P-ALCOVES AND NONEMPTINESS OF AFFINE DELIGNE-LUSZTIG VARIETIES

P-ALCÔVES ET VACUITÉ DE VARIÉTÉS DE DELIGNE-LUSZTIG AFFINES

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Abstract. We study affine Deligne-Lusztig varieties in the affine flag manifold of an algebraic group, and in particular the question, which affine Deligne-Lusztig varieties are non-empty. Under mild assumptions on the group, we provide a complete answer to this question in terms of the underlying affine root system. In particular, this proves the corresponding conjecture for split groups stated in [3]. The question of non-emptiness of affine Deligne-Lusztig varieties is closely related to the relationship between certain natural stratifications of moduli spaces of abelian varieties in positive characteristic.

Nous étudions les variétés de Deligne-Lusztig affines dans la variété de drapeaux affine d'un groups algébrique, et en particulier la question de savoir quelles variétés de Deligne-Lusztig affines sont non vides. À quelques restrictions près, nous donnons une réponse complète à cette question en termes du système de racines affine sous-jacent. Pour le cas des groupes déployés, cela résoud en particulier la conjecture énoncée dans [3]. Ces propriétés sur les variétés de Deligne-Lusztig affines reflètent les relations entre certaines stratifications naturelles d'espaces de modules des variétés abéliennes en caractéristique positive.

1. Introduction

1.1. Affine Deligne-Lusztig varieties (see below for the definition) are the analogues of Deligne-Lusztig varieties in the context of an affine root system, and hence are natural objects which deserve to be studied in their own interest. Furthermore, results about them have direct

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applications to certain questions in arithmetic geometry, specifically to moduli spaces of p-divisible groups and reductions of Shimura varieties. More concretely, if $\mathcal{M}$ is a Rapoport-Zink space, then $\mathcal{M}(\Bbbk)$ can be identified by Dieudonné theory with a (mixed-characteristic) affine Deligne-Lusztig variety. In this case, the formal scheme $\mathcal{M}$ provides a scheme structure. See [2, 5.10] for further information on this connection.

1.2. Let $\mathbb{F}_q$ be the finite field with q elements. Let $\Bbbk$ be an algebraic closure of $\mathbb{F}_q$. Consider one of the following two cases:

- Mixed characteristic case. Let $\mathbb{F}/\mathbb{Q}_p$ be a finite field extension with residue class field $\mathbb{F}_q$, and let $\mathbb{L}$ be the completion of the maximal unramified extension of $\mathbb{F}$. Denote by ε a uniformizer of $\mathbb{F}$.
- Equal characteristic case. Let $\mathbb{F} = \mathbb{F}_q((\epsilon))$, the field of Laurent series over $\mathbb{F}_q$, and $\mathbb{L} := \Bbbk((\epsilon))$, the field of Laurent series over $\Bbbk$. As in the previous case, $\mathbb{L}$ is the completion of the maximal unramified extension of $\mathbb{F}$.

Let $\mathbf{G}$ be a connected reductive group over $\mathbb{F}$ which splits over a tamely ramified extension of $\mathbb{F}$. Let σ be the Frobenius automorphism of $\mathbb{L}/\mathbb{F}$. We also denote the induced automorphism on $\mathbf{G}(\mathbb{L})$ by σ.

We fix a σ-invariant Iwahori subgroup $\mathbf{I} \subset \mathbf{G}(\mathbb{L})$. In the equal characteristic case we can view $\mathbf{G}(\mathbb{L})/\mathbf{I}$ as the $\Bbbk$-points of an ind-projective ind-scheme Flag over $\Bbbk$, the affine flag variety for $\mathbf{G}$, see [10]. The $\mathbf{I}$-double cosets in $\mathbf{G}(\mathbb{L})$ are parameterized by the extended affine Weyl group $\tilde{W}$. The automorphism on $\tilde{W}$ induced by σ is denoted by $\delta\colon \tilde{W} \to \tilde{W}$. Furthermore we denote by $\tilde{S} \subseteq \tilde{W}$ the set of simple affine reflections.

Following Rapoport [13], we define:

Definition 1.2.1. *Let $x \in \tilde{W}$, and $b \in \mathbf{G}(\mathbb{L})$. The affine Deligne-Lusztig variety attached to x and b is the subset*

$$X_x(b) = \{g\mathbf{I} \in \mathbf{G}(\mathbb{L})/\mathbf{I};\ g^{-1}b\sigma(g) \in \mathbf{I}x\mathbf{I}\}.$$

In the equal characteristic case, it is not hard to see that there exists a unique locally closed $X_x(b) \subset$ Flag whose set of $\Bbbk$-valued points is the subset $X_x(b) \subseteq \mathbf{G}(\mathbb{L})/\mathbf{I}$ defined above. Moreover, $X_x(b)$ is a finite-dimensional $\Bbbk$-scheme, locally of finite type over $\Bbbk$ (but not in general of finite type: depending on b, $X_x(b)$ may have infinitely many irreducible components). In the mixed characteristic case, the term "variety" is not really justified. More precisely one should speak about affine Deligne-Lusztig sets.

As experience and partial results show, many basic properties of affine Deligne-Lusztig varieties such as non-emptiness and dimension depend only on the underlying combinatorial structure of the (affine) root system, and therefore coincide in the mixed characteristic and equal characteristic cases.

For the remainder of the introduction, we fix a basic element $b \in \mathbf{G}(\mathbb{L})$, i.e., an element whose Newton vector is central, or equivalently, whose σ-conjugacy class can be represented by a length zero element of the extended affine Weyl group. See [8] or [12] for more details.

So far, the main questions that have been studied are

(1) For which x is $X_x(b) \neq \emptyset$?
(2) If $X_x(b) \neq \emptyset$ and $X_x(b)$ carries a scheme structure, what is $\dim X_x(b)$?

Until recently, most of the results have been established only for split groups. For tamely ramified quasi-split groups, we refer to [6, Section 12] for question 2, at least in the equal-characteristic case.

In this paper, we focus on Question 1 above and give a complete answer to this question.

We first show that it suffices to consider quasi-split, semisimple groups of adjoint type (see Sections 2.2, 2.3 for an explanation how to reduce to this case). For such groups, the answer is given in terms of the affine root system and the affine Weyl group of $\mathbf{G}$ and uses the notion of (J, w, δ)-alcove (see Section 3.3), a generalization of the notion of $\mathbf{P}$-alcove introduced in [3] for split groups.

Let $\mathbf{a}$ be the fundamental alcove and $x \in \tilde{W}$. Roughly speaking, the alcove $x\mathbf{a}$ is a (J, w, δ)-alcove if the following two conditions are met: x must satisfy a restriction on its finite part, and the alcove must lie in a certain region of the apartment, which is essentially a union of certain finite Weyl chambers. See Section 3.3 for the precise definition and [3, Section 3] for a visualization.

We denote by $\Gamma_{\mathbb{F}}$ the absolute Galois group of $\mathbb{F}$ and by $\kappa_{\mathbf{G}} \colon \mathbf{G}(\mathbb{L}) \to \pi_1(\mathbf{G})_{\Gamma_{\mathbb{F}}}$ the Kottwitz map; see [8], [12]. Note that $\kappa_{\mathbf{G}}$ also gives rise to maps with source $\tilde{W}$ and source $B(\mathbf{G})$. Likewise, for a Levi subgroup $\mathbf{M}$, we denote by $\kappa_{\mathbf{M}}$ the corresponding Kottwitz map.

Theorem A (Corollary 3.6.1, Theorem 4.4.7)**.** *Let $b \in \mathbf{G}(\mathbb{L})$ be a basic element, and let $x \in \tilde{W}$. Then $X_x(b) = \emptyset$ if and only if there exists a pair (J, w) such that $x\mathbf{a}$ is a (J, w, δ)-alcove and*

$$\kappa_{\mathbf{M}_J}(w^{-1}x\delta(w)) \notin \kappa_{\mathbf{M}_J}([b] \cap \mathbf{M}_J(\mathbb{L})).$$

We say that an element $x \in \tilde{W}$ lies in the shrunken Weyl chambers if $x\mathbf{a}$ does not lie in the same strip as the base alcove $\mathbf{a}$ with respect to any root direction (cf. Prop. 3.6.5). In this case, we have a more explicit description of the nonemptiness behavior of $X_x(b)$. The answer is given in terms of the map η_δ from $\tilde{W}$ to the finite Weyl group W defined in Section 3.6.

Theorem B (Proposition 3.6.5, Proposition 4.4.9)**.** *Let $x \in \tilde{W}$ lie in the shrunken Weyl chambers. Let $b \in \mathbf{G}(\mathbb{L})$ be a basic element. Then $X_x(b) \neq \emptyset$ if and only if $\kappa_{\mathbf{G}}(x) = \kappa_{\mathbf{G}}(b)$ and $\eta_\delta(x) \in W - \bigcup_{J \subsetneq S, \delta(J)=J} W_J$.*

Both theorems require that b is a *basic* element. As in [3, Conj. 9.5.1 (b)], we expect that this hypothesis is superfluous for elements x of sufficiently large length (depending on b). However, we are unable to make a precise statement along these lines (this could be seen as formulating a version of Mazur's inequality in the Iwahori case). In applications to the reduction of Shimura varieties, usually the basic case is the most important one: The basic locus is the unique closed Newton stratum (in many cases, this is the supersingular locus), it is the only one where one can hope for a complete geometric description, and it can sometimes be used as a starting point for understanding other Newton strata.

Let us give an overview of the paper. In Section 2 we collect some preliminaries and reduce to the case where $\mathbf{G}$ is quasi-split and semisimple of adjoint type. In Section 3 we prove, imitating the proof given in [3] in the split case, the direction of Theorem A claiming emptiness. In the final Section 4 we prove the non-emptiness statement of the theorem by employing the "reduction method" of Deligne and Lusztig. We show that the notion of (J, w, δ)-alcove is compatible with this reduction. Using some interesting combinatorial properties of affine Weyl groups established by the second-named and third-named authors [7], we are able to reduce the question to the case of $X_x(b)$, where x is of minimal length in its δ-conjugacy class. This case can be handled directly using the explicit description of minimal length elements in [7].

