_h e(\tilde\nu), \varphi_k e(\tilde\nu), x_l e(\tilde\nu); h,k \in [1,n), l \in [1,n],\\ \nu \in I^n, \tilde\nu \in \tilde I^n, \tilde\nu_h = \tilde\nu_{h+1}, \tilde\nu_k \neq \tilde\nu_{k+1}\big\}.$$

Let us concentrate on part (b). For each tuple $\tilde\nu \in \tilde I^{\tilde\alpha}$, we fix a sequence of simple reflections $s_{l_1}, s_{l_2}, \dots, s_{l_j}$ in $\mathfrak{S}_n$ such that

- $\tilde\nu = s_{l_1} s_{l_2} \cdots s_{l_j}(\tilde\alpha^+)$;
- the l_hth entry of $s_{l_{h+1}} s_{l_{h+2}} \cdots s_{l_j}(\tilde\alpha^+)$ in $\tilde I$ is smaller than the (l_h+1)th one for each $h \in [1,j]$.

In particular, $w = s_{l_1} s_{l_2} \cdots s_{l_j}$ is a reduced decomposition and w is minimal in its left coset in $\mathfrak{S}_n$ relatively to the stabilizer of $\tilde\alpha^+$. Hence, we have

$$\varphi_{l_1}\varphi_{l_2}\cdots\varphi_{l_j} = \varphi_w, \tag{15}$$

and the element $\pi_{\tilde\nu}$ in $e(\tilde\nu)R^\Lambda(\alpha,\underline{\mathbf{h}}')e(\tilde\alpha^+)$ given by

$$\pi_{\tilde\nu} = e(\tilde\nu)\varphi_w e(\tilde\alpha^+) \tag{16}$$

depends only on $\tilde\nu$ and not on the choice of $l_1, l_2, \dots, l_j$. It is invertible by Lemma 3.16. We deduce that there is an $\underline{\mathbf{h}}'$-algebra isomorphism

$$\begin{aligned}\mathrm{Mat}_{\tilde I^{\tilde\alpha}}\big(e(\tilde\alpha^+)R^\Lambda(\alpha,\underline{\mathbf{h}}')e(\tilde\alpha^+)\big) &\to e(\tilde\alpha)R^\Lambda(\alpha,\underline{\mathbf{h}}')e(\tilde\alpha),\\ E_{\tilde\nu,\tilde\nu'}(m) &\mapsto \pi_{\tilde\nu} m \pi_{\tilde\nu'}^{-1}.\end{aligned} \tag{17}$$

Here $E_{\tilde\nu,\tilde\nu'}(m)$ is the elementary matrix with an m at the spot $(\tilde\nu,\tilde\nu')$ and 0s elsewhere. Part (b) is proved.

Now, we prove part (c). Set $\{t \in \tilde I; n_t \neq 0\} = \{t_1, t_2, \dots, t_m\}$ with $t_1 < t_2 < \cdots < t_m$. Fix an $\tilde I$-tuple (α_t) of elements in Q_+ such that $\sum_t \alpha_t = \alpha$ and $\mathrm{ht}(\alpha_t) = n_t$ for each t. There is a canonical inclusion

$$\bigotimes_{t\in\tilde I} R(\alpha_t,\underline{\mathbf{h}}') = R(\alpha_{t_1},\underline{\mathbf{h}}') \otimes R(\alpha_{t_2},\underline{\mathbf{h}}') \otimes \cdots \otimes R(\alpha_{t_m},\underline{\mathbf{h}}') \to R(\alpha,\underline{\mathbf{h}}').$$

Composing it with the multiplication by the idempotent $e(\tilde\alpha^+)$, we get a map

$$\bigotimes_{t\in\tilde I} R(\alpha_t,\underline{\mathbf{h}}') \to e(\tilde\alpha^+)R(\alpha,\underline{\mathbf{h}}')e(\tilde\alpha^+) \tag{18}$$

such that, for each $\tilde{I}$-tuple (ν_t) with $\nu_t \in I^{\alpha_t}$, we have

$$e(\nu_{t_1}) \otimes e(\nu_{t_2}) \otimes \cdots \otimes e(\nu_{t_m}) \mapsto e(\nu_{t_1}\nu_{t_2}\cdots\nu_{t_m})e(\tilde{\alpha}^+).$$

To prove (c) we must check that the direct sum of all maps (18), where (α_t) runs over the set of all tuples as above, gives an $\underline{\mathbf{h}}'$-algebra isomorphism

$$\bigoplus_{(\alpha_t)} \bigotimes_{t\in\tilde{I}} R^{\omega_t}(\alpha_t, \underline{\mathbf{h}}') \xrightarrow{\sim} e(\tilde{\alpha}^+)R^{\Lambda}(\alpha, \underline{\mathbf{h}}')e(\tilde{\alpha}^+). \tag{19}$$

First, we will prove that the $\underline{\mathbf{h}}'$-algebra homomorphism (19) is well defined. To do that, it is enough to check that for each tuple (α_t) the map (18) factors to an algebra homomorphism

$$\bigotimes_{t\in\tilde{I}} R^{\omega_t}(\alpha_t, \underline{\mathbf{h}}') \to e(\tilde{\alpha}^+)R^{\Lambda}(\alpha, \underline{\mathbf{h}}')e(\tilde{\alpha}^+), \tag{20}$$

because (20) maps into orthogonal direct summands of $e(\tilde{\alpha}^+)R^{\Lambda}(\alpha, \underline{\mathbf{h}}')e(\tilde{\alpha}^+)$ for different tuples (α_t). To simplify the notation, we will check this in a particular case. The proof of the general case is very similar. Assume that

$$\Lambda = 2\omega_i + \omega_j, \qquad \tilde{I} = \{t_1, t_2, t_3\}, \quad t_1 = (i,1), t_2 = (i,2), t_3 = (j,1),$$

with $i \neq j$ and $t_1 < t_2 < t_3$. Thus, we have

$$a_i^{\Lambda}(u) = (u + y_{t_1})(u + y_{t_2}), \qquad a_j^{\Lambda}(u) = u + y_{t_3}, \qquad a_k^{\Lambda}(u) = 1, \quad \forall k \neq i, j,$$

$$\omega_{t_1} = \omega_i, \qquad \omega_{t_2} = \omega_i, \qquad \omega_{t_3} = \omega_j.$$

Next, fix a positive integer n and an element $\tilde{\alpha} \in \tilde{Q}_+$ of height n. We have

$$\tilde{\alpha} = n_{t_1} \cdot t_1 + n_{t_2} \cdot t_2 + n_{t_3} \cdot t_3,$$
$$\tilde{\alpha}^+ = (\tilde{\alpha}_1^+, \tilde{\alpha}_2^+, \ldots, \tilde{\alpha}_n^+) = \big((t_1)^{n_{t_1}}(t_2)^{n_{t_2}}(t_3)^{n_{t_3}}\big),$$

with $n_{t_1} + n_{t_2} + n_{t_3} = n$. To simplify we will assume that the integers $n_{t_1}, n_{t_2}, n_{t_3}$ are positive.

CLAIM 3.19
For any $\nu \in I^n$, the following relations hold in $e(\tilde{\alpha}^+)R^{\Lambda}(n, \underline{\mathbf{h}}')e(\tilde{\alpha}^+)$:

- $(x_1 + y_{t_1})e(\tilde{\alpha}^+)e(\nu) = 0$ *if* $\nu_1 = i$, *and* $e(\tilde{\alpha}^+)e(\nu) = 0$ *otherwise;*
- $(x_{1+n_{t_1}} + y_{t_2})e(\tilde{\alpha}^+)e(\nu) = 0$ *if* $\nu_{1+n_{t_1}} = i$, *and* $e(\tilde{\alpha}^+)e(\nu) = 0$ *otherwise;*
- $(x_{1+n_{t_1}+n_{t_2}} + y_{t_3})e(\tilde{\alpha}^+)e(\nu) = 0$ *if* $\nu_{1+n_{t_1}+n_{t_2}} = j$, *and* $e(\tilde{\alpha}^+)e(\nu) = 0$ *otherwise.*

The first relation is obvious, because

- $(x_1 + y_{t_1})(x_1 + y_{t_2})e(\nu) = 0$ if $\nu_1 = i$;
- $(x_1 + y_{t_3})e(\nu) = 0$ if $\nu_1 = j$;
- $e(\nu) = 0$ if $\nu_1 \neq i, j$;
- $(x_1 + y_{t_2})e(\tilde{\alpha}^+)$, $(x_1 + y_{t_3})e(\tilde{\alpha}^+)$ are invertible in $e(\tilde{\alpha}^+)R^\Lambda(n, \underline{\mathbf{h}}')e(\tilde{\alpha}^+)$.

To prove the second relation, note that

$$\varphi_1\varphi_2\cdots\varphi_{n_{t_1}}(x_{1+n_{t_1}} + y_{t_2})e(\tilde{\alpha}^+)e(\nu) = (x_1 + y_{t_2})e(\tilde{\nu})e(\gamma)\varphi_1\varphi_2\cdots\varphi_{n_{t_1}},$$

where $\tilde{\nu} = s_1 s_2 \cdots s_{n_{t_1}}(\tilde{\alpha}^+)$ and $\gamma = s_1 s_2 \cdots s_{n_{t_1}}(\nu)$. Since $\tilde{\nu}_1 = t_2$ and $\gamma_1 = \nu_{1+n_{t_1}}$, we deduce that

- $\varphi_1\varphi_2\cdots\varphi_{n_{t_1}}(x_{1+n_{t_1}} + y_{t_2})e(\tilde{\alpha}^+)e(\nu) = 0$ if $\nu_{1+n_{t_1}} = i$,
- $\varphi_1\varphi_2\cdots\varphi_{n_{t_1}}e(\tilde{\alpha}^+)e(\nu) = 0$ otherwise.

Further, by Lemma 3.16, the operator

$$\varphi_1\varphi_2\cdots\varphi_{n_{t_1}}e(\tilde{\alpha}^+) : e(\tilde{\alpha}^+)R^\Lambda(\alpha, \underline{\mathbf{h}}')e(\tilde{\alpha}^+) \to e(\tilde{\nu})R^\Lambda(\alpha, \underline{\mathbf{h}}')e(\tilde{\alpha}^+)$$

is invertible. Thus, we have

- $(x_{1+n_{t_1}} + y_{t_2})e(\tilde{\alpha}^+)e(\nu) = 0$ if $\nu_{1+n_{t_1}} = i$,
- $e(\tilde{\alpha}^+)e(\nu) = 0$ otherwise,

proving the second relation. The third relation is proved in a similar way, using the product of intertwiners $\varphi_1\varphi_2\cdots\varphi_{n_{t_1}+n_{t_2}}$ instead of $\varphi_1\varphi_2\cdots\varphi_{n_{t_1}}$. The claim is proved.

The claim implies that the homomorphism (20) is well defined, and so (19) is also well defined. We must check that it is invertible. To prove the surjectivity, we must check that the $\underline{\mathbf{h}}'$-algebra isomorphism (19) yields a surjective map

$$\bigotimes_{t \in \tilde{I}} R^{\omega_t}(n_t, \underline{\mathbf{h}}') \xrightarrow{\sim} e(\tilde{\alpha}^+)R^\Lambda(n, \underline{\mathbf{h}}')e(\tilde{\alpha}^+).$$

This is a consequence of the following fact.

CLAIM 3.20
The $\underline{\mathbf{h}}'$-algebra $e(\tilde{\alpha}^+)R^\Lambda(n, \underline{\mathbf{h}}')e(\tilde{\alpha}^+)$ is generated by the subset

$$\{e(\nu)e(\tilde{\alpha}^+), \tau_h e(\tilde{\alpha}^+), x_l e(\tilde{\alpha}^+); h \in [1, n), l \in [1, n], \nu \in I^n, \tilde{\alpha}_h^+ = \tilde{\alpha}_{h+1}^+\}. \tag{21}$$

Let us concentrate on the injectivity. We will construct a left inverse to the map (19). To do that, recall that parts (a) and (b) yield an $\underline{\mathbf{h}}'$-algebra isomorphism

$$\bigoplus_{\tilde{\alpha}} \mathrm{Mat}_{\tilde{I}^{\tilde{\alpha}}}\big(e(\tilde{\alpha}^+)R^\Lambda(\alpha, \underline{\mathbf{h}}')e(\tilde{\alpha}^+)\big) \xrightarrow{\sim} R^\Lambda(\alpha, \underline{\mathbf{h}}'), \tag{22}$$

where the sum is over the set of all elements $\tilde{\alpha} \in \tilde{Q}_+$ of height n.

CLAIM 3.21

(a) *Fix* $\tilde\nu \in \tilde I^{\tilde\alpha}$ *and* $w \in \mathfrak{S}_n$ *as in (16). For each* $h,k,l \in [1,n]$ *such that* $h,k \neq n$, $\tilde\nu_h = \tilde\nu_{h+1}$, *and* $\tilde\nu_k \neq \tilde\nu_{k+1}$, *the map (22) is such that*

$$\begin{aligned} E_{\tilde\nu,\tilde\nu}\big(x_{w^{-1}(l)}e(\tilde\alpha^+)\big) &\mapsto x_l e(\tilde\nu), \\ E_{s_k(\tilde\nu),\tilde\nu}\big(e(\tilde\alpha^+)\big) &\mapsto \varphi_k e(\tilde\nu), \\ E_{\tilde\nu,\tilde\nu}\big(\tau_{w^{-1}(h)}e(\tilde\alpha^+)\big) &\mapsto \tau_h e(\tilde\nu). \end{aligned}$$

(b) *Consider the assignment*

$$\begin{aligned} x_l e(\tilde\nu) &\mapsto E_{\tilde\nu,\tilde\nu}(x_{w^{-1}(l)}), \\ \varphi_k e(\tilde\nu) &\mapsto E_{s_k(\tilde\nu),\tilde\nu}(1), \\ \tau_h e(\tilde\nu) &\mapsto E_{\tilde\nu,\tilde\nu}(\tau_{w^{-1}(h)}), \end{aligned}$$

for each $h,k,l \in [1,n]$ *and* $\tilde\nu \in \tilde I^{\tilde\alpha}$ *such that* $h,k \neq n$, $\tilde\nu_h = \tilde\nu_{h+1}$, *and* $\tilde\nu_k \neq \tilde\nu_{k+1}$, *where* $w \in \mathfrak{S}_n$ *as in (16). It extends uniquely to an* $\underline{\mathbf{h}}'$*-algebra homomorphism*

$$R^\Lambda(\alpha,\underline{\mathbf{h}}') \to \bigoplus_{\tilde\alpha}\bigoplus_{(\alpha_t)} \mathrm{Mat}_{\tilde I^{\tilde\alpha}}\Big(\bigotimes_{t\in\tilde I} R^{\omega_t}(\alpha_t,\underline{\mathbf{h}}')\Big). \tag{23}$$

Proof

Part (a) follows from (17) and from the following computations:

$$\begin{aligned} x_l e(\tilde\nu) &= x_l \pi_{\tilde\nu} e(\tilde\alpha^+)\pi_{\tilde\nu}^{-1} \\ &= \pi_{\tilde\nu} x_{w^{-1}(l)} e(\tilde\alpha^+)\pi_{\tilde\nu}^{-1}, \\ \varphi_k e(\tilde\nu) &= \varphi_k \pi_{\tilde\nu} e(\tilde\alpha^+)\pi_{\tilde\nu}^{-1} \\ &= \varphi_k e(\tilde\nu)\varphi_w e(\tilde\alpha^+)\pi_{\tilde\nu}^{-1} \\ &= e\big(s_k(\tilde\nu)\big)\varphi_k\varphi_w e(\tilde\alpha^+)\pi_{\tilde\nu}^{-1} \\ &= \pi_{s_k(\tilde\nu)} e(\tilde\alpha^+)\pi_{\tilde\nu}^{-1}, \\ \tau_h e(\tilde\nu) &= \tau_h \pi_{\tilde\nu} e(\tilde\alpha^+)\pi_{\tilde\nu}^{-1} \\ &= \tau_h e(\tilde\nu)\varphi_w e(\tilde\alpha^+)\pi_{\tilde\nu}^{-1} \\ &= e(\tilde\nu)\tau_h\varphi_w e(\tilde\alpha^+)\pi_{\tilde\nu}^{-1} \\ &= e(\tilde\nu)\varphi_w \tau_{w^{-1}(h)} e(\tilde\alpha^+)\pi_{\tilde\nu}^{-1} \\ &= \pi_{\tilde\nu}\tau_{w^{-1}(h)} e(\tilde\alpha^+)\pi_{\tilde\nu}^{-1}. \end{aligned}$$

Here, we used the equality $w^{-1}(h+1) = w^{-1}(h)+1$, which follows from the definition of w in (15), and the equalities $s_h(\tilde{\nu}) = \tilde{\nu}$ and $s_{w^{-1}(h)}(\tilde{\alpha}^+) = \tilde{\alpha}^+$, which follow from the identities $\tilde{\nu}_h = \tilde{\nu}_{h+1}$ and $\tilde{\nu} = w(\tilde{\alpha}^+)$.

Now, let us concentrate on (b). Since the elements $e(\nu)e(\tilde{\nu})$, $x_l e(\tilde{\nu})$, $\varphi_k e(\tilde{\nu})$, $\tau_h e(\tilde{\nu})$ generate $R^\Lambda(n,\underline{\mathbf{h}}')$ by Claim 3.18, it is enough to check that the defining relations of $R^\Lambda(\alpha,\underline{\mathbf{h}}')$ given in Section 3.1.3 are satisfied. This is obvious. Note that the element $\tau_{w^{-1}(h)}$ belongs indeed to $\bigotimes_{t\in\tilde{I}} R^{\omega_t}(\alpha_t,\underline{\mathbf{h}}')$ because $s_{w^{-1}(h)}(\tilde{\alpha}^+) = \tilde{\alpha}^+$. □

Composing (22) and (23), we get an $\underline{\mathbf{h}}'$-algebra homomorphism

$$\bigoplus_{\tilde{\alpha}} \mathrm{Mat}_{\tilde{I}\tilde{\alpha}}\big(e(\tilde{\alpha}^+)R^\Lambda(\alpha,\underline{\mathbf{h}}')e(\tilde{\alpha}^+)\big) \to \bigoplus_{\tilde{\alpha}}\bigoplus_{(\alpha_t)} \mathrm{Mat}_{\tilde{I}\tilde{\alpha}}\Big(\bigotimes_{t\in\tilde{I}} R^{\omega_t}(\alpha_t,\underline{\mathbf{h}}')\Big).$$

For each $\tilde{\alpha}$ it restricts to an $\underline{\mathbf{h}}'$-algebra homomorphism

$$e(\tilde{\alpha}^+)R^\Lambda(\alpha,\underline{\mathbf{h}}')e(\tilde{\alpha}^+) \to \bigoplus_{(\alpha_t)}\bigotimes_{t\in\tilde{I}} R^{\omega_t}(\alpha_t,\underline{\mathbf{h}}'),$$

which is a left inverse to (19). □

□

3.2.4. The isotypic filtration

Recall that $\mathfrak{e} = \mathfrak{sl}_2$. A $(P\times\mathbb{Z})$*-graded* $\Bbbk$-category $\mathcal{C}$ is a direct sum of categories of the form $\mathcal{C} = \bigoplus_{\lambda\in P}\mathcal{C}_\lambda$, where each $\mathcal{C}_\lambda$ is a $\mathbb{Z}$-graded $\Bbbk$-category. We will call λ the *P-degree* of $\mathcal{C}_\lambda$.

Given $i\in I$ there is an $\mathfrak{sl}_2$-triple $\mathfrak{e}_i \subset \mathfrak{g}$. In this section, we study the restriction of a categorical $\mathfrak{g}_{\mathbf{k}}$-representation to such an $\mathfrak{sl}_2$-triple. To simplify the notation, in this section we fix an element $i\in I$ and identify $\mathfrak{e} = \mathfrak{e}_i$.

Recall that for each weight $\lambda\in P$ we set $\lambda_i = \langle\alpha_i^\vee,\lambda\rangle$. By a $(P\times\mathbb{Z})$*-graded* categorical $\mathfrak{e}_{\mathbf{k}}$-representation on $\mathcal{C}$, we will mean a representation such that for each $\lambda\in P$ the $\mathbf{k}$-subcategory $\mathcal{C}_\lambda$ has the weight λ_i relative to the $\mathfrak{e}$-action. In particular, if $\mathcal{C}$ is an integrable categorical representation of $\mathfrak{g}_{\mathbf{k}}$, restricting the $\mathfrak{g}$-action on $\mathcal{C}$ to $\mathfrak{e}_i$ yields an integrable $(P\times\mathbb{Z})$-graded categorical $\mathfrak{e}$-representation on $\mathcal{C}$ of degree d_i.

Now, fix an integer $k\in\mathbb{N}$. Given a $(P\times\mathbb{Z})$-graded $\Bbbk$-category $\mathcal{M}$, such that $\mathcal{M}_\mu = 0$ whenever $\mu_i \neq k$, and a $\mathbb{Z}$-graded $\Bbbk$-algebra homomorphism $\mathrm{Z}^{k,[d_i]} \to \mathrm{Z}(\mathcal{M}/\mathbb{Z})$, we equip the tensor product $\mathcal{V}^{k,[d_i]} \otimes_{\mathrm{Z}^{k,[d_i]}} \mathcal{M}$ with the $(P\times\mathbb{Z})$-graded categorical $\mathfrak{e}$-representation of degree d_i such that

- $\mathfrak{e}$ acts on the left factor;
- the summand $\mathcal{V}_n^{k,[d_i]} \otimes_{\mathrm{Z}^{k,[d_i]}} \mathcal{M}_\mu$ has the P-degree $\mu - n\alpha_i$ for each $n\in\mathbb{N}$.

The vanishing $\mathcal{M}_\mu = 0$ whenever $\mu_i \neq k$ is imposed so that the tensor product $\mathcal{V}^{k,[d_i]} \otimes_{Z^{k,[d_i]}} \mathcal{M}$ is a $(P \times \mathbb{Z})$-graded categorical $\mathfrak{e}$-representation as defined above.

PROPOSITION 3.22

Fix an integrable categorical representation of $\mathfrak{g}_\mathbf{k}$ on $\mathcal{C}$ of degree 1 *which is bounded above. For each $i \in I$ there is a decreasing filtration $\cdots \subseteq \mathcal{C}_{\geq 1} \subseteq \mathcal{C}_{\geq 0} = \mathcal{C}$ of $\mathcal{C}$ by full $(P \times \mathbb{Z})$-graded integrable categorical $\mathfrak{e}$-representations of degree d_i which are closed under taking direct summands and such that*

- *for each $k \in \mathbb{N}$ there is a $(P \times \mathbb{Z})$-graded $\Bbbk$-category $\mathcal{M}_k$ such that $\mathcal{M}_{k,\mu} = 0$ whenever $\mu_i \neq k$, with a $\mathbb{Z}$-graded $\Bbbk$-algebra homomorphism $\mathbf{k} \otimes_\Bbbk Z^{k,[d_i]} \to Z(\mathcal{M}_k/\mathbb{Z})$ and a $\mathbf{k}$-linear equivalence of $(P \times \mathbb{Z})$-graded categorical $\mathfrak{e}_{Z^k}$-representations $(\mathcal{C}_{\geq k}/\mathcal{C}_{>k})^c \simeq (\mathcal{V}^{k,[d_i]} \otimes_{Z^{k,[d_i]}} \mathcal{M}_k)^c$ of degree d_i;*
- *for each $\lambda \in P$ we have $\mathcal{C}_{\geq k} \cap \mathcal{C}_\lambda = 0$ for k large enough.*

Proof

Since $\mathcal{C}$ is bounded above, by a standard argument we may assume that the set of weights of $\mathcal{C}$ is contained in a cone $\mu - Q_+$ for some $\mu \in P$ (see, e.g., [26, Lemma 2.1.10]). Next, for each coset $\pi \in P/\mathbb{Z}\alpha_i$, the $\mathfrak{g}$-action on $\mathcal{C}$ yields a $(P \times \mathbb{Z})$-graded categorical $\mathfrak{e}$-representation on $\mathcal{C}_\pi = \bigoplus_{\lambda \in \pi} \mathcal{C}_\lambda$ of degree d_i. Since $\mathcal{C}$ decomposes as the direct sum of $\Bbbk$-categories $\mathcal{C} = \bigoplus_\pi \mathcal{C}_\pi$, it is enough to define a filtration of $\mathcal{C}_\pi$ satisfying the properties above for each π. Note that $\pi \cap (\mu - Q_+)$ is a cone of the form $\nu - \mathbb{N}\alpha_i$ for some weight $\nu \in \mu - Q_+$. Thus the claim follows from [35, Theorem 4.22], applied to the $\mathfrak{e}$-representation on $\mathcal{C}_\pi$. □

We call $(\mathcal{C}_{\geq k})$ the *ith isotypic filtration* of $\mathcal{C}$. For each $k \in \mathbb{N}$ and $\lambda \in P$, let $\mathcal{C}_{\geq k,\lambda} \subset \mathcal{C}_{\geq k}$ be the weight λ subcategory given by $\mathcal{C}_{\geq k,\lambda} = \mathcal{C}_{\geq k} \cap \mathcal{C}_\lambda$. We also define

$$\begin{aligned} \mathcal{C}_k &= (\mathcal{V}^{k,[d_i]} \otimes_{Z^{k,[d_i]}} \mathcal{M}_k)^c, \\ \mathcal{C}_{k,\lambda} &= \bigoplus_{n,\mu} (\mathcal{V}^{k,[d_i]}_{k-2n} \otimes_{Z^{k,[d_i]}} \mathcal{M}_{k,\mu})^c, \end{aligned} \tag{24}$$

where the sum runs over all $\mu \in P$ and $n \in \mathbb{N}$ such that $\lambda = \mu - n\alpha_i$. Recall that we have assumed that $\mathcal{M}_{k,\mu} = 0$ whenever $\mu_i \neq k$. Note also that $\mathcal{C}_k^{\mathrm{hw}} = \mathcal{M}_k^c$ and that $\mathcal{C}_k^{\mathrm{hw}}$ is the full subcategory of $(\mathcal{C}/\mathcal{C}_{>k})^c$ consisting of the objects M such that $E_i(M) = 0$.

3.2.5. The loop operators

Consider a categorical representation of $\mathfrak{g}_\mathbf{k}$ of degree ℓ on a $\mathbb{Z}$-graded $\Bbbk$-category $\mathcal{C}$. For each $i \in I$ and $r \in \mathbb{N}$, we define $\mathbf{k}$-linear operators $x_{ir}^\pm$ on the $\mathbb{Z}$-graded $\mathbf{k}$-module

$\mathrm{tr}(\mathcal{C}/\mathbb{Z})$ such that the maps

$$x^+_{ir}: \mathrm{tr}(\mathcal{C}_\lambda/\mathbb{Z}) \to \mathrm{tr}(\mathcal{C}_{\lambda+\alpha_i}/\mathbb{Z}), \qquad x^-_{ir}: \mathrm{tr}(\mathcal{C}_\lambda/\mathbb{Z}) \to \mathrm{tr}(\mathcal{C}_{\lambda-\alpha_i}/\mathbb{Z})$$

are given by

$$x^+_{ir} = \mathrm{tr}_{E_i}(x_i^r), \qquad x^-_{ir} = \mathrm{tr}_{F_i}(x_i^r)$$

(see Definition 2.2 for the notation). The map $x^\pm_{ir}$ is homogeneous of degree $2r\ell d_i$. Let us quote the following fact for future use.

PROPOSITION 3.23
The maps $x^\pm_{ir}$ are functorial; that is, a morphism $\mathcal{C} \to \mathcal{C}'$ of categorical representations of $\mathfrak{g}_\mathbf{k}$ yields a $\mathbf{k}$-linear map $\mathrm{tr}(\mathcal{C}/\mathbb{Z}) \to \mathrm{tr}(\mathcal{C}'/\mathbb{Z})$ which intertwines the operators $x^\pm_{ir}$ on $\mathrm{tr}(\mathcal{C}/\mathbb{Z})$ and on $\mathrm{tr}(\mathcal{C}'/\mathbb{Z})$.

3.2.6. The loop operators on the trace of the minimal categorical representation
Fix a dominant weight $\Lambda \in P_+$. For $\alpha \in Q_+$ we write $\lambda = \Lambda - \alpha$. Recall the base ring $\underline{\mathbf{k}}$ from Section 3.1.3. Let $\mathbf{k}$ be an $\mathbb{N}$-graded $\underline{\mathbf{k}}$-algebra. In this section, we consider the particular case of the minimal categorical representation $\mathcal{V}^\Lambda$.

Definition 3.24
Let $L\mathfrak{g}$ be the $\mathbb{N}$-graded Lie $\Bbbk$-algebra generated by elements $x^\pm_{ir}$, h_{ir} with $i \in I$, $r \in \mathbb{N}$ satisfying the following relations:
(a) $[h_{ir}, h_{js}] = 0$;
(b) $[x^+_{ir}, x^-_{js}] = \delta_{ij} h_{i,r+s}$;
(c) $[h_{ir}, x^\pm_{js}] = \pm a_{ij} x^\pm_{j,r+s}$;
(d) $\sum_{p=0}^m (-1)^p \binom{m}{p} [x^\pm_{i,r+p}, x^\pm_{j,s+m-p}] = 0$ with $i \neq j$ and $m = -a_{ij}$;
(e) $[x^\pm_{i,r}, x^\pm_{i,s}] = 0$;
(f) $[x^\pm_{i,r_1}, [x^\pm_{i,r_2}, \dots, [x^\pm_{i,r_m}, x^\pm_{j,r_0}] \dots]] = 0$ with $i \neq j$, $r_p \in \mathbb{N}$ and $m = 1 - a_{ij}$.
The grading is given by $\deg(x^\pm_{ir}) = \deg(h_{ir}) = 2r$.

There is a unique Lie $\Bbbk$-algebra anti-involution ϖ of $L\mathfrak{g}$ such that

$$\varpi(h_{ir}) = h_{ir}, \qquad \varpi(x^\pm_{ir}) = x^\mp_{ir}.$$

We define the *ℓ-twist* of $L\mathfrak{g}$ to be the Lie $\Bbbk$-subalgebra $L\mathfrak{g}^{[\ell]} \subset L\mathfrak{g}$ generated by the set of elements $\{x^\pm_{i,r\ell}, h_{i,r\ell}; i \in I, r \in \mathbb{N}\}$. We write $L\mathfrak{g}^{[\ell]}_\mathbf{k} = \mathbf{k}^{[\ell]} \otimes_\Bbbk L\mathfrak{g}^{[\ell]}$.

THEOREM 3.25
If $\mathfrak{g}$ is symmetric and (11) is satisfied, then the operators $x^\pm_{i,\ell r}$ with $i \in I$, $r \in \mathbb{N}$ define a $\mathbb{Z}$-graded representation of $L\mathfrak{g}^{[\ell]}_\mathbf{k}$ on $\mathrm{tr}(\mathcal{V}^{\Lambda,[\ell]}/\mathbb{Z})$.

Proof
It is enough to set $\ell = 1$. The proof is similar to a proof in [3] and [4]. However, our setting differs from that proof and cannot be reduced to it, because, in our case, $\mathfrak{g}$ may have any symmetric type and because we do not require that all axioms of a strong categorical representation be satisfied by $\mathcal{V}^\Lambda$. We have written an independent proof in Appendix B.3. □

Remark 3.26
(a) If $\mathfrak{g}$ is not symmetric or (11) is not satisfied, then a more general version of the theorem is given in Proposition B.5 below.

(b) Assume that $\mathfrak{g}$ is symmetric. If $\mathfrak{g}$ is *simply laced*, then the relations (d) and (e) are equivalent to the following : the bracket $[x^{\pm}_{i,r}, x^{\pm}_{j,s}]$ depends only on i, j and on the sum $r + s$ but not on the integers r or s. If $\mathfrak{g}$ is of finite type, then $L\mathfrak{g}$ is the *current algebra* $\mathfrak{g}[t]$ with $\deg(t) = 2$. In general, $L\mathfrak{g}$ is *bigger* than $\mathfrak{g}[t]$. For instance, if $\mathfrak{g}$ is of affine type, then $L\mathfrak{g}$ is a central extension of $\mathfrak{g}[t]$ with an infinite-dimensional center (see [30, Section 3], [39, Section 13], and [13, Section 1.3] for more details).

3.2.7. The loop operators on the center
Since the functors E_i, F_i on $\mathcal{V}^\Lambda$ are biadjoint, we also have operators

$$Z^+_{ir} : Z(\mathcal{V}^\Lambda_\lambda/\mathbb{Z}) \to Z(\mathcal{V}^\Lambda_{\lambda+\alpha_i}/\mathbb{Z}), \qquad Z^-_{ir} : Z(\mathcal{V}^\Lambda_\lambda/\mathbb{Z}) \to Z(\mathcal{V}^\Lambda_{\lambda-\alpha_i}/\mathbb{Z})$$

defined by

$$Z^+_{ir} = Z_{F_i}(x^r_i), \qquad Z^-_{ir} = Z_{E_i}(x^r_i)$$

(see Definition 2.5 for the notation).

The proposition below gives a more explicit description of the operators $x^{\pm}_{ir}$ and $Z^{\pm}_{ir}$ in terms of the algebras $R^\Lambda(\alpha)$. Fix v_k, $v^\vee_k$ such that

$$\hat{\eta}'_{i,\lambda}(1) = \sum_k v^\vee_k \otimes v_k, \qquad v^\vee_k \in R^\Lambda(\alpha)e(\alpha-\alpha_i, i), v_k \in e(\alpha-\alpha_i, i)R^\Lambda(\alpha).$$

PROPOSITION 3.27
For each $\alpha \in Q_+$ of height n and each $f \in R^\Lambda(\alpha)$, $g \in Z(R^\Lambda(\alpha))$, we have the following:

(a) $\operatorname{tr}(\mathcal{V}^\Lambda/\mathbb{Z}) = \operatorname{tr}(R^\Lambda)$, $Z(\mathcal{V}^\Lambda/\mathbb{Z}) = Z(R^\Lambda)$ *as a $\mathbb{Z}$-graded* **k***-module and* **k***-algebra, respectively;*
(b) $x^-_{ir}(\operatorname{tr}(f)) = \operatorname{tr}(e(\alpha, i)x^r_{n+1} f) \in \operatorname{tr}(R^\Lambda(\alpha+\alpha_i))$*;*
(c) $x^+_{ir}(\operatorname{tr}(f)) = \sum_k \hat{\varepsilon}'_{i,\lambda+\alpha_i}(x^r_n v_k f v^\vee_k) \in \operatorname{tr}(R^\Lambda(\alpha-\alpha_i))$*;*

(d) $Z^+_{ir}(g) = \hat{\varepsilon}'_{i,\lambda+\alpha_i}(x^r_n g e(\alpha-\alpha_i, i)) \in \mathrm{Z}(R^\Lambda(\alpha-\alpha_i))$;
(e) $Z^-_{ir}(g) = \mu((1 \otimes x^r_{n+1} g)\hat{\eta}'_{i,\lambda-\alpha_i}(1)) \in \mathrm{Z}(R^\Lambda(\alpha+\alpha_i))$.

