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Canonical bases and affine Hecke algebras of type D

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Abstract

We prove a conjecture of Miemietz and Kashiwara on canonical bases and branching rules of affine Hecke algebras of type D. The proof is similar to the proof of the type B case in Varagnolo and Vasserot (in press) [15].

Keywords: Affine Hecke algebras; Canonical bases

Introduction

Let **f** be the negative part of the quantized enveloping algebra of type $A^{(1)}$. Lusztig's description of the canonical basis of **f** implies that this basis can be naturally identified with the set of isomorphism classes of simple objects of a category of modules of the affine Hecke algebras of type A. This identification was mentioned in [6], and was used in [1]. More precisely, there is a linear isomorphism between **f** and the Grothendieck group of finite-dimensional modules of the affine Hecke algebras of type A, and it is proved in [1] that the induction/restriction functors for affine Hecke algebras are given by the action of the Chevalley generators and their transposed operators with respect to some symmetric bilinear form on **f**.

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The branching rules for affine Hecke algebras of type B have been investigated quite recently, see [2–5,11] and [15]. In particular, in [2–5] an analogue of Ariki's construction is conjectured and studied for affine Hecke algebras of type B. Here **f** is replaced by a module ${}^{\theta}\mathbf{V}(\lambda)$ over an algebra ${}^{\theta}\mathbf{B}$. More precisely it is conjectured there that ${}^{\theta}\mathbf{V}(\lambda)$ admits a canonical basis which is naturally identified with the set of isomorphism classes of simple objects of a category of modules of the affine Hecke algebras of type B. Further, in this identification the branching rules of the affine Hecke algebras of type B should be given by the ${}^{\theta}\mathbf{B}$-action on ${}^{\theta}\mathbf{V}(\lambda)$. This conjecture has been proved [15]. It uses both the geometric picture introduced in [2] (to prove part of the conjecture) and a new kind of graded algebras similar to the KLR algebras from [7,14].

A similar description of the branching rules for affine Hecke algebras of type D has also been conjectured in [8]. In this case **f** is replaced by another module ${}^{\circ}\mathbf{V}$ over the algebra ${}^{\theta}\mathbf{B}$ (the same algebra as in the type B case). The purpose of this paper is to prove the type D case. The method of the proof is the same as in [15]. First we introduce a family of graded algebras ${}^{\circ}\mathbf{R}_m$ for m a non-negative integer. They can be viewed as the Ext-algebras of some complex of constructible sheaves naturally attached to the Lie algebra of the group $SO(2m)$, see Remark 2.8. This complex enters in the Kazhdan–Lusztig classification of the simple modules of the affine Hecke algebra of the group $Spin(2m)$. Then we identify ${}^{\circ}\mathbf{V}$ with the sum of the Grothendieck groups of the graded algebras ${}^{\circ}\mathbf{R}_m$.

The plan of the paper is the following. In Section 1 we introduce a graded algebra ${}^{\circ}\mathbf{R}(\Gamma)_{\nu}$. It is associated with a quiver Γ with an involution θ and with a dimension vector ν. In Section 2 we consider a particular choice of pair (Γ,θ). The graded algebras ${}^{\circ}\mathbf{R}(\Gamma)_{\nu}$ associated with this pair (Γ,θ) are denoted by the symbol ${}^{\circ}\mathbf{R}_m$. Next we introduce the affine Hecke algebra of type D, more precisely the affine Hecke algebra associated with the group $SO(2m)$, and we prove that it is Morita equivalent to ${}^{\circ}\mathbf{R}_m$. In Section 3 we categorify the module ${}^{\circ}\mathbf{V}$ from [8] using the graded algebras ${}^{\circ}\mathbf{R}_m$, see Theorem 3.28. The main result of the paper is Theorem 3.33.

0. Notation

0.1. Graded modules over graded algebras. Let **k** be an algebraically closed field of characteristic 0. By a graded **k**-algebra $\mathbf{R}=\bigoplus_d \mathbf{R}_d$ we'll always mean a $\mathbb{Z}$-graded associative **k**-algebra. Let **R-mod** be the category of finitely generated graded **R**-modules, **R-fmod** be the full subcategory of finite-dimensional graded modules and **R-proj** be the full subcategory of **R-fmod** consisting of projective objects. Unless specified otherwise all modules are left modules. We'll abbreviate

$$K(\mathbf{R})=[\mathbf{R}\text{-}\mathbf{proj}],\qquad G(\mathbf{R})=[\mathbf{R}\text{-}\mathbf{fmod}].$$

Here $[\mathcal{C}]$ denotes the Grothendieck group of an exact category $\mathcal{C}$. Assume that the **k**-vector spaces $\mathbf{R}_d$ are finite dimensional for each d. Then $K(\mathbf{R})$ is a free Abelian group with a basis formed by the isomorphism classes of the indecomposable objects in **R-proj**, and $G(\mathbf{R})$ is a free Abelian group with a basis formed by the isomorphism classes of the simple objects in **R-fmod**. Given an object M of **R-proj** or **R-fmod** let $[M]$ denote its class in $K(\mathbf{R})$, $G(\mathbf{R})$ respectively. When there is no risk of confusion we abbreviate $M=[M]$. We'll write $[M:N]$ for the composition multiplicity of the **R**-module N in the **R**-module M. Consider the ring $\mathcal{A}=\mathbb{Z}[v,v^{-1}]$. If the grading of **R** is bounded below then the $\mathcal{A}$-modules $K(\mathbf{R})$, $G(\mathbf{R})$ are free. Here $\mathcal{A}$ acts on $G(\mathbf{R})$, $K(\mathbf{R})$ as follows

$$vM=M[1],\qquad v^{-1}M=M[-1].$$

For any M, N in **R-mod** let

$$\mathrm{hom}_{\mathbf{R}}(M,N)=\bigoplus_d \mathrm{Hom}_{\mathbf{R}}\big(M,N[d]\big)$$

be the $\mathbb{Z}$-graded **k**-vector space of all **R**-module homomorphisms. If $\mathbf{R}=\mathbf{k}$ we'll omit the subscript **R** in hom's and in tensor products. For any graded **k**-vector space $M=\bigoplus_d M_d$ we'll write

$$\mathrm{gdim}(M)=\sum_d v^d \dim(M_d),$$

where dim is the dimension over **k**.

0.2. Quivers with involutions. Recall that a quiver Γ is a tuple $(I, H, h\mapsto h', h\mapsto h'')$ where I is the set of vertices, H is the set of arrows and for each $h\in H$ the vertices $h', h''\in I$ are the origin and the target of h respectively. Note that the set I may be infinite. We'll assume that no arrow may join a vertex to itself. For each $i, j\in I$ we write

$$H_{i,j}=\big\{h\in H;\, h'=i,\, h''=j\big\}.$$

We'll abbreviate $i\to j$ if $H_{i,j}\neq\emptyset$. Let $h_{i,j}$ be the number of elements in $H_{i,j}$ and set

$$i\cdot j=-h_{i,j}-h_{j,i},\quad i\cdot i=2,\ i\neq j.$$

An involution θ on Γ is a pair of involutions on I and H, both denoted by θ, such that the following properties hold for each h in H

- $\theta(h)'=\theta(h'')$ and $\theta(h)''=\theta(h')$,
- $\theta(h')=h''$ iff $\theta(h)=h$.

We'll always assume that θ has no fixed points in I, i.e., there is no $i\in I$ such that $\theta(i)=i$. To simplify we'll say that θ has no fixed point. Let

$${}^{\theta}\mathbb{N}I=\Big\{\nu=\sum_i \nu_i i\in\mathbb{N}I\colon\ \nu_{\theta(i)}=\nu_i,\ \forall i\Big\}.$$

For any $\nu\in{}^{\theta}\mathbb{N}I$ set $|\nu|=\sum_i \nu_i$. It is an even integer. Write $|\nu|=2m$ with $m\in\mathbb{N}$. We'll denote by ${}^{\theta}I^{\nu}$ the set of sequences

$$\mathbf{i}=(i_{1-m},\dots,i_{m-1},i_m)$$

of elements in I such that $\theta(i_l)=i_{1-l}$ and $\sum_k i_k=\nu$. For any such sequence $\mathbf{i}$ we'll abbreviate $\theta(\mathbf{i})=(\theta(i_{1-m}),\dots,\theta(i_{m-1}),\theta(i_m))$. Finally, we set

$${}^{\theta}I^m=\bigcup_{\nu}{}^{\theta}I^{\nu},\quad \nu\in{}^{\theta}\mathbb{N}I,\qquad |\nu|=2m.$$

0.3. The wreath product. Given a positive integer m, let $\mathfrak{S}_m$ be the symmetric group, and $\mathbb{Z}_2 = \{-1, 1\}$. Consider the wreath product $W_m = \mathfrak{S}_m \wr \mathbb{Z}_2$. Write $s_1, \dots, s_{m-1}$ for the simple reflections in $\mathfrak{S}_m$. For each $l = 1, 2, \dots, m$ let $\varepsilon_l \in (\mathbb{Z}_2)^m$ be -1 placed at the l-th position. There is a unique action of W_m on the set $\{1-m, \dots, m-1, m\}$ such that $\mathfrak{S}_m$ permutes $1, 2, \dots, m$ and such that ε_l fixes k if $k \neq l, 1-l$ and switches l and $1-l$. The group W_m acts also on ${}^\theta I^\nu$. Indeed, view a sequence $\mathbf{i}$ as the map

$$\{1-m, \dots, m-1, m\} \to I, \qquad l \mapsto i_l.$$

Then we set $w(\mathbf{i}) = \mathbf{i} \circ w^{-1}$ for $w \in W_m$. For each ν we fix once for all a sequence

$$\mathbf{i}_e = (i_{1-m}, \dots, i_m) \in {}^\theta I^\nu.$$

Let W_e be the centralizer of $\mathbf{i}_e$ in W_m. Then there is a bijection

$$W_e \backslash W_m \to {}^\theta I^\nu, \qquad W_e w \mapsto w^{-1}(\mathbf{i}_e).$$

Now, assume that $m > 1$. We set $s_0 = \varepsilon_1 s_1 \varepsilon_1$. Let ${}^\circ W_m$ be the subgroup of W_m generated by $s_0, \dots, s_{m-1}$. We'll regard it as a Weyl group of type D_m such that $s_0, \dots, s_{m-1}$ are the simple reflections. Note that W_e is a subgroup of ${}^\circ W_m$. Indeed, if $W_e \not\subset {}^\circ W_m$ there should exist l such that ε_l belongs to W_e. This would imply that $i_l = \theta(i_l)$, contradicting the fact that θ has no fixed point. Therefore ${}^\theta I^\nu$ decomposes into two ${}^\circ W_m$-orbits. We'll denote them by ${}^\theta I^\nu_+$ and ${}^\theta I^\nu_-$. For $m = 1$ we set ${}^\circ W_1 = \{e\}$ and we choose again ${}^\theta I^\nu_+$ and ${}^\theta I^\nu_-$ in an obvious way.

1. The graded k-algebra ${}^\circ\mathbf{R}(\Gamma)_\nu$

Fix a quiver Γ with set of vertices I and set of arrows H. Fix an involution θ on Γ. Assume that Γ has no 1-loops and that θ has no fixed points. Fix a dimension vector $\nu \neq 0$ in ${}^\theta \mathbb{N}I$. Set $|\nu| = 2m$.

1.1. Definition of the graded k-algebra ${}^\circ\mathbf{R}(\Gamma)_\nu$. Assume that $m > 1$. We define a graded **k**-algebra ${}^\circ\mathbf{R}(\Gamma)_\nu$ with 1 generated by $1_\mathbf{i}$, $\varkappa_l$, σ_k, with $\mathbf{i} \in {}^\theta I^\nu$, $l = 1, 2, \dots, m$, $k = 0, 1, \dots, m-1$ modulo the following defining relations

(a) $1_\mathbf{i} 1_{\mathbf{i}'} = \delta_{\mathbf{i},\mathbf{i}'} 1_\mathbf{i}$, $\sigma_k 1_\mathbf{i} = 1_{s_k \mathbf{i}} \sigma_k$, $\varkappa_l 1_\mathbf{i} = 1_\mathbf{i} \varkappa_l$,
(b) $\varkappa_l \varkappa_{l'} = \varkappa_{l'} \varkappa_l$,
(c) $\sigma_k^2 1_\mathbf{i} = Q_{i_k, i_{s_k(k)}}(\varkappa_{s_k(k)}, \varkappa_k) 1_\mathbf{i}$,
(d) $\sigma_k \sigma_{k'} = \sigma_{k'} \sigma_k$ if $1 \leqslant k < k' - 1 < m - 1$ or $0 = k < k' \neq 2$,

(e) $$(\sigma_{s_k(k)} \sigma_k \sigma_{s_k(k)} - \sigma_k \sigma_{s_k(k)} \sigma_k) 1_\mathbf{i} = \begin{cases} \dfrac{Q_{i_k, i_{s_k(k)}}(\varkappa_{s_k(k)}, \varkappa_k) - Q_{i_k, i_{s_k(k)}}(\varkappa_{s_k(k)}, \varkappa_{s_k(k)+1})}{\varkappa_k - \varkappa_{s_k(k)+1}} 1_\mathbf{i} & \text{if } i_k = i_{s_k(k)+1}, \\ 0 & \text{else,} \end{cases}$$