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2. Preliminaries

2.1. Notation. Let $\mathbf{S} \subset \mathbf{G}$ be a maximal $\mathbb{L}$-split torus defined over $\mathbb{F}$. The centralizer $\mathbf{T}$ of $\mathbf{S}$ in $\mathbf{G}$ is a maximal torus, because over $\mathbb{L}$, $\mathbf{G}$ is quasi-split. The Frobenius automorphism σ of $\mathbb{L}/\mathbb{F}$ acts on the

Iwahori-Weyl group

$$\tilde{W} = \mathbf{N_S}(\mathbb{L})/\mathbf{T}(\mathbb{L})_1.$$

Here $\mathbf{N_S}$ denotes the normalizer of $\mathbf{S}$ in $\mathbf{G}$, and $\mathbf{T}(\mathbb{L})_1$ denotes the unique parahoric subgroup of $\mathbf{T}(\mathbb{L})$. For $w \in \tilde{W}$, we choose a representative in $\mathbf{N_S}(\mathbb{L})$ and also write it as w.

We denote by $\mathcal{A}$ the apartment of $\mathbf{G}(\mathbb{L})$ corresponding to $\mathbf{S}$. We fix a σ-invariant alcove $\mathfrak{a}$ in $\mathcal{A}$, and denote by $\mathbf{I} \subseteq \mathbf{G}(\mathbb{L})$ the Iwahori subgroup corresponding to $\mathfrak{a}$ over $\mathbb{L}$.

2.1.1. *The affine Weyl group.* Denote by $\mathbf{G}_1 \subset \mathbf{G}(\mathbb{L})$ the subgroup generated by all parahoric subgroups. We denote by

$$W_a := (\mathbf{N_S}(\mathbb{L}) \cap \mathbf{G}_1)/(\mathbf{N_S}(\mathbb{L}) \cap \mathbf{I})$$

the affine Weyl group.

The affine Weyl group acts simply transitively on the set of alcoves in $\mathcal{A}$, and our choice of base alcove gives rise to a length function and the Bruhat order on W_a. As usual, the length of an alcove is the number of "affine root hyperplanes" in the apartment separating the alcove from the base alcove.

2.1.2. *Semi-direct product representations of the Iwahori-Weyl group.* Denote by Γ the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{L}}/\mathbb{L})$ of $\mathbb{L}$. We can identify Γ with the inertia subgroup of the absolute Galois group $\Gamma_{\mathbb{F}}$ of $\mathbb{F}$. By a subscript $\bullet_\Gamma$ we denote Γ-coinvariants.

Denote by $W = \mathbf{N_S}(\mathbb{L})/\mathbf{T}(\mathbb{L})$ the (relative, finite) Weyl group of $\mathbf{G}$ with respect to $\mathbf{S}$.

We use the following important short exact sequence:

$$0 \to X_*(\mathbf{T})_\Gamma \to \tilde{W} \to W \to 1, \tag{2.1.1}$$

where the map $\tilde{W} \to W$ is the natural projection and $X_*(\mathbf{T})$ is the cocharacter group of $\mathbf{T}$. Its kernel is $\mathbf{T}(\mathbb{L})/\mathbf{T}(\mathbb{L})_1$ which can be identified with $X_*(\mathbf{T})_\Gamma$ via the Kottwitz map, see [10, Section 5]. This short exact sequence splits, and we obtain

$$\tilde{W} = X_*(\mathbf{T})_\Gamma \rtimes W = \{\epsilon^\lambda w; \lambda \in X_*(\mathbf{T})_\Gamma, w \in W\}.$$

See [5, Proposition 13].

On the other hand, the affine Weyl group naturally embeds into $\tilde{W}$, and we have an exact sequence

$$1 \to W_a \to \tilde{W} \to X^*(\mathbf{Z}(\widehat{\mathbf{G}})^\Gamma) \to 0.$$

Here $\widehat{\mathbf{G}}$ is the (connected) Langlands dual group for $\mathbf{G}$; $\mathbf{Z}(\widehat{\mathbf{G}})$ is the center of $\mathbf{G}$; $X^*(\mathbf{Z}(\widehat{\mathbf{G}})^\Gamma)$ is the character group of $\mathbf{Z}(\widehat{\mathbf{G}})^\Gamma$. We can identify $X^*(\mathbf{Z}(\widehat{\mathbf{G}})^\Gamma)$ with the stabilizer of the base alcove $\mathfrak{a}$ in $\tilde{W}$. This shows that $\tilde{W} = W_a \rtimes X^*(\mathbf{Z}(\widehat{\mathbf{G}})^\Gamma)$. See [5, Lemma 14]. Setting $\ell(x) = 0$ for $x \in X^*(\mathbf{Z}(\widehat{\mathbf{G}})^\Gamma)$, we extend the length function to $\tilde{W}$.

At the same time, we can view W_a as the Iwahori-Weyl group of the simply connected cover $\mathbf{G}_{\rm sc}$ of the derived group of $\mathbf{G}$. Denoting by $\mathbf{T}_{\rm sc} \subset \mathbf{G}_{\rm sc}$ the maximal torus given by the choice of $\mathbf{T}$, we obtain a semi-direct product decomposition

$$W_a = X_*(\mathbf{T}_{\rm sc})_\Gamma \rtimes W.$$

We can identify W_a with the group generated by the reflections with respect to the walls of $\mathfrak{a}$.

2.1.3. *Affine flag varieties.* The structure theory for $\mathbf{G}(\mathbb{L})$ established by Bruhat and Tits gives the Iwahori-Bruhat decomposition

$$\mathbf{G}(\mathbb{L}) = \bigsqcup_{w \in \tilde{W}} \mathbf{I} w \mathbf{I}, \qquad \mathbf{G}(\mathbb{L})/\mathbf{I} = \bigsqcup_{w \in \tilde{W}} \mathbf{I} w \mathbf{I}/\mathbf{I},$$

where both unions are disjoint.

2.2. Reduction to adjoint groups. Let $\mathbf{G}$ be a connected reductive group over $\mathbb{F}$, and let $\mathbf{G}_{\rm ad}$ be the corresponding group of adjoint type, i.e., the quotient of $\mathbf{G}$ by its center. The buildings of $\mathbf{G}$ and $\mathbf{G}_{\rm ad}$ coincide, so that the choice of an alcove $\mathbf{a}$ in the building of $\mathbf{G}$ determines an alcove, and hence an Iwahori group of $\mathbf{G}_{\rm ad}$. We first consider the more complicated case of equal characteristic.

Denote by Flag and $\mathrm{Flag}_{\rm ad}$ the corresponding affine flag varieties for $\mathbf{G}$ and $\mathbf{G}_{\rm ad}$.

Proposition 2.2.1. *Assume that* $\mathrm{char}\,\Bbbk$ *does not divide the order of* $\pi_1(\mathbf{G}_{\rm ad})$.

(1) *The homomorphism* $\mathbf{G} \to \mathbf{G}_{\rm ad}$ *induces an immersion*

$$\mathrm{Flag} \to \mathrm{Flag}_{\rm ad}\,.$$

(2) *Let* $\lambda \in \pi_0(\mathrm{Flag}) = \pi_1(\mathbf{G})_\Gamma$, *denote by* $\lambda_{\rm ad}$ *its image under the injective map* $\pi_0(\mathrm{Flag}) \to \pi_0(\mathrm{Flag}_{\rm ad})$, *and denote by* Flag_λ *and* $\mathrm{Flag}_{{\rm ad},\lambda_{\rm ad}}$ *the corresponding connected components. Then the above immersion induces an isomorphism*

$$\mathrm{Flag}_\lambda \xrightarrow{\cong} \mathrm{Flag}_{{\rm ad},\lambda_{\rm ad}}\,.$$

Proof. Denote by $\mathbf{G}_{\rm sc}$ the simply connected cover of $\mathbf{G}$, and by $\mathrm{Flag}_{\rm sc}$ its affine flag variety (attached to the Iwahori subgroup of $\mathbf{G}_{\rm sc}$ given by $\mathbf{a}$). It is proved in [10, 6.a] that there are natural maps

$$\mathrm{Flag}_{\rm sc} \to \mathrm{Flag} \to \mathrm{Flag}_{\rm ad}$$

and that

$$\mathrm{Flag}_{\mathrm{sc}} \to \mathrm{Flag}, \quad \text{and} \quad \mathrm{Flag}_{\mathrm{sc}} \to \mathrm{Flag}_{\mathrm{ad}}$$

are immersions which identify $\mathrm{Flag}_{\mathrm{sc}}$ with the neutral connected component of Flag and of $\mathrm{Flag}_{\mathrm{ad}}$. Now let $\lambda \in \pi_0(\mathrm{Flag}) = \pi_1(\mathbf{G})_\Gamma$ (cf. [10, Theorem 5.1]). Since $\pi_0(\mathrm{Flag}) = \pi_0(L\mathbf{G})$, we can find a representative $g \in L\mathbf{G}(\Bbbk)$ of λ. Left multiplication identifies the neutral connected component Flag_0 with Flag_λ, and likewise the image of g in $L\mathbf{G}_{\mathrm{ad}}(\Bbbk)$ identifies $\mathrm{Flag}_{\mathrm{ad},0}$ with $\mathrm{Flag}_{\mathrm{ad},\lambda_{\mathrm{ad}}}$. This proves the proposition. □

Choosing maximal tori in $\mathbf{G}$ and $\mathbf{G}_{\mathrm{ad}}$ compatibly, we obtain a map $x \mapsto x_{\mathrm{ad}}$ between the corresponding extended affine Weyl groups. For $b \in \mathbf{G}(\mathbb{L})$, we denote by b_{ad} its image in $\mathbf{G}_{\mathrm{ad}}(\mathbb{L})$. Finally, for an affine Deligne-Lusztig variety $X_x(b)$ and $\lambda \in \pi_0(\mathrm{Flag})$, we denote by $X_x(b)_\lambda$ the intersection $X_x(b) \cap \mathrm{Flag}_\lambda$, and likewise for $\mathbf{G}_{\mathrm{ad}}$.

In the mixed characteristic case, an analogous set-theoretic statement is true without any assumption on the order of $\pi_1(\mathbf{G}_{\mathrm{ad}})$. The notion of *connected component* should be replaced by *fiber of the Kottwitz homomorphism* $\mathbf{G}(\mathbb{L}) \to \pi_1(\mathbf{G})_\Gamma$. This can be shown along the same lines as above.

The proposition immediately implies the following corollary. Compare the discussion before Prop. 5.9.2 in [2] for an analogous statement for split groups and affine Grassmannians.