The proof is standard and left to the reader. We only make a few comments.

First, the element f in (b) is viewed as an element of $R^\Lambda(\alpha+\alpha_i)$ via the map $\iota_i : R^\Lambda(\alpha) \to R^\Lambda(\alpha+\alpha_i)$. Hence $e(\alpha,i)x^r_{n+1}f$ belongs to $R^\Lambda(\alpha+\alpha_i)$ and $x^-_{ir}(f)$ belongs to $\mathrm{tr}(R^\Lambda(\alpha+\alpha_i))$.

Next, the equality (c) should be interpreted in the following way: f is identified with the $R^\Lambda(\alpha)$-module endomorphism of $R^\Lambda(\alpha)$ given by $m \mapsto mf$, and E_i is represented by the bimodule $e(\alpha-\alpha_i,i)R^\Lambda(\alpha)$ which is a projective $R^\Lambda(\alpha-\alpha_i)$-module by [16, Theorem 4.5]. Then $x^+_{ir}(\mathrm{tr}(f))$ is the class in $\mathrm{tr}(R^\Lambda(\alpha-\alpha_i)\text{-proj}) \simeq \mathrm{tr}(R^\Lambda(\alpha-\alpha_i))$ of the endomorphism

$$\varphi : e(\alpha-\alpha_i,i)R^\Lambda(\alpha) \to e(\alpha-\alpha_i,i)R^\Lambda(\alpha), \qquad m \mapsto x^r_n m f.$$

Explicitly, since $(\hat{\varepsilon}'_i E_i)\circ(E_i\hat{\eta}'_i) = E_i$, we have $m = \sum_k \hat{\varepsilon}'_i(mv^\vee_k)v_k$. Hence there is a surjective $R^\Lambda(\alpha-\alpha_i)$-module morphism

$$p : \bigoplus_k R^\Lambda(\alpha-\alpha_i) \to e(\alpha-\alpha_i,i)R^\Lambda(\alpha), \qquad (m_k)_k \mapsto \sum_k m_k v_k,$$

with a splitting i given by

$$i : e(\alpha-\alpha_i,i)R^\Lambda(\alpha) \to \bigoplus_k R^\Lambda(\alpha-\alpha_i), \qquad m \mapsto \big(\hat{\varepsilon}'_i(mv^\vee_k)\big)_k;$$

that is, we have $p \circ i = 1$. Consider the endomorphism $\tilde{\varphi}$ given by

$$\tilde{\varphi} = i \circ \varphi \circ p : \bigoplus_k R^\Lambda(\alpha-\alpha_i) \to \bigoplus_k R^\Lambda(\alpha-\alpha_i),$$
$$(m_k)_k \mapsto \Big(\hat{\varepsilon}'_i\Big(x^r_n \sum_l m_l v_l f v^\vee_k\Big)\Big)_k.$$

Since $m_l \in R^\Lambda(\alpha-\alpha_i)$, it commutes with x^r_n. Since $\hat{\varepsilon}'_i$ is $R^\Lambda(\alpha-\alpha_i)$-linear, we deduce that $\hat{\varepsilon}'_i(x^r_n \sum_l m_l v_l f v^\vee_k) = \sum_l m_l \hat{\varepsilon}'_i(x^r_n v_l f v^\vee_k)$. So $\tilde{\varphi}$ is the right multiplication by the matrix $(\hat{\varepsilon}'_i(x^r_n v_l f v^\vee_k))_{l,k}$. Therefore we have

$$\mathrm{tr}(\varphi) = \mathrm{tr}(\varphi \circ p \circ i) = \mathrm{tr}(\tilde{\varphi}) = \sum_k \hat{\varepsilon}'_i(x^r_n v_k f v^\vee_k).$$

We obtain $x^+_{ir}(\mathrm{tr}(f)) = \sum_k \hat{\varepsilon}'_i(x^r_n v_k f v^\vee_k)$.

Finally, in parts (d) and (e) we have $Z^+_{ir}(g) = Z_{F_i}(x^r_n g)$ and $Z^-_{ir}(g) = Z_{F_i}(x^r_{n+1} g)$. Note that $e(\alpha-\alpha_i,i) = \eta'_{i,\lambda+\alpha_i}(1)$ and $\mu = \varepsilon'_{i,\lambda-\alpha_i}$.

Let $r(\alpha,i)$ be as in (63). The following proposition relates the operator $x_{ir}^{\pm}$ on $\mathrm{tr}(\mathcal{V}^{\Lambda}/\mathbb{Z})$ with the transpose of the operator $Z_{ir}^{\mp}$ on $\mathrm{Z}(\mathcal{V}^{\Lambda}/\mathbb{Z})$ under the pairing given by the symmetrizing form t_{Λ} in Proposition 3.10:

$$\mathrm{Z}(\mathcal{V}^{\Lambda}/\mathbb{Z})\times \mathrm{tr}(\mathcal{V}^{\Lambda}/\mathbb{Z})\to \mathbf{k},\qquad (a,b)\mapsto t_{\Lambda}(ab). \tag{25}$$

PROPOSITION 3.28
Let $f\in \mathrm{tr}(R^{\Lambda}(\alpha))$, $g\in \mathrm{Z}(R^{\Lambda}(\alpha+\alpha_i))$, and $h\in \mathrm{Z}(R^{\Lambda}(\alpha-\alpha_i))$. We have
(a) $t_{\alpha+\alpha_i}(gx_{ir}^{-}(f))=r(\alpha,i)t_{\alpha}(Z_{ir}^{+}(g)f)$;
(b) $t_{\alpha-\alpha_i}(hx_{ir}^{+}(f))=r(\alpha-\alpha_i,i)^{-1}t_{\alpha}(Z_{ir}^{-}(h)f)$.

Proof
Write $f=\mathrm{tr}(f')$ with $f'\in R^{\Lambda}(\alpha)$. Write $\hat\eta_i'(1)=\sum_k v_k^{\vee}\otimes v_k$ as above. By (64) we have

$$\begin{aligned}
t_{\alpha+\alpha_i}\big(gx_{ir}^{-}(f)\big)&=t_{\alpha+\alpha_i}\big(ge(\alpha,i)x_{n+1}^{r}f'\big)\\
&=r(\alpha,i)t_{\alpha}\hat\varepsilon'_{i,\lambda}\big(e(\alpha,i)x_{n+1}^{r}gf'\big)\\
&=r(\alpha,i)t_{\alpha}\big(Z_{ir}^{+}(g)f\big),\\
t_{\alpha-\alpha_i}\big(hx_{ir}^{+}(f)\big)&=t_{\alpha-\alpha_i}\Big(\sum_k h\hat\varepsilon_i'(x_n^{r}v_kf'v_k^{\vee})\Big)\\
&=t_{\alpha-\alpha_i}\Big(\sum_k \hat\varepsilon_i'(hx_n^{r}v_kf'v_k^{\vee})\Big)\\
&=r(\alpha-\alpha_i,i)^{-1}t_{\alpha}\Big(\sum_k hx_n^{r}v_kf'v_k^{\vee}\Big)\\
&=r(\alpha-\alpha_i,i)^{-1}t_{\alpha}\Big(\sum_k v_k^{\vee}hx_n^{r}v_kf'\Big)\\
&=r(\alpha-\alpha_i,i)^{-1}t_{\alpha}\big(\mu\big((1\otimes x_n^{r}h)\hat\eta'_{i,\lambda}(1)\big)f'\big)\\
&=r(\alpha-\alpha_i,i)^{-1}t_{\alpha}\big(Z_{ir}^{-}(h)f\big).
\end{aligned}$$
□

3.2.8. The loop operators and the isotypic filtration
In this section we consider an integrable categorical representation of $\mathfrak{g}_{\mathbf{k}}$ of degree 1 on a $\mathbb{Z}$-graded $\Bbbk$-category $\mathcal{C}$ which is bounded above. As a preparation for Section 3.3, we study the behavior of the loop operators on $\mathrm{tr}(\mathcal{C}/\mathbb{Z})$ with respect to the isotypic filtration.

Given $i\in I$ and $k\geq 0$, the ith isotypic filtration of $\mathcal{C}$ yields $(P\times\mathbb{Z})$-graded integrable $\mathfrak{e}_{Z^k}$-categorical representations $\mathcal{C}_{\geq k}$, $\mathcal{C}_k$ of degree d_i such that $\mathcal{C}_k=$

$(\mathcal{C}_{\geq k}/\mathcal{C}_{>k})^c$. By Proposition 2.1(f), Theorem 3.25, and Proposition 3.23, it also yields an exact sequence of $\mathbb{Z}$-graded $\mathbf{k}\otimes_k L\mathfrak{e}_{Z^k}^{[d_i]}$-modules:

$$\mathrm{tr}(\mathcal{C}_{>k}/\mathbb{Z})\to\mathrm{tr}(\mathcal{C}_{\geq k}/\mathbb{Z})\to\mathrm{tr}(\mathcal{C}_k/\mathbb{Z})\to 0. \tag{26}$$

Let $h_k:\mathcal{C}_{\geq k}\to\mathcal{C}$ be the canonical inclusion. Consider the $\mathbb{Z}$-graded $\mathbf{k}\otimes_{\Bbbk} L\mathfrak{e}_{Z^k}^{[d_i]}$-submodule $\mathrm{tr}(\mathcal{C}/\mathbb{Z})_{\geq k}$ of $\mathrm{tr}(\mathcal{C}/\mathbb{Z})$ given by $\mathrm{tr}(\mathcal{C}/\mathbb{Z})_{\geq k}=\mathrm{tr}(h_k)(\mathrm{tr}(\mathcal{C}_{\geq k}/\mathbb{Z}))$. Since $\mathrm{tr}(\mathcal{C}/\mathbb{Z})_{>k}\subseteq\mathrm{tr}(\mathcal{C}/\mathbb{Z})_{\geq k}$, we define $\mathrm{tr}(\mathcal{C}/\mathbb{Z})_k$ as their quotient. In other words, we have an exact sequence

$$0\to\mathrm{tr}(\mathcal{C}/\mathbb{Z})_{>k}\to\mathrm{tr}(\mathcal{C}/\mathbb{Z})_{\geq k}\to\mathrm{tr}(\mathcal{C}/\mathbb{Z})_k\to 0, \tag{27}$$

and the map $\mathrm{tr}(h_k)$ factors to a surjective $\mathbb{Z}$-graded $\mathbf{k}\otimes_{\Bbbk} L\mathfrak{e}_{Z^k}^{[d_i]}$-module homomorphism

$$\mathrm{tr}(\mathcal{C}_k/\mathbb{Z})=\mathrm{tr}(\mathcal{V}^{k,[d_i]}\otimes_{Z^{k,[d_i]}}\mathcal{M}_k/\mathbb{Z})\to\mathrm{tr}(\mathcal{C}/\mathbb{Z})_k. \tag{28}$$

Now, assume that $\mathcal{M}_{k,\mu}^c=B_{k,\mu}$-grproj for some $\mathbb{Z}$-graded $\mathbf{k}\otimes_{\Bbbk} Z^{k,[d_i]}$-algebra $B_{k,\mu}$, and set $B_k=\bigoplus_\mu B_{k,\mu}$. From (9) and (24) we deduce that

$$\mathcal{C}_k=\bigoplus_{n\in\mathbb{N}}(\underline{\mathrm{H}}_n^{k,[d_i]}\otimes_{Z^{k,[d_i]}}B_k)\text{-grproj}. \tag{29}$$

This yields a $\mathbb{Z}$-graded $\mathbf{k}$-vector space isomorphism

$$\mathrm{tr}(\mathcal{C}_k/\mathbb{Z})\simeq\bigoplus_{n\in\mathbb{N}}\mathrm{tr}(\underline{\mathrm{H}}_n^{k,[d_i]})\otimes_{Z^{k,[d_i]}}\mathrm{tr}(B_k), \tag{30}$$

and the $L\mathfrak{e}_{Z^k}^{[d_i]}$-action on $\mathrm{tr}(\mathcal{C}_k/\mathbb{Z})$ is given by explicit formulas as in Section 3.2.9 below.

Finally, for a future use we will write

$$\mathrm{tr}(\mathcal{C}_\lambda)_{\geq k}=\mathrm{tr}(\mathcal{C}_\lambda)\cap\mathrm{tr}(\mathcal{C})_{\geq k},$$

and we define $\mathrm{tr}(\mathcal{C}_\lambda)_k$ in the obvious way.

Remark 3.29

We conjecture that the left arrow in sequence (26) is injective if $\mathcal{C}=\mathcal{V}^\Lambda$. Given an idempotent e of a finite-dimensional algebra A such that AeA is a stratifying ideal of A in the sense of Cline, Parshall, and Scott, we have a long exact sequence by [19, Theorem 3.1],

$$\cdots\to HH_1(B)\to HH_0(eAe)\to HH_0(A)\to HH_0(B)\to 0,$$

where $B = A/AeA$. Now, set $\mathfrak{g} = \mathfrak{sl}_3$ and let $\Lambda = \theta = \alpha_1 + \alpha_2$ be the highest positive root. Then $A = R^{\Lambda}(\theta)$ has a primitive idempotent e such that $\mathcal{C}_{>0} = eAe$-proj, $\mathcal{C} = A$-proj, and $\mathcal{C}_0 = B$-proj. In this case, the left arrow in (26) is indeed injective, although the ideal AeA is not a stratifying ideal of A.

3.2.9. The $\mathfrak{sl}_2$-case

We will use the same notation as in Example 3.14. Thus, we have $\Lambda = k\omega_1$, $\alpha = n\alpha_1$, and $d_{k,n} = 2n(k-n)$. Let $\mathbf{k}$ be a $\mathrm{Z}^{k,[\ell]}$-algebra. Recall that (9) yields

$$\mathcal{V}^{k,[\ell]} = \bigoplus_{n\geq 0} (\mathbf{k} \otimes_{\mathrm{Z}^{k,[\ell]}} \underline{\mathrm{H}}_n^{k,[\ell]})\text{-grproj}.$$

For a future use, let us quote the following.

PROPOSITION 3.30

(a) *The $\mathbf{k} \otimes_{\Bbbk} L\mathfrak{e}^{[\ell]}_{\mathrm{Z}^k}$-module $\mathrm{tr}(\mathcal{V}^{k,[\ell]})$ is generated by $\mathrm{tr}(\mathcal{V}_k^{k,[\ell]})$.*

(b) *We have $\mathrm{tr}(\underline{\mathrm{H}}_n^{k,[\ell]}) = V \otimes_{\Bbbk} \mathrm{Z}^{k,[\ell]}$ as a $\mathbb{Z}$-graded $\mathrm{Z}^{k,[\ell]}$-module, with $V^d = 0$ if $d \notin [0, \ell d_{k,n}]$ and $V^{\ell d_{k,n}} = \Bbbk$.*

Proof

It is enough to set $\ell = 1$ and $\mathbf{k} = \mathrm{Z}^k$. Then, we have

$$\mathcal{V}^k = \bigoplus_{n=0}^{k} \mathcal{V}^k_{k-2n}, \qquad \mathcal{V}^k_{k-2n} = \underline{\mathrm{H}}_n^k\text{-grproj}.$$

Fix formal variables $y_1, \ldots, y_k$ of degree 2 such that $c_p = e_p(y_1, \ldots, y_k)$ for $p = 1, 2, \ldots, k$ and

$$\mathrm{Z}^k = \Bbbk[y_1, y_2, \ldots, y_k]^{S_k}.$$

Recall from [35, Proposition 2.21, Section 4.3.2] that the algebra $\underline{\mathrm{H}}_n^k$ is Morita-equivalent to $\Bbbk[y_1, y_2, \ldots, y_k]^{S_n \times S_{k-n}}$ as Z^k-algebras if $n \in [0,k]$, and it is zero otherwise. We deduce the Z^k-linear isomorphisms

$$\mathrm{tr}(\underline{\mathrm{H}}_n^k) \simeq \mathrm{Z}(\underline{\mathrm{H}}_n^k) \simeq \begin{cases} \Bbbk[y_1, y_2, \ldots, y_k]^{S_n \times S_{k-n}} & \text{if } n \in [0,k], \\ 0 & \text{otherwise.} \end{cases}$$

This implies part (b).

Moreover, since $\mathrm{tr}(\underline{\mathrm{H}}_n^k)$ is free over Z^k in this case and the Z^k-algebra $\underline{\mathrm{H}}_n^k$ is symmetric, we have $\mathrm{tr}(\underline{\mathrm{H}}_n^k) \simeq \mathrm{CF}(\underline{\mathrm{H}}_n^k)^* \simeq \mathrm{Z}(\underline{\mathrm{H}}_n^k)^*$, where $(\bullet)^*$ is the dual as a Z^k-module. By Proposition 3.28, under this isomorphism the operators x_{ir}^+, x_{ir}^- on $\bigoplus_n \mathrm{tr}(\underline{\mathrm{H}}_n^k)$ are identified with the transpose of the operators Z_{ir}^-, Z_{ir}^+ on $\bigoplus_n \mathrm{Z}(\underline{\mathrm{H}}_n^k)$.

For each $p = 1, \ldots, k-1$, let ∂_{s_p} be the Demazure operator on $\Bbbk[y_1, y_2, \ldots, y_k]$ associated with the simple reflection $s_p = (p, p+1)$, which is defined by $\partial_{s_p}(f) = (f - s_p(f))/(y_{p+1} - y_p)$. The formulas in [36] for the symmetrizing form of $\underline{\mathrm{H}}_n^k$ imply that the Bernstein operators are given by the following explicit formulas for all $f \in \mathrm{Z}(\underline{\mathrm{H}}_n^k)$:

$$Z_{ir}^-(f) = \partial_{s_1} \circ \cdots \circ \partial_{s_n}\Big(y_{n+1}^r f \prod_{p=n+2}^{k} (y_{n+1} - y_p)\Big) \in \mathrm{Z}(\underline{\mathrm{H}}_{n+1}^k),$$

$$Z_{ir}^+(f) = \partial_{s_{k-1}} \circ \cdots \circ \partial_{s_n}\Big(y_n^r f \prod_{p=1}^{n-1} (y_p - y_n)\Big) \in \mathrm{Z}(\underline{\mathrm{H}}_{n-1}^k).$$

More precisely, the formula for $Z_{ir}^-(f)$ is [36, Lemma 5.13] and the formula for $Z_{ir}^+(f)$ is [36, Lemma 5.12].

Now, the Z^k-algebra $\mathrm{Z}(\underline{\mathrm{H}}_n^k)$ is equipped with the symmetrizing form $\partial_{w_n} : \mathrm{Z}(\underline{\mathrm{H}}_n^k) \to \mathrm{Z}^k$, where $w_n \in S_k$ is the unique element of minimal length in the longest coset in $S_k / S_n \times S_{k-n}$. It yields a nondegenerate bilinear form $\tau_n^k : \mathrm{Z}(\underline{\mathrm{H}}_n^k) \times \mathrm{Z}(\underline{\mathrm{H}}_n^k) \to \mathrm{Z}^k$ such that $(a,b) \mapsto \partial_{w_n}(ab)$. The bilinear form τ_n^k identifies $\mathrm{Z}(\underline{\mathrm{H}}_n^k)^*$ with $\mathrm{Z}(\underline{\mathrm{H}}_n^k)$ as a Z^k-module.

Finally, taking the transpose of Z_{ir}^-, Z_{ir}^+ with respect to τ_n^k, we get the following formulas for the operators x_{ir}^+, x_{ir}^-, which are viewed as Z^k-linear operators on $\bigoplus_n \mathrm{Z}(\underline{\mathrm{H}}_n^k)$:

$$x_{ir}^+(f) = \sum_{p=n}^{k} (p,n)(y_n^r f) \in \mathrm{Z}(\underline{\mathrm{H}}_{n-1}^k),$$

$$x_{ir}^-(f) = \sum_{p=1}^{n+1} (p,n+1)(y_{n+1}^r f) \in \mathrm{Z}(\underline{\mathrm{H}}_{n+1}^k).$$

In these formulas the symbols (p,n) and $(p,n+1)$ hold for the transpositions of p, n and of p, $n+1$, respectively.

We can now prove part (a) of the proposition. For each $n \geq 0$, let

$$\Lambda^+(n) = \big\{\lambda \in \mathbb{N}^n; \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)\big\}$$

be the set of dominant weights. We abbreviate $x_{i\lambda}^- = x_{i\lambda_1}^- x_{i\lambda_2}^- \cdots x_{i\lambda_n}^-$. We must check that

$$\sum_{\lambda \in \Lambda^+(n)} x_{i\lambda}^-(\mathrm{Z}^k) = \mathrm{Z}(\underline{\mathrm{H}}_n^k). \tag{31}$$

This follows from the equality $\mathrm{Z}(\underline{\mathrm{H}}_n^k) = \sum_{\lambda \in \Lambda^+(n)} \mathrm{Z}^k \cdot h_\lambda(y_1, y_2, \ldots, y_n)$ and the formula

$$x_{i\lambda}^-(f) = f \cdot h_\lambda(y_1, y_2, \ldots, y_n), \quad \forall f \in \mathrm{Z}^k. \qquad \square$$

3.3. The center and the cocenter of the minimal categorical representation

From now on, and until the end of Section 3, we will assume that $\Bbbk$ is of characteristic 0. Let $\mathfrak{g}$ be the symmetrizable Kac–Moody algebra over $\Bbbk$ associated with the Cartan datum $(P, P^\vee, \Phi, \Phi^\vee)$. Fix a dominant weight $\Lambda \in P_+$. Let $\mathbf{k}$ be an $\mathbb{N}$-graded $\underline{\mathbf{k}}$-algebra as in Section 3.1. We equip $\mathrm{tr}(R^\Lambda)$, $\mathrm{Z}(R^\Lambda)$ with the $\mathbb{Z}$-gradings such that for each $d \in \mathbb{Z}$ we have

$$\mathrm{tr}(R^\Lambda)^d = \{\mathrm{tr}(x); x \in R^{\Lambda,d}\},$$
$$\mathrm{Z}(R^\Lambda)^d = R^{\Lambda,d} \cap \mathrm{Z}(R^\Lambda).$$

3.3.1. The grading of the cocenter

In this section we set $\mathbf{k} = \Bbbk$. For $\alpha \in Q_+$ we write $\lambda = \Lambda - \alpha$ and $R^\Lambda(\alpha) = R^\Lambda(\alpha; \Bbbk)$.

THEOREM 3.31

Assume that $\mathbf{k} = \Bbbk$. Then, for each $\alpha \in Q_+$ we have the following:

(a) *If* $\mathrm{tr}(R^\Lambda(\alpha))^d \neq 0$, *then* $d \in [0, d_{\Lambda,\alpha}]$.

(b) *If* $\mathrm{tr}(R^\Lambda(\alpha))^{d_{\Lambda,\alpha}} = 0$, *then* $V(\Lambda)_\lambda = 0$.

(c) $\dim \mathrm{tr}(R^\Lambda(\alpha))^0 = \dim V(\Lambda)_\lambda$.

Proof

For each $i \in I$ and $k \in \mathbb{N}$ the ith isotypic filtration of $\mathcal{V}^\Lambda$ yields $(P \times \mathbb{Z})$-graded categorical $\mathfrak{e}$-representations $\mathcal{V}^\Lambda_{\geq k}$ and $\mathcal{V}^\Lambda_k$ of degree d_i such that

- $\mathcal{V}^\Lambda_{\geq 0} = \mathcal{V}^\Lambda$;
- $\mathcal{V}^\Lambda_{\lambda,\geq k} = 0$ if k is large enough (depending on λ);
- $\mathcal{V}^\Lambda_k = (\mathcal{V}^\Lambda_{\geq k}/\mathcal{V}^\Lambda_{>k})^c$.

Taking the trace we get $\mathbb{Z}$-graded $L\mathfrak{e}^{[d_i]}_{\mathrm{Z}^k}$-submodules $\mathrm{tr}(R^\Lambda)_{\geq k} \subset \mathrm{tr}(R^\Lambda)$ as in (27) such that

- $\mathrm{tr}(R^\Lambda)_{\geq 0} = \mathrm{tr}(R^\Lambda)$;
- $\mathrm{tr}(R^\Lambda(\alpha))_{\geq k} = 0$ if k is large enough (depending on α).

For each $i \in I$, let ϵ_i be the Kashiwara function on the set of simple objects in $\mathcal{A}^\Lambda/\mathbb{Z}$, which is defined as in [21, Section 3.2]. For each indecomposable object $P, Q \in \mathcal{V}^\Lambda$, we have

$$\epsilon_i\big(\mathrm{top}(P)\big) > \epsilon_i\big(\mathrm{top}(Q)\big) \iff \exists k \in \mathbb{N} \quad \text{such that } P \in \mathcal{V}^\Lambda_{\geq k}, Q \notin \mathcal{V}^\Lambda_{\geq k}.$$

Now, fix an indecomposable object $P \in \mathcal{V}^\Lambda_\lambda$ and an element $v_P \in \mathrm{tr}(\mathrm{End}_{\mathcal{V}^\Lambda/\mathbb{Z}}(P))$ which is homogeneous of degree d. Let v be the image of v_P in $\mathrm{tr}(R^\Lambda(\alpha))$. We must prove that either $d \in [0, d_{\Lambda,\alpha}]$ or $v = 0$.

Since $R^\Lambda(0) \simeq \Bbbk$, we may assume that $\alpha \neq 0$. The Grothendieck group of $\mathcal{V}^\Lambda_\lambda/\mathbb{Z}$ is isomorphic to a $\mathbb{Z}$-lattice in $V(\Lambda)_\lambda$ by [16]. Since $\alpha \neq 0$, the weight subspace $V(\Lambda)_\lambda$ does not contain any highest weight vector for the $\mathfrak{g}$-action on $V(\Lambda)$. So, we may choose $i \in I$ such that the integer $m = \epsilon_i(\mathrm{top}(P))$ is positive. From now on, let $\mathcal{V}^\Lambda_{\geq k}$ denote the ith isotypic filtration of $\mathcal{V}^\Lambda$ for this choice of i.

Now, let k be maximal such that $P \in \mathcal{V}^\Lambda_{\geq k}$ and set $m = \epsilon_i(\mathrm{top}(P))$. We have

$$v \in \mathrm{tr}\big(R^\Lambda(\alpha)\big)_{\geq k}.$$

Note that $\epsilon_i(\mathrm{top}(Q))$ is bounded above as Q runs over the set of all indecomposable objects in $\mathcal{V}^\Lambda_\lambda$ and that $\epsilon_i(\mathrm{top}(Q)) > m$ if and only if $Q \in \mathcal{V}^\Lambda_{>k}$.

The proof is an increasing induction on α (or, equivalently, on $\mathrm{ht}(\alpha)$) and a decreasing induction on m (or, equivalently, on k). We will assume that

(d) $\mathrm{deg}(\mathrm{tr}(R^\Lambda(\beta)) \subseteq [0, d_{\Lambda,\beta}]$ for each $\beta \in Q_+$ such that $\beta < \alpha$;

(e) $\mathrm{deg}(\mathrm{tr}(R^\Lambda(\alpha))_{>k}) \subseteq [0, d_{\Lambda,\alpha}]$.

We must check that $\mathrm{deg}(\mathrm{tr}(R^\Lambda(\alpha)_k) \subseteq [0, d_{\Lambda,\alpha}]$. Then, from (27) and the inductive hypothesis (e), we will deduce that $\mathrm{deg}(\mathrm{tr}(R^\Lambda(\alpha)_{\geq k}) \subseteq [0, d_{\Lambda,\alpha}]$, and hence that $d \in [0, d_{\Lambda,\alpha}]$ or $v = 0$, proving the theorem.

By (28) and (30), there is a finitely generated $(P \times \mathbb{Z})$-graded $\mathrm{Z}^{k,[d_i]}$-algebra B_k such that $B_{k,\mu} = 0$ if $\mu_i \neq k$ and a surjective $\mathbb{Z}$-graded $L\mathfrak{e}^{[d_i]}_{\mathrm{Z}^k}$-module homomorphism

$$\mathrm{tr}(\mathcal{V}^\Lambda_k/\mathbb{Z}) \simeq \bigoplus_{n\in\mathbb{N}} \mathrm{tr}(\underline{\mathrm{H}}^{k,[d_i]}_n) \otimes_{\mathrm{Z}^{k,[d_i]}} \mathrm{tr}(B_k) \to \mathrm{tr}(R^\Lambda)_k.$$

Let $M_{k,\mu}$ be the image of $\mathrm{tr}(\underline{\mathrm{H}}^{k,[d_i]}_0) \otimes_{\mathrm{Z}^{k,[d_i]}} \mathrm{tr}(B_{k,\mu})$ in $\mathrm{tr}(R^\Lambda)_k$. Set $M_k = \bigoplus_\mu M_{k,\mu}$. By formula (31) in the proof of Proposition 3.30, we have

$$\mathrm{tr}(\underline{\mathrm{H}}^{k,[d_i]}_0) = \mathrm{Z}^{k,[d_i]}, \qquad \sum_{\mu\in\Lambda^+(n)} x^-_{i\mu}\big(\mathrm{tr}(\underline{\mathrm{H}}^{k,[d_i]}_0)\big) = \mathrm{tr}(\underline{\mathrm{H}}^{k,[d_i]}_n). \tag{32}$$

Thus, there is a (unique) surjective $\mathbb{Z}$-graded $\Bbbk$-vector space homomorphism

$$\bigoplus_{n\in\mathbb{N}} \mathrm{tr}(\underline{\mathrm{H}}^{k,[d_i]}_n) \otimes_{\mathrm{Z}^{k,[d_i]}} M_k \to \mathrm{tr}(R^\Lambda)_k \tag{33}$$

such that $x^-_{i\mu}(f) \otimes u \mapsto f x^-_{i\mu}(u)$ for each $f \in \Bbbk$, $\mu \in \Lambda^+(n)$, $u \in M_k$. Note that, by the definition of k, we have

$$k = \lambda_i + 2m = \langle \alpha_i^\vee, \lambda + m\alpha_i \rangle.$$

Further, for each $n \in \mathbb{N}$ the map (33) yields a surjective map

$$\mathrm{tr}(\underline{\mathrm{H}}_n^{k,[d_i]}) \otimes_{\mathrm{Z}^{k,[d_i]}} M_{k,\lambda+m\alpha_i} \to \mathrm{tr}\big(R^\Lambda\big(\alpha + (n-m)\alpha_i\big)\big)_k.$$

Now, a short computation yields

$$d_i d_{k,m} + d_{\Lambda,\alpha-m\alpha_i} = d_{\Lambda,\alpha}.$$

Thus by Proposition 3.30, to prove part (a) of the theorem we must check that we have

$$\deg(M_{k,\lambda+m\alpha_i}) \subseteq [0, d_{\Lambda,\alpha-m\alpha_i}].$$

Since $B_{k,\lambda+m\alpha_i}$ is the endomorphism ring of an object of $\mathcal{V}^\Lambda_{k,\lambda+m\alpha_i}$ and since $\mathrm{tr}(B_{k,\lambda+m\alpha_i})$ maps onto $M_{k,\lambda+m\alpha_i}$, we have

$$\deg(M_{k,\lambda+m\alpha_i}) \subseteq \deg\big(\mathrm{tr}\big(R^\Lambda(\alpha - m\alpha_i)\big)\big).$$

Since $m > 0$, the inductive hypothesis (d) implies that

$$\deg\big(\mathrm{tr}\big(R^\Lambda(\alpha - m\alpha_i)\big)\big) \subseteq [0, d_{\Lambda,\alpha-m\alpha_i}].$$

This finishes the proof of (a).

To prove part (b), note that by Proposition 3.10 the symmetrizing form on $R^\Lambda(\alpha)$ yields a $\Bbbk$-linear isomorphism $\mathrm{Z}(R^\Lambda(\alpha))^0 \simeq \mathrm{Hom}_\Bbbk(\mathrm{tr}(R^\Lambda(\alpha))^{d_{\Lambda,\alpha}}, \Bbbk)$. We deduce that

$$V(\Lambda)_\lambda \neq 0 \Rightarrow R^\Lambda(\alpha) \neq 0 \Rightarrow \mathrm{Z}\big(R^\Lambda(\alpha)\big)^0 \neq 0 \Rightarrow \mathrm{tr}\big(R^\Lambda(\alpha)\big)^{d_{\Lambda,\alpha}} \neq 0.$$

Finally, let us prove (c). We identify $\mathfrak{g}$ with its image in $L\mathfrak{g}$. The operators $x^\pm_{i0}$ with $i \in I$ define a representation of $\mathfrak{g}$ on $\mathrm{tr}(R^\Lambda)^d$ for each integer d. Now, we specialize $d = 0$ and we consider the $\mathfrak{e}$-action on $\mathrm{tr}(R^\Lambda)^0$.

Let $V(k)$ be the $(k+1)$-dimensional irreducible representation of $\mathfrak{e}$. We have $\mathrm{tr}(\underline{\mathrm{H}}^k_n)^0 = \Bbbk$ for each $n \in [0,k]$ and 0 if $n \notin [0,k]$. Further, the formulas in the proof of Proposition 3.30 show that there is an $\mathfrak{e}$-module isomorphism

$$\bigoplus_{n\in\mathbb{N}} \mathrm{tr}(\underline{\mathrm{H}}_n^{k,[d_i]})^0 \simeq V(k).$$

Now, the degree 0 part of $\mathrm{Z}^{k,[d_i]}$ is $\Bbbk$. Thus, the map (33) yields a surjective $\mathfrak{e}$-module homomorphism

$$\bigoplus_{n\in\mathbb{N}} \mathrm{tr}(\underline{\mathrm{H}}_n^{k,[d_i]})^0 \otimes_\Bbbk (M_k)^0 \to \mathrm{tr}(R^\Lambda)^0_k,$$

which can be viewed as a surjective $\mathfrak{e}$-module homomorphism

$$V(k) \otimes_{\Bbbk} (M_k)^0 \to \operatorname{tr}(R^\Lambda)^0_k.$$

The $\mathfrak{e}$-module $V(k)$ is generated by the highest weight subspace $V(k)_k$. Hence, the $\mathfrak{e}$-module $\operatorname{tr}(R^\Lambda)^0_k$ is generated by the subspace $V(k)_k \otimes_{\Bbbk} (M_k)^0$. We deduce that the same induction as in (d) above (based on the fact that for each indecomposable object $P \in \mathcal{V}^\Lambda_{\Lambda-\alpha}$ with $\alpha \neq 0$ we may choose $i \in I$ such that the integer $\epsilon_i(\operatorname{top}(P))$ is positive) implies that the representation of $\mathfrak{g}$ on $\operatorname{tr}(R^\Lambda)^0$ is cyclic. Hence $\operatorname{tr}(R^\Lambda)^0 \simeq V(\Lambda)$ as a $\mathfrak{g}$-module. □

3.3.2. The grading of the center

We use the same notation as in Section 3.3.1. The pairing (25) gives a $\Bbbk$-linear isomorphism

$$\mathrm{Z}\big(R^\Lambda(\alpha)\big)^d \xrightarrow{\sim} \operatorname{Hom}_{\Bbbk}\big(\operatorname{tr}\big(R^\Lambda(\alpha)\big)^{d_{\Lambda,\alpha}-d}, \Bbbk\big).$$

From Theorem 3.31 we deduce the following.