(f) $$(\sigma_k \varkappa_l - \varkappa_{s_k(l)} \sigma_k) 1_\mathbf{i} = \begin{cases} -1_\mathbf{i} & \text{if } l = k,\ i_k = i_{s_k(k)}, \\ 1_\mathbf{i} & \text{if } l = s_k(k),\ i_k = i_{s_k(k)}, \\ 0 & \text{else.} \end{cases}$$

Here we have set $\varkappa_{1-l} = -\varkappa_l$ and

$$Q_{i,j}(u,v) = \begin{cases} (-1)^{h_{i,j}}(u-v)^{-i\cdot j} & \text{if } i \neq j, \\ 0 & \text{else.} \end{cases} \tag{1.1}$$

If $m = 0$ we set ${}^\circ\mathbf{R}(\Gamma)_0 = \mathbf{k} \oplus \mathbf{k}$. If $m = 1$ then we have $\nu = i + \theta(i)$ for some $i \in I$. Write $\mathbf{i} = i\theta(i)$, and

$${}^\circ\mathbf{R}(\Gamma)_\nu = \mathbf{k}[\varkappa_1]1_{\mathbf{i}} \oplus \mathbf{k}[\varkappa_1]1_{\theta(\mathbf{i})}.$$

We'll abbreviate $\sigma_{\mathbf{i},k} = \sigma_k 1_{\mathbf{i}}$ and $\varkappa_{\mathbf{i},l} = \varkappa_l 1_{\mathbf{i}}$. The grading on ${}^\circ\mathbf{R}(\Gamma)_0$ is the trivial one. For $m \geqslant 1$ the grading on ${}^\circ\mathbf{R}(\Gamma)_\nu$ is given by the following rules:

$$\begin{aligned} &\deg(1_{\mathbf{i}}) = 0, \\ &\deg(\varkappa_{\mathbf{i},l}) = 2, \\ &\deg(\sigma_{\mathbf{i},k}) = -i_k \cdot i_{s_k(k)}. \end{aligned}$$

We define ω to be the unique anti-involution of the graded $\mathbf{k}$-algebra ${}^\circ\mathbf{R}(\Gamma)_\nu$ which fixes $1_{\mathbf{i}}$, $\varkappa_l$, σ_k. We set ω to be identity on ${}^\circ\mathbf{R}(\Gamma)_0$.

1.2. Relation with the graded k-algebra ${}^\theta\mathbf{R}(\Gamma)_\nu$. A family of graded $\mathbf{k}$-algebra ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$ was introduced in [15, Sec. 5.1], for λ an arbitrary dimension vector in $\mathbb{N}I$. Here we'll only consider the special case $\lambda = 0$, and we abbreviate ${}^\theta\mathbf{R}(\Gamma)_\nu = {}^\theta\mathbf{R}(\Gamma)_{0,\nu}$. Recall that if $\nu \neq 0$ then ${}^\theta\mathbf{R}(\Gamma)_\nu$ is the graded $\mathbf{k}$-algebra with 1 generated by $1_{\mathbf{i}}$, $\varkappa_l$, σ_k, π_1, with $\mathbf{i} \in {}^\theta I^\nu$, $l = 1, 2, \dots, m$, $k = 1, \dots, m-1$ such that $1_{\mathbf{i}}$, $\varkappa_l$ and σ_k satisfy the same relations as before and

$$\begin{aligned} &\pi_1^2 = 1, \qquad \pi_1 1_{\mathbf{i}} \pi_1 = 1_{\varepsilon_1 \mathbf{i}}, \qquad \pi_1 \varkappa_l \pi_1 = \varkappa_{\varepsilon_1(l)}, \\ &(\pi_1\sigma_1)^2 = (\sigma_1\pi_1)^2, \qquad \pi_1\sigma_k\pi_1 = \sigma_k \quad \text{if } k \neq 1. \end{aligned}$$

If $\nu = 0$ then ${}^\theta\mathbf{R}(\Gamma)_0 = \mathbf{k}$. The grading is given by setting $\deg(1_{\mathbf{i}})$, $\deg(\varkappa_{\mathbf{i},l})$, $\deg(\sigma_{\mathbf{i},k})$ to be as before and $\deg(\pi_1 1_{\mathbf{i}}) = 0$. *In the rest of Section* 1 *we'll assume* $m > 0$. Then there is a canonical inclusion of graded $\mathbf{k}$-algebras

$${}^\circ\mathbf{R}(\Gamma)_\nu \subset {}^\theta\mathbf{R}(\Gamma)_\nu \tag{1.2}$$

such that $1_{\mathbf{i}}, \varkappa_l, \sigma_k \mapsto 1_{\mathbf{i}}, \varkappa_l, \sigma_k$ for $\mathbf{i} \in {}^\theta I^\nu$, $l = 1, \dots, m$, $k = 1, \dots, m-1$ and such that $\sigma_0 \mapsto \pi_1\sigma_1\pi_1$. From now on we'll write $\sigma_0 = \pi_1\sigma_1\pi_1$ whenever $m > 1$. The assignment $x \mapsto \pi_1 x \pi_1$ defines an involution of the graded $\mathbf{k}$-algebra ${}^\theta\mathbf{R}(\Gamma)_\nu$ which normalizes ${}^\circ\mathbf{R}(\Gamma)_\nu$. Thus it yields an involution

$$\gamma : {}^\circ\mathbf{R}(\Gamma)_\nu \to {}^\circ\mathbf{R}(\Gamma)_\nu.$$

Let $\langle\gamma\rangle$ be the group of two elements generated by γ. The smash product ${}^\circ\mathbf{R}(\Gamma)_\nu \rtimes \langle\gamma\rangle$ is a graded $\mathbf{k}$-algebra such that $\deg(\gamma) = 0$. There is a unique isomorphism of graded $\mathbf{k}$-algebras

$${}^\circ\mathbf{R}(\Gamma)_\nu \rtimes \langle\gamma\rangle \to {}^\theta\mathbf{R}(\Gamma)_\nu \tag{1.3}$$

which is identity on ${}^\circ\mathbf{R}(\Gamma)_\nu$ and which takes γ to π_1.

1.3. The polynomial representation and the PBW theorem. For any $\mathbf{i}$ in ${}^\theta I^\nu$ let ${}^\theta\mathbf{F}_\mathbf{i}$ be the subalgebra of ${}^\circ\mathbf{R}(\Gamma)_\nu$ generated by $1_\mathbf{i}$ and $\varkappa_{\mathbf{i},l}$ with $l=1,2,\dots,m$. It is a polynomial algebra. Let

$$^\theta\mathbf{F}_\nu=\bigoplus_{\mathbf{i}\in{}^\theta I^\nu}{}^\theta\mathbf{F}_\mathbf{i}.$$

The group W_m acts on ${}^\theta\mathbf{F}_\nu$ via $w(\varkappa_{\mathbf{i},l})=\varkappa_{w(\mathbf{i}),w(l)}$ for any $w\in W_m$. Consider the fixed points set

$$^\circ\mathbf{S}_\nu=\big({}^\theta\mathbf{F}_\nu\big)^{{}^\circ W_m}.$$

Regard ${}^\theta\mathbf{R}(\Gamma)_\nu$ and $\mathrm{End}({}^\theta\mathbf{F}_\nu)$ as ${}^\theta\mathbf{F}_\nu$-algebras via the left multiplication. Proposition 5.4 in [15] shows that there is an injective graded ${}^\theta\mathbf{F}_\nu$-algebra morphism ${}^\theta\mathbf{R}(\Gamma)_\nu\to\mathrm{End}({}^\theta\mathbf{F}_\nu)$. It restricts via (1.2) to an injective graded ${}^\theta\mathbf{F}_\nu$-algebra morphism

$$^\circ\mathbf{R}(\Gamma)_\nu\to\mathrm{End}\big({}^\theta\mathbf{F}_\nu\big).$$

Next, recall that ${}^\circ W_m$ is the Weyl group of type D_m with simple reflections $s_0,\dots,s_{m-1}$. For each w in ${}^\circ W_m$ we choose a reduced decomposition $\dot w$ of w. It has the following form

$$w=s_{k_1}s_{k_2}\cdots s_{k_r},\quad 0\leqslant k_1,k_2,\dots,k_r\leqslant m-1.$$

We define an element $\sigma_{\dot w}$ in ${}^\circ\mathbf{R}(\Gamma)_\nu$ by

$$\sigma_{\dot w}=\sum_\mathbf{i}1_\mathbf{i}\sigma_{\dot w},\quad 1_\mathbf{i}\sigma_{\dot w}=\begin{cases}1_\mathbf{i} & \text{if } r=0,\\ 1_\mathbf{i}\sigma_{k_1}\sigma_{k_2}\cdots\sigma_{k_r} & \text{else.}\end{cases}\tag{1.4}$$

Observe that the element $\sigma_{\dot w}$ may depend on the choice of the reduced decomposition $\dot w$.

1.4. Proposition. *The* $\mathbf{k}$*-algebra* ${}^\circ\mathbf{R}(\Gamma)_\nu$ *is a free (left or right)* ${}^\theta\mathbf{F}_\nu$*-module with basis* $\{\sigma_{\dot w};\ w\in{}^\circ W_m\}$. *Its rank is* $2^{m-1}m!$. *The operator* $1_\mathbf{i}\sigma_{\dot w}$ *is homogeneous and its degree is independent of the choice of the reduced decomposition* $\dot w$.

Proof. The proof is the same as in [15, Prop. 5.5]. First, we filter the algebra ${}^\circ\mathbf{R}(\Gamma)_\nu$ with $1_\mathbf{i}$, $\varkappa_{\mathbf{i},l}$ in degree 0 and $\sigma_{\mathbf{i},k}$ in degree 1. The *Nil Hecke algebra* of type D_m is the $\mathbf{k}$-algebra ${}^\circ\mathbf{NH}_m$ generated by $\bar\sigma_0,\bar\sigma_1,\dots,\bar\sigma_{m-1}$ with relations

$$\bar\sigma_k\bar\sigma_{k'}=\bar\sigma_{k'}\bar\sigma_k\quad\text{if } 1\leqslant k<k'-1<m-1 \text{ or } 0=k<k'\neq 2,$$
$$\bar\sigma_{s_k(k)}\bar\sigma_k\bar\sigma_{s_k(k)}=\bar\sigma_k\bar\sigma_{s_k(k)}\bar\sigma_k,\qquad \bar\sigma_k^2=0.$$

We can form the semi-direct product ${}^\theta\mathbf{F}_\nu\rtimes{}^\circ\mathbf{NH}_m$, which is generated by $1_\mathbf{i}$, $\bar\varkappa_l$, $\bar\sigma_k$ with the relations above and

$$\bar\sigma_k\bar\varkappa_l=\bar\varkappa_{s_k(l)}\bar\sigma_k,\qquad \bar\varkappa_l\bar\varkappa_{l'}=\bar\varkappa_{l'}\bar\varkappa_{l'}.$$

One proves as in [15, Prop. 5.5] that the map

$$ {}^\theta\mathbf{F}_\nu \rtimes {}^\circ\mathbf{NH}_m \to \mathrm{gr}\big({}^\circ\mathbf{R}(\Gamma)_\nu\big), \qquad 1_\mathbf{i} \mapsto 1_\mathbf{i}, \qquad \bar{\varkappa}_l \mapsto \varkappa_l, \qquad \bar{\sigma}_k \mapsto \sigma_k, $$

is an isomorphism of $\mathbf{k}$-algebras. □

Let ${}^\theta\mathbf{F}'_\nu = \bigoplus_\mathbf{i} {}^\theta\mathbf{F}'_\mathbf{i}$, where ${}^\theta\mathbf{F}'_\mathbf{i}$ is the localization of the ring ${}^\theta\mathbf{F}_\mathbf{i}$ with respect to the multiplicative system generated by

$$ \big\{\varkappa_{\mathbf{i},l} \pm \varkappa_{\mathbf{i},l'};\ 1 \leqslant l \neq l' \leqslant m\big\} \cup \{\varkappa_{\mathbf{i},l};\ l = 1, 2, \ldots, m\}. $$

1.5. Corollary. *The inclusion ${}^\circ\mathbf{R}(\Gamma)_\nu \subset \mathrm{End}({}^\theta\mathbf{F}_\nu)$ yields an isomorphism of ${}^\theta\mathbf{F}'_\nu$-algebras ${}^\theta\mathbf{F}'_\nu \otimes_{{}^\theta\mathbf{F}_\nu} {}^\circ\mathbf{R}(\Gamma)_\nu \to {}^\theta\mathbf{F}'_\nu \rtimes {}^\circ W_m$, such that for each $\mathbf{i}$ and each $l = 1, 2, \ldots, m$, $k = 0, 1, 2, \ldots, m-1$ we have*

$$ \begin{aligned} &1_\mathbf{i} \mapsto 1_\mathbf{i}, \\ &\varkappa_{\mathbf{i},l} \mapsto \varkappa_l 1_\mathbf{i}, \\ &\sigma_{\mathbf{i},k} \mapsto \begin{cases} (\varkappa_k - \varkappa_{s_k(k)})^{-1}(s_k - 1)1_\mathbf{i} & \text{if } i_k = i_{s_k(k)}, \\ (\varkappa_k - \varkappa_{s_k(k)})^{h_{i_{s_k(k)}, i_k}} s_k 1_\mathbf{i} & \text{if } i_k \neq i_{s_k(k)}. \end{cases} \end{aligned} \tag{1.5} $$

Proof. Follows from [15, Cor. 5.6] and Proposition 1.4. □

Restricting the ${}^\theta\mathbf{F}_\nu$-action on ${}^\circ\mathbf{R}(\Gamma)_\nu$ to the $\mathbf{k}$-subalgebra ${}^\circ\mathbf{S}_\nu$ we get a structure of graded ${}^\circ\mathbf{S}_\nu$-algebra on ${}^\circ\mathbf{R}(\Gamma)_\nu$.

