FOURIER COEFFICIENTS AND CUSPIDAL SPECTRUM FOR SYMPLECTIC GROUPS

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Abstract. In [2], J. Arthur classifies the automorphic discrete spectrum of symplectic groups up to global Arthur packets. We continue with our investigation of Fourier coefficients and their implication to the structure of the cuspidal spectrum for symplectic groups ([16] and [20]). As result, we obtain certain characterization and construction of small cuspidal automorphic representations and gain a better understanding of global Arthur packets and of the structure of local unramified components of the cuspidal spectrum, which has impacts to the generalized Ramanujan problem as posted by P. Sarnak in [43].

1. Introduction

Let F be a number field and $\mathbb{A}$ be the ring of adeles of F. For an F-split classical group G, $\mathcal{A}_2(\mathrm{G})$ denotes the set of equivalence classes of all automorphic representations of $\mathrm{G}(\mathbb{A})$ that occur in the discrete spectrum of the space of all square-integrable automorphic forms on $\mathrm{G}(\mathbb{A})$. The automorphic representations π in the set $\mathcal{A}_2(\mathrm{G})$ have been classified, up to global Arthur packets, in the fundamental work of J. Arthur ([2]), via the theory of endoscopy. More precisely, for any $\pi \in \mathcal{A}_2(\mathrm{G})$, there exists a global Arthur packet, denoted by $\widetilde{\Pi}_\psi(\mathrm{G})$, such that $\pi \in \widetilde{\Pi}_\psi(\mathrm{G})$ for some global Arthur parameter $\psi \in \widetilde{\Psi}_2(\mathrm{G})$. Following [2], a global Arthur parameter $\psi \in \widetilde{\Psi}_2(\mathrm{G})$ can be written formally as

$$\psi = (\tau_1, b_1) \boxplus (\tau_2, b_2) \boxplus \cdots \boxplus (\tau_r, b_r) \tag{1.1}$$

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where $\tau_j \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_{a_j})$ and $b_j \geq 1$ are integers. We refer to Section 2 for more details. A global Arthur parameter ψ is called *generic*, following [2], if the integers b_j are one, i.e. a generic global Arthur parameter ψ can be written as

$$\psi = \phi = (\tau_1, 1) \boxplus (\tau_2, 1) \boxplus \cdots \boxplus (\tau_r, 1). \tag{1.2}$$

For a generic global Arthur parameter ϕ as in (1.2), the global Arthur packet $\widetilde{\Pi}_\phi(\mathrm{G})$ contains at least one member π from the set $\mathcal{A}_2(\mathrm{G})$. More precisely, this π must belong to the subset $\mathcal{A}_{\mathrm{cusp}}(\mathrm{G})$, i.e. it is cuspidal. This assertion follows essentially from the theory of automorphic descents of Ginzburg-Rallis-Soudry ([10]), as discussed in [20]. In fact, as in [20, Section 3.1], one can show that a global Arthur parameter $\psi = \phi$ is generic if and only if the global Arthur packet $\widetilde{\Pi}_\phi(\mathrm{G})$ contains a member $\pi \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{G})$ that has a nonzero Whittaker-Fourier coefficient (Theorem 3.4 in [20]). It is not hard to show that in such a circumstance, the following holds:

$$\widetilde{\Pi}_\phi(\mathrm{G}) \cap \mathcal{A}_2(\mathrm{G}) \subset \mathcal{A}_{\mathrm{cusp}}(\mathrm{G}).$$

All members in $\widetilde{\Pi}_\phi(\mathrm{G}) \cap \mathcal{A}_2(\mathrm{G})$ may be constructed via the *twisted automorphic descents* as developed in [24] and more generally in [26], [27], [28], [17], and [22].

In [37] and [38], C. Mœglin investigates the following problem: for a global Arthur parameter $\psi \in \widetilde{\Psi}_2(\mathrm{G})$, when does the global Arthur packet $\widetilde{\Pi}_\psi(\mathrm{G})$ contain a non-cuspidal member in $\mathcal{A}_2(\mathrm{G})$ and how can one construct such non-cuspidal members if they exist? Mœglin states her results in terms of her local and global conjectures in the papers. We refer to [37] and [38] for detailed discussions on those problems.

The objective of this paper is to investigate the following simple question: for a global Arthur parameter $\psi \in \widetilde{\Psi}_2(\mathrm{G})$, when does the global Arthur packet $\widetilde{\Pi}_\psi(\mathrm{G})$ contain no cuspidal members, i.e. when is the intersection

$$\widetilde{\Pi}_\psi(\mathrm{G}) \cap \mathcal{A}_{\mathrm{cusp}}(\mathrm{G})$$

an empty set? One closely related question is: if a $\pi \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{G})$ belongs to a global Arthur packet $\widetilde{\Pi}_\psi(\mathrm{G})$, what can one say about the simple global Arthur parameters (for definition see Section 2.2) $(\tau_1, b_1), \cdots, (\tau_r, b_r)$ as in (1.1) occurring in the ψ? In other words, can one bound the integers $b_1, \cdots, b_r$?

The approach that we are taking to investigate this problem is based on our understanding of the structure of Fourier coefficients of automoprhic forms associated to nilpotent orbits or partitions, and the notion of small cuspidal automorphic representations, following the discussions and conjectures in [16, Section 4] and [20]. This study can be regarded as an extension of the fundamental work of R. Howe on the theory of singular automorphic forms using his notion of ranks for unitary representations ([12]).

In this paper, we consider mainly the case that $\mathrm{G} = \mathrm{Sp}_{2n}$, the symplectic groups. The method is applicable to other classical groups. Due to technical reasons, we leave the discussion for other classical groups to our future work.

We start the discussion with a particular global Arthur parameter

$$\psi = (\tau, 2e) \boxplus (1,1) \in \widetilde{\Psi}_2(\mathrm{Sp}_{4e})$$

with $\tau \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_2)$ of symplectic type. When $e = 1$, the well-known example of Saito-Kurokawa provides irreducible cuspidal automorphic representations in the global packet $\widetilde{\Pi}_\psi(\mathrm{Sp}_4)$, as constructed by I. Piatetski-Shapiro in [41] using global theta correspondences. This is the first known counter-example to the generalized Ramanujan conjecture, which is not of unipotent cuspidal type. Of course, the counter-examples of unipotent cuspidal type were constructed in 1979 by Howe and Piatetski-Shapiro in [13], also using global theta correspondences. It was desirable to find such non-tempered cuspidal automorphic representations for general Sp_{2n} or even for general reductive groups. In 1996, W. Duke and Ö. Imamoglu made a conjecture in [6] that when $F = \mathbb{Q}$, there exists an analogue of the Saito-Kurokawa type cuspidal automorphic forms on Sp_{4e} for all integers $e \geq 1$. In terms of the endoscopic classification theory ([2]), the Duke-Imamoglu conjecture asserts that when $F = \mathbb{Q}$, the intersection $\widetilde{\Pi}_\psi(\mathrm{Sp}_{4e}) \cap \mathcal{A}_{\mathrm{cusp}}(\mathrm{Sp}_{4e})$ is non-empty for the global Arthur parameter $\psi = (\tau, 2e) \boxplus (1,1)$. This conjecture was confirmed by T. Ikeda in 2001 ([14]) and an extension to the case that F is totally real is in [15]. The following questions remain:

(1) What happens to the symplectic groups Sp_{4e+2}?
(2) What happens if F is not totally real?

For a general number field F, the authors jointly with L. Zhang proved in [25] that the intersection $\widetilde{\Pi}_\psi(\mathrm{Sp}_{2n}) \cap \mathcal{A}_2(\mathrm{Sp}_{2n})$ is non-empty for a family of global Arthur parameters ψ, including the case that $\psi = (\tau, 2e) \boxplus (1,1)$. We explicitly constructed non-zero square-integrable residual representations in the global Arthur packets $\widetilde{\Pi}_\psi(\mathrm{Sp}_{2n})$ for a family of global Arthur parameters and hence confirmed the conjecture of Mœglin in [37] and [38] for those cases. Our main motivation in [25] is to find *automorphic kernel functions* for the automorphic integral transforms that explicitly produce endoscopy correspondences as explained in [16].

One of the main results in this paper confirms that when F is *totally imaginary* and $n \geq 5$, the intersection $\widetilde{\Pi}_\psi(\mathrm{Sp}_{2n}) \cap \mathcal{A}_{\mathrm{cusp}}(\mathrm{Sp}_{2n})$ is empty for the global Arthur parameters $\psi = (\tau, 2e) \boxplus (1,1)$ if $n = 2e$ and $\psi = (\tau, 2e+1) \boxplus (\omega_\tau, 1)$ if $n = 2e+1$, where ω_τ is the central character of τ, and $\tau \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_2)$ is self-dual. Note that when $n = 2e$, τ is of symplectic type; and when $n = 2e+1$, τ is of orthogonal type (for definitions of symplectic and orthogonal types, see Section 2.2). This conclusion is a consequence of more general results obtained in Section 4, where three different versions of criteria for global Arthur packets containing no cuspidal members are given in Theorems 4.1, 4.2, 4.3, and 4.4; and explicit examples are also discussed in Section 4.2. However, if the number field F is neither totally real nor totally imaginary, it remains to know whether $\widetilde{\Pi}_\psi(\mathrm{Sp}_{2n}) \cap \mathcal{A}_{\mathrm{cusp}}(\mathrm{Sp}_{2n})$ is empty for the global Arthur parameters $\psi = (\tau, n) \boxplus (1,1)$. In Section 5, we are going to discuss the relation of the existence of such cuspidal automorphic representations with the Ramanujan type bound for the whole cuspidal spectrum of Sp_{2n}.

On the other hand, we discuss the characterization of cuspidal automorphic representations with smallest possible Fourier coefficients, which are called *small* cuspidal representations in Section 2. We first explain how to re-interpret the result of J.-S. Li that cuspidal automorphic representations of classical groups are non-singular, in terms of the Fourier coefficients associated to partitions or nilpotent orbits. This leads to a question about the smallest possible Fourier coefficients for the cuspidal spectrum of classical groups, which is closely related to the generalized Ramanujan problem as posted by P. Sarnak in 2005 ([43]). As a consequence of the discussion in Section 3, we find simple criterion for Sp_{4n} that determines families of global Arthur parameters of unipotent type, with which the global Arthur packets contains no cuspidal members (Theorem 3.1). Examples and the relation of Theorem 3.1 with the work of S. Kudla and S. Rallis ([30]) are also discussed briefly in Section 3.

Generally speaking, by the endoscopic classification of the discrete spectrum of Arthur ([2]), the global Arthur parameters provide the bounds for the Hecke eigenvalues or the exponents of the Satake parameters at the unramified local places for automorphic representations occurring in the discrete spectrum. Since it is not clear how to deduce directly from the endoscopic classification which global Arthur packets contains no cuspidal members, we apply the method of Fourier coefficients associated to unipotent orbits. Hence it is expected that our discussion improves those bounds for the exponents of the Satake parameters of cuspidal spectrum if we find more global Arthur packets containing no cuspidal members. In Section 5, we obtain a preliminary result towards the generalized Ramanujan problem. For general number fields, we show in Proposition 5.1 that when $n = 2e$ is even, the cuspidal automorphic representations of Sp_{4e} constructed by Piatetski-Shapiro and Rallis ([42]) achieve the worst bound, which is $\frac{n}{2} = e$, for the exponents of the Satake parameters of the cuspidal spectrum. While in Proposition 5.2, we assume that F is totally imaginary and $n = 2e + 1 \geq 5$ is odd, $\frac{n-1}{2} = e$ is an upper bound for the exponents of the Satake parameters of the cuspidal spectrum. It needs more work to understand if the bound $\frac{n-1}{2} = e$ is *sharp* when F is totally imaginary and $n = 2e + 1 \geq 5$ is odd. It is also not clear that how to construct cuspidal representations with the worst bound for the exponents of the Satake parameters. We will come back to those issues in our future work.

In the last section (Section 6), we characterize the small cuspidal automorphic representations of $\mathrm{Sp}_{2n}(\mathbb{A})$ by means of Fourier coefficients of Fourier-Jacobi type, and by the notion of hypercuspidal automorphic representations in the sense of Piatetski-Shapiro ([41]). As consequence, we prove (Theorem 6.5) that when F is totally imaginary and $n \geq 5$, there does not exist any hypercuspidal automorphic representation of $\mathrm{Sp}_{2n}(\mathbb{A})$. Hence the Ikeda construction will not exist when F is totally imaginary.

The basic facts on the endoscopic classification of the discrete spectrum and the basic conjecture on the relations between the Fourier coefficients of automorphic forms and their global Arthur parameters are recalled in Section 2. Here we also recall the recent, relevant results of the authors, which are used in the rest of this paper.

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2. Fourier Coefficients and Global Arthur Packets

2.1. **Fourier coefficients attached to nilpotent orbits.** In this section, we recall Fourier coefficients of automorphic forms attached to nilpotent orbits, following the formulation in [11], which is slightly more general and easier to use than the one taken in [16] and [20]. Let G be a reductive group defined over F, or a central extension of finite degree. Fix a nontrivial additive character ψ of $F\backslash\mathbb{A}$. Let $\mathfrak{g}$ be the Lie algebra of $\mathrm{G}(F)$ and f be a nilpotent element in $\mathfrak{g}$. The element f defines a function on $\mathfrak{g}$:

$$\psi_f : \mathfrak{g} \to \mathbb{C}^\times$$

by $\psi_f(x) = \psi(\kappa(f, x))$, where κ is the Killing form on $\mathfrak{g}$.

Given any semi-simple element $h \in \mathfrak{g}$, under the adjoint action, $\mathfrak{g}$ is decomposed to a direct sum of eigenspaces $\mathfrak{g}_i^h$ of h corresponding to eigenvalues i. For any rational number $r \in \mathbb{Q}$, let $\mathfrak{g}_{\geq r}^h = \oplus_{r' \geq r}\mathfrak{g}_{r'}^h$. The element h is called *rational semi-simple* if all its eigenvalues are in $\mathbb{Q}$. Given a nilpotent element f, a *Whittaker pair* is a pair (h, f) with $h \in \mathfrak{g}$ being a rational semi-simple element, and $f \in \mathfrak{g}_{-2}^h$. The element h in a Whittaker pair (h, f) is called a *neutral element* for f if there is a nilpotent element $e \in \mathfrak{g}$ such that (e, h, f) is an $\mathfrak{sl}_2$-triple. A Whittaker pair (h, f) with h being a neutral element for f is called a *neutral pair*. For any $X \in \mathfrak{g}$, let $\mathfrak{g}_X$ be the centralizer of X in $\mathfrak{g}$.