COROLLARY 3.32

If $\mathbf{k} = \Bbbk$, then we have $\mathrm{Z}(R^\Lambda(\alpha))^d = 0$ for any $d \notin [0, d_{\Lambda,\alpha}]$.

We have $\mathrm{Z}(R^\Lambda(\alpha))^0 \neq \{0\}$ whenever $V(\Lambda)_\lambda \neq 0$.

CONJECTURE 3.33

If $\mathbf{k} = \Bbbk$, then we have $\mathrm{Z}(R^\Lambda(\alpha))^0 \simeq \operatorname{tr}(R^\Lambda(\alpha))^{d_{\Lambda,\alpha}} \simeq \Bbbk$ whenever $V(\Lambda)_\lambda \neq 0$.

If $\mathfrak{g}$ is symmetric of finite type, then the conjecture holds (see Remark 3.41 below).

3.3.3. The cocenter is a cyclic module

Consider a categorical representation of $\mathfrak{g}_{\mathbf{k}}$ on a $\mathbb{Z}$-graded $\Bbbk$-category $\mathcal{C}$. Then, the $\mathbf{k}$-linear operator $x^\pm_{ir}$ on $\operatorname{tr}(\mathcal{C}/\mathbb{Z})$ is well defined for each $i \in I$, $r \in \mathbb{N}$. Let $h : \operatorname{tr}(\mathcal{C}^{\mathrm{hw}}/\mathbb{Z}) \to \operatorname{tr}(\mathcal{C}/\mathbb{Z})$ be the trace of the canonical embedding $\mathcal{C}^{\mathrm{hw}}/\mathbb{Z} \subset \mathcal{C}/\mathbb{Z}$. Let $\operatorname{tr}(\mathcal{C}/\mathbb{Z})^{\mathrm{cyc}} \subset \operatorname{tr}(\mathcal{C}/\mathbb{Z})$ be the $\mathbf{k}$-submodule generated by the image of h under the action of all operators $x^\pm_{ir}$.

PROPOSITION 3.34

If $\mathcal{C} = \mathcal{V}^\Lambda$, then we have $\operatorname{tr}(\mathcal{C}/\mathbb{Z}) = \operatorname{tr}(\mathcal{C}/\mathbb{Z})^{\mathrm{cyc}}$.

The ith isotypic filtration of $\mathcal{C}$ yields a categorical $\mathfrak{e}_{Z^k}$-representation $\mathcal{C}_k$ for each $k \geq 0$. Let us quote the following consequence of (30) and Proposition 3.30.

LEMMA 3.35
For all $i \in I$, $k \geq 0$ we have $\operatorname{tr}(\mathcal{C}_k/\mathbb{Z}) = \operatorname{tr}(\mathcal{C}_k/\mathbb{Z})^{\mathrm{cyc}}$.

Now, we can prove Proposition 3.34.

Proof
Fix $\alpha \in Q_+$, and set $\lambda = \Lambda - \alpha$. Fix an indecomposable object $P \in \mathcal{C}_\lambda^\Lambda$ and an element $v_P \in \operatorname{tr}(\operatorname{End}_{\mathcal{V}^\Lambda/\mathbb{Z}}(P))$. Let v be the image of v_P in $\operatorname{tr}(R^\Lambda(\alpha))$. We must prove that $v \in \operatorname{tr}(\mathcal{C}_\lambda/\mathbb{Z})^{\mathrm{cyc}}$. Since $\mathcal{C} = \mathcal{V}^\Lambda$, we have $\operatorname{tr}(\mathcal{C}_\lambda/\mathbb{Z}) = \operatorname{tr}(R^\Lambda(\alpha))$. Since $\operatorname{tr}(R^\Lambda(0)) \subseteq \operatorname{tr}(R^\Lambda)^{\mathrm{cyc}}$, we may assume that $\alpha \neq 0$. Let i, m, k_0 be as in the proof of Theorem 3.31. The proof is an induction on α and a descending induction on m.

We have $v \in \operatorname{tr}(R^\Lambda(\alpha))_{\geq k_0}$, and we will assume that

- $\deg(\operatorname{tr}(R^\Lambda(\beta))) \subset \operatorname{tr}(R^\Lambda)^{\mathrm{cyc}}$ for each $\beta \in Q_+$ such that $\beta < \alpha$;
- $\operatorname{tr}(R^\Lambda(\alpha))_{>k_0} \subset \operatorname{tr}(R^\Lambda)^{\mathrm{cyc}}$.

We must check that $\operatorname{tr}(R^\Lambda(\alpha))_{\geq k_0} \subset \operatorname{tr}(R^\Lambda)^{\mathrm{cyc}}$.

Fix β as above and fix an element $x \in U(L\mathfrak{e}_{\mathbf{k}})$ of weight $\alpha - \beta$. Consider the following commutative diagram, whose rows are exact sequences of $L\mathfrak{e}_{\mathbf{k}}$-modules:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{tr}\big(R^\Lambda(\alpha)\big)_{>k_0} & \xrightarrow{f} & \operatorname{tr}\big(R^\Lambda(\alpha)\big)_{\geq k_0} & \xrightarrow{g} & \operatorname{tr}\big(R^\Lambda(\alpha)\big)_{k_0} & \longrightarrow & 0 \\
 & & & & x\Big\uparrow & & x\Big\uparrow & & \\
 & & & & \operatorname{tr}\big(R^\Lambda(\beta)\big)_{\geq k_0} & \xrightarrow{g} & \operatorname{tr}\big(R^\Lambda(\beta)\big)_{k_0} & \longrightarrow & 0
\end{array} \tag{34}$$

Now, set $\bar{v} = g(v) \in \operatorname{tr}(R^\Lambda(\alpha))_{k_0}$. By the definition of $\operatorname{tr}(R^\Lambda)_{k_0}$, there is a surjective $U(L\mathfrak{e}_{\mathbf{k}})$-module homomorphism $\operatorname{tr}(\mathcal{V}_{k_0}^\Lambda/\mathbb{Z}) \to \operatorname{tr}(R^\Lambda)_{k_0}$. Hence, by Lemma 3.35, we can choose β, x such that there is an element $\bar{u} \in \operatorname{tr}(R^\Lambda(\beta))_{k_0}$ with $\bar{v} = x \cdot \bar{u}$. Then, fix $u \in \operatorname{tr}(R^\Lambda(\beta))_{\geq k_0}$ such that $g(u) = \bar{u}$. Then, we have $v - x \cdot u = f(v')$ for some $v' \in \operatorname{tr}(R^\Lambda(\alpha))_{>k_0}$.

Finally, we can apply both recursive hypotheses. We deduce that $u \in \operatorname{tr}(R^\Lambda(\beta)) \subset \operatorname{tr}(R^\Lambda)^{\mathrm{cyc}}$ and $v' \in \operatorname{tr}(R^\Lambda(\alpha))_{>k_0} \subset \operatorname{tr}(R^\Lambda)^{\mathrm{cyc}}$. Hence, we have $v = v - x \cdot u + x \cdot u = f(v') + x \cdot u \in \operatorname{tr}(R^\Lambda)^{\mathrm{cyc}}$. $\square$

3.4. The symmetric case

Let $\mathbf{k}$ be a commutative $\mathbb{N}$-graded ring as in Section 3.1 with $\Bbbk$ of characteristic 0. Let $\mathfrak{g}$ be the symmetrizable Kac–Moody algebra over $\Bbbk$ associated with the Cartan datum $(P, P^\vee, \Phi, \Phi^\vee)$.

3.4.1. Weyl modules

Assume that $\mathfrak{g}$ is of finite type, and fix a dominant weight Λ. The *local Weyl module* $\mathrm{W}(\Lambda)$ (over $\Bbbk$) is the $\mathbb{Z}$-graded $L\mathfrak{g}$-module generated by a nonzero element $|\Lambda\rangle$ of degree 0 with the following defining relations:

- $\mathfrak{n}^+[t] \cdot |\Lambda\rangle = 0$;
- $(f_i)^{\Lambda_i+1} \cdot |\Lambda\rangle = 0$;
- $h \cdot |\Lambda\rangle = \langle h, \Lambda\rangle |\Lambda\rangle$ for all $h \in \mathfrak{h}$;
- $t\mathfrak{h}[t] \cdot |\Lambda\rangle = 0$.

The *global Weyl module* $\underline{\mathrm{W}}(\Lambda)$ is the $\mathbb{Z}$-graded $L\mathfrak{g}$-module generated by a nonzero element $|\Lambda\rangle$ satisfying the first three relations above. Consider the formal series $\Psi_i(z) = \sum_{r\geq 0} \Psi_{ir} z^r$, $i \in I$, given by $\Psi_i(z) = \exp(-\sum_{r\geq 1} h_{ir} z^r / r)$. Then, there is a unique $\underline{\mathbf{k}}$-module structure on $\underline{\mathrm{W}}(\Lambda)$ such that the representation of $L\mathfrak{g}$ is $\underline{\mathbf{k}}$-linear and we have $\Psi_{ip} \cdot |\Lambda\rangle = c_{ip} |\Lambda\rangle$ for each $(i, p) \in I \times \mathbb{N}$. For any $\Bbbk$-algebra homomorphism $\underline{\mathbf{k}} \to \mathbf{k}$, we set $\mathrm{W}(\Lambda, \mathbf{k}) = \mathbf{k} \otimes_{\underline{\mathbf{k}}} \underline{\mathrm{W}}(\Lambda)$.

The Weyl modules are *universal* in the following sense. Let M be a $\mathbb{Z}$-graded integrable $L\mathfrak{g}_{\mathbf{k}}$-module containing an element m of weight Λ which is annihilated by $\mathfrak{n}^+[t]$. Then there is a unique $\underline{\mathbf{k}}$-algebra structure on $\mathbf{k}$ and a unique $\mathbb{Z}$-graded $L\mathfrak{g}_{\mathbf{k}}$-module homomorphism $\mathrm{W}(\Lambda, \mathbf{k}) \to M$ such that $|\Lambda\rangle \mapsto m$.

Let $\Lambda_{\min}$ be the unique minimal element in the poset $\{\lambda \in P_+; \lambda \leq \Lambda\}$. Let $\underline{\mathbf{h}}'$ be as in Section 3.2.3. The following is well known (see [31], [25, Theorem 1.1]).

PROPOSITION 3.36

(a) $\dim_{\Bbbk}(\mathrm{W}(\Lambda)) < \infty$.

(b) $\underline{\mathrm{W}}(\Lambda)$ *is a free* $\underline{\mathbf{k}}$*-module of finite rank.*

(c) $\mathrm{top}(\mathrm{W}(\Lambda)) = \mathrm{W}(\Lambda)^0 \simeq V(\Lambda)$ *as a* $\mathbb{Z}$*-graded* $\mathfrak{g}$*-module.*

(d) *If* $\mathfrak{g}$ *is symmetric, then*

$$\mathrm{soc}\big(\mathrm{W}(\Lambda)\big) = \mathrm{W}(\Lambda)^{d_{\Lambda,\Lambda_{\min}}} \simeq V(\Lambda_{\min})[-d_{\Lambda,\Lambda_{\min}}]$$

as a $\mathbb{Z}$*-graded* $\mathfrak{g}$*-module.*

(e) $\mathrm{W}(\Lambda, \Bbbk) \xrightarrow{\sim} \mathrm{W}(\Lambda)$ *as a* $\mathbb{Z}$*-graded* $L\mathfrak{g}$*-module.*

(f) $\Lambda = \omega_i \Rightarrow \underline{\mathrm{W}}(\Lambda) \simeq \underline{\mathbf{k}} \otimes_{\Bbbk} \mathrm{W}(\Lambda)$ *as a* $\mathbb{Z}$*-graded* $L\mathfrak{g}_{\underline{\mathbf{k}}}$*-module.*

(g) $\underline{\mathrm{W}}(\Lambda, \underline{\mathbf{h}}') \simeq \underline{\mathbf{h}}' \otimes_{\underline{\mathbf{h}}} \bigotimes_i \underline{\mathrm{W}}(\omega_i)^{\otimes \Lambda_i}$ *as* $L\mathfrak{g}_{\underline{\mathbf{h}}'}$*-modules such that* $|\Lambda\rangle \mapsto 1 \otimes \bigotimes_i (w_{\omega_i})^{\otimes \Lambda_i}$.

Now, assume that $\mathfrak{g}$ is symmetric and (11) is satisfied. We consider the $\mathbb{Z}$-graded representation of $L\mathfrak{g}_{\mathbf{k}}$ on $\mathrm{tr}(R^\Lambda)$ in Theorem 3.25.

THEOREM 3.37
If $\mathfrak{g}$ is symmetric of finite type and (11) is satisfied, then there is a unique $\mathbb{Z}$-graded $L\mathfrak{g}_{\mathbf{k}}$-module isomorphism $\mathrm{W}(\Lambda,\mathbf{k}) \xrightarrow{\sim} \mathrm{tr}(R^\Lambda)$ *such that* $|\Lambda\rangle \mapsto |\Lambda\rangle$.

Proof
From Proposition 3.34, we deduce that the element $|\Lambda\rangle = \mathrm{tr}(1)$ of $\mathrm{tr}(R^\Lambda(0))$ is a generator of the $L\mathfrak{g}_{\mathbf{k}}$-module $\mathrm{tr}(R^\Lambda)$. Thus, it is enough to prove that there is a $\mathbb{Z}$-graded $L\mathfrak{g}_{\underline{\mathbf{k}}}$-module isomorphism $\underline{\mathrm{W}}(\Lambda) \xrightarrow{\sim} \mathrm{tr}(\underline{R}^\Lambda)$ such that $|\Lambda\rangle \mapsto |\Lambda\rangle$. To do this, note that, since $\underline{\mathrm{W}}(\Lambda)$ is universal, there is a unique $\mathbb{Z}$-graded $L\mathfrak{g}_{\underline{\mathbf{k}}}$-module homomorphism

$$\underline{\phi}^\Lambda : \underline{\mathrm{W}}(\Lambda) \to \mathrm{tr}(\underline{R}^\Lambda), \quad |\Lambda\rangle \mapsto |\Lambda\rangle. \tag{35}$$

By Proposition 3.34, we deduce that $\underline{\phi}^\Lambda$ is surjective.

First, we consider the map $\phi^\Lambda : \mathrm{W}(\Lambda) \to \mathrm{tr}(R^\Lambda(\Bbbk))$ given by $\phi^\Lambda = 1 \otimes \underline{\phi}^\Lambda$. Since $\underline{\phi}^\Lambda$ is surjective, the map ϕ^Λ is also surjective. To prove that it is injective, we must check that $\phi^\Lambda(\mathrm{soc}(\mathrm{W}(\Lambda))) \neq 0$. Since ϕ^Λ is surjective and $\mathrm{soc}(\mathrm{W}(\Lambda)) \simeq V(\Lambda_{\min})[-d_{\Lambda,\Lambda_{\min}}]$ as a $\mathbb{Z}$-graded $L\mathfrak{g}$-module, it is enough to prove that $\mathrm{tr}(R^\Lambda(\Bbbk))^{d_{\Lambda,\Lambda_{\min}}} \neq 0$. The weight subspace $V(\Lambda)_{\Lambda_{\min}}$ is nonzero. Thus the injectivity of ϕ^Λ follows from Theorem 3.31(b).

Now, we prove that $\underline{\phi}^\Lambda$ is injective. To do so, since $\underline{\mathrm{W}}(\Lambda)$ is a free $\underline{\mathbf{k}}$-module and since ϕ^Λ is invertible, it is enough to check that $\mathrm{tr}(\underline{R}^\Lambda)$ is free as a $\underline{\mathbf{k}}$-module. To do so, note that by Theorem 3.15 and Example 3.5(d), the $\underline{\mathbf{h}}'$-algebras $R^\Lambda(\underline{\mathbf{h}}')$ and $\underline{\mathbf{h}}' \otimes_\Bbbk \bigotimes_i R^{\omega_i}(\Bbbk)^{\otimes \Lambda_i}$ are Morita-equivalent. We deduce that there is an $\underline{\mathbf{h}}'$-linear isomorphism

$$\mathrm{tr}\big(R^\Lambda(\underline{\mathbf{h}}')\big) \to \underline{\mathbf{h}}' \otimes_\Bbbk \bigotimes_i \mathrm{tr}\big(R^{\omega_i}(\Bbbk)\big)^{\otimes \Lambda_i}, \quad |\Lambda\rangle \mapsto 1 \otimes \bigotimes_i |\omega_i\rangle^{\otimes \Lambda_i}. \tag{36}$$

Further, by Theorem 3.15, the map (36) is $L\mathfrak{g}_{\underline{\mathbf{h}}'}$-linear. Next, by Proposition 3.36(g) we have an $L\mathfrak{g}_{\underline{\mathbf{h}}'}$-module isomorphism

$$\mathrm{W}(\Lambda,\underline{\mathbf{h}}') \to \underline{\mathbf{h}}' \otimes_\Bbbk \bigotimes_i \mathrm{W}(\omega_i)^{\otimes \Lambda_i}, \quad |\Lambda\rangle \mapsto 1 \otimes \bigotimes_i |\omega_i\rangle^{\otimes \Lambda_i}. \tag{37}$$

Since the maps (36) and (37) are $L\mathfrak{g}_{\underline{\mathbf{h}}'}$-linear, from the unicity of the morphism $\underline{\phi}^\Lambda$ in (35) we deduce that the following square is commutative:

$$\begin{array}{ccc} \mathrm{W}(\Lambda, \underline{\mathbf{h}}') & \xrightarrow{(37)} & \underline{\mathbf{h}}' \otimes_{\mathbb{k}} \bigotimes_i \mathrm{W}(\omega_i)^{\otimes \Lambda_i} \\ {\scriptstyle 1\otimes \underline{\phi}^{\Lambda}} \downarrow & & \downarrow {\scriptstyle 1\otimes \bigotimes_i (\phi^{\omega_i})^{\otimes \Lambda_i}} \\ \mathrm{tr}\big(R^{\Lambda}(\underline{\mathbf{h}}')\big) & \xrightarrow{(36)} & \underline{\mathbf{h}}' \otimes_{\mathbb{k}} \bigotimes_i \mathrm{tr}\big(R^{\omega_i}(\mathbb{k})\big)^{\otimes \Lambda_i} \end{array}$$

Since ϕ^{ω_i} is an isomorphism for each i, this implies that the map $1 \otimes \underline{\phi}^{\Lambda}$ is also an isomorphism.

Finally, we must check that $\underline{\phi}^{\Lambda}$ is an isomorphism. To do so, note that by construction the map $\underline{\phi}^{\Lambda}$ preserves the weight decomposition of $\underline{\mathrm{W}}(\Lambda)$, $\mathrm{tr}(\underline{R}^{\Lambda})$. Therefore, the claim follows from the following lemma.

LEMMA 3.38
Let $\underline{\psi} : M \to N$ be a $\mathbb{Z}$-graded $\underline{\mathbf{k}}$-module homomorphism such that M, N are both finitely generated. Assume that the maps $1 \otimes \underline{\psi} : \mathbb{k} \otimes_{\underline{\mathbf{k}}} M \to \mathbb{k} \otimes_{\underline{\mathbf{k}}} N$ and $1 \otimes \underline{\psi} : \underline{\mathbf{h}}' \otimes_{\underline{\mathbf{k}}} M \to \underline{\mathbf{h}}' \otimes_{\underline{\mathbf{k}}} N$ are invertible. Then $\underline{\psi}$ is also invertible. □

3.4.2. Equivariant homology
For any complex algebraic variety X and any commutative ring $\mathbb{k}$, let $H_*(X, \mathbb{k})$ be the Borel–Moore homology with coefficients in $\mathbb{k}$. Given an action of a complex linear algebraic group G on X, let $H_*^G(X, \mathbb{k})$ be the G-equivariant Borel–Moore homology. We will assume that X admits a locally closed G-equivariant embedding into a smooth projective G-variety. We define it as in [28, Section 2.8], but we assign the degree as in [12], so that the fundamental class $[X]$ of X has degree $2 \dim X$ if X is pure-dimensional.

Alternatively, let $D^G(X, \mathbb{k})$ be the equivariant derived category of constructible complexes on X with coefficients in $\mathbb{k}$ (see [6]). Let $\mathbb{k}_X$ and $\mathbb{k}_X^D$ be the constant and the dualizing sheaf on X, respectively. These are objects of $D^G(X, \mathbb{k})$. If M is in $D^G(X, \mathbb{k})$, then the ith equivariant cohomology of Y with coefficients in M is by definition $H_G^i(X, M) = \mathrm{Ext}^i(\mathbb{k}_X, M)$. In particular, the G-equivariant cohomology and Borel–Moore homology of X are defined by

$$H_G^i(X, \mathbb{k}) = H_G^i(X, \mathbb{k}_X), \qquad H_i^G(X, \mathbb{k}) = H_G^{-i}(X, \mathbb{k}_Y^D).$$

Note that with our conventions, one can have $H_i^G(X, \mathbb{k}) \neq 0$ for $i < 0$. We will abbreviate

$$H^i(X, \mathbb{k}) = H_{\{1\}}^i(X, \mathbb{k}), \qquad H_i(X, \mathbb{k}) = H_i^{\{1\}}(X, \mathbb{k}).$$

The action of the cohomology on the Borel–Moore homology yields a map

$$\cap : H^i_G(X,\Bbbk) \otimes_{\Bbbk} H^G_j(X,\Bbbk) \to H^G_{j-i}(X,\Bbbk).$$

If X is smooth and pure-dimensional, then the cap product with $[X]$ yields an isomorphism

$$H^i_G(X,\Bbbk) \to H^G_{2\dim X-i}(X,\Bbbk).$$

3.4.3. Quiver varieties

Assume that $\mathfrak{g}$ is symmetric. Following Nakajima, to each $\Lambda \in P_+$ and $\alpha \in Q_+$ we associate a quiver variety $\mathfrak{M}(\Lambda,\alpha)$. It is a complex quasiprojective variety equipped with an action of the complex algebraic group $G_\Lambda = \prod_{i\in I} \mathrm{GL}(\Lambda_i)$. Recall the following:

- $\mathfrak{M}(\Lambda,\alpha)$ is nonsingular, symplectic, possibly empty, and is equipped with a G_Λ-equivariant projective morphism to an affine variety $p : \mathfrak{M}(\Lambda,\alpha) \to \mathfrak{M}_0(\Lambda,\alpha)$.
- The variety $\mathfrak{M}_0(\Lambda,\alpha)$ has a distinguished point denoted by 0 such that the closed subvariety $\mathfrak{L}(\Lambda,\alpha) = p^{-1}(0)$ of $\mathfrak{M}(\Lambda,\alpha)$ is Lagrangian.
- If $\mathfrak{M}(\Lambda,\alpha) \neq \emptyset$, then its dimension is $d_{\Lambda,\alpha}$.

We put $\mathfrak{M}(\Lambda) = \bigsqcup_\alpha \mathfrak{M}(\Lambda,\alpha)$ and $\mathfrak{L}(\Lambda) = \bigsqcup_\alpha \mathfrak{L}(\Lambda,\alpha)$. We have a canonical $\Bbbk$-algebra isomorphism $H^*_{G_\Lambda}(\bullet,\Bbbk) = \underline{\mathbf{k}}$. Under this isomorphism, the equivariant Euler class of the pth fundamental representation of $\mathrm{GL}(\Lambda_i)$ maps to c_{ip} for any $p \geq 0$. We define

$$\begin{aligned} H^*_{G_\Lambda}\big(\mathfrak{M}(\Lambda),\Bbbk\big) &= \bigoplus_{d\geq 0} H^d_{G_\Lambda}\big(\mathfrak{M}(\Lambda),\Bbbk\big), \\ H^{G_\Lambda}_{[*]}\big(\mathfrak{L}(\Lambda),\Bbbk\big) &= \bigoplus_{d\geq 0} H^{G_\Lambda}_{[d]}\big(\mathfrak{L}(\Lambda),\Bbbk\big) \\ &= \bigoplus_{d\geq 0}\bigoplus_{\alpha} H^{G_\Lambda}_{d_{\Lambda,\alpha}-d}\big(\mathfrak{L}(\Lambda,\alpha),\Bbbk\big). \end{aligned}$$

Assume also that $\mathfrak{g}$ is of finite type. The following is well known.

PROPOSITION 3.39

We have the following:

(a) $H_{[d]}(\mathfrak{L}(\Lambda),\Bbbk) = 0$ *if* $d \notin (2\mathbb{Z}) \cap [0, d_{\Lambda,\alpha}]$.

(b) $H^{G_\Lambda}_{[*]}(\mathfrak{L}(\Lambda),\Bbbk)$ *is free over* $\underline{\mathbf{k}}$ *and* $H_{[*]}(\mathfrak{L}(\Lambda),\Bbbk) = \Bbbk \otimes_{\underline{\mathbf{k}}} H^{G_\Lambda}_{[*]}(\mathfrak{L}(\Lambda),\Bbbk)$.

(c) *There is a perfect* $\underline{\mathbf{k}}$*-bilinear pairing* $H^d_{G_\Lambda}(\mathfrak{M}(\Lambda),\Bbbk) \times H^{G_\Lambda}_{[d_{\Lambda,\alpha}-d]}(\mathfrak{L}(\Lambda),\Bbbk) \to \underline{\mathbf{k}}$.

(d) *There is a $\mathbb{Z}$-graded $L\mathfrak{g}_{\underline{\mathbf{k}}}$-representation on $H^{G_\Lambda}_{[*]}(\mathfrak{L}(\Lambda),\Bbbk)$ which is isomorphic to $\underline{\mathrm{W}}(\Lambda)$. Under this isomorphism $H^{G_\Lambda}_{[*]}(\mathfrak{L}(\Lambda,\alpha),\Bbbk)$ maps to $\underline{\mathrm{W}}(\Lambda)_{\Lambda-\alpha}$.*

Proof
Parts (b) and (c) are proved in [32, Theorem 7.3.5] for equivariant K-theory, but the same proof applies also to equivariant Borel–Moore homology. Part (d) follows from [33]. □

From Theorem 3.37 we deduce the following.

THEOREM 3.40
If $\mathfrak{g}$ is symmetric of finite type and (11) is satisfied, then there are $\mathbb{Z}$-graded $L\mathfrak{g}_{\mathbf{k}}$-module isomorphisms $\mathrm{tr}(R^\Lambda) \simeq \mathbf{k} \otimes_{\underline{\mathbf{k}}} H^{G_\Lambda}_{[*]}(\mathfrak{L}(\Lambda),\Bbbk)$.

Remark 3.41
(a) Since $\mathfrak{L}(\Lambda,\alpha)$ is connected, we deduce from Theorem 3.40 that $\mathrm{tr}(R^\Lambda(\alpha))^{d_{\Lambda,\alpha}} \simeq \Bbbk$ whenever $V(\Lambda)_\lambda \neq 0$ if $\mathfrak{g}$ is symmetric and of finite type.

(b) We equip $\mathrm{Z}(R^\Lambda)$ with the $L\mathfrak{g}_{\mathbf{k}}$-representation dual to $\mathrm{tr}(R^\Lambda)$ relatively to the pairing (25) and the anti-involution ϖ of $L\mathfrak{g}_{\mathbf{k}}$. The action of $x^{\pm}_{ir}$ on $\mathrm{Z}(R^\Lambda)$ is described in Proposition 3.28. Next, we equip $H^*_{G_\Lambda}(\mathfrak{M}(\Lambda),\Bbbk)$ with the $L\mathfrak{g}_{\underline{\mathbf{k}}}$-representation dual to $H^{G_\Lambda}_{[*]}(\mathfrak{L}(\Lambda),\Bbbk)$ relatively to the pairing in (c) above and the anti-involution ϖ. Then, from Theorem 3.40 we deduce that there is a $\mathbb{Z}$-graded $L\mathfrak{g}_{\mathbf{k}}$-module isomorphism

$$\mathbf{k} \otimes_{\underline{\mathbf{k}}} H^*_{G_\Lambda}\big(\mathfrak{M}(\Lambda),\Bbbk\big) \xrightarrow{\sim} \mathrm{Z}(R^\Lambda). \tag{38}$$

3.4.4. The multiplicative structure on $H^*_{G_\Lambda}(\mathfrak{M}(\Lambda),\Bbbk)$
In this section, we explain how to construct a $\mathbf{k}$-algebra isomorphism $\mathrm{Z}(R^\Lambda) \simeq \mathbf{k} \otimes_{\underline{\mathbf{k}}} H^*_{G_\Lambda}(\mathfrak{M}(\Lambda),\Bbbk)$ from the $L\mathfrak{g}_{\mathbf{k}}$-linear isomorphism (38). To simplify, we will assume that condition (11) holds.

We first introduce a $\mathbf{k}$-linear map

$$a : \mathrm{Z}(R) \to \mathbf{k} \otimes_{\underline{\mathbf{k}}} H^*_{G_\Lambda}\big(\mathfrak{M}(\Lambda),\Bbbk\big),$$

which is usually called the *Kirwan map*. Recall that, for each $\alpha \in Q_+$, $i \in I$, the G_Λ-variety $\mathfrak{M}(\Lambda,\alpha)$ is equipped with G_Λ-equivariant bundles $\mathcal{V}_i$, $\mathcal{W}_i$ of rank a_i, Λ_i, respectively, defined as in [32, Section 2.9], where a_i is the coordinate of α along the simple root α_i. On the other hand, if $\mathrm{ht}(\alpha) = n$, then the center of $R(\alpha)$ is given by

$$\mathrm{Z}\big(R(\alpha)\big) = \Big(\bigoplus_{\nu \in I^\alpha} \mathbf{k}[x_1,\dots,x_n]e(\nu)\Big)^{\mathfrak{S}_n},$$

where the symmetric group acts on $\mathbf{k}[x_1, \dots, x_n]$ and $e(\nu)$ in the obvious way. Then, the Kirwan map is the unique $\Bbbk$-algebra homomorphism such that

$$a(c_{ip}) = (-1)^p c_p(\mathcal{W}_i), \quad \forall (i,p) \in I_\Lambda,$$

$$a\Big(\sum_{\nu \in I^\alpha} \prod_{k=1}^{n} (1 - z x_k)^{-a_{i,\nu_k}} e(\nu)\Big) = \prod_{j \in I} c_z(\mathcal{V}_j)^{-a_{i,j}},$$

where c_p and $c_z = \sum_p (-z)^p c_p$ are the pth G_Λ-equivariant Chern class and the G_Λ-equivariant Chern polynomial, respectively.

Next, let $b : \mathrm{Z}(R) \to \mathrm{Z}(R^\Lambda)$ be the canonical map induced by the quotient map $R \to R^\Lambda$.

Finally, let ψ be the unique isomorphism as in (38), which takes the unit of the ring $\mathbf{k} \otimes_{\underline{\mathbf{k}}} H^*_{G_\Lambda}(\mathfrak{M}(\Lambda, 0), \Bbbk)$ to the unit of $\mathrm{Z}(R^\Lambda(0))$.

PROPOSITION 3.42
Assume that the Kirwan map a is surjective.
(a) *The element $\psi(1) \in \mathrm{Z}(R^\Lambda(\alpha))$ is invertible, and the map $\phi : \mathbf{k} \otimes_{\underline{\mathbf{k}}} H^*_{G_\Lambda}(\mathfrak{M}(\Lambda, \alpha), \Bbbk) \to \mathrm{Z}(R^\Lambda(\alpha))$ given by $\phi = \psi(1)^{-1} \cdot \psi$ is a $\mathbf{k}$-algebra isomorphism.*
(b) *The map b is surjective.*

Proof
Consider the formal series $\Psi_i(z) = \exp(-\sum_{r \geq 1} h_{ir} z^r / r)$ introduced in Section 3.4.1. Set $\lambda = \Lambda - \alpha$. Since condition (11) holds, the formal power series $B_{-i,\lambda}(z) \in \mathrm{Z}(R^\Lambda(\alpha))[[z]]$ in Section B.2 is given by the following formula (see Proposition B.3):

$$B_{-i,\lambda}(z) = b\Big(\sum_{\nu \in I^\alpha} \prod_{p=1}^{\Lambda} (1 + z y_{ip}) \prod_{k=1}^{n} (1 - z x_k)^{-a_{i,\nu_k}} e(\nu)\Big).$$

Here, the y_{ip}'s are formal variables as in Section 3.2.3. Further, by Lemma B.4, under the representation of $L\mathfrak{g}_{\mathbf{k}}$ on $\mathrm{Z}(R^\Lambda)$, the formal series $\Psi_i(z)$ acts on $\mathrm{Z}(R^\Lambda(\alpha))$ by multiplication by $B_{-i,\lambda}(z)$. Next, by [32, Section 9.2], under the representation of $L\mathfrak{g}_{\mathbf{k}}$ on $\mathbf{k} \otimes_{\underline{\mathbf{k}}} H^*_{G_\Lambda}(\mathfrak{M}(\Lambda), \Bbbk)$, the formal series $\Psi_i(z)$ acts by multiplication by

$$c_z(\mathcal{W}_i) \cup \prod_j c_z(\mathcal{V}_j)^{-a_{i,j}}.$$

Since, by definition, the map ψ is $L\mathfrak{g}_{\mathbf{k}}$-linear, we deduce that

$$b\Big(\sum_{\nu \in I^\alpha} \prod_{p=1}^{\Lambda} (1 + z y_{ip}) \prod_{k=1}^{n} (1 - z x_k)^{-a_{i,\nu_k}} e(\nu)\Big) \cdot \psi(\bullet)$$

$$= \psi\Big(c_z(\mathcal{W}_i) \cup \prod_j c_z(\mathcal{V}_j)^{-a_{i,j}} \cup \bullet\Big)$$

$$= \psi\Big(a\Big(\sum_{\nu \in I^\alpha} \prod_{p=1}^{\Lambda} (1 + zy_{ip}) \prod_{k=1}^{n} (1 - zx_k)^{-a_{i,\nu_k}} e(\nu)\Big) \cup \bullet\Big).$$

We deduce that, for each $z \in \mathrm{Z}(R)$, we have $b(z) \cdot \psi(\bullet) = \psi(a(z) \cup \bullet)$. Thus, we have $b(z) \cdot \psi(1) = \psi(a(z))$. Since ψ and a are surjective, there is an element z such that $b(z) \cdot \psi(1) = 1$, from which we deduce that $\psi(1)$ is invertible. Furthermore, for each $z, z' \in \mathrm{Z}(R)$, we have $b(z) \cdot b(z') \cdot \psi(1) = \psi(a(z) \cup a(z'))$. Hence, the map ϕ is an algebra isomorphism and the diagram below is commutative:

$$\begin{array}{ccc} & \mathrm{Z}\big(R(\alpha)\big) & \\ {}^{a}\swarrow & & \searrow^{b} \\ \mathbf{k} \otimes_{\underline{\mathbf{k}}} H^*_{G_\Lambda}\big(\mathfrak{M}(\Lambda,\alpha), \Bbbk\big) & \xrightarrow{\ \ \phi\ \ } & \mathrm{Z}\big(R^\Lambda(\alpha)\big) \end{array}$$

The lemma is proved. □

4. The Jordan quiver

We expect that the results above could be generalized to quivers with loops using the generalized quiver Hecke algebras introduced in [17]. In this section, we consider the particular case of the Jordan quiver. In this particular case, the quiver Hecke algebra in [17] is the degenerate affine Hecke algebra of the symmetric group.