Corollary 2.2.2. (1) *(Equal characteristic case) Assume that* $\mathrm{char}\,\Bbbk$ *does not divide the order of* $\pi_1(\mathbf{G}_{\mathrm{ad}})$.[1] *Let* $b \in \mathbf{G}(\mathbb{L})$, $x \in \tilde{W}$, *and* $\lambda \in \pi_0(\mathrm{Flag})$. *Then the isomorphism* $\mathrm{Flag}_\lambda \xrightarrow{\cong} \mathrm{Flag}_{\mathrm{ad},\lambda_{\mathrm{ad}}}$ *induces an isomorphism*

$$X_x(b)_\lambda \cong X_{x_{\mathrm{ad}}}(b_{\mathrm{ad}})_{\lambda_{\mathrm{ad}}}.$$

(2) *(Mixed characteristic case) Let* $b \in \mathbf{G}(\mathbb{L})$, $x \in \tilde{W}$, *and* $\lambda \in \pi_0(\mathrm{Flag})$. *We have a bijection*

$$X_x(b)_\lambda \cong X_{x_{\mathrm{ad}}}(b_{\mathrm{ad}})_{\lambda_{\mathrm{ad}}}.$$

2.3. Reduction to the quasi-split case. Let $\mathbf{H}$ be a connected semisimple group over $\mathbb{F}$ of adjoint type and $\mathbf{G}$ be its quasi-split inner form. As before, we denote by $\tilde{W}$ the Iwahori-Weyl group of $\mathbf{G}$ (over $\mathbb{L}$). The inner forms of $\mathbf{G}$ are parameterized by the Galois cohomology group $H^1(\mathbb{F}, \mathbf{G})$. By [12, Theorem 1.15] we have a bijection

$$H^1(\mathbb{F}, \mathbf{G}) = \pi_1(\mathbf{G})_{\Gamma_{\mathbb{F}}}. \tag{2.3.1}$$

[1] We expect that the statement is still true without this assumption.

Via the map $X^*(\mathbf{Z}(\widehat{\mathbf{G}}))_\Gamma \cong \pi_1(\mathbf{G})_\Gamma \to \pi_1(\mathbf{G})_{\Gamma_\mathbb{F}}$, we may associate to $\mathbf{H}$ some length zero element $z \in \tilde{W}$. Since by Steinberg's theorem $\mathbf{H} \otimes_\mathbb{F} \mathbb{L}$ is quasi-split, we can identify $\mathbf{H}(\mathbb{L}) = \mathbf{G}(\mathbb{L})$, and tracing through the above identifications shows that the Frobenius action induced by $\mathbf{H}$ on $\mathbf{H}(\mathbb{L}) = \mathbf{G}(\mathbb{L})$ is $\sigma_\mathbf{H} = \mathrm{Int}(\gamma) \circ \sigma_\mathbf{G}$; here $\gamma \in \mathbf{N_S}(\mathbb{L}) \subset \mathbf{G}(\mathbb{L})$ is a lift of z and $\mathrm{Int}(\gamma)$ denotes conjugation by γ. In fact, Steinberg's theorem also applies over the maximal unramified extension $\mathbb{F}^{\mathrm{nr}}$ of $\mathbb{F}$, so we can identify $\mathbf{H}(\mathbb{F}^{\mathrm{nr}}) = \mathbf{G}(\mathbb{F}^{\mathrm{nr}})$. Since conjugation by γ preserves $\mathbf{S}(\mathbb{F}^{\mathrm{nr}})$ and $\mathbf{T}(\mathbb{F}^{\mathrm{nr}})$, we see that $\mathbf{S}$ and $\mathbf{T}$ descend to tori $\mathbf{S_H}$, $\mathbf{T_H} \subset \mathbf{H}$ (over $\mathbb{F}$). The Iwahori $\mathbf{I} \subset \mathbf{G}(\mathbb{L})$ for $\mathbf{G}$ is also an Iwahori subgroup for $\mathbf{H}$.

We can naturally identify $\tilde{W}$ with the Iwahori-Weyl group of $\mathbf{H}$. This identification preserves the Coxeter structure (affine simple reflections, length, Bruhat order). Of course, the actions of $\sigma_\mathbf{G}$ and $\sigma_\mathbf{H}$ on $\tilde{W}$ will usually be different. Also note that while $\sigma_\mathbf{H}$ acts on W, the splitting of the sequence (2.1.1) is not necessarily preserved by $\sigma_\mathbf{H}$: typically the set of finite simple reflections (for $\mathbf{G}$) inside $\tilde{W}$ is not stable under $\sigma_\mathbf{H}$. This just reflects the fact that for non-quasi-split $\mathbf{H}$, there is no Borel subgroup over $\mathbb{F}$.

2.4. σ-conjugacy classes. We keep the notation of Section 2.3 and draw some conclusions from results of Kottwitz [8], [9] and of Rapoport and Richartz [12] about the classification of σ-conjugacy classes.

Denote by $B(\mathbf{H})$ and $B(\mathbf{G})$ the sets of σ-conjugacy classes in $\mathbf{H}(\mathbb{L})$ (with respect to $\sigma_\mathbf{H}$) and $\mathbf{G}(\mathbb{L})$ (with respect to $\sigma_\mathbf{G}$), respectively. The map $[b] \mapsto [b\gamma]$ is a bijection $B(\mathbf{H}) \xrightarrow{\cong} B(\mathbf{G})$. This is the map considered by Kottwitz in [9, 4.18]. We obtain the following commutative diagram

$$\begin{array}{ccc} \tilde{W} & \xrightarrow{x \mapsto x\gamma} & \tilde{W} \\ \downarrow & & \downarrow \\ B(\mathbf{H}) & \xrightarrow{[b] \mapsto [b\gamma]} & B(\mathbf{G}), \end{array}$$

where the vertical arrows arise from the natural maps $\mathbf{N_S}(\mathbb{L}) \to B(\mathbf{H})$ and $\mathbf{N_S}(\mathbb{L}) \to B(\mathbf{G})$, respectively. Note that the map in the top row clearly preserves the length. In particular the set of length zero elements is preserved, and so the map in the bottom row maps basic elements for $\mathbf{H}$ to basic elements for $\mathbf{G}$.

Proposition 2.4.1. *Let $\mathbf{H}/\mathbb{F}$ be a connected semisimple algebraic group of adjoint type, and denote by $\tilde{W}$ its Iwahori-Weyl group (over $\mathbb{L}$). Then the natural map $\tilde{W} \to B(\mathbf{H})$ is surjective.*

This is [6, Theorem 3.5]. The proposition can also be proved, after reducing to the case of a quasi-split semisimple group, along the lines of Corollary 7.2.2 in [3].

2.5. Affine Deligne-Lusztig varieties. We can also identify affine Deligne-Lusztig varieties for $\mathbf{G}$ and $\mathbf{H}$. Recall that the Iwahori $\mathbf{I} \subset \mathbf{G}(\mathbb{L})$ for $\mathbf{G}$ is at the same time an Iwahori subgroup for $\mathbf{H}$, so that we can identify the affine flag varieties for $\mathbf{G}$ and for $\mathbf{H}$. Furthermore $\mathbf{I}$ is normalized by γ, because the length zero elements stabilize the base alcove. For any $x \in \tilde{W}$ and $b \in \mathbf{G}(\mathbb{L}) = \mathbf{H}(\mathbb{L})$, the condition $g^{-1}b\sigma_{\mathbf{H}}(g) \in \mathbf{I}x\mathbf{I}$ precisely amounts to $g^{-1}b\gamma\sigma_{\mathbf{G}}(g)\gamma^{-1} \in \mathbf{I}x\mathbf{I} = \mathbf{I}x\gamma\mathbf{I}\gamma^{-1}$. Thus

Proposition 2.5.1. *Let $\mathbf{G}$, $\mathbf{H}$ and γ be as above. Let $x \in \tilde{W}$, and let $b \in \mathbf{G}(\mathbb{L}) = \mathbf{H}(\mathbb{L})$. Then*

$$X_x^{\mathbf{H}}(b) = X_{x\gamma}^{\mathbf{G}}(b\gamma).$$

3. P-alcoves and emptiness of ADLV

In the rest of this paper, we let $\mathbf{G}$ be a quasi-split connected semisimple group over $\mathbb{F}$ that *splits over a tamely ramified extension of* $\mathbb{L}$. We simply write σ for the Frobenius map $\sigma_{\mathbf{G}}$ on $\mathbf{G}(\mathbb{L})$ and write δ for the induced automorphisms on W and $\tilde{W}$. The results in this section are generalizations of results of [3] to the case of quasi-split groups.

3.1. The root system. Recall that $\mathbf{S}$ is a maximal $\mathbb{L}$-split torus of $\mathbf{G}$ and $\mathbf{T}$ the maximal torus of $\mathbf{G}$ that contains $\mathbf{S}$. Consider the real vector space $V = X_*(\mathbf{T})_\Gamma \otimes \mathbb{R}$. Let Φ be the set of (relative) roots of $\mathbf{G}$ over $\mathbb{L}$ with respect to $\mathbf{S}$ and Φ_a the set of affine roots. The roots in Φ determine hyperplanes in V and the relative Weyl group W can be identified with the group generated by the reflections through these hyperplanes.

Note that the root system Φ is not necessarily reduced. By [15, Section 1.7], there exists a unique reduced root system Σ such that the affine roots Φ_a are the functions on V of the form $y \mapsto \alpha(y) + k$ for $\alpha \in \Sigma$ and $k \in \mathbb{Z}$. Moreover, $W = W(\Sigma)$ and $W_a = Q^\vee(\Sigma) \rtimes W$. Here $W(\Sigma)$ is the Weyl group of the root system Σ and $Q^\vee(\Sigma)$ is the coroot lattice for Σ.

Note that any root of Σ is proportional to a root in Φ. However, the root system Σ is not necessarily proportional to Φ, even if Φ is reduced. See [15, Section 1.7].

Of course the length function and Bruhat order on W_a produced in these two ways are the same, since in both cases they are given by the affine root hyperplanes in V, which are the same in both cases.

The identification with the affine Weyl group of a reduced root system allows us to use the corresponding notions and results from the theory of root systems.

3.2. Parabolic subgroup. For $a \in \Phi$, we denote by $\mathbf{U}_a \subset \mathbf{G}$ the corresponding root subgroup and for $\alpha \in \Phi_a$, we denote by $\mathbf{H}_\alpha \subset \mathbf{G}(\mathbb{L})$ the corresponding root subgroup scheme over $\Bbbk$. By [10, (9.8)], $\mathbf{H}_\alpha$ is one-dimensional for all $\alpha \in \Phi_a$. [2]

Our choice of fundamental alcove determines a basis S of Φ. We choose the same normalization as in [3], which means that the fundamental alcove lies in the anti-dominant Weyl chamber. We identify S with the set of simple reflections in W and hence can also view S as a basis of the reduced root system Σ. Let Φ^+ (resp. Φ^-) be the set of positive (resp. negative) roots of Φ. For $J \subset S$, let Φ_J be the set of roots spanned by J and let $\Phi_J^\pm = \Phi_J \cap \Phi^\pm$. Then J is a basis of the subsystem Φ_J. Let $W_J \subset W$ be the corresponding standard parabolic subgroup and $Q_J^\vee$ be the corresponding coroot lattice.

