J. Math. Pures Appl. 106 (2016) 123–154

Contents lists available at ScienceDirect

Journal de Mathématiques Pures et Appliquées

www.elsevier.com/locate/matpur

Existence and blowup behavior of global strong solutions to the two-dimensional barotrpic compressible Navier–Stokes system with vacuum and large initial data [☆]

Xiangdi Huang [a], Jing Li [b,*]

[a] NCMIS, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, PR China

[b] Institute of Applied Mathematics, AMSS, & Hua Loo-Keng Key Laboratory of Mathematics, Chinese Academy of Sciences, Beijing 100190, PR China

ARTICLE INFO

Article history:
Received 14 May 2014
Available online 27 February 2016

MSC:
35Q35
35B65
76N10

Keywords:
Compressible Navier–Stokes equations
Global strong solutions
Large initial data
Vacuum
Blowup

ABSTRACT

For periodic initial data with density allowed to vanish initially, we establish the global existence of strong and weak solutions to the two-dimensional barotropic compressible Navier–Stokes equations with no restrictions on the size of initial data provided the shear viscosity is a positive constant and the bulk one is $\lambda = \rho^\beta$ with $\beta > 4/3$. These results generalize and improve the previous ones due to Vaigant–Kazhikhov [Sib. Math. J. 36 (1995) 1283–1316] who required $\beta > 3$. Moreover, we also prove that the densities for both the strong and weak solutions remain bounded from above independently of time. As a consequence, it is shown that both the strong and weak solutions converge to the equilibrium state as time tends to infinity.

RÉSUMÉ

Pour les données initiales périodiques avec une densité qui peut s'annuler à l'instant initial, on établit l'existence globale de solutions fortes et faibles dans le cas de deux dimensions d'équations de Navier–Stokes compressibles sans restriction sur la taille des données initiales pourvu que la viscosité de cisaillement soit une constante positive où $\beta > 4/3$. Ces résultats se généralisent et améliorent les résultats précédents obtenus par Vaigant–Kazhikhov [Sib. Math. J. 36 (1995) 1283–1316] qui exigaient $\beta > 3$. En outre, on démontre que les densités pour les deux solutions fortes et faibles, restent indépendantes et bornées supérieurement. Finalement, il est démontré aussi que les deux solutions fortes et faibles convergent vers les états d'équilibre lorsque le temps tend vers l'infini.

☆ X.-D. Huang is partially supported by the National Center for Mathematics and Interdisciplinary Sciences, CAS, and by President Fund of Academy of Mathematics Systems Science, CAS, No. 2014-cjrwlzx-hxd and NNSFC Grant Nos. 11471321 and 11371064. J. Li is partially supported by the National Center for Mathematics and Interdisciplinary Sciences, CAS, and NNSFC Grant Nos. 11371348 and 11525106.

* Corresponding author.
E-mail addresses: xdhuang@amss.ac.cn (X. Huang), ajingli@gmail.com (J. Li).

http://dx.doi.org/10.1016/j.matpur.2016.02.003
0021-7824/

1. Introduction and main results

We study the two-dimensional barotropic compressible Navier–Stokes equations which read as follows:

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla P = \mu \triangle u + \nabla((\mu + \lambda)\operatorname{div} u), \end{cases} \tag{1.1}$$

where $\rho = \rho(x,t)$ and $u = (u_1(x,t), u_2(x,t))$ represent the unknown density and velocity respectively, and the pressure P is given by

$$P(\rho) = a\rho^\gamma, \tag{1.2}$$

with constants $a > 0$ and $\gamma > 1$. We also have the following hypothesis on the shear viscosity μ and the bulk one λ:

$$0 < \mu = const, \quad \lambda(\rho) = b\rho^\beta, \tag{1.3}$$

for positive constants b and β. In the sequel, we set $a = b = 1$ without loss of generality.

We consider the Cauchy problem with the given initial data ρ_0 and m_0, which are periodic with period 1 in each space direction $x_i, i = 1, 2$, i.e., functions defined on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. We require that

$$\rho(x,0) = \rho_0(x), \quad \rho u(x,0) = m_0(x), \quad x \in \mathbb{T}^2. \tag{1.4}$$

There is a huge literature concerning the theory of strong and weak solutions for the system of the multidimensional compressible Navier–Stokes equations with constant viscosity coefficients. The local existence and uniqueness of classical solutions are known in [29,32] in the absence of vacuum and recently, for strong solutions also, in [3,4,31] for the case that the initial density need not be positive and may vanish in open sets. The global classical solutions were first obtained by Matsumura–Nishida [28] for initial data close to a non-vacuum equilibrium in some Sobolev space H^s. Later, Hoff [14,15] studied the problem for discontinuous initial data. For the existence of solutions for large data, the major breakthrough is due to Lions [27] (see also Feireisl [11] and Feireisl et al. [12]), where he obtained global existence of weak solutions, defined as solutions with finite energy, when the exponent γ is suitably large. The main restriction on initial data is that the initial energy is finite, so that the density is allowed to vanish initially. Recently, Huang–Li–Xin [19] established the global existence and uniqueness of classical solutions to the Cauchy problem for the barotropic compressible Navier–Stokes equations in three-dimensional space with smooth initial data which are of small energy but possibly large oscillations; in particular, the initial density is allowed to vanish, even has compact support. The compatibility conditions on the initial data of [3,4,31] are further relaxed by [16,17,23,25].

However, there are few results regarding global strong solvability for equations of multi-dimensional motions of viscous gas with no restrictions on the size of initial data. One of the first ever ones is due to Vaigant–Kazhikhov [34] who obtained a remarkable result which can be stated that the two-dimensional system (1.1)–(1.4) admits a unique global strong solution for large initial data with density away from vacuum initially provided

$$\beta > 3. \tag{1.5}$$

Recently, in addition to (1.5), under the stringent constraint that

$$\gamma = \beta, \tag{1.6}$$

Perepelitsa [30] obtained the time-independent lower and upper bounds on the density of the global solution, as well as the decay of the solution to an equilibrium state when the initial density is away from vacuum. Very recently, under some additional compatibility conditions on the initial data, Jiu–Wang–Xin [20] considered classical solutions and removed the condition that the initial density should be away from vacuum in Vaigant–Kazhikhov [34] but still under the same condition (1.5) as that in [34].

Before stating the main results, we explain the notations and conventions used throughout this paper. We denote

$$\int f dx = \int_{\mathbb{T}^2} f dx, \quad \overline{f} = \frac{1}{|\mathbb{T}^2|} \int f dx. \tag{1.7}$$

For $1 \le r \le \infty$, we also denote the standard Lebesgue and Sobolev spaces as follows:

$$L^r = L^r(\mathbb{T}^2), \quad W^{s,r} = W^{s,r}(\mathbb{T}^2), \quad H^s = W^{s,2}.$$

Then, we give the definition of weak and strong solutions to (1.1).

Definition 1.1. If (ρ, u) satisfies (1.1) in the sense of distribution, then (ρ, u) is called a weak solution to (1.1). Moreover, if, for a weak solution, all derivatives involved in (1.1) are regular distributions and equations (1.1) hold almost everywhere in $\mathbb{T}^2 \times (0, T)$, then the solution is called strong.

Thus, the first main result concerning the global existence and large-time behavior of strong solutions can be stated as follows:

Theorem 1.1. *Assume that*

$$\beta > 4/3, \quad \gamma > 1, \tag{1.8}$$

and that the initial data (ρ_0, m_0) satisfy that for some $q > 2$,

$$0 \le \rho_0 \in W^{1,q}, \quad u_0 \in H^1, \quad m_0 = \rho_0 u_0. \tag{1.9}$$

Then the problem (1.1)–(1.4) has a unique strong solution (ρ, u) in $\mathbb{T}^2 \times (0, \infty)$ satisfying

$$\begin{cases} \rho \in C([0,T]; W^{1,q}), \quad \rho_t \in L^\infty(0,T; L^2), \\ u \in L^\infty(0,T; H^1) \cap L^{(q+1)/q}(0,T; W^{2,q}), \\ t^{1/2} u \in L^2(0,T; W^{2,q}), \quad t^{1/2} u_t \in L^2(0,T; H^1), \\ \rho u \in C([0,T]; L^2), \quad \sqrt{\rho} u_t \in L^2(\mathbb{T}^2 \times (0,T)), \end{cases} \tag{1.10}$$

for any $0 < T < \infty$. Moreover, if

$$\beta > 3/2, \quad 1 < \gamma < 4\beta - 3, \tag{1.11}$$

there is a positive constant C depending only on $\mu, \beta, \gamma, \|\rho_0\|_{L^\infty}$, and $\|u_0\|_{H^1}$ such that

$$\sup_{0 \le t < \infty} \|\rho(\cdot, t)\|_{L^\infty} \le C, \tag{1.12}$$

and the following large-time behavior holds:

$$\lim_{t\to\infty}\left(\|\rho-\overline{\rho_0}\|_{L^p}+\|\nabla u\|_{L^p}\right)=0, \tag{1.13}$$

for any $p\in[1,\infty)$.

The second result gives the global existence and large-time behavior of weak solutions.

Theorem 1.2. *Assume that (1.8) holds and that the initial data* (ρ_0,m_0) *satisfy that*

$$0\le\rho_0\in L^\infty,\quad u_0\in H^1,\quad m_0=\rho_0u_0. \tag{1.14}$$

Then the problem (1.1)–(1.4) has at least one weak solution (ρ,u) *in* $\mathbb{T}^2\times(0,\infty)$ *satisfying for any* $0<\tau<T<\infty$ *and* $p\ge1$,

$$\begin{cases}\rho\in L^\infty(0,T;L^\infty)\cap C(0,T;L^p),\\ u\in L^\infty(0,T;H^1),u_t\in L^2(\tau,T;L^2),\nabla u\in L^\infty(\tau,T;L^p).\end{cases} \tag{1.15}$$

Moreover, if β *and* γ *satisfy (1.11), both (1.12) and (1.13) hold true.*

Finally, similar to Li–Xin [24, Theorem 1.2], we can obtain from (1.13) the following large-time behavior of the spatial gradient of the density for the strong solution obtained in Theorem 1.1 when vacuum states appear initially.

Theorem 1.3. *Let* β,γ *satisfy (1.11). In addition to (1.9), assume further that there exists some point* $x_0\in\mathbb{T}^2$ *such that* $\rho_0(x_0)=0$. *Then the unique global strong solution* (ρ,u) *to the Cauchy problem (1.1)–(1.4) obtained in Theorem 1.1 has to blow up as* $t\to\infty$, *in the sense that for any* $2<r\le q$ *with* q *as in Theorem 1.1,*

$$\lim_{t\to\infty}\|\nabla\rho(\cdot,t)\|_{L^r}=\infty.$$

A few remarks are in order:

Remark 1.1. Theorems 1.1 and 1.2 generalize and improve the earlier results due to Vaigant–Kazhikhov [34] where they required (1.5) and that the initial density is away from vacuum.

Remark 1.2. In Theorem 1.1, the density is allowed to vanish initially. Moreover, we only require the initial data (ρ_0,m_0) satisfy the natural compatibility condition $m_0=\rho_0u_0$ instead of the following additional compatibility one:

$$-\mu\triangle u_0-\nabla((\mu+\lambda(\rho_0))\mathrm{div}u_0)+\nabla P(\rho_0)=\rho_0^{1/2}g,\ \text{for some } g\in L^2, \tag{1.16}$$

which are required in [3,4,20,31]. In fact, our methods can be applied to obtain the existence and uniqueness of the local strong solutions to the three-dimensional system (1.1) just under the natural compatibility condition $m_0=\rho_0u_0$. This will be reported in a forthcoming paper [17].

Remark 1.3. With Theorem 1.1 at hand, one can easily check that similar to [16,25], if (ρ_0,m_0) satisfies for some $q>2$,

$$0\le\rho_0\in W^{2,q},\quad u_0\in H^2,\quad m_0=\rho_0u_0,$$

along with the additional compatibility condition (1.16), the strong solution obtained in Theorem 1.1 becomes a classical one for positive time. See [16,20,25] for details.

Remark 1.4. Theorem 1.1 implies that the density remains time-independently bounded and the large-time behavior (1.13) still holds for all $\beta > 3/2$ and $1 < \gamma \le \max\{3, \beta\}$, since $4\beta - 3 > \max\{3, \beta\}$ provided $\beta > 3/2$. Moreover, when the initial density is strictly away from vacuum, Perepelitsa [30] also obtained (1.12) and

$$\lim_{t\to\infty} (\|\rho - \bar{\rho}_0\|_{L^\infty} + \|\nabla u\|_{L^2}) = 0,$$

under the stringent conditions (1.5) and (1.6). Note that if β and γ satisfy (1.5) and (1.6), (1.11) holds. Therefore, Theorems 1.1 and 1.2 improve the results of Perepelitsa [30].

Remark 1.5. It should be mentioned here that it seems that $\beta > 1$ is the extremal case for the system (1.1)–(1.3) (see [34] or Lemma 3.6 below). Therefore, it would be interesting to study the problem (1.1)–(1.4) when $1 < \beta \le 4/3$. This is left for the future.