From now on, let $\Bbbk$ be a commutative domain, let $\mathbf{k} = \Bbbk[\hbar, y_1, \dots, y_r]$, and let $\mathbf{k}'$ be the fraction field of $\mathbf{k}$. The ring $\mathbf{k}$ is $\mathbb{Z}$-graded with $\deg(y_p) = \deg(\hbar) = 2$. For any $\mathbf{k}$-module M, write

$$M' = \mathbf{k}' \otimes_{\mathbf{k}} M.$$

If M is free $\mathbb{Z}$-graded of finite rank, then let $\mathrm{grdim}(M)$ be its *graded rank*. It is the unique element in $\mathbb{N}[t, t^{-1}]$ such that $\mathrm{grdim}(\mathbf{k}[d]) = t^{-d}$ and $\mathrm{grdim}(M \oplus N) = \mathrm{grdim}(M) + \mathrm{grdim}(N)$.

4.1. The quiver Hecke algebra

For any integer $n > 0$, the quiver Hecke algebra of rank n associated with the Jordan quiver is the degenerate affine Hecke $\mathbf{k}$-algebra $R(n)$, which is generated by elements $\tau_1, \dots, \tau_{n-1}, x_1, \dots, x_n$ with the defining relations

(a) $x_k x_l = x_l x_k$,

(b) $\tau_k \tau_l = \tau_l \tau_k$ if $|k - l| > 1$,

(c) $\tau_l^2 = 1$,

(d) $\tau_k x_l - x_{s_k(l)}\tau_k = \hbar(\delta_{l,k+1} - \delta_{l,k})$,

(e) $\tau_{k+1}\tau_k\tau_{k+1} = \tau_k\tau_{k+1}\tau_k$.

We write $R(0) = \mathbf{k}$. For any integer $r \geq 0$, the cyclotomic quiver Hecke algebra of rank n and level r is the quotient $R^r(n)$ of $R(n)$ by the two-sided ideal generated by the element $\prod_{p=1}^r (x_1 - y_p)$. The $\mathbf{k}$-algebras $R(n)$, $R^r(n)$ are $\mathbb{Z}$-graded, with $\deg(\tau_k) = 0$ and $\deg(x_k) = 2$.

The canonical map $R(n) \to R^r(n)$ yields a $\mathbb{Z}$-graded $\mathbf{k}$-algebra homomorphism $\mathrm{Z}(R(n)) \to \mathrm{Z}(R^r(n))$. Let $\mathrm{Z}(R^r(n))^{\mathrm{JM}}$ be its image.

We equip $\mathrm{Z}(R^r(n))$, $\mathrm{Z}(R^r(n))^{\mathrm{JM}}$ with the grading such that

$$\mathrm{Z}\big(R^r(n)\big)^d = R^r(n)^d \cap \mathrm{Z}\big(R^r(n)\big), \qquad \big(\mathrm{Z}\big(R^r(n)\big)^{\mathrm{JM}}\big)^d = R^r(n)^d \cap \mathrm{Z}\big(R^r(n)\big)^{\mathrm{JM}}.$$

The canonical inclusion $R(n) \to R(n+1)$ factors to a $(R^r(n), R^r(n))$-bilinear map $\iota : R^r(n) \to R^r(n+1)$. We have the following.

PROPOSITION 4.1

(a) $R^r(n) = \bigoplus_{r_i,w} \mathbf{k} x_1^{r_1} \cdots x_n^{r_n} w$, *where* $r_1 + \cdots + r_n < r$ *and* $w \in \mathfrak{S}_n$.

(b) $R^r(n+1) = \bigoplus_{k=0}^{r-1} \bigoplus_{j=1}^{n+1} R^r(n)(j, n+1) x_j^k$.

(c) *For each* $z \in R^r(n+1)$, *there are unique elements* $\pi(z) \in R^r(n) \otimes_{R^r(n-1)} R^r(n)$ *and* $p_k(z) \in R(n)$ *such that* $z = \mu_{\tau_n}(\pi(z)) + \sum_{k=0}^{r-1} p_k(z) x_{n+1}^k$, *yielding a* $(R^r(n), R^r(n))$*-bimodule isomorphism*

$$R^r(n+1) = R^r(n)\tau_n R^r(n) \oplus \bigoplus_{k=0}^{r-1} R^r(n) x_{n+1}^k.$$

(d) *The canonical map* $\mathrm{Z}(R(n))' \to \mathrm{Z}(R^r(n))'$ *is surjective and we have*

$$\sum_{n \geq 0} \dim\big(\mathrm{Z}\, R^r(n)'\big) q^{2n} = \prod_{j \geq 1} (1 - q^{2j})^{-r}.$$

(e) $\mathrm{Z}(R^r(n))^{\mathrm{JM}}$ *is a free* $\mathbf{k}$*-module of finite rank such that*

$$\sum_{n \geq 0} \mathrm{grdim}\big(\mathrm{Z}\, R^r(n)^{\mathrm{JM}}\big) q^{2n} = \prod_{p=1}^{r} \prod_{i=1}^{\infty} (1 - q^{2i} t^{2(ri+p-1-r)})^{-1}.$$

Proof

Parts (a) and (b) are well known. The proof of part (c) is similar to [24, Lemma 5.6.1], where the case $\hbar = 1$ is done. Indeed, by part (b) it is enough to show that

$$\bigoplus_{k=0}^{r-1} \bigoplus_{j=1}^{n} R^r(n)(j, n+1) x_j^k = R^r(n)\tau_n R^r(n).$$

By (b), we have $R^r(n) = \bigoplus_{k=0}^{r-1} \bigoplus_{j=1}^{n} R^r(n-1)(j,n)x_j^k$. Since τ_n commutes with $R^r(n-1)$ and $\tau_n(j,n) = (j,n)(j,n+1)$, we deduce $R^r(n)\tau_n R^r(n) = \bigoplus_{k=0}^{r-1} \bigoplus_{j=1}^{n} R^r(n)(j,n+1)x_j^k$. Part (d) is proved in [8].

Let us concentrate on (e). First, we prove that $Z(R^r(n))^{\mathrm{JM}}$ is free of finite rank as a $\mathbf{k}$-module. To do that, set $\mathbf{k}_1 = \mathbb{k}[y_1, \ldots, y_r]$ and consider the $\mathbf{k}_1$-algebras

$$R(n)_1 = R(n)/(\hbar - 1), \qquad R^r(n)_1 = R^r(n)/(\hbar - 1).$$

Then, the assignment $\tau_k \mapsto \tau_k$, $x_k \mapsto x_k \otimes \hbar$, $y_p \mapsto y_p \otimes \hbar$ yields a $\mathbf{k}$-algebra homomorphism

$$R^r(n) \to R^r(n)_1 \otimes_{\mathbf{k}_1} \mathbf{k} = R^r(n)_1[\hbar].$$

It restricts to an inclusion

$$Z\big(R^r(n)\big)^{\mathrm{JM}} \subset Z\big(R^r(n)\big) \subset Z\big(R^r(n)_1\big)[\hbar]. \tag{39}$$

Now, for any n-tuple $\mu = (\mu_1, \ldots, \mu_n)$ of nonnegative integers, let

$$p_\mu(x_1, \ldots, x_n) = \sum_\nu x_1^{\nu_1} \cdots x_n^{\nu_n} \in Z\big(R^r(n)_1\big),$$

where ν runs over the set of all n-tuples which are obtained from μ by permuting its entries. Let $\mathcal{P}_n(r)$ be the set of all partitions μ such that $\ell(\mu) + \sum_i \lfloor \mu_i/r \rfloor \leq n$. By [8, Theorem 3.2], the canonical map $R(n)_1 \to R^r(n)_1$ yields a surjection $Z(R(n)_1) \to Z(R^r(n)_1)$. Further, the elements $p_\mu(x_1, \ldots, x_n)$, where μ runs over the set $\mathcal{P}_n(r)$, form a $\mathbf{k}_1$-basis of $Z(R^r(n)_1)$. Therefore, under the inclusion (39), the elements $p_\mu(x_1, \ldots, x_n) \otimes \hbar^{|\mu|}$, where $\mu \in \mathcal{P}_n(r)$ yield a $\mathbf{k}$-basis of $Z(R^r(n))^{\mathrm{JM}}$. We deduce that $Z(R^r(n))^{\mathrm{JM}}$ is free of finite rank as a $\mathbf{k}$-module and that

$$\sum_{n \geq 0} \operatorname{grdim}\big(Z\, R^r(n)\big)^{\mathrm{JM}} q^{2n} = \sum_{n \geq 0} \sum_{\mu \in \mathcal{P}_n(r)} t^{2|\mu|} q^{2n}.$$

To compute the right-hand side, note that [8, p. 243] yields a bijection

$$\varphi : \Lambda_r^+(n) \to \mathcal{P}_n(r),$$

where $\Lambda_r^+(n)$ is the set of r-partitions $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ of n. Further, if $\varphi(\lambda) = \mu$, then

$$|\mu| = r|\lambda| - (r+1)\ell(\lambda) + \sum_{p=1}^{r} p\,\ell(\lambda^{(p)}).$$

We deduce that

$$\sum_{n\geq 0}\operatorname{grdim}\big(\mathrm{Z}\,R^r(n)\big)^{\mathrm{JM}}q^{2n}=\sum_{n\geq 0}\sum_{\lambda\in\Lambda_r^+(n)}t^{2r|\lambda|-2(r+1)\ell(\lambda)+2\sum_{p=1}^r p\ell(\lambda^{(p)})}q^{2n}$$
$$=\prod_{p=1}^r\sum_{\lambda\in\Lambda_1^+(n)}t^{2r|\lambda|+2(p-r-1)\ell(\lambda)}q^{2|\lambda|}$$
$$=\prod_{p=1}^r\prod_{i=1}^\infty(1-q^{2i}t^{2(ri+p-1-r)})^{-1}.\qquad\square$$

4.2. The Lie algebra $\mathcal{W}$

Let $\mathcal{W}$ be the Lie $\mathbf{k}'$-algebra generated by elements $C_k,\ D_{-1,k},\ D_{0,k+1},\ D_{1,k}$ with $k\geq 0$, modulo the following definition relations:

(a) $[D_{0,l+1},D_{0,k+1}]=0$,

(b) $[D_{0,l+1},D_{1,k}]=\hbar D_{1,l+k}$ and $[D_{0,l+1},D_{-1,k}]=-\hbar D_{-1,l+k}$,

(c) $3[D_{12},D_{11}]-[D_{13},D_{10}]+\hbar^2[D_{11},D_{10}]=0$,

(d) $3[D_{-1,2},D_{-1,1}]-[D_{-1,3},D_{-1,0}]+\hbar^2[D_{-1,1},D_{-1,0}]=0$,

(e) $[D_{10},[D_{10},D_{11}]]=[D_{-1,0},[D_{-1,0},D_{-1,1}]]=0$,

(f) $[D_{-1,k},D_{1,l}]=E_{k+l}$,

(g) C_k is central,

where the series $E(z)=\sum_{k\geq 0}E_kz^k$ is given by the following formula

$$E(z)=C_0+\sum_{k\geq 1}\big(\gamma_\hbar(D_{0,k+1})+\gamma_{-\hbar}(D_{0,k+1})+C_k\big)z^k,$$
$$\gamma_t(D_{0,k+1})=\sum_{p=1}^k\binom{k}{p}D_{0,k-p+1}t^{p-1}.\tag{40}$$

Let $\mathcal{W}_{<0},\mathcal{W}_{\geq 0},\mathcal{W}_0\subset\mathcal{W}$ be the Lie subalgebras generated by $\{D_{1,k};k\geq 0\}$, $\{D_{-1,k};k\geq 0\}$, and $\{D_{0,k+1},C_k;k\geq 0\}$, respectively.

There is a unique Lie $\mathbf{k}'$-algebra anti-involution ϖ of $\mathcal{W}$ such that

$$\varpi(C_k)=C_k,\qquad \varpi(D_{l,k})=D_{-l,k}.\tag{41}$$

The Lie $\mathbf{k}'$-algebra $\mathcal{W}$ is $\mathbb{Z}$-graded with $\deg(D_{l,k})=2l$ and $\deg(C_k)=0$. A representation V is *diagonalizable* if the operator $D_{0,1}/\hbar$ is diagonalizable with integral eigenvalues. Then V is $\mathbb{Z}$-graded and its degree $2n$ component V_n is the eigenspace associated with the eigenvalue n. We will say that a diagonalizable representation is *quasifinite* if the degree $2n$ component is finite-dimensional for each n. Finally, we define the *character* of a quasifinite representation V to be the formal series in $\mathbb{N}[[q,q^{-1}]]$ given by

$$\operatorname{ch}(V)(q)=\sum_{n\in\mathbb{Z}}q^{2n}\dim(V_n).$$

Given a linear form $\Lambda : \mathcal{W}_0 \to \mathbf{k}'$ and a module V, an element $v \in V$ is *primitive* of weight Λ if $\mathcal{W}_0$ acts on v by Λ and $\varpi(\mathcal{W}_{<0})$ by zero. We call $\Lambda(C_0)$ the *level* of Λ.

Let $M(\Lambda)$ be the Verma module with the lowest weight Λ. It is the diagonalizable module induced from the 1-dimensional $\mathcal{W}_{\geq 0}$-module spanned by a primitive vector $|\Lambda\rangle$. Let $V(\Lambda)$ be the top of $M(\Lambda)$. It is an irreducible diagonalizable module. We call Λ the *lowest weight* and $\Lambda(C_0)$ the *level* of $M(\Lambda)$, $V(\Lambda)$. We call $|\Lambda\rangle$ the highest weight vectors of $M(\Lambda)$, $V(\Lambda)$.

4.3. The loop operators on the center and the cocenter

For each $r, n \in \mathbb{N}$ with $r \neq 0$, we set $\mathcal{C}^r_n = R^r(n)$-grproj. Write $\mathcal{C}^r = \bigoplus_n \mathcal{C}^r_n$. The restriction and induction functors form an adjoint pair (F, E) with

$$E : R^r(n+1)\text{-grmod} \to R^r(n)\text{-grmod}, \qquad N \mapsto N,$$
$$F : R^r(n)\text{-grmod} \to R^r(n+1)\text{-grmod}, \qquad M \mapsto R^r(n+1) \otimes_{R^r(n)} M.$$

Let $\varepsilon : FE \to 1$ and $\eta : 1 \to EF$ be the counit and unit, respectively, of the adjoint pair (F, E). They are represented by the multiplication map μ and the canonical map ι, respectively:

$$\varepsilon : R^r(n) \otimes_{R^r(n-1)} R^r(n) \to R^r(n),$$
$$\eta : R^r(n) \to R^r(n+1).$$

PROPOSITION 4.2

(a) *The pair (E, F) is adjoint with the counit $\hat{\varepsilon} : EF \to 1$ and the unit $\hat{\eta} : 1 \to FE$ represented by the morphisms $\hat{\varepsilon} : R^r(n+1) \to R^r(n)$, $\hat{\eta} : R^r(n) \to R^r(n) \otimes_{R^r(n-1)} R^r(n)$ such that $\hat{\varepsilon}(z) = p_{r-1}(z)$ and $\hbar\hat{\eta}(1) = \pi(x^r_{n+1})$.*

(b) *The $\mathbf{k}$-algebra $R^r(n)$ is a symmetric algebra. The symmetrizing form $t_{r,n} : R^r(n) \to \mathbf{k}$ is the unique $\mathbf{k}$-linear map sending the element $x_1^{r_1} \cdots x_n^{r_n} w$ to 1 if $r_1 = \cdots = r_n = r - 1$ and $w = 1$ and to 0 otherwise. We have $t_{r,n} = \hat{\varepsilon} \circ \cdots \circ \hat{\varepsilon}$ (n times).*

Proof

See [24, Lemma 5.7.2] for (a) and [9, Theorem A.2] for (b). □

We have $\operatorname{tr}(\mathcal{C}^r/\mathbb{Z}) = \bigoplus_n \operatorname{tr}(R^r(n))$. We equip $\operatorname{tr}(\mathcal{C}^r/\mathbb{Z})$ with the $\mathbb{Z}^2$-grading such that $\operatorname{tr}(R^r(n))$ has the weight $2n$ and the order given by the degree of elements of $R^r(n)$. For each $k \in \mathbb{N}$, we define $\mathbf{k}$-linear operators $x^\pm_k$ on $\operatorname{tr}(\mathcal{C}^r/\mathbb{Z}) = \bigoplus_n \operatorname{tr}(R^r(n))$ of weights ± 2 and order $2k$ such that the maps

$$x^+_k : \operatorname{tr}\big(R^r(n)\big) \to \operatorname{tr}\big(R^r(n-1)\big), \qquad x^-_k : \operatorname{tr}\big(R^r(n)\big) \to \operatorname{tr}\big(R^r(n+1)\big)$$

are given, for each $f \in \mathrm{End}_{\mathcal{C}^r}(a)$ and $a \in \mathcal{C}^r$, by

$$x_k^+\big(\mathrm{tr}(f)\big) = \mathrm{tr}\big(x^k(a) \circ \big(E(f)\big)\big), \qquad x_k^-\big(\mathrm{tr}(f)\big) = \mathrm{tr}\big(x^k(a) \circ \big(F(f)\big)\big).$$

Here $x \in \mathrm{End}(F)$ is represented by the right multiplication by x_{n+1} on $R^r(n+1)$, and $x \in \mathrm{End}(E)$ is the left-transposed endomorphism.

THEOREM 4.3
The assignment $C_k \mapsto p_k(y_1, \dots, y_r)$, $D_{\mp 1,k} \mapsto x_k^{\pm}$ defines a representation of level r of $\mathcal{W}$ on $\mathrm{tr}(\mathcal{C}^r/\mathbb{Z})'$.

Proof
First, we check that the operators x_0^-, x_1^-, x_2^-, x_3^- satisfy relations (c) and (e) for D_{10}, D_{11}, D_{12}, D_{13} in the definition of $\mathcal{W}$. For each $k \in \mathbb{N}$ and $f \in R^r(n)$, we have $x_k^-(\mathrm{tr}(f)) = \mathrm{tr}(f x_{n+1}^k)$, where the element f on the right-hand side is identified with its image by the map

$$\iota : R^r(n) \to R^r(n+1).$$

Thus, using the relations $\tau_{n+1} x_{n+2} \tau_{n+1} = x_{n+1} + \hbar \tau_{n+1}$ and $\tau_{n+1}^2 = 1$, we get

$$\begin{aligned}
[x_1^-, x_0^-]\big(\mathrm{tr}(f)\big) &= \mathrm{tr}(f x_{n+2} - f x_{n+1}) \\
&= \mathrm{tr}(f x_{n+2} \tau_{n+1}^2 - f x_{n+1}) \\
&= \mathrm{tr}(f \tau_{n+1} x_{n+2} \tau_{n+1} - f x_{n+1}) \\
&= \hbar\, \mathrm{tr}(f \tau_{n+1}).
\end{aligned}$$

Similarly, using the relations

$$\begin{aligned}
\tau_{n+1} x_{n+1} x_{n+2}^2 \tau_{n+1} &= x_{n+1}^2 x_{n+2} + \hbar x_{n+1} x_{n+2} \tau_{n+1}, \\
\tau_{n+1} x_{n+2}^3 \tau_{n+1} &= x_{n+1}^3 + \hbar (x_{n+2}^2 + x_{n+1} x_{n+2} + x_{n+1}^2) \tau_{n+1},
\end{aligned}$$

we deduce that

$$\begin{aligned}
[x_2^-, x_1^-]\big(\mathrm{tr}(f)\big) &= \hbar\, \mathrm{tr}(f x_{n+1} x_{n+2} \tau_{n+1}), \\
[x_3^-, x_0^-]\big(\mathrm{tr}(f)\big) &= \hbar\, \mathrm{tr}\big(f (x_{n+2}^2 + x_{n+1} x_{n+2} + x_{n+1}^2) \tau_{n+1}\big).
\end{aligned}$$

Relation (e) follows from

$$\begin{aligned}
\big[x_0^-, [x_0^-, x_1^-]\big]\big(\mathrm{tr}(f)\big) &= \hbar\, \mathrm{tr}\big(f (\tau_{n+2} - \tau_{n+1})\big) \\
&= \hbar\, \mathrm{tr}\big(f (\tau_{n+2} \tau_{n+1}^2 - \tau_{n+1} \tau_{n+2}^2)\big) \\
&= \hbar\, \mathrm{tr}\big(f (\tau_{n+1} \tau_{n+2} \tau_{n+1} - \tau_{n+2} \tau_{n+1} \tau_{n+2})\big) \\
&= 0.
\end{aligned}$$

To prove relation (c), we introduce the element $\varphi_l = (x_l - x_{l+1})\tau_l + \hbar$. We have

$$\varphi_l^2 = \hbar^2 - (x_l - x_{l+1})^2, \qquad \varphi_l x_k = x_{s_l(k)}\varphi_l, \qquad \varphi_l \tau_l = -\tau_l \varphi_l.$$

And we deduce that

$$\begin{aligned}
&\big(3[x_2^-, x_1^-] - [x_3^-, x_0^-] + \hbar^2[x_1^-, x_0^-]\big)\big(\mathrm{tr}(f)\big)\\
&\quad = \hbar\,\mathrm{tr}\big(f(3x_{n+1}x_{n+2} - x_{n+2}^2 - x_{n+1}x_{n+2} - x_{n+1}^2 - \hbar^2)\tau_{n+1}\big)\\
&\quad = \hbar\,\mathrm{tr}\big(f\big((x_{n+1} - x_{n+2})^2 - \hbar^2\big)\tau_{n+1}\big)\\
&\quad = -\hbar\,\mathrm{tr}(f\varphi_{n+1}^2\tau_{n+1})\\
&\quad = -\hbar\,\mathrm{tr}(f\varphi_{n+1}\tau_{n+1}\varphi_{n+1})\\
&\quad = \hbar\,\mathrm{tr}(f\varphi_{n+1}^2\tau_{n+1})\\
&\quad = 0.
\end{aligned}$$

We prove that the operators x_0^+, x_1^+, x_2^+, x_3^+ satisfy relations (d) and (e) for $D_{-1,0}$, $D_{-1,1}$, $D_{-1,2}$, $D_{-1,3}$ in a similar way.

Next, we prove relations (a), (b), and (f). To do so, for each $l \geq 0$, consider the element $p_l(x_1, \dots, x_n) \in Z(R^r(n))$ given by $p_l(x_1, \dots, x_n) = \sum_{i=1}^n x_i^l$ if $n > 0$ and 0 if $n = 0$. Let x_{l+1}^0 be the $\mathbf{k}$-linear operator on $\mathrm{tr}(\mathcal{C}^r/\mathbb{Z})$ given by

$$x_{l+1}^0\big(\mathrm{tr}(f)\big) = \hbar\,\mathrm{tr}\big(fp_l(x_1, \dots, x_n)\big), \quad \forall f \in R^r(n). \tag{42}$$

Then, under the assignment $D_{0,k+1} \mapsto x_{k+1}^0$, the defining relation (a) of $\mathcal{W}$ is obviously satisfied. Let us concentrate on (b). We have

$$\begin{aligned}
[x_{l+1}^0, x_k^-]\big(\mathrm{tr}(f)\big) &= \hbar\,\mathrm{tr}\big(f\big(x_{n+1}^k p_l(x_1, \dots, x_{n+1}) - x_{n+1}^k p_l(x_1, \dots, x_n)\big)\big)\\
&= \hbar x_{l+k}^-\big(\mathrm{tr}(f)\big).
\end{aligned}$$

The relation $[x_{l+1}^0, x_k^+] = \hbar x_{l+k}^+$ is proved in a similar way.

Finally, let us prove relation (f) in the definition of the Lie algebra $\mathcal{W}$. Let

$$\hat{\varepsilon} : R^r(n+1) \to R^r(n), \qquad \hat{\eta} : R^r(n) \to R^r(n) \otimes_{R^r(n-1)} R^r(n)$$

be as above. For each $k \in \mathbb{N}$ we consider the following elements in $Z(R^r(n))$:

$$\begin{aligned}
B_{+,n}^k &= \hat{\varepsilon}(x_{n+1}^{r-1+k}),\\
B_{-,n}^k &= \begin{cases} -\hbar^2 \mu_{x_n^{-r-1+k}}(\hat{\eta}(1)) & \text{if } k \geq r+1,\\ -p_{r-k}(x_{n+1}^r) & \text{if } 1 \leq k \leq r,\\ 1 & \text{if } k = 0. \end{cases}
\end{aligned}$$

We may abbreviate $B^k_{\pm} = B^k_{\pm,n}$. Consider the formal series $B_{\pm}(z) = \sum_{k\in\mathbb{N}} B^k_{\pm} z^k$. For each $k, l \in \mathbb{N}$, we define the operator $E_{k,l} = [x^+_k, x^-_l]$ on $\mathrm{tr}(\mathcal{C}^r/\mathbb{Z})'$.

LEMMA 4.4
The following hold:

(a) $\hat{\varepsilon}(\tau_n a \tau_n) = \hat{\varepsilon}(a)$ *for each* $a \in R^r(n)$*;*

(b) $B_+(z)B_-(z) = 1$*;*

(c) $E_{k+l} := E_{k,l}$ *depends only on* $k+l$ *and we have* $E_l = \sum_{k=0}^{l}(r-k)B^{l-k}_+ B^k_-$*;*

(d) $E(z) = r - z\,\mathrm{d}/\mathrm{d}z \log B_-(z)$*;*

(e) $B_-(z) = \prod_{k=1}^{n}(1 - \hbar^2 z^2/(1-zx_k)^2)\prod_{p=1}^{r}(1 - zy_p)$.

Proof
For part (a), note that $a = \mu_{\tau_{n-1}}(\pi(a)) + \sum_{k=0}^{r-1} p_k(a)x^k_n$. Thus, a direct computation yields

$$\tau_n a \tau_n = \mu_{\tau_n\tau_{n-1}\tau_n}\big(\pi(a)\big) - \sum_{k=0}^{r-1} p_k(a)\hbar \sum_{s+t=k-1} x^s_n \tau_n x^t_n + \sum_{k=0}^{r-1} p_k(a) x^k_{n+1}$$
$$- \hbar^2 \sum_{k=0}^{r-1} p_k(a) \sum_{t=0}^{k-1}\sum_{a=0}^{t-1} x^{k-1-t+a}_n x^{t-1-a}_{n+1}.$$

We deduce that $p_{r-1}(\tau_n a \tau_n) = p_{r-1}(a)$.

Next, a similar computation as in the proof of parts (a) and (c) of Lemma B.1 yields

$$\pi(x^k_{n+1}) = \sum_{s=0}^{k-r}\big(\hat{\varepsilon}(x^{k-s}_{n+1}) \otimes x^s_n \otimes 1 \otimes 1\big)\pi(x^r_{n+1}), \tag{43}$$

$$p_a(x^k_{n+1}) = \sum_{s=0}^{r-a-1} B^{k-a-s}_{+,n} B^s_{-,n} \tag{44}$$

and the equality in part (b).

The proof of part (c) is similar to Proposition B.5(b), so we briefly indicate the key steps. First, the isomorphism in Proposition 4.1(c) represents an isomorphism of functors $\rho : EF1_n \to G$ with $G = FE1_n \oplus 1^{\oplus r}_n$. By Lemma 2.3 we have

$$x^+_k x^-_l = \mathrm{tr}_{EF}(x^k_i x^l_i) = \mathrm{tr}_G(\rho^{-1}(x^k x^l)\rho,$$

and it is equal to the sum of the trace of $\rho^{-1}(x^k x^l)\rho$ restricted to each direct factor of G. The restriction of $\rho^{-1}(x^k x^l)\rho$ to $FE1_n$ is represented by

$$R^r(n) \otimes_{R^r(n-1)} R^r(n) \to R^r(n) \otimes_{R^r(n-1)} R^r(n), \qquad z \mapsto \pi\big(x^k_{n+1}\mu_{\tau_n}(z)x^l_{n+1}\big).$$

We have

$$\pi\left(x_{n+1}^k \mu_{\tau_n}(z) x_{n+1}^l\right) = \mu_{x_n^l \tau_n x_n^k}(z) + \hbar \sum_{a=0}^{k+l-1} \mu_{x_n^a}(z) \pi(x_{n+1}^{k+l-1-a}).$$

Hence the restriction of $\rho^{-1}(x^k x^l)\rho$ to $FE1_n$ is the endomorphism

$$x^l x^k + \hbar \sum_{a=0}^{k+l-1} \pi(x_{n+1}^{k+l-1-a}) \varepsilon(1 \otimes x_n^a),$$

and by (43) its trace is equal to

$$x_l^- x_k^+ + \sum_{s=r+1}^{k+l} (r-s) B_{+,n}^{l+k-s} B_{-,n}^s.$$

The restriction of $\rho^{-1}(x^k x^l)\rho$ to the ath copy of 1_n is represented by the map $R^r(n) \to R^r(n)$, $z \mapsto z p_a(x_{n+1}^{k+l+a})$. By (44), it is equal to $\sum_{s=0}^{r-1-a} B_{+,n}^{k+l-s} B_{-,n}^s$. We conclude that

$$\begin{aligned} x_k^+ x_l^- &= x_l^- x_k^+ + \sum_{s=r+1}^{k+l} (r-s) B_{+,n}^{l+k-s} B_{-,n}^s + \sum_{a=0}^{r-1} \sum_{s=0}^{r-1-a} B_{+,n}^{k+l-s} B_{-,n}^s \\ &= x_l^- x_k^+ + \sum_{s=0}^{k+l} (r-s) B_{+,n}^{l+k-s} B_{-,n}^s. \end{aligned}$$

To prove (d), note that, using (b), we get

$$\begin{aligned} -z\mathrm{d}/\mathrm{d}z \log B_-(z) &= -\Big(\sum_{k\geq 0} k B_-^k z^k\Big)\Big(\sum_{k\geq 0} B_+^k z^k\Big) \\ &= \sum_{l\geq 0} \sum_{k=0}^{l} (-k B_-^k B_+^{l-k}) z^l = -r + E(z). \end{aligned}$$

Finally, we concentrate on (e). We have

$$\tau_n x_n^k \tau_n = x_{n+1}^k - \hbar \sum_{p+q=k-1} x_n^p \tau_n x_n^q - \hbar^2 \sum_{a=0}^{k-2} (a+1) x_n^a x_{n+1}^{k-2-a}.$$

Using (a), we deduce that

$$\hat{\varepsilon}(x_n^k) = \hat{\varepsilon}(\tau_n x_n^k \tau_n) = \hat{\varepsilon}(x_{n+1}^k) - \hbar^2 \sum_{a=0}^{k-2} (a+1) x_n^a \hat{\varepsilon}(x_{n+1}^{k-2-a}).$$

Thus, we have

$$B_{+,n-1}^{k-r+1} = B_{+,n}^{k-r+1} - \hbar^2 \sum_{a=0}^{k-2} (a+1) x_n^a B_{+,n}^{k-2-a-r+1}.$$

This yields

$$B_{+,n-1}(z) = \big(1 - \hbar^2 z^2 (1 - z x_n)^{-2}\big) B_{+,n}(z).$$

Hence by (b) we get

$$B_{-,n}(z) = \big(1 - \hbar^2 z^2 (1 - z x_n)^{-2}\big) B_{-,n-1}(z)$$

for $n \geq 1$. By induction it remains to compute $B_{-,0}(z)$. Since

$$x_1^r = -\sum_{a=1}^{r} (-1)^a e_a(y_1, \dots, y_r) x_1^{r-a},$$

we have

$$\pi(x_1^r) = 0, \qquad B_{-,0}^a = (-1)^a e_a(y_1, \dots, y_r) \quad \forall a \in [1, r], \qquad B_{-,0}^a = 0 \quad \forall a > r.$$

Therefore, we have $B_{-,0}(z) = \prod_{a=1}^{r} (1 - z y_a)$. □

We can now finish the proof of relation (f) of $\mathcal{W}$. According to Lemma 4.4, the formal series $E(z) = \sum_{l \geq 0} E_l z^l$ is given by

$$E(z) = r - z \mathrm{d}/\mathrm{d}z \log\Big(\prod_{p=1}^{r} (1 - z y_p) \prod_{k=1}^{n} \big(1 - z(x_k - \hbar)\big) \times \big(1 - z(x_k + \hbar)\big)(1 - z x_k)^{-2} \Big).$$

Comparing this with formula (42) and the identity

$$\sum_{k \geq 1} (za)^k = -z(\mathrm{d}/\mathrm{d}z) \log(1 - za), \tag{45}$$

we deduce that

$$E(z) = r - \sum_{k \geq 1} \sum_{p=1}^{n} \big(2x_p^k - (x_p - \hbar)^k - (x_p + \hbar)^k\big) z^k + \sum_{k \geq 1} \sum_{p=1}^{r} y_p^k z^k.$$

This implies that

$$E(z) = r + \sum_{k\geq 1}\big(\gamma_{\hbar}(D_{0,k+1}) + \gamma_{-\hbar}(D_{0,k+1}) + p_k(y_1,\dots,y_r)\big)z^k,$$
$$\gamma_t(D_{0,k+1}) = \sum_{p=1}^{k}\binom{k}{p} D_{0,k-p+1}t^{p-1}. \tag{46}$$

This finishes the proof of the theorem. □

Set $\mathrm{Z}(\mathcal{C}^r/\mathbb{Z}) = \bigoplus_n \mathrm{Z}(R^r(n))$. The symmetrizing form $t_r = \bigoplus_n t_{r,n}$ on $\bigoplus_n R^r(n)'$ yields a $\mathbf{k}'$-bilinear form

$$\mathrm{Z}(\mathcal{C}^r/\mathbb{Z})' \times \mathrm{tr}(\mathcal{C}^r/\mathbb{Z})' \to \mathbf{k}', \qquad (a,b) \mapsto t_r(ab),$$

which induces an isomorphism $\mathrm{Z}(\mathcal{C}^r/\mathbb{Z})' \simeq \mathrm{Hom}_{\mathbf{k}'}(\mathrm{tr}(\mathcal{C}^r/\mathbb{Z})', \mathbf{k}')$. Taking the transpose with respect to this bilinear form and twisting the action by the anti-involution ϖ in (41), we get a representation of $\mathcal{W}$ on $\mathrm{Z}(\mathcal{C}^r/\mathbb{Z})'$ of level r. Let $|r\rangle \in \mathrm{Z}(\mathcal{C}^r/\mathbb{Z})'$ denote the unit of $\mathrm{Z}(R^r(0))' = \mathbf{k}'$. We define a weight Λ_r of level r of $\mathcal{W}$ by the formula

$$\Lambda_r(C_k) = p_k(y_1,\dots,y_r), \qquad \Lambda_r(D_{0,k+1}) = 0, \quad \forall k \geq 0. \tag{47}$$

PROPOSITION 4.5
The following hold:
(a) $|r\rangle$ *is a primitive vector of* $\mathrm{Z}(\mathcal{C}^r/\mathbb{Z})'$ *of weight* Λ_r*;*
(b) $\mathrm{Z}(\mathcal{C}^r/\mathbb{Z})'$ *is quasifinite of character* $\prod_{j\geq 1}(1-q^{2j})^{-r}$.

