NEARLY CIRCULAR DOMAINS WHICH ARE INTEGRABLE CLOSE TO THE BOUNDARY ARE ELLIPSES

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Abstract. The Birkhoff conjecture says that the boundary of a strictly convex integrable billiard table is necessarily an ellipse. In this article, we consider a stronger notion of integrability, namely integrability close to the boundary, and prove a local version of this conjecture: a small perturbation of an ellipse of small eccentricity which preserves integrability near the boundary, is itself an ellipse. This extends the result in [1], where integrability was assumed on a larger set. In particular, it shows that (local) integrability near the boundary implies global integrability. One of the crucial ideas in the proof consists in analysing expansion of the corresponding action-angle coordinates with respect to the eccentricity parameter near the origin, deriving and studying a higher order condition.

1. Introduction

A *mathematical billiard* is a system describing the inertial motion of a point mass inside a domain, with elastic reflections at the boundary (which is assumed to have infinite mass). This simple model has been first proposed by G.D. Birkhoff as a mathematical playground where "*the formal side, usually so formidable in dynamics, almost completely disappears and only the interesting qualitative questions need to be considered* ", [5, pp. 155-156].

Since then billiards have captured much attention in many different contexts, becoming a very popular subject of investigation. Not only is their law of motion very physical and intuitive, but billiard-type dynamics is ubiquitous. Mathematically, they offer models in every subclass of dynamical systems (integrable, regular, chaotic, etc.); more importantly, techniques initially devised for billiards have often been applied and adapted to other systems, becoming standard tools and having ripple effects beyond the field.

Let us first recall some properties of the billiard map. We refer to [31, 35] for a more comprehensive introduction to the study of billiards.

Let Ω be a strictly convex domain in $\mathbb{R}^2$ with C^r boundary $\partial\Omega$, with $r \geq 3$. The phase space M of the billiard map consists of unit vectors (x, v) whose foot points x are on $\partial\Omega$ and which have inward directions. The billiard ball map $f : M \longrightarrow M$ takes (x, v) to (x', v'), where x' represents the point where the trajectory starting at x with velocity v hits the boundary $\partial\Omega$ again, and v' is the *reflected velocity*, according to the standard reflection law: angle of incidence is equal to the angle of reflection (Figure 1).

Remark 1.1. Observe that if Ω is not convex, then the billiard map is not continuous; in this article we shall be interested only in strictly convex domains (see Remark 1.4). Moreover, as pointed out by Halpern [19], if the boundary is not at least C^3, then the (continuous) Billiard flow might not be complete (or, equivalently, there might be non-trivial orbits with finite total length).

Let us introduce coordinates on M. We suppose that $\partial\Omega$ is parametrized by arc-length s and let $\gamma : [0, |\partial\Omega|] \longrightarrow \mathbb{R}^2$ denote such a parametrisation, where $|\partial\Omega|$ denotes the length of $\partial\Omega$. Let θ be the angle between v and the positive tangent to $\partial\Omega$ at x. Hence, M can be identified with the annulus $\mathbb{A} = (0, |\partial\Omega|) \times (0, \pi)$ and the billiard map f can be described as

$$\begin{aligned} f : (0, |\partial\Omega|) \times (0, \pi) &\longrightarrow (0, |\partial\Omega|) \times (0, \pi) \\ (s, \theta) &\longmapsto (s', \theta'). \end{aligned}$$

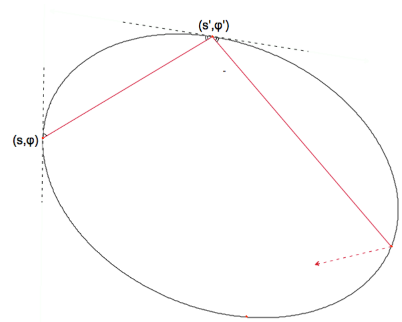

FIGURE 1.

In particular f can be extended to $\bar{\mathbb{A}} = [0, |\partial\Omega|] \times [0, \pi]$ by fixing $f(s, 0) = f(s, \pi) = s$ for all s.

It is easy to check that the billiard map f preserves the area form $\sin\theta \, ds \wedge d\theta$. If we denote by

$$\ell(s, s') := \|\gamma(s) - \gamma(s')\|$$

the Euclidean distance between two points on $\partial\Omega$, then one can check that

$$\begin{cases} \dfrac{\partial \ell}{\partial s}(s,s') = -\cos\theta \\ \dfrac{\partial \ell}{\partial s'}(s,s') = \cos\theta' . \end{cases} \tag{1.1}$$

Remark 1.2. If we lift everything to the universal cover and introduce new coordinates $(x,y) = (s, -\cos\theta) \in \mathbb{R} \times (-1,1)$, then the billiard map is a twist map with ℓ as generating function and it preserves the area form $dx \wedge dy$. See [31, 35].

Despite the apparently simple (local) dynamics, the qualitative dynamical properties of billiard maps are extremely non-local. This global influence on the dynamics translates into several intriguing *rigidity phenomena*, which are at the basis of several unanswered questions and conjectures (see for example [1, 14, 20, 22, 31, 32, 35]). Amongst many, in this article we shall address the question of classifying *integrable billiards*, also known as *Birkhoff conjecture*.

1.1. **Integrable billiards and Birkhoff conjecture.** The easiest example of billiard is given by a billiard in a disc $\mathcal{D}$ (for example of radius R). It is easy to check in this case that the angle of reflection remains constant at each reflection (see also [35, Chapter 2]). If we denote by s the arc-length parameter (*i.e.*, $s \in \mathbb{R}/2\pi R\mathbb{Z}$) and by $\theta \in (0, \pi/2]$ the angle of reflection, then the billiard map has a very simple form:

$$f(s,\theta) = (s + 2R\,\theta,\ \theta).$$

In particular, θ stays constant along the orbit and it represents an *integral of motion* for the map. Moreover, this billiard enjoys the peculiar property of having the phase space – which is topologically a cylinder – completely foliated by homotopically non-trivial invariant curves $\Gamma_{\theta_0} = \{\theta \equiv \theta_0\}$. These curves correspond to concentric circles of radii $\rho_0 = R\cos\theta_0$ and are examples of what are called *caustics*, which are defined as follows:

A smooth convex curve $\Gamma \subset \Omega$ is called a caustic*, if whenever a trajectory is tangent to it, then it remains tangent after each reflection (see figure 2).*

Notice that in the circular case, each caustic Γ corresponds to an invariant curve of the associated billiard map f and, therefore, has a well-defined rotation number.

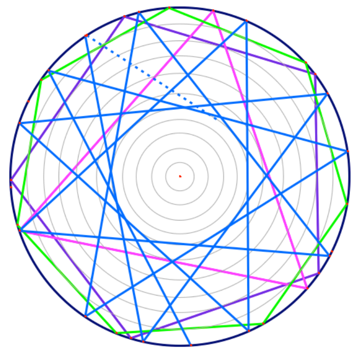

FIGURE 2. Billiard in a disc

A billiard in a disc is an example of an *integrable billiard*. There are different ways to define global/local integrability for billiards (the equivalence of these notions is an interesting problem itself):

- either through the existence of an integral of motion, globally or locally near the boundary (in the circular case an integral of motion is given by $I(s, \theta) = \theta$),
- or through the existence of a (smooth) foliation of the whole phase space or of an open subset (for example, of a neighbourhood of the boundary $\{\theta = 0\}$), consisting of invariant curves of the billiard map; for example, in the circular case these are given by Γ_θ. This property translates (under suitable assumptions) into the existence of a (smooth) family of caustics, globally or locally near the boundary (in the circular case, the concentric circles of radii $R\cos\theta$).

In [3], Misha Bialy proved the following result concerning global integrability (see also [38]):

Theorem (Bialy). *If the phase space of the billiard ball map is globally foliated by continuous invariant curves which are not null-homotopic, then it is a circular billiard.*

However, while circular billiards are the only examples of global integrable billiards, non-global integrability itself is still an intriguing open question. One could consider a billiard in an ellipse: this is in fact integrable, yet the dynamical picture is very distinct from the circular case: as it is showed in figure 3, each trajectory which does not pass through a focal point, is always tangent to precisely one confocal conic section, either a confocal ellipse or the two branches of a confocal hyperbola (see for example [35, Chapter 4]). Thus, the confocal ellipses inside an elliptical billiards are convex caustics, but they do not foliate the whole domain: the segment between the two foci is left out (describing the dynamics explicitly is much more complicated: see for example [36]).

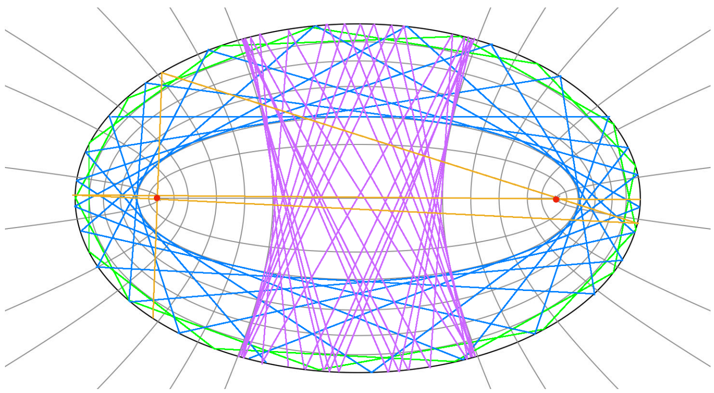

FIGURE 3. Billiard in an ellipse

Question (Birkhoff). *Are there other examples of integrable billiards?*

Remark 1.3. Although some vague indications of this question can be found in [5], to the best of our knowledge, its first appearance as a conjecture was in a paper by Poritsky [29],[1] which was published several years after Birkhoff's death. Thereafter, references to this conjecture (either as *Birkhoff conjecture* or *Birkhoff-Poritsky conjecture*) repeatedly appeared in the literature: see, for example, Gutkin [18, Section 1], Moser [27, Appendix A], Tabachnikov [34, Section 2.4], etc.

Remark 1.4. In [24] Mather proved the non-existence of caustics (hence, the non-integrability) if the curvature of the boundary vanishes at one point. This observation justifies the restriction of our attention to strictly convex domains.

Remark 1.5. Interestingly, Treschev in [37] gives indication that there might exist analytic billiards, different from ellipses, for which the dynamics in a neighborhood of the elliptic period-2 orbit is conjugate to a rigid rotation. These billiards can be seen as an instance of *local integrability*; however, this regime is somehow complementary to the one conjectured by Birkhoff: one has local integrability in a neighborhood of an elliptic periodic orbit of period 2, while Birkhoff conjecture is

[1]Poritsky was presumably Birkhoff's postdoctoral student at Harvard and cites several papers of Birkhoff on the topic in [29].

related to integrability in a neighborhood of the boundary. This gives an indication that these two notions of integrability might differ.

Remark 1.6. Birkhoff conjecture can be also thought as an analogue, in the case of billiards, of the following task: classifying *integrable* (Riemannian) geodesic flows on $\mathbb{T}^2$. The complexity of this question, of course, depends on the notion of integrability that one considers. If one assumes that the whole space space is foliated by invariant Lagrangian graphs (*i.e.*, the system is C^0-*integrable*), then it follows from Hopf conjecture [9] that the associated metric must be flat. However, the question becomes more challenging – and it is still open – if one considers integrability only on an open and dense set (*global integrability*), or assumes the existence of an open set foliated by invariant Lagrangian graphs (*local integrability*). Example of globally integrable (non-flat) geodesic flows on $\mathbb{T}^2$ are those associated to *Liouville-type metrics*, namely metrics of the form

$$ds^2 = (f_1(x_1) + f_2(x_2))\,(dx_1^2 + dx_2^2).$$

A folklore conjecture states that these metrics are the only globally (resp. locally) integrable metrics on $\mathbb{T}^2$. A partial answer to this conjecture (global case) is provided in [6], where the authors prove it under the assumption that the system admits an integral of motion which is quadratic in the momenta. The question to which we provide an affirmative answer in this article (local Birkhoff conjecture), can be considered as an analogue, in the billiard setting, of the above conjecture (local case). It is interesting to point out, however, that - contrarily to what happens with billiards – there is evidence that this local conjecture might be false for geodesic flows (see [11]).