We now comment on the analysis of this paper. Note that for smooth initial data away from vacuum, the local existence and uniqueness of strong solutions to the Cauchy problem (1.1)–(1.4) have been established in [31,33]. Thus, to extend the strong solutions globally in time and allow the density to vanish initially, one needs global a priori estimates, which are independent of the lower bound of the initial density, on smooth solutions to (1.1)–(1.4) in suitable higher norms. Motivated by our recent studies [18] on the blow-up criteria of strong solutions to (1.1), it turns out that the key issue in this paper is to derive the upper bound of the density which is independent of the lower one of the initial density just under the condition $\beta > 4/3$. To do so, first, similar to [27,30], we rewrite $(1.1)_2$ as (3.42) in terms of a sum of commutators of Riesz transforms and the operators of multiplication by u_i (see (3.32)). Then, by energy type estimates and the compensated compactness analysis [7, Theorem II.1], we show that $\log(1+\|\nabla u\|_{L^2})$ does not exceed a polynomial function of $\|\rho\|_{L^\infty}$ (see (3.7) and (3.52)). Finally, using the $W^{1,p}$-estimate of the commutator due to Coifman–Meyer [6] (see (2.8)) and the Brezis–Wainger inequality (see (2.5)), we obtain an estimate on the L^∞-norm of the commutators in terms of L^∞ norm of the density (see (3.33) and (3.51)). Both estimates yield the key a priori upper bound of the density which is independent of the lower one of the initial density provided $\beta > 4/3$. See Proposition 3.5 and its proof.

The next main step is to bound the spatial gradient of the density just under the natural compatibility condition $m_0 = \rho_0 u_0$. We first obtain the spatial weighted mean estimate on the material derivatives of the velocity which is achieved by modifying the basic one due to Hoff [14]. Then, following [18], the L^p-bound of the spatial gradient of the density can be obtained by solving a logarithm Gronwall inequality based on a Beale–Kato–Majda type inequality (see Lemma 2.4), the a priori estimates we have just derived and some careful initial layer analysis; and moreover, such a derivation yields simultaneously also the bound for $L^1(0,T;L^\infty(\mathbb{T}^2))$-norm of the spatial gradient of the velocity. See Proposition 4.3 and its proof.

The rest of the paper is organized as follows: In Section 2, we collect some elementary facts and inequalities which will be needed in later analysis. Section 3 is devoted to deriving the upper bound of the density which are independent of the lower one of the initial density and needed to extend the local solution to all time. Based on the previous estimates, bounds on higher-order derivatives are established in Section 4. Then finally, the main results, Theorems 1.1–1.3, are proved in Section 5.

2. Preliminaries

The following well-known local existence theory, where the initial density is strictly away from vacuum, can be found in [31,33].

Lemma 2.1. *Assume that* (ρ_0, m_0) *satisfies*

$$\rho_0 \in H^2, \quad u_0 \in H^2, \quad \inf_{x\in\mathbb{T}^2} \rho_0(x) > 0, \quad m_0 = \rho_0 u_0. \tag{2.1}$$

Then there are a small time $T > 0$ and a constant $C_0 > 0$ both depending only on $\|\rho_0\|_{H^2}, \|u_0\|_{H^2}$, and $\inf_{x\in\mathbb{T}^2} \rho_0(x)$ such that there exists a unique strong solution (ρ, u) to the problem (1.1)–(1.4) in $\mathbb{T}^2 \times (0,T)$ satisfying

$$\begin{cases} \rho \in C([0,T]; H^2), \quad \rho_t \in C([0,T]; H^1), \\ u \in L^2(0,T; H^3), \quad u_t \in L^2(0,T; H^1), \\ u_t \in L^2(0,T; H^2), \quad u_{tt} \in L^2((0,T)\times\mathbb{T}^2), \end{cases} \tag{2.2}$$

and

$$\inf_{(x,t)\in\mathbb{T}^2\times(0,T)} \rho(x,t) \geq C_0 > 0. \tag{2.3}$$

Remark 2.1. It should be mentioned that [31,33] dealt with the case that $\lambda = const$. However, after some slight modifications, their methods can also be applied to the problem (1.1)–(1.4).

Remark 2.2. In [31,33], instead of $(2.2)_1$, it was shown that

$$\rho \in L^\infty(0,T; H^2), \quad \rho_t \in L^\infty(0,T; H^1).$$

However, one can use [26, Lemma 2.3] to derive $(2.2)_1$ by standard arguments (see [3] for details). Moreover, one can also obtain $(2.2)_3$ by standard arguments due to $(2.2)_1$, $(2.2)_2$, and (2.3).

The following Poincaré–Sobolev and Brezis–Wainger inequalities will be used frequently.

Lemma 2.2. *(See [22,2,9].) There exists a positive constant C depending only on $\mathbb{T}^2$ such that every function $u \in H^1(\mathbb{T}^2)$ satisfies for $2 < p < \infty$,*

$$\|u - \bar{u}\|_{L^p} \leq Cp^{1/2} \|u - \bar{u}\|_{L^2}^{2/p} \|\nabla u\|_{L^2}^{1-2/p}, \ \|u\|_{L^p} \leq Cp^{1/2} \|u\|_{L^2}^{2/p} \|u\|_{H^1}^{1-2/p}. \tag{2.4}$$

Moreover, for $q > 2$, there exists some positive constant C depending only on q and $\mathbb{T}^2$ such that every function $v \in W^{1,q}(\mathbb{T}^2)$ satisfies

$$\|v\|_{L^\infty} \leq C\|\nabla v\|_{L^2} \ln^{1/2}(e + \|\nabla v\|_{L^q}) + C\|v\|_{L^2} + C. \tag{2.5}$$

The following Poincaré type inequality can be found in [11, Lemma 3.2].

Lemma 2.3. *Let $v \in H^1(\mathbb{T}^2)$, and let ρ be a non-negative function such that*

$$0 < M_1 \leq \int_{\mathbb{T}^2} \rho dx, \quad \int_{\mathbb{T}^2} \rho^\gamma dx \leq M_2,$$

with $\gamma > 1$. Then there is a constant C depending solely on M_1, M_2, and γ such that

$$\|v\|_{L^2(\mathbb{T}^2)}^2 \leq C \int_{\mathbb{T}^2} \rho v^2 dx + C\|\nabla v\|_{L^2(\mathbb{T}^2)}^2. \tag{2.6}$$

Then, we state the following Beale–Kato–Majda type inequality which was proved in [21] (see also [1]) when $\mathrm{div} u \equiv 0$ and will be used later to estimate $\|\nabla u\|_{L^\infty}$ and $\|\nabla\rho\|_{L^p}$.

Lemma 2.4. *(See [1,21,18].) For $2 < q < \infty$, there is a constant $C(q)$ such that the following estimate holds for all $\nabla u \in W^{1,q}(\mathbb{T}^2)$,*

$$\|\nabla u\|_{L^\infty} \le C\left(\|\mathrm{div} u\|_{L^\infty} + \|\mathrm{rot} u\|_{L^\infty}\right)\log(e + \|\nabla^2 u\|_{L^q}) + C\|\nabla u\|_{L^2} + C.$$

Next, let $\triangle^{-1}$ denote the Laplacian inverse with zero mean on $\mathbb{T}^2$ and $R_i = (-\triangle)^{-1/2}\partial_i$ be the usual Riesz transform on $\mathbb{T}^2$. Moreover, let $\mathcal{H}^1(\mathbb{T}^2)$ and $\mathcal{BMO}(\mathbb{T}^2)$ stand for the usual Hardy and BMO spaces:

$$\mathcal{H}^1 = \{f \in L^1(\mathbb{T}^2) : \|f\|_{\mathcal{H}^1} = \|f\|_{L^1} + \|R_1 f\|_{L^1} + \|R_2 f\|_{L^1} < \infty, \quad \bar{f} = 0\},$$
$$\mathcal{BMO} = \{f \in L^1_{loc}(\mathbb{T}^2) : \|f\|_{\mathcal{BMO}} < \infty\}$$

with

$$\|f\|_{\mathcal{BMO}} = \sup_{x\in\mathbb{T}^2, r\in(0,d)} \frac{1}{|\Omega_r(x)|}\int_{\Omega_r(x)} \left| f(y) - \frac{1}{|\Omega_r(x)|}\int_{\Omega_r(x)} f(z)dz \right| dy,$$

where d is the diameter of $\mathbb{T}^2$, $\Omega_r(x) = \mathbb{T}^2 \cap B_r(x)$, and $B_r(x)$ is a ball with center x and radius r. Consider the composition of two Riesz transforms, $R_i \circ R_j (i,j = 1,2)$. There is a representation of this operator as a singular integral

$$R_i \circ R_j(f)(x) = \text{p.v.}\int K_{ij}(x-y)f(y)dy,$$

where the kernel $K_{ij}(x)(i,j = 1,2)$ has a singularity of the second order at 0 and

$$|K_{ij}(x)| \le C|x|^{-2}, \ x \in \mathbb{T}^2.$$

Given a function b, define the linear operator

$$[b, R_iR_j](f) \triangleq bR_i \circ R_j(f) - R_i \circ R_j(bf), \ i,j = 1,2.$$

This operator can be written as a convolution with the singular kernel K_{ij},

$$[b, R_iR_j](f)(x) \triangleq \text{p.v.}\int K_{ij}(x-y)(b(x)-b(y))f(y)dy, \ i,j = 1,2.$$

The following properties of the commutator $[b, R_iR_j](f)$ will be useful for our analysis.

Lemma 2.5. *Let $b, f \in C^\infty(\mathbb{T}^2)$. Then for $p \in (1,\infty)$, there is $C(p)$ such that*

$$\|[b, R_iR_j](f)\|_{L^p} \le C(p)\|b\|_{\mathcal{BMO}}\|f\|_{L^p}. \tag{2.7}$$

Moreover, for $q_i \in (1,\infty)(i = 1,2,3)$ with $q_1^{-1} = q_2^{-1} + q_3^{-1}$, there is a positive constant C depending only on $q_i(i = 1,2,3)$ such that

$$\|\nabla[b, R_iR_j](f)\|_{L^{q_1}} \le C\|\nabla b\|_{L^{q_2}}\|f\|_{L^{q_3}}. \tag{2.8}$$

Remark 2.3. Properties (2.7) and (2.8) are due to Coifman–Rochberg–Weiss [5] and Coifman–Meyer [6] respectively.

Finally, the following Zlotnik inequality will be used to get the time-independent upper bound of the density ρ.

Lemma 2.6. *(See [35].) Let the function $y \in W^{1,1}(0,T)$ satisfy*

$$y'(t) = g(y) + h'(t) \text{ on } [0,T], \quad y(0) = y^0$$

with $g \in C(R)$ and $h \in W^{1,1}(0,T)$. If $g(\infty) = -\infty$ and

$$h(t_2) - h(t_1) \le N_0 + N_1(t_2 - t_1) \tag{2.9}$$

for all $0 \le t_1 < t_2 \le T$ with some $N_0 \ge 0$ and $N_1 \ge 0$, then

$$y(t) \le \max\left\{y^0, \tilde{\zeta}\right\} + N_0 < \infty \text{ on } [0,T],$$

where $\tilde{\zeta}$ is a constant such that

$$g(\zeta) \le -N_1 \quad \text{for} \quad \zeta \ge \tilde{\zeta}.$$

3. A priori estimates (I): upper bound of the density

In this section and the next, we will always assume that (ρ_0, m_0) satisfies (2.1) and (ρ, u) is the strong solution to (1.1)–(1.4) on $\mathbb{T}^2 \times (0,T]$ obtained by Lemma 2.1. Moreover, without loss of generality, we assume that

$$\int \rho_0 dx = 1.$$

3.1. Time-independent upper bound of the density

We denote

$$\nabla^\perp \triangleq (\partial_2, -\partial_1), \quad \frac{\mathrm{D}}{\mathrm{D}t} f = \dot{f} \triangleq f_t + u \cdot \nabla f,$$

where $\frac{\mathrm{D}}{\mathrm{D}t} f$ is the material derivative of f. Let G and ω be the effective viscous flux and the vorticity respectively as follows:

$$G \triangleq (2\mu + \lambda(\rho))\mathrm{div} u - (P - \bar{P}), \quad \omega \triangleq \nabla^\perp \cdot u = \partial_2 u_1 - \partial_1 u_2. \tag{3.1}$$

We thus define

$$(A_1(t))^2 \triangleq \int_{\mathbb{T}^2} \left((\omega(t))^2 + \frac{(G(t))^2}{2\mu + \lambda(\rho(t))} \right) dx, \tag{3.2}$$

$$(A_2(t))^2 \triangleq \int_{\mathbb{T}^2} \rho(t) |\dot{u}(t)|^2 dx, \tag{3.3}$$

$$(A_3(t))^2 \triangleq \int_{\mathbb{T}^2} \left((2\mu + \lambda(\rho(t)))(\mathrm{div} u(t))^2 + \mu(\omega(t))^2 \right) dx, \tag{3.4}$$

and

$$R_T \triangleq \sup_{0 \le t \le T} \|\rho(\cdot,t)\|_{L^\infty}. \tag{3.5}$$

Since $\rho \geq 0$ due to (2.3), standard calculations give

$$\mu\|\nabla u\|_{L^2}^2 \leq A_3^2(t) \leq \left(2\mu + R_T^\beta\right)\|\nabla u\|_{L^2}^2. \tag{3.6}$$

Then we have the following estimate on the upper bound of $\log(e + A_1^2(t) + A_3^2(t))$ in terms of R_T which will play an important role in obtaining the upper bound of the density.

Lemma 3.1. *For any $\alpha \in (0,1)$, there is a constant $C(\alpha)$ depending only on $\alpha, \mu, \beta, \gamma$, $\|\rho_0\|_{L^\infty}$, and $\|u_0\|_{H^1}$ such that*

$$\sup_{0\leq t\leq T} \log(e + A_1^2(t) + A_3^2(t)) + \int_0^T \frac{A_2^2(t)}{e + A_1^2(t)} dt \leq C(\alpha) R_T^{1+\kappa+\alpha\beta}, \tag{3.7}$$

with

$$\kappa = \max\{0,\ \gamma - 2\beta,\ \beta - \gamma - 2\}. \tag{3.8}$$

Since Lemma 3.1 is a direct consequence of the following Lemma 3.2, we will postpone its proof until we complete that of Lemma 3.2.