1.6. Proposition.

(a) *${}^\circ\mathbf{S}_\nu$ is isomorphic to the center of ${}^\circ\mathbf{R}(\Gamma)_\nu$.*
(b) *${}^\circ\mathbf{R}(\Gamma)_\nu$ is a free graded module over ${}^\circ\mathbf{S}_\nu$ of rank $(2^{m-1}m!)^2$.*

Proof. Part (a) follows from Corollary 1.5. Part (b) follows from (a) and Proposition 1.4. □

2. Affine Hecke algebras of type D

2.1. Affine Hecke algebras of type D. Fix p in $\mathbf{k}^\times$. For any integer $m \geqslant 0$ we define the extended affine Hecke algebra $\mathbf{H}_m$ of type D_m as follows. If $m > 1$ then $\mathbf{H}_m$ is the $\mathbf{k}$-algebra with 1 generated by

$$ T_k, \quad X_l^{\pm 1}, \quad k = 0, 1, \ldots, m-1,\ l = 1, 2, \ldots, m $$

satisfying the following defining relations:

(a) $X_l X_{l'} = X_{l'} X_l$,
(b) $T_k T_{s_k(k)} T_k = T_{s_k(k)} T_k T_{s_k(k)}$, $T_k T_{k'} = T_{k'} T_k$ if $1 \leqslant k < k' - 1$ or $k = 0$, $k' \neq 2$,
(c) $(T_k - p)(T_k + p^{-1}) = 0$,
(d) $T_0 X_1^{-1} T_0 = X_2$, $T_k X_k T_k = X_{s_k(k)}$ if $k \neq 0$, $T_k X_l = X_l T_k$ if $k \neq 0, l, l-1$ or $k = 0$, $l \neq 1, 2$.

Finally, we set $\mathbf{H}_0 = \mathbf{k} \oplus \mathbf{k}$ and $\mathbf{H}_1 = \mathbf{k}[X_1^{\pm 1}]$.

2.2. Remarks. (a) The extended affine Hecke algebra $\mathbf{H}_m^{\mathrm{B}}$ of type B_m with parameters $p, q \in \mathbf{k}^\times$ such that $q = 1$ is generated by P, T_k, $X_l^{\pm 1}$, $k = 1, \dots, m-1$, $l = 1, \dots, m$ such that T_k, $X_l^{\pm 1}$ satisfy the relations as above and

$$P^2 = 1, \qquad (PT_1)^2 = (T_1 P)^2, \qquad PT_k = T_k P \quad \text{if } k \neq 1,$$
$$PX_1^{-1}P = X_1, \qquad PX_l = X_l P \quad \text{if } l \neq 1.$$

See e.g., [15, Sec. 6.1]. There is an obvious $\mathbf{k}$-algebra embedding $\mathbf{H}_m \subset \mathbf{H}_m^{\mathrm{B}}$. Let γ denote also the involution $\mathbf{H}_m \to \mathbf{H}_m$, $a \mapsto PaP$. We have a canonical isomorphism of $\mathbf{k}$-algebras

$$\mathbf{H}_m \rtimes \langle \gamma \rangle \simeq \mathbf{H}_m^{\mathrm{B}}.$$

Compare Section 1.2.

(b) Given a connected reductive group G we call *affine Hecke algebra of G* the Hecke algebra of the extended affine Weyl group $W \ltimes P$, where W is the Weyl group of (G, T), P is the group of characters of T, and T is a maximal torus of G. Then $\mathbf{H}_m$ is the affine Hecke algebra of the group $SO(2m)$. Let $\mathbf{H}_m^e$ be the affine Hecke algebra of the group $Spin(2m)$. It is generated by $\mathbf{H}_m$ and an element X_0 such that

$$X_0^2 = X_1 X_2 \cdots X_m, \qquad T_k X_0 = X_0 T_k \quad \text{if } k \neq 0, \qquad T_0 X_0 X_1^{-1} X_2^{-1} T_0 = X_0.$$

Thus $\mathbf{H}_m$ is the fixed point subset of the $\mathbf{k}$-algebra automorphism of $\mathbf{H}_m^e$ taking T_k, X_l to $T_k, (-1)^{\delta_{l,0}} X_l$ for all k, l. Therefore, if p is not a root of 1 the simple $\mathbf{H}_m$-modules can be recovered from the Kazhdan–Lusztig classification of the simple $\mathbf{H}_m^e$-modules via Clifford theory, see e.g., [13].

2.3. Intertwiners and blocks of $\mathbf{H}_m$. We define

$$\mathbf{A} = \mathbf{k}\big[X_1^{\pm 1}, X_2^{\pm 1}, \dots, X_m^{\pm 1}\big], \qquad \mathbf{A}' = \mathbf{A}\big[\Sigma^{-1}\big], \qquad \mathbf{H}'_m = \mathbf{A}' \otimes_{\mathbf{A}} \mathbf{H}_m,$$

where Σ is the multiplicative set generated by

$$1 - X_l X_{l'}^{\pm 1}, \quad 1 - p^2 X_l^{\pm 1} X_{l'}^{\pm 1}, \quad l \neq l'.$$

For $k = 0, \dots, m-1$ the intertwiner φ_k is the element of $\mathbf{H}'_m$ given by the following formulas

$$\varphi_k - 1 = \frac{X_k - X_{s_k(k)}}{pX_k - p^{-1}X_{s_k(k)}}(T_k - p). \tag{2.1}$$

The group ${}^\circ W_m$ acts on $\mathbf{A}'$ as follows

$$(s_k a)(X_1, \dots, X_m) = a(X_1, \dots, X_{k+1}, X_k, \dots, X_m) \quad \text{if } k \neq 0,$$
$$(s_0 a)(X_1, \dots, X_m) = a\big(X_2^{-1}, X_1^{-1}, \dots, X_m\big).$$

There is an isomorphism of $\mathbf{A}'$-algebras

$$\mathbf{A}' \rtimes {}^\circ W_m \to \mathbf{H}'_m, \qquad s_k \mapsto \varphi_k.$$

The semi-direct product group $\mathbb{Z} \rtimes \mathbb{Z}_2 = \mathbb{Z} \rtimes \{-1, 1\}$ acts on $\mathbf{k}^\times$ by $(n, \varepsilon) : z \mapsto z^\varepsilon p^{2n}$. Given a $\mathbb{Z} \rtimes \mathbb{Z}_2$-invariant subset I of $\mathbf{k}^\times$ we denote by $\mathbf{H}_m\text{-}\mathbf{Mod}_I$ the category of all $\mathbf{H}_m$-modules such that the action of $X_1, X_2, \ldots, X_m$ is locally finite with eigenvalues in I. We associate to the set I and to the element $p \in \mathbf{k}^\times$ a quiver Γ as follows. The set of vertices is I, and there is one arrow $p^2 i \to i$ whenever i lies in I. We equip Γ with an involution θ such that $\theta(i) = i^{-1}$ for each vertex i and such that θ takes the arrow $p^2 i \to i$ to the arrow $i^{-1} \to p^{-2} i^{-1}$. *We'll assume that the set I does not contain* 1 *nor* -1 *and that* $p \neq 1, -1$. Thus the involution θ has no fixed points and no arrow may join a vertex of Γ to itself.

2.4. Remark. We may assume that $I = \pm\{p^n;\ n \in \mathbb{Z}_{\mathrm{odd}}\}$. See the discussion in [8]. Then Γ is of type A_∞ if p has infinite order and Γ is of type $\mathrm{A}_r^{(1)}$ if p^2 is a primitive r-th root of unity.

2.5. $\mathbf{H}_m$-modules versus ${}^\circ\mathbf{R}_m$-modules. Assume that $m \geqslant 1$. We define the graded $\mathbf{k}$-algebra

$$^\theta\mathbf{R}_{I,m} = \bigoplus_\nu {}^\theta\mathbf{R}_{I,\nu}, \qquad {}^\theta\mathbf{R}_{I,\nu} = {}^\theta\mathbf{R}(\Gamma)_\nu, \qquad {}^\circ\mathbf{R}_{I,m} = \bigoplus_\nu {}^\circ\mathbf{R}_{I,\nu}, \qquad {}^\circ\mathbf{R}_{I,\nu} = {}^\circ\mathbf{R}(\Gamma)_\nu,$$

$$^\theta I^m = \bigsqcup_\nu {}^\theta I^\nu,$$

where ν runs over the set of all dimension vectors in ${}^\theta\mathbb{N}I$ such that $|\nu| = 2m$. When there is no risk of confusion we abbreviate

$$^\theta\mathbf{R}_\nu = {}^\theta\mathbf{R}_{I,\nu}, \qquad {}^\theta\mathbf{R}_m = {}^\theta\mathbf{R}_{I,m}, \qquad {}^\circ\mathbf{R}_\nu = {}^\circ\mathbf{R}_{I,\nu}, \qquad {}^\circ\mathbf{R}_m = {}^\circ\mathbf{R}_{I,m}.$$

Note that ${}^\theta\mathbf{R}_\nu$ and ${}^\theta\mathbf{R}_m$ are the same as in [15, Sec. 6.4], with $\lambda = 0$. Note also that the $\mathbf{k}$-algebra ${}^\circ\mathbf{R}_m$ may not have 1, because the set I may be infinite. We define ${}^\circ\mathbf{R}_m\text{-}\mathbf{Mod}_0$ as the category of all (non-graded) ${}^\circ\mathbf{R}_m$-modules such that the elements $\varkappa_1, \varkappa_2, \ldots, \varkappa_m$ act locally nilpotently. Let ${}^\circ\mathbf{R}_m\text{-}\mathbf{fMod}_0$ and $\mathbf{H}_m\text{-}\mathbf{fMod}_I$ be the full subcategories of finite-dimensional modules in ${}^\circ\mathbf{R}_m\text{-}\mathbf{Mod}_0$ and $\mathbf{H}_m\text{-}\mathbf{Mod}_I$ respectively. Fix a formal series $f(\varkappa)$ in $\mathbf{k}[[\varkappa]]$ such that $f(\varkappa) = 1 + \varkappa$ modulo $(\varkappa^2)$.

2.6. Theorem. *We have an equivalence of categories*

$$^\circ\mathbf{R}_m\text{-}\mathbf{Mod}_0 \to \mathbf{H}_m\text{-}\mathbf{Mod}_I, \qquad M \mapsto M$$

which is given by

(a) *X_l acts on $1_\mathbf{i}M$ by $i_l^{-1} f(\varkappa_l)$ for each $l = 1, 2, \ldots, m$,*
(b) *if $m > 1$ then T_k acts on $1_\mathbf{i}M$ as follows for each $k = 0, 1, \ldots, m-1$,*

$$\frac{(pf(\varkappa_k) - p^{-1} f(\varkappa_{s_k(k)}))(\varkappa_k - \varkappa_{s_k(k)})}{f(\varkappa_k) - f(\varkappa_{s_k(k)})}\sigma_k + p \quad \text{if } i_{s_k(k)} = i_k,$$

$$\frac{f(\varkappa_k)-f(\varkappa_{s_k(k)})}{(p^{-1}f(\varkappa_k)-pf(\varkappa_{s_k(k)}))(\varkappa_k-\varkappa_{s_k(k)})}\sigma_k+\frac{(p^{-2}-1)f(\varkappa_{s_k(k)})}{pf(\varkappa_k)-p^{-1}f(\varkappa_{s_k(k)})} \quad \text{if } i_{s_k(k)}=p^2 i_k,$$

$$\frac{pi_k f(\varkappa_k)-p^{-1}i_{s_k(k)}f(\varkappa_{s_k(k)})}{i_k f(\varkappa_k)-i_{s_k(k)}f(\varkappa_{s_k(k)})}\sigma_k+\frac{(p^{-1}-p)i_k f(\varkappa_{s_k(k)})}{i_{s_k(k)}f(\varkappa_k)-i_k f(\varkappa_{s_k(k)})} \quad \text{if } i_{s_k(k)}\neq i_k, p^2 i_k.$$

Proof. This follows from [15, Thm. 6.5] by Section 1.2 and Remark 2.2(a). One can also prove it by reproducing the arguments in [15] by using (1.5) and (2.1). □

2.7. Corollary. *There is an equivalence of categories*

$$\Psi : {}^{\circ}\mathbf{R}_m\text{-}\mathbf{fMod}_0 \to \mathbf{H}_m\text{-}\mathbf{fMod}_I, \qquad M \mapsto M.$$