Despite its long history and the amount of attention that this conjecture has captured, it remains still open. As far as our understanding of integrable billiards is concerned, the most important related results are the above–mentioned theorem by Bialy [3] (see also [38]), a result by Delshams and Ramírez-Ros [13] in which they study entire perturbations of elliptic billiards and prove that any nontrivial symmetric perturbation of the elliptic billiard is not integrable, a result by Innami[2] [21], in which he shows that the existence of caustics for all rotation numbers in $(0, 1/2)$ implies that the billiard must be an ellipse, and a more recent result by Avila, De Simoi and Kaloshin [1] in which they

[2]We are grateful to M. Bialy for pointing out this reference.

show a perturbative version of this conjecture for ellipses of small eccentricity, assuming the existence of caustics for all rotation numbers in $(0, 1/3]$. The latter result was generalised to ellipses of any eccentricity by Kaloshin and Sorrentino [22].

Let us introduce an important notion for this paper.

Definition 1.7. (i) *We say* Γ *is an* integrable rational caustic *for the billiard map in* Ω*, if the corresponding (non-contractible) invariant curve* Γ *consists of periodic points; in particular, the corresponding rotation number is rational.*
(ii) *Let* $q_0 \geqslant 2$*. If the billiard map inside* Ω *admits integrable rational caustics of rotation number* p/q *for all* $0 < p/q < 1/q_0$*, we say that* Ω *is* q_0–rationally integrable.

Remark 1.8. A simple sufficient condition for rational integrability is the following (see [1, Lemma 1]). Let $\mathcal{C}_\Omega$ denote the union of all smooth convex caustics of the billiard in Ω; if the interior of $\mathcal{C}_\Omega$ contains caustics of rotation number p/q for all $0 < p/q < 1/q_0$, then Ω is q_0-rationally integrable.

Let us denote with $\mathcal{E}_{e,c} \subset \mathbb{R}^2$ an ellipse of eccentricity e and semifocal distance c. We state the following local version of Birkhoff conjecture.

Conjecture 1.9. *For any integer* $q_0 \geq 3$*, there exist* $e_0 = e_0(q_0) \in (0, 1)$*,* $m_0 = m_0(q_0)$*,* $n_0 = n_0(q_0) \in \mathbb{N}$ *such that the following holds. For each* $0 < e \leqslant e_0$ *and* $c \geqslant 0$*, there exists* $\varepsilon = \varepsilon(e, c, q_0) > 0$ *such that any* q_0*-rationally integrable* C^{m_0}*–smooth domain* Ω*, whose boundary* $\partial\Omega$ *is* C^{n_0}*–*ε*-close to an ellipse* $\mathcal{E}_{e,c}$*, is an ellipse.*

In this paper we prove this conjecture in some cases and provide a proof for the remaining ones based on certain non-degeneracy conditions. These non-degeneracy conditions are explicit and computable: in Section 7 we provide a description of how to implement an algorithm to verify them by means of symbolic computations.

More precisely, our main results are the following.

Theorem 1.1. *Conjecture 1.9 holds true for* $q_0 = 2, 3, 4, 5$*, with* $m_0 = 40q_0$ *and* $n_0 = 3q_0$*.*

Theorem 1.2. *For any integer* $q_0 \geqslant 6$*, Conjecture 1.9 holds true for* q_0 *with* $m_0 = 40q_0$ *and* $n_0 = 3q_0$*, provided that the* $q_0 - 2$ *matrices* (7.11)-(7.17) *are non-degenerate.*

Remark 1.10. (i) Case $q_0 = 2$ was proven in [1] (see also [21]).

(ii) Notice that $\varepsilon(e, c, q_0) \to 0$ as $e \to 0$. Non-zero e produces asymetry and is fundamental for our arguments. The less e is, the smaller is the perturbation so that we stay in asymmetric regeme.

(iii) The smoothness exponent is probably not optimal. In the proof of one of the key lemmata (Lemma 3.3), we have directly used certain C^1-estimates from [1]. One may improve the smoothness exponent by deriving C^n estimates instead.

(iv) Notice that we actually do not need the existence of all caustics of rotation number less than $1/q_0$; in fact, we only use caustics of rotation numbers of the form $j/q < 1/q_0$ for $j = 1, 2, 3$.

(v) Analysis of caustics of rotation numbers $\frac{2}{2q+1}$ is fairly delicate [3]. For either a domain near the circle or an arbitrary smooth enough domain with q being large, the condition of preservation of caustics of rotation numbers $\frac{2}{2q+1}$ and $\frac{1}{2q+1}$ **are the same** to the leading order! Thus, to obtain a new condition from caustics of rotation numbers $\frac{2}{2q+1}$ we need a precise information about *higher order dependence on the rotation number*. For small eccentricity e this can be extracted from expansion of action-angle variables in eccentricity (see Appendix C for details). Without this precise information our method would not work! This analysis can be considered as the main novel feature of the present paper compares to [1] and [22].

(vi) The coefficients of matrices (7.11)-(7.17) are completely determined by the e-expansion of the action-angle parametrisation of the ellipse, which, in turn, is explicitly given by elliptic integrals (see (3.2) and Appendix C). In particular, the entries of these matrices are either 0, 1 or of the form $\xi \cos^{-2j}(w\pi)e^{2j}$, where $\xi \in \mathbb{Q}$, $j \in \mathbb{N}$, $w \in \{\frac{1}{2k+1}, \frac{2}{2k+1}, \frac{1}{2k}, \frac{3}{2k} : \ k > j\}$. See also Remarks 7.5 and 7.9.

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[3]the same remark applies to rotation numbers $\frac{3}{2q}$ where q is not divisible by 3

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2. The strategy of the proof

Let us consider the ellipse

$$\mathcal{E}_{e,c} = \left\{ (x,y) \in \mathbb{R}^2 : \ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right\},$$

centered at the origin and with semiaxes of lengths, respectively, $0 < b \leqslant a$; in particular e denotes its eccentricity, given by $e = \sqrt{1 - \frac{b^2}{a^2}} \in [0,1)$ and $c = \sqrt{a^2 - b^2}$ the semi-focal distance. Observe that when $e = 0$, then $c = 0$ and $\mathcal{E}_{0,0}$ degenerates to a 1-parameter family of circles centered at the origin.

The family of confocal elliptic caustics in $\mathcal{E}_{e,c}$ is given by (see also Figure 3):

$$C_\lambda = \left\{ (x,y) \in \mathbb{R}^2 : \ \frac{x^2}{a^2 - \lambda^2} + \frac{y^2}{b^2 - \lambda^2} = 1 \right\} \qquad 0 < \lambda < b. \tag{2.1}$$

Observe that the boundary corresponds to $\lambda = 0$, while the limit case $\lambda = b$ corresponds to the the two foci $\mathcal{F}_\pm = (\pm\sqrt{a^2 - b^2}, 0)$. Clearly, for $e = 0$ we recover the family of concentric circles described in Figure 2.

Denote $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. A more convenient coordinate frame for addressing our question is provided by the so-called *elliptic-polar coordinates* (or, simply, *elliptic coordinates*) $(\mu, \varphi) \in \mathbb{R}_{\geq 0} \times \mathbb{T}$, given by:

$$\begin{cases} x = c \cosh \mu \, \cos \varphi \\ y = c \sinh \mu \, \sin \varphi, \end{cases}$$

where $c = \sqrt{a^2 - b^2} > 0$ represents the semi-focal distance (in the case $e = 0$, this parametrisation degenerates to the usual polar coordinates). Observe that for each $\mu_* > 0$, the equation $\mu \equiv \mu_*$ represents a confocal ellipse.

Therefore, in these elliptic polar coordinates $\mathcal{E}_{e,c}$ becomes:

$$\mathcal{E}_{e,c} = \left\{ (\mu_0, \varphi), \ \varphi \in \mathbb{T} \right\},$$

where $\mu_0 = \mu_0(e) := \cosh^{-1}(1/e)$.Then, any smooth perturbation Ω of the ellipse $\mathcal{E}_{e,c}$ can be written in this elliptic coordinate frame as

$$\partial\Omega = \{(\mu_0 + \mu(\varphi), \varphi) : \varphi \in \mathbb{T}\},$$

where $\mu(\varphi)$ is a small smooth 2π-periodic function; hereafter, we shall adopt this shorthand notation and write

$$\partial\Omega = \mathcal{E}_{e,c} + \mu(\varphi).$$

Before describing the strategy of our proof, let us first recall the scheme in [1], and then describe the needed adjustments.

2.1. **A preliminary scheme of proving Theorem 1.1 for $q_0 = 2$.** In this form Theorem 1.1 was proven in [1] and we now describe the proof therein. In order to get a clearer idea, let us start from the simplified case of integrable infinitesimal deformations of a circle.

Let Ω_0 be an circle centered at the origin. Let Ω_ε be a one-parameter family of deformations given in polar coordinates by

$$\partial\Omega_\varepsilon = \{(\mu, \varphi) = (\mu_0 + \varepsilon\mu(\varphi) + O(\varepsilon^2), \varphi)\}.$$

Fix a parametrisation of the boundary $\varphi : \mathbb{T} \to \mathbb{T}$. Consider the Fourier expansion of $\mu \circ \varphi$:

$$\mu \circ \varphi(\theta) = \mu'_0 + \sum_{k>0} \mu'_{k,\varphi} \sin(k\theta) + \mu''_{k,\varphi} \cos(k\theta).$$

Theorem 2.1 (Ramírez-Ros [30])**.** *If Ω_ε has an integrable rational caustic $\Gamma_{1/q}$ of rotation number $1/q$, for any ε sufficiently smooth, then for certain parametrisation of the boundary*[4] *$\varphi_{1/q}(\theta)$ we have $\mu'_{q,\varphi_{1/q}} = \mu''_{q,\varphi_{1/q}} = 0$.*

Notice that in the case of the circle $\varphi_{1/q}(\theta) \equiv \theta$ for all $q > 2$, but this is violated away from the circle, see (3.2) for more general parametrisation of the boundary ellipses. This more general framework allows us to explain our strategy better.

Let us now assume that the domains Ω_ε are 2-rationally integrable for all sufficiently small ε and ignore for a moment dependence on the parametrisation: then the above theorem implies that $\mu'_k = \mu''_k = 0$ for $k > 2$, *i.e.*,

$$\begin{aligned}\mu(\varphi) &= \mu'_0 + \mu'_1 \cos\varphi + \mu''_1 \sin\varphi + \mu'_2 \cos 2\varphi + \mu''_2 \sin 2\varphi \\ &= \mu'_0 + \mu^*_1 \cos(\varphi - \varphi_1) + \mu^*_2 \cos 2(\varphi - \varphi_2),\end{aligned}$$

where φ_1 and φ_2 are appropriately chosen phases.

Remark 2.1. Observe that

- μ'_0 corresponds to an homothety;

[4]This parametrisation of the boundary can be found in Lemma 3.3

- $\mu_1^* \cos(\varphi - \varphi_1)$ corresponds to a translation in the direction forming an angle φ_1 with the polar semi-axis $\{\varphi = 0\}$;
- $\mu_2^* \cos 2(\varphi - \varphi_2)$ corresponds to a deformation of the disc into an ellipse of small eccentricity, whose major axis forms an angle φ_2 with the polar semi-axis.

This implies that, infinitesimally (as $\varepsilon \to 0$), 2-rationally integrable deformations of a circle are tangent to the 5-parameter family of ellipses.

Observe that in principle, in the above theorem, one may need to take $\varepsilon \to 0$ as $q \to \infty$. However, note that the cases we have to deal with correspond to $\varepsilon > 0$ small, but not infinitesimal; hence one cannot use directly the above scheme to prove the result and a more elaborate strategy needs to be adopted. Let us describe it more precisely.