Lemma 3.2. *For any $\alpha \in (0,1)$, there is a positive constant $C(\alpha)$ depending only on $\alpha, \mu, \beta, \gamma, \|\rho_0\|_{L^\infty}$, and $\|u_0\|_{H^1}$ such that*

$$\frac{d}{dt} A_1^2(t) + A_2^2(t) \leq C(\alpha)\left(R_T \varphi_\alpha^2 + \|\rho\|_{L^\beta}^{\beta/2} \varphi_\alpha\right) A_3^2, \tag{3.9}$$

where φ_α is defined by

$$\varphi_\alpha(t) \triangleq 1 + \left\|P(2\mu + \lambda)^{-1}\right\|_{L^2} + A_1 R_T^{\alpha\beta/2}. \tag{3.10}$$

Proof. First, the mass conservation equation $(1.1)_1$ leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \rho dx = 0,$$

which directly gives

$$R_T \geq \|\rho(\cdot,t)\|_{L^\infty} \geq \int \rho(x,t)dx = \int \rho_0 dx = 1. \tag{3.11}$$

Moreover, the standard energy inequality reads:

$$\sup_{0\leq t\leq T} \int \left(\rho|u|^2 + \rho^\gamma\right) dx + \int_0^T A_3^2(t)dt \leq C, \tag{3.12}$$

which together with (2.6) and (3.6) yields that for $t \in [0,T]$,

$$\|u\|_{H^1} \leq C + C\|\nabla u\|_{L^2} \leq C + CA_3. \tag{3.13}$$

Next, direct calculations show that

$$\nabla^{\perp}\cdot\dot u=\frac{\mathrm{D}}{\mathrm{D}t}\omega-(\partial_1 u\cdot\nabla)u_2+(\partial_2 u\cdot\nabla)u_1=\frac{\mathrm{D}}{\mathrm{D}t}\omega+\omega\mathrm{div}u, \tag{3.14}$$

and that

$$\begin{aligned}\mathrm{div}\dot u&=\frac{\mathrm{D}}{\mathrm{D}t}\mathrm{div}u+(\partial_1 u\cdot\nabla)u_1+(\partial_2 u\cdot\nabla)u_2\\&=\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{G}{2\mu+\lambda}\right)+\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{P-\bar P}{2\mu+\lambda}\right)-2\nabla u_1\cdot\nabla^{\perp}u_2+(\mathrm{div}u)^2.\end{aligned} \tag{3.15}$$

Then, we rewrite the momentum equations as

$$\rho\dot u=\nabla G+\mu\nabla^{\perp}\omega. \tag{3.16}$$

Multiplying (3.16) by $2\dot u$ and integrating the resulting equality over $\mathbb{T}^2$, we obtain after using (3.14) and (3.15) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}A_1^2+2A_2^2=&-\int\omega^2\mathrm{div}udx+4\int G\nabla u_1\cdot\nabla^{\perp}u_2dx\\&-2\int G(\mathrm{div}u)^2dx-\int\frac{(\beta-1)\lambda-2\mu}{(2\mu+\lambda)^2}G^2\mathrm{div}udx\\&+2\beta\int\frac{\lambda(P-\bar P)}{(2\mu+\lambda)^2}G\mathrm{div}udx-2\gamma\int\frac{P}{2\mu+\lambda}G\mathrm{div}udx\\&+2(\gamma-1)\int P\mathrm{div}udx\int\frac{G}{2\mu+\lambda}dx\triangleq\sum_{i=1}^{7}I_i.\end{aligned} \tag{3.17}$$

Each I_i can be estimated as follows:

First, it follows from (2.4) that

$$\begin{aligned}\|\omega\|_{L^4}&\le C\|\omega\|_{L^2}^{1/2}\|\nabla\omega\|_{L^2}^{1/2}\\&\le C\varphi_\alpha^{1/2}\|\omega\|_{H^1}^{1/2},\end{aligned} \tag{3.18}$$

for φ_α as in (3.10). Combining this, (3.6), and the Hölder inequality leads to

$$\begin{aligned}|I_1|&\le C\|\omega\|_{L^4}^2\|\mathrm{div}u\|_{L^2}\\&\le CA_3\|\omega\|_{H^1}\varphi_\alpha.\end{aligned} \tag{3.19}$$

Next, we will use an idea due to [8] (see also [30]) to estimate I_2. Noticing that

$$\mathrm{rot}\nabla u_1=0,\quad \mathrm{div}\nabla^{\perp}u_2=0,$$

one thus derives from [7, Theorem II.1] that

$$\|\nabla u_1\cdot\nabla^{\perp}u_2\|_{\mathcal{H}^1}\le C\|\nabla u\|_{L^2}^2.$$

This combined with the fact that $\mathcal{BMO}$ is the dual space of $\mathcal{H}^1$ (see [10]) gives

$$\begin{aligned}|I_2|&\le C\|G\|_{\mathcal{BMO}}\|\nabla u_1\cdot\nabla^{\perp}u_2\|_{\mathcal{H}^1}\\&\le C\|\nabla G\|_{L^2}\|\nabla u\|_{L^2}^2\\&\le CA_3\|G\|_{H^1}\varphi_\alpha,\end{aligned} \tag{3.20}$$

where in the last inequality, we have used (3.6) and the following simple fact:

$$\begin{aligned}\|\nabla u\|_{L^2} &\le C\|\omega\|_{L^2} + C\|\mathrm{div}u\|_{L^2} \\ &\le C\|\omega\|_{L^2} + C\left\|\frac{G}{2\mu+\lambda}\right\|_{L^2} + C\left\|\frac{P-\bar P}{2\mu+\lambda}\right\|_{L^2} \le C\varphi_\alpha.\end{aligned}$$

Next, the Hölder inequality yields that for $0<\alpha<1$,

$$\begin{aligned}\sum_{i=3}^{6}|I_i| &\le C\int\frac{G^2|\mathrm{div}u|}{2\mu+\lambda}dx + C\int\frac{P+\bar P}{2\mu+\lambda}|G||\mathrm{div}u|dx \\ &\le CA_3\left\|\frac{G^2}{2\mu+\lambda}\right\|_{L^2} + CA_3\|G\|_{L^{2+4\gamma/\beta}}\left\|\frac{P+1}{(2\mu+\lambda)^{3/2}}\right\|_{L^{2+\beta/\gamma}} \\ &\le C(\alpha)A_3\|G\|_{H^1}\varphi_\alpha, \end{aligned}\tag{3.21}$$

where in the last inequality we have used the following simple fact:

$$\begin{aligned}\left\|\frac{G^2}{\sqrt{2\mu+\lambda}}\right\|_{L^2} &\le C\left\|\frac{G}{\sqrt{2\mu+\lambda}}\right\|_{L^2}^{1-\alpha}\|G\|_{L^{2(1+\alpha)/\alpha}}^{1+\alpha} \\ &\le C(\alpha)A_1^{1-\alpha}\|G\|_{L^2}^{\alpha}\|G\|_{H^1} \\ &\le C(\alpha)A_1R_T^{\alpha\beta/2}\|G\|_{H^1} \\ &\le C(\alpha)\|G\|_{H^1}\varphi_\alpha, \end{aligned}\tag{3.22}$$

due to (2.4) and $\|G\|_{L^2}\le CR_T^{\beta/2}A_1$.

Next, it follows from (3.1), (3.12), and the Hölder inequality that

$$\begin{aligned}|I_7| &= 2(\gamma-1)\left|\int(P-\overline{P})\mathrm{div}u dx\int\left(\mathrm{div}u-\frac{P-\overline{P}}{2\mu+\lambda}\right)dx\right| \\ &= 2(\gamma-1)\left|\int(G-(2\mu+\lambda)\mathrm{div}u)\mathrm{div}u dx\int\frac{P-\overline{P}}{2\mu+\lambda}dx\right| \\ &\le CA_3\|G\|_{L^2} + CA_3^2. \end{aligned}\tag{3.23}$$

Substituting (3.19)–(3.21) and (3.23) into (3.17), we obtain that for any $\alpha\in(0,1)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}A_1^2(t) + 2A_2^2(t) \le C(\alpha)A_3\left(\|G\|_{H^1}+\|\omega\|_{H^1}\right)\varphi_\alpha + CA_3^2. \tag{3.24}$$

Finally, it follows from (3.16) that

$$\triangle G = \mathrm{div}(\rho\dot u), \quad \mu\triangle\omega = \nabla^\perp\cdot(\rho\dot u) \tag{3.25}$$

which together with the standard L^p-estimate of elliptic equations implies that for $p\in(1,\infty)$,

$$\|\nabla G\|_{L^p} + \|\nabla\omega\|_{L^p} \le C(p,\mu)\|\rho\dot u\|_{L^p}. \tag{3.26}$$

This combined with the Poincaré–Sobolev inequality gives

$$\|\omega\|_{H^1}+\|G\|_{H^1}\le C\|\nabla\omega\|_{L^2}+C\|G-\bar G\|_{L^2}+C|\bar G|+C\|\nabla G\|_{L^2}$$
$$\le CR_T^{1/2}A_2+C\|\rho\|_{L^\beta}^{\beta/2}A_3, \tag{3.27}$$

where in the second inequality we have used the following simple fact:

$$|\bar G|\le C\|\rho\|_{L^\beta}^{\beta/2}A_3. \tag{3.28}$$

Putting (3.27) into (3.24), we directly obtain (3.9) after using Cauchy's inequality. The proof of Lemma 3.2 is finished. □

Now, we can use Lemma 3.2 to prove Lemma 3.1 as follows:

Proof of Lemma 3.1. It follows from (3.10) and (3.12) that

$$\varphi_\alpha(t)\le C+CA_1R_T^{\alpha\beta/2}+CR_T^{(\gamma-2\beta)/2}, \tag{3.29}$$

which together with (3.9) and (3.12) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}A_1^2(t)+A_2^2(t)\le C(\alpha)R_T\left(\varphi_\alpha^2+\|\rho\|_{L^\beta}^{\beta}R_T^{-2}\right)A_3^2$$
$$\le C(\alpha)R_T\left(1+R_T^{\alpha\beta}A_1^2+R_T^{\gamma-2\beta}+R_T^{\beta-\gamma-2}\right)A_3^2. \tag{3.30}$$

Dividing (3.30) by $e+A_1^2(t)$ and integrating the resulting inequality over $(0,T)$, we obtain (3.7) after using (3.11), (3.12), and the following simple fact that

$$CA_3^2(t)-C-CR_T^{\gamma-\beta}\le A_1^2(t)\le CA_3^2(t)+C+CR_T^{\gamma-\beta}, \tag{3.31}$$

due to (3.2), (3.4), and (3.12). We thus finish the proof of Lemma 3.1. □

With Lemma 3.1 at hand, to obtain the time-independent upper bound of the density, we still need the following key estimate on the L^∞-norm of the commutator F defined by

$$F\triangleq\sum_{i,j=1}^{2}[u_i,R_iR_j](\rho u_j). \tag{3.32}$$

Lemma 3.3. *For any $\varepsilon>0$, there is a positive constant $C(\varepsilon)$ depending only on ε, μ, β, γ, $\|\rho_0\|_{L^\infty}$, and $\|u_0\|_{H^1}$ such that*

$$\|F\|_{L^\infty}\le\frac{C(\varepsilon)R_T^{-1-\kappa}A_2^2}{e+A_1^2}+C(\varepsilon)A_3^2R_T^{(3+\kappa)/2+\varepsilon}+C(\varepsilon)R_T^{1+\varepsilon}, \tag{3.33}$$

with κ as in (3.8).

Proof. First, we deduce from (2.7) and (3.13) that

$$\|F\|_{L^q}\le C(q)\|\nabla u\|_{L^2}\|\rho u\|_{L^q}\le C(q)R_T(A_3^2+1),$$

which together with the Gagliardo–Nirenberg inequality and (2.8) thus gives that for $q\in(8,\infty)$,

$$
\begin{aligned}
\|F\|_{L^\infty} &\le C(q)\|F\|_{L^q} + C(q)\|F\|_{L^q}^{1-4/q}\|\nabla F\|_{L^{4q/(q+4)}}^{4/q} \\
&\le C(q)R_T(A_3^2+1) + C(q)\|\nabla u\|_{L^2}^{1-4/q}\|\rho u\|_{L^q}^{1-4/q}\|\nabla u\|_{L^4}^{4/q}\|\rho u\|_{L^q}^{4/q} \\
&\le C(q)R_T(A_3^2+1) + C(q)A_3^{1-4/q}\|\nabla u\|_{L^4}^{4/q}\|\rho u\|_{L^q}. \qquad (3.34)
\end{aligned}
$$

Next, noticing that (3.31) shows

$$
e + A_1 \le C R_T^{\max\{0,(\gamma-\beta)/2\}}(e + A_3),
$$

we obtain from (3.18), (3.22), (3.27), and (3.29) that

$$
\begin{aligned}
\|\nabla u\|_{L^4} &\le C(\|\mathrm{div} u\|_4 + \|\omega\|_4) \\
&\le C\left\|\frac{G^2}{\sqrt{2\mu+\lambda}}\right\|_{L^2}^{1/2} + C\left\|\frac{P-\bar P}{2\mu+\lambda}\right\|_{L^4} + C\varphi_\alpha^{1/2}\|\omega\|_{H^1}^{1/2} \\
&\le C(\alpha)\left(\|G\|_{H^1} + \|\omega\|_{H^1}\right)^{1/2}\varphi_\alpha^{1/2} + CR_T^{(3\gamma-4\beta)/4} \\
&\le CR_T^{\tilde C}(e+A_3)\left(\frac{R_T^{-4-\kappa}A_2^2}{e+A_1^2}\right)^{1/4} + CR_T^{\tilde C}(e+A_3), \qquad (3.35)
\end{aligned}
$$

for some constant $\tilde C > 1$ depending only on β and γ. Combining this and (3.7) implies that for $\alpha \in (0,1)$,

$$
\log\left(e + \|\nabla u\|_{L^4}\right) \le C(\alpha)R_T^{1+\kappa+\alpha\beta} + \frac{CR_T^{-4-\kappa}A_2^2}{(e+A_1^2)(e+A_3)^6},
$$

which together with (2.5) and (3.13) gives that for $\alpha \in (0,1)$,

$$
\begin{aligned}
\|u\|_{L^\infty} &\le C\|\nabla u\|_{L^2}\log^{1/2}\left(e + \|\nabla u\|_{L^4}\right) + C\|\nabla u\|_{L^2} + C \\
&\le C(\alpha)A_3R_T^{(1+\kappa+\alpha\beta)/2} + C\left(\frac{R_T^{-4-\kappa}A_2^2}{(e+A_1^2)(e+A_3)^4}\right)^{1/2} + C. \qquad (3.36)
\end{aligned}
$$