Given any Whittaker pair (h, f), define an anti-symmetric form ω_f on $\mathfrak{g}$ by $\omega_f(X, Y) := \kappa(f, [X, Y])$, as above, κ is the Killing form. We denote by $\omega = \omega_f$ when there is no confusion. Let $\mathfrak{u}_h = \mathfrak{g}_{\geq 1}^h$ and let $\mathfrak{n}_h = \ker(\omega)$ be the radial of $\omega|_{\mathfrak{u}_h}$. Then $[\mathfrak{u}_h, \mathfrak{u}_h] \subset \mathfrak{g}_{\geq 2}^h \subset \mathfrak{n}_h$. By [11, Lemma 3.2.6], $\mathfrak{n}_h = \mathfrak{g}_{\geq 2}^h + \mathfrak{g}_1^h \cap \mathfrak{g}_f$. Note that if the Whittaker pair (h, f) comes from an $\mathfrak{sl}_2$-triple (e, h, f), then $\mathfrak{n}_h = \mathfrak{g}_{\geq 2}^h$. Let $U_h = \exp(\mathfrak{u}_h)$ and $N_h = \exp(\mathfrak{n}_h)$ be the corresponding unipotent subgroups of G. Define a character of N_h by $\psi_f(n) = \psi(\kappa(f, \log(n)))$. Let $N_h' = N_h \cap \ker(\psi_f)$. Then U_h/N_h' is a Heisenberg group with center N_h/N_h'. It follows that for each Whittaker pair (h, f), ψ_f defines a character of $N_h(\mathbb{A})$ which is trivial on $N_h(F)$.

Assume that π be an automorphic representation of $\mathrm{G}(\mathbb{A})$. Define a *degenerate Whittaker-Fourier coefficient* of $\varphi \in \pi$ by

$$\mathcal{F}_{h,f}(\varphi)(g) = \int_{N_h(F)\backslash N_h(\mathbb{A})} \varphi(ng)\overline{\psi}_f(n)dn, g \in \mathrm{G}(\mathbb{A}). \tag{2.1}$$

Let $\mathcal{F}_{h,f}(\pi) = \{\mathcal{F}_{h,f}(\varphi) | \varphi \in \pi\}$. If h is a neutral element for f, then $\mathcal{F}_{h,f}(\varphi)$ is also called a *generalized Whittaker-Fourier coefficient* of φ. The (global) *wave-front set* $\mathfrak{n}(\pi)$ of π is defined to the set of nilpotent orbits $\mathcal{O}$ such that $\mathcal{F}_{h,f}(\pi)$ is nonzero, for some Whittaker pair (h, f) with $f \in \mathcal{O}$ and h being a neutral element for f. Note that if $\mathcal{F}_{h,f}(\pi)$ is nonzero for some Whittaker pair (h, f) with $f \in \mathcal{O}$ and h being a neutral element for f, then it is nonzero for any such Whittaker pair (h, f), since the non-vanishing property of such Fourier coefficients does not depends on the choices of representatives of $\mathcal{O}$. Let $\mathfrak{n}^m(\pi)$ be the set of maximal elements in $\mathfrak{n}(\pi)$ under the natural order of nilpotent orbits. The following theorem is one of the main results in [11].

Theorem 2.1 (Theorem C, [11])**.** *Let π be an automorphic representation of $\mathrm{G}(\mathbb{A})$. Given two Whittaker pairs (h, f) and (h', f), with h being a neutral element for f, if $\mathcal{F}_{h',f}(\pi)$ is nonzero, then $\mathcal{F}_{h,f}(\pi)$ is nonzero.*

When G is a quasi-split classical group, it is known that the nilpotent orbits are parametrized by pairs $(\underline{p}, \underline{q})$, where $\underline{p}$ is a partition and $\underline{q}$ is a set of non-degenerate quadratic forms (see [46]). When $\mathrm{G} = \mathrm{Sp}_{2n}$, $\underline{p}$ is a symplectic partition, namely, odd parts occur with even multiplicities. When $\mathrm{G} = \mathrm{SO}_{2n}^{\alpha}, \mathrm{SO}_{2n+1}$, $\underline{p}$ is an orthogonal partition, namely, even parts occur with even multiplicities. Note that if α is not a square in $F^\times$, $\mathrm{SO}_{2n}^{\alpha}$ denotes the quasi-split orthogonal group, corresponding to the quadratic form in $2n$ variables, with Witt index $n-1$ and discriminant $(-1)^n\alpha$. In these cases, let $\mathfrak{p}^m(\pi)$ be the partitions corresponding to nilpotent orbits in $\mathfrak{n}^m(\pi)$, that is, the maximal nilpotent orbits in the wave-front set $\mathfrak{n}(\pi)$ of the automorphic representation π.

Convention. *Let* G *be a quasi-split classical group and π be an automorphic representation of $\mathrm{G}(\mathbb{A})$. For any symplectic/orthogonal partition $\underline{p}$, by a Fourier coefficient attached to $\underline{p}$, we mean a generalized Whittaker-Fourier coefficient $\mathcal{F}_{h,f}(\varphi)$ attached to an orbit $\mathcal{O}$ parametrized by a pair $(\underline{p}, \underline{q})$ for some $\underline{q}$, where $\varphi \in \pi$, $f \in \mathcal{O}$ and h is a neutral element for f. Sometimes, for convenience, we also write a Fourier coefficient attached to $\underline{p}$ as $\mathcal{F}^{\psi_{\underline{p}}}(\varphi)$ without specifying the F-rational orbit $\mathcal{O}$ and Whittaker pairs.*

Next, we recall the following result of [18], which is one of the main ingredients of this paper.

Theorem 2.2 (Theorem 5.3, [18]). *Assume that F is a totally imaginary number field. Let π be a cuspidal automorphic representation of $\mathrm{Sp}_{2n}(\mathbb{A})$ or the metaplectic double cover $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$. Then there exists an even partition (that is, consisting of only even parts) in $\mathfrak{p}^m(\pi)$, as constructed in* [8], *with the property that*

$$\underline{p}_\pi := [(2n_1)^{s_1}(2n_2)^{s_2}\cdots(2n_r)^{s_r}],$$

with $2n_1 > 2n_2 > \cdots > 2n_r$ and $s_i \leq 4$ for $1 \leq i \leq r$.

In this paper, we will consider two orders on the set of all partitions as follows. For a given partition $\underline{p} = [p_1p_2\cdots p_r]$, define $|\underline{p}| = \sum_{i=1}^r p_i$.

Definition 2.3. (1). **Lexicographical order**. *Given two partitions $\underline{p} = [p_1p_2\cdots p_r]$ with $p_1 \geq p_2 \geq \cdots \geq p_r$, and $\underline{q} = [q_1q_2\cdots q_r]$ with $q_1 \geq q_2 \geq \cdots \geq q_r$, (add zeros at the end if needed) which may not be partitions of the same positive integer, i.e., $|\underline{p}|$ and $|\underline{q}|$ may not be equal. If there exists $1 \leq i \leq r$ such that $p_j = q_j$ for $1 \leq j \leq i-1$, and $p_i < q_i$, then we say that $\underline{p} < \underline{q}$ under the lexicographical order of partitions. Lexicographical order is a total order.*

(2). **Dominance order**. *Given two partitions $\underline{p} = [p_1p_2\cdots p_r]$ with $p_1 \geq p_2 \geq \cdots \geq p_r$, and $\underline{q} = [q_1q_2\cdots q_r]$ with $q_1 \geq q_2 \geq \cdots \geq q_r$ (add zeros at the end if needed), which again may not be partitions of the same positive integer, i.e., $|\underline{p}|$ and $|\underline{q}|$ may not be equal. If for any $1 \leq i \leq r$, $\sum_{j=1}^i p_j \leq \sum_{j=1}^i q_j$, then we say that $\underline{p} \leq \underline{q}$ under the dominance order of partitions. Dominance order is a partial order.*

Remark 2.4. *Given two partitions $\underline{p}$ and $\underline{q}$, if we do not specify which order of partitions, by $\underline{p} \leq \underline{q}$, we mean that it is under the dominance order of partitions.*

2.2. **Automorphic discrete spectrum and Fourier coefficients.** In this paper, we consider mainly the symplectic groups. Although the methods are expected to work for all quasi-split classical groups, due to the state of art in the current development of the theory, one knows much less when the classical groups are not of symplectic type. Hence we will concentrate on symplectic groups here and leave the discussion for other classical groups in the future.

For the symplectic group Sp_{2n}, the endoscopic classification of the discrete spectrum was obtained by Arthur in [2]. A preliminary statement of the endoscopic classification is recalled below.

Theorem 2.5 (Arthur [2]). *For any $\pi \in \mathcal{A}_2(\mathrm{Sp}_{2n})$, there exists a global Arthur parameter*

$$\psi = \psi_1 \boxplus \cdots \boxplus \psi_r,$$

such that $\pi \in \widetilde{\Pi}_\psi(\mathrm{Sp}_{2n})$, the global Arthur packet associated to ψ.

The notation used in this theorem can be explained as follows. Each $\psi_i = (\tau_i, b_i)$ is called a *simple Arthur parameter*, where $\tau_i \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_{a_i})$ with central character ω_{τ_i}, and $b_i \in \mathbb{Z}_{\geq 1}$. Every simple Arthur parameter ψ_i is of orthogonal type. This means that if τ_i is of symplectic type, that is, $L(s, \tau_i, \wedge^2)$ has a pole at $s = 1$, then b_i must be even; and if τ_i is of orthogonal type, that is, $L(s, \tau_i, \mathrm{Sym}^2)$ has a pole at $s = 1$, then b_i must be odd. In order for the formal sum $\psi = \psi_1 \boxplus \cdots \boxplus \psi_r$ to be a global Arthur parameter in $\widetilde{\Psi}_2(\mathrm{Sp}_{2n})$, one requires that $2n+1 = \sum_{i=1}^r a_i b_i$, $\prod_{i=1}^r \omega_{\tau_i}^{b_i} = 1$, and the simple parameters ψ_i are pair-wise different.

A global Arthur parameter ψ is called *generic*, following [2], if the integers b_i are one. The set of generic global Arthur parameters is denoted by $\widetilde{\Phi}_2(\mathrm{Sp}_{2n})$. A generic global Arthur parameter ϕ can be written as $\phi = (\tau_1, 1) \boxplus (\tau_2, 1) \boxplus \cdots \boxplus (\tau_r, 1)$.

Theorem 2.6 (Theorem 3.3, [20])**.** *For any generic global Arthur parameter $\phi = \boxplus_{i=1}^r (\tau_i, 1) \in \widetilde{\Phi}_2(\mathrm{Sp}_{2n})$, there is an irreducible generic cuspidal automorphic representation π of $\mathrm{Sp}_{2n}(\mathbb{A})$ belonging to $\widetilde{\Pi}_\psi(\mathrm{Sp}_{2n})$, and hence $\mathfrak{p}^m(\pi) = \{[(2n)]\}$.*

Theorem 2.6 was proved in [20] by using the automorphic descent of Ginzburg, Rallis and Soudry ([10]). Following the endoscopic classification of Arthur ([2]), Theorem 2.6 implies that every tempered global L-packet has a generic member, i.e. the global Shahidi conjecture holds. Note that by analyzing constant terms of residual representations, Mœglin ([37, Proposition 1.2.1]) shows that if there is a residual representation occurring in $\widetilde{\Pi}_\psi(\mathrm{Sp}_{2n})$, then the Arthur parameter is never generic. Hence we have

$$\widetilde{\Pi}_\phi(\mathrm{Sp}_{2n}) \cap \mathcal{A}_2(\mathrm{Sp}_{2n}) \subset \mathcal{A}_{\mathrm{cusp}}(\mathrm{Sp}_{2n}),$$

for all generic global Arthur parameters $\phi \in \widetilde{\Phi}_2(\mathrm{Sp}_{2n})$. For general Arthur parameters $\psi = \boxplus_{i=1}^r (\tau_i, b_i) \in \widetilde{\Psi}_2(\mathrm{Sp}_{2n})$, [16, Conjecture 4.2] extends Theorem 2.6 naturally. Before stating this conjecture, we first recall the definition of Barbasch-Vogan dual of partitions in the following remark.

Remark 2.7. *Given a partition $\underline{p} = [p_1 p_2 \cdots p_r]$ of $2n+1$ with $p_1 \geq p_2 \geq \cdots \geq p_r > 0$, even parts occurring with even multiplicities. By* [3, Definition A1] *and* [1, Section 3.5]*, Barbasch-Vogan dual $\eta(\underline{p})$ is defined to be $((\underline{p}^-)_{\mathrm{Sp}})^t$, which is a partition of $2n$. More precisely, one has that $\underline{p}^- = [p_1 p_2 \cdots (p_r - 1)]$ and $(\underline{p}^-)_{\mathrm{Sp}}$ is the biggest symplectic partition that is less than or equal to $\underline{p}^-$. We refer to* [5, Lemma 6.3.8] *for the recipe* *of obtaining* $(\underline{p}^-)_{\mathrm{Sp}}$ *from* $\underline{p}^-$. $(\underline{p}^-)_{\mathrm{Sp}}$ *is called the symplectic collapse of* $\underline{p}^-$. *Finally,* $((\underline{p}^-)_{\mathrm{Sp}})^t$ *is the transpose of* $(\underline{p}^-)_{\mathrm{Sp}}$. *By* [1, Lemma 3.3], *one has that* $\eta(\underline{p}) = ((\underline{p}^t)^-)_{\mathrm{Sp}}$.

We note that when G *and its complex dual of* G *are of the same type, Barbasch-Vogan duality defined in* [3, Definition A1] *is the same as Lusztig-Spaltenstein duality as discussed* [36] *and* [45].

Conjecture 2.8 ([16]). *For a given global Arthur parameter* $\psi = \boxplus_{i=1}^r (\tau_i, b_i) \in \widetilde{\Psi}_2(\mathrm{Sp}_{2n})$, *the partition* $\eta(\underline{p}_\psi)$, *which is Barbasch-Vogan dual of the partition* $\underline{p}_\psi = [b_1^{a_1} b_2^{a_2} \cdots b_r^{a_r}]$ *associated to the parameter* ψ, *has the following properties:*

1. $\eta(\underline{p}_\psi)$ *is greater than or equal to any* $\underline{p} \in \mathfrak{p}^m(\pi)$ *for all* $\pi \in \widetilde{\Pi}_\psi(\mathrm{Sp}_{2n}) \cap \mathcal{A}_2(\mathrm{Sp}_{2n})$, *under the dominance order of partitions as in Definition 2.3; and*
2. *there exists a* $\pi \in \widetilde{\Pi}_\psi(\mathrm{Sp}_{2n}) \cap \mathcal{A}_2(\mathrm{Sp}_{2n})$ *with* $\eta(\underline{p}_\psi) \in \mathfrak{p}^m(\pi)$.