2.2. **An actual scheme of the proof of Theorem 1.1 for $q_0 = 2$.** Let $\mathcal{E}_{e,c}$ be an ellipse of small eccentricity e and semifocal distance c. Let (μ, φ) be the associated elliptic coordinate frame. Any domain Ω whose boundary is close to $\mathcal{E}_{e,c}$ in the elliptic coordinates associated to $\mathcal{E}_{e,c}$ can be written in the form

$$\partial\Omega = \{(\mu_0 + \mu(\varphi), \varphi) : \varphi \in \mathbb{T}\},$$

where $\mu(\varphi)$ is a (small) smooth 2π-periodic function. The strategy used in [1] proceeds as follows (keep in mind that the ellipse $\mathcal{E}_{e.c}$ admits all integrable rational caustics of rotation number $1/q$ for $q > 2$).

Step 1: Derive a quantitative necessary condition for the preservation of an integrable rational caustic of a given rotation number (see [1, Theorem 3] or Lemma 3.3 below).

Step 2: Define the *Deformed Fourier modes*

$$\{c_0, c_q, c_{-q}\}_{q>0}$$

associated to the ellipse $\mathcal{E}_{e,c}$. They satisfy the following properties:

- *(Relation with Fourier Modes)* There exist (see [1, Lemma 20]) $C^*(e,c) > 0$ with $C^*(e,c) \to 0$ as $e \to 0$, and a properly chosen parametrisation of the boundary such that

$$\|c_0 - 1\|_{C^0} \leqslant C^*(e,c)$$

and for any $q \geq 1$

$$\begin{cases} \|c_q - \cos(q\,\cdot)\|_{C^0} \leqslant q^{-1}\, C^*(e,c) \\ \|c_{-q} - \sin(q\,\cdot)\|_{C^0} \leqslant q^{-1}\, C^*(e,c). \end{cases} \tag{2.2}$$

- (*Transformations preserving integrability*) The first five functions

$$c_0,\ c_1,\ c_{-1},\ c_2,\ c_{-2}$$

correspond to infinitesimal generators of deformations preserving the class of ellipses: namely, homotheties, translations and hyperbolic rotations about an arbitrary axis.

- (*Annihilation of inner products*) Consider the one-parameter family of domains Ω_ε, $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, written in the elliptic coordinate frame associated to the ellipse $\mathcal{E}_{e,c}$,

$$\partial\Omega_\varepsilon := \mathcal{E}_{e,c} + \varepsilon\mu.$$

For any $q > 2$, if Ω_ε admits an integrable rational caustic of rotation number $1/q$ for all sufficiently small ε, then

$$\langle \mu, c_q \rangle = 0, \quad \langle \mu, c_{-q} \rangle = 0, \tag{2.3}$$

where $\langle \cdot, \cdot \rangle$ is a suitably weighted L^2 inner product.

Notice that the functions $c_{\pm q}$ can be explicitly defined using elliptic integrals via action-angle coordinates.

- (*Linear independence and Basis property*) For sufficiently small eccentricity the functions $\{c_0, c_q, c_{-q} : q > 0\}$ form a (non-orthogonal) basis of $L^2(\mathbb{T})$.

Step 3 (*Approximation*): Using the annihilation of the inner products, for the domain $\partial\Omega = \mathcal{E}_{e,c} + \mu$ with small eccentricity e, one could always find another ellipse $\mathcal{E}'$ such that

$$\partial\Omega = \mathcal{E}' + \mu' \qquad \text{and} \qquad \|\mu'\|_{C^1} \leqslant \frac{1}{2}\|\mu\|_{C^1}.$$

Apply this result to the best approximation of Ω by an ellipse, then argue by contradiction, and conclude that Ω itself must be an ellipse.

2.3. **The adjusted scheme for the case $q_0 > 2$.** Now we describe how to modify the above strategy to deal with the case $q_0 > 2$. Fix an ellipse $\mathcal{E}_{e,c}$ of eccentricity $e > 0$ and semifocal distance c. In Section 3.1 we shall introduce the action-angle coordinates associated to the billiard problem in $\mathcal{E}_{e,c}$. It turns out that for $e = 0$ these action-angle coordinates degenerate into the polar coordinates (ρ, θ).

Step 1′: For small $e > 0$, we study the expansions of the action-angle coordinates in the polar coordinates with respect to e. Using this expansion, we derive the necessary condition for the preservation of smooth convex rational caustics, in terms of the Fourier coefficients of the function μ, up to the precision of order e^{2N}, for some positive integer $N = N(q_0)$. See Section 3 and equality (3.8) for more details.

Step 2′: We define the *deformed Fourier modes* $\{\mathcal{C}_0,\ \mathcal{C}_q,\ \mathcal{C}_{-q}\}_{q>0}$, similarly to what described before. These functions satisfy the following properties: fix some $r \in \mathbb{N}$,

(1) *(Relation with Fourier mode)* We have $\mathcal{C}_0 = 1$,

$$\mathcal{C}_q(\cdot) = \mathcal{V}_q(\cdot), \quad \mathcal{C}_{-q}(\cdot) = \mathcal{V}_{-q}(\cdot), \quad 0 < q \leqslant q_0,$$

and there exists $C^*(e) >$ with $C^*(e) \to 0$, as $e \to 0$, such that

$$\begin{cases} \|\, \mathcal{C}_q(\cdot) - \mathcal{V}_q(\cdot)\, \|_r \ \leqslant\ C^*(e)/q, \\ \|\mathcal{C}_{-q}(\cdot) - \mathcal{V}_{-q}(\cdot)\|_r \leqslant C^*(e)/q, \end{cases} \qquad q > q_0,$$

where $\|\cdot\|_r$ is the norm in the Sobolev space $H^r(\mathbb{T})$, and $\mathcal{V}_q$ are the zero average functions on $\mathbb{T}$, such that

$$\begin{cases} \mathcal{V}_q^{(r)}(\cdot) = \quad \cos(q\ \cdot), & q > 0 \\ \mathcal{V}_q^{(r)}(\cdot) = -\sin(q\ \cdot), & q < 0, \end{cases}$$

where $\mathcal{V}_q^{(r)}$ denotes the r-th derivative of $\mathcal{V}_q$. The constant $C^*(e)$ here can be chosen as in (2.2).

(2) *(Linear independence and Basis property)* For small eccentricity, the set of functions $\{\mathcal{C}_0,\ \mathcal{C}_q,\ \mathcal{C}_{-q},\ q \in \mathbb{Z}_+\}$ form a (non-orthogonal) basis of the Hilbert space H^r (see Lemma 8.3).

(3) *(Annihilation of inner products)* From the existence of integrable rational caustics with rotation numbers $1/q$, $q > q_0$, we deduce the following relations:

$$\langle \mu, \mathcal{C}_{\pm q}\rangle_r = O(q^7 \|\mu\|_{C^1}^{1+\beta}), \quad \beta > 0,$$

where $\langle\cdot,\cdot\rangle_r$ is the inner product of the Hilbert space $H^r(\mathbb{T})$ (see Lemma 8.9).

Observe that since $q_0 \geqslant 3$, with respect to the previous scheme we have lost finitely many annihilation conditions:

$$\langle \mu, \mathcal{C}_q\rangle_r = \langle \mu, \mathcal{C}_{-q}\rangle_r = 0, \quad 3 \leqslant q \leqslant q_0. \tag{2.4}$$

Hence, we need to find a way to recover them. Our goal becomes then to show:

$$\langle \mu, \mathcal{C}_q\rangle_r = O(e^2) \quad , \quad \langle \mu, \mathcal{C}_{-q}\rangle_r = O(e^2), \quad 3 \leqslant q \leqslant q_0. \tag{2.5}$$

In particular, here it is how we manage to prove this.

- **Case $q_0 = 3$:** We lose a pair of conditions (2.4), corresponding to $q = 3$. In Section 4 we study the necessary conditions for the existence of integrable rational caustics of rotation numbers $1/7, 1/5, 2/7$. We use the expansions, with respect to e, of the resulting equalities, up to the precision $O(e^6)$, to derive a system of linear equations (see (4.8)) for the 3^{th}, 5^{th}, 7^{th} Fourier coefficients. Solving this linear system will provide us with (2.5) for $q_0 = 3$.

- **Case $q_0 = 4$:** In this case we lose two pairs of conditions (2.4), corresponding to $q = 3, 4$. In Section 5 we derive (2.5) for $q = 3, 4$; this will be achieved in two steps:
  - To recover (2.5) for $q = 3$, we study the necessary conditions for the existence of integrable rational caustics of rotation numbers $p/q = 1/5$, $1/7$, $1/9$, $2/9$, written in terms of the Fourier coefficients of μ, and considering their expansions, with respect to e, up to order $O(e^8)$. We then derive a linear system for the 3^{th}, 5^{th}, 7^{th}, 9^{th} Fourier coefficients, whose solution will provide us with (2.5) for $q = 3$.
  - To recover (2.5) for $q = 4$, we study the necessary conditions for the existence of integrable rational caustics of rotation numbers $p/q = 1/6$, $1/8$, $1/10$, $1/12$, $1/14$, $3/14$, which give rise to a system of linear equation for the 4^{th}, 6^{th}, 8^{th}, 10^{th}, 12^{th}, 14^{th} Fourier coefficients; similarly as above, the solution of this system will prove (2.5) for $q = 4$.

- **Case $q_0 = 5$ and the general case:** Along the same lines described in the previous two items, the case $q_0 = 5$ will be discussed in Section 6. Moreover, in Section 7 we shall outline a general (conditional) procedure to derive (2.5) for any $q_0 \geqslant 6$; the implementation of this scheme is based on the assumption that certain explicit non-degeneracy conditions for the corresponding linear systems hold (see Remarks 7.5 and 7.9).

Step 3′: Finally, once the previous steps are completed, we adapt the approximation arguments from [1] and show that Ω must be an ellipse; see Section 9 for more details.

3. Necessary conditions for the existence of a caustic with rational rotation number

3.1. **Elliptic billiard dynamics and caustics.** Now we want to provide a more precise description of the billiard dynamics in $\mathcal{E}_{e,c}$. We rely on notations of Section 2. In addition we need the following notations.

Let $0 \leqslant k < 1$, we define elliptic integrals and Jacobi Elliptic functions:

- Incomplete elliptic integral of the first kind:
$$F(\varphi; k) := \int_0^{\varphi} \frac{1}{\sqrt{1 - k^2 \sin^2 \tau}} d\tau.$$
- Complete elliptic integral of the first kind:
$$K(k) = F\left(\frac{\pi}{2}; k\right).$$
- Jacobi Elliptic functions are obtained by inverting incomplete elliptic integrals of the first kind. Precisely, if
$$u = F(\varphi; k) = \int_0^{\varphi} \frac{1}{\sqrt{1 - k^2 \sin^2 \tau}} \, d\tau,$$
then by definition
$$\varphi := \operatorname{am}(u; k),$$
and the Jacobi elliptic functions are defined as:
$$\begin{aligned} \operatorname{sn}(u; k) &:= \sin(\operatorname{am}(u; k)) = \sin(\varphi), \\ \operatorname{cn}(u; k) &:= \cos(\operatorname{am}(u; k)) = \cos(\varphi). \end{aligned}$$

The following result has been proven in [10] (see also [12, Lm. 2.1]).

Proposition 3.1. *Let $\lambda \in (0, b)$ and let*
$$k_\lambda^2 := \frac{a^2 - b^2}{a^2 - \lambda^2} \quad \text{and} \quad \delta_\lambda := 2\, F(\arcsin(\lambda/b); k_\lambda).$$
Let us denote, in cartesian coordinates, $q_\lambda(t) := (a \operatorname{cn}(t; k_\lambda), b \operatorname{sn}(t; k_\lambda))$. Then, for every $t \in [0, 4K(k_\lambda))$ the segment joining $q_\lambda(t)$ and $q_\lambda(t+\delta_\lambda)$ is tangent to the caustic C_λ, defined in (2.1).

Observe that:

- k_λ is a strictly increasing function of $\lambda \in (0, b)$; in particular $k_\lambda \to e$ as $\lambda \to 0^+$, while $k_\lambda \to 1$ as $\lambda \to b^-$. Observe that k_λ represents the eccentricity of the ellipse C_λ.
- δ_λ is also a strictly increasing function of $\lambda \in (0, b)$; in fact, $F(\varphi; k)$ is clearly strictly increasing in both φ and $k \in [0, 1)$. Moreover, $\delta_\lambda \to 0$ as $\lambda \to 0^+$, and $\delta_\lambda \to +\infty$ as $\lambda \to b^-$.