Proof
Part (a) follows from formula (42), which implies that $x^0_{l+1}(|r\rangle) = 0$ for all $l \geq 0$. Part (b) follows from Proposition 4.1(d). □

4.4. The cohomology ring of the moduli space of framed instantons
Let $\mathfrak{M}(r,n)$ be the moduli space of framed rank r torsion-free sheaves on $\mathbb{P}^2$ with fixed second Chern class n. Set $\mathfrak{M}(r) = \bigsqcup_n \mathfrak{M}(r,n)$. First, let us review a few basic facts on $\mathfrak{M}(r)$ (see [34, Section 3] for more details).

The group $\mathrm{GL}(r) \times \mathrm{GL}(2)$ acts on $\mathfrak{M}(r)$ in the obvious way : $\mathrm{GL}(r)$ acts by changing the framing and $\mathrm{GL}(2)$ via the tautological action on $\mathbb{P}^2$ which preserves the line at infinity. Let $T \subset \mathrm{GL}(r)$, $A \subset \mathrm{GL}(2)$ be the maximal tori, and let $\mathbb{C}^\times \subset A$ be the *hyperbolic* torus $\{\mathrm{diag}(t,t^{-1}); t \in \mathbb{C}^\times\}$. Set $G = T \times \mathbb{C}^\times$ and $G_A = T \times A$.

We identify the $\mathbb{Z}$-graded $\Bbbk$-algebra $H^*_G(\bullet,\Bbbk)$ with $\mathbf{k}$ in the obvious way. Let $\mathbf{h} = H^*_{G_A}(\bullet,\Bbbk)$, and write $\mathbf{h}'$ for the fraction field of $\mathbf{h}$. From now on, let $\Bbbk$ be a field of characteristic 0.

The G_A-variety $\mathfrak{M}(r,n)$ is equivariantly formal and smooth of dimension $d_{r,n} = 2rn$. Thus $H^*_G(\mathfrak{M}(r),\mathbb{k})$ is a free $\mathbb{Z}$-graded $\mathbf{k}$-module isomorphic to $H^*(\mathfrak{M}(r),\mathbb{k}) \otimes \mathbf{k}$. The G_A-action yields an α-partition of $\mathfrak{M}(r)$ into affine spaces in the sense of De Concini, Lusztig, and Procesi [11]. We deduce that

$$\begin{aligned}\sum_{d,n\geq 0} &\dim H^d\big(\mathfrak{M}(r,n),\mathbb{k}\big)q^{2n}t^d \\ &= \sum_{n\geq 0} \mathrm{grdim} H^*_G\big(\mathfrak{M}(r,n),\mathbb{k}\big)q^{2n} \\ &= \prod_{p=1}^{r}\prod_{i=1}^{\infty}(1-q^{2i}t^{2(ri+p-1-r)})^{-1}. \end{aligned} \tag{48}$$

In particular, note that the odd cohomology of $\mathfrak{M}(r)$ vanishes, and hence the $\mathbf{k}$-algebra $H^*_G(\mathfrak{M}(r),\mathbb{k})$ is commutative. Let $|r\rangle$ be the unit of $H^*_G(\mathfrak{M}(r,0),\mathbb{k}) \simeq \mathbf{k}$. First, we prove the following.

PROPOSITION 4.6

(a) *There is a representation of the Lie $\mathbf{k}'$-algebra $\mathcal{W}$ on $H^*_G(\mathfrak{M}(r),\mathbb{k})'$ which is isomorphic to $V(\Lambda_r)$. This representation is quasifinite of character $\prod_{j\geq 1}(1-q^{2j})^{-r}$.*

(b) *There is a unique isomorphism $\psi : \mathrm{Z}(\mathcal{C}^r/\mathbb{Z})' \to H^*_G(\mathfrak{M}(r),\mathbb{k})'$ of representations of $\mathcal{W}'$ which takes the element $|r\rangle$ to $|r\rangle$.*

(c) *For each $n \in \mathbb{N}$, the element $\psi(1) \in H^*_G(\mathfrak{M}(r,n),\mathbb{k})'$ is invertible, and the map $\phi' : \mathrm{Z}(R^r(n))' \to H^*_G(\mathfrak{M}(r,n),\mathbb{k})'$ given by $\phi'(\bullet) = \psi(1)^{-1} \cup \psi(\bullet)$ is a $\mathbf{k}'$-algebra isomorphism.*

Proof

For each $n,k \in \mathbb{N}$, we consider the locus

$$\mathfrak{B}(r,n+k,n) \subset \mathfrak{M}(r,n+k) \times \mathbb{C}^2 \times \mathfrak{M}(r,n)$$

of triples $(\mathcal{E},x,\mathcal{F})$ such that $\mathcal{E} \subset \mathcal{F}$ and $\mathcal{F}/\mathcal{E}$ is a length k sheaf supported at x. For each $\gamma \in H^*_G(\mathfrak{B}(r,n+k,n),\mathbb{k})$ the correspondence $\mathfrak{B}(r,n+k,n)$ defines two maps

$$\begin{aligned}\Theta_+(\gamma) &: H^*_G\big(\mathfrak{M}(r,n),\mathbb{k}\big) \to H^*_G\big(\mathfrak{M}(r,n+k),\mathbb{k}\big), \\ \Theta_-(\gamma) &: H^*_G\big(\mathfrak{M}(r,n+k),\mathbb{k}\big) \to H^*_G\big(\mathfrak{M}(r,n),\mathbb{k}\big)'.\end{aligned}$$

The map Θ_- uses localized equivariant cohomology, because the projection $\mathfrak{B}(r,n+k,n) \to \mathfrak{M}(r,n)$ is not proper.

Let $\tau_{n+k,n}$, τ_n be the *tautological bundles* on $\mathfrak{M}(r,n+k) \times \mathfrak{M}(r,n)$, $\mathfrak{M}(r,n)$. Write c_i for the ith equivariant Chern class. The obvious map

$$H^*_G\big(\mathfrak{M}(r,n+k)\times\mathfrak{M}(r,n),\Bbbk\big)\times H^*_G(\mathbb{C}^2,\Bbbk)\to H^*_G\big(\mathfrak{B}(r,n+k,n),\Bbbk\big)$$

is denoted by $(a,b)\mapsto a\otimes b$.

Now, we define the action of C_k, $D_{1,k}$, $D_{-1,k}$, $D_{0,k+1}$ on an element of $H^*_G(\mathfrak{M}(r,n),\Bbbk)'$ by

$$\begin{aligned}
D_{0,1}&=\hbar n,\\
C_k&=p_k(y_1,\dots,y_r),\\
D_{1,k}&=-\hbar^2\Theta_+\big(c_1(\tau_{n+1,n})^k\otimes 1\big),\\
D_{-1,k}&=(-1)^{r-1}\Theta_-\big(c_1(\tau_{n+1,n})^k\otimes 1\big),\\
\sum_{k\geq 0}D_{0,k+2}z^k&=-\hbar(\mathrm{d}/\mathrm{d}z)\log\Big(1+\sum_{k\geq 1}c_k(\tau_n)(-z)^k\Big)\cup\bullet.
\end{aligned}\tag{49}$$

The operators $D_{-1,k}$, $D_{1,k}$, $D_{0,k+1}$ above are equal to the operators $\hbar^k D_{-1,k}$, $\hbar^k D_{1,k}$, $\hbar^{k+1}D_{0,k+1}$ in [37, (3.17)], respectively. The reason for this normalization by powers of $\hbar$ is to give the term $\hbar$ in relations (b), (c), and (d) of $\mathcal{W}$ in Section 4.2, which does not appear in the corresponding relations in [2].

By [37, Corollary 3.3] and [2], the formulas (49) yield a representation of $\mathbf{h}'\otimes_{\mathbf{k}'}\mathcal{W}$ on

$$H^*_{G_A}\big(\mathfrak{M}(r),\Bbbk\big)'=\mathbf{h}'\otimes_{\mathbf{h}}H^*_{G_A}\big(\mathfrak{M}(r),\Bbbk\big).$$

Note that [37] uses equivariant homology rather than equivariant cohomology, but since $\mathfrak{M}(r)$ is smooth, its equivariant homology and cohomology are isomorphic by Poincaré duality. We must check that the formulas (49) give indeed a representation of $\mathcal{W}$ on $H^*_G(\mathfrak{M}(r),\Bbbk)'$.

The representation of $\mathbf{h}'\otimes_{\mathbf{k}'}\mathcal{W}$ in [37] depends on parameters $y_1,\dots,y_r,x,y$ which are generators of the field extension $\mathbf{h}'$ of $\Bbbk$. Note that y_p is denoted by the symbol e_p in [37]. The representation of $\mathcal{W}$ we consider here is a specialization along the hyperplane $x=-y=\hbar$ of some integral form of the representation in [37]. See [7] for more details on this integral form. We must check that the representation in [37] specializes effectively.

To prove this, note that the main results of [37] are obtained by explicit computations in the $\mathbf{h}'$-basis of $H^*_{G_A}(\mathfrak{M}(r),\Bbbk)'$ formed by the fundamental classes of the fixed points of $\mathfrak{M}(r)$ under the action of the torus G_A. For these computations it is essential that the fixed points are isolated. Now, it is well known, and easy to prove, that the fixed-point sets $\mathfrak{M}(r)^G$ and $\mathfrak{M}(r)^{G_A}$ are the same. Indeed, one can easily check that the explicit formulas for the representation of $\mathbf{h}'\otimes_{\mathbf{k}'}\mathcal{W}$ in [37, Corollary 3.3] in the basis of $H^*_{G_A}(\mathfrak{M}(r),\Bbbk)'$ have no poles along the hyperplane $x=y$. This follows from formulas (3.17) and (D.1)–(D.3) in [37].

Note, however, that the series $E(z)$ in (40) differs from the corresponding one in [37, (1.70)]. We must check that these formulas are compatible. To do that, let us review quickly the proof in [37, pp. 326–327]. Our setting differs from [37] because there we assumed that both parameters x, y are generic, while here we have $x = \hbar = -y$ with $\hbar$ generic.

Let $\{a_i ; i \in I\}$ and $\{b_j ; j \in J\}$ be as in [37, p. 327]. Then, the computation there implies that the element $[D_{-1,k}, D_{1,l}] = E_{k+l}$ depends only on $k + l$ and yields the following identity:

$$E(z) = \sum_{k \geq 0} \Big(\sum_{i \in I} a_i - \sum_{j \in J} b_j \Big) z^k.$$

Fix some formal variables $x_1, x_2, \ldots, x_n$ such that $c_i(\tau_n) = e_i(x_1, x_2, \ldots, x_n)$ for each $i \in [1, n]$. Then, the same argument as in [37, p. 327] using [37, Lemma D.1] implies that

$$E(z) = r - \sum_{k \geq 1} \sum_{p=1}^{n} \big(2x_p^k - (x_p - \hbar)^k - (x_p + \hbar)^k \big) z^k + \sum_{k \geq 1} \sum_{p=1}^{r} y_p^k z^k.$$

We deduce that (see (46))

$$\begin{aligned} E(z) &= r + \sum_{k \geq 1} \big(\gamma_\hbar(D_{0,k+1}) + \gamma_{-\hbar}(D_{0,k+1}) + p_k(y_1, \ldots, y_r) \big) z^k, \\ \gamma_t(D_{0,k+1}) &= \sum_{p=1}^{k} \binom{k}{p} D_{0,k-p+1} t^{p-1}. \end{aligned} \tag{50}$$

We have proved that the formulas in (49) define a representation of $\mathcal{W}$ on $H_G^*(\mathfrak{M}(r), \Bbbk)'$. Now, we must check that this representation is irreducible and is isomorphic to $V(\Lambda_r)$.

The irreducibility follows from the main result of [37]. More precisely, it is proved in [37, Theorem 8.33] that the representation of $\mathcal{W}$ on $H_{G_A}^*(\mathfrak{M}(r), \Bbbk)'$ gives rise to a representation of the W-algebra of the affine Kac–Moody algebra $\widehat{\mathfrak{gl}}_r$ on $H_{G_A}^*(\mathfrak{M}(r), \Bbbk)'$ (see, e.g., [1] and [15] for some background on $W(\widehat{\mathfrak{gl}}_r)$). It is also proved there that the $W(\widehat{\mathfrak{gl}}_r)$-module $H_{G_A}^*(\mathfrak{M}(r), \Bbbk)'$ is isomorphic to the Verma module with highest weight and level given, respectively, by

$$a/x - \rho(1 + y/x) \qquad \text{and} \qquad -y/x - r. \tag{51}$$

Here we have set $a = (y_1, y_2, \ldots, y_r)$ and $\rho = (0, -1, \ldots, 1 - r)$ (see also [7]). Then, the irreducibility of $H_{G_A}^*(\mathfrak{M}(r), \Bbbk)'$ as a $W(\widehat{\mathfrak{gl}}_r)$-module is well known, because a Verma module with a generic highest weight is irreducible. To prove that it is also irreducible as a $\mathcal{W}$-module, use [37, Theorem 8.22] as in [37, Corollary 8.29].

The same argument proves that $H_G^*(\mathfrak{M}(r),\Bbbk)'$ is irreducible as a $\mathcal{W}$-module.

Next, we must identify the representation of $\mathcal{W}$ on $H_G^*(\mathfrak{M}(r),\Bbbk)'$ with $V(\Lambda_r)$. To do that, it is enough to prove that the element $|r\rangle$ of $H_G^*(\mathfrak{M}(r),\Bbbk)'$ is primitive of weight Λ_r. The equality [37, (3.9)] yields

$$c_k(\tau_n) \cup |r\rangle = 0, \quad \forall k \geq 1.$$

Thus, from (49) we deduce that $D_{0,k+1}(|r\rangle) = 0$ for each $k \geq 0$.

Finally, we must check the character formula in (a). It is well known, and follows easily by counting the (isolated) fixed points in $\mathfrak{M}(r,n)$. This finishes the proof of (a).

Now, let us concentrate on (b). Since the element $|r\rangle$ of $\mathrm{Z}(\mathcal{C}^r/\mathbb{Z})'$ is primitive of weight Λ_r and since $M(\Lambda_r)$ has a simple top isomorphic to $V(\Lambda_r)$, there is a unique surjective $\mathcal{W}$-module homomorphism from the submodule $M \subseteq \mathrm{Z}(\mathcal{C}^r/\mathbb{Z})'$ generated by $|r\rangle$ to $H_G^*(\mathfrak{M}(r),\Bbbk)'$ such that $|r\rangle \mapsto |r\rangle$. Since $H_G^*(\mathfrak{M}(r),\Bbbk)'$ and $\mathrm{Z}(\mathcal{C}^r/\mathbb{Z})'$ have the same character, we deduce that

$$\mathrm{Z}(\mathcal{C}^r/\mathbb{Z})' = M = H_G^*\big(\mathfrak{M}(r),\Bbbk\big)'.$$

Let ψ be the unique $\mathcal{W}'$-module isomorphism

$$\psi : \mathrm{Z}(\mathcal{C}^r/\mathbb{Z})' \to H_G^*\big(\mathfrak{M}(r),\Bbbk\big)', \quad |r\rangle \mapsto |r\rangle.$$

Finally, let us prove part (c). By restriction, the map ψ yields a $\mathbf{k}'$-linear isomorphism

$$\psi : \mathrm{Z}\big(R^r(n)\big)' \to H_G^*\big(\mathfrak{M}(r,n),\Bbbk\big)'$$

for each $n \in \mathbb{N}$. We must prove that $\psi(1)$ is invertible in $H_G^*(\mathfrak{M}(r,n),\Bbbk)'$ and the map

$$\phi' : \mathrm{Z}\big(R^r(n)\big)' \to H_G^*\big(\mathfrak{M}(r,n),\Bbbk\big)', \qquad \phi'(\bullet) = \psi(1)^{-1} \cup \psi(\bullet)$$

is a $\mathbf{k}'$-algebra isomorphism. To do so, we consider the diagram

$$\begin{array}{ccc} & \mathrm{Z}\big(R(n)\big)' & \\ {}^{a'}\swarrow & & \searrow^{b'} \\ H_G^*\big(\mathfrak{M}(r,n),\Bbbk\big)' & & \mathrm{Z}\big(R^r(n)\big)' \end{array}$$

The map b' is the $\mathbf{k}'$-algebra homomorphism induced by the canonical map $R(n) \to R^r(n)$. It is surjective by Proposition 4.1. The map a' is the $\mathbf{k}'$-algebra homomorphism given by

$$a'\big(e_i(x_1,\dots,x_n)\big) = c_i(\tau_n), \quad \forall i \in [1,n].$$

Note that, by the definition of the representation of $\mathcal{W}$ on $H_G^*(\mathfrak{M}(r,n),\Bbbk)'$, formula (49) yields

$$\hbar^{-1} D_{0,k+1} = a'\big(p_k(x_1,\dots,x_n)\big) \cup \bullet \quad \text{on } H_G^*\big(\mathfrak{M}(r,n),\Bbbk\big)'. \tag{52}$$

Next, by the definition of the representation of $\mathcal{W}$ on $\mathrm{Z}(\mathcal{C}^r/\mathbb{Z})'$, the formula (42) in the proof of Theorem 4.3 yields

$$\hbar^{-1} D_{0,k+1} = b'\big(p_k(x_1,\dots,x_n)\big) \cdot \bullet \quad \text{on } \mathrm{Z}\big(R^r(n)\big)'. \tag{53}$$

From (52) and (53), since ψ is $\mathcal{W}$-linear, we deduce that

$$\psi\big(b'\big(p_k(x_1,\dots,x_n)\big)\cdot \bullet\big) = a'\big(p_k(x_1,\dots,x_n)\big) \cup \psi(\bullet). \tag{54}$$

Now, an easy induction using (54) yields

$$\psi b'(z) = a'(z) \cup \psi(1), \quad \forall z \in \mathrm{Z}\big(R(n)\big)'. \tag{55}$$

We also deduce that

$$\psi(zz') \cup \psi(1) = \psi(z) \cup \psi(z'), \quad \forall z, z' \in \mathrm{Z}\big(R^r(n)\big)'. \tag{56}$$

Now, since b' and ψ are surjective, equality (55) implies that the element $\psi(1)$ is invertible in the (commutative) $\mathbf{k}'$-algebra $H_G^*(\mathfrak{M}(r,n),\Bbbk)'$. Thus, the map ϕ' above is well defined and it is a $\mathbf{k}'$-algebra homomorphism by (56). It is clearly bijective because it is injective and both sides are finite-dimensional of the same dimension over $\mathbf{k}'$. Further, we have a commutative diagram:

$$\begin{array}{ccc} & \mathrm{Z}\big(R(n)\big)' & \\ {\scriptstyle a'}\swarrow & & \searrow{\scriptstyle b'} \\ H_G^*\big(\mathfrak{M}(r,n),\Bbbk\big)' & \xleftarrow{\ \phi'\ } & \mathrm{Z}\big(R^r(n)\big)' \end{array} \tag{57}$$

Part (c) of the proposition is proved. □

We can now prove the following, which is one of the main results of this paper.

THEOREM 4.7

The canonical map $\mathrm{Z}(R(n)) \to H_G^*(\mathfrak{M}(r,n),\Bbbk)$ *is a surjective* $\mathbf{k}$*-algebra homomorphism. It factors to a* $\mathbf{k}$*-algebra isomorphism* $\mathrm{Z}(R^r(n))^{\mathrm{JM}} \to H_G^*(\mathfrak{M}(r,n),\Bbbk)$.

Proof
We define the maps $a : \mathrm{Z}(R(n)) \to H_G^*(\mathfrak{M}(r,n),\Bbbk)$ and $b : \mathrm{Z}(R(n)) \to \mathrm{Z}(R^r(n))$ as in the triangle (57) above. Thus, we have $a' = \mathbf{k}' \otimes a$ and $b' = \mathbf{k}' \otimes b$. We claim that there is a $\mathbf{k}$-linear map ϕ making the following triangle commute:

$$\begin{array}{ccc} & \mathrm{Z}\big(R(n)\big) & \\ {}^{a}\swarrow & & \searrow^{b} \\ H_G^*\big(\mathfrak{M}(r,n),\Bbbk\big) & \xleftarrow{\quad\phi\quad} & \mathrm{Z}\big(R^r(n)\big)^{\mathrm{JM}} \end{array} \tag{58}$$

To prove this, it is enough to check that $\mathrm{Ker}(b) \subset \mathrm{Ker}(a)$. Since the triangle (57) commutes, we have $\mathrm{Ker}(b') \subset \mathrm{Ker}(a')$. Thus, since $\mathrm{Z}(R(n))$ is free as a $\mathbf{k}$-module, the map $x \mapsto 1 \otimes x$ yields an inclusion

$$\mathrm{Ker}(b) \subset \big(1 \otimes \mathrm{Z}\big(R(n)\big)\big) \cap \mathrm{Ker}(a').$$

Finally, since $\mathrm{Z}(R(n))$ is free as a $\mathbf{k}$-module, the map $x \mapsto 1 \otimes x$ yields an isomorphism

$$\begin{aligned} \mathrm{Ker}(a) &\simeq 1 \otimes \mathrm{Ker}(a) \\ &\simeq \big(1 \otimes \mathrm{Z}\big(R(n)\big)\big) \cap \mathrm{Ker}(a'). \end{aligned}$$

The claim is proved. Note that, since the map b' is surjective and the triangles (57) and (58) commute, we have $\phi' = \mathbf{k}' \otimes \phi$. Thus, since ϕ' is injective, we deduce that ϕ is also injective. Now, recall that Proposition 4.1 and (48) yield

$$\sum_{n \geq 0} \mathrm{grdim}\big(\mathrm{Z}\, R^r(n)\big)^{\mathrm{JM}} q^{2n} = \sum_{n \geq 0} \mathrm{grdim}\, H_G^*\big(\mathfrak{M}(r,n),\Bbbk\big) q^{2n}.$$

We deduce that the map ϕ is an isomorphism $\mathrm{Z}(R^r)^{\mathrm{JM}} \to H_G^*(\mathfrak{M}(r),\Bbbk)$. □

The theorem above can be reformulated in the following way. Set $\mathbf{k}_1 = \Bbbk[y_1,\dots,y_r]$ and consider the $\mathbf{k}_1$-algebras

$$R(n)_1 = R(n)/(\hbar - 1), \qquad R^r(n)_1 = R^r(n)/(\hbar - 1).$$

Recall the inclusion $\mathrm{Z}(R^r(n))^{\mathrm{JM}} \subset \mathrm{Z}(R^r(n)_1)[\hbar]$ in (39). By [8], the canonical map $R(n)_1 \to R^r(n)_1$ yields a surjection $\mathrm{Z}(R(n)_1) \to \mathrm{Z}(R^r(n)_1)$. Since $\mathrm{Z}(R(n)_1)$ is $\mathbb{N}$-graded, this yields an increasing separated and exhaustive $\mathbb{N}$-filtration $F_\bullet$ of $\mathrm{Z}(R^r(n)_1)$. Let $\mathrm{Rees}(\mathrm{Z}(R^r(n)_1))$ be the corresponding Rees algebra; that is,

$$\mathrm{Rees}\big(\mathrm{Z}\big(R^r(n)_1\big)\big) = \sum_{d \geq 0} F_{2d}\big(\mathrm{Z}\big(R^r(n)_1\big)\big) \otimes \hbar^d \subset \mathrm{Z}\big(R^r(n)_1\big)[\hbar].$$

By construction, the map (39) identifies the $\mathbf{k}$-algebras $Z(R^r(n))^{\mathrm{JM}}$ and $\mathrm{Rees}(Z(R^r(n)_1))$. We deduce the following

COROLLARY 4.8
There is a $\mathbf{k}$*-algebra isomorphism* $\mathrm{Rees}(Z(R^r(n)_1)) \simeq H^*_G(\mathfrak{M}(r,n),\Bbbk)$.

Remark 4.9
In the particular case $r = 1$, the corollary was already known and follows from [38].

Appendices

Appendix A. The symmetrizing form

Fix a dominant weight $\Lambda \in P_+$. Let $\alpha \in Q_+$ and $i, j \in I$. Set $\lambda = \Lambda - \alpha$ and $\lambda_i = \langle \alpha_i^\vee, \lambda \rangle$.

A.1. Bubbles

Assume that α has the height n.

Definition A.1
For each $k \in \mathbb{N}$ the *bubble* $B^k_{\pm i,\lambda}$ is the element of $R^\Lambda(\alpha)$ given by the following:

- If $\lambda_i \geq 0$, we set

$$B^k_{+i,\lambda} = \begin{cases} \hat{\varepsilon}'_{i,\lambda}(x_{n+1}^{\lambda_i - 1 + k} e(\alpha, i)) & \text{if } k \geq -\lambda_i + 1, \\ 1 & \text{if } k = \lambda_i = 0, \end{cases}$$

$$B^k_{-i,\lambda} = \begin{cases} \mu_{x_n^{-\lambda_i - 1 + k}}(\hat{\eta}'_{i,\lambda}(1)) & \text{if } k \geq \lambda_i + 1, \\ -p_{\lambda_i - k}(x_{n+1}^{\lambda_i} e(\alpha, i)) & \text{if } 1 \leq k \leq \lambda_i, \\ 1 & \text{if } k = 0. \end{cases}$$

- If $\lambda_i \leq 0$, we set

$$B^k_{+i,\lambda} = \begin{cases} \hat{\varepsilon}'_{i,\lambda}(x_{n+1}^{\lambda_i - 1 + k} e(\alpha, i)) & \text{if } k \geq -\lambda_i + 1, \\ -\mu_{x_n^{-\lambda_i}}(\tilde{\pi}_{-\lambda_i - k}) & \text{if } 1 \leq k \leq -\lambda_i, \\ 1 & \text{if } k = 0, \end{cases}$$

$$B^k_{-i,\lambda} = \begin{cases} \mu_{x_n^{-\lambda_i - 1 + k}}(\hat{\eta}'_{i,\lambda}(1)) & \text{if } k \geq \lambda_i + 1, \\ 1 & \text{if } k = \lambda_i = 0. \end{cases}$$

Note that $B^0_{\pm i,\lambda} = 1$ in all cases. We set by convention $B^k_{\pm,\lambda} = 0$ if $k < 0$.

LEMMA A.2
The elements $B^k_{\pm i,\lambda}$ are homogenous central elements in $R^\Lambda(\alpha)$ of degree $2k$.

Proof
The central and homogenous property follows from the fact that ε'_i, p_k, η'_i are homogenous $R^\Lambda(\alpha)$-bilinear morphisms and that the element x_{n+1} centralizes $R^\Lambda(\alpha)$ in $R^\Lambda(\alpha+\alpha_i)$. The degree is given by an explicit computation. □

Remark A.3
In Khovanov and Lauda's diagrammatic categorification, the element $B^k_{+i,\lambda}$ corresponds to a clockwise *bubble* with a *dot* of multiplicity $\lambda_i - 1 + k$, and $B^k_{-i,\lambda}$ corresponds to a clockwise bubble with a dot of multiplicity $-\lambda_i - 1 + k$.

A.2. A useful lemma

Assume that α has the height $n-1$. Let $\lambda' = \lambda - \alpha_j$ and $\lambda'_i = \langle \alpha_i^\vee, \lambda' \rangle = \lambda_i - a_{ij}$. Consider the morphisms

$$\mathbb{X}_{i,j,\lambda} : E'_i F'_i 1_\lambda \xrightarrow{E'_i \eta'_j F'_i} E'_i E'_j F'_j F'_i 1_\lambda \xrightarrow{\tau\tau} E'_j E'_i F'_i F'_j 1_\lambda \xrightarrow{E'_j \hat{\varepsilon}'_{i,\lambda'} F'_j} E'_j F'_j 1_\lambda,$$

$$\mathbb{I}_{i,j,\lambda} : E'_i F'_i 1_\lambda \xrightarrow{\hat{\varepsilon}'_{i,\lambda}} 1_\lambda \xrightarrow{\eta'_j} E'_j F'_j 1_\lambda.$$

The morphism $\mathbb{X}_{i,j,\lambda}$ is represented by the composition

$$\begin{array}{c} e(\alpha,i)R^\Lambda(\alpha+\alpha_i)e(\alpha,i) \xrightarrow{\iota_j} e(\alpha,ij)R^\Lambda(\alpha+\alpha_i+\alpha_j)e(\alpha,ij) \\ \downarrow \tau_n(\bullet)\tau_n \\ e(\alpha,ji)R^\Lambda(\alpha+\alpha_i+\alpha_j)e(\alpha,ji) \xrightarrow{\hat{\varepsilon}'_{i,\lambda'}} e(\alpha,j)R^\Lambda(\alpha+\alpha_j)e(\alpha,j), \end{array}$$

and $\mathbb{I}_{i,j,\lambda}$ is represented by

$$e(\alpha,i)R^\Lambda(\alpha+\alpha_i)e(\alpha,i) \xrightarrow{\hat{\varepsilon}'_{i,\lambda}} R^\Lambda(\alpha) \xrightarrow{\iota_j} e(\alpha,i)R^\Lambda(\alpha+\alpha_i)e(\alpha,i).$$

In other words, given $a \in e(\alpha,i)R^\Lambda(\alpha+\alpha_i)e(\alpha,i)$, we have

$$\mathbb{X}_{i,j,\lambda}(a) = \hat{\varepsilon}'_{i,\lambda'}\big(\tau_n \iota_j(a)\tau_n\big),$$
$$\mathbb{I}_{i,j,\lambda}(a) = \iota_j \hat{\varepsilon}'_{i,\lambda}(a).$$

Note that, since $\iota_j : R^\Lambda(\beta) \to R^\Lambda(\beta+\alpha_j)$ is the canonical embedding for any $\beta \in Q_+$, we write $\iota_j(b) = be(\beta, j)$ or simply $\iota_j(b) = b$ for any $b \in R^\Lambda(\beta)$. Note also that since $B^k_{\pm i,\lambda} \in \mathrm{Z}(R^\Lambda(\alpha))$, it can be viewed as an element in $\mathrm{End}(1_\lambda)$. Thus $x^r B^s_{\pm i,\lambda} x^t$ defines an endomorphism of $E'_i 1_\lambda F'_i = E'_i F'_i$ for each $r, s, t \in \mathbb{N}$.

LEMMA A.4

The following hold:

(a) *If $i \neq j$, then $\mathbb{X}_{i,j,\lambda} = c_{i,j,-a_{ij},0}\mathbb{I}_{i,j,\lambda}$.*

(b) $\mathbb{X}_{i,i,\lambda} = -\mathbb{I}_{i,i,\lambda} + \sum_{g_1+g_2+g_3=-\lambda'_i-1} x^{g_1} B^{g_2}_{+i,\lambda'} x^{g_3}$.

Note that if $\lambda'_i \geq 0$ the sum over g_1, g_2, g_3 is empty, and hence $\mathbb{X}_{i,i,\lambda} = -\mathbb{I}_{i,i,\lambda}$.

Proof

Let us prove part (a). First, assume $\lambda_i > 0$. Then

$$a = \mu_{\tau_{n-1}}\big(\pi(a)\big) + \sum_{k=0}^{\lambda_i-1} p_k(a) x_n^k$$

with $\pi(a) \in R^\Lambda(\alpha)e(\alpha-\alpha_i, i) \otimes_{R^\Lambda(\alpha-\alpha_i)} e(\alpha-\alpha_i, i)R^\Lambda(\alpha)$ and $p_k(a) \in R^\Lambda(\alpha)$. We have

$$\tau_n \iota_j(a)\tau_n = \mu_{\tau_n\tau_{n-1}\tau_n e(\alpha-\alpha_i, iji)}\big(\pi(a)\big) + \sum_{k=0}^{\lambda_i-1} p_k(a)\tau_n x_n^k e(\alpha, ij)\tau_n.$$

The relations (d), (e), and (f) in Definition 3.1 yield

$$\begin{aligned}
&\mu_{\tau_n\tau_{n-1}\tau_n e(\alpha-\alpha_i, iji)}\big(\pi(a)\big)\\
&\quad = \mu_{(\tau_{n-1}\tau_n\tau_{n-1} + \frac{Q_{i,j}(x_{n-1},x_n)-Q_{i,j}(x_{n+1},x_n)}{x_{n-1}-x_{n+1}})e(\alpha-\alpha_i, iji)}\big(\pi(a)\big),\\
&\tau_n x_n^k e(\alpha, ij)\tau_n = x_{n+1}^k \tau_n^2 e(\alpha, ji) = x_{n+1}^k Q_{j,i}(x_n, x_{n+1})e(\alpha, ji).
\end{aligned}$$

Since $\lambda'_i = \lambda_i - a_{ij} \geq \lambda_i > 0$, we have $\mathbb{X}_{i,j,\lambda}(a) = p_{\lambda'_i-1}(\tau_n \iota_j(a)\tau_n)$, the coefficient of $x_{n+1}^{\lambda_i-a_{ij}-1}$. Since the degree of x_{n+1} in $\frac{Q_{i,j}(x_{n-1},x_n)-Q_{i,j}(x_{n+1},x_n)}{x_{n-1}-x_{n+1}}$ is at most $-a_{ij}-1$, which is less than $\lambda'_i - 1 = \lambda_i - a_{ij} - 1$, and since $p_{\lambda'_i-1} \times (\mu_{\tau_{n-1}\tau_n\tau_{n-1}}(\pi(a))) = 0$, we deduce that $p_{\lambda'_i-1}$ kills $\mu_{\tau_n\tau_{n-1}\tau_n e(\alpha-\alpha_i, iji)}(\pi(a))$.