2.8. Remarks. (a) Let $\mathfrak{g}$ be the Lie algebra of $G = SO(2m)$. Fix a maximal torus $T \subset G$. The group of characters of T is the lattice $\bigoplus_{l=1}^m \mathbb{Z}\varepsilon_l$, with Bourbaki's notation. Fix a dimension vector $\nu \in {}^\theta\mathbb{N}I$. Recall the sequence $\mathbf{i}_e = (i_{1-m}, \ldots, i_{m-1}, i_m)$ from Section 0.3. Let $g \in T$ be the element such that $\varepsilon_l(g) = i_l^{-1}$ for each $l = 1, 2, \ldots, m$. Recall also the notation ${}^\theta\boldsymbol{\mathcal{V}}_\nu$, $\mathbf{V}$, ${}^\theta E_\mathbf{V}$, and ${}^\theta G_\mathbf{V}$ from [15]. Then $\mathbf{V}$ is an object of ${}^\theta\boldsymbol{\mathcal{V}}_\nu$, ${}^\theta G_\mathbf{V} = G_g$ is the centralizer of g in G, and

$${}^\theta E_\mathbf{V} = \mathfrak{g}_{g,p}, \quad \mathfrak{g}_{g,p} = \{x \in \mathfrak{g};\ \mathrm{ad}_g(x) = p^2 x\}.$$

Let F_g be the set of all Borel Lie subalgebras of $\mathfrak{g}$ fixed by the adjoint action of g. It is a non-connected manifold whose connected components are labeled by ${}^\theta I_+^\nu$. In Section 3.14 we'll introduce two central idempotents $1_{\nu,+}$, $1_{\nu,-}$ of ${}^\circ\mathbf{R}_\nu$. This yields a graded $\mathbf{k}$-algebra decomposition

$${}^\circ\mathbf{R}_\nu = {}^\circ\mathbf{R}_\nu 1_{\nu,+} \oplus {}^\circ\mathbf{R}_\nu 1_{\nu,-}.$$

By [15, Thm. 5.8] the graded $\mathbf{k}$-algebra ${}^\circ\mathbf{R}_\nu 1_{\nu,+}$ is isomorphic to

$$\mathrm{Ext}^*_{G_g}(\mathcal{L}_{g,p}, \mathcal{L}_{g,p}),$$

where $\mathcal{L}_{g,p}$ is the direct image of the constant perverse sheaf by the projection

$$\{(\mathfrak{b}, x) \in F_g \times \mathfrak{g}_{g,p};\ x \in \mathfrak{b}\} \to \mathfrak{g}_{g,p}, \qquad (\mathfrak{b}, x) \mapsto x.$$

The complex $\mathcal{L}_{g,p}$ has been extensively studied by Lusztig, see e.g., [9,10]. We hope to come back to this elsewhere.

(b) The results in Section 2.5 hold true if $\mathbf{k}$ is any field. Set $f(\varkappa) = 1 + \varkappa$ for instance.

2.9. Induction and restriction of $\mathbf{H}_m$-modules. For $i \in I$ we define functors

$$E_i : \mathbf{H}_{m+1}\text{-}\mathbf{fMod}_I \to \mathbf{H}_m\text{-}\mathbf{fMod}_I,$$
$$F_i : \mathbf{H}_m\text{-}\mathbf{fMod}_I \to \mathbf{H}_{m+1}\text{-}\mathbf{fMod}_I, \tag{2.2}$$

where $E_i M \subset M$ is the generalized i^{-1}-eigenspace of the X_{m+1}-action, and where

$$F_i M = \mathrm{Ind}_{\mathbf{H}_m \otimes \mathbf{k}[X_{m+1}^{\pm 1}]}^{\mathbf{H}_{m+1}} (M \otimes \mathbf{k}_i).$$

Here $\mathbf{k}_i$ is the 1-dimensional representation of $\mathbf{k}[X_{m+1}^{\pm 1}]$ defined by $X_{m+1} \mapsto i^{-1}$.

3. Global bases of °V and projective graded °R-modules

3.1. The Grothendieck groups of $^\circ\mathbf{R}_m$. The graded $\mathbf{k}$-algebra $^\circ\mathbf{R}_m$ is free of finite rank over its center by Proposition 1.6, a commutative graded $\mathbf{k}$-subalgebra. Therefore any simple object of $^\circ\mathbf{R}_m$**-mod** is finite-dimensional and there is a finite number of isomorphism classes of simple modules in $^\circ\mathbf{R}_m$**-mod**$_0$. The Abelian group $G(^\circ\mathbf{R}_m)$ is free with a basis formed by the classes of the simple objects of $^\circ\mathbf{R}_m$**-mod**. The Abelian group $K(^\circ\mathbf{R}_m)$ is free with a basis formed by the classes of the indecomposable projective objects. Both $G(^\circ\mathbf{R}_m)$ and $K(^\circ\mathbf{R}_m)$ are free $\mathcal{A}$-modules, where v shifts the grading by 1. We consider the following $\mathcal{A}$-modules

$$^\circ\mathbf{K}_I = \bigoplus_{m\geqslant 0} {}^\circ\mathbf{K}_{I,m}, \quad {}^\circ\mathbf{K}_{I,m} = K\big(^\circ\mathbf{R}_m\big),$$

$$^\circ\mathbf{G}_I = \bigoplus_{m\geqslant 0} {}^\circ\mathbf{G}_{I,m}, \quad {}^\circ\mathbf{G}_{I,m} = G\big(^\circ\mathbf{R}_m\big).$$

We'll also abbreviate

$$^\circ\mathbf{K}_{I,*} = \bigoplus_{m>0} {}^\circ\mathbf{K}_{I,m}, \qquad {}^\circ\mathbf{G}_{I,*} = \bigoplus_{m>0} {}^\circ\mathbf{G}_{I,m}.$$

From now on, to unburden the notation we may abbreviate $^\circ\mathbf{R} = {}^\circ\mathbf{R}_m$, hoping it will not create any confusion. For any M, N in $^\circ\mathbf{R}$**-mod** we set

$$(M : N) = \mathrm{gdim}\big(M^\omega \otimes_{^\circ\mathbf{R}} N\big), \qquad \langle M : N\rangle = \mathrm{gdim}\,\mathrm{hom}_{^\circ\mathbf{R}}(M, N),$$

where ω is the involution defined in Section 1.1. The Cartan pairing is the perfect $\mathcal{A}$-bilinear form

$$^\circ\mathbf{K}_I \times {}^\circ\mathbf{G}_I \to \mathcal{A}, \qquad (P, M) \mapsto \langle P : M\rangle.$$

First, we concentrate on the $\mathcal{A}$-module $^\circ\mathbf{G}_I$. Consider the duality

$$^\circ\mathbf{R}\textbf{-fmod} \to {}^\circ\mathbf{R}\textbf{-fmod}, \qquad M \mapsto M^\flat = \mathrm{hom}(M, \mathbf{k}),$$

with the action and the grading given by

$$(xf)(m) = f\big(\omega(x)m\big), \qquad \big(M^\flat\big)_d = \mathrm{Hom}(M_{-d}, \mathbf{k}).$$

This duality functor yields an $\mathcal{A}$-antilinear map

$$^\circ\mathbf{G}_I \to {}^\circ\mathbf{G}_I, \qquad M \mapsto M^\flat.$$

Let $^\circ B$ denote the set of isomorphism classes of simple objects of $^\circ\mathbf{R}$**-fMod**$_0$. We can now define the upper global basis of $^\circ\mathbf{G}_I$ as follows. The proof is given in Section 3.21.

3.2. Proposition/Definition. *For each b in ${}^\circ B$ there is a unique selfdual irreducible graded ${}^\circ\mathbf{R}$-module ${}^\circ G^{\mathrm{up}}(b)$ which is isomorphic to b as a (non graded) ${}^\circ\mathbf{R}$-module. We set ${}^\circ G^{\mathrm{up}}(0)=0$ and ${}^\circ\mathbf{G}^{\mathrm{up}}=\{{}^\circ G^{\mathrm{up}}(b);\ b\in {}^\circ B\}$. Hence ${}^\circ\mathbf{G}^{\mathrm{up}}$ is an $\mathcal{A}$-basis of ${}^\circ\mathbf{G}_I$.*

Now, we concentrate on the $\mathcal{A}$-module ${}^\circ\mathbf{K}_I$. We equip ${}^\circ\mathbf{K}_I$ with the symmetric $\mathcal{A}$-bilinear form

$$ {}^\circ\mathbf{K}_I\times {}^\circ\mathbf{K}_I\to\mathcal{A},\qquad (M,N)\mapsto (M:N). \tag{3.1}$$

Consider the duality

$$ {}^\circ\mathbf{R}\text{-}\mathbf{proj}\to {}^\circ\mathbf{R}\text{-}\mathbf{proj},\qquad P\mapsto P^\sharp=\hom_{{}^\circ\mathbf{R}}\big(P,{}^\circ\mathbf{R}\big),$$

with the action and the grading given by

$$ (xf)(p)=f(p)\omega(x),\qquad \big(P^\sharp\big)_d=\mathrm{Hom}_{{}^\circ\mathbf{R}}\big(P[-d],{}^\circ\mathbf{R}\big).$$

This duality functor yields an $\mathcal{A}$-antilinear map

$$ {}^\circ\mathbf{K}_I\to {}^\circ\mathbf{K}_I,\qquad P\mapsto P^\sharp.$$

Set $\mathcal{K}=\mathbb{Q}(v)$. Let $\mathcal{K}\to\mathcal{K}$, $f\mapsto\bar f$ be the unique involution such that $\bar v=v^{-1}$.

3.3. Definition. For each b in ${}^\circ B$ let ${}^\circ G^{\mathrm{low}}(b)$ be the unique indecomposable graded module in ${}^\circ\mathbf{R}$-**proj** whose top is isomorphic to ${}^\circ G^{\mathrm{up}}(b)$. We set ${}^\circ G^{\mathrm{low}}(0)=0$ and ${}^\circ\mathbf{G}^{\mathrm{low}}=\{{}^\circ G^{\mathrm{low}}(b);\ b\in {}^\circ B\}$, an $\mathcal{A}$-basis of ${}^\circ\mathbf{K}_I$.

3.4. Proposition. (a) *We have $\langle {}^\circ G^{\mathrm{low}}(b):{}^\circ G^{\mathrm{up}}(b')\rangle=\delta_{b,b'}$ for each b,b' in ${}^\circ B$.*
(b) *We have $\langle P^\sharp:M\rangle=\overline{\langle P:M^\flat\rangle}$ for each P, M.*
(c) *We have ${}^\circ G^{\mathrm{low}}(b)^\sharp={}^\circ G^{\mathrm{low}}(b)$ for each b in ${}^\circ B$.*

The proof is the same as in [15, Prop. 8.4].

3.5. Example. Set $\nu=i+\theta(i)$ and $\mathbf{i}=i\theta(i)$. Consider the graded ${}^\circ\mathbf{R}_\nu$-modules

$$ {}^\circ\mathbf{R}_{\mathbf{i}}={}^\circ\mathbf{R}1_{\mathbf{i}}=1_{\mathbf{i}}{}^\circ\mathbf{R},\qquad {}^\circ\mathbf{L}_{\mathbf{i}}=\mathrm{top}\big({}^\circ\mathbf{R}_{\mathbf{i}}\big).$$

The global bases are given by

$$ {}^\circ\mathbf{G}^{\mathrm{low}}_\nu=\big\{{}^\circ\mathbf{R}_{\mathbf{i}},\ {}^\circ\mathbf{R}_{\theta(\mathbf{i})}\big\},\qquad {}^\circ\mathbf{G}^{\mathrm{up}}_\nu=\big\{{}^\circ\mathbf{L}_{\mathbf{i}},\ {}^\circ\mathbf{L}_{\theta(\mathbf{i})}\big\}.$$

For $m=0$ we have ${}^\circ\mathbf{R}_0=\mathbf{k}\oplus\mathbf{k}$. Set $\phi_+=\mathbf{k}\oplus 0$ and $\phi_-=0\oplus\mathbf{k}$. We have

$$ {}^\circ\mathbf{G}^{\mathrm{low}}_0={}^\circ\mathbf{G}^{\mathrm{up}}_0=\{\phi_+,\ \phi_-\}.$$

3.6. Definition of the operators e_i, f_i, e'_i, f'_i. In this section we'll always assume $m > 0$ unless specified otherwise. First, let us introduce the following notation. Let $D_{m,1}$ be the set of minimal representative in ${}^\circ W_{m+1}$ of the cosets in ${}^\circ W_m \backslash {}^\circ W_{m+1}$. Write

$$D_{m,1;m,1} = D_{m,1} \cap (D_{m,1})^{-1}.$$

For each element w of $D_{m,1;m,1}$ we set

$$W(w) = {}^\circ W_m \cap w\big({}^\circ W_m\big)w^{-1}.$$

Let $\mathbf{R}_1$ be the $\mathbf{k}$-algebra generated by elements 1_i, $\varkappa_i$, $i \in I$, satisfying the defining relations $1_i\,1_{i'} = \delta_{i,i'}1_i$ and $\varkappa_i = 1_i\varkappa_i 1_i$. We equip $\mathbf{R}_1$ with the grading such that $\deg(1_i) = 0$ and $\deg(\varkappa_i) = 2$. Let

$$\mathbf{R}_i = 1_i\mathbf{R}_1 = \mathbf{R}_1 1_i, \qquad \mathbf{L}_i = \operatorname{top}(\mathbf{R}_i) = \mathbf{R}_i/(\varkappa_i).$$

Then $\mathbf{R}_i$ is a graded projective $\mathbf{R}_1$-module and $\mathbf{L}_i$ is simple. We abbreviate

$${}^\theta\mathbf{R}_{m,1} = {}^\theta\mathbf{R}_m \otimes \mathbf{R}_1, \qquad {}^\circ\mathbf{R}_{m,1} = {}^\circ\mathbf{R}_m \otimes \mathbf{R}_1.$$

There is a unique inclusion of graded $\mathbf{k}$-algebras

$$\begin{aligned}
&{}^\theta\mathbf{R}_{m,1} \to {}^\theta\mathbf{R}_{m+1},\\
&1_{\mathbf{i}} \otimes 1_i \mapsto 1_{\mathbf{i}'},\\
&1_{\mathbf{i}} \otimes \varkappa_{i,l} \mapsto \varkappa_{\mathbf{i}',m+l},\\
&\varkappa_{\mathbf{i},l} \otimes 1_i \mapsto \varkappa_{\mathbf{i}',l},\\
&\pi_{\mathbf{i},1} \otimes 1_i \mapsto \pi_{\mathbf{i}',1},\\
&\sigma_{\mathbf{i},k} \otimes 1_i \mapsto \sigma_{\mathbf{i}',k},
\end{aligned} \tag{3.2}$$

where, given $\mathbf{i} \in {}^\theta I^m$ and $i \in I$, we have set $\mathbf{i}' = \theta(i)\mathbf{i}i$, a sequence in ${}^\theta I^{m+1}$. This inclusion restricts to an inclusion ${}^\circ\mathbf{R}_{m,1} \subset {}^\circ\mathbf{R}_{m+1}$.