It thus follows from the Hölder inequality, (3.36), and (3.12) that for $\alpha \in (0,1)$ and $q \in (8,\infty)$,

$$
\begin{aligned}
\|\rho u\|_{L^q} &\le CR_T^{1-1/q}\|\rho^{1/2}u\|_{L^2}^{2/q}\|u\|_{L^\infty}^{1-2/q} \\
&\le C(\alpha)A_3^{1-2/q}R_T^{(3+\kappa+\alpha\beta)/2} + C\left(\frac{R_T^{-1-\kappa}A_2^2}{(e+A_1^2)(e+A_3)^4}\right)^{1/2-1/q} + CR_T. \qquad (3.37)
\end{aligned}
$$

Putting (3.37) and (3.35) into (3.34) yields that for $\alpha \in (0,1)$ and $q \in (8,\infty)$,

$$\begin{aligned}
\|F\|_{L^\infty} \le\, & C(q)R_T(A_3^2+1)+C(q,\alpha)\|\nabla u\|_{L^4}^{4/q}A_3^{(2q-6)/q}R_T^{(3+\kappa+\alpha\beta)/2}\\
&+C(q,\alpha)A_3^{(q-4)/q}\|\nabla u\|_{L^4}^{4/q}\left(\frac{R_T^{-1-\kappa}A_2^2}{(e+A_1^2)(e+A_3)^4}\right)^{1/2-1/q}\\
&+C(q)R_TA_3^{1-4/q}\|\nabla u\|_{L^4}^{4/q}\\
\le\, & C(q)R_T(A_3^2+1)+C(q,\alpha)A_3^{2-6/q}(e+A_3)^{4/q}R_T^{(3+\kappa+\alpha\beta)/2+4\tilde C/q}\\
&+C(q,\alpha)A_3^{2-6/q}(e+A_3)^{4/q}\left(\frac{R_T^{-1-\kappa}A_2^2}{e+A_1^2}\right)^{1/q}R_T^{(3+\kappa+\alpha\beta)/2+4\tilde C/q}\\
&+C(q,\alpha)\|\nabla u\|_{L^4}^{4/q}\left(\frac{R_T^{-1-\kappa}A_2^2}{e+A_1^2}\right)^{1/2-1/q}\\
&+C(q)R_T\left(A_3+1\right)\|\nabla u\|_{L^4}^{4/q}\triangleq C(q)R_T(A_3^2+1)+\sum_{i=1}^4 J_i.
\end{aligned}\tag{3.38}$$

Then, on the one hand, the Hölder inequality implies that

$$\begin{aligned}
|J_1|+|J_2| \le\, & C(q,\alpha)\left(R_T^{(3+\kappa+\alpha\beta)/2+4\tilde C/q}A_3^{2-6/q}\right)^{q/(q-3)}\\
&+C(q,\alpha)(e+A_3^2)+\frac{R_T^{-1-\kappa}A_2^2}{e+A_1^2}\\
\le\, & C(q,\alpha)+C(q,\alpha)A_3^2R_T^{\tilde\kappa(\alpha,q)}+\frac{R_T^{-1-\kappa}A_2^2}{e+A_1^2},
\end{aligned}\tag{3.39}$$

with

$$\tilde\kappa(\alpha,q)\triangleq\left(\frac{3}{2}+\frac{\kappa}{2}+\frac{\alpha\beta}{2}+\frac{4\tilde C}{q}\right)\frac{q}{q-3}.$$

On the other hand, the Hölder inequality gives

$$\begin{aligned}
|J_3|+|J_4| \le\, & CR_T\|\nabla u\|_{L^4}^{8/q}+\frac{R_T^{-1-\kappa}A_2^2}{e+A_1^2}+C(q,\alpha)R_T+C(q)R_TA_3^2\\
\le\, & C(\alpha,q)R_T^{1+16\tilde C/q}+C(\alpha,q)R_T^{1+16\tilde C/q}A_3^2+\frac{CR_T^{-1-\kappa}A_2^2}{e+A_1^2},
\end{aligned}\tag{3.40}$$

where in the second inequality we have used:

$$\|\nabla u\|_{L^4}^{8/q}\le C(q)R_T^{16\tilde C/q}(A_3^2+1)+\frac{R_T^{-4-\kappa}A_2^2}{e+A_1^2},$$

due to (3.35). Thus, putting (3.39) and (3.40) into (3.38), we obtain (3.33) after choosing q suitably large and then α suitably small. The proof of Lemma 3.3 is completed. □

Now, we are in a position to state the main result of this subsection, that is, we will establish the following time-independent upper bound of the density provided (1.11) holds.

Proposition 3.4. *If (1.11) holds, there is a positive constant C depending only on $\mu,\beta,\gamma,\|\rho_0\|_{L^\infty}$, and $\|u_0\|_{H^1}$ such that*

$$\sup_{0\le t\le T}(\|\rho\|_{L^\infty}+\|u\|_{H^1})+\int_0^T\left(\|\omega\|_{H^1}^2+\|G\|_{H^1}^2+A_2^2(t)\right)dt\le C. \tag{3.41}$$

Proof. It follows from (3.16) that G solves

$$\triangle G=\partial_t(\mathrm{div}(\rho u))+\mathrm{divdiv}(\rho u\otimes u),$$

which implies

$$G-\bar G+\frac{\mathrm{D}}{\mathrm{D}t}\left((-\triangle)^{-1}\mathrm{div}(\rho u)\right)=F,$$

with F as in (3.32). This combined with the mass equation $(1.1)_1$ leads to

$$\frac{\mathrm{D}}{\mathrm{D}t}\theta(\rho)+P=\frac{\mathrm{D}}{\mathrm{D}t}\psi+\bar P-\bar G-F, \tag{3.42}$$

with

$$\theta(\rho)\triangleq 2\mu\log\rho+\beta^{-1}\rho^\beta,\quad \psi\triangleq(-\triangle)^{-1}\mathrm{div}(\rho u).$$

Since the function $y=\theta(\rho)$ is strictly increasing for $\rho\in(0,\infty)$, the inverse function $\rho=\theta^{-1}(y)$ exists for $y\in(-\infty,\infty)$. We rewrite (3.42) as

$$\frac{\mathrm{D}}{\mathrm{D}t}y=g(y)+\frac{\mathrm{D}}{\mathrm{D}t}h,$$

with

$$y=\theta(\rho),\quad g(y)=-P(\theta^{-1}(y)),\quad h=\psi+\int_0^t\left(\bar P-\bar G-F\right)dt. \tag{3.43}$$

To apply Lemma 2.6, noticing that

$$\lim_{y\to\infty}g(y)=-\infty,$$

we need to estimate h. First, it follows from (2.5) and (3.13) that

$$\begin{aligned}\|\psi\|_{L^\infty}&\le C\|\nabla\psi\|_{L^2}\log^{1/2}(e+\|\nabla\psi\|_{L^3})+C\|\psi\|_{L^2}+C\\&\le C\|\rho u\|_{L^2}\log^{1/2}(e+\|\rho u\|_{L^3})+C\|\rho u\|_{L^{2\gamma/(\gamma+1)}}+C\\&\le CR_T^{1/2}\log^{1/2}(e+R_T(1+\|\nabla u\|_{L^2}))+C\\&\le CR_T^{1/2}\log^{1/2}(e+A_3^2)+CR_T,\end{aligned} \tag{3.44}$$

where in the third inequality we have used

$$\|\rho u\|_{L^{2\gamma/(\gamma+1)}}\le\|\rho\|_{L^\gamma}^{1/2}\|\rho^{1/2}u\|_{L^2}\le C,$$

due to (3.12). Combining (3.44) and (3.7) gives that for κ as in (3.8)

$$\sup_{0\le t\le T}\|\psi\|_{L^\infty}\le CR_T^{(3+\kappa)/2}. \tag{3.45}$$

Next, on the one hand, (3.12) and (3.28) lead to

$$|\bar{P}-\bar{G}| \le C + CR_T^{\max\{\beta-\gamma,0\}}A_3^2(t). \tag{3.46}$$

On the other hand, one deduces from (3.33), (3.7), and (3.12) that for any $\varepsilon > 0$ and all $0 \le t_1 \le t_2 \le T$

$$\int_{t_1}^{t_2} \|F\|_{L^\infty} ds \le C(\varepsilon)R_T^{(3+\kappa)/2+\varepsilon} + C(\varepsilon)R_T^{1+\varepsilon}(t_2-t_1).$$

This combined with (3.45) and (3.46) implies that for all $0 \le t_1 \le t_2 \le T$ and any $\varepsilon > 0$,

$$|h(t_2)-h(t_1)| \le C(\varepsilon)R_T^{\max\{(3+\kappa)/2+\varepsilon,\beta-\gamma\}} + C(\varepsilon)R_T^{1+\varepsilon}(t_2-t_1).$$

Therefore, one can choose N_0 and N_1 in (2.9) as:

$$N_0 = C(\varepsilon)R_T^{\max\{(3+\kappa)/2+\varepsilon,\beta-\gamma\}}, \quad N_1 = C(\varepsilon)R_T^{1+\varepsilon}.$$

For $g(y)$ as in (3.43), we have

$$g(\zeta) = -(\theta^{-1}(\zeta))^\gamma \le -N_1 = -C(\varepsilon)R_T^{1+\varepsilon},$$

for all $\zeta \ge \tilde{\zeta} \triangleq C(\varepsilon)R_T^{\beta(1+\varepsilon)/\gamma}$. Lemma 2.6 thus yields that

$$R_T^\beta \le C(\varepsilon)R_T^{\max\{(3+\kappa)/2+\varepsilon,\ \beta-\gamma,\ (1+\varepsilon)\beta/\gamma\}},$$

which together with (1.11) and (3.8) gives

$$\sup_{0\le t\le T} \|\rho\|_{L^\infty} \le C.$$

This combined with (3.30), (3.12), (3.13), (3.27), and Gronwall's inequality yields (3.41) and finishes the proof of Proposition 3.4. □

3.2. Time-dependent upper bound of the density

If (1.11) fails, we still have the following Proposition 3.5 which gives a time-dependent upper bound of the density provided (1.8) holds.

Proposition 3.5. *Assume that (1.8) holds. Then there is a positive constant $C(T)$ depending only on T, μ, β, $\gamma, \|\rho_0\|_{L^\infty}$, and $\|u_0\|_{H^1}$ such that*

$$\sup_{0\le t\le T} (\|\rho\|_{L^\infty} + \|u\|_{H^1}) + \int_0^T \left(\|\omega\|_{H^1}^2 + \|G\|_{H^1}^2 + A_2^2(t)\right) dt \le C(T). \tag{3.47}$$

Before proving Proposition 3.5, we establish some a priori estimates, Lemmas 3.7 and 3.8.

We first state the estimate on the L^p-norm of the density due to Vaigant–Kazhikhov [34].

Lemma 3.6. *(See [34].) Let $\beta > 1$. For any $1 \le p < \infty$, there is a positive constant $C(T)$ depending only on $T, \mu, \beta, \gamma, \|\rho_0\|_{L^\infty}$, and $\|u_0\|_{H^1}$ such that*

$$\sup_{0\le t\le T} \|\rho(\cdot,t)\|_{L^p} \le C(T)p^{\frac{2}{\beta-1}}. \tag{3.48}$$

Then Lemma 3.6 directly yields the following L^p-estimate of the momentum which plays an important role in obtaining the upper bound of the density.

Lemma 3.7. *Let $\beta > 1$. For any $q > 4$, there is a positive constant $C(q,T)$ depending only on $q, T, \mu, \beta, \gamma, \|\rho_0\|_{L^\infty}$, and $\|u_0\|_{H^1}$ such that*

$$\|\rho u\|_{L^q} \le C(q,T) R_T^{1+\beta(q-2)/(4q)} (1 + A_3)^{1-2/q}. \tag{3.49}$$

Proof. First, we claim that there is a positive constant $\nu_0 \le 1/2$ depending only on μ such that

$$\sup_{0\le t\le T} \int \rho |u|^{2+\nu} dx \le C(T), \tag{3.50}$$

with

$$\nu \triangleq R_T^{-\beta/2} \nu_0 \in (0, 1/2].$$

In fact, multiplying $(1.1)_2$ by $(2+\nu)|u|^\nu u$, we get after integrating the resulting equation over $\mathbb{T}^2$ that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \int \rho |u|^{2+\nu} dx + (2+\nu) \int |u|^\nu \left(\mu |\nabla u|^2 + (\mu + \rho^\beta)(\mathrm{div} u)^2\right) dx \\
&\le (2+\nu)\nu \int (\mu + \rho^\beta) |\mathrm{div} u| |u|^\nu |\nabla u| dx + C \int \rho^\gamma |u|^\nu |\nabla u| dx \\
&\le \frac{2+\nu}{2} \int (\mu + \rho^\beta)(\mathrm{div} u)^2 |u|^\nu dx + \frac{2+\nu}{2} \nu_0^2 (\mu + 1) \int |u|^\nu |\nabla u|^2 dx \\
&\quad + \mu \int |u|^\nu |\nabla u|^2 dx + C \int \rho |u|^{2+\nu} dx + C \int \rho^{(2+\nu)\gamma - \nu/2} dx,
\end{aligned}$$

which, after choosing $\nu_0(\mu)$ suitably small, together with Gronwall's inequality and (3.48) thus gives (3.50).