Next, the degree of x_{n+1} in $x_{n+1}^k Q_{j,i}(x_n, x_{n+1}) = x_{n+1}^k Q_{i,j}(x_{n+1}, x_n)$ is less than or equal to $k - a_{ij}$ with the coefficient of $x_{n+1}^{k-a_{ij}}$ given by $c_{i,j,-a_{ij},0}$; therefore,

$$p_{\lambda_i'-1}\Big(\sum_{k=0}^{\lambda_i-1} p_k(a)\tau_n x_n^k e(\alpha,ij)\tau_n\Big) = \iota_j\big(p_{\lambda_i-1}(a)\big)c_{i,j,-a_{ij},0}$$
$$= c_{i,j,-a_{ij},0}\big(\iota_j \circ \hat{\varepsilon}_i(a)\big).$$

It follows that $\mathbb{X}_{i,j,\lambda} = c_{i,j,-a_{ij},0}\mathbb{I}_{i,j,\lambda}$.

Now, we consider the case $\lambda_i \le 0$. Let $\tilde{a} \in R^\Lambda(\alpha)e(\alpha-\alpha_i,i) \otimes_{R^\Lambda(\alpha-\alpha_i)} e(\alpha-\alpha_i,i)R^\Lambda(\alpha)$ such that $\mu_{\tau_{n-1}}(\tilde{a}) = a$, $\mu_{x_{n-1}^k}(\tilde{a}) = 0$ for $k \in [0,-\lambda_i-1]$. We have

$$\tau_n\iota_j(a)\tau_n = \mu_{\tau_n\tau_{n-1}\tau_n e(\alpha-\alpha_i,iji)}(\tilde{a})$$
$$= \mu_{(\tau_{n-1}\tau_n\tau_{n-1}+\frac{Q_{i,j}(x_{n-1},x_n)-Q_{i,j}(x_{n+1},x_n)}{x_{n-1}-x_{n+1}})e(\alpha-\alpha_i,iji)}(\tilde{a}).$$

Assume that $\lambda_i,\lambda_i' \le 0$. Then $\mathbb{X}_{i,j,\lambda}(a) = \mu_{x_n^{-\lambda_i'}}(\tau_n\widetilde{\iota_j(a)}\tau_n)$. We claim that

$$\tau_n\widetilde{\iota_j(a)}\tau_n = (1\otimes\tau_{n-1}\otimes\tau_{n-1}\otimes 1)(\iota_j\otimes\iota_j)\tilde{a}. \tag{59}$$

Denote the right-hand side by b. Concretely write $\tilde{a} = \sum_r \tilde{a}_r' \otimes \tilde{a}_r''$ with $\tilde{a}_r' \in R^\Lambda(\alpha)e(\alpha-\alpha_i,i)$, $\tilde{a}_r'' \in (\alpha-\alpha_i,i)R^\Lambda(\alpha)$; then $b = \sum_r \tilde{a}_r' e(\alpha-\alpha_i,ij)\tau_{n-1} \otimes \tau_{n-1}e(\alpha-\alpha_i,ij)\tilde{a}_r''$.

To prove (59), note that the degree of x_{n-1} in $\frac{Q_{i,j}(x_{n-1},x_n)-Q_{i,j}(x_{n+1},x_n)}{x_{n-1}-x_{n+1}}$ is less than or equal to $-a_{ij}-1 \le -\lambda_i-1$; hence

$$\mu_{\frac{Q_{i,j}(x_{n-1},x_n)-Q_{i,j}(x_{n+1},x_n)}{x_{n-1}-x_{n+1}}e(\alpha-\alpha_i,iji)}(\tilde{a}) = 0.$$

We deduce

$$\mu_{\tau_n}(b) = \mu_{\tau_{n-1}\tau_n\tau_{n-1}e(\alpha-\alpha_i,iji)}(\tilde{a}) = \tau_n\iota_j(a)\tau_n,$$
$$\mu_{x_n^k}(b) = \mu_{\tau_{n-1}x_n^k\tau_{n-1}e(\alpha-\alpha_i,ij)}(\tilde{a}) = \mu_{x_{n-1}^k Q_{ij}(x_{n-1},x_n)e(\alpha-\alpha_i,ij)}(\tilde{a}).$$

Next, using the fact that $\mu_{x_{n-1}^k}(\tilde{a}) = 0$ for $k \in [0,-\lambda_i-1]$, $\hat{\varepsilon}_{i,\lambda}'(a) = \mu_{x_{n-1}^{-\lambda_i}}(\tilde{a})$, and the degree of x_{n-1} in $Q_{ij}(x_{n-1},x_n)$ is at most $-a_{ij}$ where the coefficient of $x_{n-1}^{-a_{ij}}$ equals $c_{i,j,-a_{ij},0}$, we get $\mu_{x_{n-1}^k Q_{ij}(x_{n-1},x_n)e(\alpha-\alpha_i,ij)}(\tilde{a}) = 0$ if $k \in [0,-\lambda_i-1+a_{ij}]$ and

$$\mu_{x_{n-1}^{-\lambda_i'}Q_{ij}(x_{n-1},x_n)e(\alpha-\alpha_i,ij)}(\tilde{a}) = c_{i,j,-a_{ij},0}\iota_j\big(\mu_{x_{n-1}^{-\lambda_i}}(\tilde{a})\big) = c_{i,j,-a_{ij},0}\iota_j\circ\hat{\varepsilon}_{i,\lambda}'(a).$$

Formula (59) follows, and we have $\mathbb{X}_{i,j,\lambda}(a) = \mu_{x_n^{-\lambda_i'}}(b) = c_{i,j,-a_{ij},0}\mathbb{I}_{i,j,\lambda}(a)$.

Now, assume that $\lambda_i \le 0$ and $\lambda_i' > 0$. Then $\mathbb{X}_{i,j,\lambda}(a) = p_{\lambda_i'-1}(\tau_n\iota_j(a)\tau_n)$. We have $p_{\lambda_i'-1}(\mu_{\tau_{n-1}\tau_n\tau_{n-1}}(\tilde{a})) = 0$. Recall that $Q_{i,j}(u,v) = \sum_{p,q\ge 0} c_{i,j,p,q}u^p v^q$, with $p \le -a_{ij}$. We have

$$\frac{Q_{i,j}(x_{n-1},x_n)-Q_{i,j}(x_{n+1},x_n)}{x_{n-1}-x_{n+1}} = \sum_{p,q\geq 0} c_{i,j,p,q}\frac{x_{n-1}^p - x_{n+1}^p}{x_{n-1}-x_{n+1}}x_n^q$$
$$= \sum_{p,q\geq 0} c_{i,j,p,q}\Big(\sum_{r_1+r_2=p-1} x_{n-1}^{r_1}x_{n+1}^{r_2}\Big)x_n^q.$$

The height of α is $n-1$, and hence x_n, x_{n+1} centralize $R^\Lambda(\alpha)$. We deduce that

$$\mu_{\frac{Q_{i,j}(x_{n-1},x_n)-Q_{i,j}(x_{n+1},x_n)}{x_{n-1}-x_{n+1}}e(\alpha-\alpha_i,iji)}(\tilde{a})$$
$$= \sum_{p,q\geq 0} c_{i,j,p,q}\Big(\sum_{r_1+r_2=p-1} \mu_{x_{n-1}^{r_1}}(\tilde{a})x_{n+1}^{r_2}\Big)x_n^q e(\alpha,ji).$$

Now $\mu_{x_{n-1}^{r_1}}(\tilde{a})\neq 0$ only if $r_1\geq -\lambda_i$ and $p_{\lambda_i'-1}(\mu_{x_{n-1}^{r_1}}(\tilde{a})x_{n+1}^{r_2}x_n^q)\neq 0$ only if $r_2\geq \lambda_i'-1=\lambda_i-a_{ij}-1$. Hence $r_1+r_2=p-1\leq -a_{ij}-1$ implies that

$$p_{\lambda_i'-1}\big(\mu_{\frac{Q_{i,j}(x_{n-1},x_n)-Q_{i,j}(x_{n+1},x_n)}{x_{n-1}-x_{n+1}}e(\alpha-\alpha_i,iji)}(\tilde{a})\big) = c_{i,j,-a_{ij},0}\mu_{x_{n-1}^{-\lambda_i}}(\tilde{a})e(\alpha,j)$$
$$= c_{i,j,-a_{ij},0}\iota_j\circ\hat{\varepsilon}'_{i,\lambda}(a).$$

We get $\mathbb{X}_{i,j,\lambda}(a)=c_{i,j,-a_{ij},0}\mathbb{I}_{i,j,\lambda}$.

Now, we concentrate on part (b) in the case $\lambda_i'\geq 0$. Since $\lambda_i'=\lambda_i-2$, we have $\lambda_i\geq 2$. Thus

$$a=\mu_{\tau_{n-1}}\big(\pi(a)\big)+\sum_{k=0}^{\lambda_i-1}p_k(a)x_n^k \tag{60}$$

with $\pi(a)\in R^\Lambda(\alpha)e(\alpha-\alpha_i,i)\otimes_{R^\Lambda(\alpha-\alpha_i)}e(\alpha-\alpha_i,i)R^\Lambda(\alpha)$ and $p_k(a)\in R^\Lambda(\alpha)$. We have

$$\tau_n\iota_i(a)\tau_n=\mu_{\tau_n\tau_{n-1}\tau_n e(\alpha-\alpha_i,i^3)}\big(\pi(a)\big)+\sum_{k=0}^{\lambda_i-1}p_k(a)\tau_n x_n^k\tau_n e(\alpha,i^2),$$
$$\mathbb{X}_{i,i,\lambda}(a)=p_{\lambda_i'-1}\big(\tau_n\iota_i(a)\tau_n\big).$$

Since $Q_{ii}=0$, the relation (f) in Definition 3.1 yields

$$\mu_{\tau_n\tau_{n-1}\tau_n e(\alpha-\alpha_i,i^3)}\big(\pi(a)\big)=\mu_{\tau_{n-1}\tau_n\tau_{n-1}e(\alpha-\alpha_i,i^3)}\big(\pi(a)\big).$$

Hence it is killed by $p_{\lambda_i'-1}$ when $\lambda_i'>0$. The relations (d) and (e) in Definition 3.1 imply

$$\begin{aligned}\tau_n x_n^k \tau_n e(\alpha, i^2) &= x_{n+1}^k \tau_n^2 e(\alpha, i^2) - \frac{x_n^k - x_{n+1}^k}{x_n - x_{n+1}} \tau_n e(\alpha, i^2) \\ &= -\sum_{r_1+r_2=k-1} x_n^{r_1} x_{n+1}^{r_2} \tau_n e(\alpha, i^2) \\ &= -\sum_{r_1+r_2=k-1} x_n^{r_1} \Big(\tau_n x_n^{r_2} + \sum_{g_1+g_2=r_2-1} x_n^{g_1} x_{n+1}^{g_2} \Big) \\ &= -\sum_{r_1+r_2=k-1} x_n^{r_1} \tau_n x_n^{r_2} - \sum_{g_1+g_2=k-2} (g_1+1) x_n^{g_1} x_{n+1}^{g_2}. \end{aligned} \tag{61}$$

If $\lambda'_i > 0$, then we deduce that

$$\begin{aligned} p_{\lambda'_i-1}\big(\tau_n x_n^k \tau_n e(\alpha, i^2)\big) &= -p_{\lambda_i-3}\Big(\sum_{g_1+g_2=k-2} (g_1+1) x_n^{g_1} x_{n+1}^{g_2} \Big) \\ &= \begin{cases} 0 & \text{if } k < \lambda_i - 1, \\ -1 & \text{if } k = \lambda_i - 1. \end{cases} \end{aligned}$$

By consequence, $\mathbb{X}_{i,i,\lambda}(a) = -p_{\lambda_i-1}(a) e(\alpha, i) = -\iota_i \circ \hat{\varepsilon}'_{i,\lambda}(a) = -\mathbb{I}_{i,i,\lambda}(a)$.

If $\lambda'_i = 0$, then we deduce that

$$\begin{aligned} \tau_n \iota_i(a) \tau_n &= \mu_{\tau_{n-1}\tau_n\tau_{n-1} e(\alpha-\alpha_i, i^3)}\big(\pi(a)\big) - p_1(a) \tau_n e(\alpha, i^2) \\ &= \mu_{\tau_n}\big((1 \otimes \tau_{n-1} \otimes \tau_{n-1} \otimes 1)(\iota_i \otimes \iota_i)\pi(a) - \iota_i\big(p_1(a)\big) \otimes e(\alpha, i^2)\big). \end{aligned}$$

Hence

$$\begin{aligned} \hat{\varepsilon}'_{i,\lambda'}\big(\tau_n \iota_i(a) \tau_n\big) &= \mu_1\big((1 \otimes \tau_{n-1} \otimes \tau_{n-1} \otimes 1)(\iota_i \otimes \iota_i)\pi(a) - \iota_i\big(p_1(a)\big) \otimes e(\alpha, i^2)\big) \\ &= -\iota_i\big(p_1(a)\big). \end{aligned}$$

Here in the second equality we used the fact that $\tau_{n-1}^2 e(\alpha - \alpha_i, i^3) = 0$. Since $\lambda_i = 2$, we have $\hat{\varepsilon}'_{i,\lambda}(a) = p_1(a)$. So we get again $\mathbb{X}_{i,i,\lambda}(a) = -\mathbb{I}_{i,i,\lambda}(a)$.

Finally, we prove part (b) for $\lambda'_i < 0$. By assumption, we have $\lambda'_i = \lambda_i - 2 < 0$. First, if $\lambda_i = 1$, then $a = \mu_{\tau_{n-1}}(\pi(a)) + p_0(a)$ and $\hat{\varepsilon}'_{i,\lambda}(a) = p_0(a)$ as in (60). The same computation as in the previous lemma yields

$$\tau_n \iota_i(a) \tau_n = \mu_{\tau_{n-1}\tau_n\tau_{n-1} e(\alpha-\alpha_i, i^3)}\big(\pi(a)\big).$$

Let $b = (1 \otimes \tau_{n-1} \otimes \tau_{n-1} \otimes 1)(\iota_i \otimes \iota_i)\pi(a)$. Then $\tau_n \iota_i(a) \tau_n = \mu_{\tau_n}(b)$ and $\mu_{x_n^0}(b) = 0$. Since $\lambda'_i = -1$, we deduce $b = \widetilde{\tau_n \iota_i(a) \tau_n}$ and

$$\begin{aligned}\hat{\varepsilon}'_{i,\lambda'}\big(\tau_n\iota_i(a)\tau_n\big) &= \mu_{x_n}(b)\\ &= \mu_{\tau_{n-1}x_n\tau_{n-1}e(\alpha-\alpha_i,i^2)}\big(\pi(a)\big)\\ &= \mu_{\tau_{n-1}e(\alpha-\alpha_i,i^2)}\big(\pi(a)\big)\\ &= a-\iota_i\big(p_0(a)\big).\end{aligned}$$

We conclude that $\mathbb{X}_{i,i,\lambda}(a)=-\mathbb{I}_{i,i,\lambda}(a)+a=-\mathbb{I}_{i,i,\lambda}(a)+B^0_{+i,\lambda'}a$.

It remains to consider the case $\lambda_i\le 0$. Let $\tilde{a}\in R^\Lambda(\alpha)e(\alpha-\alpha_i,i)\otimes_{R^\Lambda(\alpha-\alpha_i)} e(\alpha-\alpha_i,i)R^\Lambda(\alpha)$ such that $\mu_{\tau_{n-1}}(\tilde{a})=a$, $\mu_{x^k_{n-1}}(\tilde{a})=0$ for $k\in[0,-\lambda_i-1]$. We have

$$\begin{aligned}\tau_n\iota_i(a)\tau_n &= \mu_{\tau_n\tau_{n-1}\tau_n e(\alpha-\alpha_i,i^3)}(\tilde{a})\\ &= \mu_{\tau_{n-1}\tau_n\tau_{n-1}e(\alpha-\alpha_i,i^3)}(\tilde{a}).\end{aligned}$$

Let $b=(1\otimes\tau_{n-1}\otimes\tau_{n-1}\otimes 1)(\iota_i\otimes\iota_i)(\tilde{a})\in R^\Lambda(\alpha+\alpha_i)e(\alpha,i)\otimes_{R^\Lambda(\alpha)} e(\alpha,i)\times R^\Lambda(\alpha+\alpha_i)$. Then $\mu_{\tau_n}(b)=\tau_n\iota_i(a)\tau_n$ and $\mu_{x^k_n}(b)=\mu_{\tau_{n-1}x^k_n\tau_{n-1}e(\alpha-\alpha_i,i^2)}(\tilde{a})$. A computation similar to (61) yields

$$\begin{aligned}&\tau_{n-1}x_n^k\tau_{n-1}e(\alpha-\alpha_i,i^2)\\ &\quad=\Big(\sum_{g_1+g_2=k-1}x_n^{g_1}\tau_{n-1}x_n^{g_2}-\sum_{g_1+g_2=k-2}(g_2+1)x_{n-1}^{g_1}x_n^{g_2}\Big)e(\alpha-\alpha_i,i^2).\end{aligned}$$

Therefore, for $0\le k\le -\lambda'_i$, we have

$$\begin{aligned}\mu_{x_n^k}(b) &= \sum_{g_1+g_2=k-1}x_n^{g_1}\mu_{\tau_{n-1}}(\tilde{a})x_n^{g_2}-\sum_{g_1+g_2=k-2}(g_2+1)\mu_{x_{n-1}^{g_1}}(\tilde{a})x_n^{g_2}\\ &= \sum_{g_1+g_2=k-1}x_n^{g_1}ax_n^{g_2}-\delta_{k=-\lambda'_i}\mu_{x_{n-1}^{-\lambda_i}}(\tilde{a}).\end{aligned}$$

Here we used the fact that $\mu_{\tau_{n-1}}(\tilde{a})=a$ and $\mu_{x^k_{n-1}}(\tilde{a})=0$ for $0\le k\le -\lambda_i-1$ in the second equality. Finally, recall from (7) that there are elements $\tilde{\pi}_\ell\in R^\Lambda(\alpha+\alpha_i)e(\alpha,i)\otimes_{R^\Lambda(\alpha)} e(\alpha,i)R^\Lambda(\alpha+\alpha_i)$ for $0\le\ell\le-\lambda'_i-1$ such that $\mu_{\tau_n}(\tilde{\pi}_\ell)=0$ and $\mu_{x_n^k}(\tilde{\pi}_\ell)=\delta_{k,l}$. Set

$$c=b-\sum_{k=0}^{-\lambda'_i-1}\Big(\sum_{g_1+g_2=k-1}x_n^{g_1}\tilde{\pi}_k x_n^{g_2}\Big).$$

Then $\mu_{\tau_n}(c)=\mu_{\tau_n}(b)=\tau_n\iota_i(a)\tau_n$ and $\mu_{x^k_n}(c)=0$ for $0\le k\le-\lambda'_i-1$. Hence $c=\widetilde{\tau_n\iota_i(a)\tau_n}$, and $\mathbb{X}_{i,i,\lambda}(a)$ is equal to

$$\mu_{x_n^{-\lambda'_i}}(c) = \mu_{x_n^{-\lambda'_i}}(b) - \sum_{k=0}^{-\lambda'_i-1}\Big(\sum_{g_1+g_2=k-1} x_n^{g_1} a \mu_{x_n^{-\lambda'_i}}(\tilde{\pi}_k) x_n^{g_2}\Big)$$

$$= -\mu_{x_{n-1}^{-\lambda_i}}(\tilde{a})e(\alpha,i) + \sum_{g_1+g_2=-\lambda'_i-1} x_n^{g_1} a x_n^{g_2}$$

$$- \sum_{g_3=1}^{-\lambda'_i} \sum_{g_1+g_2=-\lambda'_i-1-g_3} x_n^{g_1} a \mu_{x_n^{-\lambda'_i}}(\tilde{\pi}_{-\lambda'_i-g_3}) x_n^{g_2}$$

$$= -\mathbb{I}_{i,i,\lambda}(a) + \sum_{g_1+g_2+g_3=-\lambda'_i-1} x_n^{g_1} B^{g_3}_{+i,\lambda'} x_n^{g_2}(a).$$

In the second equality, we substituted $g_3 = -\lambda'_i - k$. In the third equality, we used the definition of $B^{g_3}_{+i,\lambda'}$ for $0 \le g_3 \le -\lambda'_i$ in Definition A.1. □

Remark A.5
Assume that the *Q-cyclicity* condition in [10, (2.4), (2.5)] holds for $\mathcal{V}^\Lambda$; that is, the endomorphisms $x_i \in \operatorname{End}(E'_i)$, $\tau_{ij} \in \operatorname{End}(E'_i E'_j)$ are such that

$$x_i^\vee = {}^\vee x_i, \qquad \tau_{ij}^\vee = {}^\vee \tau_{ij}.$$

See the notation in Section 2.1.4. Then, under the adjunction isomorphism $\operatorname{Hom}(E'_i F'_i, E'_j F'_j) = \operatorname{Hom}(F_j E_i, F_j E_i)$, part (a) of the previous lemma gives the second equality in the mixed relation [10, (2.16)]. For $i = j$, under the same adjunction, part (b) gives the second equalities in [10, (2.22), (2.24), (2.26)]. The other relations in [10, Section 2.6.3] can be checked similarly. Since the computations are quite lengthy and will not be needed, we omit the details here. Finally, the *fake bubbles* relation [10, (2.20)] is proved in Lemma B.2(a) below. Therefore, assuming the Q-cyclicity condition, we have proved that $\mathcal{V}^\Lambda$ carries a representation of Khovanov and Lauda's 2-Kac–Moody algebra. The Q-cyclicity condition can probably be proved by similar computations as in [18]. We have not checked this.

A.3. *Proof of Proposition 3.10*

Assume that $\operatorname{ht}(\alpha) = n$. For $\nu = (\nu_1, \dots, \nu_n) \in I^\alpha$ and $k \in [1, n]$, we set $\nu^{(k)} = (\nu_1, \dots, \nu_k)$. Consider the map

$$\hat{\varepsilon}_\nu = \hat{\varepsilon}_{\nu_1} \circ \cdots \circ \hat{\varepsilon}_{\nu_n} : e(\nu) R^\Lambda(\alpha) e(\nu) \to \mathbf{k}.$$

Recall that $\hat{\varepsilon}_{\nu_k}$ is a map

$$\hat{\varepsilon}_{\nu_k} : e(\nu^{(k)}) R^\Lambda\Big(\alpha - \sum_{j=k+1}^{n} \alpha_{\nu_j}\Big) e(\nu^{(k)}) \to e(\nu^{(k-1)}) R^\Lambda\Big(\alpha - \sum_{j=k}^{n} \alpha_{\nu_j}\Big) e(\nu^{(k-1)}).$$

Next, we define an invertible element $r_\nu \in \mathbf{k}_0^\times$ by

$$r_\nu = \prod_{k<l} r_{\nu_k,\nu_l}, \quad \text{where } r_{ij} = \begin{cases} c_{i,j,-a_{ij},0} & \text{if } j \neq i, \\ 1 & \text{if } j = i. \end{cases} \tag{62}$$

Definition A.6
We define a $\mathbf{k}$-linear map $t_{\Lambda,\alpha} : R^\Lambda(\alpha) \to \mathbf{k}$ by setting, for all $\nu, \nu' \in I^\alpha$,

$$t_{\Lambda,\alpha}\big(e(\nu) \bullet e(\nu')\big) = \begin{cases} 0 & \text{if } \nu \neq \nu', \\ r_\nu \hat{\varepsilon}_\nu(e(\nu) \bullet e(\nu')) & \text{if } \nu = \nu'. \end{cases}$$

We will abbreviate $t_\alpha = t_{\Lambda,\alpha}$ and $t_\Lambda = \sum_\alpha t_\alpha$. For $\nu \in I^\alpha$, we set

$$r(\alpha, \nu_n) = \prod_{k=1}^{n-1} r_{\nu_k,\nu_n}. \tag{63}$$

By Corollary 3.9, the map t_α is homogenous of degree $-d_{\Lambda,\alpha}$. Note that $r_\nu = r(\alpha, \nu_n) r_{\nu^{(n-1)}}$. Therefore we have

$$t_\alpha(a) = r(\alpha - \alpha_{\nu_n}, \nu_n) t_{\alpha-\alpha_{\nu_n}}\big(\hat{\varepsilon}_{\nu_n}(a)\big), \quad \forall a \in e(\nu) R^\Lambda(\alpha) e(\nu). \tag{64}$$

We will prove that t_α is a symmetric form. First, note that the form t_α is Frobenius. Indeed, let $E' = \bigoplus_{\nu \in I^n} E'_\nu$ and $F' = \bigoplus_{\nu \in I^n} F'_\nu$. By Theorem 3.8 we have an adjoint pair (E', F') relative to the counit $\sum_\nu \hat{\varepsilon}_\nu$, where $\hat{\varepsilon}_\nu : E'F' \to 1$ is the morphism $E'_\nu F'_\nu \to 1$ represented by the linear map $\hat{\varepsilon}_\nu$ above and extended by 0. From the discussion in Section 2.2.3 applied with $B = \mathbf{k}$ and $A = R^\Lambda(n)$, we deduce that the linear form $\sum_\nu \hat{\varepsilon}_\nu : R^\Lambda(n) \to \mathbf{k}$ is Frobenius, and hence $t_\alpha : R^\Lambda(\alpha) \to \mathbf{k}$ is also Frobenius. We must prove that for each $w, z \in R^\Lambda(\alpha)$ we have

$$t_\alpha(zw) = t_\alpha(wz). \tag{65}$$

Without loss of generality, we may assume that

$$z \in e(\nu) R^\Lambda(\alpha) e(\mu), \qquad w \in e(\mu) R^\Lambda(\alpha) e(\nu), \tag{66}$$

the other cases being trivial. We will prove (65) by induction on the height of α. Assuming it holds for all α's of height $n-1$, let us prove it for α of height n. We will write $\nu_n = i$, $\mu_n = j$, $\beta = \alpha - \alpha_i - \alpha_j$, and $\lambda = \Lambda - (\alpha - \alpha_i)$.

First, consider the case when z belongs to the image of the map

$$\begin{aligned} \sigma_{ij} = \mu_{\tau_{n-1}} : R^\Lambda(\alpha - \alpha_i) e(\beta, j) \otimes_{R^\Lambda(\beta)} e(\beta, i) R^\Lambda(\alpha - \alpha_j) \\ \to e(\alpha - \alpha_i, i) R^\Lambda(\alpha) e(\alpha - \alpha_j, j) \end{aligned}$$

for $\beta = \alpha - \alpha_i - \alpha_j$, and w belongs to the image of σ_{ji}. Note that this is always the case if $i \neq j$ or if $i = j$ and $\lambda_i \leq 0$. In this situation, up to taking a linear combination, we may write

$$z = \iota_i(z')\tau_{n-1}\iota_j(z''), \qquad w = \iota_j(w')\tau_{n-1}\iota_i(w''), \tag{67}$$

where ι_s is the canonical morphism $R^\Lambda(\alpha - \alpha_s) \to e(\alpha - \alpha_s, s)R^\Lambda(\alpha)e(\alpha - \alpha_s, s)$ for $s = i, j$ and

$$z' \in e(\nu^{(n-1)})R^\Lambda(\alpha - \alpha_i)e(\xi, j), \qquad z'' \in e(\xi, i)R^\Lambda(\alpha - \alpha_j)e(\mu^{(n-1)}),$$
$$w' \in e(\mu^{(n-1)})R^\Lambda(\alpha - \alpha_j)e(\eta, i), \qquad w'' \in e(\eta, j)R^\Lambda(\alpha - \alpha_i)e(\nu^{(n-1)})$$

for some $\xi, \eta \in I^\beta$. By (64) and $R^\Lambda(\alpha - \alpha_i)$-bilinearity of $\hat{\varepsilon}_i$, we have

$$\begin{aligned} t_\alpha(zw) &= r_\nu \hat{\varepsilon}_\nu\big(\iota_i(z')\tau_{n-1}\iota_j(z''w')\tau_{n-1}\iota_i(w'')\big) \\ &= r(\alpha - \alpha_i, i)t_{\alpha-\alpha_i}\big(z'\hat{\varepsilon}_i\big(\tau_{n-1}\iota_j(z''w')\tau_{n-1}\big)w''\big). \end{aligned}$$

Next, Lemma A.4 yields

$$\hat{\varepsilon}_i\big(\tau_{n-1}\iota_j(z''w')\tau_{n-1}\big) = r_{ij}\iota_j\hat{\varepsilon}_i(z''w') + \delta_{ij} \sum_{g_1+g_2+g_3=-\lambda_i-1} x_{n-1}^{g_1} B_{+i,\lambda}^{g_2} z''w'x_{n-1}^{g_3}.$$

Therefore $t_\alpha(zw) = A(z,w) + B(z,w)$, where

$$\begin{aligned} A(z,w) &= r(\alpha - \alpha_i, i)r_{ij}t_{\alpha-\alpha_i}\big(z'\iota_j \circ \hat{\varepsilon}_i(z''w')w''\big), \\ B(z,w) &= \delta_{ij}r(\alpha - \alpha_i, i)t_{\alpha-\alpha_i}\Big(\sum_{g_1+g_2+g_3=-\lambda_i-1} z'x_{n-1}^{g_1} B_{+i,\lambda}^{g_2} z''w'x_{n-1}^{g_3}w''\Big). \end{aligned}$$

Thus, the formula (65) follows from the identities

$$A(z,w) = A(w,z), \qquad B(z,w) = B(w,z). \tag{68}$$

Let us first prove (68) for $A(z,w)$. We have

$$\begin{aligned} t_{\alpha-\alpha_i}\big(z'\iota_j \circ \hat{\varepsilon}_i(z''w')w''\big) &= t_{\alpha-\alpha_i}\big(\iota_j \circ \hat{\varepsilon}_i(z''w')w''z'\big) \\ &= \Big(\prod_{p=1}^{n-2} r_{\xi_p,j}\Big)t_\beta\big(\hat{\varepsilon}_j\big(\iota_j \circ \hat{\varepsilon}_i(z''w')w''z'\big)\big) \\ &= \Big(\prod_{p=1}^{n-2} r_{\xi_p,j}\Big)t_\beta\big(\hat{\varepsilon}_i(z''w')\hat{\varepsilon}_j(w''z')\big). \end{aligned}$$

Here, the first equality is because $t_{\alpha-\alpha_i}$ is symmetric by induction; the second one is given by (64) since $\hat{\varepsilon}_i(z''w')w''z' \in e(\xi, j)R^\Lambda(\alpha - \alpha_i)e(\xi, j)$; and the third one is the $R^\Lambda(\beta)$-bilinearity of $\hat{\varepsilon}_j$. Now, observe that $r_{ij}(\prod_{p=1}^{n-2} r_{\xi_p,j}) = \prod_{p=1}^{n-1} r_{\nu'_p,j}$. We deduce that

$$A(z,w) = r(\alpha-\alpha_i,i)r(\alpha-\alpha_j,j)t_\beta\big(\hat{\varepsilon}_i(z''w')\hat{\varepsilon}_j(w''z')\big).$$

Exchanging z and w means also exchanging i and j, and ν and μ. So the right-hand side is symmetric with respect to z and w. We deduce that $A(z,w) = A(w,z)$.

Next, since $t_{\alpha-\alpha_i}$ is symmetric by the inductive hypothesis and since $B^{g_2}_{+i,\lambda}$ belongs to the center of $R^\Lambda(\alpha-\alpha_i)$, we have

$$B(z,w) = \delta_{ij} r(\alpha-\alpha_i,i)t_{\alpha-\alpha_i}\Big(\sum_{g_1+g_2+g_3=-\lambda_i-1} B^{g_2}_{+i,\lambda} x^{g_1}_{n-1} z''w' x^{g_3}_{n-1} w''z'\Big).$$

Exchanging z and w means also exchanging i and j, and ν and μ. But here $B(z,w) \neq 0$ only when $i = j$. In this situation, $r(\alpha-\alpha_i,i) = r(\alpha-\alpha_j,j)$. We conclude that $B(z,w) = B(w,z)$. Hence, we have proved (65) when z and w are both of the form (67).

If z and w are not of the form (67), then we must have $i = j$ and $\lambda_i > 0$ as discussed in the paragraph before (67). In this situation, $z \in e(\alpha-\alpha_i,i)R^\Lambda(\alpha)e(\alpha-\alpha_i,i)$ can be uniquely written as $z = \mu_{\tau_{n-1}}(\pi(z)) + \sum_{k=0}^{\lambda_i-1} p_k(z)x_n^k$ (see Theorem 3.7), and similarly for w. By the linearity of t_α the remaining cases to be considered are

(1) $z = z_k x_n^k, w = w'\tau_{n-1}w''$,

(2) $z = z_k x_n^k, w = w_l x_n^l$,

for $z_k, w_l \in R^\Lambda(\alpha-\alpha_i)$, $k, l \in [0, \lambda_i - 1]$, and $w' \in R^\Lambda(\alpha-\alpha_i)e(\beta,i), w'' \in e(\beta,i)R^\Lambda(\alpha-\alpha_i)$. Note that in both cases we have

$$\begin{aligned} t_\alpha(zw-wz) &= r_\nu\hat{\varepsilon}_\nu(zw) - r_\mu\hat{\varepsilon}_\mu(wz) \\ &= r(\alpha,i)t_{\alpha-\alpha_i}\big(\hat{\varepsilon}_i(zw-wz)\big). \end{aligned}$$

Here, in the last equality, we used (64) and the fact that $i = j$. By induction, to prove that t_α is symmetric, it is enough to prove that $\hat{\varepsilon}_i(zw-wz)$ belongs to the commutator of $R^\Lambda(\alpha-\alpha_i)$. We do this case by case:

- We have $zw = z_k w'(x_n^k\tau_{n-1})e(\beta,i^2)w''$. Now

$$x_n^k\tau_{n-1}e(\beta,i^2) = \Big(\tau_{n-1}x_{n-1} + \sum_{p+q=k-1} x^p_{n-1}x^q_n\Big)e(\beta,i^2).$$

  Hence $\hat{\varepsilon}_i(zw) = p_{\lambda_i-1}(zw) = 0$. Similarly, $\hat{\varepsilon}_i(wz) = 0$.
- We have $zw - wz = (z_k w_l - w_l z_k)x_n^{k+l}$. Hence $\hat{\varepsilon}_i(zw-wz) = [z_k,w_l] \times \hat{\varepsilon}_i(x_n^{k+l})$. Note that $\hat{\varepsilon}_i(x_n^{k+l})$ belongs to the center of $R^\Lambda(\alpha-\alpha_i)$. So $\hat{\varepsilon}_i(zw-wz)$ belongs to the commutator of $R^\Lambda(\alpha-\alpha_i)$.