3.7. Lemma. *The graded ${}^\circ\mathbf{R}_{m,1}$-module ${}^\circ\mathbf{R}_{m+1}$ is free of rank $2(m+1)$.*

Proof. For each w in $D_{m,1}$ we have the element $\sigma_{\dot w}$ in ${}^\circ\mathbf{R}_{m+1}$ defined in (1.5). Using filtered/graded arguments it is easy to see that

$${}^\circ\mathbf{R}_{m+1} = \bigoplus_{w \in D_{m,1}} {}^\circ\mathbf{R}_{m,1}\sigma_{\dot w}. \qquad \square$$

We define a triple of adjoint functors $(\psi_!, \psi^*, \psi_*)$ where

$$\psi^* : {}^\circ\mathbf{R}_{m+1}\text{-}\mathbf{mod} \to {}^\circ\mathbf{R}_m\text{-}\mathbf{mod} \times \mathbf{R}_1\text{-}\mathbf{mod}$$

is the restriction and $\psi_!$, ψ_* are given by

$$\psi_! : \begin{cases} {}^\circ\mathbf{R}_m\textbf{-mod} \times \mathbf{R}_1\textbf{-mod} \to {}^\circ\mathbf{R}_{m+1}\textbf{-mod}, \\ (M, M') \mapsto {}^\circ\mathbf{R}_{m+1} \otimes_{{}^\circ\mathbf{R}_{m,1}} (M \otimes M'), \end{cases}$$

$$\psi_* : \begin{cases} {}^\circ\mathbf{R}_m\textbf{-mod} \times \mathbf{R}_1\textbf{-mod} \to {}^\circ\mathbf{R}_{m+1}\textbf{-mod}, \\ (M, M') \mapsto \hom_{{}^\circ\mathbf{R}_{m,1}}({}^\circ\mathbf{R}_{m+1}, M \otimes M'). \end{cases}$$

First, note that the functors $\psi_!$, ψ^*, ψ_* commute with the shift of the grading. Next, the functor ψ^* is exact, and it takes finite-dimensional graded modules to finite-dimensional ones. The right graded ${}^\circ\mathbf{R}_{m,1}$-module ${}^\circ\mathbf{R}_{m+1}$ is free of finite rank. Thus $\psi_!$ is exact, and it takes finite-dimensional graded modules to finite-dimensional ones. The left graded ${}^\circ\mathbf{R}_{m,1}$-module ${}^\circ\mathbf{R}_{m+1}$ is also free of finite rank. Thus the functor ψ_* is exact, and it takes finite-dimensional graded modules to finite-dimensional ones. Further $\psi_!$ and ψ^* take projective graded modules to projective ones, because they are left adjoint to the exact functors ψ^*, ψ_* respectively. To summarize, the functors $\psi_!$, ψ^*, ψ_* are exact and take finite-dimensional graded modules to finite-dimensional ones, and the functors $\psi_!$, ψ^* take projective graded modules to projective ones.

For any graded ${}^\circ\mathbf{R}_m$-module M we write

$$\begin{aligned} f_i(M) &= {}^\circ\mathbf{R}_{m+1} 1_{m,i} \otimes_{{}^\circ\mathbf{R}_m} M, \\ e_i(M) &= {}^\circ\mathbf{R}_{m-1} \otimes_{{}^\circ\mathbf{R}_{m-1,1}} 1_{m-1,i} M. \end{aligned} \tag{3.3}$$

Let us explain these formulas. The symbols $1_{m,i}$ and $1_{m-1,i}$ are given by

$$1_{m-1,i} M = \bigoplus_{\mathbf{i}} 1_{\theta(i)\mathbf{i}i} M, \quad \mathbf{i} \in {}^\theta I^{m-1}$$

and similarly for ${}^\circ\mathbf{R}_{m+1} 1_{m,i}$. Note that $f_i(M)$ is a graded ${}^\circ\mathbf{R}_{m+1}$-module, while $e_i(M)$ is a graded ${}^\circ\mathbf{R}_{m-1}$-module. The tensor product in the definition of $e_i(M)$ is relative to the graded $\mathbf{k}$-algebra homomorphism

$${}^\circ\mathbf{R}_{m-1,1} \to {}^\circ\mathbf{R}_{m-1} \otimes \mathbf{R}_1 \to {}^\circ\mathbf{R}_{m-1} \otimes \mathbf{R}_i \to {}^\circ\mathbf{R}_{m-1} \otimes \mathbf{L}_i = {}^\circ\mathbf{R}_{m-1}.$$

In other words, let $e'_i(M)$ be the graded ${}^\circ\mathbf{R}_{m-1}$-module obtained by taking the direct summand $1_{m-1,i} M$ and restricting it to ${}^\circ\mathbf{R}_{m-1}$. Observe that if M is finitely generated then $e'_i(M)$ may not lie in ${}^\circ\mathbf{R}_{m-1}\textbf{-mod}$. To remedy this, since $e'_i(M)$ affords a ${}^\circ\mathbf{R}_{m-1} \otimes \mathbf{R}_i$-action we consider the graded ${}^\circ\mathbf{R}_{m-1}$-module

$$e_i(M) = e'_i(M) / \varkappa_i e'_i(M).$$

3.8. Definition. The functors e_i, f_i preserve the category ${}^\circ\mathbf{R}\textbf{-proj}$, yielding $\mathcal{A}$-linear operators on ${}^\circ\mathbf{K}_I$ which act on ${}^\circ\mathbf{K}_{I,*}$ by the formula (3.3) and satisfy also

$$f_i(\phi_+) = {}^\circ\mathbf{R}_{\theta(i)i}, \qquad f_i(\phi_-) = {}^\circ\mathbf{R}_{i\theta(i)}, \qquad e_i(\mathbf{R}_{\theta(j)j}) = \delta_{i,j}\phi_+ + \delta_{i,\theta(j)}\phi_-.$$

Let e_i, f_i denote also the $\mathcal{A}$-linear operators on ${}^\circ\mathbf{G}_I$ which are the transpose of f_i, e_i with respect to the Cartan pairing.

Note that the symbols $e_i(M)$, $f_i(M)$ have a different meaning if M is viewed as an element of ${}^\circ\mathbf{K}_I$ or if M is viewed as an element of ${}^\circ\mathbf{G}_I$. We hope this will not create any confusion. The proof of the following lemma is the same as in [15, Lem. 8.9].

3.9. Lemma. (a) *The operators e_i, f_i on ${}^\circ\mathbf{G}_I$ are given by*

$$e_i(M) = 1_{m-1,i}M \quad f_i(M) = \hom_{{}^\circ\mathbf{R}_{m,1}}\big({}^\circ\mathbf{R}_{m+1}, M \otimes \mathbf{L}_i\big), \quad M \in {}^\circ\mathbf{R}_m\text{-}\mathbf{fmod}.$$

(b) *For each $M \in {}^\circ\mathbf{R}_m$-**mod**, $M' \in {}^\circ\mathbf{R}_{m+1}$-**mod** we have*

$$\big(e_i'(M') : M\big) = \big(M' : f_i(M)\big).$$

(c) *We have $f_i(P)^\sharp = f_i(P^\sharp)$ for each $P \in {}^\circ\mathbf{R}$-**proj**.*
(d) *We have $e_i(M)^\flat = e_i(M^\flat)$ for each $M \in {}^\circ\mathbf{R}$-**fmod**.*

3.10. Induction of $\mathbf{H}_m$-modules versus induction of ${}^\circ\mathbf{R}_m$-modules. Recall the functors E_i, F_i on $\mathbf{H}$-$\mathbf{fMod}_I$ defined in (2.2). We have also the functors

$$\mathbf{for} : {}^\circ\mathbf{R}_m\text{-}\mathbf{fmod} \to {}^\circ\mathbf{R}_m\text{-}\mathbf{fMod}_0, \qquad \Psi : {}^\circ\mathbf{R}_m\text{-}\mathbf{fMod}_0 \to \mathbf{H}_m\text{-}\mathbf{fMod}_I,$$

where **for** is the forgetting of the grading. Finally we define functors

$$\begin{aligned} E_i &: {}^\circ\mathbf{R}_m\text{-}\mathbf{fMod}_0 \to {}^\circ\mathbf{R}_{m-1}\text{-}\mathbf{fMod}_0, \qquad E_i M = 1_{m-1,i}M, \\ F_i &: {}^\circ\mathbf{R}_m\text{-}\mathbf{fMod}_0 \to {}^\circ\mathbf{R}_{m+1}\text{-}\mathbf{fMod}_0, \qquad F_i M = \psi_!(M, \mathbf{L}_i). \end{aligned} \tag{3.4}$$

3.11. Proposition. *There are canonical isomorphisms of functors*

$$E_i \circ \Psi = \Psi \circ E_i, \qquad F_i \circ \Psi = \Psi \circ F_i, \qquad E_i \circ \mathbf{for} = \mathbf{for} \circ e_i, \qquad F_i \circ \mathbf{for} = \mathbf{for} \circ f_{\theta(i)}.$$

Proof. The proof is the same as in [15, Prop. 8.17]. □

3.12. Proposition. (a) *The functor Ψ yields an isomorphism of Abelian groups*

$$\bigoplus_{m \geqslant 0}\big[{}^\circ\mathbf{R}_m\text{-}\mathbf{fMod}_0\big] = \bigoplus_{m \geqslant 0}[\mathbf{H}_m\text{-}\mathbf{fMod}_I].$$

The functors E_i, F_i yield endomorphisms of both sides which are intertwined by Ψ.
(b) *The functor* **for** *factors to a group isomorphism*

$${}^\circ\mathbf{G}_I/(v-1) = \bigoplus_{m \geqslant 0}\big[{}^\circ\mathbf{R}_m\text{-}\mathbf{fMod}_0\big].$$

Proof. Claim (a) follows from Corollary 2.7 and Proposition 3.11. Claim (b) follows from Proposition 3.2. □

3.13. Type D versus type B. We can compare the previous constructions with their analogues in type B. Let ${}^\theta\mathbf{K}$, ${}^\theta B$, ${}^\theta G^{\mathrm{low}}$, etc., denote the type B analogues of ${}^\circ\mathbf{K}$, ${}^\circ B$, ${}^\circ G^{\mathrm{low}}$, etc. See [15] for details. We'll use the same notation for the functors ψ^*, $\psi_!$, ψ_*, e_i, f_i, etc., on the type B side and on the type D side. Fix $m > 0$ and $\nu \in {}^\theta\mathbb{N}I$ such that $|\nu| = 2m$. The inclusion of graded $\mathbf{k}$-algebras ${}^\circ\mathbf{R}_\nu \subset {}^\theta\mathbf{R}_\nu$ in (1.2) yields a restriction functor

$$\mathrm{res} : {}^\theta\mathbf{R}_\nu\text{-}\mathbf{mod} \to {}^\circ\mathbf{R}_\nu\text{-}\mathbf{mod}$$

and an induction functor

$$\mathrm{ind} : {}^\circ\mathbf{R}_\nu\text{-}\mathbf{mod} \to {}^\theta\mathbf{R}_\nu\text{-}\mathbf{mod}, \qquad M \mapsto {}^\theta\mathbf{R}_\nu \otimes_{{}^\circ\mathbf{R}_\nu} M.$$

Both functors are exact, they map finite-dimensional graded modules to finite-dimensional ones, and they map projective graded modules to projective ones. Thus, they yield morphisms of $\mathcal{A}$-modules

$$\begin{aligned} &\mathrm{res} : {}^\theta\mathbf{K}_{I,m} \to {}^\circ\mathbf{K}_{I,m}, \qquad &&\mathrm{res} : {}^\theta\mathbf{G}_{I,m} \to {}^\circ\mathbf{G}_{I,m}, \\ &\mathrm{ind} : {}^\circ\mathbf{K}_{I,m} \to {}^\theta\mathbf{K}_{I,m}, \qquad &&\mathrm{ind} : {}^\circ\mathbf{G}_{I,m} \to {}^\theta\mathbf{G}_{I,m}. \end{aligned}$$

Moreover, for any $P \in {}^\theta\mathbf{K}_{I,m}$ and any $L \in {}^\theta\mathbf{G}_{I,m}$ we have

$$\begin{aligned} &\mathrm{res}\big(P^\sharp\big) = (\mathrm{res}\, P)^\sharp, \qquad &&\mathrm{ind}\big(P^\sharp\big) = (\mathrm{ind}\, P)^\sharp \\ &\mathrm{res}\big(L^\flat\big) = (\mathrm{res}\, L)^\flat, \qquad &&\mathrm{ind}\big(L^\flat\big) = (\mathrm{ind}\, L)^\flat. \end{aligned} \tag{3.5}$$

Note also that ind and res are left and right adjoint functors, because

$${}^\theta\mathbf{R}_\nu \otimes_{{}^\circ\mathbf{R}_\nu} M = \hom_{{}^\circ\mathbf{R}_\nu}\big({}^\theta\mathbf{R}_\nu, M\big)$$

as graded ${}^\theta\mathbf{R}_\nu$-modules.