We denote by $\mathbf{M}_J$ the Levi subgroup of $\mathbf{G}$ generated by $\mathbf{T}$ and $\mathbf{U}_a$ for $a \in \Phi_J$ and by $\mathbf{N}_J$ the subgroups generated by $\mathbf{U}_a$ for $a \in \Phi^+ - \Phi_J^+$. Then $\mathbf{P}_J = \mathbf{M}_J \mathbf{N}_J$ is a parabolic subgroup of $\mathbf{G}$. If moreover $\delta(J) = J$, then $\mathbf{P}_J, \mathbf{M}_J$ and $\mathbf{N}_J$ are defined over $\mathbb{F}$. The Iwahori-Weyl group of $\mathbf{M}_J$ is $\tilde{W}_J = X_*(\mathbf{T})_\Gamma \rtimes W_J$. We simply write κ_J instead of $\kappa_{\mathbf{M}_J}$.

3.3. (J, w, δ)**-alcoves.** As in [3], we use the notation ${}^x g := x g x^{-1}$ and ${}^\sigma g := \sigma(g)$ for $g \in \mathbf{G}(\mathbb{L})$, and similarly for subsets of $\mathbf{G}(\mathbb{L})$.

Recall that $\mathbf{a}$ is the fundamental alcove. Let $J \subset S$ with $\delta(J) = J$ and $w \in W$. Let $x \in \tilde{W}$. We say $x\mathbf{a}$ is a (J, w, δ)-*alcove*, if

(1) $w^{-1} x \delta(w) \in \tilde{W}_J$, and
(2) For any $a \in w(\Phi^+ - \Phi_J^+)$, $\mathbf{U}_a \cap {}^x\mathbf{I} \subseteq \mathbf{U}_a \cap \mathbf{I}$, or equivalently, $\mathbf{U}_{-a} \cap {}^x\mathbf{I} \supseteq \mathbf{U}_{-a} \cap \mathbf{I}$.

We say $x\mathbf{a}$ is a *strict* (J, w, δ)-alcove if instead of (2) we have

(3) For any $a \in w(\Phi^+ - \Phi_J^+)$, $\mathbf{U}_a \cap {}^x\mathbf{I} \subsetneqq \mathbf{U}_a \cap \mathbf{I}$, or equivalently, $\mathbf{U}_{-a} \cap {}^x\mathbf{I} \supsetneqq \mathbf{U}_{-a} \cap \mathbf{I}$.

In the split case, $x\mathbf{a}$ is a (J, w, δ)-alcove if and only if it is a ${}^w\mathbf{P}_J$-alcove in the sense of [3].

[2] This is the place where we use the "tamely ramified" hypothesis. We also suspect that this hypothesis is not essential. However, as the scheme structure of affine flag varieties (and hence affine Deligne-Lusztig varieties) are not much studied for wildly ramified groups even in equal characteristic, we only consider tamely ramified groups in this paper.

Condition (1) implies that ${}^{x\sigma}({}^{w}\mathbf{M}_J) = {}^{w}\mathbf{M}_J$. If we pass to the (non-connected) group $\mathbf{G} \rtimes \langle\sigma\rangle$, then we can reformulate condition (1) above as $x\delta \in {}^{w}(\tilde{W}_J \rtimes \langle\delta\rangle)$.

Now we state the main result of this section, which generalizes [3, Theorem 2.1.2].

Theorem 3.3.1. *Suppose $J \subset S$ with $\delta(J) = J$ and $w \in W$, and $x\mathbf{a}$ is a (J, w, δ)-alcove. Set $\mathbf{I_M} = {}^{w}\mathbf{M}_J \cap \mathbf{I}$. Then the map*

$$\phi : \mathbf{I} \times^{\mathbf{I_M}} \mathbf{I_M} x\sigma(\mathbf{I_M}) \to \mathbf{I}x\mathbf{I}$$

induced by $(i, m) \mapsto im\sigma(i)^{-1}$, is surjective. If $x\mathbf{a}$ is a strict (J, w, δ)-alcove, then ϕ is injective. In general, ϕ is not injective, but if $[i, m]$ and $[i', m']$ belong to the same fiber of ϕ, the elements m and m' are σ-conjugate by an element of $\mathbf{I_M}$.

Similarly to [3, Lemma 4.1.1], the theorem is equivalent to the following statement: the map

$$\phi : \ ({}^{\delta^{-1}(x)^{-1}}\mathbf{I} \cap \mathbf{I}) \times^{{}^{\delta^{-1}(x)^{-1}}\mathbf{I_M} \cap \mathbf{I_M}} \mathbf{I_M} x \to \mathbf{I}x$$

given by $(i, m) \mapsto im\,\sigma(i)^{-1}$ is surjective, and is bijective if $x\mathbf{a}$ is a strict (J, w, δ)-alcove. In general, if $[i, xj]$ and $[i', xj']$ belong to the same fiber of ϕ, then xj and xj' are σ-conjugate by an element of ${}^{x}\mathbf{I_M} \cap \mathbf{I_M}$.

The proof of the portion relating to the fiber of ϕ is just the same as in [3, Section 4]. For the proof of surjectivity, we follow the strategy of [3, Section 6].

3.4. For $n \in \mathbb{N}$, let $\mathbf{T}(\mathbb{L})_n$ be the corresponding congruence subgroup of $\mathbf{T}(\mathbb{L}) \cap \mathbf{I}$ (see [11, 2.6]). For any $r \geq 0$, let $\mathbf{I}_r \subset \mathbf{I}$ be the subgroup generated by $\mathbf{T}(\mathbb{L})_n$ for $n \geq r$ and $\mathbf{H}_{a+m}$ for $a \in \Phi$ and $m \geq r$ such that $a + m$ is a positive affine root. Let $\mathbf{I}_{r^+} = \cup_{s>r}\mathbf{I}_s$. Then $\mathbf{I}_r$ and $\mathbf{I}_{r^+}$ are normal subgroups of $\mathbf{I}$ for all $r \geq 0$.

Recall that $x\mathbf{a}$ is a (J, w, δ)-alcove. Let $\mathbf{M} = {}^{w}\mathbf{M}_J$. Let $\mathbf{N} \subset \mathbf{G}$ be the subgroup generated by $\mathbf{U}_a$ for $a \in w(\Phi^+ - \Phi_J^+)$ and $\overline{\mathbf{N}} \subset \mathbf{G}$ be the subgroup generated by $\mathbf{U}_{-a}$ for $a \in w(\Phi^+ - \Phi_J^+)$.

For $r \geq 0$, consider the normal subgroups $\mathbf{N}_r = \mathbf{N}(\mathbb{L}) \cap \mathbf{I}_r$ and $\mathbf{N}_{r^+} = \mathbf{N}(\mathbb{L}) \cap \mathbf{I}_{r^+}$ of $\mathbf{N}(\mathbb{L}) \cap \mathbf{I}$. Similarly, let $\overline{\mathbf{N}}_r = \overline{\mathbf{N}}(\mathbb{L}) \cap \mathbf{I}_r$ and $\overline{\mathbf{N}}_{r^+} = \overline{\mathbf{N}}(\mathbb{L}) \cap \mathbf{I}_{r^+}$. Since $x\mathbf{a}$ is a (J, w, δ)-alcove, we have ${}^{x\sigma}\mathbf{N}_r \subseteq \mathbf{N}_r$ and ${}^{x\sigma}\overline{\mathbf{N}}_r \supseteq \overline{\mathbf{N}}_r$.

Lemma 3.4.1. *Fix an element $m \in \mathbf{I_M}$ and $r \geq 0$.*

(i) *Given $i_- \in \sigma(\overline{\mathbf{N}}_r)$, there exists $b_- \in \overline{\mathbf{N}}_r$ such that ${}^{(mx)^{-1}}b_- i_- {}^{\sigma}b_-^{-1} \in \sigma(\overline{\mathbf{N}}_{r^+})$.*

(ii) *Given $i_+ \in \mathbf{N}_r$, there exists $b_+ \in \mathbf{N}_r$ such that $b_+ i_+ {}^{mx\sigma}b_+^{-1} \in \mathbf{N}_{r^+}$.*

Proof. To the Borel subgroup ${}^w\mathbf{P}_\emptyset$ of $\mathbf{G}$, we associate a finite separating filtration by normal subgroups

$$\mathbf{N}_{\mathbb{L}} = \mathbf{N}[1] \supset \mathbf{N}[2] \supset \cdots$$

as in [3, proof of Lemma 6.1.1]. It has the following properties:

(1) For each i, $\mathbf{N}[i] \subset \mathbf{N}_{\mathbb{L}}$ is normal, and stable under conjugation with elements of $\mathbf{M}$.
(2) For each i, ${}^{x\sigma}\mathbf{N}[i] \subseteq \mathbf{N}[i]$.
(3) For each i, the quotient $\mathbf{N}\langle i\rangle := \mathbf{N}[i]/\mathbf{N}[i+1]$ is abelian.

We define $\mathbf{N}_r[i] := \mathbf{N}_r \cap \mathbf{N}[i]$, and $\mathbf{N}_r\langle i\rangle := \mathbf{N}_r[i]/\mathbf{N}_r[i+1]$, and define $\mathbf{N}_{r^+}\langle i\rangle$ analogously. Then $\mathbf{N}_r\langle i\rangle/\mathbf{N}_{r^+}\langle i\rangle$ is a vector group over $\Bbbk$. We define the groups $\overline{\mathbf{N}}[i]$, $\overline{\mathbf{N}}\langle i\rangle$, $\overline{\mathbf{N}}_r[i]$, $\overline{\mathbf{N}}_r\langle i\rangle$ and $\overline{\mathbf{N}}_{r^+}\langle i\rangle$ in an analogous manner. It is easy to see from the definition that ${}^{(mx)^{-1}}\overline{\mathbf{N}}_r[i] \subset \sigma(\overline{\mathbf{N}}_r[i])$ and ${}^{mx\sigma}\mathbf{N}_r[i] \subset \mathbf{N}_r[i]$.