Let us now consider the parametrisation of the boundary induced by the dynamics on the caustic C_λ:

$$\begin{aligned} Q_\lambda : \mathbb{R}/2\pi\mathbb{Z} &\longrightarrow \mathbb{R}^2 \\ \theta &\longmapsto q_\lambda\left(\frac{4K(k_\lambda)}{2\pi}\,\theta\right). \end{aligned}$$

We define the *rotation number* associated to the caustic C_λ to be

$$\omega(\lambda, e) := \frac{\delta_\lambda}{4K(k_\lambda)} = \frac{F(\arcsin(\lambda/b); k_\lambda)}{2K(k_\lambda)}. \tag{3.1}$$

In particular $\omega(\lambda, e)$ is strictly increasing in λ and $\omega(\lambda, e) \longrightarrow 0$ as $\lambda \to 0^+$, while $\omega(\lambda, e) \to \frac{1}{2}$ as $\lambda \to b^-$.In addition, for every $\theta \in \mathbb{T}$, the orbit starting at $Q_\lambda(\theta)$ and tangent to C_λ, hits the boundary at $Q_\lambda(\theta + 2\pi\,\omega_\lambda)$.

Denote the inverse of the function $\omega(\lambda, e)$, by $\lambda_\omega = \lambda(e, \omega)$. Notice that $\omega(\lambda, e)$ is even[5] with respect to e, hence so is $\lambda(e, \omega)$. Moreover,

$$\omega(\lambda, 0) = \frac{\arcsin(\lambda/b)}{\pi},$$

and it is straightforward to show that the following estimate holds.

Lemma 3.2. *There exists $C > 0$ such that for each $e \in [0, \frac{1}{2}]$ and $\omega \in (0, 1/2)$, we have*

$$|\lambda(e, \omega) - b \sin \omega\pi| \leqslant Ce^2.$$

We want to write the boundary parametrisation induced by the caustic C_λ, expressed in elliptic coordinates (μ, φ), namely determine the function $S_\lambda(\theta) = (\mu_\lambda(\theta), \varphi_\lambda(\theta)) = (\mu_0, \varphi_\lambda(\theta))$ such that the orbit starting at $S_\lambda(\theta)$ (in elliptic coordinates) and tangent to C_λ, hits the boundary at $S_\lambda(\theta + 2\pi\,\omega_\lambda)$.

It is easy to deduce from the above expression that

$$\varphi_\lambda(\theta) = \varphi(\theta, \lambda, e) := \mathrm{am}\left(\frac{4K(k_\lambda)}{2\pi}\,\theta; k_\lambda\right). \tag{3.2}$$

Therefore, we have $S_\lambda(\theta) = \left(\mu_0, \mathrm{am}\left(\frac{4K(k_\lambda)}{2\pi}\,\theta; k_\lambda\right)\right)$. The parametrisation given in (3.2) is called the *action-angle parametrisation* of the boundary, associated to the caustic C_λ. Below, if we need to emphasize the rotation number of the associated caustic we write C_{λ_ω}.

[5]Notice that e enters the definition of the function in the form e^2.

Fix a positive integer m and consider a perturbation of the ellipse $\mathcal{E}_{e,c}$, denoted Ω_ε, *i.e.*, in the elliptic coordinates

$$\partial\Omega_\varepsilon = \{(\mu,\varphi):\ \mu = \mu_0 + \mu_\varepsilon(\varphi),\ \varphi \in [0,2\pi)\},$$

where $\mu_\varepsilon(\varphi)$ is a 2π-periodic C^m-function with $\|\mu_\varepsilon\|_{C^m} \leqslant M$ and $\|\mu_\varepsilon\|_{C^1} \leq \varepsilon$ and is small enough.

Lemma 3.3. *For any rational number $p/q \in (0,1/2)$ in lowest terms, if the billiard inside Ω_ε admits an integrable rational caustic $C^\varepsilon_{\lambda_{p/q}}$ of rotation number p/q, then there exist constants $c_{p/q}$ and $C = C(M)$ such that*

$$\lambda_{p/q}\sum_{k=1}^{q}\mu_\varepsilon(\varphi_{\lambda_{p/q}}(\theta + \frac{kp}{q}2\pi)) = c_{p/q} + \Upsilon_{p/q}(\theta), \tag{3.3}$$

where[6] $\Upsilon_{p/q} \in C^{m-1}(\mathbb{T})$ *and*

$$\|\Upsilon_{p/q}\|_{C^{m-1}} \leqslant qC \qquad \|\Upsilon_{p/q}\|_{C^0} \leqslant Cq^7\|\mu_\varepsilon\|^2_{C^1}.$$

Remark 3.4. We need higher regularity estimates proven in this lemma to prove Lemma 8.9.

Proof. Let φ_k, $k = 0,\dots,q-1$, be the vertices of the maximal (p,q)-gon inscribed in the ellipse $\mathcal{E}_{e,c}$, tangent to the caustic $C^0_{p/q}$ of the billiard map in $\mathcal{E}_{e,c}$, with φ_0 being θ-dependent. Let φ^ε_k, $k = 0,\dots,q-1$, be the vertices of the maximal (p,q)-gon inscribed in Ω, tangent to the caustic $C^\varepsilon_{p/q}$, with $\varphi^\varepsilon_0 = \varphi_0$. Then by Lemma 5 in [1], we have that there exists $C > 0$, independent of q, such that

$$|\varphi_k - \varphi^\varepsilon_k| \leqslant Cq^3\|\mu_\varepsilon\|^2_{C^1}, \quad k = 0,\dots,q-1. \tag{3.4}$$

For $\|\mu_\epsilon\|_{C^1}$ small enough, the generating function of the billiard dynamics inside Ω_ϵ is given by

$$h_\varepsilon(\varphi,\varphi') = h_0(\varphi,\varphi') + h_1(\varphi,\varphi') + h_2(\varphi,\varphi'),$$

where $h_0(\varphi,\varphi')$ is the generating function of the billiard dynamics inside the ellipse $\mathcal{E}_{e,c}$ and

$$\|h_1(\varphi,\varphi')\|_{C^1} \leqslant 2\|\mu_\varepsilon\|_{C^1} \qquad \|h_2(\varphi,\varphi')\|_{C^0} < C\|\mu_\varepsilon\|^2_{C^1},$$

and

$$\|h_1\|_{C^m} + \|h_2\|_{C^m} \leqslant C.$$

Using [28, Proposition 4.1], we deduce

$$\sum_{k=0}^{q-1} h_1(\varphi_k,\varphi_{k+1}) = 2\lambda_{p/q}\sum_{k=0}^{q-1}\mu_\varepsilon(\varphi_k). \tag{3.5}$$

[6]we drop dependence on ε in the notations

By the existence of an integrable rational caustic with rotation number p/q for the billiard dynamics inside Ω_ε, we have

$$L(\varphi) := \sum_{k=0}^{q-1} h_\varepsilon(\varphi_k^\varepsilon, \varphi_{k+1}^\varepsilon) = const, \quad \text{where } \varphi_0^\varepsilon = \varphi. \tag{3.6}$$

Since

$$\begin{aligned}
&\sum_{k=1} h_0(\varphi_k^\varepsilon, \varphi_{k+1}^\varepsilon) = \sum_{k=0}^{q-1} \left[h_0(\varphi_k^\varepsilon, \varphi_{k+1}^\varepsilon) - h_0(\varphi_k, \varphi_{k+1}) + h_0(\varphi_k, \varphi_{k+1})\right] \\
&= \sum_{k=1}^{q-1} \left[(\partial_1 h_0(\varphi_k, \varphi_{k+1}) + \partial_2 h_0(\varphi_{k-1}, \varphi_k))(\varphi_k^\varepsilon - \varphi_k) + h_0(\varphi_k, \varphi_{k+1})\right. \\
&\qquad \left. + \; O(|\varphi_k^\varepsilon - \varphi_k|^2)\right] \\
&= \sum_{k=0}^{q-1} h_0(\varphi_k, \varphi_{k+1}) + \Upsilon_{p/q}^0,
\end{aligned}$$

with

$$\|\Upsilon_{p/q}^0\|_{C^0} = O(q^7 \|\mu_\varepsilon\|_{C^1}^2), \quad \|\Upsilon_{p/q}^0\|_{C^m} \leqslant qC_0(M),$$

and

$$\sum_{k=0}^{q-1} h_1(\varphi_k^\varepsilon, \varphi_{k+1}^\varepsilon) = \sum_{k=0}^{q-1} h_1(\varphi_k, \varphi_{k+1}) + \Upsilon_{p/q}^1,$$

with

$$\|\Upsilon_{p/q}^1\|_{C^0} = O(q^7 \|\mu_\varepsilon\|_{C^1}^2), \quad \|\Upsilon_{p/q}^1\|_{C^{m-1}} \leqslant qC_1(M),$$

using the fact that

$$\varphi_k = \varphi_{\lambda_{p/q}}\left(\theta + \frac{pk}{q} 2\pi\right), \quad k = 0, \dots, q-1,$$

the assertion of the lemma follows from (3.5) and (3.6). □

Let us consider the Fourier series of $\mu_\varepsilon(\varphi)$,

$$\mu_\varepsilon(\varphi) = \mu_0' + \sum_{k=1}^{+\infty} a_k \cos k\varphi + b_k \sin k\varphi.$$

Now substitute into $\mu_\varepsilon(\varphi)$ the action-angle parametrisation $\varphi = \varphi_\lambda(\theta)$, and expand it with respect to the eccentricity e. By Lemma C.1 we obtain, for any positive integer $N \in \mathbb{N}$, $N \leqslant m - 1$,

$$\mu_\varepsilon(\varphi(\theta,\lambda,e)) = \mu_\varepsilon(\theta) + \sum_{n=1}^{N} P_n(\theta)\frac{a^n e^{2n}}{(a^2-\lambda^2)^n} + O(\|\mu_\varepsilon\|_{C^{N+1}} e^{2N+2}), \tag{3.7}$$

where

$$P_n(\theta) = \sum_{k=1}^{+\infty}\sum_{l=-n}^{n} \xi_{n,l}(k)\big(a_k\cos(k+2l)\theta + b_k \sin(k+2l)\theta\big),$$

and $\xi_{n,l}(k)$ are polynomials of k (see Appendix C). Let us recall the following elementary identities (we leave the proof to the reader).

Lemma 3.5. *Let $0 < p/q \in \mathbb{Q}$ in lowest terms. If $n \in \mathbb{N}\setminus q\mathbb{N}$, then*

$$\sum_{m=1}^{q}\cos(n(\theta+\frac{pm}{q}2\pi)) \equiv 0 \qquad \textit{and} \qquad \sum_{m=1}^{q}\sin(n(\theta+\frac{pm}{q}2\pi)) \equiv 0.$$

If $n \in q\mathbb{N}$, then

$$\sum_{m=1}^{q}\cos(n(\theta+\frac{pm}{q}2\pi)) \equiv q\cos n\theta \quad , \quad \sum_{m=1}^{q}\sin(n(\theta+\frac{pm}{q}2\pi)) \equiv q\sin n\theta.$$

If we apply the above equalities to (3.3) and (3.7), we obtain

$$\begin{aligned}&\sum_{j=1}^{+\infty} a_j\cos jq\theta + b_j \sin jq\theta \\ &+ \sum_{n=1}^{N}\sum_{l=-n}^{n}\xi_{n,l}(jq-2l)(a_{jq-2l}\cos jq\theta + b_{jq-2l}\sin jq\theta)\frac{a^n e^{2n}}{(a^2-\lambda_{p/q}^2)^n} \\ &= O(\|\mu_\varepsilon\|_{C^{N+1}}e^{2N+2} + \lambda_{p/q}^{-1}q^7\|\mu_\varepsilon\|_{C^1}^2) + \frac{c_{p/q}}{q} - \mu_0'.\end{aligned}$$

Multiplying both sides by $\cos q\theta$ and integrating with respect to θ from 0 to 2π, we get the following proposition.