3.14. Definition. For any graded ${}^\circ\mathbf{R}_\nu$-module M we define the graded ${}^\circ\mathbf{R}_\nu$-module M^γ with the same underlying graded $\mathbf{k}$-vector space as M such that the action of ${}^\circ\mathbf{R}_\nu$ is twisted by γ, i.e., the graded $\mathbf{k}$-algebra ${}^\circ\mathbf{R}_\nu$ acts on M^γ by $a\,m = \gamma(a)m$ for $a \in {}^\circ\mathbf{R}_\nu$ and $m \in M$. Note that $(M^\gamma)^\gamma = M$, and that M^γ is simple (resp. projective, indecomposable) if M has the same property.

For any graded ${}^\circ\mathbf{R}_m$-module M we have canonical isomorphisms of ${}^\circ\mathbf{R}$-modules

$$\big(f_i(M)\big)^\gamma = f_i\big(M^\gamma\big), \qquad \big(e_i(M)\big)^\gamma = e_i\big(M^\gamma\big).$$

The first isomorphism is given by

$${}^\circ\mathbf{R}_{m+1}1_{m,i} \otimes_{{}^\circ\mathbf{R}_m} M \to {}^\circ\mathbf{R}_{m+1}1_{m,i} \otimes_{{}^\circ\mathbf{R}_m} M, \qquad a \otimes m \mapsto \gamma(a) \otimes m.$$

The second one is the identity map on the vector space $1_{m,i}M$.

Recall that ${}^\theta I^\nu$ is the disjoint union of ${}^\theta I^\nu_+$ and ${}^\theta I^\nu_-$. We set

$$1_{\nu,+} = \sum_{\mathbf{i}\in {}^\theta I^\nu_+} 1_{\mathbf{i}}, \qquad 1_{\nu,-} = \sum_{\mathbf{i}\in {}^\theta I^\nu_-} 1_{\mathbf{i}}.$$

3.15. Lemma. *Let M be a graded ${}^\circ\mathbf{R}_\nu$-module.*

(a) *Both $1_{\nu,+}$ and $1_{\nu,-}$ are central idempotents in ${}^\circ\mathbf{R}_\nu$. We have $1_{\nu,+} = \gamma(1_{\nu,-})$.*
(b) *There is a decomposition of graded ${}^\circ\mathbf{R}_\nu$-modules $M = 1_{\nu,+}M \oplus 1_{\nu,-}M$.*
(c) *We have a canonical isomorphism of ${}^\circ\mathbf{R}_\nu$-modules* $\mathrm{res}\circ\mathrm{ind}(M) = M \oplus M^\gamma$.
(d) *If there exists $a \in \{+,-\}$ such that $1_{\nu,-a}M = 0$, then there are canonical isomorphisms of graded ${}^\circ\mathbf{R}_\nu$-modules*

$$M = 1_{\nu,a}M, \qquad 0 = 1_{\nu,a}M^\gamma, \qquad M^\gamma = 1_{\nu,-a}M^\gamma.$$

Proof. Part (a) follows from Proposition 1.6 and the equality $\varepsilon_1({}^\theta I^\nu_+) = {}^\theta I^\nu_-$. Part (b) follows from (a), (c) is given by definition, and (d) follows from (a), (b). □

Now, let m and ν be as before. Given $i \in I$, we set $\nu' = \nu + i + \theta(i)$. There is an obvious inclusion $W_m \subset W_{m+1}$. Thus the group W_m acts on ${}^\theta I^{\nu'}$, and the map

$${}^\theta I^\nu \to {}^\theta I^{\nu'}, \qquad \mathbf{i} \mapsto \theta(i)\mathbf{i}i$$

is W_m-equivariant. Thus there is $a_i \in \{+,-\}$ such that the image of ${}^\theta I^\nu_+$ is contained in ${}^\theta I^{\nu'}_{a_i}$, and the image of ${}^\theta I^\nu_-$ is contained in ${}^\theta I^{\nu'}_{-a_i}$.

3.16. Lemma. *Let M be a graded ${}^\circ\mathbf{R}_\nu$-module such that $1_{\nu,-a}M = 0$, with $a = +,-$. Then we have*

$$1_{\nu',-a_i a} f_i(M) = 0, \qquad 1_{\nu',a_i a} f_{\theta(i)}(M) = 0.$$

Proof. We have

$$\begin{aligned} 1_{\nu',-a_i a} f_i(M) &= 1_{\nu',-a_i a}{}^\circ\mathbf{R}_{\nu'} 1_{\nu,i} \otimes_{{}^\circ\mathbf{R}_\nu} M \\ &= {}^\circ\mathbf{R}_{\nu'} 1_{\nu',-a_i a} 1_{\nu,i} 1_{\nu,a} \otimes_{{}^\circ\mathbf{R}_\nu} M. \end{aligned}$$

Here we have identified $1_{\nu,a}$ with the image of $(1_{\nu,a}, 1_i)$ via the inclusion (3.2). The definition of this inclusion and that of a_i yield that

$$1_{\nu',a_i a} 1_{\nu,i} 1_{\nu,a} = 1_{\nu,a}, \qquad 1_{\nu',-a_i a} 1_{\nu,i} 1_{\nu,a} = 0.$$

The first equality follows. Next, note that for any $\mathbf{i} \in {}^\theta I^\nu$, the sequences $\theta(i)\mathbf{i}i$ and $i\mathbf{i}\theta(i) = \varepsilon_{m+1}(\theta(i)\mathbf{i}i)$ always belong to different ${}^\circ W_{m+1}$-orbits. Thus, we have $a_{\theta(i)} = -a_i$. So the second equality follows from the first. □

Consider the following diagram

$$\begin{array}{ccc} {}^{\circ}\mathbf{R}_{\nu}\text{-}\mathbf{mod} \times \mathbf{R}_i\text{-}\mathbf{mod} & \underset{\psi^*}{\overset{\psi_!}{\rightleftarrows}} & {}^{\circ}\mathbf{R}_{\nu'}\text{-}\mathbf{mod} \\ \text{res}\times\text{id} \uparrow\downarrow \text{ind}\times\text{id} & & \text{res} \uparrow\downarrow \text{ind} \\ {}^{\theta}\mathbf{R}_{\nu}\text{-}\mathbf{mod} \times \mathbf{R}_i\text{-}\mathbf{mod} & \underset{\psi^*}{\overset{\psi_!}{\rightleftarrows}} & {}^{\theta}\mathbf{R}_{\nu'}\text{-}\mathbf{mod}. \end{array}$$

3.17. Lemma. *There are canonical isomorphisms of functors*

$$\text{ind} \circ \psi_! = \psi_! \circ (\text{ind} \times \text{id}), \qquad \psi^* \circ \text{ind} = (\text{ind} \times \text{id}) \circ \psi^*, \qquad \text{ind} \circ \psi_* = \psi_* \circ (\text{ind} \times \text{id}),$$
$$\text{res} \circ \psi_! = \psi_! \circ (\text{res} \times \text{id}), \qquad \psi^* \circ \text{res} = (\text{res} \times \text{id}) \circ \psi^*, \qquad \text{res} \circ \psi_* = \psi_* \circ (\text{res} \times \text{id}).$$

Proof. The functor ind is left and right adjoint to res. Therefore it is enough to prove the first two isomorphisms in the first line. The isomorphism

$$\text{ind} \circ \psi_! = \psi_! \circ (\text{ind} \times \text{id})$$

comes from the associativity of the induction. Let us prove that

$$\psi^* \circ \text{ind} = (\text{ind} \times \text{id}) \circ \psi^*.$$

For any M in ${}^{\circ}\mathbf{R}_{\nu'}$-**mod**, the obvious inclusion ${}^{\theta}\mathbf{R}_{\nu} \otimes \mathbf{R}_i \subset {}^{\theta}\mathbf{R}_{\nu'}$ yields a map

$$(\text{ind} \times \text{id})\, \psi^*(M) = \left({}^{\theta}\mathbf{R}_{\nu} \otimes \mathbf{R}_i\right) \otimes_{{}^{\circ}\mathbf{R}_{\nu} \otimes \mathbf{R}_i} \psi^*(M) \to \psi^*\left({}^{\theta}\mathbf{R}_{\nu'} \otimes_{{}^{\circ}\mathbf{R}_{\nu} \otimes \mathbf{R}_i} M\right).$$

Combining it with the obvious map

$${}^{\theta}\mathbf{R}_{\nu'} \otimes_{{}^{\circ}\mathbf{R}_{\nu} \otimes \mathbf{R}_i} M \to {}^{\theta}\mathbf{R}_{\nu'} \otimes_{{}^{\circ}\mathbf{R}_{\nu'}} M$$

we get a morphism of ${}^{\theta}\mathbf{R}_{\nu} \otimes \mathbf{R}_i$-modules

$$(\text{ind} \times \text{id})\, \psi^*(M) \to \psi^*\, \text{ind}(M).$$

We need to show that it is bijective. This is obvious because at the level of vector spaces, the map above is given by

$$M \oplus (\pi_{1,\nu} \otimes M) \to M \oplus (\pi_{1,\nu'} \otimes M), \qquad m + \pi_{1,\nu} \otimes n \mapsto m + \pi_{1,\nu'} \otimes n.$$

Here $\pi_{1,\nu}$ and $\pi_{1,\nu'}$ denote the element π_1 in ${}^{\theta}\mathbf{R}_{\nu}$ and ${}^{\theta}\mathbf{R}_{\nu'}$ respectively. □

3.18. Corollary. (a) *The operators e_i, f_i on ${}^{\circ}\mathbf{K}_{I,*}$ and on ${}^{\theta}\mathbf{K}_{I,*}$ are intertwined by the maps* ind*,* res*, i.e., we have*

$$e_i \circ \text{ind} = \text{ind} \circ e_i, \qquad f_i \circ \text{ind} = \text{ind} \circ f_i, \qquad e_i \circ \text{res} = \text{res} \circ e_i, \qquad f_i \circ \text{res} = \text{res} \circ f_i.$$

(b) *The same result holds for the operators e_i, f_i on ${}^{\circ}\mathbf{G}_{I,*}$ and on ${}^{\theta}\mathbf{G}_{I,*}$.*

3.19. Now, we concentrate on non-graded irreducible modules. First, let

$$\mathrm{Res}: {}^{\theta}\mathbf{R}_{\nu}\text{-}\mathbf{Mod} \to {}^{\circ}\mathbf{R}_{\nu}\text{-}\mathbf{Mod}, \qquad \mathrm{Ind}: {}^{\circ}\mathbf{R}_{\nu}\text{-}\mathbf{Mod} \to {}^{\theta}\mathbf{R}_{\nu}\text{-}\mathbf{Mod}$$

be the (non-graded) restriction and induction functors. We have

$$\mathbf{for} \circ \mathrm{res} = \mathrm{Res} \circ \mathbf{for}, \qquad \mathbf{for} \circ \mathrm{ind} = \mathrm{Ind} \circ \mathbf{for}.$$

3.20. Lemma. *Let L, L' be irreducible ${}^{\circ}\mathbf{R}_{\nu}$-modules.*

(a) *The ${}^{\circ}\mathbf{R}_{\nu}$-modules L and L^{γ} are not isomorphic.*
(b) $\mathrm{Ind}(L)$ *is an irreducible ${}^{\theta}\mathbf{R}_{\nu}$-module, and every irreducible ${}^{\theta}\mathbf{R}_{\nu}$-module is obtained in this way.*
(c) $\mathrm{Ind}(L) \simeq \mathrm{Ind}(L')$ *iff $L' \simeq L$ or L^{γ}.*

Proof. For any ${}^{\theta}\mathbf{R}_{\nu}$-module $M \neq 0$, both $1_{\nu,+}M$ and $1_{\nu,-}M$ are nonzero. Indeed, we have $M = 1_{\nu,+}M \oplus 1_{\nu,-}M$, and we may suppose that $1_{\nu,+}M \neq 0$. The automorphism $M \to M$, $m \mapsto \pi_1 m$ takes $1_{\nu,+}M$ to $1_{\nu,-}M$ by Lemma 3.15(a). Hence $1_{\nu,-}M \neq 0$.