By [3, Lemma 5.1.1], the map $b_- \mapsto {}^{(mx)^{-1}}b_-\,{}^{\sigma}b_-^{-1}$ is surjective from the vector group $\overline{\mathbf{N}}_r\langle i\rangle/\overline{\mathbf{N}}_{r^+}\langle i\rangle$ to $\sigma(\overline{\mathbf{N}}_r\langle i\rangle/\overline{\mathbf{N}}_{r^+}\langle i\rangle)$ and the map $b_+ \mapsto b_+\,{}^{mx\sigma}b_+^{-1}$ is surjective on each vector group $\mathbf{N}_r\langle i\rangle/\mathbf{N}_{r^+}\langle i\rangle$. Applying it repeatedly on these quotients in a suitable order, we may find $b_- \in \overline{\mathbf{N}}_r$ such that

$${}^{(xm)^{-1}}b_- i_-\,{}^{\sigma}b_-^{-1} \in \overline{\mathbf{N}}_{r^+},$$

and $b_+ \in \mathbf{N}_r$ such that $b_+ i_+\,{}^{mx\sigma}b_+^{-1} \in \mathbf{N}_{r^+}$. □

Corollary 3.4.2. *Let $m \in \mathbf{I}_{\mathbf{M}}$ and $r \geq 0$. Given $i_- \in \overline{\mathbf{N}}_r$, there exists $b_- \in \sigma^{-1}({}^{(mx)^{-1}}\overline{\mathbf{N}}_r)$ such that $b_- i_-\,{}^{mx\sigma}b_-^{-1} \in \overline{\mathbf{N}}_{r^+}$.*

Proof. By Lemma 3.4.1, there exists $b \in \overline{\mathbf{N}}_r$ such that ${}^{(mx)^{-1}}b\sigma(i_-)\,{}^{\sigma}b^{-1} \in \sigma(\overline{\mathbf{N}}_{r^+})$. Set $b_- = \sigma^{-1}({}^{(mx)^{-1}}b)$. Then $b_- i_-\,{}^{mx\sigma}b_-^{-1} \in \overline{\mathbf{N}}_{r^+}$. □

As explained in [3, Section 6], a generic Moy-Prasad filtration gives a filtration $\mathbf{I} = \cup_{r\geqslant 0}\mathbf{I}[r]$ with $\mathbf{I}[r] \supset \mathbf{I}[s]$ for $r < s$ such that each $\mathbf{I}[r]$ is normal in $\mathbf{I}$, and each $\mathbf{I}[r]$ is a semidirect product $\mathbf{I}\langle r\rangle\mathbf{I}[r^+]$, where $\mathbf{I}\langle r\rangle$ is either an affine root subgroup or contained in $\mathbf{T}(\mathfrak{o})$.

Let $y \in \mathbf{I}x$. By the same argument as in [3, Section 6], for any $i \geqslant 0$, there exists $h_i \in {}^{\delta^{-1}(x)^{-1}}\mathbf{I} \cap \mathbf{I}$ (suitably small when i is large) such that

$$h_i h_{i-1} \cdots h_0 y \sigma(h_i h_{i-1} \cdots h_0)^{-1} \in \mathbf{I}[i^+]\mathbf{I}_{\mathbf{M}}x.$$

Let $g = \cdots h^{(2)}h^{(1)}h^{(0)}$ be the convergent product. Then $gy\sigma(g)^{-1} \in x\mathbf{I}_{\mathbf{M}}$. This proves the surjectivity.

By the same argument as in [3, Section 6], we also have the following result.

Proposition 3.4.3. *Suppose $J \subset S$ with $\delta(J) = J$ and $w \in W$, and $x\mathbf{a}$ is a (J, w, δ)-alcove. Set $\mathbf{I}_{\mathbf{M}} = {}^w\mathbf{M}_J \cap \mathbf{I}$. If moreover, ${}^{x\sigma}\mathbf{I}_{\mathbf{M}} = \mathbf{I}_{\mathbf{M}}$,* *then we may σ-conjugate any element of* $\mathbf{I}x$ *to* x*, using an element of* ${}^{\delta^{-1}(x)^{-1}}\mathbf{I} \cap \mathbf{I}$.

3.5. Some properties on Newton points. We recall Kottwitz's classification of $B(\mathbf{G})$ in [9, §4.13].

A σ-conjugacy class $[b]$ is determined by two invariants. One is given by the image of $[b]$ under the Kottwitz map $\kappa_{\mathbf{G}} : B(\mathbf{G}) \to \pi_1(\mathbf{G})_\Gamma$. The other is given by the Newton point $\bar{\nu}_b$, i.e., the image of $[b]$ under the Newton map $B(\mathbf{G}) \to X_*(\mathbf{T})^+_{\mathbb{Q}}$. We do not recall the original definition of the Newton map here. Instead, we give an explicit description of the restriction of the Newton map to $\tilde{W}$. This is enough for our purposes in this paper since any σ-conjugacy class in $\mathbf{G}(\mathbb{L})$ is represented by an element in $\tilde{W}$ (see Proposition 2.4.1).

Let n be the order of $W \rtimes \langle \delta \rangle$ (we consider δ as an element of the automorphism group of W). For $x \in \tilde{W}$, $x\delta(x) \cdots \delta^{n-1}(x)$ is a translation element because of the choice of n; it equals ϵ^μ for some $\mu \in X_*(\mathbf{T})^\delta_\Gamma$. We set $\nu_x = \mu/n \in X_*(\mathbf{T})_\Gamma \otimes \mathbb{Q}$. Let $\bar{\nu}_x \in X_*(\mathbf{T})^\delta_\Gamma \otimes \mathbb{Q}$ be the unique dominant element in the W-orbit of ν_x. This is the Newton point of x if we regard x as an element in $\mathbf{G}(\mathbb{L})$. For any $\lambda \in X_*(\mathbf{T})_\Gamma$, we sometimes simply write ν_λ for ν_{ϵ^λ}.

We say that the Dynkin diagram of $\mathbf{G}$ is δ-connected if it cannot be written as a union of two proper δ-stable subdiagrams that are not connected to each other.

The following properties are easy to verify and we omit the details.

(1) Let $J \subset S$ with $\delta(J) = J$ and $x = \epsilon^\lambda w \in \tilde{W}_J$. Then $\nu_x - \nu_\lambda \in Q_J^\vee \otimes_{\mathbb{Z}} \mathbb{Q}$.

(2) Assume that the Dynkin diagram of $\mathbf{G}$ is δ-connected. Let $J \subsetneq S$ with $\delta(J) = J$. If $\lambda, \lambda' \in V$ such that $\langle \lambda, \alpha \rangle \geqslant 0$ for all $\alpha \in J$, λ' is central and $\lambda - \lambda' \in Q_J^\vee \otimes_{\mathbb{Z}} \mathbb{Q}$, then $\lambda = \lambda'$.

The following proposition says that σ-conjugacy classes never fuse.

Proposition 3.5.1. *Let* $[b]$ *be a* σ*-conjugacy class in* $\mathbf{G}(\mathbb{L})$ *and* $J \subset S$ *with* $\delta(J) = J$*. Then* $[b] \cap \mathbf{M}_J(\mathbb{L})$ *contains at most one* σ*-conjugacy class of* $\mathbf{M}_J(\mathbb{L})$*.*

Proof. By Proposition 2.4.1, any σ-conjugacy class of $\mathbf{M}_J(\mathbb{L})$ is represented by some element in $\tilde{W}_J$. Let $x = \epsilon^\lambda w, x' = \epsilon^{\lambda'} w' \in \tilde{W}_J$ such that x and x' are in the same σ-conjugacy class of $\mathbf{G}(\mathbb{L})$. By Kottwitz [8] and [9], $\nu_x = \nu_{x'}$ and $\kappa_{\mathbf{G}}(x) = \kappa_{\mathbf{G}}(x')$. By the definition of $\kappa_{\mathbf{G}}$, $\lambda' = \lambda + \theta - \delta(\theta) + r$ for some coweight θ and $r \in Q^\vee$. We write r as $r = r_J + r'_J$, where $r_J \in Q_J^\vee$ and $r'_J \in Q_{S-J}^\vee$.

By Section 3.5 (1), $\nu_{\lambda'} - \nu_\lambda \in Q_J^\vee \otimes_{\mathbb{Z}} \mathbb{Q}$. Hence

$$\nu_{r'_J} \in Q_J^\vee \otimes_{\mathbb{Z}} \mathbb{Q} \cap Q_{S-J}^\vee \otimes_{\mathbb{Z}} \mathbb{Q} = \{0\}.$$

In other words, $\sum_{i=0}^{n-1} \delta^i(r'_J) = 0$, where n is the order of $W \rtimes \langle\delta\rangle$. Since δ permutes simple coroots of $S-J$, we can assume, without loss of generality, that $r'_J = \sum_{j=0}^{s-1} b_j \delta^j(\alpha^\vee)$, where $b_j \in \mathbb{Z}$, $\alpha^\vee$ is a simple coroot of $S-J$ and s is the smallest positive integer with $\delta^s(\alpha^\vee) = \alpha^\vee$. The equality $\sum_{i=0}^{n-1} \delta^i(r'_J) = 0$ is equivalent to $\sum_{j=0}^{s-1} b_j = 0$. Let $c_j = \sum_{k=0}^{j} b_k$ and $v = \sum_{j=0}^{s-1} c_j \delta^j(\alpha^\vee)$. Then $r'_J = v - \delta(v)$. Hence $\lambda' - \lambda = \theta' - \delta(\theta') + r_J$ for some coweight θ'.

Therefore $\kappa_J(x) = \kappa_J(x')$. By [9, 4.13], x and x' are in the same σ-conjugacy class of $\mathbf{M}_J(\mathbb{L})$. □

3.6. Applications to affine Deligne-Lusztig varieties. For $x \in \tilde{W}$, we denote by $x \mapsto \bar{x}$ the projection $\tilde{W} = X_*(\mathbf{T})_\Gamma \rtimes W \to W$ and $\eta_l(x)$ the unique element $w \in W$ such that $w^{-1}x \in {}^S\tilde{W}$. Here ${}^S\tilde{W}$ is the set of $x \in \tilde{W}$ such that $x\mathbf{a}$ lies in the dominant chamber.

Set

$$\eta_\delta(x) = \delta^{-1}(\eta_l(x)^{-1}\bar{x})\eta_l(x).$$

So if $x = v\epsilon^\mu w$ with $\epsilon^\mu w\mathbf{a}$ contained in the dominant chamber, $v, w \in W$, then $\bar{x} = vw$, $\eta_l(x) = v$, and $\eta_\delta(x) = \delta^{-1}(w)v$.

Now we discuss some consequences of Theorem 3.3.1 on affine Deligne-Lusztig varieties. For analogues in the split case, see [3, Section 9].