Proposition 3.6. *Let $0 < p/q \in \mathbb{Q}\cap(0,1)$ and Ω admit an integrable rational caustic of rotation number p/q. Let $N \in \mathbb{N}$ such that $q > 2N$. Then*

$$\begin{aligned}a_q + \sum_{n=1}^{N}\sum_{|l|\le n}\xi_{n,l}(q-2l)\,a_{q-2l}\,\frac{a^n e^{2n}}{(a^2-\lambda_{p/q}^2)^n} \\ = O(e^{2N+2}\|\mu_\varepsilon\|_{C^{N+1}} + \lambda_{p/q}^{-1}q^7\|\mu_\varepsilon\|_{C^1}^2).\end{aligned} \tag{3.8}$$

Similarly, if we multiply both sides by $\sin q\theta$ and integrate with respect to θ from 0 to 2π, we obtain the analogous equality for b_q.

4. The case $q_0 = 3$

In this section we consider a 3-rationally integrable domain Ω, whose boundary is C^2–close to an ellipse $\mathcal{E}_{e,c}$, *i.e.*, for a C^2–small function $\mu(\varphi)$ we have

$$\partial\Omega = \mathcal{E}_{e,c} + \mu(\varphi).$$

Let

$$\mu(\varphi) = \mu_0' + \sum_{k=1}^{+\infty} a_k \cos k\varphi + b_k \sin k\varphi,$$

and assume

$$\|\mu\|_{C^1} \leqslant e^6. \tag{4.1}$$

We will show that the higher order relations on the existence of smooth convex integrable caustics of rotation numbers $\frac{1}{2k+1}$, $k \geqslant 1$, and $\frac{2}{7}$ imply that

$$a_3 = O(e^2\|\mu\|_{C^3}) \quad , \quad b_3 = O(e^2\|\mu\|_{C^3}). \tag{4.2}$$

Remark 4.1. The proof in this case consists of one step and does not require any other iteration of the same arguments (compare also with Remarks 5.1 and 6.1).

For simplicity, we assume that the semi-major axis of $\mathcal{E}_{e,c}$ equals to 1, *i.e.*, $c = e$, and we denote it simply by $\mathcal{E}_e$.

Let us start by observing the following lemma, which is a special case of Lemma fourier41 (and of Lemma 7.1 for $k_0 = 2$).

Lemma 4.2. $a_5 = O(e^2\|\mu\|_{C^1})$, $a_7,\ a_9,\ a_{11} = O(e^4\|\mu\|_{C^2})$.

Remark 4.3. Although we do not provide a direct proof of this lemma, let us point out that it exploits the existence of smooth convex integrable caustics of rotation numbers $\frac{1}{5}$, $\frac{1}{7}$, $\frac{1}{9}$, $\frac{1}{11}$, and $\frac{1}{13}$.

Let us now show how property (4.2) follows from this lemma.

- From the existence of an integrable rational caustic with rotation number $1/5$, using (3.8) and (4.1) with $N = 1$, we deduce that

  $$a_5 + \left[\xi_{1,-1}(7)\, a_7 + \xi_{1,1}(3)\, a_3\,\right] \frac{e^2}{1-\lambda_{1/5}^2} + O(e^4\|\mu\|_{C^2}) = 0,$$

  where $\xi_{1,\pm 1}(k) = \pm\frac{k}{16}$ (see Appendix C). Hence, it follows from Lemmata 3.2 and 4.2 that

  $$a_5 + \frac{3a_3}{16}\frac{e^2(1+O(e^2))}{\cos^2\frac{\pi}{5}} = O(e^4\|\mu\|_{C^2}). \tag{4.3}$$

which implies

$$a_5 + \frac{3a_3}{16}\frac{e^2}{\cos^2\frac{\pi}{5}} = O(e^4\|\mu\|_{C^2}). \tag{4.4}$$

- From the existence of an integrable rational caustic with rotation number 1/7, using (3.8) and (4.1) with $N = 2$, we obtain that

$$a_7 + \sum_{n=1}^{2}\sum_{l=-n}^{n} \xi_{n,l}(7-2l)\, a_{7-2l}\, \frac{e^{2n}}{(1-\lambda_{1/7}^2)^n} + O(e^6\|\mu\|_{C^3}) = 0,$$

  where $\xi_{2,2}(k) = \frac{k^2+k}{512}$ (see Appendix C). It follows from Lemma 3.2 that

$$\frac{1}{(1-\lambda_\omega^2)^n} = \frac{1+O(e^2)}{\cos^{2n}(\pi\omega)}. \tag{4.5}$$

  Hence, we obtain:

$$a_7 + \sum_{n=1}^{2}\sum_{l=-n}^{n} \xi_{n,l}(7-2l)\, a_{7-2l}\, \frac{(1+O(e^2))}{\cos^{2n}(\frac{\pi}{7})}e^{2n} = O(e^6\|\mu\|_{C^3}),$$

  which implies, using the estimates in Lemma 4.2, that

$$a_7 + \frac{5a_5}{16}\frac{e^2}{\cos^2(\frac{\pi}{7})} + \frac{12a_3}{512}\frac{e^4}{\cos^4(\frac{\pi}{7})} = O(e^6\|\mu\|_{C^3}). \tag{4.6}$$

  In fact, for $n = 2$ the terms $e^{2n}O(e^2) = O(e^6)$. For $n = 1$, the same is true, observing that $a_5 = O(e^2\|\mu\|_{C^1})$, $a_7 = O(e^4\|\mu\|_{C^2})$ and $a_9 = O(e^4\|\mu\|_{C^2})$, as it follows from Lemma 4.2 (see also Sections 7.1 and 7.2 for more precise computations)
- Similarly, from the existence of an integrable rational caustic with rotation number 2/7, we get

$$a_7 + \frac{5a_5}{16}\frac{e^2(1+O(e^2))}{\cos^2\frac{2\pi}{7}} + \frac{12a_3}{512}\frac{e^4}{\cos^4\frac{2\pi}{7}} = O(e^6\|\mu\|_{C^3}). \tag{4.7}$$

- Combining (4.4), (4.6) and (4.7), we obtain the linear system

$$\begin{pmatrix} \frac{3e^2}{16\cos^2\frac{\pi}{5}} & 1 & 0 \\ \frac{12e^4}{512\cos^4\frac{\pi}{7}} & \frac{5e^2}{16\cos^2\frac{\pi}{7}} & 1 \\ \frac{12e^4}{512\cos^4\frac{2\pi}{7}} & \frac{5e^2}{16\cos^2\frac{2\pi}{7}} & 1 \end{pmatrix} \begin{pmatrix} a_3 \\ a_5 \\ a_7 \end{pmatrix} = \begin{pmatrix} O(e^4\|\mu\|_{C^2}) \\ O(e^6\|\mu\|_{C^3}) \\ O(e^6\|\mu\|_{C^3}) \end{pmatrix}. \tag{4.8}$$

  Observe that coefficient matrix is invertible[7]; moreover, using Theorem D.1 in Appendix D we can compute the first row of

[7]By means of Mathematica, one can compute that its determinant is $-4.182 \times 10^{-4}e^4$.

its inverse, which has the form

$$\left(\begin{array}{ccc} O(e^{-2}) & O(e^{-4}) & O(e^{-4}) \end{array} \right). \tag{4.9}$$

This allows us to conclude that $a_3 = O(e^2\|\mu\|_{C^3})$.

Similarly, one can prove that $b_3 = O(e^2\|\mu\|_{C^3})$.

5. The case $q_0 = 4$

In this section we consider a 4-rationally integrable domain Ω, whose boundary is C^6–close to an ellipse $\mathcal{E}_e$ (also here we assume $c = e$), *i.e.*

$$\partial\Omega = \mathcal{E}_e + \mu(\varphi),$$

where μ is a C^6–small function. Let

$$\mu(\varphi) = \mu_0' + \sum_{k=1}^{+\infty} a_k \cos k\varphi + b_k \sin k\varphi,$$

and assume

$$\|\mu\|_{C^1} \leqslant e^{12}. \tag{5.1}$$

We will show that the higher order conditions on the existence of integrable rational caustics of rotation numbers $\frac{1}{2k+1}$, $\frac{1}{2k+2}$, $k \geqslant 2$, along with $\frac{2}{9}$, and $\frac{3}{14}$ imply that

$$a_k = O(e^2\|\mu\|_{C^6}) \quad , \quad b_k = O(e^2\|\mu\|_{C^6}), \quad k = 3, 4, \tag{5.2}$$

where group (i) of caustics leads to the result for $k = 3$, while group (ii) leads to the result for $k = 4$.

Remark 5.1. The proof in this case consists of two steps, related to the odd and even cases, and does not require any iteration (compare also with Remarks 4.1 and 6.1). We give a detailed account of the odd case below.

Let us start by stating the following lemma, which is also a special case of Lemma 7.1 with $k_0 = 2$.

Lemma 5.2.

$$a_5 = O(e^2\|\mu\|_{C^3}),\ a_7 = O(e^4\|\mu\|_{C^3}),\ a_9,\ a_{11},\ a_{13} = O(e^6\|\mu\|_{C^3}).$$

Remark 5.3. Although an independent proof of this lemma is not given, let us observe that it exploits the existence of smooth convex integrable caustics of rotation numbers $\frac{1}{5}$, $\frac{1}{7}$, $\frac{1}{9}$, $\frac{1}{11}$, $\frac{1}{13}$, $\frac{1}{15}$, and $\frac{1}{17}$.

Let us now describe how to prove (5.2).

Let us first show that $a_3 = O(e^2\|\mu\|_{C^4})$.

- From the existence of an integrable rational caustic with rotation number 1/5, using equality (3.8) (with $N = 1$) together with Lemmata 3.2 and 5.2, we deduce that (see also the analogous discussion for (4.4))

$$a_5 + a_3\,\xi_{1,1}(3)\,\frac{e^2}{\cos^2(\frac{\pi}{5})}(1+O(e^2)) = O(e^4\|\mu\|_{C^2}).$$

  Since $|a_3| \leqslant \|\mu\|_{C^2}$, the term $a_3\xi_{1,1}(3)e^2O(e^2)$ could be put into the error term $O(e^4\|\mu\|_{C^2})$ on the right hand side.
- Similarly to what done in (4.6), from the existence of integrable rational caustic with rotation numbers 1/7, we get

$$a_7 + a_5\,\frac{\xi_{1,1}(5)e^2}{\cos^2(\frac{\pi}{7})} + a_3\,\frac{\xi_{2,2}(3)e^4}{\cos^4(\frac{\pi}{7})} = O(e^6\|\mu\|_{C^3}).$$

- From the existence of integrable rational caustic with rotation number 1/9, using (3.8) (with $N = 3$), (4.5) and (5.1), we obtain that

$$a_9 + \sum_{n=1}^{3}\sum_{l=-n}^{n}\xi_{n,l}(9-2l)\,a_{9-2l}\,\frac{(1+O(e^2))}{\cos^{2n}(\frac{\pi}{9})}e^{2n} + O(e^8\|\mu\|_{C^4}) = 0,$$

  which implies, using the estimates in Lemma 5.2, that

$$a_9 + a_7\,\frac{\xi_{1,1}(7)e^2}{\cos^2(\frac{\pi}{9})} + a_5\,\frac{\xi_{2,2}(5)e^4}{\cos^4(\frac{\pi}{9})} + a_3\,\frac{\xi_{3,3}(3)e^6}{\cos^6(\frac{\pi}{9})} = O(e^8\|\mu\|_{C^4}).$$

  In fact, for $n = 3$ the terms $e^{2n}O(e^2) = O(e^8)$. For $n = 1, 2$, the same is true, observing that $a_5 = O(e^2\|\mu\|_{C^3})$, $a_7 = O(e^4\|\mu\|_{C^3})$, $a_9 = a_{11} = a_{13} = O(e^6\|\mu\|_{C^3})$, as it follows from Lemma 5.2 (see also Sections 7.1 and 7.2 for more precise computations).