We recall the following result from [21], which is also a main ingredient of this paper.

Theorem 2.9 (Theorem 1.3 and Proposition 6.4, [21]). *For a given global Arthur parameter* $\psi = \boxplus_{i=1}^r (\tau_i, b_i) \in \widetilde{\Psi}_2(\mathrm{Sp}_{2n})$, *Barbasch-Vogan dual* $\eta(\underline{p}_\psi)$ *is greater than or equal to any* $\underline{p} \in \mathfrak{p}^m(\pi)$ *for every* $\pi \in \widetilde{\Pi}_\psi(\mathrm{Sp}_{2n}) \cap \mathcal{A}_2(\mathrm{Sp}_{2n})$, *under the lexicographical order of partitions as in Definition 2.3.*

It is clear that when the global Arthur parameter $\psi = \phi$ is generic, the partition $\underline{p}_\phi = [1^{2n+1}]$, and hence the partition $\eta(\underline{p}_\phi) = [(2n)]$, which corresponds to the regular nilpotent orbit in $\mathfrak{sp}_{2n}$. Since any symplectic partition is less than or equal to $[(2n)]$, it follows that Conjecture 2.8 holds for all generic Arthur parameters $\phi \in \widetilde{\Phi}_2(\mathrm{Sp}_{2n})$. Hence, it is more delicate to understand the lower bound for partitions $\underline{p} \in \mathfrak{p}^m(\pi)$ for all $\pi \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{Sp}_{2n})$. It is even harder to understand the lower bound for partitions $\underline{p} \in \mathfrak{p}^m(\pi)$ when $\pi \in \widetilde{\Pi}_\psi(\mathrm{Sp}_{2n}) \cap \mathcal{A}_{\mathrm{cusp}}(\mathrm{Sp}_{2n})$ for a given global Arthur parameter $\psi \in \widetilde{\Psi}_2(\mathrm{Sp}_{2n})$.

Problem 2.10. *Find symplectic partitions* $\underline{p}_0$ *of* $2n$ *with the property that*

1. *there exists a* $\pi \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{Sp}_{2n})$ *such that* $\underline{p}_0 \in \mathfrak{p}^m(\pi)$, *but*
2. *for any* $\pi \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{Sp}_{2n})$, *there does not exist a partition* $\underline{p} \in \mathfrak{p}^m(\pi)$ *such that* $\underline{p} < \underline{p}_0$, *under the dominance order of partitions as in Definition 2.3.*

This problem was motivated by the theory of singular automorphic representations of $\mathrm{Sp}_{2n}(\mathbb{A})$, which is briefly recalled in the following section.

2.3. **Singular automorphic representations.** In this section, consider $\mathrm{G}_n = \mathrm{Sp}_{2n}, \mathrm{SO}_{2n+1}$ or SO_{2n} to be split classical groups. The theory of singular automorphic representations of $\mathrm{G}_n(\mathbb{A})$ has been developed based on the notion of ranks for unitary representations of Howe ([12]) and by the fundamental work of Li ([33]).

When $\mathrm{G}_n = \mathrm{Sp}_{2n}$ is the symplectic group, defined by the skew-symmetric matrix $J_n = \begin{pmatrix} 0 & w \\ -w & 0 \end{pmatrix}$, with $w = (w_{ij})_{n\times n}$ anti-diagonal, and $w_{ij} = 0$ or 1. Take $P_n = M_n U_n$ to be the Siegel parabolic subgroup of Sp_{2n}. Hence $M_n \cong \mathrm{GL}_n$ and the elements of U_n are of form

$$u(X) = \begin{pmatrix} I_n & X \\ 0 & I_n \end{pmatrix}.$$

Pontryagin duality asserts that the group of unitary characters $U_n(\mathbb{A})$ which are trivial on $U_n(F)$ is isomorphic to $\mathrm{Sym}^2(F^n)$, i.e.

$$\widehat{U_n(F)\backslash U_n}(\mathbb{A}) \cong \mathrm{Sym}^2(F^n).$$

The explicit isomorphism is given as follows. Take ψ_F to be a nontrivial additive character of $F\backslash\mathbb{A}$. For any $T \in \mathrm{Sym}^2(F^n)$, i.e. any $n\times n$ symmetric matrix T, the corresponding character ψ_T is given by

$$\psi_T(u(X)) := \psi_F(\mathrm{tr}(TwX)).$$

The adjoint action of the Levi subgroup GL_n on U_n induces an action of GL_n on $\mathrm{Sym}^2(F^n)$. For an automorphic form φ on $\mathrm{Sp}_{2n}(\mathbb{A})$, the ψ_T-Fourier coefficient is defined by

$$\mathcal{F}^{\psi_T}(\varphi)(g) := \int_{U_m(F)\backslash U_n(\mathbb{A})} \varphi(u(X)g)\psi_T^{-1}(u(X))du(X). \tag{2.2}$$

An automorphic form φ on $\mathrm{Sp}_{2n}(\mathbb{A})$ is called *non-singular* if φ has a nonzero ψ_T-Fourier coefficient for some T with maximal F-rank, which is n, and *singular* otherwise. In other words, an automorphic form φ on $\mathrm{Sp}_{2n}(\mathbb{A})$ is called *singular* if φ has the property that if a ψ_T-Fourier coefficient $\mathcal{F}^{\psi_T}(\varphi)$ is nonzero, then $\det(T) = 0$.

Based on his notion of ranks for unitary representations, Howe shows in [12] that if an automorphic form φ on $\mathrm{Sp}_{2n}(\mathbb{A})$ is singular, then φ can be expressed as a linear combination of certain theta functions. Li in [32] shows that a cuspidal automorphic form of $\mathrm{Sp}_{2n}(\mathbb{A})$ with n even is distinguished, i.e. φ has a nonzero ψ_T-Fourier coefficient with only one GL_n-orbit of non-degenerate T if and only if φ is in the image of the theta lifting from the orthogonal group O_T defined by T. A family of explicit examples of such distinguished cuspidal automorphic representations of $\mathrm{Sp}_{2n}(\mathbb{A})$ with n even was constructed by Piatetski-Shapiro and Rallis in [42]. Furthermore, Li proves in [33] the following theorem.

Theorem 2.11 ([33])**.** *For any classical group* G_n*, cuspidal automorphic forms on* $\mathrm{G}_n(\mathbb{A})$ *are non-singular.*

For orthogonal groups G_n, the singularity of automorphic forms can be defined as follows, following [33]. Let (V,q) be a non-degenerate quadratic space defined over F of dimension m with Witt index $n = [\frac{m}{2}]$. Let X^+ be a maximal totally isotropic subspace of V, which has dimension n, and let X^- be the maximal totally isotropic subspace of V dual to X^+ with respect to q. Hence we have the polar decomposition

$$V = X^- + V_0 + X^+$$

with V_0 being the orthogonal complement of $X^- + X^+$, which has dimension zero or one. The generalized flag

$$\{0\} \subset X^+ \subset V$$

defines a maximal parabolic subgroup P_{X^+}, whose Levi part M_{X^+} is isomorphic to GL_n and whose unipotent radical N_{X^+} is abelian if m is even; and is a two-step unipotent subgroup with its center Z_{X^+} if m is odd. When m is even, we set $Z_{X^+} = N_{X^+}$. Again, by Pontryagin duality, we have

$$Z_{X^+}\widehat{(F)\backslash Z}_{X^+}(\mathbb{A}) \cong \wedge^2(F^n),$$

which is given explicitly, as in the case Sp_{2n}, by the following formula: For any $T \in \wedge^2(F^{[\frac{m}{2}]})$,

$$\psi_T(z(X)) := \psi_F(\mathrm{tr}(TwX)).$$

The adjoint action of the Levi subgroup GL_n on Z_{X^+} induces an action of GL_n on the space $\wedge^2(F^n)$. For an automorphic form φ on $\mathrm{G}(\mathbb{A})$, the ψ_T-Fourier coefficient is defined by

$$\mathcal{F}^{\psi_T}(\varphi)(g) := \int_{Z_{X^+}(F)\backslash Z_{X^+}(\mathbb{A})} \varphi(z(X)g)\psi_T^{-1}(z(X))dz(X). \tag{2.3}$$

An automorphic form φ on $\mathrm{G}(\mathbb{A})$ is called *non-singular* if φ has a non-zero ψ_T-Fourier coefficient for some $T \in \wedge^2(F^n)$ of maximal rank.

Following Section 2.1, we may reformulate the maximal rank Fourier coefficients of automorphic forms in terms of partitions, and denote by $\underline{p}_{\mathrm{ns}}$ the partition corresponding to the non-singular Fourier coefficients.

It is easy to figure out the following (for definitions of special partitions, see [5, Section 6.3]):

(1) When $\mathrm{G}_n = \mathrm{Sp}_{2n}$, one has $\underline{p}_{\mathrm{ns}} = [2^n]$. This is a special partition for Sp_{2n}.
(2) When $\mathrm{G}_n = \mathrm{SO}_{2n+1}$, one has
$$\underline{p}_{\mathrm{ns}} = \begin{cases} [2^{2e}1] & \text{if } n = 2e; \\ [2^{2e}1^3] & \text{if } n = 2e+1. \end{cases}$$
This is not a special partition of SO_{2n+1}.
(3) When $\mathrm{G}_n = \mathrm{SO}_{2n}$, one has
$$\underline{p}_{\mathrm{ns}} = \begin{cases} [2^{2e}] & \text{if } n = 2e; \\ [2^{2e}1^2] & \text{if } n = 2e+1. \end{cases}$$
This is a special partition of SO_{2n}.

According to [23], for any automorphic representation π, the set $\mathfrak{p}^m(\pi)$ contains only special partitions. Since the non-singular partition $\underline{p}_{\mathrm{ns}}$ is not special when $\mathrm{G}_n = \mathrm{SO}_{2n+1}$, the partitions contained in $\mathfrak{p}^m(\pi)$ as π runs over the cuspidal spectrum of G_n should be greater than or equal to the following partition
$$\underline{p}_{\mathrm{ns}}^{\mathrm{G}_n} = \begin{cases} [32^{2e-2}1^2] & \text{if } n = 2e; \\ [32^{2e-2}1^4] & \text{if } n = 2e+1. \end{cases}$$
Following [5], $\underline{p}_{\mathrm{ns}}^{\mathrm{G}_n}$ denotes the G_n-expansion of the partition $\underline{p}_{\mathrm{ns}}$, i.e., the smallest special partition which is greater than or equal to $\underline{p}_{\mathrm{ns}}$. Of course, when $\mathrm{G}_n = \mathrm{Sp}_{2n}$ or SO_{2n}, one has that $\underline{p}_{\mathrm{ns}}^{\mathrm{G}_n} = \underline{p}_{\mathrm{ns}}$.

Proposition 2.12. *For split classical group G_n, the G_n-expansion of the non-singular partition, $\underline{p}_{\mathrm{ns}}^{\mathrm{G}_n}$, is a lower bound for partitions in the set $\mathfrak{p}^m(\pi)$ as π runs over the cuspidal spectrum of G_n.*

It is natural to ask whether the lower bound $\underline{p}_{\mathrm{ns}}^{\mathrm{G}_n}$ is sharp. This is to construct or find an irreducible cuspidal automorphic representation π of $\mathrm{G}_n(\mathbb{A})$ with the property that $\underline{p}_{\mathrm{ns}}^{\mathrm{G}_n} \in \mathfrak{p}^m(\pi)$.

When $\mathrm{G}_n = \mathrm{Sp}_{4e}$ with $n = 2e$ even, and when F is totally real, the examples constructed by Ikeda ([14] and [15]) are irreducible cuspidal automorphic representations π of $\mathrm{Sp}_{4e}(\mathbb{A})$ with the global Arthur parameter $\psi = (\tau, 2e) \boxplus (1,1)$, where $\tau \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_2)$ is of symplectic type. By Theorem 2.9, for any partition $\underline{p} \in \mathfrak{p}^m(\pi)$, we should have
$$\underline{p} \leq \eta(\underline{p}_{\psi}) = [2^{2e}] = \underline{p}_{\mathrm{ns}}^{\mathrm{Sp}_{4e}},$$

under the lexicographical order of partitions, which automatically implies that $\underline{p} \leq [2^{2e}] = \underline{p}^{\mathrm{Sp}_{4e}}_{\mathrm{ns}}$ under the dominance order of partitions. On the other hand, by Theorem 2.11, for any partition $\underline{p} \in \mathfrak{p}^m(\pi)$, we must have

$$\underline{p}^{\mathrm{Sp}_{4e}}_{\mathrm{ns}} = [2^{2e}] \leq \underline{p},$$

under the dominance order of partitions. It follows that $\mathfrak{p}^m(\pi) = \{[2^{2e}] = \underline{p}^{\mathrm{Sp}_{4e}}_{\mathrm{ns}}\}$.

Proposition 2.13. *When F is totally real, the non-singular partition $\underline{p}^{\mathrm{Sp}_{4e}}_{\mathrm{ns}} = \underline{p}_{\mathrm{ns}} = [2^{2e}]$ is the sharp lower bound in the sense that for all $\pi \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{Sp}_{4e})$, the partition $\underline{p}^{\mathrm{Sp}_{4e}}_{\mathrm{ns}} \in \mathfrak{p}(\pi)$ and there exists a $\pi \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{Sp}_{4e})$, as constructed in [14] and [15], such that $\underline{p}^{\mathrm{Sp}_{4e}}_{\mathrm{ns}} \in \mathfrak{p}^m(\pi)$.*

It is clear that the assumption that F must be totally real is substantial in the construction of Ikeda in [14] and [15]. However, there is no known approach to carry out a similar construction when F is not totally real. We are going to discuss the situation in the following sections when F is totally imaginary, which leads to a totally different conclusion.

Also the situation is different when we consider orthogonal groups. For G_n to be SO_{2n+1} or SO_{2n}, in the spirit of a conjecture of Ginzburg ([7]), any partition $\underline{p}$ in $\mathfrak{p}(\pi)$ with $\pi \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{G}_n)$ should contain only odd parts. Hence it is reasonable to conjecture the existence of a lower bound which is better than the one determined by non-singularity of cuspidal automorphic representations.