Now, to prove part (a), suppose that $\phi: L \to L^{\gamma}$ is an isomorphism of ${}^{\circ}\mathbf{R}_{\nu}$-modules. We can regard ϕ as a γ-antilinear map $L \to L$. Since L is irreducible, by Schur's lemma we may assume that $\phi^2 = \mathrm{id}_L$. Then L admits a ${}^{\theta}\mathbf{R}_{\nu}$-module structure such that the ${}^{\circ}\mathbf{R}_{\nu}$-action is as before and π_1 acts as ϕ. Thus, by the discussion above, neither $1_{\nu,+}L$ nor $1_{\nu,-}L$ is zero. This contradicts the fact that L is an irreducible ${}^{\circ}\mathbf{R}_{\nu}$-module.

Parts (b), (c) follow from (a) by Clifford theory, see e.g. [12, Appendix]. □

We can now prove Proposition 3.2.

3.21. Proof of Proposition 3.2. Let $b \in {}^{\circ}B$. We may suppose that $b = 1_{\nu,+}b$. By Lemma 3.20(b) the module ${}^{\theta}b = \mathrm{Ind}(b)$ lies in ${}^{\theta}B$. By [15, Prop. 8.2] there exists a unique selfdual irreducible graded ${}^{\theta}\mathbf{R}$-module ${}^{\theta}G^{\mathrm{up}}({}^{\theta}b)$ which is isomorphic to ${}^{\theta}b$ as a non-graded module. Set

$${}^{\circ}G^{\mathrm{up}}(b) = 1_{\nu,+}\,\mathrm{res}\big({}^{\theta}G^{\mathrm{up}}\big({}^{\theta}b\big)\big).$$

By Lemma 3.15(d) we have ${}^{\circ}G^{\mathrm{up}}(b) = b$ as a non-graded ${}^{\circ}\mathbf{R}$-module, and by (3.5) it is selfdual. This proves existence part of the proposition. To prove the uniqueness, suppose that M is another module with the same properties. Then $\mathrm{ind}(M)$ is a selfdual graded ${}^{\theta}\mathbf{R}$-module which is isomorphic to ${}^{\theta}b$ as a non-graded ${}^{\theta}\mathbf{R}$-module. Thus we have $\mathrm{ind}(M) = {}^{\theta}G^{\mathrm{up}}({}^{\theta}b)$ by [15]. By Lemma 3.15(d) we have also

$$M = 1_{\nu,+}\,\mathrm{res}\big({}^{\theta}G^{\mathrm{up}}\big({}^{\theta}b\big)\big).$$

So M is isomorphic to ${}^{\circ}G^{\mathrm{up}}(b)$. □

3.22. The crystal operators on ${}^{\circ}\mathbf{G}_I$ and ${}^{\circ}B$. Fix a vertex i in I. For each irreducible graded ${}^{\circ}\mathbf{R}_m$-module M we define

$$\tilde{e}_i(M) = \mathrm{soc}\big(e_i(M)\big), \qquad \tilde{f}_i(M) = \mathrm{top}\,\psi_!(M, \mathbf{L}_i), \qquad \varepsilon_i(M) = \max\big\{n \geqslant 0;\ e_i^n(M) \neq 0\big\}.$$

3.23. Lemma. *Let M be an irreducible graded ${}^\circ\mathbf{R}$-module such that $e_i(M)$, $f_i(M)$ belong to ${}^\circ\mathbf{G}_{I,*}$. We have*

$$\operatorname{ind}\big(\tilde{e}_i(M)\big) = \tilde{e}_i\big(\operatorname{ind}(M)\big), \qquad \operatorname{ind}\big(\tilde{f}_i(M)\big) = \tilde{f}_i\big(\operatorname{ind}(M)\big), \qquad \varepsilon_i(M) = \varepsilon_i\big(\operatorname{ind}(M)\big).$$

In particular, $\tilde{e}_i(M)$ is irreducible or zero and $\tilde{f}_i(M)$ is irreducible.

Proof. By Corollary 3.18 we have $\operatorname{ind}(e_i(M)) = e_i(\operatorname{ind}(M))$. Thus, since ind is an exact functor we have $\operatorname{ind}(\tilde{e}_i(M)) \subset e_i(\operatorname{ind}(M))$. Since ind is an additive functor, by Lemma 3.20(b) we have indeed

$$\operatorname{ind}\big(\tilde{e}_i(M)\big) \subset \tilde{e}_i\big(\operatorname{ind}(M)\big).$$

Note that $\operatorname{ind}(M)$ is irreducible by Lemma 3.20(b), thus $\tilde{e}_i(\operatorname{ind}(M))$ is irreducible by [15, Prop. 8.21]. Since $\operatorname{ind}(\tilde{e}_i(M))$ is nonzero, the inclusion is an isomorphism. The fact that $\operatorname{ind}(\tilde{e}_i(M))$ is irreducible implies in particular that $\tilde{e}_i(M)$ is simple. The proof of the second isomorphism is similar. The third equality is obvious. □

Similarly, for each irreducible ${}^\circ\mathbf{R}$-module b in ${}^\circ B$ we define

$$\tilde{E}_i(b) = \operatorname{soc}\big(E_i(b)\big), \qquad \tilde{F}_i(b) = \operatorname{top}\big(F_i(b)\big), \qquad \varepsilon_i(b) = \max\big\{n \geqslant 0;\ E_i^n(b) \neq 0\big\}.$$

Hence we have

$$\mathbf{for} \circ \tilde{e}_i = \tilde{E}_i \circ \mathbf{for}, \qquad \mathbf{for} \circ \tilde{f}_i = \tilde{F}_i \circ \mathbf{for}, \qquad \varepsilon_i = \varepsilon_i \circ \mathbf{for}.$$

3.24. Proposition. *For each b, b' in ${}^\circ B$ we have*

(a) $\tilde{F}_i(b) \in {}^\circ B$,
(b) $\tilde{E}_i(b) \in {}^\circ B \cup \{0\}$,
(c) $\tilde{F}_i(b) = b' \iff \tilde{E}_i(b') = b$,
(d) $\varepsilon_i(b) = \max\{n \geqslant 0; \tilde{E}_i^n(b) \neq 0\}$,
(e) $\varepsilon_i(\tilde{F}_i(b)) = \varepsilon_i(b) + 1$,
(f) *if $\tilde{E}_i(b) = 0$ for all i then $b = \phi_\pm$.*

Proof. Part (c) follows from adjunction. The other parts follow from [15, Prop. 3.14] and Lemma 3.23. □

3.25. Remark. For any $b \in {}^\circ B$ and any $i \neq j$ we have $\tilde{F}_i(b) \neq \tilde{F}_j(b)$. This is obvious if $j \neq \theta(i)$. Because in this case $\tilde{F}_i(b)$ and $\tilde{F}_j(b)$ are ${}^\circ\mathbf{R}_\nu$-modules for different ν. Now, consider the case $j = \theta(i)$. We may suppose that $\tilde{F}_i(b) = 1_{\nu,+}\tilde{F}_i(b)$ for certain ν. Then by Lemma 3.16 we have $1_{\nu,+}\tilde{F}_{\theta(i)}(b) = 0$. In particular $\tilde{F}_i(b)$ is not isomorphic to $\tilde{F}_{\theta(i)}(b)$.

3.26. The algebra ${}^\theta\mathbf{B}$ and the ${}^\theta\mathbf{B}$-module ${}^\circ\mathbf{V}$. Following [3–5] we define a $\mathcal{K}$-algebra ${}^\theta\mathbf{B}$ as follows.

3.27. Definition. Let ${}^\theta\mathbf{B}$ be the $\mathcal{K}$-algebra generated by e_i, f_i and invertible elements t_i, $i \in I$, satisfying the following defining relations

(a) $t_i t_j = t_j t_i$ and $t_{\theta(i)} = t_i$ for all i, j,
(b) $t_i e_j t_i^{-1} = v^{i\cdot j+\theta(i)\cdot j} e_j$ and $t_i f_j t_i^{-1} = v^{-i\cdot j-\theta(i)\cdot j} f_j$ for all i, j,
(c) $e_i f_j = v^{-i\cdot j} f_j e_i + \delta_{i,j} + \delta_{\theta(i),j} t_i$ for all i, j,
(d) $$\sum_{a+b=1-i\cdot j} (-1)^a e_i^{(a)} e_j e_i^{(b)} = \sum_{a+b=1-i\cdot j} (-1)^a f_i^{(a)} f_j f_i^{(b)} = 0 \quad \text{if } i \neq j.$$

Here and below we use the following notation

$$\theta^{(a)} = \theta^a / \langle a \rangle!, \qquad \langle a \rangle = \sum_{l=1}^{a} v^{a+1-2l}, \qquad \langle a \rangle! = \prod_{l=1}^{m} \langle l \rangle.$$

We can now construct a representation of ${}^\theta\mathbf{B}$ as follows. By base change, the operators e_i, f_i in Definition 3.8 yield $\mathcal{K}$-linear operators on the $\mathcal{K}$-vector space

$${}^\circ\mathbf{V} = \mathcal{K} \otimes_{\mathcal{A}} {}^\circ\mathbf{K}_I.$$

We equip ${}^\circ\mathbf{V}$ with the $\mathcal{K}$-bilinear form given by

$$(M : N)_{KE} = \left(1 - v^2\right)^m (M : N), \quad \forall M, N \in {}^\circ\mathbf{R}_m\text{-}\mathbf{proj}.$$

3.28. Theorem. (a) *The operators e_i, f_i define a representation of ${}^\theta\mathbf{B}$ on ${}^\circ\mathbf{V}$. The ${}^\theta\mathbf{B}$-module ${}^\circ\mathbf{V}$ is generated by linearly independent vectors ϕ_+ and ϕ_- such that for each $i \in I$ we have*

$$e_i \phi_\pm = 0, \qquad t_i \phi_\pm = \phi_\mp, \qquad \left\{x \in {}^\circ\mathbf{V};\ e_j x = 0,\ \forall j\right\} = \mathbf{k}\phi_+ \oplus \mathbf{k}\phi_-.$$

(b) *The symmetric bilinear form on ${}^\circ\mathbf{V}$ is non-degenerate. We have $(\phi_a : \phi_{a'})_{KE} = \delta_{a,a'}$ for $a, a' = +, -$, and $(e_i x : y) = (x : f_i y)_{KE}$ for $i \in I$ and $x, y \in {}^\circ\mathbf{V}$.*

Proof. For each i in I we define the $\mathcal{A}$-linear operator t_i on ${}^\circ\mathbf{K}_I$ by setting

$$t_i \phi_\pm = \phi_\mp \quad \text{and} \quad t_i P = v^{-\nu\cdot(i+\theta(i))} P^\gamma, \quad \forall P \in {}^\circ\mathbf{R}_\nu\text{-}\mathbf{proj}.$$

We must prove that the operators e_i, f_i, and t_i satisfy the relations of ${}^\theta\mathbf{B}$. The relations (a), (b) are obvious. The relation (d) is standard. It remains to check (c). For this we need a version of the Mackey's induction–restriction theorem. Note that for $m > 1$ we have

$$D_{m,1;m,1} = \{e, s_m, \varepsilon_{m+1}\varepsilon_1\},$$
$$W(e) = {}^\circ W_m, \qquad W(s_m) = {}^\circ W_{m-1}, \qquad W(\varepsilon_{m+1}\varepsilon_1) = {}^\circ W_m.$$

Recall also that for $m = 1$ we have set ${}^\circ W_1 = \{e\}$.

3.29. Lemma. *Fix i, j in I. Let μ, ν in ${}^\theta\mathbb{N}I$ be such that $\nu + i + \theta(i) = \mu + j + \theta(j)$. Put $|\nu| = |\mu| = 2m$. The graded $({}^\circ\mathbf{R}_{m,1}, {}^\circ\mathbf{R}_{m,1})$-bimodule $1_{\nu,i}{}^\circ\mathbf{R}_{m+1}1_{\mu,j}$ has a filtration by graded bimodules whose associated graded is isomorphic to*

$$\delta_{i,j}\big({}^\circ\mathbf{R}_\nu \otimes \mathbf{R}_i\big) \oplus \delta_{\theta(i),j}\big(\big({}^\circ\mathbf{R}_\nu\big)^\gamma \otimes \mathbf{R}_{\theta(i)}\big)\big[d'\big] \oplus A[d],$$

where A is equal to

$$\big({}^\circ\mathbf{R}_m 1_{\nu',i} \otimes \mathbf{R}_i\big) \otimes_{\mathbf{R}'} \big(1_{\nu',i}{}^\circ\mathbf{R}_m \otimes \mathbf{R}_i\big) \quad \text{if } m > 1,$$
$$\big({}^\circ\mathbf{R}_{\theta(\mathbf{j})} \otimes \mathbf{R}_i \otimes_{{}^\circ\mathbf{R}_1\otimes\mathbf{R}_1} {}^\circ\mathbf{R}_{\theta(\mathbf{i})} \otimes \mathbf{R}_j\big) \oplus \big({}^\circ\mathbf{R}_{\mathbf{j}} \otimes \mathbf{R}_i \otimes_{{}^\circ\mathbf{R}_1\otimes\mathbf{R}_1} {}^\circ\mathbf{R}_{\mathbf{i}} \otimes \mathbf{R}_j\big) \quad \text{if } m = 1.$$

Here we have set $\nu' = \nu - j - \theta(j)$, $\mathbf{R}' = {}^\circ\mathbf{R}_{m-1,1} \otimes \mathbf{R}_1$, $\mathbf{i} = i\theta(i)$, $\mathbf{j} = j\theta(j)$, $d = -i \cdot j$, and $d' = -\nu \cdot (i + \theta(i))/2$.