- Similarly, from the existence of an integrable rational caustic with rotation number 2/9, we get

$$a_9 + a_7\,\frac{\xi_{1,1}(7)e^2}{\cos^2(\frac{2\pi}{9})} + a_5\,\frac{\xi_{2,2}(5)e^4}{\cos^4(\frac{2\pi}{9})} + a_3\,\frac{\xi_{3,3}(3)e^6}{\cos^6(\frac{2\pi}{9})} = O(e^8\|\mu\|_{C^4}).$$

- Therefore, we obtain the following system of linear equations in the variables $a_3, \dots, a_9$:

$$\begin{pmatrix} \frac{\xi_{1,1}(3)e^2}{\cos^2(\frac{\pi}{5})} & 1 & 0 & 0 \\ \frac{\xi_{2,2}(3)e^4}{\cos^4(\frac{\pi}{7})} & \frac{\xi_{1,1}(5)e^2}{\cos^2(\frac{\pi}{7})} & 1 & 0 \\ \frac{\xi_{3,3}(3)e^6}{\cos^6(\frac{\pi}{9})} & \frac{\xi_{2,2}(5)e^4}{\cos^4(\frac{\pi}{9})} & \frac{\xi_{1,1}(7)e^2}{\cos^2(\frac{\pi}{9})} & 1 \\ \frac{\xi_{3,3}(3)e^6}{\cos^6(\frac{2\pi}{9})} & \frac{\xi_{2,2}(5)e^4}{\cos^4(\frac{2\pi}{9})} & \frac{\xi_{1,1}(7)e^2}{\cos^2(\frac{2\pi}{9})} & 1 \end{pmatrix} \begin{pmatrix} a_3 \\ a_5 \\ a_7 \\ a_9 \end{pmatrix} = \begin{pmatrix} O(e^4\|\mu\|_{C^2}) \\ O(e^6\|\mu\|_{C^3}) \\ O(e^8\|\mu\|_{C^4}) \\ O(e^8\|\mu\|_{C^4}) \end{pmatrix}. \quad (5.3)$$

Observe that the coefficient matrix of this linear system is invertible[8]; moreover, using Theorem D.1 in Appendix D we can compute the first row of its inverse, which has the form

$$\begin{pmatrix} O(e^{-2}) & O(e^{-4}) & O(e^{-6}) & O(e^{-6}) \end{pmatrix}. \quad (5.4)$$

All of this is enough to conclude that

$$a_3 = O(e^2\|\mu\|_{C^4}).$$

Let us now show that $a_4 = (e^2\|\mu\|_{C^6})$. In the same way as before, from the existence of integrable rational caustics with rotation numbers 1/6, 1/8, 1/10, 1/12, 1/14 and 3/14, we obtain a linear system of

[8]By means of Mathematica, one can compute that its determinant is $-4.02 \times 10^{-6}e^6$

equations in the variables a_4, $a_6, \dots, a_{14}$.

$$\begin{pmatrix} \frac{\xi_{1,1}(4)e^2}{\cos^2(\frac{\pi}{6})} & 1 & 0 & 0 & 0 & 0 \\ \frac{\xi_{2,2}(4)e^4}{\cos^4(\frac{\pi}{8})} & \frac{\xi_{1,1}(6)e^2}{\cos^2(\frac{\pi}{8})} & 1 & 0 & 0 & 0 \\ \frac{\xi_{3,3}(4)e^6}{\cos^6(\frac{\pi}{10})} & \frac{\xi_{2,2}(6)e^4}{\cos^4(\frac{\pi}{10})} & \frac{\xi_{1,1}(8)e^2}{\cos^2(\frac{\pi}{10})} & 1 & 0 & 0 \\ \frac{\xi_{4,4}(4)e^8}{\cos^8(\frac{\pi}{12})} & \frac{\xi_{3,3}(6)e^6}{\cos^6(\frac{\pi}{12})} & \frac{\xi_{2,2}(8)e^4}{\cos^4(\frac{\pi}{12})} & \frac{\xi_{1,1}(10)e^2}{\cos^2(\frac{\pi}{12})} & 1 & 0 \\ \frac{\xi_{5,5}(4)e^{10}}{\cos^{10}(\frac{\pi}{14})} & \frac{\xi_{4,4}(6)e^8}{\cos^8(\frac{\pi}{14})} & \frac{\xi_{3,3}(8)e^6}{\cos^6(\frac{\pi}{14})} & \frac{\xi_{2,2}(10)e^4}{\cos^4(\frac{\pi}{14})} & \frac{\xi_{1,1}(12)e^2}{\cos^2(\frac{\pi}{14})} & 1 \\ \frac{\xi_{5,5}(4)e^{10}}{\cos^{10}(\frac{3\pi}{14})} & \frac{\xi_{4,4}(6)e^8}{\cos^8(\frac{3\pi}{14})} & \frac{\xi_{3,3}(8)e^6}{\cos^6(\frac{3\pi}{14})} & \frac{\xi_{2,2}(10)e^4}{\cos^4(\frac{3\pi}{14})} & \frac{\xi_{1,1}(12)e^2}{\cos^2(\frac{3\pi}{14})} & 1 \end{pmatrix} \tag{5.5}$$

$$\times \begin{pmatrix} a_4 \\ a_6 \\ a_8 \\ a_{10} \\ a_{12} \\ a_{14} \end{pmatrix} = \begin{pmatrix} O(e^4\|\mu\|_{C^2}) \\ O(e^6\|\mu\|_{C^3}) \\ O(e^8\|\mu\|_{C^4}) \\ O(e^{10}\|\mu\|_{C^5}) \\ O(e^{12}\|\mu\|_{C^6}) \\ O(e^{12}\|\mu\|_{C^6}) \end{pmatrix}.$$

Also in this case the coefficient matrix is non-degenerate[9]; moreover, using Theorem D.1 in Appendix D we can compute the first row of its inverse, which has the form

$$\begin{pmatrix} O(e^{-2}) & O(e^{-4}) & O(e^{-6}) & O(e^{-8}) & O(e^{-10}) & O(e^{-10}) \end{pmatrix}. \tag{5.6}$$

Hence, we can conclude that

$$a_4 = O(e^2\|\mu\|_{C^6}).$$

Repeating the same arguments as before, one can show the analogue equalities for b_k's, namely

$$b_3 = O(e^2\|\mu\|_{C^4}), \quad b_4 = O(e^2\|\mu\|_{C^6}).$$

[9]By means of Mathematica, one can compute that its determinant is $7.1437 \times 10^{-5}e^{10}$

6. The case $q_0 = 5$

In this section we consider a 5-rationally integrable domain Ω, whose boundary is C^7–close to an ellipse $\mathcal{E}_e$ (we continue to assume that $c = e$), *i.e.*, for a C^7–small function $\mu(\varphi)$ we have

$$\partial\Omega = \mathcal{E}_e + \mu(\varphi).$$

Let

$$\mu(\varphi) = \mu_0' + \sum_{k=1}^{+\infty} a_k \cos k\varphi + b_k \sin k\varphi,$$

and assume

$$\|\mu\|_{C^1} \leqslant e^{14}. \tag{6.1}$$

We shall show that the higher order conditions on the existence of integrable rational caustics of rotation numbers $\frac{1}{2k+1}$, $\frac{1}{2k+2}$, $k \geqslant 2$, along with $\frac{2}{11}$, $\frac{2}{13}$ and $\frac{3}{16}$ implies that

$$a_k, b_k = O(e^2 \|\mu\|_{C^7}), \quad k = 3, 4, 5.$$

Remark 6.1. The proof in this case consists of three steps: we start analyzing the odd and even cases and in the odd case we do need to iterate the argument once (an inductive step). In the general case $q_0 > 5$, we shall need the number of inductive steps to be $[q_0/2] - 1$; see beginning of Section 7.

Let us start by stating the following lemma, which is also a special case of Lemma 7.1 for $k_0 = 3$.

Lemma 6.2.

$$a_7 = O(e^2\|\mu\|_{C^1}), \quad a_9 = O(e^4\|\mu\|_{C^2}), \quad a_{11}, a_{13}, a_{15} = O(e^6\|\mu\|_{C^3}).$$

Remark 6.3. Although an independent proof of this lemma is not given, let us observe that it exploits the existence of smooth convex integrable caustics of rotation numbers $\frac{1}{5}$, $\frac{1}{7}$, $\frac{1}{9}$, $\frac{1}{11}$, $\frac{1}{13}$, $\frac{1}{15}$, $\frac{1}{17}$, and $\frac{1}{19}$.

Proof. See the proof of Lemma 7.1. □

Let us now show how property (6.3) follows from this lemma.

- From the existence of an integrable rational caustic with rotation number 1/7, using equality (3.8) with $N = 1$, we have

$$a_7 + (\xi_{1,1}(5)a_5 + \xi_{1,-1}(9)a_9)\frac{e^2}{1-\lambda_{1/7}^2} + O(e^4\|\mu\|_{C^2}) = 0.$$

By Lemmata 3.2 and 6.2, it follows that

$$a_7 + \xi_{1,1}(5) a_5 \frac{e^2}{\cos^2(\pi/7)} = O(e^4 \|\mu\|_{C^2}).$$

- From the existence of an integrable rational caustic with rotation number 1/9, using (3.8) (with $N = 2$) and (6.1), we obtain that

$$a_9 + \sum_{n=1}^{2} \sum_{l=-n}^{n} \xi_{n,l}(9-2l)\, a_{9-2l}\, \frac{e^{2n}}{(1-\lambda_{1/9}^2)^n} + O(e^6 \|\mu\|_{C^3}) = 0.$$

  Using (4.5), we obtain

$$a_9 + \sum_{n=1}^{2} \sum_{l=-n}^{n} \xi_{n,l}(9-2l)\, a_{9-2l}\, \frac{(1+O(e^2))}{\cos^{2n}(\frac{\pi}{9})} e^{2n} = O(e^6 \|\mu\|_{C^3}),$$

  which implies, using the estimates in Lemma 6.2, that

$$a_9 + a_7 \frac{\xi_{1,1}(7) e^2}{\cos^2(\pi/9)} + a_5 \frac{\xi_{2,2}(5) e^4}{\cos^4(\pi/9)} = O(e^6 \|\mu\|_{C^3}).$$

  In fact, clearly for $n = 2$ the terms $e^{2n} O(e^2) = O(e^6)$. For $n = 1$, the same is true, observing that $a_7 = O(e^2 \|\mu\|_{C^1})$, $a_9 = O(e^4 \|\mu\|_{C^2})$, and $a_{11}, a_{13} = O(e^6 \|\mu\|_{C^3})$, as it follows from Lemma 6.2 (see also Sections 7.1 and 7.2 for more precise computations).

- From the existence of integrable rational caustic with rotation number 1/11, using (3.8) (with $N = 3$), (6.1) and (4.5), we obtain that

$$a_{11} + \sum_{n=1}^{3} \sum_{l=-n}^{n} \xi_{n,l}(11-2l)\, a_{11-2l}\, \frac{(1+O(e^2))}{\cos^{2n}(\frac{\pi}{11})} e^{2n} + O(e^8 \|\mu\|_{C^4}) = 0,$$

  which implies, again using the estimates in Lemma 6.2, that

$$a_{11} + a_9 \frac{\xi_{1,1}(9) e^2}{\cos^2(\pi/11)} + a_7 \frac{\xi_{2,2}(7) e^4}{\cos^4(\pi/11)} + a_5 \frac{\xi_{3,3}(5) e^6}{\cos^6(\pi/11)} = O(e^8 \|\mu\|_{C^4});$$

  see also Sections 7.1 and 7.2 for more precise computations.
- Similarly, from the existence of an integrable rational caustic with rotation number 2/11, we obtain:

$$a_{11} + a_9 \frac{\xi_{1,1}(9) e^2}{\cos^2(2\pi/11)} + a_7 \frac{\xi_{2,2}(7) e^4}{\cos^4(2\pi/11)} + a_5 \frac{\xi_{3,3}(5) e^6}{\cos^6(2\pi/11)} = O(e^8 \|\mu\|_{C^4}).$$

- Putting all of this information together, we obtain a system of linear equations with unknowns a_5, a_7, a_9, a_{11} and the following coefficient matrix:

$$\begin{pmatrix} \frac{\xi_{1,1}(5)e^2}{\cos^2(\pi/7)} & 1 & 0 & 0 \\ \frac{\xi_{2,2}(5)e^4}{\cos^4(\pi/9)} & \frac{\xi_{1,1}(7)e^2}{\cos^2(\pi/9)} & 1 & 0 \\ \frac{\xi_{3,3}(5)e^6}{\cos^6(\pi/11)} & \frac{\xi_{2,2}(7)e^4}{\cos^4(\pi/11)} & \frac{\xi_{1,1}(9)e^2}{\cos^2(\pi/11)} & 1 \\ \frac{\xi_{3,3}(5)e^6}{\cos^6(2\pi/11)} & \frac{\xi_{2,2}(7)e^4}{\cos^4(2\pi/11)} & \frac{\xi_{1,1}(9)e^2}{\cos^2(2\pi/11)} & 1 \end{pmatrix}$$

This matrix is invertible[10] and, using Theorem D.1 in Appendix D, we can compute the first row of its inverse, which has the form

$$\big(O(e^{-2}),\ O(e^{-4}),\ O(e^{-6}),\ O(e^{-6})\big). \tag{6.2}$$

Hence, we conclude that

$$a_5 = O(e^2\|\mu\|_{C^4}), \tag{6.3}$$

Now, using this new estimate, we obtain the following improvement of Lemma 6.2. This is the second inductive step aforementioned in Remark 6.1. This Lemma follows from (7.7) for $k_0 = 3$ and $m = 2$.