Then, since $q > 4$, we have $r \triangleq (q-2)(2+\nu)/\nu > 2$. It follows from the Hölder inequality, (3.50), (2.4), and (3.13) that

$$\begin{aligned}
\|\rho u\|_{L^q} &\le C \|\rho u\|_{L^{2+\nu}}^{2/q} \|\rho u\|_{L^r}^{1-2/q} \\
&\le C(T) R_T \|u\|_{L^r}^{1-2/q} \\
&\le C(T) R_T \left(r^{1/2} \|u\|_{H^1}\right)^{1-2/q} \\
&\le C(q,T) R_T^{1+\beta(q-2)/(4q)} (1 + \|\nabla u\|_{L^2})^{1-2/q},
\end{aligned}$$

which together with (3.6) shows (3.49) and finishes the proof of Lemma 3.7. □

The next lemma will deal with the time-dependent estimate on the spatial L^∞-norm of the commutator F defined by (3.32).

Lemma 3.8. *Let $\beta > 1$. For any $\varepsilon > 0$, there is a positive constant $C(\varepsilon, T)$ depending only on $\varepsilon, T, \mu, \beta, \gamma$, $\|\rho_0\|_{L^\infty}$, and $\|u_0\|_{H^1}$ such that*

$$\|F\|_{L^\infty} \le \frac{C(\varepsilon,T)A_2^2}{e+A_1^2} + C(\varepsilon,T)(1+A_3^2)R_T^{1+\beta/4+\varepsilon}. \tag{3.51}$$

Proof. It follows from (3.35) and (3.49) that for $q > 8$,

$$\begin{aligned}
&A_3^{(q-4)/q}\|\nabla u\|_{L^4}^{4/q}\|\rho u\|_{L^q} \\
&\le C(q,T)R_T^{1+\beta(q-2)/(4q)}\left(A_3^{2-6/q}+1\right)\|\nabla u\|_{L^4}^{4/q} \\
&\le C(q,T)R_T^{1+\beta(q-2)/(4q)+4\tilde C/q}\left(A_3^{2-2/q}+1\right)\left(\frac{A_2^2}{e+A_1^2}\right)^{1/q} \\
&\quad + C(q,T)R_T^{1+\beta(q-2)/(4q)+4\tilde C/q}\left(A_3^2+1\right) \\
&\le C(q,T)R_T^{1+\beta/4+8\tilde C/(q-1)}(1+A_3^2) + \frac{A_2^2}{e+A_1^2},
\end{aligned}$$

which combined with (3.34) yields that

$$\|F\|_{L^\infty} \le C(q,T)R_T^{1+\beta/4+8\tilde C/(q-1)}(1+A_3^2) + C(q,T)\frac{A_2^2}{e+A_1^2}.$$

This directly gives (3.51) after choosing q suitably large. The proof of Lemma 3.8 is completed. □

Now we will use Lemmas 3.2, 3.7, and 3.8 to prove Proposition 3.5.

Proof of Proposition 3.5. We deduce from (3.10) and (3.48) that for any $\alpha \in (0,1)$,

$$\varphi_\alpha(t) \le C(T,\alpha) + C(T,\alpha)A_1R_T^{\alpha\beta/2},$$

which together with Lemmas 3.2 and 3.6 gives that for any $\alpha \in (0,1)$,

$$\sup_{0\le t\le T}\log(e+A_1^2(t)) + \int_0^T \frac{A_2^2(t)}{e+A_1^2(t)}dt \le C(T,\alpha)R_T^{1+\alpha\beta}. \tag{3.52}$$

Then, it follows from (3.42), (3.46), (3.48), (3.51), and (3.52) that for $\varepsilon \in (0,1)$,

$$R_T^\beta \le C(\varepsilon,T)R_T^{\max\{1+\beta/4+\varepsilon,\,\beta-\gamma,\,4/3\}}, \tag{3.53}$$

where we have used

$$\|\psi\|_{L^\infty} \le C(T)R_T^{4/3},$$

due to (3.44), (3.52), and (3.31). Since $\beta > 4/3$, choosing ε suitably small in (3.53) implies

$$\sup_{0\le t\le T}\|\rho\|_{L^\infty} \le C(T)$$

which combined with (3.30), (3.12), (3.13), (3.27), and Gronwall's inequality gives (3.47) and finishes the proof of Proposition 3.5. □

4. A priori estimates (II): higher order estimates

Lemma 4.1. *Assume that*

$$\sup_{0\le t\le T}\|\rho\|_{L^\infty}\le M, \tag{4.1}$$

for some positive constant M. Then there is a positive constant $C(M)$ depending only on M,μ,β,γ, and $\|u_0\|_{H^1}$ such that

$$\sup_{0\le t\le T}\sigma\int\rho|\dot u|^2dx+\int_0^T\sigma\|\nabla\dot u\|_{L^2}^2dt\le C(M), \tag{4.2}$$

with $\sigma(t)\triangleq\min\{1,t\}$. Moreover, for any $p\in[1,\infty)$, there is a positive constant $C(p,M)$ depending only on p,M,μ,β,γ, $\|\rho_0\|_{L^\infty}$, and $\|u_0\|_{H^1}$ such that

$$\sup_{0\le t\le T}\sigma^{1/2}\|\nabla u\|_{L^p}\le C(p,M). \tag{4.3}$$

Proof. First, it follows from (3.30), (3.12), (3.13), and (4.1) that

$$\sup_{0\le t\le T}\|u\|_{H^1}+\int_0^T\left(\|\nabla u\|_{L^2}^2+\|\rho^{1/2}\dot u\|_{L^2}^2\right)dt\le C(M). \tag{4.4}$$

Next, we will adapt an idea due to Hoff [14] to prove (4.2). Operating $\dot u^j[\partial/\partial t+\mathrm{div}(u\cdot)]$ to $(1.1)_2^j$, summing with respect to j, and integrating the resulting equation over $\mathbb{T}^2$, one obtains after integration by parts that

$$\begin{aligned}&\left(\frac12\int\rho|\dot u|^2dx\right)_t\\&=-\int\dot u_j[\partial_jP_t+\mathrm{div}(\partial_jPu)]dx+\mu\int\dot u_j[\partial_t\triangle u_j+\mathrm{div}(u\triangle u_j)]dx\\&\quad+\int\dot u_j[\partial_{jt}((\mu+\lambda)\mathrm{div}u)+\mathrm{div}(u\partial_j((\mu+\lambda)\mathrm{div}u))]dx\triangleq\sum_{i=1}^3N_i.\end{aligned} \tag{4.5}$$

First, using the equation $(1.1)_1$, we obtain after integration by parts that

$$\begin{aligned}N_1&=-\int\dot u_j[\partial_jP_t+\mathrm{div}(\partial_jPu)]dx\\&=\int[-P'\rho\mathrm{div}u\partial_j\dot u_j+\partial_k(\partial_j\dot u_ju_k)P-P\partial_j(\partial_k\dot u_ju_k)]dx\\&\le C(M)\|\nabla u\|_{L^2}\|\nabla\dot u\|_{L^2}\\&\le\frac{\mu}{8}\|\nabla\dot u\|_{L^2}^2+C(M)\|\nabla u\|_{L^2}^2.\end{aligned} \tag{4.6}$$

Then, integration by parts leads to

$$
\begin{aligned}
N_2 &= \mu \int \dot{u}_j [\partial_t \triangle u_j + \operatorname{div}(u \triangle u_j)] dx \\
&= -\mu \int \left(|\nabla \dot{u}|^2 + \partial_i \dot{u}_j \partial_k u_k \partial_i u_j - \partial_i \dot{u}_j \partial_i u_k \partial_k u_j - \partial_i u_j \partial_i u_k \partial_k \dot{u}_j \right) dx \\
&\le -\frac{3\mu}{4} \int |\nabla \dot{u}|^2 dx + C(M) \int |\nabla u|^4 dx.
\end{aligned} \tag{4.7}
$$

Similarly,

$$
\begin{aligned}
N_3 &= \int \dot{u}_j [\partial_{jt}((\mu+\lambda)\operatorname{div}u) + \operatorname{div}(u \partial_j((\mu+\lambda)\operatorname{div}u))] dx \\
&= -\int \partial_j \dot{u}_j [((\mu+\lambda)\operatorname{div}u)_t + \operatorname{div}(u(\mu+\lambda)\operatorname{div}u)] dx \\
&\quad - \int \dot{u}_j \operatorname{div}(\partial_j u (\mu+\lambda)\operatorname{div}u) dx \\
&= -\int \left(\frac{\mathrm{D}}{\mathrm{D}t}\operatorname{div}u + \partial_j u_i \partial_i u_j \right) \left[(\mu+\lambda) \frac{\mathrm{D}}{\mathrm{D}t}\operatorname{div}u - \rho \lambda'(\rho) \operatorname{div}u \right] dx \\
&\quad + \int \nabla \dot{u}_j \cdot \partial_j u (\mu+\lambda) \operatorname{div}u dx \\
&\le -\frac{\mu}{2} \int \left(\frac{\mathrm{D}}{\mathrm{D}t}\operatorname{div}u \right)^2 dx + \frac{\mu}{8} \|\nabla \dot{u}\|_{L^2}^2 + C(M)\|\nabla u\|_{L^4}^4 + C(M)\|\nabla u\|_{L^2}^2,
\end{aligned} \tag{4.8}
$$

where in the third equality we have used the following simple fact:

$$
\begin{aligned}
&((\mu+\lambda)\operatorname{div}u)_t + \operatorname{div}(u(\mu+\lambda)\operatorname{div}u) \\
&= (\mu+\lambda)\operatorname{div}u_t + (\mu+\lambda)(u\cdot\nabla)\operatorname{div}u + \lambda_t \operatorname{div}u + (u \cdot \nabla\lambda)\operatorname{div}u \\
&= (\mu+\lambda)\frac{\mathrm{D}}{\mathrm{D}t}\operatorname{div}u - \rho\lambda'(\rho)\operatorname{div}u,
\end{aligned}
$$

due to $(1.1)_1$.

Finally, substituting (4.6)–(4.8) into (4.5) shows that

$$
\begin{aligned}
&\left(\int \rho |\dot{u}|^2 dx \right)_t + \mu \int |\nabla \dot{u}|^2 dx + \mu \int \left(\frac{\mathrm{D}}{\mathrm{D}t}\operatorname{div}u \right)^2 dx \\
&\le C(M)|\nabla u\|_{L^4}^4 + C(M)\|\nabla u\|_{L^2}^2 \\
&\le C(M)(\|G\|_{L^4}^4 + \|\omega\|_{L^4}^4 + \|P - \overline{P}\|_{L^4}^4 + \|\nabla u\|_{L^2}^2) \\
&\le C(M)(\|G\|_{L^2}^2 \|G\|_{H^1}^2 + \|\omega\|_{L^2}^2 \|\nabla\omega\|_{L^2}^2 + \|P - \overline{P}\|_{L^2}^2 + \|\nabla u\|_{L^2}^2) \\
&\le C(M)\|\rho^{1/2}\dot{u}\|_{L^2}^2 + C(M)\|\nabla u\|_{L^2}^2,
\end{aligned} \tag{4.9}
$$

where in the last inequality we have used (3.27), (3.6), and (4.4). Multiplying (4.9) by σ and integrating the resulting inequality over $(0, T)$, we obtain (4.2) after using (4.4).

It remains to prove (4.3). Direct calculations show that for $p \geq 2$,

$$\begin{aligned}\|\nabla u\|_{L^p} &\leq C(p)\|\mathrm{div}u\|_{L^p} + C(p)\|\omega\|_{L^p} \\ &\leq C(p)\|G\|_{L^p} + C(p)\|P - \overline{P}\|_{L^p} + C(p)\|\omega\|_{L^p} \\ &\leq C(p)\|G\|_{H^1} + C(p)\|\omega\|_{H^1} + C(p, M) \\ &\leq C(p, M)\|\rho^{1/2}\dot{u}\|_{L^2} + C(p, M),\end{aligned}$$

where in the last inequality we have used (3.27) and (3.6). This combined with (4.2) gives (4.3). We finish the proof of Lemma 4.1. □

Lemma 4.2. *Assume that (1.8) holds. Then for any $p > 2$, there is a positive constant C depending only on p, T, μ, β, γ, $\|\rho_0\|_{L^\infty}$, and $\|u_0\|_{H^1}$ such that*

$$\begin{aligned}&\int_0^T (\|G\|_{L^\infty} + \|\nabla G\|_{L^p} + \|\nabla\omega\|_{L^p} + \|\rho\dot{u}\|_{L^p})^{1+1/p}\, dt \\ &+ \int_0^T t\left(\|\nabla G\|_{L^p}^2 + \|\nabla\omega\|_{L^p}^2 + \|\dot{u}\|_{H^1}^2\right) dt \leq C.\end{aligned} \tag{4.10}$$