Corollary 3.6.1. *Let $[b]$ be a basic σ-conjugacy class in $\mathbf{G}(\mathbb{L})$. Suppose $J \subset S$ with $\delta(J) = J$ and $w \in W$, and $x\mathbf{a}$ is a (J, w, δ)-alcove. Then $X_x(b) = \emptyset$, unless $\kappa_J(w^{-1}x\delta(w)) \in \kappa_J([b] \cap \mathbf{M}_J(\mathbb{L}))$.*

Remark 3.6.2. By Proposition 3.5.1, $[b] \cap \mathbf{M}_J(\mathbb{L})$ is empty or a single σ-conjugacy class of $\mathbf{M}_J(\mathbb{L})$ and hence $\kappa_J([b] \cap \mathbf{M}_J(\mathbb{L}))$ consists of at most one element.

Lemma 3.6.3. *Let $J \subset S$ with $\delta(J) = J$. Let $x \in \tilde{W}$, and write $w = \eta_l(x) \in W$. If $\eta_\delta(x) \in W_J$, then $x\mathbf{a}$ is a (J, w, δ)-alcove.*

Proof. First note that $w^{-1}x\delta(w)$ and $\delta(\eta_\delta(x))$ have that same finite part, so the assumption implies that $w^{-1}x\delta(w) \in \tilde{W}_J$.

Let $\mathbf{U}$ be the subgroup of $\mathbf{G}$ generated by $\mathbf{U}_\alpha$ for $\alpha \in \Phi^+$. Then for any $\beta \in w(\Phi^+ - \Phi_J^+)$,

$$\mathbf{U}_\beta \cap {}^x\mathbf{I} \subseteq {}^w\mathbf{U} \cap {}^x\mathbf{I} \subseteq {}^w(\mathbf{U} \cap \mathbf{I}) \subseteq \mathbf{I}.$$

The second inclusion follows from the assumption that $w^{-1}x\mathbf{a}$ lies in the dominant chamber. □

Proposition 3.6.4. *Assume that the Dynkin diagram of* $\mathbf{G}$ *is* δ*-connected. Let* b *be basic. Let* $x \in \tilde{W}$*, and write* $x = \epsilon^\lambda u$*,* $u \in W$*. Assume that* $\overline{\nu}_{\eta_l(x)^{-1}\lambda} \neq \overline{\nu}_b$ *and that* $\eta_\delta(x) \in \bigcup_{J \subsetneq S, \delta(J)=J} W_J$*. Then* $X_x(b) = \emptyset$*.*

Proof. Write $w = \eta_l(x) \in W$. By Lemma 3.6.3 and our hypothesis, $x\mathbf{a}$ is a (J, w, δ)-alcove for some δ-stable proper subset $J \subsetneq S$. In order to apply Corollary 3.6.1 we only need that $\kappa_J(w^{-1}x\delta(w)) \notin \kappa_J([b] \cap \mathbf{M}_J(\mathbb{L}))$. Here we denote by $[b] \subset \mathbf{G}(\mathbb{L})$ the σ-conjugacy class of b. Otherwise, there exists $b_J \in \mathbf{M}_J(\mathbb{L})$ which is σ-conjugate to b, and such that $\kappa_J(w^{-1}x\delta(w)) = \kappa_J(b_J)$. We may and will assume that $b_J \in \tilde{W}_J$. If we write $w^{-1}x\delta(w) = \epsilon^{\lambda'}u'$, $b_J = \epsilon^\mu v$, $u', v \in W_J$, then $\lambda' = w^{-1}\lambda$ and by the definition of κ_J, for a suitable coweight θ,

$$\lambda' - \mu + \theta - \delta(\theta) \in Q_J^\vee.$$

Thus $\nu_{\lambda'} - \nu_\mu \in Q_J^\vee \otimes_{\mathbb{Z}} \mathbb{Q}$. By Section 3.5 (1),

$$\nu_{\lambda'} - \nu_{b_J} = \nu_{\lambda'} - \nu_\mu + \nu_\mu - \nu_{b_J} \in Q_J^\vee \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Note that λ' is dominant and $\nu_{b_J} = \bar{\nu}_b$ is central since b is basic. By Section 3.5 (2), $\nu_{\lambda'} = \bar{\nu}_b$, which contradicts our assumption. □

Following [3], for any $a \in \Sigma$ and alcove $\mathbf{b}$, let $k(a, \mathbf{b})$ be the unique integer k such that $\mathbf{b}$ lies in the region between the hyperplanes $H_{a,k}$ and $H_{a,k-1}$. Note that here we work with the root system Σ, so it seems less natural to rewrite this in terms of root subgroups (as was done in [3]). But see Section 4.1.

Proposition 3.6.5. *Let* $x = \epsilon^\lambda u$ *lie in the shrunken Weyl chambers, i.e.* $k(a, x\mathbf{a}) \neq k(a, \mathbf{a})$ *for all* $a \in \Sigma$*. Assume that* $\eta_\delta(x) \in \bigcup_{J \subsetneq S, \delta(J)=J} W_J$*. Then* $\overline{\nu}_{\eta_l(x)^{-1}\lambda}$ *is not central.*

In particular, $X_x(b) = \emptyset$ *for any basic element* $b \in \mathbf{G}(\mathbb{L})$*.*

Proof. We may assume that $\mathbf{G}$ is adjoint and that $\mathbf{G} = \mathbf{G}_1 \times \cdots \times \mathbf{G}_r$, where each $\mathbf{G}_i$ is quasi-split over $\mathbb{F}$ with δ-connected Dynkin diagram. We may then write x as $x = (x_1, \dots, x_r)$, where x_i is in the Iwahori-Weyl group $\tilde{W}_i$ of $\mathbf{G}_i$ for each i. By the definition of shrunken Weyl chamber, each x_i lies in the shrunken Weyl chamber of $\tilde{W}_i$.

To prove the Proposition, it suffices to consider the case where the Dynkin diagram of $\mathbf{G}$ is δ-connected.

Let n be the order of $W \rtimes \langle\delta\rangle$. Let $\lambda' = \eta_l(x)^{-1}\lambda$. Suppose that $\overline{\nu}_{\lambda'}$ is central. Then $\lambda' + \delta(\lambda') + \cdots + \delta^{n-1}(\lambda')$ is central and

$$\langle \lambda' + \delta(\lambda') + \cdots + \delta^{n-1}(\lambda'), \beta\rangle = n\langle \lambda', \beta\rangle = 0,$$

where β is the unique maximal root. As λ' is dominant, λ' is central. Hence $x = \eta_l(x)\epsilon^{\lambda'}\eta_l(x)^{-1}u = \epsilon^{\lambda'}u = u\epsilon^{\lambda'}$. Thus $x\mathbf{a} = u\mathbf{a}$. This alcove belongs to the shrunken Weyl chambers only if $u = w_0$. This contradicts our assumption that $\eta_\delta(x) \in \bigcup_{J \subsetneq S, \delta(J)=J} W_J$.

The "moreover" part follows from Proposition 3.6.4. □

4. Reduction method and nonemptiness of ADLV

4.1. Condition (2) of (J, w, δ)-alcoves. By abuse of notation, we continue to use S for the set of simple root in Σ and the set of simple reflections in W. We denote by $\tilde{S} \supset S$ the set of simple reflections in W_a. For any $J \subset S$, we denote by Σ_J the set of roots in Σ spanned by J, and let $\Sigma_J^+ = \Sigma^+ \cap \Sigma_J$.

For any $a \in \Sigma$ and alcoves $\mathbf{b}_1$ and $\mathbf{b}_2$, we say that $\mathbf{b}_1 \geqslant_a \mathbf{b}_2$ if $k(a, \mathbf{b}_1) \geqslant k(a, \mathbf{b}_2)$.

Condition (2) in the definition of (J, w, δ)-alcoves is equivalent to saying that for any $a \in w(\Phi^+ - \Phi_J^+)$ and an affine root $\alpha = a + m$ (with $m \in \mathbb{Q}$), if $x\mathbf{a}$ is in the half-apartment $\alpha^{-1}([-\infty, 0])$, then so is $\mathbf{a}$. We may then reformulate this definition as follows.

(2') For any $a \in w(\Sigma^+ - \Sigma_J^+)$, $x\mathbf{a} \geqslant_a \mathbf{a}$.

In particular, this condition is just a condition on the relative position between certain alcoves and walls. Thus it only depends on the affine Weyl group and does not depend on the set of affine roots.

Proposition 4.1.1. *Let $x \in \tilde{W}$ lie in the shrunken Weyl chambers. If $x\mathbf{a}$ is a (J, w, δ)-alcove for $J \subseteq S$ with $\delta(J) = J$ and $w \in W$, then $\eta_\delta(x) \in W_J$.*

Proof. By the definition of the shrunken Weyl chambers and of (J, w, δ)-alcoves, for any $a \in w(\Sigma^+ - \Sigma_J^+)$,

$$k(\eta_l(x)^{-1}a, \eta_l(x)^{-1}x\mathbf{a}) = k(a, x\mathbf{a}) > k(a, \mathbf{a}) \geqslant 0.$$

Since $\eta_l(x)^{-1}x\mathbf{a}$ lies in the dominant chamber, $\eta_l(x)^{-1}a \in \Sigma^+$ for all $a \in w(\Sigma^+ - \Sigma_J^+)$. Therefore $\eta_l(x)^{-1}w \in W_J$. By the definition of (J, w, δ)-alcoves, $w^{-1}\bar{x}\delta(w) \in W_J$. Thus $\eta_\delta(x) \in W_J$. □

4.2. Reduction method. In this section, we will recall the reduction method in [6] and prove that $\mathbf{P}$-alcoves are "compatible" with the reduction. As a consequence, we prove that an affine Deligne-Lusztig variety $X_w(b)$ for basic b is nonempty exactly when the $\mathbf{P}$-alcoves predict it to be. See Theorem 4.4.7 for the precise formulation; compare also with Corollary 3.6.1.

We first recall a "reduction method" à la Deligne and Lusztig [1, proof of Theorem 1.6], compare also [4].

Proposition 4.2.1. *Let $b \in \mathbf{G}(\mathbb{L})$, $x \in \tilde{W}$ and $s \in \tilde{S}$.*

(1) If $\ell(sx\delta(s)) = \ell(x)$, then $X_x(b) \neq \emptyset$ if and only if $X_{sx\delta(s)}(b) \neq \emptyset$.

(2) If $\ell(sx\delta(s)) = \ell(x) - 2$, then $X_x(b) \neq \emptyset$ if and only if $X_{sx\delta(s)}(b) \neq \emptyset$ or $X_{sx}(b) \neq \emptyset$.

4.3. Minimal length elements. Let $x, x' \in \tilde{W}$ and $s \in \tilde{S}$. We write $x \xrightarrow{s}_\delta x'$ if $x' = sx\delta(s)$ and $\ell(x) \geq \ell(x')$ and write $x \xrightharpoonup{s} x'$ if either $x \xrightarrow{s}_\delta x'$ or $x' = sx$ and $\ell(x) > \ell(x')$.