Lemma 6.4.

$$a_7 = O(e^4\|\mu\|_{C^4}),\ a_9 = O(e^6\|\mu\|_{C^4}),\ a_{11} = O(e^8\|\mu\|_{C^4}),\ a_{13} = O(e^{10}\|\mu\|_{C^5})$$

and

$$a_{15}, a_{17}, a_{19}, a_{21} = O(e^{10}\|\mu\|_{C^5}).$$

Proceeding as before, using Lemmata 3.2 and 6.4 and the equality (3.8), from the higher order relations on the existence of integrable rational caustics of rotation numbers 1/7, 1/9, 1/11, 2/11, 1/13, and 2/13, we obtain the following linear system (see Sections 7.1 and 7.2 for more precise computations):

[10]By means of Mathematica, one can compute that its determinant is $1.4 \times 10^{-5}e^6$.

$$\begin{pmatrix} \frac{\xi_{2,2}(3)e^4}{\cos^4(\pi/7)} & \frac{\xi_{1,1}(5)e^2}{\cos^2(\pi/7)} & 1 & 0 & 0 & 0 \\ \frac{\xi_{3,3}(3)e^6}{\cos^6(\pi/9)} & \frac{\xi_{2,2}(5)e^4}{\cos^4(\pi/9)} & \frac{\xi_{1,1}(7)e^2}{\cos^2(\pi/9)} & 1 & 0 & 0 \\ \frac{\xi_{4,4}(3)e^8}{\cos^8(\pi/11)} & \frac{\xi_{3,3}(5)e^6}{\cos^6(\pi/11)} & \frac{\xi_{2,2}(7)e^4}{\cos^4(\pi/11)} & \frac{\xi_{1,1}(9)e^2}{\cos^2(\pi/11)} & 1 & 0 \\ \frac{\xi_{4,4}(3)e^8}{\cos^8(2\pi/11)} & \frac{\xi_{3,3}(5)e^6}{\cos^6(2\pi/11)} & \frac{\xi_{2,2}(7)e^4}{\cos^4(2\pi/11)} & \frac{\xi_{1,1}(9)e^2}{\cos^2(2\pi/11)} & 1 & 0 \\ \frac{\xi_{5,5}(3)e^{10}}{\cos^{10}(\pi/13)} & \frac{\xi_{4,4}(5)e^8}{\cos^8(\pi/13)} & \frac{\xi_{3,3}(7)e^6}{\cos^6(\pi/13)} & \frac{\xi_{2,2}(9)e^4}{\cos^4(\pi/13)} & \frac{\xi_{1,1}(11)e^2}{\cos^2(\pi/13)} & 1 \\ \frac{\xi_{5,5}(3)e^{10}}{\cos^{10}(2\pi/13)} & \frac{\xi_{4,4}(5)e^8}{\cos^8(2\pi/13)} & \frac{\xi_{3,3}(7)e^6}{\cos^6(2\pi/13)} & \frac{\xi_{2,2}(9)e^4}{\cos^4(2\pi/13)} & \frac{\xi_{1,1}(11)e^2}{\cos^2(2\pi/13)} & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} a_3 \\ a_5 \\ a_7 \\ a_9 \\ a_{11} \\ a_{13} \end{pmatrix} = \begin{pmatrix} O(e^6\|\mu\|_{C^4}) \\ O(e^8\|\mu\|_{C^4}) \\ O(e^{10}\|\mu\|_{C^5}) \\ O(e^{10}\|\mu\|_{C^5}) \\ O(e^{12}\|\mu\|_{C^6}) \\ O(e^{12}\|\mu\|_{C^6}) \end{pmatrix}. \tag{6.4}$$

This matrix of coefficients is invertible[11] and, using Theorem D.1 in Appendix D, we can compute the first two rows of its inverse, which have the form

$$\begin{pmatrix} O(e^{-4}) & O(e^{-6}) & O(e^{-8}) & O(e^{-8}) & O(e^{-10}) & O(e^{-10}) \\ O(e^{-2}) & O(e^{-4}) & O(e^{-6}) & O(e^{-6}) & O(e^{-8}) & O(e^{-8}) \end{pmatrix}. \tag{6.5}$$

Therefore, we conclude that

$$a_3 = O(e^2\|\mu\|_{C^6}), \quad a_5 = O(e^4\|\mu\|_{C^6}).$$

Then, we want to show that

$$a_4 = O(e^2\|\mu\|_{C^7}).$$

For this, we exploit relations (3.8), obtained from the higher order conditions on the existence caustics with rotation numbers $1/6,\ 1/8,\ 1/10,\ 1/12,\ 1/14,\ 1/16$ and $3/16$.

[11] By means of Mathematica, one can compute that its determinant is $6.86498 \times 10^{-15}e^{16}$.

In the same spirit as before (see Sections 7.1 and 7.2 for more precise computations), we get the following linear system:

$$\begin{pmatrix} \frac{\xi_{1,1}(4)e^2}{\cos^2(\pi/6)} & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{\xi_{2,2}(4)e^4}{\cos^4(\pi/8)} & \frac{\xi_{1,1}(6)e^2}{\cos^2(\pi/8)} & 1 & 0 & 0 & 0 & 0 \\ \frac{\xi_{3,3}(4)e^6}{\cos^6(\pi/10)} & \frac{\xi_{2,2}(6)e^4}{\cos^4(\pi/10)} & \frac{\xi_{1,1}(8)e^2}{\cos^2(\pi/10)} & 1 & 0 & 0 & 0 \\ \frac{\xi_{4,4}(4)e^8}{\cos^8(\pi/12)} & \frac{\xi_{3,3}(6)e^6}{\cos^6(\pi/12)} & \frac{\xi_{2,2}(8)e^4}{\cos^4(\pi/10)} & \frac{\xi_{1,1}(10)e^2}{\cos^2(\pi/12)} & 1 & 0 & 0 \\ \frac{\xi_{5,5}(4)e^{10}}{\cos^{10}(\pi/14)} & \frac{\xi_{4,4}(6)e^8}{\cos^8(\pi/14)} & \frac{\xi_{3,3}(8)e^6}{\cos^6(\pi/14)} & \frac{\xi_{2,2}(10)e^4}{\cos^4(\pi/14)} & \frac{\xi_{1,1}(12)e^2}{\cos^2(\pi/14)} & 1 & 0 \\ \frac{\xi_{6,6}(4)e^{12}}{\cos^{12}(\pi/16)} & \frac{\xi_{5,5}(6)e^{10}}{\cos^{10}(\pi/16)} & \frac{\xi_{4,4}(8)e^8}{\cos^8(\pi/16)} & \frac{\xi_{3,3}(10)e^6}{\cos^6(\pi/16)} & \frac{\xi_{2,2}(12)e^4}{\cos^4(\pi/16)} & \frac{\xi_{1,1}(14)e^2}{\cos^2(\pi/16)} & 1 \\ \frac{\xi_{6,6}(4)e^{12}}{\cos^{12}(3\pi/16)} & \frac{\xi_{5,5}(6)e^{10}}{\cos^{10}(3\pi/16)} & \frac{\xi_{4,4}(8)e^8}{\cos^8(3\pi/16)} & \frac{\xi_{3,3}(10)e^6}{\cos^6(3\pi/16)} & \frac{\xi_{2,2}(12)e^4}{\cos^4(3\pi/16)} & \frac{\xi_{1,1}(14)e^2}{\cos^2(3\pi/16)} & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} a_4 \\ a_6 \\ a_8 \\ a_{10} \\ a_{12} \\ a_{14} \\ a_{16} \end{pmatrix} = \begin{pmatrix} O(e^4\|\mu\|_{C^2}) \\ O(e^6\|\mu\|_{C^3}) \\ O(e^8\|\mu\|_{C^4}) \\ O(e^{10}\|\mu\|_{C^5}) \\ O(e^{12}\|\mu\|_{C^6}) \\ O(e^{14}\|\mu\|_{C^7}) \\ O(e^{14}\|\mu\|_{C^7}) \end{pmatrix}.$$

This matrix is invertible[12] and, using Theorem D.1 in Appendix D, we can compute the first row of its inverse, which has the form

$$\begin{pmatrix} O(e^{-2}) & O(e^{-4}) & O(e^{-6}) & O(e^{-8}) & O(e^{-10}) & O(e^{-12}) & O(e^{-12}) \end{pmatrix}. \tag{6.6}$$

Hence, we obtain

$$a_4 = O(e^2\|\mu\|_{C^7}).$$

In the same way as before (see again Sections 7.1 and 7.2 for more precise computations), one proves that

$$b_k = O(e^2\|\mu\|_{C^7}), \quad k = 3, 4, 5.$$

7. The general cases

In the previous sections, we have described how to recover the missing relations in the cases $q_0 = 3, 4, 5$. Clearly, the same set of ideas can be implemented for any $q_0 \geqslant 6$. In this section we aim to outline the procedure for proving these results in the general case.

[12] By means of Mathematica, one can compute that its determinant is $-2.5 \times 10^{-6} e^{12}$.

Let $q_0 \geqslant 6$ and let us consider a q_0-rationally integrable domain Ω, whose boundary is close to an ellipse $\mathcal{E}_e$ (we use the normalization $c = e$)

$$\partial\Omega = \mathcal{E}_e + \mu(\varphi).$$

Let

$$\mu(\varphi) = \mu_0' + \sum_{k=1}^{+\infty} a_k \cos k\varphi + b_k \sin k\varphi,$$

and assume

$$\|\mu\|_{C^1} \leqslant e^{6q_0}.$$

Without loss of generality, we assume that q_0 is an even integer, *i.e.*,

$$q_0 = 2k_0, \quad \text{with} \quad k_0 \geqslant 3.$$

The arguments below is a finite induction on m; the basis step is $m = k_0$ and the final step is $m = 2$.

Now, we outline this inductive procedure to show that

$$a_k, b_k = O(e^2\|\mu\|_{C^{5k_0}}), \quad k = 3, \dots, q_0 = 2k_0.$$

The proof of this claim will be detailed in the following subsections (see Proposition 7.10 for a more precise statement).

Let us start with the following Lemma.