Conjecture 2.14. *For G_n an F-split SO_{2n+1} or SO_{2n}, the sharp lower bound partition $\underline{p}^{\mathrm{G}_n}_0$ for $\underline{p} \in \mathfrak{p}(\pi)$, as π runs over $\mathcal{A}_{\mathrm{cusp}}(\mathrm{G}_n)$, is given as follows:*

(1) *When $\mathrm{G}_n = \mathrm{SO}_{2n+1}$,*

$$\underline{p}^{\mathrm{SO}_{2n+1}}_0 = \begin{cases} [3^e 1^{e+1}] & \text{if } n = 2e; \\ [3^{e+1} 1^e] & \text{if } n = 2e+1. \end{cases}$$

(2) *When $\mathrm{G}_n = \mathrm{SO}_{2n}$,*

$$\underline{p}^{\mathrm{SO}_{2n}}_0 = \begin{cases} [3^e 1^e] & \text{if } n = 2e; \\ [5 3^{e-1} 1^e] & \text{if } n = 2e+1. \end{cases}$$

We note that a sharp lower bound partition for the Fourier coefficients of all irreducible cuspidal representations of $\mathrm{G}_n(\mathbb{A})$ involves deep arithmetic of the base field F, which is one of the main concerns in our investigation. Following the line of ideas in [12] and [33], we define the following set of *small partitions* for the cuspidal spectrum of $\mathrm{G}_n(\mathbb{A})$:

$$\mathfrak{p}_{\mathrm{sm}}^{\mathrm{G}_n,F} := \min \cup_{\pi\in\mathcal{A}_{\mathrm{cusp}}(\mathrm{G}_n)} \mathfrak{p}^m(\pi), \tag{2.4}$$

where the minimums are taken under the dominance order of partitions. Note that the set $\mathfrak{p}_{\mathrm{sm}}^{\mathrm{G}_n,F}$ may not be singleton. We call a $\pi \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{G}_n)$ *small* if $\mathfrak{p}^m(\pi)\cap\mathfrak{p}_{\mathrm{sm}}^{\mathrm{G}_n,F}$ is not empty. Our discussion for small cuspidal automorphic representations will resume in Section 6.

3. On Cuspidality for General Number Fields

In this section, we assume that F is a general number field. We mainly consider the *cuspidality problem* for the global Arthur packets with a family of global Arthur parameters of form:

$$\psi = (\chi, b) \boxplus (\tau_2, b_2) \boxplus \cdots \boxplus (\tau_r, b_r) \in \widetilde{\Psi}_2(\mathrm{Sp}_{2n}),$$

where χ is Hecke character, and for $2 \le i \le r$, τ_i is a cuspidal representation of $\mathrm{GL}_{a_i}(\mathbb{A})$, $b + \sum_{i=2}^r a_i b_i = 2n+1$. When b is large, it is most likely that the corresponding global Arthur packet $\widetilde{\Pi}_\psi(\mathrm{Sp}_{2n})$ contains no cuspidal members.

Recall from Section 2.2 that by Conjecture 2.8 for $\mathrm{G}_n = \mathrm{Sp}_{2n}$, for any $\pi \in \widetilde{\Pi}_\psi(\mathrm{Sp}_{2n}) \cap \mathcal{A}_2(\mathrm{Sp}_{2n})$, it is expected that for any partition $\underline{p} \in \mathfrak{p}^m(\pi)$, one should have

$$\underline{p} \le \eta(\underline{p}_\psi), \tag{3.1}$$

under the dominance order of partitions. We will take this as an *assumption* for the discussion in this section.

For $\psi = (\chi, b)\boxplus(\tau_2, b_2)\boxplus\cdots\boxplus(\tau_r, b_r) \in \widetilde{\Psi}_2(\mathrm{Sp}_{2n})$, with χ a quadratic character, the partition associated to ψ is

$$\underline{p}_\psi = [(b)^1(b_2)^{a_2}\cdots(b_r)^{a_r}].$$

By the definition of Arthur parameters for Sp_{2n}, b is automatically odd. As explained in Remark 2.7, $\eta(\underline{p}_\psi) = ((\underline{p}_\psi^t)^-)_{\mathrm{Sp}}$. Assume that $b > b_0 := \max(b_2, \ldots, b_r)$, then

$$\underline{p}_\psi^t = [(1)^b] + [(a_2)^{b_2}] + \cdots + [(a_r)^{b_r}]$$

has the form $[(1 + \sum_{i=2}^r a_i)p_2\cdots p_{b_0}(1)^{b-b_0}]$, and

$$(\underline{p}_\psi^t)^- = [(1 + \sum_{i=2}^r a_i)p_2\cdots p_{b_0}(1)^{b-b_0-1}].$$

After taking the symplectic collapse, $\eta(\underline{p}_\psi) = ((\underline{p}_\psi^t)^-)_{\mathrm{Sp}}$ has the form

$$[q_1q_2\cdots q_k(1)^m],$$

with $m \leq b-1-\sum_{i=2}^{r} b_i$, and $k+m=b-1$.

If there is a $\pi \in \widetilde{\Pi}_\psi(\mathrm{Sp}_{2n}) \cap \mathcal{A}_{\mathrm{cusp}}(\mathrm{Sp}_{2n})$, by Theorem 2.11, π has a nonzero Fourier coefficient attached to the partition $[2^n]$. It is clear that $b > n+1$ if and only if $[2^n]$ is either greater than or not related to the above partition $[q_1 q_2 \cdots q_k (1)^m]$. Hence, we have the following result.

Theorem 3.1. *Assume that* (3.1) *holds. For*

$$\psi = (\chi, b) \boxplus (\tau_2, b_2) \boxplus \cdots \boxplus (\tau_r, b_r) \in \widetilde{\Psi}_2(\mathrm{Sp}_{2n})$$

with χ a quadratic character, if $b > n+1$, then the intersection

$$\widetilde{\Pi}_\psi(\mathrm{Sp}_{2n}) \cap \mathcal{A}_{\mathrm{cusp}}(\mathrm{Sp}_{2n})$$

is empty.

Here is an example illustrating the theorem.

Example 3.2. *Consider $\psi = (\chi, 7) \boxplus (\tau, 2) \in \widetilde{\Psi}_2(\mathrm{Sp}_{10})$, where $\chi = 1_{\mathrm{GL}_1(\mathbb{A})}$, and $\tau \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_2)$ with $L(s, \tau, \wedge^2)$ having a pole at $s=1$. $\underline{p}_\psi = [72^2]$ and $\eta(\underline{p}_\psi) = [3^2 1^4]$, which is not related to $[2^5]$. Hence, by the assumption that* (3.1) *holds, there are no cuspidal members in the global Arthur packet $\widetilde{\Pi}_\psi(\mathrm{Sp}_{10})$.*

Remark 3.3. *In* [30, Theorem 7.2.5]*, Kudla and Rallis show that for a given $\pi \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{Sp}_{2n})$ and a quadratic character χ, the L-function $L(s, \pi \times \chi)$ has its right-most possible pole at $s = 1 + [\frac{n}{2}]$. This implies that the simple global Arthur parameter of type (χ, b) occurring in the global Arthur parameter of π must satisfy the condition that b is at most $2[\frac{n}{2}]+1$. Because b has to be odd in this case, it follows that b is at most $n+1$ if n is even, and b is at most n if n is odd. In any case, one obtains that if $b > n+1$, then the simple global Arthur parameter of type (χ, b) can not occur in the global Arthur parameter of π for any $\pi \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{Sp}_{2n})$. This matches the result in the above theorem.*

Corollary 3.4. *Assume that* (3.1) *holds. For*

$$\psi = (\tau_1, b_1) \boxplus (\tau_2, b_2) \boxplus \cdots \boxplus (\tau_r, b_r) \in \widetilde{\Psi}_2(\mathrm{Sp}_{2n}),$$

if the set $\widetilde{\Pi}_\psi(\mathrm{Sp}_{2n}) \cap \mathcal{A}_{\mathrm{cusp}}(\mathrm{Sp}_{2n})$ is not empty, then b_i is bounded above by $2[\frac{n}{2}]+1$ for $i = 1, 2, \cdots, r$.

We will discuss the sharpness of the upper bound $2[\frac{n}{2}]+1$ in Section 5.

4. On Cuspidality for Totally Imaginary Fields

In this section, we assume that F is a totally imaginary number field. We show that there are more global Arthur packets that contain no cuspidal members in this situation. It is an interesting question to discover the significance of such a difference depending on the arithmetic of the ground field F.

4.1. **On criteria for cuspidality.** For any $\underline{a}=(a_1,a_2,\dots,a_r)\in\mathbb{Z}_{\geq 1}^r$, define a set $B_{\underline{a}}$, depending only on $\underline{a}$, to be the subset of $\mathbb{Z}_{\geq 1}^r$ that consists of all r-tuples $\underline{b}=(b_1,b_2,\dots,b_r)$ with the property: There are some self-dual $\tau_i\in\mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_{a_i})$ for $1\leq i\leq r$, such that

$$\psi=(\tau_1,b_1)\boxplus(\tau_2,b_2)\boxplus\cdots\boxplus(\tau_r,b_r)$$

belongs to $\widetilde{\Psi}_2(\mathrm{Sp}_{2n})$ for some $n\geq 1$ with $2n+1=\sum_{i=1}^r a_ib_i$. We define an integer $N_{\underline{a}}$ that depends only on $\underline{a}$ by

$$N_{\underline{a}}=\begin{cases}(\sum_{i=1}^r a_i)^2+2(\sum_{i=1}^r a_i) & \textit{if } \sum_{i=1}^r a_i \textit{is even;}\\ (\sum_{i=1}^r a_i)^2-1 & \textit{otherwise.}\end{cases} \tag{4.1}$$

Theorem 4.1. *Assume that F is a totally imaginary number field. Given an $\underline{a}=(a_1,a_2,\dots,a_r)\in\mathbb{Z}_{\geq 1}^r$ that defines the set $B_{\underline{a}}$ and the integer $N_{\underline{a}}$ as above. For any $\underline{b}=(b_1,b_2,\dots,b_r)\in B_{\underline{a}}$, write $2n+1=\sum_{i=1}^r a_ib_i$. If the condition*

$$2n=(\sum_{i=1}^r a_ib_i)-1>N_{\underline{a}}$$

holds, then for any global Arthur parameter ψ of the form

$$\psi=(\tau_1,b_1)\boxplus(\tau_2,b_2)\boxplus\cdots\boxplus(\tau_r,b_r)\in\widetilde{\Psi}_2(\mathrm{Sp}_{2n}),$$

with $\tau_i\in\mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_{a_i})$ for $i=1,2,\cdots,r$, the set $\widetilde{\Pi}_\psi(\mathrm{Sp}_{2n})\cap\mathcal{A}_2(\mathrm{Sp}_{2n})$ contains no cuspidal members.

Proof. By assumption, $\psi=\boxplus_{i=1}^r(\tau_i,b_i)$ belongs to $\widetilde{\Psi}_2(\mathrm{Sp}_{2n})$. Recall that $\underline{p}_\psi=[(b_1)^{a_1}(b_2)^{a_2}\cdots(b_r)^{a_r}]$ is the partition of $2n+1$ attached to ψ. By Remark 2.7, $\eta(\underline{p}_\psi)=((\underline{p}_\psi^t)^-)_{\mathrm{Sp}}$. Then Barbasch-Vogan dual $\eta(\underline{p}_\psi)$ has the following form

$$[(\sum_{i=1}^r a_i)p_2\cdots p_s]_{\mathrm{Sp}}, \tag{4.2}$$

where $\sum_{i=1}^r a_i\geq p_2\geq\cdots\geq p_s$. After taking the symplectic collapse of the partition in (4.2), one obtains that $\eta(\underline{p}_\psi)$ must be one of the following three possible forms:

(1) It equals $[(\sum_{i=1}^r a_i)p_2\cdots p_s]$, where $\sum_{i=1}^r a_i$ is even and

$$\sum_{i=1}^r a_i \geq p_2 \geq \cdots \geq p_s.$$

(2) It equals $[(\sum_{i=1}^r a_i)p_2\cdots p_s]$, where $\sum_{i=1}^r a_i$ is odd and

$$\sum_{i=1}^r a_i \geq p_2 \geq \cdots \geq p_s.$$

(3) It equals $[((\sum_{i=1}^r a_i)-1)p_2\cdots p_s]$, where $(\sum_{i=1}^r a_i)$ is odd and

$$(\sum_{i=1}^r a_i) - 1 \geq p_2 \geq \cdots \geq p_s.$$

Assume that π belongs to $\widetilde{\Pi}_\psi(\mathrm{Sp}_{2n}) \cap \mathcal{A}_{\mathrm{cusp}}(\mathrm{Sp}_{2n})$. By Theorem 2.2, one may assume that

$$\underline{p}_\pi = [(2n_1)^{s_1}(2n_2)^{s_2}\cdots(2n_k)^{s_k}] \in \mathfrak{p}^m(\pi)$$

with $n_1 > n_2 > \cdots > n_k \geq 1$ and with the property that $1 \leq s_i \leq 4$ holds for $1 \leq i \leq k$.

Case 1: By Theorem 2.9, we have $2n_1 \leq \sum_{i=1}^r a_i$. It follows that

$$\begin{aligned} 2n &= \sum_{i=1}^k 2n_i s_i \\ &\leq 4(2+4+6+\cdots+\sum_{i=1}^r a_i) \\ &= (\sum_{i=1}^r a_i)^2 + 2(\sum_{i=1}^r a_i) = N_{\underline{a}}. \end{aligned}$$

Cases 2 and 3: By Theorem 2.9, we have $2n_1 \leq (\sum_{i=1}^r a_i) - 1$. It follows that

$$\begin{aligned} 2n &= \sum_{i=1}^k 2n_i s_i \\ &\leq 4(2+4+6+\cdots+(\sum_{i=1}^r a_i)-1) \\ &= (\sum_{i=1}^r a_i)^2 - 1 = N_{\underline{a}}. \end{aligned}$$

Now it is easy to check that for any r-tuple $\underline{b} = (b_1, b_2, \ldots, b_r) \in B_{\underline{a}}$, if $2n = (\sum_{i=1}^r a_i b_i) - 1 > N_{\underline{a}}$, then the global Arthur packets $\widetilde{\Pi}_\psi(\mathrm{Sp}_{2n})$ associated to any global Arthur parameters of the form

$$\psi = (\tau_1, b_1) \boxplus (\tau_2, b_2) \boxplus \cdots \boxplus (\tau_r, b_r)$$

contain no cuspidal members. This completes the proof of the theorem. □

Note that in Theorem 4.1, for a given $\underline{a} = (a_1, a_2, \ldots, a_r) \in \mathbb{Z}^r_{\geq 1}$, the integer n defining the group Sp_{2n} depends on the choice of $\underline{b} = (b_1, b_2, \ldots, b_r) \in B_{\underline{a}}$. We may reformulate the result for a given group Sp_{2n} as follows.