The proof is standard and is left to the reader. Now, recall that for $m > 1$ we have

$$f_j(P) = {}^\circ\mathbf{R}_{m+1}1_{m,j} \otimes_{{}^\circ\mathbf{R}_{m,1}} (P \otimes \mathbf{R}_1), \qquad e_i'(P) = 1_{m-1,i}P,$$

where $1_{m-1,i}P$ is regarded as a ${}^\circ\mathbf{R}_{m-1}$-module. Therefore we have

$$e_i' f_j(P) = 1_{m,i}{}^\circ\mathbf{R}_{m+1}1_{m,j} \otimes_{{}^\circ\mathbf{R}_{m,1}} (P \otimes \mathbf{R}_1),$$
$$f_j e_i'(P) = {}^\circ\mathbf{R}_m 1_{m-1,j} \otimes_{{}^\circ\mathbf{R}_{m-1,1}} (1_{m-1,i}P \otimes \mathbf{R}_1).$$

Therefore, up to some filtration we have the following identities

- $e_i' f_i(P) = P \otimes \mathbf{R}_i + f_i e_i'(P)[-2]$,
- $e_i' f_{\theta(i)}(P) = P^\gamma \otimes \mathbf{R}_{\theta(i)}[-\nu \cdot (i + \theta(i))/2] + f_{\theta(i)}e_i'(P)[-i \cdot \theta(i)]$,
- $e_i' f_j(P) = f_j e_i'(P)[-i \cdot j]$ if $i \neq j, \theta(j)$.

These identities also hold for $m = 1$ and $P = {}^\circ\mathbf{R}_{\theta(i)i}$ for any $i \in I$. The first claim of part (a) follows because $\mathbf{R}_i = \mathbf{k} \oplus \mathbf{R}_i[2]$. The fact that ${}^\circ\mathbf{V}$ is generated by $\phi_\pm$ is a corollary of Proposition 3.31 below. Part (b) of the theorem follows from [8, Prop. 2.2(ii)] and Lemma 3.9(b). □

3.30. Remarks. (a) The ${}^\theta\mathbf{B}$-module ${}^\circ\mathbf{V}$ is the same as the ${}^\theta\mathbf{B}$-module V_θ from [8, Prop. 2.2]. The involution $\sigma : {}^\circ\mathbf{V} \to {}^\circ\mathbf{V}$ in [8, Rem. 2.5(ii)] is given by $\sigma(P) = P^\gamma$. It yields an involution of ${}^\circ B$ in the obvious way. Note that Lemma 3.20(a) yields $\sigma(b) \neq b$ for any $b \in {}^\circ B$.

(b) Let ${}^\theta\mathbf{V}$ be the ${}^\theta\mathbf{B}$-module $\mathcal{K} \otimes_{\mathcal{A}} {}^\theta\mathbf{K}_I$ and let ϕ be the class of the trivial ${}^\theta\mathbf{R}_0$-module $\mathbf{k}$, see [15, Thm. 8.30]. We have an inclusion of ${}^\theta\mathbf{B}$-modules

$${}^\theta\mathbf{V} \to {}^\circ\mathbf{V}, \qquad \phi \mapsto \phi_+ \oplus \phi_-, \qquad P \mapsto \operatorname{res}(P).$$

3.31. Proposition. *For any $b \in {}^\circ B$ the following holds*

$$\text{(a)} \qquad \begin{cases} f_i\big({}^\circ G^{\mathrm{low}}(b)\big) = \big\langle \varepsilon_i(b) + 1\big\rangle {}^\circ G^{\mathrm{low}}(\tilde{F}_i b) + \sum_{b'} f_{b,b'}{}^\circ G^{\mathrm{low}}\big(b'\big), \\ b' \in {}^\circ B, \quad \varepsilon_i\big(b'\big) > \varepsilon_i(b) + 1, \quad f_{b,b'} \in v^{2-\varepsilon_i(b')}\mathbb{Z}[v], \end{cases}$$

(b) $$\begin{cases} e_i\big({}^\circ G^{\mathrm{low}}(b)\big) = v^{1-\varepsilon_i(b)}\,{}^\circ G^{\mathrm{low}}(\tilde{E}_i b) + \sum_{b'} e_{b,b'}\,{}^\circ G^{\mathrm{low}}\big(b'\big), \\ b' \in {}^\circ B, \quad \varepsilon_i\big(b'\big) \geqslant \varepsilon_i(b), \quad e_{b,b'} \in v^{1-\varepsilon_i(b')}\mathbb{Z}[v]. \end{cases}$$

Proof. We prove part (a), the proof for (b) is similar. If ${}^\circ G^{\mathrm{low}}(b) = \phi_\pm$ this is obvious. So we assume that ${}^\circ G^{\mathrm{low}}(b)$ is a ${}^\circ\mathbf{R}_m$-module for $m \geqslant 1$. Fix $\nu \in {}^\theta\mathbb{N}I$ such that $f_i({}^\circ G^{\mathrm{low}}(b))$ is a ${}^\circ\mathbf{R}_\nu$-module. We'll abbreviate $1_{\nu,a} = 1_a$ for $a \in \{+,-\}$. Since ${}^\circ G^{\mathrm{low}}(b)$ is indecomposable, it fulfills the condition of Lemma 3.16. So there exists $a \in \{+,-\}$ such that $1_{-a} f_i({}^\circ G^{\mathrm{low}}(b)) = 0$. Thus, by Lemma 3.15(c), (d) and Corollary 3.18 we have

$$f_i\big({}^\circ G^{\mathrm{low}}(b)\big) = 1_a \operatorname{res}\operatorname{ind} f_i\big({}^\circ G^{\mathrm{low}}(b)\big) = 1_a \operatorname{res} f_i \operatorname{ind}\big({}^\circ G^{\mathrm{low}}(b)\big).$$

Note that ${}^\theta b = \mathrm{Ind}(b)$ belongs to ${}^\theta B$ by Lemma 3.20(b). Hence (3.5) yields

$$\operatorname{ind}\big({}^\circ G^{\mathrm{low}}(b)\big) = {}^\theta G^{\mathrm{low}}\big({}^\theta b\big).$$

We deduce that

$$f_i\big({}^\circ G^{\mathrm{low}}(b)\big) = 1_a \operatorname{res} f_i\big({}^\theta G^{\mathrm{low}}\big({}^\theta b\big)\big).$$

Now, write

$$f_i\big({}^\theta G^{\mathrm{low}}\big({}^\theta b\big)\big) = \sum f_{{}^\theta b,{}^\theta b'}\,{}^\theta G^{\mathrm{low}}\big({}^\theta b'\big), \quad {}^\theta b' \in {}^\theta B.$$

Then we have

$$f_i\big({}^\circ G^{\mathrm{low}}(b)\big) = \sum f_{{}^\theta b,{}^\theta b'} 1_a \operatorname{res}\big({}^\theta G^{\mathrm{low}}\big({}^\theta b'\big)\big).$$

For any ${}^\theta b' \in {}^\theta B$ the ${}^\circ\mathbf{R}$-module $1_a \mathrm{Res}({}^\theta b')$ belongs to ${}^\circ B$. Thus we have

$$1_a \operatorname{res}\big({}^\theta G^{\mathrm{low}}\big({}^\theta b'\big)\big) = {}^\circ G^{\mathrm{low}}\big(1_a \mathrm{Res}\big({}^\theta b'\big)\big).$$

If ${}^\theta b' \neq {}^\theta b''$ then $1_a \mathrm{Res}({}^\theta b') \neq 1_a \mathrm{Res}({}^\theta b'')$, because ${}^\theta b' = \mathrm{Ind}(1_a \mathrm{Res}({}^\theta b'))$. Thus

$$f_i\big({}^\circ G^{\mathrm{low}}(b)\big) = \sum f_{{}^\theta b,{}^\theta b'}\,{}^\circ G^{\mathrm{low}}\big(1_a \mathrm{Res}\big({}^\theta b'\big)\big),$$

and this is the expansion of the left-hand side in the lower global basis. Finally, we have

$$\varepsilon_i\big(1_a \mathrm{Res}\big({}^\theta b'\big)\big) = \varepsilon_i\big({}^\theta b'\big)$$

by Lemma 3.23. So part (a) follows from [15, Prop. 10.11(b), 10.16]. □

3.32. The global bases of ${}^\circ\mathbf{V}$. Since the operators e_i, f_i on ${}^\circ\mathbf{V}$ satisfy the relations $e_i f_i = v^{-2} f_i e_i + 1$, we can define the modified root operators $\tilde{\mathbf{e}}_i$, $\tilde{\mathbf{f}}_i$ on the ${}^\theta\mathbf{B}$-module ${}^\circ\mathbf{V}$ as follows. For each u in ${}^\circ\mathbf{V}$ we write

$$u = \sum_{n \geqslant 0} f_i^{(n)} u_n \quad \text{with } e_i u_n = 0,$$

$$\tilde{\mathbf{e}}_i(u) = \sum_{n \geqslant 1} f_i^{(n-1)} u_n, \qquad \tilde{\mathbf{f}}_i(u) = \sum_{n \geqslant 0} f_i^{(n+1)} u_n.$$

Let $\mathcal{R} \subset \mathcal{K}$ be the set of functions which are regular at $v = 0$. Let ${}^\circ\mathbf{L}$ be the $\mathcal{R}$-submodule of ${}^\circ\mathbf{V}$ spanned by the elements $\tilde{\mathbf{f}}_{i_1} \dots \tilde{\mathbf{f}}_{i_l}(\phi_\pm)$ with $l \geqslant 0$, $i_1, \dots, i_l \in I$. The following is the main result of the paper.

3.33. Theorem. (a) *We have*

$$ {}^\circ\mathbf{L} = \bigoplus_{b \in {}^\circ B} \mathcal{R}\, {}^\circ G^{\mathrm{low}}(b), \qquad \tilde{\mathbf{e}}_i\big({}^\circ\mathbf{L}\big) \subset {}^\circ\mathbf{L}, \qquad \tilde{\mathbf{f}}_i\big({}^\circ\mathbf{L}\big) \subset {}^\circ\mathbf{L},$$

$$\tilde{\mathbf{e}}_i\big({}^\circ G^{\mathrm{low}}(b)\big) = {}^\circ G^{\mathrm{low}}\big(\tilde{E}_i(b)\big) \bmod v\, {}^\circ\mathbf{L}, \qquad \tilde{\mathbf{f}}_i\big({}^\circ G^{\mathrm{low}}(b)\big) = {}^\circ G^{\mathrm{low}}\big(\tilde{F}_i(b)\big) \bmod v\, {}^\circ\mathbf{L}.$$

(b) *The assignment* $b \mapsto {}^\circ G^{\mathrm{low}}(b) \bmod v\, {}^\circ\mathbf{L}$ *yields a bijection from* ${}^\circ B$ *to the subset of* ${}^\circ\mathbf{L}/v{}^\circ\mathbf{L}$ *consisting of the* $\tilde{\mathbf{f}}_{i_1} \dots \tilde{\mathbf{f}}_{i_l}(\phi_\pm)$*'s. Further* ${}^\circ G^{\mathrm{low}}(b)$ *is the unique element* $x \in {}^\circ\mathbf{V}$ *such that* $x^\sharp = x$ *and* $x = {}^\circ G^{\mathrm{low}}(b) \bmod v\, {}^\circ\mathbf{L}$.

(c) *For each* b, b' *in* ${}^\circ B$ *let* $E_{i,b,b'}, F_{i,b,b'} \in \mathcal{A}$ *be the coefficients of* ${}^\circ G^{\mathrm{low}}(b')$ *in* $e_{\theta(i)}({}^\circ G^{\mathrm{low}}(b))$, $f_i({}^\circ G^{\mathrm{low}}(b))$ *respectively. Then we have*

$$E_{i,b,b'}|_{v=1} = \big[F_i \Psi \mathbf{for}\big({}^\circ G^{\mathrm{up}}\big(b'\big)\big) : \Psi \mathbf{for}\big({}^\circ G^{\mathrm{up}}(b)\big)\big],$$

$$F_{i,b,b'}|_{v=1} = \big[E_i \Psi \mathbf{for}\big({}^\circ G^{\mathrm{up}}\big(b'\big)\big) : \Psi \mathbf{for}\big({}^\circ G^{\mathrm{up}}(b)\big)\big].$$

Proof. Part (a) follows from [5, Thm. 4.1, Cor. 4.4], [2, Section 2.3], and Proposition 3.31. The first claim in (b) follows from (a). The second one is obvious. Part (c) follows from Proposition 3.11. More precisely, by duality we can regard $E_{i,b,b'}$, $F_{i,b,b'}$ as the coefficients of ${}^\circ G^{\mathrm{up}}(b)$ in $f_{\theta(i)}({}^\circ G^{\mathrm{up}}(b'))$ and $e_i({}^\circ G^{\mathrm{up}}(b'))$ respectively. Therefore, by Proposition 3.11 we can regard $E_{i,b,b'}|_{v=1}$, $F_{i,b,b'}|_{v=1}$ as the coefficients of $\Psi\mathbf{for}({}^\circ G^{\mathrm{up}}(b))$ in $F_i\Psi\mathbf{for}({}^\circ G^{\mathrm{up}}(b'))$ and $E_i\Psi\mathbf{for}({}^\circ G^{\mathrm{up}}(b'))$ respectively. □

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