Proof. It follows from (2.6), (2.4), and (3.47) that

$$\begin{aligned}\|\rho\dot{u}\|_{L^p} &\leq C\|\rho\dot{u}\|_{L^2}^{2(p-1)/(p^2-2)}\|\dot{u}\|_{L^{p^2}}^{p(p-2)/(p^2-2)} \\ &\leq C\|\rho\dot{u}\|_{L^2}^{2(p-1)/(p^2-2)}\|\dot{u}\|_{H^1}^{p(p-2)/(p^2-2)} \\ &\leq C\|\rho\dot{u}\|_{L^2} + C\|\rho\dot{u}\|_{L^2}^{2(p-1)/(p^2-2)}\|\nabla\dot{u}\|_{L^2}^{p(p-2)/(p^2-2)},\end{aligned}$$

which together with (3.47), (4.2), and (2.6) implies that

$$\begin{aligned}&\int_0^T \left(\|\rho\dot{u}\|_{L^p}^{1+1/p} + t\|\dot{u}\|_{H^1}^2\right) dt \\ &\leq C\int_0^T \left(\|\rho^{1/2}\dot{u}\|_{L^2}^2 + t\|\nabla\dot{u}\|_{L^2}^2 + t^{-1+2/(p^3-p^2-2p+2)}\right) dt \\ &\leq C.\end{aligned} \tag{4.11}$$

Noticing that the Gargliardo–Nirenberg inequality and (3.47) yield that

$$\begin{aligned}&\|\mathrm{div}u\|_{L^\infty} + \|\omega\|_{L^\infty} + \|G\|_{L^\infty} \\ &\leq C + C\|\nabla G\|_{L^p}^{p/(2(p-1))} + C\|\nabla\omega\|_{L^p}^{p/(2(p-1))} \\ &\leq C + C\|\rho\dot{u}\|_{L^p}^{p/(2(p-1))},\end{aligned} \tag{4.12}$$

we directly derive (4.10) from (4.11) and (3.26). The proof of Lemma 4.2 is finished. □

Proposition 4.3. *Assume that (1.8) holds. Then, for $q > 2$, there is a constant C depending only on $T, q, \mu, \gamma, \beta, \|u_0\|_{H^1}$, and $\|\rho_0\|_{W^{1,q}}$ such that*

$$\sup_{0\le t\le T}\left(\|\rho\|_{W^{1,q}}+\|u\|_{H^1}+t\|u\|_{H^2}\right) + \int_0^T\left(\|\nabla^2 u\|_{L^q}^{(q+1)/q}+t\|\nabla^2 u\|_{L^q}^2+t\|u_t\|_{H^1}^2\right)dt\le C. \tag{4.13}$$

Proof. Following [18], we will prove (4.13). First, denoting by $\Phi \triangleq (\Phi^1, \Phi^2)$ with $\Phi^i \triangleq (2\mu + \lambda(\rho))\partial_i\rho$ $(i = 1, 2)$, one deduces from $(1.1)_1$ that Φ^i satisfies

$$\Phi^i_t+(u\cdot\nabla)\Phi^i+(2\mu+\lambda(\rho))\partial_i u^j\partial_j\rho+\rho\partial_i G+\rho\partial_i P+\Phi^i \mathrm{div}u=0. \tag{4.14}$$

For $q > 2$, multiplying (4.14) by $|\Phi|^{q-2}\Phi^i$ and integrating the resulting equation over $\mathbb{T}^2$, we obtain after integration by parts and using (3.26) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Phi\|_{L^q}\le C(1+\|\nabla u\|_{L^\infty})\|\nabla\rho\|_{L^q}+C\|\nabla G\|_{L^q}. \tag{4.15}$$

Next, we deduce from standard L^p-estimate for elliptic system, (4.12), and (3.26) that

$$\begin{aligned}\|\nabla^2 u\|_{L^q}&\le C\|\nabla\mathrm{div}u\|_{L^q}+C\|\nabla\omega\|_{L^q}\\&\le C\|\nabla((2\mu+\lambda)\mathrm{div}u)\|_{L^q}+C\|\mathrm{div}u\|_{L^\infty}\|\nabla\rho\|_{L^q}+C\|\nabla\omega\|_{L^q}\\&\le C(\|\mathrm{div}u\|_{L^\infty}+1)\|\nabla\rho\|_{L^q}+C\|\nabla G\|_{L^q}+C\|\nabla\omega\|_{L^q}\\&\le C(\|\rho\dot u\|_{L^q}^{q/(2(q-1))}+1)\|\nabla\rho\|_{L^q}+C\|\rho\dot u\|_{L^q}\\&\le C\|\nabla\rho\|_{L^q}^{(2q-2)/(q-2)}+C\|\rho\dot u\|_{L^q}+C.\end{aligned} \tag{4.16}$$

Then, it follows from Lemma 2.4, (4.12), and (4.16) that

$$\begin{aligned}\|\nabla u\|_{L^\infty}&\le C\left(\|\mathrm{div}u\|_{L^\infty}+\|\omega\|_{L^\infty}\right)\log(e+\|\nabla^2 u\|_{L^q})+C\|\nabla u\|_{L^2}+C\\&\le C\left(1+\|\rho\dot u\|_{L^q}^{q/(2(q-1))}\right)\log(e+\|\rho\dot u\|_{L^q}+\|\nabla\rho\|_{L^q})+C\\&\le C\left(1+\|\rho\dot u\|_{L^q}\right)\log(e+\|\nabla\rho\|_{L^q}).\end{aligned} \tag{4.17}$$

Substituting (4.17) into (4.15), we deduce from Gronwall's inequality and (4.10) that

$$\sup_{0\le t\le T}\|\nabla\rho\|_{L^q}\le C, \tag{4.18}$$

which combined with (4.16) and (4.10) shows

$$\int_0^T\left(\|\nabla^2 u\|_{L^q}^{(q+1)/q}+t\|\nabla^2 u\|_{L^q}^2\right)dt\le C. \tag{4.19}$$

Finally, it follows from (2.6), (3.47), (4.2), and (4.19) that

$$
\begin{aligned}
&\int_0^T t\|u_t\|_{H^1}^2 dt \\
&\le C\int_0^T t\left(\|\rho^{1/2}u_t\|_{L^2}^2+\|\nabla u_t\|_{L^2}^2\right)dt \\
&\le C\int_0^T t\left(\|\rho^{1/2}\dot u\|_{L^2}^2+\|u\cdot\nabla u\|_{L^2}^2+\|\nabla\dot u\|_{L^2}^2+\|\nabla(u\cdot\nabla u)\|_{L^2}^2\right)dt \\
&\le C+C\int_0^T t\|\nabla u\|_{L^4}^4 dt+C\int_0^T t\|u\|_{H^1}^2\|\nabla^2 u\|_{L^q}^2 dt \\
&\le C+C\int_0^T t\|\nabla u\|_{L^2}^2\|\nabla^2 u\|_{L^2}^2 dt+C\int_0^T t\|\nabla^2 u\|_{L^q}^2 dt \\
&\le C.
\end{aligned} \tag{4.20}
$$

We obtain from (3.47), (3.26), and (4.18) that

$$
\begin{aligned}
\|\nabla^2 u\|_{L^2} &\le C\|\nabla\omega\|_{L^2}+C\|\nabla\mathrm{div}u\|_{L^2} \\
&\le C\|\nabla\omega\|_{L^2}+C+C\|\nabla G\|_{L^2}+C\|\mathrm{div}u\|_{L^{2q/(q-2)}}\|\nabla\rho\|_{L^q} \\
&\le C\|\nabla\omega\|_{L^2}+C\|\nabla G\|_{L^2}+C+C\|\nabla u\|_{L^2}^{(q-2)/q}\|\nabla^2 u\|_{L^2}^{2/q} \\
&\le C+\frac{1}{2}\|\nabla^2 u\|_{L^2}+C\|\rho\dot u\|_{L^2},
\end{aligned} \tag{4.21}
$$

which together with (4.2) gives

$$
\sup_{0\le t\le T} t\|\nabla^2 u\|_{L^2}\le C.
$$

This combined with (4.18)–(4.20) and (3.47) yields (4.13). The proof of Proposition 4.3 is completed. □

5. Proofs of Theorems 1.1–1.3

With all the a priori estimates in Sections 3 and 4 at hand, we are ready to prove the main results of this paper in this section. We first state the global existence of strong solution (ρ, u) provided that (1.8) holds and that (ρ_0, m_0) satisfies (2.1).

Proposition 5.1. *Assume that (1.8) holds and that (ρ_0, m_0) satisfies (2.1). Then there exists a unique strong solution (ρ, u) to (1.1)–(1.4) in $\mathbb{T}^2\times(0,\infty)$ satisfying (2.2) for any $T\in(0,\infty)$. In addition, for any $q>2$, (ρ, u) satisfies (4.13) with some positive constant C depending only on $T, q, \mu, \gamma, \beta, \|u_0\|_{H^1}$, and $\|\rho_0\|_{W^{1,q}}$. Moreover, if (1.11) holds, there exists some positive constant C depending only on $\mu, \beta, \gamma, \|\rho_0\|_{L^\infty}$, and $\|u_0\|_{H^1}$ such that (3.41) and (4.3) hold.*

Proof. Standard local existence result, Lemma 2.1, applies to show that the problem (1.1)–(1.4) with initial data (ρ_0, m_0) has a unique local solution (ρ, u), defined up to a positive time T_0 which may depend on $\inf_{x\in\mathbb{T}^2}\rho_0(x)$, and satisfying (2.2) and (2.3). We set

$$
T^*=\sup\left\{T \;\middle|\; \sup_{0\le t\le T}\|(\rho,u)\|_{H^2}<\infty\right\}. \tag{5.1}
$$

Clearly, $T^* \geq T_0$. If $T^* < \infty$, we claim that there exists a positive constant $\hat{C}$ which may depend on T^* and $\inf\limits_{x\in\mathbb{T}^2} \rho_0(x)$ such that, for all $0 < T < T^*$,

$$\sup_{0\leq t\leq T} \|\rho\|_{H^2} \leq \hat{C}. \tag{5.2}$$

This together with (4.13) contradicts (5.1). Thus, we have

$$T^* = \infty. \tag{5.3}$$

The estimates (4.13), (4.3), and (3.41) directly follow from (2.2), Lemma 4.1, and Propositions 3.4 and 4.3.

To finish the proof of Proposition 5.1, it only remains to prove (5.2).

First, standard calculations together with (4.13) imply that for any $T \in (0, T^*)$,

$$\inf_{(x,t)\in\mathbb{T}^2\times(0,T)} \rho(x,t) \geq \inf_{x\in\mathbb{T}^2} \rho_0(x) \exp\left\{ -\int_0^T \|\mathrm{div} u\|_{L^\infty} dt \right\} \geq \hat{C}^{-1}, \tag{5.4}$$

where and in what follows, $\hat{C}$ denotes some generic positive constant depending on $\inf\limits_{x\in\mathbb{T}^2} \rho_0(x)$ and T^* but independent of T. Because of (2.1), we define

$$\sqrt{\rho}\dot{u}(x, t = 0) = \rho_0^{-1/2} \left(\mu\triangle u_0 + \nabla((\mu + \lambda(\rho_0))\mathrm{div} u_0) - \nabla P(\rho_0)\right). \tag{5.5}$$

Integrating (4.9) with respect to t over $(0, T)$ together with (2.1), (3.47), and (5.5) yields

$$\sup_{0\leq t\leq T} \int \rho|\dot{u}|^2 dx + \int_0^T \|\nabla \dot{u}\|_{L^2}^2 dt \leq \hat{C}. \tag{5.6}$$

This combined with (3.25), (4.21), and (4.13) leads to

$$\begin{aligned}
&\sup_{0\leq t\leq T} \left(\|\nabla^2 u\|_{L^2} + \|\nabla G\|_{L^2} + \|\nabla\omega\|_{L^2}\right) + \int_0^T (\|\nabla^2 G\|_{L^2}^2 + \|\nabla^2\omega\|_{L^2}^2) dt \\
&\leq \hat{C} \sup_{0\leq t\leq T} \|\rho\dot{u}\|_{L^2} + \hat{C}\int_0^T \|\nabla(\rho\dot{u})\|_{L^2}^2 dt \\
&\leq \hat{C} + \hat{C}\int_0^T \left(\|\nabla\rho\|_{L^q}^2 \|\dot{u}\|_{L^{2q/(q-2)}}^2 + \|\nabla\dot{u}\|_{L^2}^2\right) dt \\
&\leq \hat{C} + \hat{C}\int_0^T \|\dot{u}\|_{H^1}^2 dt \leq \hat{C},
\end{aligned} \tag{5.7}$$

where in the last inequality, we have used (2.6) and (5.6).