For any r-tuple $\underline{a} = (a_1, a_2, \ldots, a_r) \in \mathbb{Z}^r_{\geq 1}$, define $B^{2n}_{\underline{a}}$ to be the subset of $\mathbb{Z}^r_{\geq 1}$, consisting of r-tuples $\underline{b} = (b_1, b_2, \ldots, b_r)$ such that

$$\psi = (\tau_1, b_1) \boxplus (\tau_2, b_2) \boxplus \cdots \boxplus (\tau_r, b_r) \in \widetilde{\Psi}_2(\mathrm{Sp}_{2n})$$

for some self-dual $\tau_i \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_{a_i})$ with $1 \leq i \leq r$. Note that this set $B^{2n}_{\underline{a}}$ could be empty in this formulation. The integer $N_{\underline{a}}$ is defined to be the same as in (4.1). Theorem 4.1 can be reformulated as follows.

Theorem 4.2. *Assume that F is a totally imaginary number field and that $\underline{a} = (a_1, a_2, \ldots, a_r) \in \mathbb{Z}^r_{\geq 1}$ has a non-empty $B^{2n}_{\underline{a}}$. If $2n > N_{\underline{a}}$, then for any global Arthur parameter ψ of the form*

$$\psi = (\tau_1, b_1) \boxplus (\tau_2, b_2) \boxplus \cdots \boxplus (\tau_r, b_r) \in \widetilde{\Psi}_2(\mathrm{Sp}_{2n}),$$

with $\tau_i \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_{a_i})$ for $i = 1, 2, \cdots, r$, and $\underline{b} = (b_1, b_2, \ldots, b_r) \in B^{2n}_{\underline{a}}$, the set $\widetilde{\Pi}_\psi(\mathrm{Sp}_{2n}) \cap \mathcal{A}_2(\mathrm{Sp}_{2n})$ contains no cuspidal members.

On the one hand, the integer $N_{\underline{a}}$ is not hard to calculate. This makes Theorems 4.1 and 4.2 easy to use. On the other hand, the integer $N_{\underline{a}}$ depends only on $\underline{a}$, and hence may not carry enough information for some applications. Next, we try to improve the above bound $N_{\underline{a}}$, by defining a new bound $N^{(1)}_{\underline{a},\underline{b}}$, depending on both $\underline{a}$ and $\underline{b}$.

For a partition $\underline{p} = [p_1 p_2 \cdots p_k]$, set $|\underline{p}| = \sum_{i=1}^k p_i$. Given $\underline{a} = (a_1, a_2, \ldots, a_r)$ and $\underline{b} = (b_1, b_2, \ldots, b_r) \in B_{\underline{a}}$ as above, let

$$2n + 1 = \sum_{i=1}^r a_i b_i.$$

Then the new bound $N^{(1)}_{\underline{a},\underline{b}}$ is defined to be maximal value of $|\underline{p}|$ for all symplectic partitions $\underline{p}$, which may not be a partition of $2n$, satisfying the following conditions:

(1) $\underline{p} \leq \eta(\underline{p}_{\psi})$ under the *lexicographical order* of partitions as in Definition 2.3, and
(2) $\underline{p}$ has the form $[(2n_1)^{s_1}(2n_2)^{s_2}\cdots(2n_k)^{s_k}]$ with $2n_1 > 2n_2 > \cdots > 2n_k$ and $1 \leq s_i \leq 4$ for $1 \leq i \leq k$.

Note that the integer $N^{(1)}_{\underline{a},\underline{b}}$ depends on $\underline{b}$ through Condition (1) above. For this new bound, we have the following result.

Theorem 4.3. *Assume that F is a totally imaginary number field. Given an $\underline{a} = (a_1, a_2, \ldots, a_r) \in \mathbb{Z}^r_{\geq 1}$ that defines the set $B_{\underline{a}}$. For any $\underline{b} = (b_1, b_2, \ldots, b_r) \in B_{\underline{a}}$, if $2n = (\sum_{i=1}^r a_i b_i) - 1 > N^{(1)}_{\underline{a},\underline{b}}$, then for any global Arthur parameter ψ of the form*

$$\psi = (\tau_1, b_1) \boxplus (\tau_2, b_2) \boxplus \cdots \boxplus (\tau_r, b_r) \in \widetilde{\Psi}_2(\mathrm{Sp}_{2n}),$$

with $\tau_i \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_{a_i})$ for $i = 1, 2, \cdots, r$, the set $\widetilde{\Pi}_{\psi}(\mathrm{Sp}_{2n}) \cap \mathcal{A}_2(\mathrm{Sp}_{2n})$ contains no cuspidal members.

Proof. Assume that there is a $\pi \in \widetilde{\Pi}_{\psi}(\mathrm{Sp}_{2n}) \cap \mathcal{A}_{\mathrm{cusp}}(\mathrm{Sp}_{2n})$. By Theorem 2.9, for any $\underline{p} \in \mathfrak{p}^m(\pi)$, which is a partition of $2n$, we must have that $\underline{p} \leq \eta(\underline{p}_{\psi})$ under the lexicographical order of partitions. In particular, the even partition $\underline{p}_{\pi} \in \mathfrak{p}^m(\pi)$, constructed in [8], enjoys this property. On the other hand, since F is totally imaginary, by Theorem 2.2, $\underline{p}_{\pi}$ has the form $[(2n_1)^{s_1}(2n_2)^{s_2}\cdots(2n_k)^{s_k}]$ with $2n_1 > 2n_2 > \cdots > 2n_k$ and $s_i \leq 4$ for $1 \leq i \leq k$. Hence, $\underline{p}_{\pi}$ satisfies the above two conditions defining the bound $N^{(1)}_{\underline{a},\underline{b}}$. It follows that $N^{(1)}_{\underline{a},\underline{b}} \geq 2n = |\underline{p}_{\pi}|$. This contradicts the assumption that $2n > N^{(1)}_{\underline{a},\underline{b}}$. □

If we assume that Part (1) of Conjecture 2.8 holds, namely, $\eta(\underline{p}_{\psi})$ is greater than or equal to any $\underline{p} \in \mathfrak{p}^m(\pi)$, under the dominance order of partitions, for all $\pi \in \widetilde{\Pi}_{\psi}(\mathrm{Sp}_{2n}) \cap \mathcal{A}_2(\mathrm{Sp}_{2n})$, we may replace the bound $N^{(1)}_{\underline{a},\underline{b}}$ by an even better bound $N^{(2)}_{\underline{a},\underline{b}}$ as follows.

Given an $\underline{a} = (a_1, a_2, \ldots, a_r) \in \mathbb{Z}^r_{\geq 1}$ that defines the set $B_{\underline{a}}$. For any $\underline{b} = (b_1, b_2, \ldots, b_r) \in B_{\underline{a}}$ that defines the integer n with

$$2n + 1 = \sum_{i=1}^r a_i b_i,$$

the new bound $N^{(2)}_{\underline{a},\underline{b}}$ is defined to be the maximal value of $|\underline{p}|$ for all symplectic partitions $\underline{p}$, which may not be a partition of $2n$, satisfying the following conditions:

(1) $\underline{p} \leq \eta(\underline{p}_{\psi})$ under the *dominance order* of partitions, as in Definition 2.3, and

(2) $\underline{p}$ has the form $[(2n_1)^{s_1}(2n_2)^{s_2}\cdots(2n_k)^{s_k}]$ with $2n_1 > 2n_2 > \cdots > 2n_k$ and $1 \leq s_i \leq 4$ holds for $1 \leq i \leq k$.

It is clear that the integer $N^{(2)}_{\underline{a},\underline{b}}$ depends on $\underline{b}$ through Condition (1) above. By assuming Part (1) of Conjecture 2.8, we can prove the following with this new bound.

Theorem 4.4. *Assume that F is a totally imaginary number field, and that Part (1) of Conjecture 2.8 is true. Given an $\underline{a} = (a_1, a_2, \dots, a_r) \in \mathbb{Z}^r_{\geq 1}$ that defines the set $B_{\underline{a}}$. For any $\underline{b} = (b_1, b_2, \dots, b_r) \in B_{\underline{a}}$, if $2n = (\sum_{i=1}^r a_i b_i) - 1 > N^{(2)}_{\underline{a},\underline{b}}$, then for any global Arthur parameter ψ of the form*

$$\psi = (\tau_1, b_1) \boxplus (\tau_2, b_2) \boxplus \cdots \boxplus (\tau_r, b_r) \in \widetilde{\Psi}_2(\mathrm{Sp}_{2n}),$$

with $\tau_i \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_{a_i})$ for $i = 1, 2, \cdots, r$, the set $\widetilde{\Pi}_\psi(\mathrm{Sp}_{2n}) \cap \mathcal{A}_2(\mathrm{Sp}_{2n})$ contains no cuspidal members.

Proof. The proof is the same as that of Theorem 4.3, with Theorem 2.9 replaced by Part (1) of Conjecture 2.8, and the lexicographical order of partitions replaced by the dominance order of partitions. □

First, it is clear that $N_{\underline{a}} \geq N^{(1)}_{\underline{a},\underline{b}} \geq N^{(2)}_{\underline{a},\underline{b}}$. We expect that the bound $N^{(2)}_{\underline{a},\underline{b}}$ is sharp. Namely, for any $\underline{b} = (b_1, b_2, \dots, b_r) \in B_{\underline{a}}$ with $\sum_{i=1}^r a_i b_i = N^{(2)}_{\underline{a},\underline{b}} + 1$, we expect that any global packet $\widetilde{\Pi}_\psi(\mathrm{Sp}_{N^{(2)}_{\underline{a},\underline{b}}})$ associated to any global Arthur parameter ψ of the form

$$\psi = (\tau_1, b_1) \boxplus (\tau_2, b_2) \boxplus \cdots \boxplus (\tau_r, b_r) \in \widetilde{\Psi}_2(\mathrm{Sp}_{N^{(2)}_{\underline{a},\underline{b}}}),$$

with $\tau_i \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_{a_i})$ for $i = 1, 2, \cdots, r$, contains a cuspidal member. An interesting problem is to figure out the explicit formula of the bounds $N^{(1)}_{\underline{a},\underline{b}}$ and $N^{(2)}_{\underline{a},\underline{b}}$ as functions of $\underline{a}$ and $\underline{b}$. Secondly, one may easily write down the corresponding analogues of Theorem 4.2 for bounds $N^{(1)}_{\underline{a},\underline{b}}$ and $N^{(2)}_{\underline{a},\underline{b}}$, we omit them here. Finally, we give examples to indicate that $N_{\underline{a}} > N^{(1)}_{\underline{a},\underline{b}} > N^{(2)}_{\underline{a},\underline{b}}$.

Consider $\psi = (\tau_1, 1) \boxplus (\tau_2, 8)$, where $\tau_1 \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_5)$ of orthogonal type, and $\tau_2 \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_2)$ of symplectic type. By Remark 2.7,

$$\eta(\underline{p}_\psi) = (([1^5 8^2]^t)^-)_{\mathrm{Sp}} = [72^6 1]_{\mathrm{Sp}} = [62^7].$$

In this case, one has that $N_{\underline{a}} = (5+2)^2 - 1 = 48$. On the other hand, one has that $N^{(1)}_{\underline{a},\underline{b}} = 24$ and $N^{(2)}_{\underline{a},\underline{b}} = 16$.

In fact, $[4^4 2^4]$ is the only partition $\underline{p}$ that gives maximal $|\underline{p}|$, and satisfies the conditions: $\underline{p} \leq \eta(\underline{p}_\psi)$ under the lexicographical order of partitions, and $\underline{p}$ has the form $[(2n_1)^{s_1}(2n_2)^{s_2}\cdots(2n_k)^{s_k}]$ with $2n_1 > 2n_2 > \cdots 2n_k$ and $1 \leq s_i \leq 4$ for $1 \leq i \leq k$. This shows that $N^{(1)}_{\underline{a},\underline{b}} = 24$.

Also, $[4^2 2^4]$ is the only partition $\underline{p}$ that gives maximal $|\underline{p}|$, and satisfies the conditions: $\underline{p} \leq \eta(\underline{p}_\psi)$ under the dominance order of partitions, and $\underline{p}$ has the form $[(2n_1)^{s_1}(2n_2)^{s_2}\cdots(2n_k)^{s_k}]$ with $2n_1 > 2n_2 > \cdots 2n_k$ and $1 \leq s_i \leq 4$ for $1 \leq i \leq k$. This shows that $N^{(2)}_{\underline{a},\underline{b}} = 16$.

Note that the bound $N^{(1)}_{\underline{a},\underline{b}}$ uses Theorem 2.9, while the bound $N^{(2)}_{\underline{a},\underline{b}}$ needs the assumption that Part (1) of Conjecture 2.8 holds.

4.2. **Examples.** We give examples of Arthur parameters ψ such that $\widetilde{\Pi}_\psi(\mathrm{Sp}_{2n}) \cap \mathcal{A}_2(\mathrm{Sp}_{2n})$ contains no cuspidal members.

Example 1: Let $\tau \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_{2l})$ be such that $L(s, \tau, \wedge^2)$ has a pole at $s = 1$. Consider the Arthur parameter $\psi = (\tau, 2m) \boxplus (1_{\mathrm{GL}_1(\mathbb{A})}, 1)$. In this case, we have that $\underline{a} = (2l, 1)$ and $\underline{b} = (2m, 1)$. Since $a_1 + a_2 = 2l + 1$ is odd, we have that

$$N_{\underline{a}} = (a_1 + a_2)^2 - 1 = (2l + 1)^2 - 1.$$

If $m > l + 1$, then we have

$$4ml = a_1 b_1 + a_2 b_2 - 1 = 2l(2m) + 1 - 1 > (2l + 1)^2 - 1 = N_{\underline{a}},$$

and hence, by Theorem 4.1 or Theorem 4.2, $\widetilde{\Pi}_\psi(\mathrm{Sp}_{4ml}) \cap \mathcal{A}_2(\mathrm{Sp}_{4ml})$ contains no cuspidal members.

But, if in addition, $L(\frac{1}{2}, \tau) \neq 0$, we can construct a residual representation in $\widetilde{\Pi}_\psi(\mathrm{Sp}_{4ml}) \cap \mathcal{A}_2(\mathrm{Sp}_{4ml})$ as follows. Let $P_{2ml} = M_{2ml} N_{2ml}$ be the parabolic subgroup of Sp_{4ml} with Levi subgroup $M_{2ml} \cong \mathrm{GL}_{2l}^{\times m}$. For any

$$\phi \in \mathcal{A}(N_{2ml}(\mathbb{A}) M_{2ml}(F) \backslash \mathrm{Sp}_{4ml}(\mathbb{A}))_{\Delta(\tau, m)},$$

following [31] and [40, Chapter VI], a residual Eisenstein series can be defined by

$$E(\phi, s)(g) = \sum_{\gamma \in P_{2ml}(F) \backslash \mathrm{Sp}_{4ml}(F)} \lambda_s \phi(\gamma g).$$

It converges absolutely for real part of s large and has meromorphic continuation to the whole complex plane $\mathbb{C}$. Since $L(\frac{1}{2}, \tau) \neq 0$, by [25], this Eisenstein series has a simple pole at $\frac{m}{2}$, which is the right-most one. Denote the representation generated by these residues at $s = \frac{m}{2}$ by $\mathcal{E}_{\Delta(\tau, m)}$, which is square-integrable. By [25, Section 6.2], $\mathcal{E}_{\Delta(\tau, m)}$ has the global Arthur parameter $\psi = (\tau, 2m) \boxplus (1_{\mathrm{GL}_1(\mathbb{A})}, 1)$, and hence belongs to $\widetilde{\Pi}_\psi(\mathrm{Sp}_{4ml}) \cap \mathcal{A}_2(\mathrm{Sp}_{4ml})$.