The proof of Proposition 3.10 is now complete.

Appendix B. Relations

In this section we prove Theorem 3.25.

B.1. A useful lemma

Let $z \in e(\alpha, i)R^\Lambda(\alpha + \alpha_i)e(\alpha, i)$. Recall the following:

- If $\lambda_i \geq 0$, then z can be uniquely written as

$$z = \mu_{\tau_n}\big(\pi(z)\big) + \sum_{k=0}^{\lambda_i - 1} p_k(z) x_{n+1}^k.$$

- If $\lambda_i \leq 0$, then there are $\tilde{z}, \tilde{\pi}_k \in R^\Lambda(\alpha)e(\alpha - \alpha_i, i) \otimes_{R^\Lambda(\alpha-\alpha_i)} e(\alpha - \alpha_i, i) \times R^\Lambda(\alpha)$ such that $\mu_{\tau_n}(\tilde{z}) = z$, $\mu_{x_n^p}(\tilde{z}) = \mu_{\tau_n}(\tilde{\pi}_k) = 0$ and $\mu_{x_n^p}(\tilde{\pi}_k) = \delta_{k,p}$ for $k, p \in [0, -\lambda_i - 1]$.

To prove Theorem 3.25, we will need the following technical result.

LEMMA B.1

For each $r \in \mathbb{N}$, we have the following:

(a) *If $\lambda_i > 0$, then*

$$\pi\big(x_{n+1}^r e(\alpha, i)\big) = \sum_{a=0}^{r-\lambda_i} (B_{+i,\lambda}^{r-\lambda_i - a} \otimes x_n^a \otimes 1 \otimes 1)\big(-\hat{\eta}_i'(1)\big),$$

$$p_k\big(x_{n+1}^r e(\alpha, i)\big) = \sum_{a=0}^{\lambda_i - k - 1} B_{+i,\lambda}^{r-k-a} B_{-i,\lambda}^a, \quad \forall k \in [0, \lambda_i - 1].$$

(b) *If $\lambda_i \leq 0$, then*

$$\mu_{x_n^r}(\tilde{z}) = \sum_{p=0}^{r+\lambda_i} \hat{\varepsilon}_{i,\lambda}'(z x_{n+1}^p) B_{-i,\lambda}^{r+\lambda_i - p}, \quad \forall z \in e(\alpha, i)R^\Lambda(\alpha + \alpha_i)e(\alpha, i),$$

$$\mu_{x_n^r}(\tilde{\pi}_k) = \sum_{a=k}^{-\lambda_i - 1} B_{+i,\lambda}^{a-k} B_{-i,\lambda}^{r-a}, \quad \forall k \in [0, -\lambda_i - 1].$$

(c) $\sum_{a=0}^r B_{+i,\lambda}^{r-a} B_{-i,\lambda}^a = \delta_{r,0}$.

Proof

First, assume that $\lambda_i > 0$. To simplify notation, we write $x_{n+1}^r = x_{n+1}^r e(\alpha, i)$. We have

$$x_{n+1}^{r+1} = x_{n+1}\Big(\mu_{\tau_n}\big(\pi(x_{n+1}^r)\big) + \sum_{k=0}^{\lambda_i - 1} p_k(x_{n+1}^r) x_{n+1}^k\Big)$$

$$= \mu_{x_{n+1}\tau_n}\big(\pi(x_{n+1}^r)\big) + \sum_{k=0}^{\lambda_i-1} p_k(x_{n+1}^r)x_{n+1}^{k+1}$$

$$= \mu_{\tau_n x_n}\big(\pi(x_{n+1}^r)\big) + \varepsilon_i'\big(\pi(x_{n+1}^r)\big) + \sum_{k=0}^{\lambda_i-1} p_k(x_{n+1}^r)x_{n+1}^{k+1}.$$

Here, in the last equality, we used $(x_{n+1}\tau - \tau_n x_n - 1)e(\alpha - \alpha_i, i^2) = 0$ and $\mu_1 = \varepsilon_i'$. It follows that

$$\pi(x_{n+1}^{r+1}) = (1 \otimes x_n \otimes 1 \otimes 1)\pi(x_{n+1}^r) + p_{\lambda_i-1}(x_{n+1}^r)\pi(x_{n+1}^{\lambda_i}), \tag{69}$$

$$p_0(x_{n+1}^{r+1}) = \varepsilon_i'\big(\pi(x_{n+1}^r)\big) + p_{\lambda_i-1}(x_{n+1}^r)\,p_0(x_{n+1}^{\lambda_i}), \tag{70}$$

$$p_k(x_{n+1}^{r+1}) = p_{k-1}(x_{n+1}^r) + p_{\lambda_i-1}(x_{n+1}^r)\,p_k(x_{n+1}^{\lambda_i}), \quad \forall k \in [1, \lambda_i - 1]. \tag{71}$$

Now, recall that $p_{\lambda_i-1}(x_{n+1}^{r+\lambda_i-1}) = \hat{\varepsilon}_{i,\lambda}'(x_{n+1}^{r+\lambda_i-1}) = B_{+i,\lambda}^r$, $p_k(x_{n+1}^{\lambda_i}) = B_{-i,\lambda}^{\lambda_i-k}$ and $\hat{\eta}_{i,\lambda}' = -\pi(x_{n+1}^{\lambda_i})$.

If $r < \lambda_i$, then the first equality in part (a) is trivial with both sides being zero. If $r \geq \lambda_i$, then it follows recursively from (69).

Next, by applying recursively (71), we obtain

$$p_k(x_{n+1}^{r+\lambda_i}) = p_0(x_{n+1}^{r+\lambda_i-k}) - \sum_{a=0}^{k-1} B_{+i,\lambda}^{r-a} B_{-i,\lambda}^{\lambda_i-k+a}.$$

Substituting $p_0(x_{n+1}^{r+\lambda_i-k})$ using (70) gives

$$p_k(x_{n+1}^{r+\lambda_i}) = \varepsilon_i'\big(\pi(x_{n+1}^{r+\lambda_i-k-1})\big) - \sum_{a=0}^{k} B_{+i,\lambda}^{r-a} B_{-i,\lambda}^{\lambda_i-k+a}.$$

Apply this to the special case $k = \lambda_i - 1$ and we get

$$\varepsilon_i'\big(\pi(x_{n+1}^r)\big) = \sum_{a=0}^{\lambda_i} B_{+i,\lambda}^{r+1-a} B_{-i,\lambda}^a. \tag{72}$$

Therefore, we deduce that

$$p_k(x_{n+1}^{r+\lambda_i}) = \sum_{a=0}^{\lambda_i} B_{+i,\lambda}^{r+\lambda_i-k-a} B_{-i,\lambda}^a - \sum_{a=\lambda_i-k}^{\lambda_i} B_{+i,\lambda}^{r+\lambda_i-k-a} B_{-i,\lambda}^a$$

$$= \sum_{a=0}^{\lambda_i-1-k} B_{+i,\lambda}^{r+\lambda_i-k-a} B_{-i,\lambda}^a.$$

This proves the second equality in part (a) for $r \geq \lambda_i$.

On the other hand, we have

$$\varepsilon_i'\big(\pi(x_{n+1}^r)\big)=\sum_{a=0}^{r-\lambda_i}B_{+i,\lambda}^{r-\lambda_i-a}\mu_{x_n^a}\big(-\hat{\eta}_i'(1)\big)=-\sum_{a=\lambda_i+1}^{r+1}B_{+i,\lambda}^{r+1-a}B_{-i,\lambda}^{a}$$

by the first equality in part (a). Combined with (72) it gives part (c) for $r>0$. The case $r=0$ is obvious.

Finally, for $r\le\lambda_i-1$ we have $p_k(x_{n+1}^r e(\alpha,i))=\delta_{k,r}$ and

$$\sum_{a=0}^{\lambda_i-k-1}B_{+i,\lambda}^{r-k-a}B_{-i,\lambda}^{a}=\sum_{a=0}^{r-k}B_{+i,\lambda}^{r-k-a}B_{-i,\lambda}^{a}=\delta_{r,k}$$

by part (c). We deduce the second equality in part (a) for $r\le\lambda_i-1$.

The case $\lambda_i\le 0$ is proved by a computation of similar style. We only indicate some key steps. First, one checks by a direct computation that

$$\widetilde{zx_{n+1}}=(1\otimes x_n\otimes 1\otimes 1)\tilde{z}-\hat{\varepsilon}_{i,\lambda}'(z)\hat{\eta}_i'(1).$$

Applying it recursively, we get

$$\widetilde{zx_{n+1}^r}=(1\otimes x_n^r\otimes 1\otimes 1)\tilde{z}-\sum_{p=0}^{r-1}\hat{\varepsilon}_{i,\lambda}'(zx_{n+1}^p)(1\otimes x_n^{r-1-p}\otimes 1\otimes 1)\hat{\eta}_i'(1).$$

The first equality in (b) is obtained by applying $\mu_{x_n^{-\lambda_i}}$ to both sides of the above equality with r replaced by $r+\lambda_i$. To prove the second equality, observe that for $k,p\in[0,-\lambda_i-2]$ we write $A=(1\otimes x_n\otimes 1\otimes 1)\tilde{\pi}_{k+1}$, and we have $\mu_{\tau_n}(A)=0$, $\mu_{x_n^p}(A)=\delta_{k,p}$, and $\mu_{x_n^{-\lambda_i-1}}(A)=-B_{-i,\lambda}^{-\lambda_i-k-1}$. We deduce that

$$\tilde{\pi}_k=\sum_{p=0}^{-\lambda_i-1-k}B_{-i,\lambda}^{-\lambda_i-k-1-p}(1\otimes x_n^p\otimes 1\otimes 1)\hat{\eta}_{i,\lambda}'(1).$$

Now, apply $\mu_{x_n^r}$ to both sides, and we get the second equality in (b). Finally, to get (c), observe that the first equality applied to $z=1$ yields

$$\mu_{x_n^r}(\tilde{1})=\sum_{p=-\lambda_i+1}^{r+1}B_{+i,\lambda}^{p}B_{-i,\lambda}^{r+1-p}.$$

On the other hand, it is easy to check that $-\tilde{1}=(1\otimes x_n\otimes 1\otimes 1)\tilde{\pi}_0+B_{+i,\lambda}^{-\lambda_i}\hat{\eta}_{i,\lambda}'(1)$. Hence

$$-\mu_{x_n^r}(\tilde{1})=\mu_{x_n^{r+1}}(\tilde{\pi}_0)+B_{+i,\lambda}^{-\lambda_i}B_{-i,\lambda}^{r+1+\lambda_i}=\sum_{p=0}^{-\lambda_i}B_{+i,\lambda}^{p}B_{-i,\lambda}^{r+1-p}$$

by the second equality in (b). Combining the two equalities gives (c). □

B.2. The Cartan loop operators

Consider the following formal power series:

$$B_{\pm i,\lambda}(z)=\sum_{k\geq 0}B^k_{\pm i,\lambda}z^k\in \mathrm{Z}\big(R^\Lambda(\alpha)\big)\big[[z]\big].$$

The aim of this section is to express the coefficients of the formal series $B_{\pm i,\lambda}(z)$ as elements in the image of the canonical map $Z(R(\alpha))\to Z(R^\Lambda(\alpha))$ (see Proposition B.3 for details). To do that, fix formal variables $y_{i,1},\dots,y_{i,\Lambda_i}$ of degree 2 such that

$$a_i^\Lambda(u)=\prod_{p=1}^{\Lambda_i}(u+y_{ip})\quad\text{with } c_{ip}=e_p(y_{i,1},\dots,y_{i,\Lambda_i}). \tag{73}$$

Consider the following formal series in $\mathbf{k}[[u,v]]$:

$$q_{ij}(u,v)=\begin{cases}u^{-a_{ij}}Q_{ij}(u^{-1},v)/c_{i,j,-a_{ij},0} & \text{if } i\neq j,\\ (1-uv)^{-2} & \text{else.}\end{cases} \tag{74}$$

LEMMA B.2

For each $\alpha\in Q_+$ of height n, we have

(a) $B_{+i,\lambda}(z)B_{-i,\lambda}(z)=1$;

(b) $(B_{\pm i,\lambda}(z)-q_{ij}(z,x_n)^{\mp 1}B_{\pm i,\lambda+\alpha_j}(z))e(\alpha-\alpha_j,j)=0$ *for each* i,j.

Proof

Part (a) will be proved in Lemma B.1(c) below. Now, we concentrate on (b). By (a), it is enough to prove it for $B_{+i,\lambda}(z)$. Write $\beta=\alpha-\alpha_j$ and $\lambda'=\Lambda-\beta=\lambda+\alpha_j$.

First, assume that $i=j$. Recall that

$$\varphi_n e(\beta,i^2)=(x_n\tau_n-\tau_n x_n)e(\beta,i^2)\in R^\Lambda(\alpha+\alpha_i).$$

We have

$$\begin{aligned}&x^k_{n+1}e(\beta,i^2)\\&\quad=\varphi_n x_n^k\varphi_n e(\beta,i^2)\\&\quad=(x_n\tau_n-\tau_n x_n)x_n^k(x_n\tau_n-\tau_n x_n)e(\beta,i^2)\\&\quad=\big(x_n(\tau_n x_n^{k+1}\tau_n)-\tau_n x_n^{k+2}\tau_n-x_n(\tau_n x_n^k\tau_n)x_n+(\tau_n x_n^{k+1}\tau_n)x_n\big)e(\beta,i^2).\end{aligned}$$

By Lemma A.4(b), for all $\ell\in\mathbb{N}$, we have

$$
\begin{aligned}
\hat{\varepsilon}'_{i,\lambda}\big(\tau_n x_n^{\ell}\tau_n e(\beta,i^2)\big) &= \mathbb{X}_{i,i,\lambda'}\big(x_n^{\ell}e(\beta,i)\big)\\
&= \Big(-\mathbb{I}_{i,i,\lambda'} + \sum_{g_1+g_2+g_3=-\lambda_i-1} x^{g_1}B^{g_2}_{+i,\lambda}x^{g_3}\Big)\big(x_n^{\ell}e(\beta,i)\big)\\
&= -\hat{\varepsilon}'_{i,\lambda'}\big(x_n^{\ell}e(\beta,i)\big) + \sum_{g_1+g_2+g_3=-\lambda_i-1} x_n^{\ell+g_1}B^{g_2}_{+i,\lambda}x_n^{g_3}.
\end{aligned}
$$

If $k \geq -\lambda_i + 1$, then $k-2 \geq -\lambda'_i + 1$, and we have

$$
B^k_{+i,\lambda} = \hat{\varepsilon}'_{i,\lambda}\big(x_{n+1}^{\lambda_i-1+k}e(\alpha,i)\big), \qquad B^l_{+i,\lambda'} = \hat{\varepsilon}'_{i,\lambda'}\big(x_n^{\lambda_i-3+l}e(\beta,i)\big), \quad \forall l \geq k-2.
$$

In this situation,

$$
\begin{aligned}
B^k_{+i,\lambda}e(\beta,i) &= \hat{\varepsilon}'_{i,\lambda}\big(x_{n+1}^{\lambda_i-1+k}e(\alpha,i)\big)e(\beta,i)\\
&= \hat{\varepsilon}'_{i,\lambda}\big(x_{n+1}^{\lambda_i-1+k}e(\beta,i^2)\big)\\
&= -2x_n\hat{\varepsilon}'_{i,\lambda'}\big(x_n^{\lambda_i+k}e(\beta,i)\big) + \hat{\varepsilon}'_{i,\lambda'}\big(x_n^{\lambda_i+k+1}e(\beta,i)\big)\\
&\quad + x_n^2\hat{\varepsilon}'_{i,\lambda'}\big(x_n^{\lambda_i+k-1}e(\beta,i)\big)\\
&= (B^k_{+i,\lambda'} - 2x_nB^{k-1}_{+i,\lambda'} + x_n^2B^{k-2}_{+i,\lambda'})e(\beta,i).
\end{aligned} \tag{75}
$$

In particular, this yields

$$
B_{+i,\lambda}(z)e(\beta,i) = (1-x_nz)^2B_{+i,\lambda'}(z)e(\beta,i) \quad \text{if } \lambda_i \geq 0.
$$

If $\lambda_i < 0$, then we must also check (75) for $k \leq -\lambda_i$. Consider the element

$$
A_k = (1\otimes\tau_{n-1}\otimes\tau_{n-1}\otimes 1)(\iota_i\otimes\iota_i)(\tilde{\pi}_{-\lambda'_i-k} - 2x_n\tilde{\pi}_{-\lambda'_i-(k-1)} + x_n^2\tilde{\pi}_{-\lambda'_i-(k-2)})
$$

in $R^{\Lambda}(\alpha)e(\beta,i)\otimes_{R^{\Lambda}(\beta)}e(\beta,i)R^{\Lambda}(\alpha)$. Here $\tilde{\pi}_{-\lambda'_i-l} \in R^{\Lambda}(\beta)e(\beta-\alpha_i,i)\otimes_{R^{\Lambda}(\beta-\alpha_i)} e(\beta-\alpha_i,i)R^{\Lambda}(\beta)$ is the element defined in (8) for $l \in [1,-\lambda'_i]$, and we set $\tilde{\pi}_{-1} = -\widetilde{e(\beta,i)}$ and $\tilde{\pi}_{-2} = -x_n\widetilde{e(\beta,i)}$, where $e(\beta,i)$ and $x_ne(\beta,i)$ are viewed as elements in $e(\beta,i)R(\alpha)e(\beta,i)$. One can check by direct computation that $\mu_{\tau_n}(A_k) = 0$, $\mu_{x_n^a}(A_k) = \delta_{a,-\lambda_i-k}e(\beta,i)$ for $a \in [0,-\lambda_i-1]$, and that

$$
\mu_{x_n^{-\lambda_i}}(A_k) = (B^k_{+i,\lambda'} - 2x_nB^{k-1}_{+i,\lambda'} + x_n^2B^{k-2}_{+i,\lambda'})e(\beta,i).
$$

It follows that $A_k = \tilde{\pi}_{-\lambda_i-k}e(\beta,i)$ and

$$
B^k_{+i,\lambda}e(\beta,i) = \mu_{x_n^{-\lambda_i}}(A_k) = (B^k_{+i,\lambda'} - 2x_nB^{k-1}_{+i,\lambda'} + x_n^2B^{k-2}_{+i,\lambda'})e(\beta,i).
$$

The proof for part (b) in the case $i = j$ is complete.

Finally, assume that $i \neq j$. By relation (d) in Definition 3.1, we have

$$\tau_n x_n^k \tau_n e(\beta, ji) = x_{n+1}^k \tau_n^2 e(\beta, ji) = \sum_{p,q} c_{i,j;p,q} x_{n+1}^{p+k} x_n^q e(\beta, ji).$$

Applying $\hat{\varepsilon}'_{i,\lambda}$ to both sides of the equality, we get

$$c_{i,j,-a_{ij},0} \hat{\varepsilon}'_{i,\lambda'}(x_n^k) e(\beta, j) = \sum_{p,q} c_{i,j,p,q} \hat{\varepsilon}'_{i,\lambda}\big(x_{n+1}^{p+k} e(\alpha, i)\big) x_n^q e(\beta, j)$$

by Lemma A.4(a). Hence, if $k \geq -\lambda_i + 1$, then we get

$$c_{i,j,-a_{ij},0} B_{+i,\lambda'}^{k-a_{ij}} e(\beta, j) = \sum_{p,q} c_{i,j,p,q} B_{+i,\lambda}^{k+p} x_n^q e(\beta, j).$$

Now, assume that $k \leq -\lambda_i$. Then $k - a_{ij} \leq -\lambda'_i$. In this case, $B_{+i,\lambda'}^{k-a_{ij}} = -\mu_{x_{n-1}^{-\lambda'_i}}(\tilde{\pi}_{-\lambda_i-k})$ with $\tilde{\pi}_{-\lambda_i-k} \in R^\Lambda(\beta)e(\beta-\alpha_i, i) \otimes_{R^\Lambda(\beta-\alpha_i)} e(\beta-\alpha_i, i)R^\Lambda(\beta)$. Set

$$A_k = (1 \otimes \tau_{n-1} \otimes \tau_{n-1} \otimes 1)(\iota_j \otimes \iota_j)(\tilde{\pi}_{-\lambda_i-k}).$$

Then we have the following equalities in $e(\alpha, i)R^\Lambda(\alpha+\alpha_i)e(\alpha, i)$:

$$\begin{aligned}
\mu_{\tau_n}(A_k) &= \mu_{\tau_{n-1}\tau_n\tau_{n-1}e(\beta-\alpha_i, iji)}(\tilde{\pi}_{-\lambda_i-k}) \\
&= \mu_{(\tau_n\tau_{n-1}\tau_n - \sum_{p,q} c_{i,j,p,q}(\sum_{a=0}^{p-1} x_{n-1}^a x_{n+1}^{p-1-a}) x_n^q) e(\beta-\alpha_i, iji)}(\tilde{\pi}_{-\lambda_i-k}) \\
&= -\sum_{p,q} c_{i,j,p,q}\Big(\sum_{a=0}^{p-1} \mu_{x_{n-1}^a}(\tilde{\pi}_{-\lambda_i-k}) x_{n+1}^{p-1-a}\Big) x_n^q e(\beta, ji),
\end{aligned}$$

because $\mu_{\tau_n\tau_{n-1}\tau_n}(\tilde{\pi}_{-\lambda_i-k}) = \tau_n \mu_{\tau_{n-1}}(\tilde{\pi}_{-\lambda_i-k})\tau_n = 0$. Since $\mu_{x_{n-1}^a}(\tilde{\pi}_{-\lambda_i-k}) = \delta_{a,-\lambda_i-k}$ for $a \in [0, -\lambda'_i - 1]$, for any $p \in [0, -a_{ij}]$, we have

$$\begin{aligned}
&\sum_{a=0}^{p-1} \mu_{x_{n-1}^a}(\tilde{\pi}_{-\lambda_i-k}) x_{n+1}^{p-1-a} \\
&\quad = \begin{cases} -\sum_q c_{i,j,p,q} x_n^q x_{n+1}^{p-1+\lambda_i+k} e(\beta, ji) & \text{if } p > -\lambda_i - k + 1, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Next, for any positive integer l, we have

$$\begin{aligned}
\mu_{x_n^l}(A_k) &= \mu_{\tau_{n-1} x_n^l \tau_{n-1} e(\beta-\alpha_i, ij)}(\tilde{\pi}_{-\lambda_i-k}) \\
&= \mu_{(x_{n-1}^l \sum_{p,q} c_{i,j,p,q} x_{n-1}^p x_n^q) e(\beta-\alpha_i, ij)}(\tilde{\pi}_{-\lambda_i-k}) \\
&= \sum_{p,q} c_{i,j,p,q} \mu_{x_{n-1}^{l+p}}(\tilde{\pi}_{-\lambda_i-k}) x_n^q e(\beta, j).
\end{aligned}$$

In particular, since $\mu_{x_{n-1}^a}(\tilde{\pi}_{-\lambda_i-k}) = \delta_{a,-\lambda_i-k}$ for $a \in [0, -\lambda'_i - 1]$, we get

$$\mu_{x_n^{-\lambda_i}}(A_k) = c_{i,j,-a_{ij},0}\,\mu_{x_{n-1}^{-\lambda'_i}}(\tilde{\pi}_{-\lambda_i-k})e(\beta, j) = -c_{i,j,-a_{ij},0}\,B_{+i,\lambda'}^{k-a_{ij}}\,e(\beta, j) \quad (76)$$

and for $l \in [0, -\lambda_i - 1]$ we have

$$\mu_{x_n^l}(A_k) = \begin{cases} \sum_q c_{i,j,(-\lambda_i-k-l),q}\,x_n^q e(\beta, j) & \text{if } l \in [\min\{0, -\lambda_i - k + a_{ij}\}, -\lambda_i - k], \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$\begin{aligned} A_k = &-\sum_{p=-\lambda_i-k+1}^{-a_{ij}} \sum_q c_{i,j,p,q}\,x_n^q (\widetilde{x_{n+1}^{p-1+\lambda_i+k}})e(\beta, ji) \\ &+ \sum_{p=0}^{-\lambda_i-k} \sum_q c_{i,j,p,q}\,x_n^q e(\beta, j)\tilde{\pi}_{-\lambda_i-p-k} \end{aligned}$$

in $R^\Lambda(\alpha)e(\alpha-\alpha_i, i) \otimes_{R^\Lambda(\alpha-\alpha_i)} e(\alpha-\alpha_i, i)R^\Lambda(\alpha)$. By applying $\mu_{x_n^{-\lambda_i}}$ to both sides of the equation and by (76) and Definition A.1, we get

$$c_{i,j,-a_{ij},0}\,B_{+i,\lambda'}^{k-a_{ij}}\,e(\beta, j) = \sum_{p,q} c_{i,j,p,q}\,B_{+i,\lambda}^{k+p}\,x_n^q e(\beta, j).$$

The proof for part (b) is now complete. □

We can now prove the main result of this section.

PROPOSITION B.3

For each $\alpha \in Q_+$ of height n, we have

$$B_{\pm i,\lambda}(z) = z^{\mp\Lambda_i} a_i^\Lambda(z^{-1})^{\mp 1} \sum_{\nu \in I^\alpha} \prod_{k=1}^n q_{i\nu_k}(z, x_k)^{\mp 1} e(\nu).$$

Proof

The relation $a_i^\Lambda(x_1) = 0$ yields $x_1^{\Lambda_i} = -\sum_{k=0}^{\Lambda_i-1} e_k(y_{i,1}, \ldots, y_{i,\Lambda_i})x_1^{\Lambda_i-k}$. Therefore, for $k \in [1, \Lambda_i]$ we have $B_{-i,\Lambda}^k = -p_{\Lambda_i-k}(x_1^{\Lambda_i}) = e_k(y_{i,1}, \ldots, y_{i,\Lambda_i})$, and for $k > \Lambda_i$, since $\eta'_{i,\Lambda} = -\pi(x_1^{\Lambda_i}) = 0$, we deduce that $B_{-i,\Lambda}^k = 0$. So we have

$$B_{-i,\Lambda}(z) = \sum_{p=0}^{\Lambda_i} B_{-i,\Lambda}^p z^p = \sum_{p=0}^{\Lambda_i} e_p(y_{i,1}, \ldots, y_{i,\Lambda_i})z^p = \prod_{p=1}^{\Lambda_i}(1 + y_{ip}z).$$

We deduce that $B_{\pm i,\Lambda}(z) = \prod_{p=1}^{\Lambda_i}(1 + y_{ip}z)^{\mp 1}$. Now, by Lemma B.2, we have

$$\begin{aligned} B_{\pm i,\lambda}(z) &= B_{\pm i,\lambda'}(z) \sum_j q_{ij}(z, x_n)^{\mp 1} e(\alpha - \alpha_j, j) \\ &= B_{\pm i,\Lambda}(z) \sum_{\nu \in I^\alpha} \prod_{k=1}^{n} q_{i\nu_k}(z, x_k)^{\mp 1} e(\nu). \end{aligned}$$

□

B.3. Proof of Theorem 3.25

Consider the operator $h_{ir} \in \mathrm{End}_{\mathbf{k}}(\mathrm{tr}(\mathcal{C}/\mathbb{Z}))$ which acts on $\mathrm{tr}(R^\Lambda(\alpha))$ by multiplication by the central element

$$h_{ir,\lambda} = \sum_{k=0}^{r} (\lambda_i - k) B_{+i,\lambda}^{r-k} B_{-i,\lambda}^{k}$$

in $R^\Lambda(\alpha)$. Then, define the following formal series:

$$\Psi_i(z) = \sum_{r \geq 0} \psi_{ir} z^r = \exp\Big(-\sum_{r \geq 1} h_{ir} z^r / r\Big), \qquad H_i(z) = \sum_{r \geq 0} h_{ir} z^r. \tag{77}$$

The following holds.

LEMMA B.4

The operator ψ_{ir} acts on $\mathrm{tr}(R^\Lambda(\alpha))$ by multiplication by $B_{-i,\lambda}^r$.

Proof

By definition, the operator h_{ir} acts by multiplication by the element

$$h_{ir,\lambda} = \sum_{k=0}^{r} (\lambda_i - k) B_{+i,\lambda}^{r-k} B_{-i,\lambda}^{k}.$$

Since $B_{\pm i,\lambda}(z) = \sum_{k \geq 0} B_{\pm i,\lambda}^k z^k$ and since $B_{+i,\lambda}(z) B_{-i,\lambda}(z) = 1$ by Lemma B.2, we deduce that

$$H_{i,\lambda}(z) = \lambda_i - z\mathrm{d}/\mathrm{d}z \log B_{-i,\lambda}(z).$$

Now, from (77) we get

$$H_i(z) = h_{i0} - z\mathrm{d}/\mathrm{d}z \log \Psi_i(z).$$

Hence, the formal series $\Psi_i(z)$ acts on $\mathrm{tr}(R^\Lambda(\alpha))$ by multiplication by $B_{-i,\lambda}(z)$. □

Finally, let $a_{i,j,p,q} \in \mathbf{k}$ be such that

$$q_{ij}(u,v) = \sum_{p,q\geq 0} a_{i,j,p,q} u^p v^q.$$

We can now prove the following.

PROPOSITION B.5
For each $i, j \in I$, $r, s \in \mathbb{N}$, we have
(a) $[h_{ir}, h_{js}] = 0$;
(b) $[x^+_{ir}, x^-_{js}] = \delta_{ij} h_{i,r+s}$;
(c) $\psi_{ir} x^-_{js} = \sum_{p,q\geq 0} a_{i,j,p,q} x^-_{j,s+q} \psi_{i,r-p}$ *and* $x^+_{js}\psi_{ir} = \sum_{p,q\geq 0} a_{i,j,p,q} \times \psi_{i,r-p} x^+_{j,s+q}$;
(d) $\sum_{p,q\geq 0} c_{i,j,p,q} [x^\pm_{i,r+p}, x^\pm_{j,s+q}] = 0$ *if* $i \neq j$;
(e) $[x^\pm_{i,r}, x^\pm_{i,s}] = 0$;
(f) $[x^\pm_{i,r_1}, [x^\pm_{i,r_2}, \ldots, [x^\pm_{i,r_m}, x^\pm_{j,s}] \ldots]] = 0$ *with* $i \neq j$, $r_p \in \mathbb{N}$, $m = 1 - a_{ij}$.

Proof
The first relation is obvious. Let us concentrate on part (b). If $i \neq j$, then we have an isomorphism $\sigma_{ji,\lambda} : F_j E_i 1_\lambda \simeq E_i F_j 1_\lambda$ such that $\sigma_{ji,\lambda}(x^s_j x^r_i)\sigma^{-1}_{ji,\lambda} = x^r_i x^s_j$. Hence, by Lemma 2.3 we have $\mathrm{tr}_{F_j E_i 1_\lambda}(x^s_j x^r_i) = \mathrm{tr}_{E_i F_j 1_\lambda}(x^r_i x^s_j)$, which is $[x^+_{ir}, x^-_{js}] = 0$.