Next, operating ∇ to (4.14) and multiplying the resulting equality by $\nabla\Phi^i$, we obtain after integration by parts and using (5.4) and (5.7) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Phi\|_{L^2} \le \hat{C}(1+\|\nabla u\|_{L^\infty})\left(1+\|\nabla\Phi\|_{L^2}+\|\nabla\rho\|_{L^4}^2+\|\nabla^2\rho\|_{L^2}\right) \\ +\hat{C}\||\nabla\rho||\nabla^2 u|\|_{L^2}+\hat{C}\||\nabla\rho||\nabla G|\|_{L^2}+\hat{C}\|\nabla^2 G\|_{L^2}. \tag{5.8}$$

Note that (2.4) and (4.13) lead to

$$\begin{aligned}\|\nabla^2\rho\|_{L^2}+\|\nabla\rho\|_{L^4}^2 &\le \hat{C}\|\nabla\Phi\|_{L^2}+\hat{C}\|\nabla\rho\|_{L^4}^2 \\ &\le \hat{C}\|\nabla\Phi\|_{L^2}+\hat{C}\|\nabla\rho\|_{L^{\min\{4,q\}}}^{\min\{4,q\}/2}\|\nabla^2\rho\|_{L^2}^{(4-\min\{4,q\})/2} \\ &\le \hat{C}\|\nabla\Phi\|_{L^2}+\frac{1}{2}\|\nabla^2\rho\|_{L^2}+\hat{C}.\end{aligned} \tag{5.9}$$

Then, it follows from the Hölder inequality, (2.4), (4.13), and (5.7) that

$$\begin{aligned}&\||\nabla\rho||\nabla^2 u|\|_{L^2}+\||\nabla\rho||\nabla G|\|_{L^2} \\ &\le \hat{C}\|\nabla\rho\|_{L^q}\|\nabla^2 u\|_{L^2}^{1-2/q}\|\nabla^3 u\|_{L^2}^{2/q}+\hat{C}\|\nabla\rho\|_{L^q}\|\nabla G\|_{L^2}^{1-2/q}\|\nabla^2 G\|_{L^2}^{2/q} \\ &\le \hat{C}(\varepsilon)+\varepsilon\|\nabla^3 u\|_{L^2}+\hat{C}\|\nabla^2 G\|_{L^2}.\end{aligned} \tag{5.10}$$

Moreover, the L^2-estimate of elliptic system leads to

$$\begin{aligned}\|\nabla^3 u\|_{L^2} &\le \hat{C}\|\nabla^2\mathrm{div}u\|_{L^2}+\hat{C}\|\nabla^2\omega\|_{L^2} \\ &\le \hat{C}\|\nabla^2((2\mu+\lambda(\rho))\mathrm{div}u)\|_{L^2}+\hat{C}\||\nabla\rho||\nabla^2 u|\|_{L^2} \\ &\quad +\hat{C}\||\nabla^2\rho||\nabla u|\|_{L^2}+\hat{C}\||\nabla\rho|^2|\nabla u|\|_{L^2}+\hat{C}\|\nabla^2\omega\|_{L^2} \\ &\le \hat{C}\|\nabla^2 G\|_{L^2}+\hat{C}\|\nabla^2\rho\|_{L^2}+\hat{C}\|\nabla\rho\|_{L^4}^2+\hat{C}\||\nabla\rho||\nabla^2 u|\|_{L^2} \\ &\quad +\hat{C}\left(\|\nabla^2\rho\|_{L^2}+\|\nabla\rho\|_{L^4}^2\right)\|\nabla u\|_{L^\infty}+\hat{C}\|\nabla^2\omega\|_{L^2}.\end{aligned}$$

Substituting this and (5.9) into (5.10) shows

$$\begin{aligned}&\||\nabla\rho||\nabla^2 u|\|_{L^2}+\||\nabla\rho||\nabla G|\|_{L^2} \\ &\le \hat{C}\|\nabla^2 G\|_{L^2}+\hat{C}\|\nabla^2\omega\|_{L^2}+\hat{C}(1+\|\nabla\Phi\|_{L^2})(1+\|\nabla u\|_{L^\infty}),\end{aligned}$$

which together with (5.8) and (5.9) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Phi\|_{L^2} \le \hat{C}(1+\|\nabla u\|_{L^\infty})(\|\nabla\Phi\|_{L^2}+1)+\hat{C}\|\nabla^2 G\|_{L^2}+\hat{C}\|\nabla^2\omega\|_{L^2}.$$

This combined with (5.7) and Gronwall's inequality yields

$$\sup_{0\le t\le T}\|\nabla\Phi\|_{L^2}\le\hat{C},$$

which together with (5.9) implies (5.2). The proof of Proposition 5.1 is finished. □

Proof of Theorem 1.1. Let (ρ_0, m_0) satisfying (1.9) be the initial data as described in Theorem 1.1. For constant $\delta\in(0,1)$, we define

$$\rho_0^\delta \triangleq j_\delta*\rho_0+\delta\ge\delta>0,\quad u_0^\delta\triangleq j_\delta*u_0,\quad m_0^\delta=\rho_0^\delta u_0^\delta, \tag{5.11}$$

where j_δ is the standard mollifying kernel of width δ. Hence, we have $\rho_0^\delta, u_0^\delta\in H^\infty$, and

$$\lim_{\delta\to 0}\left(\|\rho_0^\delta-\rho_0\|_{W^{1,q}}+\|u_0^\delta-u_0\|_{H^1}\right)=0.$$

Proposition 5.1 thus yields that the problem (1.1)–(1.4) with (ρ_0, m_0) being replaced by $(\rho_0^\delta, m_0^\delta)$ has a unique global strong solution (ρ^δ, u^δ) satisfying (4.13) for any $T > 0$ and for some C independent of δ. Moreover, if (1.11) holds, there exists some positive constant C depending only on $\mu, \beta, \gamma, \|\rho_0\|_{L^\infty}$, and $\|u_0\|_{H^1}$ such that (ρ^δ, u^δ) satisfies (3.41) and (4.3). Letting $\delta \to 0$, standard arguments (see [30,34,16,25]) thus show that the problem (1.1)–(1.4) has a global strong solution (ρ, u) satisfying the properties listed in Theorem 1.1 except (1.13) and the uniqueness of (ρ, u) satisfying (1.10). Moreover, (ρ, u) satisfies (3.41) and (4.3) for some positive constant C depending only on $\mu, \beta, \gamma, \|\rho_0\|_{L^\infty}$, and $\|u_0\|_{H^1}$ provided (1.11) holds.

Since the proof of the uniqueness of (ρ, u) satisfying (1.10) is similar to that of Germain [13] and (1.13) will be proved in Theorem 1.2, we finish the proof of Theorem 1.1. □

Proof of Theorem 1.2. Let (ρ_0, m_0) satisfying (1.14) be the initial data as described in Theorem 1.2. For constant $\delta \in (0, 1)$, let $(\rho_0^\delta, u_0^\delta)$ be as in (5.11). Hence, we have $\rho_0^\delta, u_0^\delta \in H^\infty$, and for any $p > 1$,

$$\lim_{\delta\to 0}\left(\|\rho_0^\delta-\rho_0\|_{L^p}+\|u_0^\delta-u_0\|_{H^1}\right)=0.$$

Moreover,

$$\rho_0^\delta \rightharpoonup \rho_0 \text{ weakly * in } L^\infty,\ \text{ as } \delta\to 0.$$

Proposition 5.1 thus yields that the problem (1.1)–(1.4) with (ρ_0, m_0) being replaced by $(\rho_0^\delta, \rho_0^\delta u_0^\delta)$ has a unique global strong solution (ρ^δ, u^δ) satisfying (3.47), (4.2), (4.3), and (4.10), for any $T > 0$ and for some C independent of δ. Moreover, if (1.11) holds, there exists some positive constant C depending only on $\mu, \beta, \gamma, \|\rho_0\|_{L^\infty}$, and $\|u_0\|_{H^1}$ such that (ρ^δ, u^δ) satisfies (3.41) and (4.3).

We modify the compactness arguments in [34,30] to obtain the compactness results of (ρ^δ, u^δ).

First, it follows from (3.47) and (4.10) that

$$\sup_{0\le t\le T}\|u^\delta\|_{H^1}+\int_0^T t\|u_t^\delta\|_{L^2}^2dt\le C,$$

which together with the Aubin–Lions lemma gives that, up to a subsequence,

$$\begin{cases} u^\delta \rightharpoonup u \text{ weakly * in } L^\infty(0,T;H^1),\\ u^\delta\to u \text{ strongly in } C([\tau,T];L^p),\end{cases}$$

for any $\tau \in (0, T)$ and $p \in [1, \infty)$.

Next, let $A^\delta \triangleq (2\mu + \lambda(\rho^\delta))\mathrm{div}u^\delta - P(\rho^\delta)$. One thus deduces from (3.14), (3.15), (3.47), and (4.10) that

$$\int_0^T\left(\|A^\delta\|_{L^\infty}^{4/3}+\|\omega^\delta\|_{H^1}^2+\|A^\delta\|_{H^1}^2+t\|\omega_t^\delta\|_{L^2}^2+t\|A_t^\delta\|_{L^2}^2\right)dt\le C, \tag{5.12}$$

which implies that, up to a subsequence,

$$\begin{cases} A^\delta\rightharpoonup A \text{ weakly * in } L^{4/3}(0,T;L^\infty),\\ \omega^\delta\to\omega=\mathrm{curl}u,\ A^\delta\to A \text{ strongly in } L^2(\tau,T;L^p),\end{cases} \tag{5.13}$$

for any $\tau \in (0, T)$ and $p \in [1, \infty)$.

Next, to obtain the strong limit of ρ^δ, we deduce from (3.47) that, up to a subsequence,

$$\rho^\delta \rightharpoonup \rho \text{ weakly * in } L^\infty(0,T;L^\infty).$$

Let $f(s)$ be an arbitrary continuous function on $[0,C]$ with C as in (3.47). Then, we have that, up to a subsequence, $f(\rho^\delta)$ converges weakly $*$ in $L^\infty(0,T;L^\infty)$. Denote the weak-$*$ limit by $\overline{f(\rho)}$:

$$f(\rho^\delta) \rightharpoonup \overline{f(\rho)} \text{ weakly * in } L^\infty(0,T;L^\infty).$$

Noticing that,

$$\mathrm{div} u^\delta = \phi(\rho^\delta)A^\delta + \phi(\rho^\delta)P(\rho^\delta),$$

with $\phi(s) \triangleq 1/(2\mu + \lambda(s))$, we have

$$\mathrm{div} u = \overline{\phi(\rho)}A + \overline{\phi(\rho)P(\rho)}, \quad \text{a.e. in } \mathbb{T}^2 \times (0,T).$$

From (1.1), we obtain

$$(\overline{\rho^2})_t + \mathrm{div}(\overline{\rho^2}u) + A\overline{\rho^2\phi(\rho)} + \overline{\rho^2\phi(\rho)P(\rho)} = 0, \text{ in } \mathcal{D}'(\mathbb{T}^2 \times (0,\infty)).$$

Using [26, Lemma 2.3], we get by standard arguments that

$$(\rho^2)_t + \mathrm{div}(\rho^2 u) + A\rho^2\overline{\phi(\rho)} + \rho^2\overline{\phi(\rho)P(\rho)} = 0, \text{ in } \mathcal{D}'(\mathbb{T}^2 \times (0,\infty)).$$

Thus, for $\Psi \triangleq \overline{\rho^2} - \rho^2 \geq 0$, we have

$$\begin{cases} \Psi_t + \mathrm{div}(\Psi u) + A(\overline{\rho^2\phi(\rho)} - \rho^2\phi(\rho)) + A\rho^2(\phi(\rho) - \overline{\phi(\rho)}) + \overline{\rho^2\phi(\rho)P(\rho)} \\ \quad - \rho^2\phi(\rho)P(\rho) + \rho^2\left(\phi(\rho)P(\rho) - \overline{\phi(\rho)P(\rho)}\right) = 0, \quad \text{in } \mathcal{D}'(\mathbb{T}^2 \times (0,\infty)), \\ \Psi(x, t=0) = 0, \quad \text{a.e. } x \in \mathbb{T}^2. \end{cases} \tag{5.14}$$

By writing $(\rho^\delta)^2 - \rho^2 = 2\rho(\rho^\delta - \rho) + (\rho^\delta - \rho)^2$, we see that, up to a subsequence,

$$\overline{\lim_{\delta\to 0}}\|\rho^\delta - \rho\|_{L^2}^2(t) \leq \int \Psi(x,t)dx, \quad \text{a.e. } t > 0.$$

Also, for any $f(s) \in C^2([0,C])$ and $h(x) \in L^\infty(\mathbb{T}^2)$, noticing that

$$f(\rho^\delta) - f(\rho) = f'(\rho)(\rho^\delta - \rho) + \int_0^1 \theta \int_0^1 f''(\rho + \theta\alpha(\rho^\delta - \rho))d\alpha d\theta(\rho^\delta - \rho)^2,$$

we deduce from (3.47) that

$$\left|\int h(x)(\overline{f(\rho)} - f(\rho))dx\right| \leq \sup_{0\leq s\leq C}|f''(s)|\|h\|_{L^\infty}\int \Psi dx. \tag{5.15}$$

In particular, since

$$f_1(s) \triangleq s^2\phi(s) \in C^2([0,C]), \quad f_2(s) \triangleq s^2\phi(s)P(s) \in C^2([0,C]),$$

we have

$$M \triangleq \sup_{0\le s\le C}(|f_1''(s)|+|f_2''(s)|)<\infty.$$

It thus follows from (5.15) that

$$\left|\int A\left(\overline{\rho^2\phi(\rho)}-\rho^2\phi(\rho)\right)dx\right|\le M\|A\|_{L^\infty}\int \Psi dx, \tag{5.16}$$

and that

$$\left|\int (\overline{\rho^2\phi(\rho)P(\rho)}-\rho^2\phi(\rho)P(\rho))dx\right|\le M\int \Psi dx. \tag{5.17}$$

Next, for $g(s)\in C^1([0,C])\cap C^2((0,C])$, simple calculations show

$$\begin{aligned}&\rho^2(g(\rho^\delta)-g(\rho))-\rho^2 g'(\rho)(\rho^\delta-\rho)\\ &=\rho^2\int_0^1\theta\int_0^1 g''(\rho+\theta\alpha(\rho^\delta-\rho))d\alpha d\theta(\rho^\delta-\rho)^2,\end{aligned}$$

which yields that for any $h(x)\in L^\infty(\mathbb{T}^2)$,

$$\left|\int h(x)\rho^2(\overline{g(\rho)}-g(\rho))dx\right|\le M_g\|h\|_{L^\infty}\int\Psi dx, \tag{5.18}$$

provided

$$M_g\triangleq\sup_{0\le\rho,s\le C}\left|\rho^2\int_0^1\theta\int_0^1 g''(\rho+\theta\alpha(s-\rho))d\alpha d\theta\right|<\infty. \tag{5.19}$$

Let $g_1(s)\triangleq\phi(s)$ and $g_2(s)\triangleq\phi(s)P(s)$. Since $g_i\in C^1([0,C])\cap C^2((0,C])(i=1,2)$ satisfy (5.19), from (5.18) we obtain that

$$\left|\int A\rho^2(\overline{\phi(\rho)}-\phi(\rho))dx\right|\le M_{g_1}\|A\|_{L^\infty}\int\Psi dx, \tag{5.20}$$

and that

$$\left|\int \rho^2(\overline{\phi(\rho)P(\rho)}-\phi(\rho)P(\rho))dx\right|\le M_{g_2}\int\Psi dx. \tag{5.21}$$

Substituting (5.16), (5.17), (5.20), and (5.21) into (5.14), after using Gronwall's inequality and (5.12), we arrive at

$$\Psi=0 \text{ a.e. in } \mathbb{T}^2\times(0,T),$$

which gives that, up to a subsequence,

$$\rho^\delta\to\rho \text{ strongly in } L^p(\mathbb{T}^2\times(0,T)), \tag{5.22}$$

for any $p \in [1, \infty)$. This combined with (5.13) implies that, up to a subsequence,

$$A^\delta \to A = (2\mu + \lambda(\rho))\mathrm{div}u - P(\rho), \quad \text{strongly in } L^2(\mathbb{T}^2 \times (\tau, T)), \tag{5.23}$$

for any $\tau \in (0, T)$. Standard arguments thus show that the limit (ρ, u) is a global weak solution of (1.1)–(1.4).