Example 2: Consider a family of Arthur parameters of symplectic groups of the form $\psi = (1_{\mathrm{GL}_1(\mathbb{A})}, b_1) \boxplus (\tau, b_2)$, where $b_1 \geq 1$ is odd,

$\tau \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_2)$ is of symplectic type and $b_2 \geq 1$ is even. By definition, $\underline{p}_\psi = [b_1 b_2^2]$, and

$$\eta(\underline{p}_\psi) = ((\underline{p}_\psi^-)_{\mathrm{Sp}})^t = ((\underline{p}_\psi^t)^-)_{\mathrm{Sp}} = (([1^{b_1}] + [2^{b_2}])^-)_{\mathrm{Sp}}.$$

It is clear that the biggest part occurring in the partition $\eta(\underline{p}_\psi)$ is at most 3. Note that $2n = a_1 b_1 + a_2 b_2 - 1 = b_1 + 2b_2 - 1$.

Assume that π belongs to $\widetilde{\Pi}_\psi(\mathrm{Sp}_{2n}) \cap \mathcal{A}_{\mathrm{cusp}}(\mathrm{Sp}_{2n})$ with the above given global Arthur parameter ψ. By Theorem 2.9, for any $\underline{p} \in \mathfrak{p}^m(\pi)$, its biggest part is less than or equal to 3. On the other hand, the partition $\underline{p}_\pi \in \mathfrak{p}^m(\pi)$ constructed in [8] is even. Hence, $\underline{p}_\pi = \{[2^n]\}$. Since F is totally imaginary, by Theorem 2.2, we must have that $n \leq 4$. Hence, one can see that $N_{\underline{a}} = N^{(1)}_{\underline{a},\underline{b}} = N^{(2)}_{\underline{a},\underline{b}} = 8$, where $\underline{a} = \{1, 2\}$, $\underline{b} = \{b_1, b_2\}$. It follows from Theorems 4.1–4.4 that $\widetilde{\Pi}_\psi(\mathrm{Sp}_{2n}) \cap \mathcal{A}_2(\mathrm{Sp}_{2n})$ contains no cuspidal members except possibly the following cases (see Figure 1 below)

$$(b_1, b_2) = (1, 2), (1, 4), (3, 2), (5, 2).$$

In particular, the global Arthur packet $\widetilde{\Pi}_\psi(\mathrm{Sp}_{2n})$ contains no cuspidal members if $n \geq 5$.

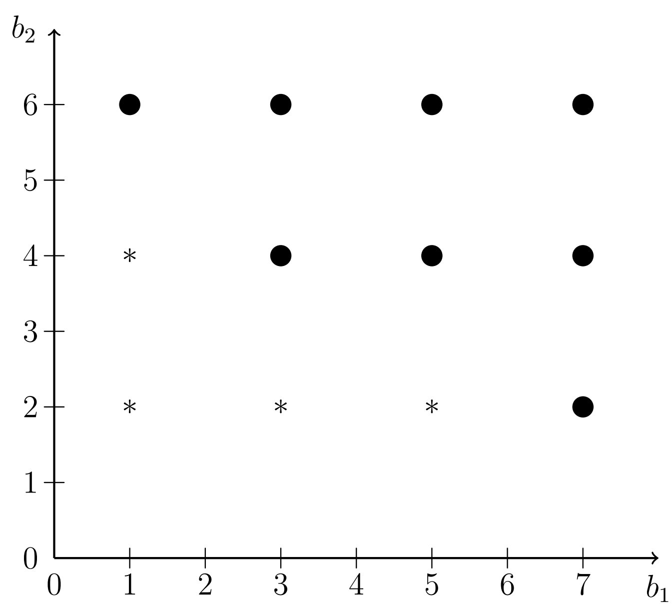

FIGURE 1. The *'s indicate that the Arthur packets may possibly contain cuspidal members.

As we mentioned before, for generic global Arthur parameters $\phi \in \widetilde{\Phi}_2(\mathrm{G})$, one must have

$$\widetilde{\Pi}_\phi(\mathrm{G}) \cap \mathcal{A}_2(\mathrm{G}) \subset \mathcal{A}_{\mathrm{cusp}}(\mathrm{G}).$$

In [37] and [38], Mœglin considers the problem of which non-generic global Arthur packets contains non-cuspidal members, i.e. the square-integrable residual representations of $\mathrm{G}(\mathbb{A})$. She gives a conjecture on necessary and sufficient conditions for this problem and proves the conjecture when the square-integral representations have cohomology at infinity. Moreover, in [37, Section 4.6], Mœglin predicts that her conjecture implies that for a given global Arthur parameter $\psi = \boxplus_{i=1}^r (\tau_i, b_i)$ of a symplectic group Sp_{2n}, where $\tau_i \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_{a_i})$ is self-dual, if there exist $1 \leq j_1 \leq r$ such that $b_{j_1} \geq a_{j_1} + a_{j_2} + b_{j_2}$, for any $1 \leq j_2 \neq j_1 \leq r$, then $\widetilde{\Pi}_\psi(\mathrm{Sp}_{2n}) \cap \mathcal{A}_2(\mathrm{Sp}_{2n})$ contains no cuspidal members. Comparing to our discussions and examples above, one may easily find that **Example 1** gives examples that $\widetilde{\Pi}_\psi(\mathrm{Sp}_{2n}) \cap \mathcal{A}_2(\mathrm{Sp}_{2n})$ contains no cuspidal members, which matches her prediction. But, our **Example 2** contains many more cases that $\widetilde{\Pi}_\psi(\mathrm{Sp}_{2n}) \cap \mathcal{A}_2(\mathrm{Sp}_{2n})$ contains no cuspidal members, which can not be determined by the condition suggested by Mœglin. We remark that **Example 2** also includes cases that can not be decided by the discussion in Section 3. One of such cases is that given by $(b_1, b_2) = (5, 6)$.

5. On the generalized Ramanujan problem

The generalized Ramanujan problem as proposed by P. Sarnak in [43, Section 2] is to understand the behavior of the local components of irreducible cuspidal automorphic representations of $\mathrm{G}(\mathbb{A})$ for general reductive algebraic group G defined over a number field F. The generalized Ramanujan conjecture asserts that all local components of irreducible generic cuspidal representations are tempered. When the group G is not a general linear group, an irreducible cuspidal automorphic representation π of $\mathrm{G}(\mathbb{A})$ may have non-tempered local components. Examples are those cuspidal members in a global Arthur packet with a non-generic global Arthur parameters. Hence it is important also from this prospective to determine which non-generic global Arthur packets have no cuspidal members.

More precisely, the endoscopic classification of Arthur provides certain bounds for the exponents of the unramified local components of the irreducible automorphic representations occurring in the discrete spectrum. It is clear that if one is able to determine which non-generic global packets have no cuspidal members, the bounds of the exponents of the unramified local components of the cuspidal spectrum would be much improved, which definitely helps us to the understanding of the generalized Ramanujan problem.

In this section, we take a preliminary step to understand the bounds of exponents of the unramified local components of the cuspidal spectrum of Sp_{2n} based on the results obtained in Section 4.

For $\pi \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{Sp}_{2n})$, we write each of its unramified components π_v as the unique unramified subquotient of the induced representation

$$\mathrm{Ind}_{B(F_v)}^{\mathrm{Sp}_{2n}(F_v)} \chi_1|\cdot|^{\alpha_1} \otimes \chi_2|\cdot|^{\alpha_2} \otimes \cdots \otimes \chi_n|\cdot|^{\alpha_n},$$

where B is the standard Borel subgroup of Sp_{2n}, with the property that for $1 \leq i \leq n$, χ_i are unitary unramified characters of F_v^*. For $\theta \in \mathbb{R}_{\geq 0}$, we say that π *satisfies* $R(\theta)$ if $0 \leq \alpha_i \leq \theta$ holds for $i = 1, 2, \cdots, n$.

By the discussion in Remark 3.3, if there is a simple global Arthur parameter (χ, b) occurring as a formal summand in the global Arthur parameter ψ of π, where χ is a quadratic automorphic character of $\mathrm{GL}_1(\mathbb{A})$, one must have that $b \leq n+1$ if n is even, and that $b \leq n$ if n is odd. In order to find an upper bound θ such that every $\pi \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{Sp}_{2n})$ satisfies $R(\theta)$, one only needs to consider simple global Arthur parameters (τ, b) that may occur in the global Arthur parameter ψ of π, where $\tau \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_2)$ is self-dual.

First, assume that n is even. Consider a global Arthur parameter of $\mathrm{Sp}_{2n}(\mathbb{A})$, $\psi = (1_{\mathrm{GL}_1(\mathbb{A})}, 1) \boxplus (\tau, n)$, with $\tau \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_2)$ of symplectic type. By using the bound of Kim-Sarnak ([29]) and Blomer-Brumley ([4]) towards the Ramanujan conjecture for GL_2, which is $R(\frac{7}{64})$, one may easily show that any $\pi \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{Sp}_{2n}) \cap \widetilde{\Pi}_\psi(\mathrm{Sp}_{2n})$ satisfies $R(\frac{7}{64} + \frac{n-1}{2})$. By the result of Kudla and Rallis ([30]), for any $\pi \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{Sp}_{2n}) \cap \widetilde{\Pi}_\psi(\mathrm{Sp}_{2n})$ (with n even), if a simple global Arthur parameter (χ, b) occurs in the global Arthur parameter ψ of π, one must have that b is at most $n+1$, and hence satisfies $R(\frac{n}{2})$. Note that $\frac{7}{64} + \frac{n-1}{2} < \frac{n}{2}$. It follows that $\frac{n}{2}$ is a possible upper bound for all $\pi \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{Sp}_{2n})$. On the other hand, Piatetski-Shapiro and Rallis ([42]) construct a cuspidal member $\pi \in \widetilde{\Pi}_\psi(\mathrm{Sp}_{2n})$ (with n even) that has the simple global Arthur parameter $(\chi, n+1)$ occurring in the ψ. Therefore, we obtain that $\frac{n}{2}$ is the sharp upper bound for all $\pi \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{Sp}_{2n})$ when n is even. We state the conclusion of the above discussion as

Proposition 5.1. *Let F be a number field. When n is an even integer, all $\pi \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{Sp}_{2n})$ satisfy $R(\frac{n}{2})$, and the bound $\frac{n}{2}$ is achieved by the $\pi \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{Sp}_{2n})$ constructed by Piatetski-Shapiro and Rallis in* [42].

Next, assume that n is odd. Consider a global Arthur parameter of $\mathrm{Sp}_{2n}(\mathbb{A})$, $\psi = (\omega_\tau, 1) \boxplus (\tau, n)$, with $\tau \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_2)$ of orthogonal type and ω_τ the central character of τ. By the same reason, one has that all $\pi \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{Sp}_{2n}) \cap \widetilde{\Pi}_\psi(\mathrm{Sp}_{2n})$ satisfy $R(\frac{7}{64} + \frac{n-1}{2})$. Again by [30], for any $\pi \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{Sp}_{2n}) \cap \widetilde{\Pi}_\psi(\mathrm{Sp}_{2n})$ (with n odd), if a simple global Arthur parameter (χ, b) occurs in the global Arthur parameter ψ of π, one must have that b is at most n, and hence satisfies $R(\frac{n-1}{2})$. Because $\frac{n-1}{2} < \frac{7}{64} + \frac{n-1}{2}$, we obtain that $\frac{7}{64} + \frac{n-1}{2}$ is a possible upper bound for any $\pi \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{Sp}_{2n})$.

However, by Theorem 4.1, if we assume that F is totally imaginary and $n \geq 5$, then for the Arthur parameters $\psi = (\omega_\tau, 1) \boxplus (\tau, n)$ given above, there does not exist any cuspidal member in $\widetilde{\Pi}_\psi(\mathrm{Sp}_{2n}) \cap \mathcal{A}_2(\mathrm{Sp}_{2n})$. Hence, we obtain the following conclusion.

Proposition 5.2. *Assume that F is totally imaginary and $n \geq 5$ is odd. Any $\pi \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{Sp}_{2n})$ satisfies $R(\frac{n-1}{2})$.*

We may expect that a simple global Arthur parameter $(\tau, n-1)$ with n odd and $\tau \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_2)$ of symplectic type could have cuspidal members in the global Arthur packet $\widetilde{\Pi}_\psi(\mathrm{Sp}_{2n})$, although we do not know how to construct them for the moment. However, in that case the bound is $\frac{7}{64} + \frac{n-2}{2}$, which is less than $\frac{n-1}{2}$. Also, for $\tau \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_a)$ (self-dual) with $a \geq 3$, the simple global Arthur parameters of type (τ, b) produce naturally a better bound than that obtained above, and hence are omitted for further consideration.

It is a very interesting problem to determine the sharp upper bound θ for the cuspidal spectrum of $\mathrm{Sp}_{2n}(\mathbb{A})$ when n is odd. This would involve a generalization or extension of the constructions by Piatetski-Shapiro and Rallis ([42]) and by Ikeda ([14] and [15]). We will get back to this issue in our future work.

6. Small Cuspidal Automorphic Representations

In this section, we discuss some criteria on the *smallness* of cuspidal automorphic representations of $\mathrm{Sp}_{2n}(\mathbb{A})$ and give examples of small cuspidal automorphic representations, in addition to these constructed by Ikeda in [14]. From now on, we assume that F is a number field.

6.1. **Characterization of small cuspidal representations.** The characterization of small cuspidal automorphic representations will be given in terms of a vanishing condition on Fourier coefficients related to the automorphic descent method ([10]), and also in terms of the notion of hypercuspidality in the sense of Piatetski-Shapiro ([41]). Also, our discussions cover the case of symplectic group $\mathrm{Sp}_{2n}(\mathbb{A})$ and the case of the metaplectic double cover $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$ of $\mathrm{Sp}_{2n}(\mathbb{A})$ together.