Now consider the case $i = j$. First, assume that $\lambda_i > 0$. Let $G = F_i E_i 1_\lambda \oplus 1_\lambda^{\oplus\lambda_i}$. Recall the isomorphism of functors $\rho_{i,\lambda} : G \to E_i F_i 1_\lambda$. By Lemma 2.3, we have

$$x^+_{ir} x^-_{is} = \mathrm{tr}_{E_i F_i}(x^r_i x^s_i) = \mathrm{tr}_G\big(\rho^{-1}_{i,\lambda}(x^r_i x^s_i)\rho_{i,\lambda}\big)$$

and it is equal to the sum of the trace of $\rho^{-1}_{i,\lambda}(x^r_i x^s_i)\rho_{i,\lambda}$ restricted to each direct factor of G. The restriction of $\rho^{-1}_{i,\lambda}(x^r_i x^s_i)\rho_{i,\lambda}$ to $F_i E_i 1_\lambda$ is represented by

$$\begin{aligned}
&R^\Lambda(\alpha)e(\alpha-\alpha_i,i) \otimes_{R^\Lambda(\alpha-\alpha_i)} e(\alpha-\alpha_i,i)R^\Lambda(\alpha) \\
&\quad \to R^\Lambda(\alpha)e(\alpha-\alpha_i,i) \otimes_{R^\Lambda(\alpha-\alpha_i)} e(\alpha-\alpha_i,i)R^\Lambda(\alpha), \\
&z \mapsto \pi\big(x^r_{n+1}\mu_{\tau_n}(z)x^s_{n+1}\big).
\end{aligned}$$

Now, we have

$$\begin{aligned}
\pi\big(x^r_{n+1}\mu_{\tau_n}(z)x^s_{n+1}\big) &= \pi\big(\mu_{x^r_{n+1}\tau_n x^s_{n+1}}(z)\big) \\
&= \pi\Big(\mu_{x^s_n\tau_n x^r_n}(z) + \sum_{p=0}^{r+s-1} \mu_{x_n^{r+s-1}}(z)x^p_{n+1}\Big)
\end{aligned}$$

$$= (1 \otimes x_n^s \otimes x_n^r \otimes 1)(z) + \sum_{p=\lambda_i}^{r+s-1} \mu_{x_n^{r+s-1-p}}(z)\pi(x_{n+1}^p)$$
$$= (1 \otimes x_n^s \otimes x_n^r \otimes 1)(z)$$
$$- \sum_{p=\lambda_i}^{r+s-1} \sum_{a=0}^{p-\lambda_i} \mu_{x_n^{r+s-1-p}}(z)(B_{+i,\lambda}^{p-\lambda_i-a} \otimes x_n^a \otimes 1 \otimes 1)\hat{\eta}_i'(1).$$

Here we used the relation (e) in Definition 3.1 to get the second equality and Lemma B.1 for the last equality. It yields that the restriction of $\rho_{i,\lambda}^{-1}(x_i^r x_i^s)\rho_{i,\lambda}$ to $F_i E_i 1_\lambda$ is the endomorphism

$$x_i^s x_i^r - \sum_{p=\lambda_i}^{r+s-1} \sum_{a=0}^{p-\lambda_i} (B_{+i,\lambda}^{p-\lambda_i-a} F_i x_i^a) \circ \hat{\eta}_i' \circ \varepsilon_i' \circ (F_i x_i^{r+s-1-p}),$$

and its trace is equal to

$$\operatorname{tr}_{F_i E_i 1_\lambda}(x_i^s x_i^r) - \operatorname{tr}_{1_\lambda}\Big(\sum_{p=\lambda_i}^{r+s-1} \sum_{a=0}^{p-\lambda_i} B_{+i,\lambda}^{p-\lambda_i-a} \varepsilon_i' \circ (F_i x_i^{r+s-1-p+a}) \circ \hat{\eta}_i'\Big)$$
$$= x_{is}^- x_{ir}^+ - \sum_{p=\lambda_i}^{r+s-1} \sum_{a=0}^{p-\lambda_i} B_{+i,\lambda}^{p-\lambda_i-a} B_{-i,\lambda}^{r+s-p+a+\lambda_i}$$
$$= x_{is}^- x_{ir}^+ + \sum_{a=\lambda_i+1}^{r+s} (\lambda_i - a) B_{+i,\lambda}^{r+s-a} B_{-i,\lambda}^a. \tag{78}$$

The restriction of $\rho_{i,\lambda}^{-1}(x_i^r x_i^s)\rho_{i,\lambda}$ to the kth copy of 1_λ is represented by the map $R^\Lambda(\alpha) \to R^\Lambda(\alpha)$, $z \mapsto p_k(x_{n+1}^r z x_{n+1}^{s+k}) = z p_k(x_{n+1}^{r+s+k})$. By the second equality in Lemma B.1(a), it is equal to $\sum_{a=0}^{\lambda_i-k-1} B_{+i,\lambda}^{r+s-a} B_{-i,\lambda}^a$. Combined with (78) we obtain

$$x_{ir}^+ x_{is}^- = \operatorname{tr}_G\big(\rho_{i,\lambda}^{-1}(x_i^r x_i^s)\rho_{i,\lambda}\big)$$
$$= x_{is}^- x_{ir}^+ + \sum_{a=\lambda_i+1}^{r+s} (\lambda_i - a) B_{+i,\lambda}^{r+s-a} B_{-i,\lambda}^a + \sum_{k=0}^{\lambda_i-1} \sum_{a=0}^{\lambda_i-k-1} B_{+i,\lambda}^{r+s-a} B_{-i,\lambda}^a$$
$$= x_{is}^- x_{ir}^+ + \sum_{a=\lambda_i+1}^{r+s} (\lambda_i - a) B_{+i,\lambda}^{r+s-a} B_{-i,\lambda}^a + \sum_{a=0}^{\lambda_i-1} (\lambda_i - a) B_{+i,\lambda}^{r+s-a} B_{-i,\lambda}^a$$
$$= x_{is}^- x_{ir}^+ + h_{i,r+s}.$$

Now, assume that $\lambda_i < 0$. Let $G = E_i F_i 1_\lambda \oplus 1_\lambda^{\oplus(-\lambda_i)}$, and consider the isomorphism $\rho_{i,\lambda} : F_i E_i 1_\lambda \to G$. By Lemma 2.3, we have

$$x_{is}^- x_{ir}^+ = \mathrm{tr}_{F_i E_i}(x_i^s x_i^r) = \mathrm{tr}_G\big(\rho_{i,\lambda}(x_i^s x_i^r)\rho_{i,\lambda}^{-1}\big)$$

and it is equal to the sum of the trace of $\rho_{i,\lambda}(x_i^s x_i^r)\rho_{i,\lambda}^{-1}$ restricted to each direct factor of G. The restriction of $\rho_{i,\lambda}(x_i^s x_i^r)\rho_{i,\lambda}^{-1}$ to $E_i F_i 1_\lambda$ is represented by

$$e(\alpha, i) R^\Lambda(\alpha+\alpha_i) e(\alpha, i) \to e(\alpha, i) R^\Lambda(\alpha+\alpha_i) e(\alpha, i),$$
$$z \mapsto \mu_{x_n^s \tau_n x_n^r}(\tilde{z}).$$

By relation (e) in Definition 3.1 and the definition of $\tilde{z}$, we have

$$\mu_{x_n^s \tau_n x_n^r e(\alpha-\alpha_i, i^2)}(\tilde{z}) = x_{n+1}^r z x_{n+1}^s - \sum_{p=-\lambda_i}^{r+s-1} \mu_{x_n^p}(\tilde{z}) x_{n+1}^{r+s-1-p}.$$

The restriction of $\rho_{i,\lambda}(x_i^s x_i^r)\rho_{i,\lambda}^{-1}$ to the kth copy of 1_λ is represented by

$$R^\Lambda(\alpha) \to R^\Lambda(\alpha), \qquad a \mapsto a\mu_{x_n^{r+k+s}}(\tilde{\pi}_k).$$

Now, relation (b) follows from Lemma B.1(b) and the fact that

$$\mathrm{tr}_{E_i F_i 1_\lambda}\big((E_i x_i^a) \circ \eta'_{i,\lambda} \circ \hat{\varepsilon}'_{i,\lambda} \circ (E_i x_i^b)\big) = \mathrm{tr}_{1_\lambda}\big(\hat{\varepsilon}'_{i,\lambda}(E_i x_i^{a+b}) \circ \eta'_{i,\lambda}\big)$$

using similar computation as in the previous case. We leave the details to the reader.

Next, we prove (c). Consider the formal series $X_j^-(w) = \sum_{s\geq 0} x_{js}^- w^s$. From Proposition B.3 and Lemma B.4 we deduce that, for each $f \in R^\Lambda(\alpha)$, we have

$$X_j^-(w)\big(\mathrm{tr}(f)\big) = \mathrm{tr}\Big(\sum_{s\geq 0} x_{n+1}^s w^s f e(\alpha, j)\Big),$$
$$\Psi_i(z) X_j^-(w) \Psi_i(z)^{-1}\big(\mathrm{tr}(f)\big) = \mathrm{tr}\Big(q_{ij}(z, x_{n+1}) \sum_{s\geq 0} x_{n+1}^s w^s f e(\alpha, j)\Big).$$

This yields the first equation of (c). The second one is obtained in a similar way.

Let us now prove relation (d). Consider the endomorphism $x_i^r x_j^s \tau_{ji}\tau_{ij}$ on $E_i E_j$. We have $x_i^r x_j^s \tau_{ji}\tau_{ij} = \tau_{ji} x_j^s x_i^r \tau_{ij}$. Therefore,

$$\mathrm{tr}_{E_i E_j}(x_i^r x_j^s \tau_{ji}\tau_{ij}) = \mathrm{tr}_{E_i E_j}(\tau_{ji} x_j^s x_i^r \tau_{ij}) = \mathrm{tr}_{E_j E_i}(x_j^s x_i^r \tau_{ij}\tau_{ji}).$$

Next, by relation (d) in Definition 3.1, we have

$$\tau_{ji}\tau_{ij} = Q_{ij}(x_i, x_j) = \sum_{p,q} c_{i,j,p,q} x_i^p x_j^q = Q_{ji}(x_j, x_i) = \tau_{ij}\tau_{ji}.$$

Put this back into the equation above and we get

$$\sum_{p,q} c_{i,j,p,q}\big(\mathrm{tr}_{E_iE_j}(x_i^{r+p}x_j^{s+q}) - \mathrm{tr}_{E_jE_i}(x_j^{s+q}x_i^{r+s})\big) = 0,$$

which is the relation (d) .

Next, let us prove (e). The functor E_i^2 acting on $R^\Lambda(\alpha)$ is represented by the bimodule $e(\alpha - 2\alpha_i, i^2)R^\Lambda(\alpha)$. A morphism $x_i^r x_i^s$ on E_i^2 is represented by the left multiplication by $x_{n-1}^r x_n^s$ if α has height n. It is enough to consider the case $n = 2$. The intertwiner $\varphi_1 = (x_1 - x_2)\tau_1 + 1$ is such that $\varphi_1(x_1^r x_2^s)e(i^2) = (x_2^r x_1^s)\varphi_1 e(i^2)$ and $\varphi_1^2 e(i^2) = e(i^2)$. It follows that $\mathrm{tr}_{E_i^2}(x_i^r x_i^s) = \mathrm{tr}_{E_i^2}(x_i^s x_i^r)$. Hence $x_{ir}^+ x_{is}^+ = x_{is}^+ x_{ir}^+$. The proof for x^- is similar.

Finally, let us prove relation (f). By (e), it is equivalent to prove the following relation:

$$\sum_{w\in S_m}\big[x^{\pm}_{i,r_{w(1)}},\big[x^{\pm}_{i,r_{w(2)}},\dots,[x^{\pm}_{i,r_{w(m)}},x^{\pm}_{j,r_0}]\dots\big]\big] = 0$$

$$\text{with } i \neq j, r_p \in \mathbb{N} \text{ and } m = 1 - a_{ij}.$$

We will prove it for x^+; the proof for x^- is similar. First, recall that the functor $E_i^{(a)}$ is the image of the divided power operator

$$e_{i,a} = x_1^{a-1}\cdots x_{a-1}\tau_{w_0} \in R(a\alpha_i)$$

by the canonical morphism $R(a\alpha_i) \to \mathrm{End}(E_i^a)$. The element $e_{i,a}$ is an idempotent, and we have $E_i^a \simeq (E_i^{(a)})^{\oplus a!}$ (see, e.g., [22] for details).

Given $i, j \in I$ with $i \neq j$, we write $m = 1 - a_{ij}$. In [22, Proposition 6], an isomorphism of functors

$$\alpha' : \bigoplus_{a=0}^{[\frac{m}{2}]} E_i^{(2a)}E_jE_i^{(m-2a)} \xrightarrow{\sim} \bigoplus_{a=0}^{[\frac{m-1}{2}]} E_i^{(2a+1)}E_jE_i^{(m-2a-1)}$$

and a quasi-inverse α'' are constructed.

If $a_{ij} = 0$, then we have $\alpha' = \tau_{ji} : E_jE_i \to E_iE_j$ and $\alpha'' = c_{ij,0,0}^{-1}\tau_{ij}$. Since $i \neq j$, we have $\alpha'(x_j^s x_i^r) = (x_i^r x_j^s)\alpha'$. Hence

$$\begin{aligned} x_{j,s}^+ x_{i,r}^+ &= \mathrm{tr}_{E_jE_i}(x_j^s x_i^s) \\ &= \mathrm{tr}_{E_iE_j}\big(\alpha''(x_i^r x_j^s)\alpha'\big) \\ &= \mathrm{tr}_{E_iE_j}(x_i^r x_j^s) = x_{i,r}^+ x_{j,s}^+, \end{aligned}$$

yielding relation (f) in this case.

Assume now that $a_{ij} < 0$. Then the maps α', α'' are given by

$$\alpha' = \sum_{a=0}^{[\frac{m-1}{2}]} \alpha^+_{(2a,m-2a)} + \sum_{a=0}^{[\frac{m}{2}]} \alpha^-_{(2a,m-2a)},$$

$$\alpha'' = \sum_{a=0}^{[\frac{m}{2}]} \alpha^-_{(2a+1,m-1-2a)} - \sum_{a=0}^{[\frac{m-1}{2}]} \alpha^-_{(2a+1,m-1-2a)},$$

where for $a \in [0,m]$ and $b = m-a$, we have

$$\alpha^+_{a,b} = (e_{i,a+1} E_j e_{i,b-1}) \circ \tau_{a+1} \circ \tau_{a+2} \cdots \circ \tau_{a+b} \circ \iota_a :$$
$$E_i^{(a)} E_j E_i^{(b)} \to E_i^{(a+1)} E_j E_i^{(b-1)},$$
$$\alpha^-_{a,b} = (e_{i,a-1} E_j e_{i,b+1}) \circ \tau_a \circ \tau_{a-1} \cdots \circ \tau_1 \circ \iota_a :$$
$$E_i^{(a)} E_j E_i^{(b)} \to E_i^{(a-1)} E_j E_i^{(b+1)}.$$

Here $\iota_a : E_i^{(a)} E_j E_i^{(b)} \to E_i^a E_j E_i^b$ is the canonical embedding, and τ_k acts by τ on the kth and $(k+1)$th copy of E in the sequence $E_i^a E_j E_i^b$.

Given any integers $r_1, \dots, r_m, s \in \mathbb{N}$, we define for each $a \in [0,m]$ a morphism

$$\Xi_a = \sum_{w \in \mathfrak{S}_m} x_i^{r_{w(1)}} \cdots x_i^{r_{w(a)}} x_j^s x_i^{r_{w(a+1)}} \cdots x_i^{r_{w(m)}} \in \operatorname{End}(E_i^a E_j E_i^b). \tag{79}$$

Note that Ξ_a is symmetric in the first a-tuple of x_i's and also in the last b-tuple of x_i's; hence it commutes with the divided power operator $e_{i,a} E_j e_{i,b}$. By consequence, Ξ_a restricts to a well-defined endomorphism of $E_i^{(a)} E_j E_i^{(b)}$, which we denote again by Ξ_a. Further, since $E_i^a E_j E_i^b \simeq (E_i^{(a)} E_j E_i^{(b)})^{\oplus a!b!}$, we have $\operatorname{tr}_{E_i^a E_j E_i^b}(\Xi_a) = (a!b!) \operatorname{tr}_{E_i^{(a)} E_j E_i^{(b)}}(\Xi_a)$.

CLAIM B.6
We have
(a)

$$\sum_{w \in \mathfrak{S}_m} \big[x^+_{i,r_{w(1)}}, \big[\dots, [x^+_{i,r_{w(m)}}, x^+_{j,s}] \dots\big]\big] = m! \sum_{a+b=m} (-1)^b \operatorname{tr}_{E_i^{(a)} E_j E_i^{(b)}}(\Xi_a);$$

(b) $\alpha^+_{a,b} \Xi_a = \Xi_{a+1} \alpha^+_{a,b}$, $\alpha^-_{a,b} \Xi_a = \Xi_{a-1} \alpha^-_{a,b}$.

Let us show how to deduce relation (f) from this claim. Part (b) implies that the following diagram commutes:

$$\begin{array}{ccc}
\bigoplus_{a=0}^{[\frac{m}{2}]} E_i^{(2a)} E_j E_i^{(m-2a)} & \xrightarrow[\sim]{\alpha'} & \bigoplus_{a=0}^{[\frac{m-1}{2}]} E_i^{(2a+1)} E_j E_i^{(m-2a-1)} \\
\Big\downarrow \bigoplus_a \Xi_{2a} & & \Big\downarrow \bigoplus_a \Xi_{2a+1} \\
\bigoplus_{a=0}^{[\frac{m}{2}]} E_i^{(2a)} E_j E_i^{(m-2a)} & \xrightarrow[\sim]{\alpha'} & \bigoplus_{a=0}^{[\frac{m-1}{2}]} E_i^{(2a+1)} E_j E_i^{(m-2a-1)}
\end{array}$$

Therefore,

$$\sum_{a=0}^{[\frac{m}{2}]} \mathrm{tr}_{E_i^{(2a)} E_j E_i^{(m-2a)}}(\Xi_{2a}) = \sum_{a=0}^{[\frac{m-1}{2}]} \mathrm{tr}_{E_i^{(2a+1)} E_j E_i^{(m-2a-1)}}(\Xi_{2a+1}).$$

Hence by part (a) of the claim we get $\sum_{w\in\mathfrak{S}_m}[x^+_{i,r_{w(1)}},[\dots,[x^+_{i,r_{w(m)}},x^+_{j,s}]\dots]]=0$ as desired.

It remains to prove the claim. Let us introduce some more notation. For $a\in[0,m]$ let $\Gamma^a=\{\underline{k}=(k_1,\dots,k_a)\in\mathbb{N}^a \mid 1\le k_1<k_2<\dots<k_a\le m\}$, and for $\underline{k}\in\Gamma^a$ we set $\underline{k}^\circ=(l_1,\dots,l_b)\in\Gamma^b$ such that $\{k_1,\dots,k_a\}\cup\{l_1,\dots,l_b\}=\{r_1,\dots,r_m\}$. Let $\mathfrak{S}^{(a,b)}$ be the set of minimal representatives of the left cosets $\mathfrak{S}_m/\mathfrak{S}_a\times\mathfrak{S}_b$. Then the map $w\mapsto w(1,2,\dots,m)$ yields a bijection $\mathfrak{S}^{(a,b)}\simeq\Gamma^a$. Let w_b be the longest element in $\mathfrak{S}_b$. Given any sequence $r_1,\dots,r_m$ and $s\in\mathbb{N}$ we have

$$\begin{aligned}
&\big[x^+_{i,r_1},\big[x^+_{i,r_2},\dots,[x^+_{i,r_m},x^+_{j,s}]\dots\big]\big] \\
&= \sum_{a=0}^{m}(-1)^b \sum_{\underline{k}\in\Gamma^a,\underline{l}=\underline{k}^\circ} x^+_{i,r_{k_1}}\dots x^+_{i,r_{k_a}} x^+_{j,s} x^+_{i,r_{l_b}}\dots x^+_{i,r_{l_1}} \\
&= \sum_{a=0}^{m}(-1)^b \sum_{y\in\mathfrak{S}^{(a,b)}w_b} x^+_{i,r_{y(1)}}\cdots x^+_{i,r_{y(a)}} x^+_{j,s} x^+_{i,r_{y(a+1)}}\cdots x^+_{i,r_{y(m)}}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\sum_{w\in\mathfrak{S}_m}\big[x^+_{i,r_{w(1)}},\big[x^+_{i,r_{w(2)}},\dots,[x^+_{i,r_{w(m)}},x^+_{j,s}]\dots\big]\big] = \\
&= \sum_{a=0}^{m}(-1)^b \sum_{y\in\mathfrak{S}^{(a,b)}w_b}\ \sum_{w\in\mathfrak{S}_m} x^+_{i,r_{wy(1)}}\cdots x^+_{i,r_{wy(a)}} x^+_{j,s} x^+_{i,r_{wy(a+1)}}\cdots x^+_{i,r_{wy(m)}} \\
&= \sum_{a=0}^{m}(-1)^b \frac{m!}{a!b!} \sum_{w\in\mathfrak{S}_m} x^+_{i,r_{w(1)}}\cdots x^+_{i,r_{w(a)}} x^+_{j,s} x^+_{i,r_{w(a+1)}}\cdots x^+_{i,r_{w(m)}}
\end{aligned}$$

$$= \sum_{a=0}^{m} (-1)^b \frac{m!}{a!b!} \operatorname{tr}_{E_i^a E_j E_i^b}(\Xi_a)$$

$$= m! \sum_{a=0}^{m} (-1)^b \operatorname{tr}_{E_i^{(a)} E_j E_i^{(b)}}(\Xi_a).$$

This proves part (a) of the claim.

Part (b) is a direct computation. On each $R^\Lambda(\alpha)$, the functor $E_i^{(a)} E_j E_i^{(b)}$ is represented by the $(R^\Lambda(\alpha - m\alpha_i - \alpha_j), R^\Lambda(\alpha))$-bimodule

$$_{i^{(a)} j i^{(m-a)}} P = (1 \otimes e_{i,a} \otimes 1 \otimes e_{i,m-a}) e(\alpha - m\alpha_i - \alpha_j, i^a, j, i^{m-a}) R^\Lambda(\alpha).$$

To check the relation in (b), without loss of generality we may assume that $\alpha = m\alpha_i + \alpha_j$. Then Ξ_a is represented by the left multiplication on $_{i^{(a)} j i^{(b)}} P$ by

$$\sum_{w \in \mathfrak{S}_m} x_1^{r_{w(1)}} \cdots x_a^{r_{w(a)}} x_{a+1}^s x_{a+2}^{r_{w(a+1)}} \cdots x_{m+1}^{r_{w(m)}},$$

and $\alpha_{a,b}^+$ is represented by $(1 \otimes e_{i,a+1} \otimes 1 \otimes e_{i,b-1}) e(i^{a+1} j i^{b-1}) \tau_{a+1} \tau_{a+2} \cdots \tau_m$. Since Ξ_a is symmetric in $\{x_{a+2}, \ldots, x_{m+1}\}$, we have $e(i^a j i^b) \tau_k \Xi_a = e(i^a j i^b) \Xi_a \tau_k$ for any $k \in [a+2, m]$. Next,

$$\begin{aligned}
& e(i^{a+1} j i^{b-1}) \tau_{a+1} \Xi_a \\
& = \tau_{a+1} e(i^a j i^b) \Xi_a \\
& = \sum_{w \in \mathfrak{S}_m} x_1^{r_{w(1)}} \cdots x_a^{r_{w(a)}} \tau_{a+1} (x_{a+1}^s x_{a+2}^{r_{w(a+1)}}) \cdots x_{m+1}^{r_{w(m)}} e(i^a j i^b) \\
& = \sum_{w \in \mathfrak{S}_m} x_1^{r_{w(1)}} \cdots x_a^{r_{w(a)}} (x_{a+2}^s x_{a+1}^{r_{w(a+1)}}) \tau_{a+1} \cdots x_{m+1}^{r_{w(m)}} e(i^a j i^b) \\
& = \Xi_{a+1} \tau_{a+1} e(i^a j i^b) = \Xi_{a+1} e(i^{a+1} j i^{b-1}) \tau_{a+1}.
\end{aligned}$$

Finally, since Ξ_{a+1} is symmetric in $\{x_1, \ldots, x_{a+1}\}$ and in $\{x_{a+3}, \ldots, x_{a+b+1}\}$, the divided power operator $1 \otimes e_{i,a+1} \otimes 1 \otimes e_{i,b-1}$ commutes with Ξ_{a+1}. To summarize, we have

$$\begin{aligned}
\alpha_{a,b}^+ \Xi_a & = (1 \otimes e_{i,a+1} \otimes 1 \otimes e_{i,b-1}) e(i^{a+1} j i^{b-1}) \tau_{a+1} \tau_{a+2} \cdots \tau_{a+b} \Xi_a \\
& = (1 \otimes e_{i,a+1} \otimes 1 \otimes e_{i,b-1}) e(i^{a+1} j i^{b-1}) \tau_{a+1} \Xi_a \tau_{a+2} \cdots \tau_{a+b} \\
& = (1 \otimes e_{i,a+1} \otimes 1 \otimes e_{i,b-1}) \Xi_{a+1} e(i^{a+1} j i^{b-1}) \tau_{a+1} \tau_{a+2} \cdots \tau_{a+b} \\
& = \Xi_{a+1} \alpha_{a,b}^+,
\end{aligned}$$

which is the first equality in part (b); the proof for the second one is similar. □

Finally, we have the following.

COROLLARY B.7
Assume that $\mathfrak{g}$ is symmetric and that (11) holds. Then, Theorem 3.25 holds.

Proof
We can assume that $\ell = 1$. We must prove that the relations in Definition 3.24 follow from the relations in Proposition B.5. From (74) we deduce that

$$q_{ij}(u,v) = (1-uv)^{-a_{ij}}, \quad \forall i,j.$$

We must check that relations (c) and (d) in Proposition B.5 can be rewritten as
(c) $[h_{ir}, x^{\pm}_{js}] = \pm a_{ij} x^{\pm}_{j,r+s}$;
(d) $\sum_{p=0}^{m} (-1)^p \binom{m}{p} [x^{\pm}_{i,r+p}, x^{\pm}_{j,s+m-p}] = 0$ with $i \neq j$ and $m = -a_{ij}$.
Relation (d) is obvious, because (11) implies that

$$c_{i,j,p,q} = \delta_{q,-a_{ij}-p} r_{ij} (-1)^p \binom{-a_{ij}}{p}, \quad \forall p,q,$$

and r_{ij} is invertible. Let us prove (c). Given α of height n and $\lambda = \Lambda - \alpha$, by Proposition B.3 we have

$$B_{-i,\lambda}(z) = \sum_{\nu \in I^\alpha} \prod_{p=1}^{\Lambda_i} (1 + y_{ip} z) \prod_{k=1}^{n} (1 - z x_k)^{-a_{i\nu_k}} e(\nu).$$

By Lemma B.4, we have

$$\exp\Big(-\sum_{r\geq 1} h_{ir,\lambda} z^r / r\Big) = B_{-i,\lambda}(z).$$

Since the $e(\nu)$'s are orthogonal idempotents in $R^\Lambda(\alpha)$, we have

$$\exp\Big(-\sum_{r\geq 1} h_{ir,\lambda} z^r / r\Big) = \sum_{\nu \in I^\alpha} \exp\Big(-\sum_{r\geq 1} h_{ir,\lambda} e(\nu) z^r / r\Big) e(\nu).$$

Therefore, we deduce that

$$\sum_{r\geq 1} h_{ir,\lambda} e(\nu) z^r = z(\mathrm{d}/\mathrm{d}z) \log\Big(\prod_{p=1}^{\Lambda_i} (1 + y_{ip} z) \prod_{k=1}^{n} (1 - z x_k)^{-a_{i\nu_k}}\Big) e(\nu).$$

Hence we have that

$$h_{ir,\lambda} e(\nu) = \Big(\sum_{p=1}^{\Lambda_i} (-y_{ip})^r - \sum_{k=1}^{n} a_{i,\nu_k} x_k^r\Big) e(\nu), \quad \forall r \geq 1.$$

In particular, we have

$$h_{ir,\lambda-\alpha_j}e(\alpha,j)=(h_{ir,\lambda}-a_{ij}x_{n+1}^r)e(\alpha,j).$$

Hence, for any $f\in R^\Lambda(\alpha)$, we have

$$\begin{aligned}h_{ir}x_{js}^-\operatorname{tr}(f)&=\operatorname{tr}\big(h_{ir,\lambda-\alpha_j}x_{n+1}^s fe(\alpha,j)\big)\\&=\operatorname{tr}\big(h_{ir,\lambda}x_{n+1}^s fe(\alpha,j)\big)-a_{ij}\operatorname{tr}(x_{n+1}^{r+s}fe(\alpha,j)\\&=x_{js}^-h_{ir}-a_{ij}x_{j,r+s}^-.\end{aligned}$$

The proof of Theorem 3.25 is complete. □

Index of notations

2.1.1	$\mathcal{C}$, $\mathcal{C}^c$, $\mathcal{C}/\ell\mathbb{Z}$, $\mathcal{C}^{[\ell]}$,
	$1_{\mathcal{C}}$, 1_F,
	Center $Z(\mathcal{C})$,
	Shift $[\ell]$,
	M^d, $M^{[\ell]}$
2.1.2	A, $Z(A)$,
	$\operatorname{tr}(\mathcal{C})$, $\operatorname{CF}(\mathcal{C})$, $HH_*(\mathcal{C})$,
	$\operatorname{tr}(A)$, $\operatorname{CF}(A)$, $HH_*(A)$,
	Base change RM
2.1.3	Operator $\operatorname{tr}_F(x)$
2.1.4	(E,F), η_E, ε_E, ${}^\vee x$, $y^\vee$
2.1.5	Operators $Z_E(x)$, $Z_F(x)$
2.2.2	Res_B^A, Ind_B^A, A^B,
	μ, $\mu_f: A\otimes_B A\to A$
2.2.3	Frobenius form $t: A\to B$,
	$\hat{t}: A\to \operatorname{Hom}_B(A,B)$,
	Casimir element $\pi\in(A\otimes_B A)^A$
3.1	$\mathbf{k}=\bigoplus_{n\in\mathbb{N}}\mathbf{k}^n$, $\Bbbk=\mathbf{k}^0$
3.1.1	Root datum P, $P^\vee$, Φ, $\Phi^\vee$, Q_+, P_+,
	I, (a_{ij}), d_i, $(\bullet\vert\bullet)$,
	$\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{n}^+$,
	Simple module $V(\Lambda)$
3.1.2	$c_{i,j,p,q}$, $Q=(Q_{ij})_{i,j\in I}$, I^α,
	$R(n;Q,\mathbf{k})$, generators $e(\nu)$, x_k, τ_l,
	Demazure operator $\partial_{k,l}$,
	Idempotents $e(\alpha)$, $e(\alpha,\nu')$, $e(n,\nu')$

3.1.3	I_Λ with map $I_\Lambda \to I$, $\underline{\mathbf{k}} = \underline{\mathbf{k}}^\Lambda = \Bbbk[c_t; t \in I_\Lambda]$, $a_i^\Lambda(u)$, $R^\Lambda(\alpha)$, $R^\Lambda(\alpha;\mathbf{k})$, $R^\Lambda(\alpha;Q,\mathbf{k})$, R^Λ, $\underline{R}^\Lambda(\alpha)$, $R(\alpha)$, $R(\alpha;\mathbf{k})$, $R(\alpha;Q,\mathbf{k})$, R, Intertwiner φ_k
3.1.4	$\lambda_i = \langle \alpha_i^\vee, \lambda\rangle$, $\iota_i : R(\alpha) \hookrightarrow R(\alpha+\alpha_i)$, $E_i', F_i', \varepsilon'_{i,\lambda}, \eta'_{i,\lambda}, \sigma'_{ij,\lambda}, \rho'_{i,\lambda}, \hat{\eta}'_{i,\lambda}, \hat{\varepsilon}'_{i,\lambda}$, $\pi(z)$, $p_k(z)$, $\tilde{z}$, $\tilde{\pi}$
3.1.5	$d_{\Lambda,\alpha}$
3.2	$\mathfrak{g}_\mathbf{k}$
3.2.1	$\mathcal{C}_\lambda$, $\mathcal{C}^{\mathrm{hw}}$
3.2.2	Minimal categorifications $\mathcal{A}^\Lambda$, $\mathcal{V}^\Lambda$, $\mathfrak{e} = \mathfrak{sl}_2$, $\mathcal{V}^k$, Z^k, $\underline{\mathrm{H}}_n^k$, H_n^k
3.2.3	$\underline{\mathbf{h}} = \Bbbk[y_t; t \in I_\Lambda]$, $\underline{\mathbf{h}}'$
3.2.4	$\mathfrak{e}_i (= \mathfrak{e})$, $\mathcal{C}_{\geq k}$, $\mathcal{C}_{\geq k,\lambda}$, $\mathcal{C}_k$, $\mathcal{C}_{k,\lambda}$
3.2.5	Operators $x_{ir}^\pm$ on the trace
3.2.6	$L\mathfrak{g}$, generators $x_{ir}^\pm$, h_{ir}, Anti-involution ϖ
3.2.7	Operators $Z_{ir}^\pm$ on the center
3.3	Kashiwara operator ϵ_i
3.3.3	$\mathrm{tr}(\mathcal{C}/\mathbb{Z})^{\mathrm{cyc}}$
3.4.1	Weyl modules $\mathrm{W}(\Lambda)$, $\underline{\mathrm{W}}(\Lambda)$, $\lvert\Lambda\rangle$, $\Lambda_{\min}$, Ψ_{ir}, $\Psi_i(z)$
3.4.2	$H_*(X,\Bbbk)$, $H_i^G(X,\Bbbk)$, $[X]$, $H_G^i(X,\Bbbk)$, $D^G(X,\Bbbk)$, $\cap$, G_Λ, $\mathfrak{M}(\Lambda,\alpha)$, $\mathfrak{M}(\Lambda)$, $\mathfrak{L}(\Lambda,\alpha)$, $\mathfrak{L}(\Lambda)$, $H^*_{G_\Lambda}(\mathfrak{M}(\Lambda),\Bbbk)$, $H^{G_\Lambda}_{[*]}(\mathfrak{L}(\Lambda),\Bbbk)$
3.4.4	$a : \mathrm{Z}(R) \to \mathbf{k} \otimes_{\underline{\mathbf{k}}} H^*_{G_\Lambda}(\mathfrak{M}(\Lambda),\Bbbk)$, $\mathcal{V}_i$, $\mathcal{W}_i$, $b : \mathrm{Z}(R) \to \mathrm{Z}(R^\Lambda)$
4	$\mathbf{k} = \Bbbk[\hbar, y_1, \dots, y_r]$, $\mathbf{k}'$
4.1	$R(n)$, $R^r(n)$, $\mathrm{Z}(R^r(n))^{\mathrm{JM}}$
4.2	$\mathcal{W}$, generators C_k, $D_{-1,k}$, $D_{0,k+1}$, $D_{1,k}$, $E(z)$, $\gamma_t(D_{0,k+1})$, $\mathcal{W}_{<0}$, $\mathcal{W}_{\geq 0}$, $\mathcal{W}_0$,

	Anti-involution ϖ,
	$\Lambda : \mathcal{W}_0 \to \mathbf{k}'$,
	$M(\Lambda)$, $V(\Lambda)$, $\lvert\Lambda\rangle$
4.3	$\mathcal{C}^r_n$, $\mathcal{C}^r$,
	Operators $x^{\pm}_k$, $E_{k,l} = [x^+_k, x^-_l]$,
	$B^k_{\pm,n} \in \mathrm{Z}(R^r(n))$, $B_{\pm}(z)$,
	$\lvert r\rangle$, Λ_r
4.4	$\mathfrak{M}(r,n)$, $\mathfrak{M}(r)$,
	A, G, G_A,
	$\mathbf{h}$, $\mathbf{h}'$, $d_{r,n}$, $\mathbf{k}_1 = \Bbbk[y_1, \dots, y_r]$,
	$R(n)_1$, $R^r(n)_1$, $\mathrm{Rees}(\mathrm{Z}(R^r(n)_1))$
A.1	$B^k_{\pm i,\lambda} \in R^{\Lambda}(\alpha)$
A.2	$\mathbb{X}_{i,j,\lambda}$, $\mathbb{I}_{i,j,\lambda}$
A.3	$\nu^{(k)}$, $\hat{\varepsilon}_{\nu}$, r_{ν}, r_{ij}, $r(\alpha, \nu_n)$,
	$t_{\alpha} = t_{\Lambda,\alpha} : R^{\Lambda}(\alpha) \to \mathbf{k}$
B.2	$q_{ij}(u,v)$
B.3	$h_{ir,\lambda}$, ψ_{ir}, $\Psi_i(z)$, $H_i(z)$

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Shan
Laboratoire de Mathématiques d'Orsay, Université Paris-Sud, CNRS, Orsay, France;
peng.shan@math.u-psud.fr

Varagnolo
Département de Mathématiques, Université de Cergy-Pontoise, Cergy-Pontoise, France;
michela.varagnolo@u-cergy.fr

Vasserot
Institut de Mathématiques de Jussieu, Université Paris Diderot, Paris, France;
eric.vasserot@imj-prg.fr