To finish the proof of Theorem 1.2, it only remains to prove (1.13).

First, it follows from (3.41) that

$$\int_1^\infty \|P(\rho^\delta) - \overline{P(\rho^\delta)}\|_{L^2}^2 dt \le C \int_1^\infty \left(\|G^\delta\|_{L^2}^2 + (A_3^\delta(t))^2\right) dt \le C,$$

where and in what follows, $\overline{f}$ denotes the mean value of f over $\mathbb{T}^2$ as in (1.7). This combined with (5.22), (5.13), and (5.23) gives

$$\int_1^\infty \left(\|P(\rho) - \overline{P}\|_{L^2}^2 + \|G\|_{L^2}^2 + \|\omega\|_{L^2}^2\right) dt \le C. \tag{5.24}$$

Simple calculations lead to

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|P(\rho^\delta) - \overline{P(\rho^\delta)}\|_{L^2}^2) \\
&= 2\int (P(\rho^\delta) - \overline{P(\rho^\delta)})(P(\rho^\delta) - \overline{P(\rho^\delta)})_t dx \\
&= -2\int (P(\rho^\delta) - \overline{P(\rho^\delta)})(u^\delta \cdot \nabla(P(\rho^\delta) - \overline{P(\rho^\delta)}) + \rho^\delta P'(\rho^\delta)\mathrm{div}u^\delta) dx \\
&\quad + 2\int (P(\rho^\delta) - \overline{P(\rho^\delta)}) dx \int (\rho^\delta P'(\rho^\delta) - P(\rho^\delta))\mathrm{div}u^\delta dx \\
&\le C\|P(\rho^\delta) - \overline{P(\rho^\delta)}\|_{L^2}^2 + C\|\nabla u^\delta\|_{L^2}^2 \\
&\le C\|P(\rho^\delta) - \overline{P(\rho^\delta)}\|_{L^2}^2 + C\|G^\delta\|_{L^2}^2 + C\|\omega^\delta\|_{L^2}^2,
\end{aligned}$$

which gives that, for any $s, t \in [N, N+1]$,

$$\begin{aligned}
&\|P(\rho^\delta) - \overline{P(\rho^\delta)}\|_{L^2}^2(t) - \|P(\rho^\delta) - \overline{P(\rho^\delta)}\|_{L^2}^2(s) \\
&\le C \int_N^{N+1} \left(\|P(\rho^\delta) - \overline{P(\rho^\delta)}\|_{L^2}^2 + \|G^\delta\|_{L^2}^2 + \|\omega^\delta\|_{L^2}^2\right) dt.
\end{aligned} \tag{5.25}$$

Integrating (5.25) with respect to s over $(N, N+1)$ yields that

$$\begin{aligned}
&\sup_{N \le t \le N+1} \|P(\rho^\delta) - \overline{P(\rho^\delta)}\|_{L^2}^2(t) \\
&\le C \int_N^{N+1} \left(\|P(\rho^\delta) - \overline{P(\rho^\delta)}\|_{L^2}^2 + \|G^\delta\|_{L^2}^2 + \|\omega^\delta\|_{L^2}^2\right) dt.
\end{aligned}$$

From (5.22), (5.23), and (5.13), we have

$$\sup_{N \le t \le N+1} \|P(\rho) - \overline{P}\|_{L^2}^2(t) \le C \int_N^{N+1} \left(\|P(\rho) - \overline{P}\|_{L^2}^2 + \|G\|_{L^2}^2 + \|\omega\|_{L^2}^2\right) dt.$$

Letting $N \to \infty$, this combined with (5.24) yields that

$$\lim_{t\to\infty} \|P(\rho) - \overline{P}\|_{L^2}^2(t) = 0. \tag{5.26}$$

Next, standard arguments together with [26, Lemma 2.3] and $(1.1)_1$ yield that $P(\rho)$ satisfies

$$(P(\rho))_t + \mathrm{div}(P(\rho)u) + (\gamma - 1)P(\rho)\mathrm{div}u = 0, \quad \text{in } \mathcal{D}'(\mathbb{T}^2 \times (0,\infty)),$$

which gives that

$$\begin{aligned}\int_1^\infty \left|\frac{\mathrm{d}}{\mathrm{dt}}\overline{P}\right| dt &\le C \int_1^\infty \left|\int (P - \bar{P})\mathrm{div}u dx\right| dt \\ &\le C \int_1^\infty \left(\|P - \bar{P}\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\right) dt \le C,\end{aligned}$$

due to (5.24). Hence, there exists some positive constant ρ_s such that

$$\lim_{t\to\infty} \bar{P}(t) = \rho_s^\gamma,$$

due to $0 < \bar{\rho}_0^\gamma \le \bar{P} \le C$. This combined with (5.26) and (3.11) shows

$$\lim_{t\to\infty} \|\rho - \bar{\rho}_0\|_{L^p}(t) = 0, \tag{5.27}$$

for any $p \in [1, \infty)$.

Finally, similar to (5.25), from (3.30) and (3.41), we have

$$(A_1^\delta(t))^2 \le (A_1^\delta(s))^2 + C \int_N^{N+1} (A_3^\delta(t))^2 dt,$$

for any $s, t \in [N, N+1]$. This gives

$$\begin{aligned}\sup_{N\le t\le N+1} (A_1^\delta(t))^2 &\le C \int_N^{N+1} \left((A_1^\delta(t))^2 + (A_3^\delta(t))^2\right) dt \\ &\le C \int_N^{N+1} \left(\|P(\rho^\delta) - \overline{P(\rho^\delta)}\|_{L^2}^2 + \|G^\delta\|_{L^2}^2 + \|\omega^\delta\|_{L^2}^2\right) dt,\end{aligned}$$

which together with (5.13), (5.23), (5.24), and the fact that

$$\|G^\delta\|_{L^2}^2 + \|\omega^\delta\|_{L^2}^2 \le C(A_1^\delta(t))^2,$$

leads to

$$\lim_{t\to\infty} (\|G\|_{L^2}^2 + \|\omega\|_{L^2}^2)(t) = 0.$$

Because of (5.26), this shows

$$\lim_{t\to\infty}\|\nabla u\|_{L^2}\le C\lim_{t\to\infty}(\|G\|_{L^2}+\|\omega\|_{L^2}+\|P-\bar{P}\|_{L^2})(t)=0,$$

which combined with (4.3) and (5.27) directly yields (1.13). The proof of Theorem 1.2 is completed. □

Proof of Theorem 1.3. Since the proof of Theorem 1.3 is similar to that of [24, Theorem 1.2], we omit it here. □

References

[1] J.T. Beale, T. Kato, A. Majda, Remarks on the breakdown of smooth solutions for the 3-D Euler equations, Commun. Math. Phys. 94 (1984) 61–66.
[2] H. Brézis, S. Wainger, A note on limiting cases of Sobolev embeddings and convolution inequalities, Commun. Partial Differ. Equ. 5 (7) (1980) 773–789.
[3] Y. Cho, H.J. Choe, H. Kim, Unique solvability of the initial boundary value problems for compressible viscous fluids, J. Math. Pures Appl. 83 (2004) 243–275.
[4] H.J. Choe, H. Kim, Strong solutions of the Navier–Stokes equations for isentropic compressible fluids, J. Differ. Equ. 190 (2003) 504–523.
[5] R. Coifman, R. Rochberg, G. Weiss, Factorization theorems for Hardy spaces in several variables, Ann. Math. 103 (1976) 611–635.
[6] R.R. Coifman, Y. Meyer, On commutators of singular integrals and bilinear singular integrals, Trans. Am. Math. Soc. 212 (1975) 315–331.
[7] R. Coifman, P.L. Lions, Y. Meyer, S. Semmes, Compensated compactness and Hardy spaces, J. Math. Pures Appl. 72 (1993) 247–286.
[8] B. Desjardins, Regularity results for two-dimensional flows of multiphase viscous fluids, Arch. Ration. Mech. Anal. 137 (2) (1997) 135–158.
[9] H. Engler, An alternative proof of the Brezis–Wainger inequality, Commun. Partial Differ. Equ. 14 (1989) 541–544.
[10] C. Fefferman, Characterizations of bounded mean oscillation, Bull. Am. Math. Soc. 77 (1971) 587–588.
[11] E. Feireisl, Dynamics of Viscous Compressible Fluids, Oxford University Press, 2004.
[12] E. Feireisl, A. Novotny, H. Petzeltová, On the existence of globally defined weak solutions to the Navier–Stokes equations, J. Math. Fluid Mech. 3 (4) (2001) 358–392.
[13] P. Germain, Weak-strong uniqueness for the isentropic compressible Navier–Stokes system, J. Math. Fluid Mech. 13 (1) (2011) 137–146.
[14] D. Hoff, Global existence of the Navier–Stokes equations for multidimensional compressible flow with discontinuous initial data, J. Differ. Equ. 120 (1995) 215–254.
[15] D. Hoff, Strong convergence to global solutions for multidimensional flows of compressible, viscous fluids with polytropic equations of state and discontinuous initial data, Arch. Ration. Mech. Anal. 132 (1995) 1–14.
[16] X. Huang, J. Li, Global classical and weak solutions to the three-dimensional full compressible Navier–Stokes system with vacuum and large oscillations, http://arxiv.org/abs/1107.4655.
[17] X. Huang, J. Li, A. Matsumura, On the strong solutions to the three-dimensional barotropic compressible Navier–Stokes equations with vacuum, Preprint.
[18] X. Huang, J. Li, Z.P. Xin, Serrin type criterion for the three-dimensional viscous compressible flows, SIAM J. Math. Anal. 43 (4) (2011) 1872–1886.
[19] X. Huang, J. Li, Z.P. Xin, Global well-posedness of classical solutions with large oscillations and vacuum to the three-dimensional isentropic compressible Navier–Stokes equations, Commun. Pure Appl. Math. 65 (2012) 549–585.
[20] Q. Jiu, Y. Wang, Z.P. Xin, Global well-posedness of 2D compressible Navier–Stokes equations with large data and vacuum, J. Math. Fluid Mech. 16 (3) (2014) 483–521.
[21] T. Kato, Remarks on the Euler and Navier–Stokes equations in R^2, Proc. Symp. Pure Math. 45 (1986) 1–7.
[22] O.A. Ladyzenskaja, V.A. Solonnikov, N.N. Ural'ceva, Linear and Quasilinear Equations of Parabolic Type, American Mathematical Society, Providence, RI, 1968.
[23] J. Li, Z. Liang, On classical solutions to the Cauchy problem of the two-dimensional barotropic compressible Navier–Stokes equations with vacuum, J. Math. Pures Appl. 102 (2014) 640–671.
[24] J. Li, Z. Xin, Some uniform estimates and blowup behavior of global strong solutions to the Stokes approximation equations for two-dimensional compressible flows, J. Differ. Equ. 221 (2) (2006) 275–308.
[25] J. Li, J.W. Zhang, J.N. Zhao, On the global motion of three-dimensional compressible isentropic flows with large external potential forces and vacuum, SIAM J. Math. Anal. 47 (2) (2015) 1121–1153.
[26] P.L. Lions, Mathematical Topics in Fluid Mechanics, vol. 1, Incompressible Models, Oxford University Press, New York, 1996.
[27] P.L. Lions, Mathematical Topics in Fluid Mechanics, vol. 2, Compressible Models, Oxford University Press, New York, 1998.
[28] A. Matsumura, T. Nishida, The initial value problem for the equations of motion of viscous and heat-conductive gases, J. Math. Kyoto Univ. 20 (1) (1980) 67–104.
[29] J. Nash, Le problème de Cauchy pour les équations différentielles d'un fluide général, Bull. Soc. Math. Fr. 90 (1962) 487–497.

[30] M. Perepelitsa, On the global existence of weak solutions for the Navier–Stokes equations of compressible fluid flows, SIAM J. Math. Anal. 38 (1) (2006) 1126–1153.

[31] R. Salvi, I. Straškraba, Global existence for viscous compressible fluids and their behavior as $t \to \infty$, J. Fac. Sci., Univ. Tokyo, Sect. 1A, Math. 40 (1) (1993) 17–51.

[32] J. Serrin, On the uniqueness of compressible fluid motion, Arch. Ration. Mech. Anal. 3 (1959) 271–288.

[33] V.A. Solonnikov, Solvability of the initial-boundary-value problem for the equation of a viscous compressible fluid, J. Math. Sci. 14 (1980) 1120–1133.

[34] V.A. Vaigant, A.V. Kazhikhov, On existence of global solutions to the two-dimensional Navier–Stokes equations for a compressible viscous fluid, Sib. Math. J. 36 (6) (1995) 1283–1316.

[35] A.A. Zlotnik, Uniform estimates and stabilization of symmetric solutions of a system of quasilinear equations, J. Differ. Equ. 36 (2000) 701–716.