Theorem 6.1. *Assume that π is an irreducible cuspidal automorphic representation of $\mathrm{Sp}_{2n}(\mathbb{A})$ or $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$. Then $\mathfrak{p}^m(\pi) = \{[2^n]\}$ if and only if π has no nonzero Fourier coefficients attached to the partition $[41^{2n-4}]$.*

Proof. First, assume that $\mathfrak{p}^m(\pi) = \{[2^n]\}$. Since the partition $[41^{2n-4}]$ is either greater than or not related to the partition $[2^n]$, by definition of $\mathfrak{p}^m(\pi)$, π has no nonzero Fourier coefficients attached to the partition $[41^{2n-4}]$.

Next, assume that π has no nonzero Fourier coefficients attached to the partition $[41^{2n-4}]$. By Lemma 6.3 below, π has no nonzero Fourier coefficients attached to the partition $[(2k)1^{2n-2k}]$, for any $2 \leq k \leq n$. Assume that $\underline{p} = [p_1p_2\cdots p_s] \in \mathfrak{p}^m(\pi)$, with $p_1 \geq p_2 \geq \cdots \geq p_s$. If p_1 is odd, then one must have that $p_1 \geq 3$. By [18, Lemma 3.3], π has a nonzero Fourier coefficient attached to the partition $[(p_1)^2 1^{2n-2p_1}]$. Then [8, Lemma 2.4] shows that π must have a nonzero Fourier coefficient attached to the partition $[(2r)1^{2n-2r}]$ for some $2r > 2p_1 \geq 6$, which contradicts the assumption of the theorem. Now, if p_1 is even, then by [8, Lemma 2.6] or [18, Lemma 3.1], π has a nonzero Fourier coefficient attached to the partition $[(p_1)1^{2n-p_1}]$. By the assumption of the theorem, we must have that $p_1 = 2$. Hence we obtain that $2 = p_1 \geq p_2 \geq \cdots \geq p_s$, which implies that $\underline{p} \leq [2^n]$. On the other hand, by Theorem 2.11, the cuspidal π must have a nonzero Fourier coefficient attached to the partition $[2^n]$. It follows that for any $\underline{p} \in \mathfrak{p}^m(\pi)$, the case that $\underline{p} < [2^n]$ can not happen. Therefore, we conclude that $\underline{p} = [2^n]$, and hence $\mathfrak{p}^m(\pi) = \{[2^n]\}$. This completes the proof of the theorem. □

Let T be the subgroup of Sp_{2n} consists of all diagonal elements. Given $t = \mathrm{diag}(t_1, \ldots, t_n, t_n^{-1}, \ldots, t_1^{-1})$, for $1 \leq i \leq n$, let e_i be the character defined by $e_i(t) = t_i$. Let $\alpha = 2e_1$, the highest positive root of Sp_{2n}, and let X_α be the corresponding one-dimensional root subgroup. Recall from [41, Section 6] that an automorphic function φ is called *hypercuspidal* if

$$\int_{X_\alpha(F)\backslash X_\alpha(\mathbb{A})} \varphi(xg)dx \equiv 0.$$

It is clear that any hypercuspidal function is automatically cuspidal. An automorpic representation π of $\mathrm{Sp}_{2n}(\mathbb{A})$ or $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$ is called *hypercuspidal* if every $\varphi \in \pi$ is hypercuspidal.

For $0 \leq i \leq n-1$, let $P_i = M_i N_i$ be the parabolic subgroup of Sp_{2n} with Levi subgroup $M \cong \mathrm{GL}_1^i \times \mathrm{Sp}_{2n-2i}$. Define a character of N_i by $\psi_i(n) = \psi(\sum_{j=1}^{i} n_{j,j+1})$. Let π be an automorphic representation of $\mathrm{Sp}_{2n}(\mathbb{A})$ or $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$. For any $\varphi \in \pi$, let

$$\mathcal{F}_i(\varphi)(g) = \int_{N_i(F)\backslash N_i(\mathbb{A})} \varphi(ng)\psi_i^{-1}(n)dn.$$

Lemma 6.2. *Let π be a cuspidal automorphic representation of $\mathrm{Sp}_{2n}(\mathbb{A})$ or $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$. For any $\varphi \in \pi$, $\mathcal{F}_i(\varphi)$ is a linear combination of $\mathcal{F}_{i+1}(\varphi)$ and Fourier coefficients attached to the partition $[(2i+2)1^{2n-2i-2}]$.*

Proof. Let α be the root $2e_{i+1}$ and let X_α be the corresponding one-dimensional root subgroup. Since X_α normalizes N_i and preserves the character ψ_i, one can take the Fourier expansion of $\mathcal{F}_i(\varphi)$ along $X_\alpha(F)\backslash X_\alpha(\mathbb{A})$. The non-constant terms give us exactly Fourier coefficients attached to the partition $[(2i+2)1^{2n-2i-2}]$. Now consider the constant term, that is $\int_{X_\alpha(F)\backslash X_\alpha(\mathbb{A})} \mathcal{F}_i(\varphi)(xg)dx$.

For $i+2 \leq j \leq n$, let α_j be the root $e_{i+1} - e_j$, and for $n+1 \leq j \leq 2n-i-1$, let α_j be the root $e_{i+1} + e_{2n+1-j}$. For $i+2 \leq j \leq 2n-i-1$, let X_{α_j} be the corresponding one-dimensional root subgroup. Let $X = \prod_{j=i+2}^{2n-i-1} X_{\alpha_j}$. Then, one can see that X normalizes $N_i X_\alpha$ and preserves the character ψ_i. Here ψ_i is extended trivially to $N_i X_\alpha$. Hence, one can take the Fourier expansion of $\int_{X_\alpha(F)\backslash X_\alpha(\mathbb{A})} \mathcal{F}_i(\varphi)(xg)dx$ along $X(F)\backslash X(\mathbb{A})$, and obtain that

$$\begin{aligned}&\int_{X_\alpha(F)\backslash X_\alpha(\mathbb{A})} \mathcal{F}_i(\varphi)(xg)dx \\ =& \sum_{\xi \in X(F)} \int_{X(F)\backslash X(\mathbb{A})} \int_{X_\alpha(F)\backslash X_\alpha(\mathbb{A})} \mathcal{F}_i(\varphi)(xx'g)\psi_\xi^{-1}(x')dxdx'.\end{aligned}$$

Note that the constant term corresponding to $\xi = 0$ is identically zero, since $\varphi \in \pi$ is cuspidal. Also note that $\mathrm{Sp}_{2n-2i-2}(F)$ acts on $X(F)\backslash\{0\}$ transitively, and one can take a representative $\xi_0 = (1, 0, \ldots, 0)$. Denote the stabilizer of ξ_0 in $\mathrm{Sp}_{2n-2i-2}(F)$ by $H(F)$, which is a Jacobi group $\mathcal{H}_{2n-2i-4}(F) \rtimes \mathrm{Sp}_{2n-2i-4}(F)$. Embed $\mathrm{Sp}_{2n-2i-2}$ into Sp_{2n} via $g \to \begin{pmatrix} I_{i+1} & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & I_{i+1} \end{pmatrix}$, and identify it with its image under this embedding. Then the above Fourier expansion can be rewritten as

$$\int_{X_\alpha(F)\backslash X_\alpha(\mathbb{A})} \mathcal{F}_i(\varphi)(xg)dx$$

$$= \sum_{\gamma\in H(F)\backslash \mathrm{Sp}_{2n-2i-2}(F)} \int_{X(F)\backslash X(\mathbb{A})} \int_{X_\alpha(F)\backslash X_\alpha(\mathbb{A})} \mathcal{F}_i(\varphi)(xx'\gamma g)\psi_{\xi_0}^{-1}(x')dxdx',$$

which is exactly $\sum_{\gamma\in H(F)\backslash \mathrm{Sp}_{2n-2i-2}(F)} \mathcal{F}_{i+1}(\varphi)(\gamma g)$. Therefore, $\mathcal{F}_i(\varphi)$ is a linear combination of $\mathcal{F}_{i+1}(\varphi)$ and Fourier coefficients attached to the partition $[(2i+2)1^{2n-2i-2}]$. This completes the proof of the lemma. □

Next, we recall a lemma as follows.

Lemma 6.3 (Key Lemma 3.3, [9])**.** *Let π be any automorphic representation of $\mathrm{G}(\mathbb{A}) = \mathrm{Sp}_{2n}(\mathbb{A})$ or $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$. If π has no nonzero Fourier coefficients attached to the partition $[(2k)1^{2n-2k}]$, then π has no nonzero Fourier coefficients attached to the partition $[(2k+2)1^{2n-2k-2}]$.*

Theorem 6.4. *For an irreducible cuspidal automorphic representation π of $\mathrm{Sp}_{2n}(\mathbb{A})$ or $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$, $\mathfrak{p}^m(\pi) = \{[2^n]\}$ if and only if π is hypercuspidal.*

Proof. By Theorem 6.1, we just need to show that π is hypercuspidal if and only if π has no nonzero Fourier coefficients attached to the partition $[41^{2n-4}]$. First, it is clear that if π is hypercuspidal, then π has no nonzero Fourier coefficients attached to partition $[41^{2n-4}]$, since X_α for the longest root α, is the center of the standard maximal unipotent subgroup of Sp_{2n}. Now assume that π has no nonzero Fourier coefficients attached to the partition $[41^{2n-4}]$. By Lemma 6.3, π has no nonzero Fourier coefficients attached to the partition $[(2k)1^{2n-2k}]$, for any $2 \le k \le n$.

Let Y be the unipotent subgroup of Sp_{2n} consisting of elements $y = \begin{pmatrix} 1 & x & * \\ 0 & I_{2n-2} & x^* \\ 0 & 0 & 1 \end{pmatrix}$, where $x \in \mathrm{Mat}_{1\times(2n-2)}$. It is clear that Y normalizes X_α. Hence, $f(g) := \int_{X_\alpha(F)\backslash X_\alpha(\mathbb{A})} \phi(xg)dx$ can be viewed as an automorphic function over $Y(F)\backslash Y(\mathbb{A})$. After taking Fourier expansion along $Y(F)\backslash Y(\mathbb{A})$,

$$f(g) = \sum_{\xi\in F^{2n-2}\backslash\{0\}} \int_{Y(F)\backslash Y(\mathbb{A})} f(yg)\psi_\xi^{-1}(y)dy, \tag{6.1}$$

since π is a cuspidal.

Note that the action of $\mathrm{Sp}_{2n-2}(F)$ on $F^{2n-2}\backslash\{0\}$ via conjugation is transitive. Take a representative $\xi_0 = (1, 0, \ldots, 0)$. Then its stabilizer in $\mathrm{Sp}_{2n-2}(F)$ is a subgroup (denoted by H) consisting of elements $\begin{pmatrix} 1 & x & y \\ 0 & g' & x^* \\ 0 & 0 & 1 \end{pmatrix}$, where $x \in \mathrm{Mat}_{1\times 2n-4}$, $y \in F$, $g' \in \mathrm{Sp}_{2n-4}$. Embed Sp_{2n-2} into Sp_{2n} via the map $g \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and identify Sp_{2n-2} with its image under this embedding. Then, after changing of variables, the Fourier expansion in (6.1) can be rewritten as

$$f(g) = \sum_{\gamma \in H\backslash \mathrm{Sp}_{2n-2}(F)} \int_{Y(F)\backslash Y(\mathbb{A})} f(y\gamma g)\psi_{\xi_0}^{-1}(y)dy, \tag{6.2}$$

which is exactly $\sum_{\gamma\in H\backslash \mathrm{Sp}_{2n-2}(F)} \mathcal{F}_1(f)(\gamma g)$. Hence, to show that f is identically zero, it is enough to show that $\mathcal{F}_1(f)$ is identically zero.

Applying Lemma 6.2 repeatedly, $\mathcal{F}_1(f)$ is a linear combination of Fourier coefficients attached to the partitions $[(2k)1^{2n-2k}]$, $2 \le k \le n$, which are all identically zero, by the above discussion. Therefore, f is identically zero, i.e., π is hypercuspidal.

This completes the proof of the theorem. □

Combining Theorems 6.1, 6.4 with Theorem 2.2, we have the following corollary.

Theorem 6.5. *Assume that F is a totally imaginary number field and $n \ge 5$. Then $\mathrm{Sp}_{2n}(\mathbb{A})$ and $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$ have no cuspidal automorphic representations having nonzero Fourier coefficients attached to the partition $[41^{4n-4}]$, and equivalently, have no nonzero hypercuspidal representations.*

Proof. Assume that $\mathrm{Sp}_{2n}(\mathbb{A})$ and $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$ has a nonzero cuspidal representation π that has nonzero Fourier coefficients attached to the partition $[41^{4n-4}]$, which is equivalent to saying that π is hypercuspidal. Then, by Theorems 6.1, 6.4, $\mathfrak{p}^m(\pi) = \{[2^n]\}$. In particular, the even partition $\underline{p}_\pi$ constructed in [8] is exactly $[2^n]$. On the other hand, since F is totally imaginary, by Theorem 2.2, $\underline{p}_\pi$ can not be $[2^n]$ because $n \ge 5$. This contradiction proves the theorem. □

6.2. **Examples of small cuspidal representations.** In this section, we assume that F is not a totally imaginary number field if $n \ge 5$. In order to provide examples of global Arthur packets of Sp_{2n} whose cuspidal automorphic members π have the property that $\mathfrak{p}^m(\pi) = \{[2^n]\}$, we discuss specific congruence classes of n modulo 2 and modulo 3.

Case of $n = 2e$.

Proposition 6.6. *Any $\pi \in \widetilde{\Pi}_\psi(\mathrm{Sp}_{4e}) \cap \mathcal{A}_{\mathrm{cusp}}(\mathrm{Sp}_{4e})$ with*

$$\psi = (\tau, 2i) \boxplus (1_{\mathrm{GL}_1(\mathbb{A})}, 4e - 4i + 1), e \leq 2i \leq 2e,$$

and $\tau \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_2)$ of symplectic type, has the property that $\mathfrak{p}^m(\pi) = \{[2^{2e}]\}$, and hence is small.

Proof. For $\psi = (\tau, 2i) \boxplus (1_{\mathrm{GL}_1(\mathbb{A})}, 4e - 4i + 1)$, with $e \leq 2i \leq 2e$, we must have that $\underline{p}_\psi = [(2i)^2(4e - 4i + 1)]$ and $\eta(\underline{p}_\psi)$ has largest part at most 3. Any $\pi \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{Sp}_{4e}) \cap \widetilde{\Pi}_\psi(\mathrm{Sp}_{4e})$, by Theorem 2.9, any partition $\underline{p} \in \mathfrak{p}^m(\pi)$ satisfies the property that $\underline{p} \leq \underline{p}_\psi$ under the lexicographical order of partitions. Hence, any partition $\underline{p} = [p_1 p_2 \cdots p_r] \in \mathfrak{p}^m(\pi)$ has largest part $p_1 \leq 3$.