We write $x \rightarrow_\delta x'$ if there exists a sequence $x_0, x_1, \dots, x_r$ in $\tilde{W}$ and a sequence $s_1, s_2, \dots, s_r$ in $\tilde{S}$ such that $x = x_0 \xrightarrow{s_1}_\delta x_1 \xrightarrow{s_2}_\delta \cdots \xrightarrow{s_r}_\delta x_r = x'$. Similarly, we may define $x \rightharpoonup x'$.

We define the *δ-conjugation action* of $\tilde{W}$ on itself by $w \cdot_\delta w' = ww'\delta(w)^{-1}$. Any orbit is called a *δ-conjugacy class* of $\tilde{W}$. For any δ-conjugacy class $\mathcal{O}$ of $\tilde{W}$, we denote by $\mathcal{O}_{\min}$ the set of minimal length elements in $\mathcal{O}$.

One of the main results in [7] is

Theorem 4.3.1. *Let $\mathcal{O}$ be a δ-conjugacy class of $\tilde{W}$. Then for any $x \in \mathcal{O}$, there exists $x' \in \mathcal{O}_{\min}$ such that $x \rightarrow_\delta x'$.*

Note that [7] does also include the twisted case; there the action of δ is incorporated by replacing $\tilde{W}$ by a semi-direct product of the form $\tilde{W} \rtimes \langle\delta\rangle$.

The following result is a consequence of the "degree=dimension" theorem in [6]. We include a proof for completeness.

Theorem 4.3.2. *Let $x \in \tilde{W}$ and $D_{x,\delta}$ be the set of elements $y \in \tilde{W}$ such that y is of minimal length in its δ-conjugacy class and $x \rightharpoonup y$. Then for any $b \in \mathbf{G}(\mathbb{L})$, $X_x(b) \neq \emptyset$ if and only if $X_y(b) \neq \emptyset$ for some $y \in D_{x,\delta}$.*

Proof. If $y \in D_{x,\delta}$ and $X_y(b) \neq \emptyset$, then by Proposition 4.2.1 and the definition of $D_{x,\delta}$, $X_x(b) \neq \emptyset$. Now we assume that $X_x(b) \neq \emptyset$. We proceed by induction on the length of x.

If x is a minimal length element in its δ-conjugacy class $\mathcal{O}$, then $x \in D_{x,\delta}$. The statement is obvious.

Suppose that x is not a minimal length element in its δ-conjugacy class. By Theorem 4.3.1 there exists $x' \in \tilde{W}$ and $s \in \tilde{S}$ such that $x \rightarrow_\delta x'$, $\ell(x) = \ell(x')$ and $\ell(sx'\delta(s)) = \ell(x') - 2$. By Proposition 4.2.1, $X_{x'}(b) \neq \emptyset$ and $X_{sx'\delta(s)}(b) \neq \emptyset$ or $X_{sx'}(b) \neq \emptyset$. Since $\ell(sx'\delta(s)), \ell(sx') < \ell(x)$, by induction hypothesis, there exists $y \in D_{sx'\delta(s),\delta} \cup D_{sx',\delta}$ such that $X_y(b) \neq \emptyset$. By definition, $D_{sx'\delta(s),\delta} \cup D_{sx',\delta} \subset D_{x,\delta}$. So $y \in D_{x,\delta}$. The statement holds for x. □

4.4. Property (NLO). We now fix a basic element $b \in \tilde{W}$.

Definition 4.4.1. *We say that $y \in \tilde{W}$ has property (NLO) (with respect to b), if for every pair (J, w) with $J \subset S$, $\delta(J) = J$ and $w \in W$, such that $y\mathbf{a}$ is a (J, w, δ)-alcove, there exists $b_J \in w\tilde{W}_J\delta(w)^{-1}$ such that*

1. $\kappa_{\mathbf{G}}(b) = \kappa_{\mathbf{G}}(b_J)$,
2. $\nu_{b_J} = \nu_b$,
3. $\kappa_J(w^{-1}b_J\delta(w)) = \kappa_J(w^{-1}y\delta(w))$.

Here (NLO) stands for *no Levi obstruction*: Heuristically, affine Deligne-Lusztig varieties should be non-empty, unless there is an evident obstruction. For instance, if $\kappa_{\mathbf{G}}(b) \neq \kappa_{\mathbf{G}}(x)$, then $X_x(b) = \emptyset$, as is easily checked. Moreover, as the previous results show, an obstruction of a similar kind can originate from other Levi subgroups of $\mathbf{G}$. This kind of obstruction is formalized in the above definition, and we will see that it is in fact the only obstruction to non-emptiness.

A special case is that if $y\mathbf{a}$ is not a (J, w, δ)-alcove for any proper subset J of S, then y satisifies the NLO condition (with respect to b) if and only if $\kappa_{\mathbf{G}}(b) = \kappa_{\mathbf{G}}(y)$. This simple observation will be used in Proposition 4.4.9.

By Theorem 4.3.2, to prove the nonemptiness, one only needs to examine the claim for the reduction step and for minimal length elements.

Lemma 4.4.2. *Denote by $x \mapsto \bar{x}$ the projection $\tilde{W} \to W$. Let $y \in \tilde{W}$ and $s \in \tilde{S}$. Assume that $s = s_H$ for some affine root hyperplane $H = H_{\alpha,k}$ with $\alpha \in \Sigma$ and $k \in \mathbb{Z}$. Let $\beta \in \Sigma$.*

1. *If $\beta \notin \{\pm\alpha, \pm\bar{y}\delta(\alpha)\}$, then $sy\delta(s)\mathbf{a} \geqslant_{\bar{s}(\beta)} \mathbf{a}$ if and only if $y\mathbf{a} \geqslant_\beta \mathbf{a}$.*
2. *If $\beta \neq \pm\bar{y}\delta(\alpha)$, then $y\mathbf{a} \geqslant_\beta \mathbf{a}$ if and only if $y\delta(s)\mathbf{a} \geqslant_\beta \mathbf{a}$.*

Proof. We only prove (1). (2) can be proved in the same way.

Note that $sy\delta(s)\mathbf{a} \geqslant_{\bar{s}(\beta)} \mathbf{a}$ if and only if $y\delta(s)\mathbf{a} \geqslant_\beta s\mathbf{a}$. By the assumption on β, there exists a point $e \in \bar{\mathbf{a}} \cap s\bar{\mathbf{a}} \subset H$ such that, with $e' := y\delta(e) \in y\bar{\mathbf{a}} \cap y\delta(s)\bar{\mathbf{a}} \subset y\delta H$, we have $\langle e, \beta\rangle, \langle e', \beta\rangle \notin \mathbb{Z}$. Here $\langle -, -\rangle$ is the natural pairing between V and its dual V^*.

The statement follows from the following fact which is easily checked:

Let $\mathbf{c} \neq \mathbf{c}', \mathbf{d} \neq \mathbf{d}'$ be alcoves such that there exist $e \in \overline{\mathbf{c}} \cap \overline{\mathbf{c}'}$, $e' \in \overline{\mathbf{d}} \cap \overline{\mathbf{d}'}$, and let $\beta \in \Sigma$ with $\langle e, \beta\rangle, \langle e', \beta\rangle \notin \mathbb{Z}$. Then $\mathbf{c} \geqslant_\beta \mathbf{d}$ if and only if $\mathbf{c}' \geqslant_\beta \mathbf{d}'$. □

Lemma 4.4.3. *Let $y \in \tilde{W}$ and $s \in \tilde{S}$ with $\ell(sy\delta(s)) = \ell(y)$. If $y\mathbf{a}$ is a (J, w, δ)-alcove, then $sy\delta(s)\mathbf{a}$ is a $(J, \bar{s}w, \delta)$-alcove.*

Proof. It suffices to show that $sy\delta(s)\mathbf{a} \geqslant_\beta \mathbf{a}$ for $\beta \in \bar{s}w(\Sigma^+ - \Sigma_J^+)$.

Assume that $s = s_H$ for some affine root hyperplane $H = H_{\alpha,k}$ with $\alpha \in \Sigma$ and $k \in \mathbb{Z}$. Since $w^{-1}\bar{y}\delta(w) \in W_J$, $\alpha \notin w(\Sigma_J)$ if and only if $\bar{y}\delta(\alpha) \notin w(\Sigma_J)$. In this case, $w^{-1}(\alpha)$ and $w^{-1}\bar{y}\delta(\alpha)$ are both positive or both negative roots.

If $\beta \notin \{\pm\alpha, \pm\bar{s}\bar{y}\delta(\alpha)\}$, the statement follows from Lemma 4.4.2.

Now suppose that $\beta \in \{\pm\alpha, \pm\bar{s}\bar{y}\delta(\alpha)\}$. Without loss of generality, we assume that $-\alpha, \bar{s}\bar{y}\delta(\alpha) \in \bar{s}w(\Sigma^+ - \Sigma_J^+)$.

It remains to show that $sy\delta(s)\mathbf{a} \geqslant_{-\alpha} \mathbf{a}$ and $sy\delta(s) \geqslant_{\bar{s}\bar{y}\delta(\alpha)} \mathbf{a}$. There are two cases.

Case 1: $H = y\delta(H)$. Then $-\alpha = \bar{s}\bar{y}\delta(\alpha)$. So $\mathbf{a}, sy\delta(s)\mathbf{a}$ are on the same side of H and their closures intersect with H. Hence $sy\delta(s)\mathbf{a} =_\alpha \mathbf{a}$.

Case 2: $H \neq y\delta(H)$. Without loss of generality, we assume that $\ell(y\delta(s)) < \ell(y)$ (arguments for the case $\ell(y\delta(s)) > \ell(y)$ are similar). In this case, $y\delta H$ separates $y\mathbf{a}$ from $y\delta(s)\mathbf{a}$ and $\mathbf{a}$. Since $y\mathbf{a}$ is a (J, w, δ)-alcove, $y\mathbf{a} \geqslant_{\bar{y}\delta(\alpha)} \mathbf{a}$. Hence $y\mathbf{a} \geqslant_{\bar{y}\delta(\alpha)} y\delta(s)\mathbf{a}$ and $\mathbf{a} \geqslant_\alpha s\mathbf{a}$. Since $\ell(y\delta(s)) < \ell(sy\delta(s)) = \ell(y)$, $s\mathbf{a}, sy\delta(s)\mathbf{a}$ are on the same side of H, therefore $sy\delta(s)\mathbf{a} \geqslant_{-\alpha} \mathbf{a}$.