If $p_1 = 3$, then by [18, Lemma 3.3], π has a nonzero Fourier coefficient attached to the partition $[(p_1)^2 1^{4e-2p_1}]$. Furthermore, by [8, Lemma 2.4], π has a nonzero Fourier coefficient attached to the partition $[(2r)1^{4e-2r}]$ for some $2r > p_1 = 3$, which contradicts Theorem 2.9. Hence we have that $p_1 = 2$ and $\underline{p} \leq [2^{2e}]$ under the dominance order of partitions. In this case, by Theorem 2.11, π is non-singular. It follows again that any $\underline{p} \in \mathfrak{p}^m(\pi)$ satisfies the property that $\underline{p} \geq [2^{2e}]$ under the dominance order of partitions. Therefore, we must have that $\mathfrak{p}^m(\pi) = \{[2^{2e}]\}$. □

Note that if $2i < e$, then $4e - 4i + 1 > 2e + 1$. By Remark 3.3, the global Arthur packet $\widetilde{\Pi}_\psi(\mathrm{Sp}_{4e})$ corresponding to the global Arthur parameter

$$\psi = (\tau, 2i) \boxplus (1_{\mathrm{GL}_1(\mathbb{A})}, 4e - 4i + 1)$$

contains no cuspidal automorphic representations.

In the case of $2i = 2e$, $\psi = (\tau, 2e) \boxplus (1_{\mathrm{GL}_1(\mathbb{A})}, 1)$, where $\tau \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_2)$ is of symplectic type. If in addition $L(\frac{1}{2}, \tau) \neq 0$, then we can construct a residual representation in $\widetilde{\Pi}_\psi(\mathrm{Sp}_{4e}) \cap \mathcal{A}_2(\mathrm{Sp}_{4e})$ as follows.

Let $\Delta(\tau, e)$ be a Speh residual representation in the discrete spectrum of $\mathrm{GL}_{2e}(\mathbb{A})$. For more information about the Speh residual representations, we refer to [39], or [25, Section 1.2]. Let $P_r = M_r N_r$ be the maximal parabolic subgroup of Sp_{2l} with Levi subgroup M_r isomorphic to $\mathrm{GL}_r \times \mathrm{Sp}_{2l-2r}$. Using the normalization in [44], the group $X_{M_r}^{\mathrm{Sp}_{2l}}$ of all continuous homomorphisms from $M_r(\mathbb{A})$ to $\mathbb{C}^\times$, which is trivial on $M_r(\mathbb{A})^1$ (see [40]), will be identified with $\mathbb{C}$ by $s \to \lambda_s$.

For any $\phi \in \mathcal{A}(N_{2e}(\mathbb{A})M_{2e}(F)\backslash \mathrm{Sp}_{4e}(\mathbb{A}))_{\Delta(\tau,e)}$, following [31] and [40, Chapter VI], a residual Eisenstein series can be defined by

$$E(\phi, s)(g) = \sum_{\gamma \in P_{2e}(F)\backslash \mathrm{Sp}_{4e}(F)} \lambda_s \phi(\gamma g).$$

It converges absolutely for real part of s large and has meromorphic continuation to the whole complex plane $\mathbb{C}$. Since $L(\frac{1}{2}, \tau) \neq 0$, by [25], this Eisenstein series has a simple pole at $\frac{e}{2}$, which is the right-most one. Denote by $\mathcal{E}_{\Delta(\tau,e)}$ the representation generated by these residues at $s = \frac{e}{2}$. This residual representation is square-integrable. By [25, Section 6.2], the global Arthur parameter of $\mathcal{E}_{\Delta(\tau,e)}$ is $\psi = (\tau, 2e) \boxplus (1_{\mathrm{GL}_1(\mathbb{A})}, 1)$. Hence $\mathcal{E}_{\Delta(\tau,e)} \in \widetilde{\Pi}_\psi(\mathrm{Sp}_{4e}) \cap \mathcal{A}_2(\mathrm{Sp}_{4e})$.

By [34, Theorem 1.3], $\mathfrak{p}^m(\mathcal{E}_{\Delta(\tau,e)}) = \{[2^{2e}]\}$. For ψ above, $\underline{p}_\psi = [(2e)^2 1]$ and $\eta(\underline{p}_\psi) = [2^{2e}]$. Hence, as mentioned in [34], combining with Theorem 2.9, all parts of Conjecture 2.8 have been proved for the Arthur parameter $\psi = (\tau, 2e) \boxplus (1_{\mathrm{GL}_1(\mathbb{A})}, 1)$ above.

Case of $n = 2e + 1$.

Proposition 6.7. *Any $\pi \in \widetilde{\Pi}_\psi(\mathrm{Sp}_{4e+2}) \cap \mathcal{A}_{\mathrm{cusp}}(\mathrm{Sp}_{4e+2})$ with $\psi = (\tau, 2i+1) \boxplus (\omega_\tau, 4e-4i+1)$, $e \leq 2i \leq 2e$, and $\tau \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_2)$ of orthogonal type, has the property that $\mathfrak{p}^m(\pi) = \{[2^{2e+1}]\}$, and hence is small.*

The proof of this proposition is similar to that of Proposition 6.6, and is omitted here. Note that if $2i < e$, then $4e - 4i + 1 > 2e + 1$. By Remark 3.3, the global Arthur packet $\widetilde{\Pi}_\psi(\mathrm{Sp}_{4e+2})$ associated to the global Arthur parameter

$$\psi = (\tau, 2i+1) \boxplus (1_{\mathrm{GL}_1(\mathbb{A})}, 4e - 4i + 1)$$

contains no cuspidal automorphic representations.

In the case of $2i = 2e$, we can also construct a residual representation in $\widetilde{\Pi}_\psi(\mathrm{Sp}_{4e+2}) \cap \mathcal{A}_2(\mathrm{Sp}_{4e+2})$ as follows.

Since $\tau \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_2)$ is of orthogonal type, by the theory of automorphic descent of Ginzburg, Rallis and Soudry, there is a cuspidal representation π' of $\mathrm{SO}_2^\alpha(\mathbb{A})$ which is anisotropic with discriminant $-\alpha$, such that π' lifts to τ by automorphic induction. Assume that there is an irreducible generic cuspidal representation π of $\mathrm{Sp}_2(\mathbb{A})$ corresponding to π' under the theta correspondence. Then the global Langlands functorial transfer from Sp_2 to GL_3 takes π to $\tau \boxplus 1$.

For any $\phi \in \mathcal{A}(N_{2e}(\mathbb{A})M_{2e}(F)\backslash \mathrm{Sp}_{4e+2}(\mathbb{A}))_{\Delta(\tau,e)\otimes\pi}$, a residual Eisenstein series can be defined as before by

$$E(\phi,s)(g) = \sum_{\gamma\in P_{2e}(F)\backslash \mathrm{Sp}_{4e+2}(F)} \lambda_s\phi(\gamma g).$$

It converges absolutely for real part of s large and has meromorphic continuation to the whole complex plane $\mathbb{C}$. By [25], this Eisenstein series has a simple pole at $\frac{e+1}{2}$, which is the right-most one. Denote by $\mathcal{E}_{\Delta(\tau,e)\otimes\pi}$ the representation generated by these residues at $s=\frac{e+1}{2}$. This residual representation is square-integrable. By [25, Section 6.2], the global Arthur parameter of $\mathcal{E}_{\Delta(\tau,e)\otimes\pi}$ is $\psi=(\tau,2e+1)\boxplus(\omega_\tau,1)$. Hence $\mathcal{E}_{\Delta(\tau,e)\otimes\pi}\in\widetilde{\Pi}_\psi(\mathrm{Sp}_{4e+2})\cap\mathcal{A}_2(\mathrm{Sp}_{4e+2})$.

By [19, Theorem 2.1], $\mathfrak{p}^m(\mathcal{E}_{\Delta(\tau,e)\otimes\pi})=\{[2^{2e+1}]\}$. For $\psi=(\tau,2e+1)\boxplus(\omega_\tau,1)$ above, $\underline{p}_\psi=[(2e+1)^2 1]$ and $\eta(\underline{p}_\psi)=[2^{2e+1}]$. Hence, combining with Theorem 2.9, all parts of Conjecture 2.8 have been proved for the Arthur parameter $\psi=(\tau,2e+1)\boxplus(\omega_\tau,1)$ above.

Case $n=3e+1$**.**

Proposition 6.8. *Any* $\pi\in\widetilde{\Pi}_\psi(\mathrm{Sp}_{6e+2})\cap\mathcal{A}_{\mathrm{cusp}}(\mathrm{Sp}_{6e+2})$ *with* $\psi=(\tau,2e+1)$*, and* $\tau\in\mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_3)$ *of orthogonal type and with trivial central character, has the property that* $\mathfrak{p}^m(\pi)=\{[2^{3e+1}]\}$*, and hence is small.*

Proof. For $\psi=(\tau,2e+1)$, we must have that $\underline{p}_\psi=[(2e+1)^3]$ and $\eta(\underline{p}_\psi)=[3^{2e}2]$. Take any $\pi\in\widetilde{\Pi}_\psi(\mathrm{Sp}_{6e+2})\cap\mathcal{A}_{\mathrm{cusp}}(\mathrm{Sp}_{6e+2})$. By Theorem 2.9, for any $\underline{p}=[p_1p_2\cdots p_r]\in\mathfrak{p}^m(\pi)$, we have that $\underline{p}\le[3^{2e}2]$ under the lexicographical order of partitions. It follows that

$$3\ge p_1\ge\cdots\ge p_r.$$

If $p_1=3$, then by [18, Lemma 3.3], π has a nonzero Fourier coefficient attached to the partition $[(p_1)^2 1^{6e+2-2p_1}]$. Then, by [8, Lemma 2.4], π has a nonzero Fourier coefficient attached to the partition $[(2r)1^{6e+2-2r}]$ for some $2r>p_1=3$, which contradicts Theorem 2.9. Hence $p_1=2$, and $\underline{p}\le[2^{3e+1}]$ under the dominance order of partitions. On the other hand, by Theorem 2.11, π is non-singular. Hence, any $\underline{p}\in\mathfrak{p}^m(\pi)$ also satisfies the property that $\underline{p}\ge[2^{3e+1}]$ under the dominance order of partitions. Therefore, we have proved that $\mathfrak{p}^m(\pi)=\{[2^{3e+1}]\}$.

This completes the proof of the proposition. □

We can also construct a residual representation in $\widetilde{\Pi}_\psi(\mathrm{Sp}_{6e+2})\cap\mathcal{A}_2(\mathrm{Sp}_{6e+2})$ as follows. Since $\tau\in\mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_3)$ has trivial central character, and $L(s,\tau,\mathrm{Sym}^2)$ has a pole at $s=1$, by the theory of automorphic descent ([10]), there is an irreducible generic cuspidal automorphic representation π of $\mathrm{Sp}_2(\mathbb{A})$ that lifts to τ.

For any $\phi \in \mathcal{A}(N_{3e}(\mathbb{A})M_{3e}(F)\backslash \mathrm{Sp}_{6e+2}(\mathbb{A}))_{\Delta(\tau,e)\otimes\pi}$, a residual Eisenstein series can also be defined by

$$E(\phi,s)(g) = \sum_{\gamma\in P_{3e}(F)\backslash \mathrm{Sp}_{6e+2}(F)} \lambda_s \phi(\gamma g).$$

It converges absolutely for real part of s large and has meromorphic continuation to the whole complex plane $\mathbb{C}$. By [25], this Eisenstein series has a simple pole at $\frac{e+1}{2}$, which is the right-most one. Denote by $\mathcal{E}_{\Delta(\tau,e)\otimes\pi}$ the representation generated by these residues at $s = \frac{e+1}{2}$. This residual representation is square-integrable. By [25, Section 6.2], the global Arthur parameter of $\mathcal{E}_{\Delta(\tau,e)\otimes\pi}$ is $\psi = (\tau, 2e+1)$. Hence $\mathcal{E}_{\Delta(\tau,e)\otimes\pi} \in \widetilde{\Pi}_\psi(\mathrm{Sp}_{6e+2}) \cap \mathcal{A}_2(\mathrm{Sp}_{6e+2})$.

For $\psi = (\tau, 2e+1)$ as above, $\underline{p}_\psi = [(2e+1)^3]$, and $\eta(\underline{p}_\psi) = [3^{2e}2]$. Hence, by Theorem 2.9, for any $\pi \in \widetilde{\Pi}_\psi(\mathrm{Sp}_{6e+2}) \cap \mathcal{A}_{\mathrm{cusp}}(\mathrm{Sp}_{6e+2})$, we have that for any $\underline{p} = [p_1p_2\cdots p_r] \in \mathfrak{p}^m(\pi)$, $\underline{p} \leq [3^{2e}2]$ under the lexicographical order of partitions, and hence, $\underline{p} \leq [3^{2e}2]$ under the dominance order of partitions also. By [19, Theorem 2.1], $\mathfrak{p}^m(\mathcal{E}_{\Delta(\tau,e)\otimes\pi}) = \{[3^{2e}2]\}$. Therefore, all parts of Conjecture 2.8 have been proved for the global Arthur parameter $\psi = (\tau, 2e+1)$ as above.

6.3. **Small cuspidal representations over totally imaginary number fields.** In this section, let F be a totally imaginary number field. Assume that $\mathfrak{p}^m(\pi)$ is a singleton. Then $\mathfrak{p}^m(\pi)$ consists of exactly the partition $\underline{p}_\pi$ constructed in [8].

Let $\underline{p}(\mathrm{Sp}_{2n}, F)$ be the smallest even partition of $2n$ of the form

$$[(2n_1)^{s_1}(2n_2)^{s_2}\cdots(2n_r)^{s_r}],$$

with $2n_1 > 2n_2 > \cdots > 2n_r$ and $s_i \leq 4$ for $1 \leq i \leq r$.

Example 6.9. $\underline{p}(\mathrm{Sp}_8, F) = [2^4]$, $\underline{p}(\mathrm{Sp}_{10}, F) = [42^3]$, $\underline{p}(\mathrm{Sp}_{12}, F) = [42^4]$, $\underline{p}(\mathrm{Sp}_{14}, F) = [4^22^3]$ *and* $\underline{p}(\mathrm{Sp}_{26}, F) = [64^32^4]$.

The following theorem follows easily from Theorem 2.2 and the definition of $\mathfrak{p}_{sm}^{\mathrm{Sp}_{2n},F}$ (see (2.4)).

Theorem 6.10. *Let F be a totally imaginary number field. Assume that $\mathfrak{p}^m(\pi)$ is a singleton for every $\pi \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{Sp}_{2n})$. Then any partition $\underline{p} \in \mathfrak{p}_{sm}^{\mathrm{Sp}_{2n},F}$ is greater than or equal to $\underline{p}(\mathrm{Sp}_{2n}, F)$.*

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