Since $y\mathbf{a} \geqslant_{\bar{y}\delta(\alpha)} \mathbf{a}$, $sy\mathbf{a} \geqslant_{\bar{s}\bar{y}\delta(\alpha)} sy\delta(s)\mathbf{a}$. As $\ell(sy) > \ell(sy\delta(s))$, $\mathbf{a}, sy\delta(s)\mathbf{a}$ are on the same side of $sy\delta H$. Moreover, the closure $sy\delta(s)\bar{\mathbf{a}}$ intersects with $sy\delta H$. Therefore $sy\delta(s)\mathbf{a} \geqslant_{\bar{s}\bar{y}\delta(\alpha)} \mathbf{a}$. ☐

Lemma 4.4.4. *Let $J, J' \subset S$ with $\delta(J) = J$, $\delta(J') = J'$. Let $y \in \tilde{W}$ and $\alpha \in \Sigma$. If there exist $w, w' \in W$ such that $w^{-1}y\delta(s_\alpha)\delta(w) \in \tilde{W}_J$ and $(w')^{-1}y\delta(w') \in \tilde{W}_{J'}$, then $w^{-1}(\alpha) \in \Sigma_J$ or $(w')^{-1}(\alpha) \in \Sigma_{J'}$.*

Proof. Let V be the real vector space spanned by the coweights. Let $v_0, v_0' \in V^\delta$ be dominant coweights such that for any $u \in W$, $u(v_0) = v_0$ (resp. $u(v_0') = v_0'$) if and only if $u \in W_J$ (resp. $u \in W_{J'}$). In particular, $\langle v_0, \beta\rangle = 0$ if and only if $\beta \in \Sigma_J$.

Set $v = w(v_0)$ and $v' = w'(v_0')$. Then $y\delta(v') = y\delta(w'v_0') = w'(v_0') = v'$ and $y\delta(s_\alpha)\delta(v) = y\delta(s_\alpha)\delta(w)(v_0) = w(v_0) = v$. Let $(-,-)$ be the Killing form on V. Now

$$\begin{aligned}(v' - s_\alpha(v), v' - s_\alpha(v)) &= (y\delta(v' - s_\alpha(v)), y\delta(v' - s_\alpha(v))) \\ &= (v' - v, v' - v).\end{aligned}$$

Hence $(v', s_\alpha(v)) = (v', v)$. If $w^{-1}(\alpha) \notin \Sigma_J$, then $\langle v, \alpha\rangle = \langle v_0, w^{-1}(\alpha)\rangle \neq 0$. So $\langle v', \alpha\rangle = 0$ and $(w')^{-1}(\alpha) \in \Sigma_{J'}$. ☐

Theorem 4.4.5. *Let $y \in \tilde{W}$ such that property (NLO) holds for y. Let $s \in \tilde{S}$.*

(1) If $\ell(sy\delta(s)) = \ell(y)$, then property (NLO) holds for $sy\delta(s)$;

(2) If $\ell(sy\delta(s)) = \ell(y) - 2$, then property (NLO) holds for $sy\delta(s)$ or $y\delta(s)$.

Proof. Case 1: Assume that for any $J \subset S$ with $\delta(J) = J$ and $w \in W$ such that $sy\delta(s)\mathbf{a}$ is a (J, w, δ)-alcove, $y\mathbf{a}$ is also a $(J, \bar{s}w, \delta)$-alcove. This in particular includes part (1) of the theorem by Lemma 4.4.3.

So assume that y satisfies property (NLO), and that $sy\delta(s)\mathbf{a}$ is a (J, w, δ)-alcove. In this case, by assumption there is an element $b_J \in \bar{s}w\tilde{W}_J\delta(\bar{s}w)^{-1}$ satisfying conditions (1)-(3) in the definition of property (NLO) for $(y, J, \bar{s}w, \delta)$.

Set $b'_J = sb_J\delta(s) \in w\tilde{W}_J\delta(w)^{-1}$. Then b'_J satisfies conditions (1)-(3) in the definition of property (NLO) for $(sy\delta(s), J, w, \delta)$.

Case 2: There exists $J \subset S$ with $\delta(J) = J$ and $w \in W$ such that $sy\delta(s)\mathbf{a}$ is a (J, w, δ)-alcove, but $y\mathbf{a}$ is not a $(J, \bar{s}w, \delta)$-alcove. Let $\alpha \in \Sigma$ with $\bar{s} = s_\alpha$. By Lemma 4.4.2, we have $w^{-1}\bar{y}\delta(\alpha) \notin \Sigma_J$ or $w^{-1}(\alpha) \notin \Sigma_J$, which are equivalent to each other since $w^{-1}\bar{y}\delta(w) \in W_J$. Therefore, we always have $w^{-1}(\alpha) \notin \Sigma_J$. We show that $y\delta(s)$ satisfies property (NLO).

Assume $y\delta(s)\mathbf{a}$ is a (J', w', δ)-alcove for some $w' \in W$ and $J' \subset S$ with $\delta(J') = J'$. By Lemma 4.4.4, ${w'}^{-1}(\alpha) \in \Sigma_{J'}$. Hence by Lemma 4.4.2, $y\mathbf{a}$ is a (J', w', δ)-alcove. Since y satisfies property (NLO), there exists $b_{J'} \in w'\tilde{W}_{J'}\delta(w')^{-1}$ such that $\kappa_{\mathbf{G}}(b) = \kappa_{\mathbf{G}}(b_{J'})$, $\nu_b = \nu_{b_{J'}}$ and

$$\begin{aligned}\kappa_{J'}({w'}^{-1}b_{J'}\delta(w')) &= \kappa_{J'}({w'}^{-1}y\delta(w')) \\ &= \kappa_{J'}({w'}^{-1}y\delta(w')\delta({w'}^{-1}sw')) = \kappa_{J'}({w'}^{-1}y\delta(s)w'),\end{aligned}$$

where the second equality follows from ${w'}^{-1}(\alpha) \in \Sigma_{J'}$. Hence property (NLO) holds for $y\delta(s)$. □

Next we consider the case of minimal length elements:

Proposition 4.4.6. *If y is a minimal length element in its δ-conjugacy class and y satisfies property (NLO), then $X_y(b) \neq \emptyset$.*

Proof. It suffices to show that y and b are in the same σ-conjugacy class. By Kottwitz [8] and [9], this is equivalent to show that $\bar{\nu}_y = \nu_b$ and $\kappa_{\mathbf{G}}(y) = \kappa_{\mathbf{G}}(b)$. Since $y\mathbf{a}$ is automatically a $(S, 1, \delta)$-alcove, by our assumption $\kappa_{\mathbf{G}}(y) = \kappa_{\mathbf{G}}(b)$.

By [7, Proposition 2.4], there exists a minimal length element y' in the δ-conjugacy class containing y such that $\bar{\mathbf{a}} \cap V_{y'} \neq \emptyset$. Here $V_{y'} = \{v \in V; y'\delta(v) = v + \nu_{y'}\}$. We may then assume that $y = y'$ (use Theorem 4.4.5 (1)).

Let $w \in W$ such that $\bar{\nu} = w^{-1}(\nu_y)$ is dominant. Then $w(\bar{\nu}) = \nu_y = \bar{y}\delta(\nu_y) = \bar{y}\delta(w)\delta(\bar{\nu})$. In other words, $\bar{\nu} = w^{-1}\bar{y}\delta(w)\delta(\bar{\nu})$. Since $\delta(\bar{\nu})$ is the unique dominant coweight in the W-orbit of $\bar{\nu}$, we have $\bar{\nu} = \delta(\bar{\nu})$. Set $J = \{s \in S; s(\bar{\nu}) = \bar{\nu}\}$. Then $\delta(J) = J$ and $w^{-1}\bar{y}\delta(w) \in W_J$.

For any $\beta \in \Sigma$ with $w^{-1}(\beta) \in \Sigma^+ - \Sigma_J^+$, $\langle \nu_y, \beta \rangle = \langle \bar{\nu}, w^{-1}(\beta) \rangle > 0$. Hence $y\mathbf{a} \geqslant_\beta \mathbf{a}$ as $\bar{\mathbf{a}}$ intersects with V_y. Therefore $y\mathbf{a}$ is a (J, w, δ)-alcove.

Since y satisfies property (NLO), there exists $b_J \in w\tilde{W}_J\delta(w)^{-1}$ such that $\nu_{b_J} = \nu_b$ and $\kappa_J(w^{-1}b_J\delta(w)) = \kappa_J(w^{-1}y\delta(w))$. If we write $w^{-1}y\delta(w) = \epsilon^\lambda u$ and $w^{-1}b_J\delta(w) = \epsilon^{\lambda'}u'$, $u, u' \in W_J$, then for a suitable coweight θ,

$$\lambda - \lambda' + \theta - \delta(\theta) \in Q_J^\vee.$$

Thus $\nu_\lambda - \nu_{\lambda'} \in Q_J^\vee \otimes_{\mathbb{Z}} \mathbb{Q}$. By Section 3.5 (1), $\nu_{w^{-1}y\delta(w)} - \nu_{w^{-1}b_J\delta(w)} \in Q_J^\vee \otimes_{\mathbb{Z}} \mathbb{Q}$. Since $\nu_{w^{-1}y\delta(w)} = \bar{\nu}$ is orthogonal to all the roots in J and $\nu_{w^{-1}b_J\delta(w)} = \nu_b$ is central since b is basic, by Section 3.5 (2), $\bar{\nu}_y = \nu_{w^{-1}y\delta(w)} = \nu_b$. □

Altogether, we have now proved:

Theorem 4.4.7. *Let $b \in \mathbf{G}(\mathbb{L})$ be basic and $x \in \tilde{W}$. If x satisfies property (NLO), then $X_x(b) \neq \emptyset$.*

Remark 4.4.8. For split groups, this was conjectured in [3, Conjecture 9.4.2].

Proof. Since x satisfies property (NLO), there exists $y \in D_{x,\delta}$ also satisfies property (NLO). By Proposition 4.4.6, $X_y(b) \neq \emptyset$. Hence by Theorem 4.3.2, $X_x(b) \neq \emptyset$. □

Now combining Theorem 4.4.7 with Proposition 4.1.1, we have

Proposition 4.4.9. *Let $b \in \mathbf{G}(\mathbb{L})$ be basic and $x \in \tilde{W}$ lie in the shrunken Weyl chambers such that $\kappa_{\mathbf{G}}(b) = \kappa_{\mathbf{G}}(x)$. If $\eta_\delta(x) \in W - \bigcup_{J \subsetneq S, \delta(J)=J} W_J$, then $X_x(b) \neq \emptyset$.*

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