Lemma 7.1.

$$a_{2k+1} = \begin{cases} O(e^{2(k-k_0)+2}\|\mu\|_{C^{k_0+1}}) & \textit{if } k = k_0, \dots, 2k_0 - 1, \\ O(e^{2k_0+2}\|\mu\|_{C^{k_0+1}}) & \textit{if } \ k = 2k_0, \dots 3k_0, \\ O(e^{2(4k_0-k)+2}\|\mu\|_{C^{k_0+1}}) & \textit{if } k = 3k_0 + 1, \dots, 4k_0. \end{cases}$$

and

$$a_{2k} = \begin{cases} O(e^{2(k-k_0)}\|\mu\|_{C^{2k_0+1}}) & \textit{if } k = k_0 + 1, \dots, 3k_0 + 1, \\ O(e^{4k_0+2}\|\mu\|_{C^{2k_0+1}}) & \textit{if } \ k = 3k_0 + 2, \dots, 6k_0 + 1, \\ O(e^{2(8k_0-k)+4}\|\mu\|_{C^{2k_0+1}}) & \textit{if } k = 6k_0 + 2, \dots, 8k_0 + 1. \end{cases}$$

Remark 7.2. Notice that only the first two items in each bracket are used for the proof.

Proof. Let us start by proving the estimates for the Fourier coefficients of odd order. The proof consists in an iterative application of equality (3.8).

From the existence of integrable rational caustics with rotation numbers $\frac{1}{2k+1}$, with $k = k_0, \ldots, 4k_0$ (observe that this choice ensures that $\frac{1}{2k+1} < \frac{1}{q_0}$), using equality (3.8) with $N = 0$, we easily get:

$$a_{2k+1} = O(e^2 \|\mu_\varepsilon\|_{C^1}), \quad k = k_0, \ldots, 4k_0. \tag{7.1}$$

Let us now consider (3.8) with $N = 1$, for rotation numbers $\frac{1}{2k+1}$, where $k = k_0 + 1, \ldots, 4k_0 - 1$:

$$\begin{aligned} a_{2k+1} &= -\sum_{l=-1}^{1} \xi_{1,l}(2(k-l)+1)\, a_{2(k-l)+1} \frac{e^2}{1-\lambda^2_{1/2k+1}} + O(e^4\|\mu_\varepsilon\|_{C^2}) \\ &= -\sum_{l=-1}^{1} \xi_{1,l}(2(k-l)+1)\, a_{2(k-l)+1} \frac{e^2}{\cos^2 \frac{\pi}{2k+1}} + O(e^4\|\mu_\varepsilon\|_{C^2}), \end{aligned}$$

where in the last equality we have used Lemma 3.2 which implies

$$1 - \lambda^2_{1/2k+1} = \cos^2 \frac{\pi}{2k+1} + O(e^2). \tag{7.2}$$

Observe now that, since $|l| \leqslant 1$, then $k_0 \leqslant k - l \leqslant 4k_0$, hence we can use estimates (7.1) and obtain:

$$a_{2k+1} = O(e^4\|\mu\|_{C^2}), \quad k = k_0 + 1, \ldots, 4k_0 - 1.$$

In order to prove the claim, we need to iterate the same argument until $N = k_0$.

Let us describe how the inductive procedure work. Suppose that we have already iterated the same argument for $N = 1, \ldots, N_0 < k_0$; then, we have obtained:

$$a_{2k+1} = O(e^{2N_0+2}\|\mu\|_{C^{N_0+1}}), \quad k = k_0 + N_0, \ldots, 4k_0 - N_0. \tag{7.3}$$

Observe that if $k_0 \leqslant k < k_0 + N_0$, then the index k has been involved until the iteration corresponding to $N = k - k_0$; hence

$$a_{2k+1} = O(e^{2(k-k_0)+2}\|\mu\|_{C^{(k-k_0)+1}}), \quad k_0 \leqslant k < k_0 + N_0. \tag{7.4}$$

Similary, if $4k_0 - N_0 < k \leqslant 4k_0$, then the index k has been involved until the iteration corresponding to $N = 4k_0 - k$; hence

$$a_{2k+1} = O(e^{2(4k_0-k)+2}\|\mu\|_{C^{(4k_0-k)+1}}), \quad 4k_0 - N_0 < k \leqslant 4k_0. \tag{7.5}$$

Apply now (3.8) with $N = N_0 + 1$ and rotation numbers $\frac{1}{2k+1}$, with $k = k_0 + N_0 + 1, \dots, 4k_0 - N_0 - 1$. Then:

$$\begin{aligned} a_{2k+1} &= -\sum_{n=1}^{N_0+1} \sum_{|l|\le n} \xi_{n,l}(2(k-l)+1)\, a_{2(k-l)+1} \frac{e^{2n}}{(1-\lambda_{1/2k+1}^2)^n} \\ &\quad + O(e^{2(N_0+1)+2}\|\mu_\varepsilon\|_{C^{N_0+2}}). \end{aligned} \tag{7.6}$$

We want to show that all terms in this sum can be included in the remainder. Let us distinguish several cases:

- $l = 0$ appears for all $n \geqslant 1$ and, using (7.3), we conclude:

$$a_{2k+1} \frac{e^{2n}}{(1-\lambda_{1/2k+1}^2)^n} = O(e^{2(N_0+1)+2}\|\mu\|_{C^{N_0+1}}).$$

- $0 < l \leqslant N_0 + 1$ appears for all $n \geqslant l$; using (7.3), (7.4) and (7.2), we can conclude:

$$\begin{aligned} & a_{2(k-l)+1} \frac{e^{2n}}{(1-\lambda_{1/2k+1}^2)^n} \\ &= O(e^{2(k-l-k_0)+2}\|\mu\|_{C^{N_0+1}}) \cdot O(e^{2l}) \cdot \big(1 + O(e^2)\big) \\ &= O(e^{2(k-k_0)+2}\|\mu\|_{C^{N_0+1}}) \cdot \big(1 + O(e^2)\big) \\ &= O(e^{2(N_0+1)+2}\|\mu\|_{C^{N_0+1}}) \cdot \big(1 + O(e^2)\big) \\ &= O(e^{2(N_0+1)+2}\|\mu\|_{C^{N_0+1}}), \end{aligned}$$

  where, in the second-last equality, we have used that $k \geqslant k_0 + N_0 + 1$.
- $0 < -l \leqslant N_0 + 1$ appears for all $n \geqslant l$; using (7.3), (7.5) and (7.2), we can conclude:

$$\begin{aligned} & a_{2(k+l)+1} \frac{e^{2n}}{(1-\lambda_{1/2k+1}^2)^n} \\ &= O(e^{2(4k_0-k-l)+2}\|\mu\|_{C^{N_0+1}}) \cdot O(e^{2l}) \cdot \big(1 + O(e^2)\big) \\ &= O(e^{2(4k_0-k)+2}\|\mu\|_{C^{N_0+1}}) \cdot \big(1 + O(e^2)\big) \\ &= O(e^{2(N_0+1)+2}\|\mu\|_{C^{N_0+1}}) \cdot \big(1 + O(e^2)\big) \\ &= O(e^{2(N_0+1)+2}\|\mu\|_{C^{N_0+1}}), \end{aligned}$$

  where, in the second-last equality, we have used that $k \leqslant 4k_0 - N_0 - 1$.

It follows from these estimates and (7.6) that

$$a_{2k+1} = O(e^{2(N_0+1)+2}\|\mu_\varepsilon\|_{C^{N_0+2}}) \qquad \text{for } k = k_0+N_0+1, \dots, 4k_0-N_0-1.$$

The claim of the theorem then follows by taking $N_0 = k_0$ in (7.3), (7.4) and (7.5).

Similarly, one proves the relations corresponding to Fourier coefficients of even order. More specifically, one considers integrable rational caustics with rotation numbers $\frac{1}{2k}$, with $k = k_0+1, \dots, 8k_0+1$. As in the previous part, the proof consists in an iterative application of (3.8); in particular, in this case the number of needed iterations equals $2k_0$ (from which the appearance of the C^{2k_0+1}-norm).

□

Now, we want to describe how to recover the missing relations. We distinguish between Fourier coefficients corresponding to Fourier modes of, respectively, odd and even order.

7.1. **Fourier coefficients of odd order Fourier modes.** Let us prove that every integer $1 \leqslant m \leqslant k_0$, we have that

$$a_{2k+1} = \begin{cases} O(e^{2(k-m)+2}\|\mu\|_{C^{3k_0}}), & k \in [m, 3k_0 - m), \\ O(e^{2k_0+4(k_0-m)+2}\|\mu\|_{C^{3k_0}}), & k \in [3k_0 - m, 6k_0 - 3m]. \end{cases} \tag{7.7}$$

Remark 7.3. The above estimates hold with sharper choices of the norms $\|\cdot\|_{C^k}$ (see (3.8) and Lemma 7.1). However, for the sake of simplicity we have opted for common choice that is suitable for all steps involved in the needed algorithm (see Remark 7.4).

Observe, in fact, that Lemma 7.1 implies (7.7) for $m = k_0$. We proceed by (backwards) induction. Let us assume that (7.7) holds for a given $2 \le m \le k_0$ (inductive hypothesis) and let us prove it for $m-1$.

We denote $N(k) := k - m + 1$.

Let us fix $k \in \{k_0, \dots, 3k_0 - m\}$; observe that for such a choice of k, there exists a caustic with rotation number $\frac{1}{2k+1}$. Let us now apply (3.8) with $N = N(k)$:

$$a_{2k+1} + \sum_{n=1}^{N(k)} \sum_{|l|\le n} \frac{\xi_{n,l}(2(k-l)+1)\, e^{2n}}{(1-\lambda^2_{\frac{1}{2k+1}})^n} a_{2(k-l)+1} = O(e^{2N(k)+2}\|\mu\|_{C^{N(k)+1}}). \tag{7.8}$$

Remark 7.4. Notice that all estimates involve $\|\mu\|_{C^{k+1}}$ and $\|\mu\|_{C^{N(k)+1}}$, for some $k \leqslant 3k_0 - m$ and $m \geqslant 1$; in particular, $N(k) \leqslant 3k_0 - m + 1 < 3k_0$. Hence, we can choose to bound all terms with respect to $\|\mu\|_{C^{3k_0}}$. Hereafter, in order to simplify the notation, we shall neglect this term and concentrate on the part involving powers of the eccentricity e.

Now we want to show that in (7.8) the only terms in the sum that are not of the same order as the remainder are the ones corresponding to $l = n$.

- Observe that if $0 \leqslant l \leqslant n-1$, then
$$k - l \geqslant k - (N(k) - 1) = m$$
and, since $k_0 \leqslant k \leqslant 3k_0 - m$, we also have
$$k - l < 3k_0 - m.$$
Using the inductive hypothesis, the fact that $0 \leqslant l \leqslant n-1$ and (7.2), we get:
$$\begin{aligned}\frac{a_{2(k-l)+1}}{(1-\lambda^2_{\frac{1}{2k+1}})^n} e^{2n} &= O(e^{2(k-l-m)+2}) \cdot O(e^{2l+2}) \cdot (1 + O(e^2)) \\ &= O(e^{2(k-m+1)+2}) \cdot (1 + O(e^2)) \\ &= O(e^{2N(k)+2}).\end{aligned}$$
- Let us now consider negative l.
First observe that if $l = -N(k)$, then clearly
$$\frac{a_{2(k+N(k))+1}}{(1-\lambda^2_{\frac{1}{2k+1}})^{N(k)}} e^{2N(k)} = O(e^{2N(k)+2}),$$
where we have used that $a_{2(k+N(k))+1} = O(e^2)$, as it follows applying (3.8) with $N = 0$ (in fact, since $k + N(k) \leqslant k_0$, there exists by assumption a caustic of rotation number $\frac{1}{k+N(k)}$).
Let us now assume that $-N(k) + 1 \leqslant l < 0$.

  If $k + m - 3k_0 < l < 0$, then $k_0 < k - l < 3k_0 - m$; hence, using the inductive hypothesis we get:
$$\begin{aligned}\frac{a_{2(k-l)+1}}{(1-\lambda^2_{\frac{1}{2k+1}})^n} e^{2n} &= O(e^{2(k-l-m)+2}) \cdot O(e^2) \cdot (1 + O(e^2)) \\ &= O(e^{2(k-m+1)+2}) \cdot (1 + O(e^2)) \\ &= O(e^{2N(k)+2}).\end{aligned}$$

  On the other hand, if $-n \leqslant l \leqslant k + m - 3k_0$, then
$$k - l \geq 3k_0 - m \quad \text{and} \quad k - l \leqslant k + N(k) - 1 \leqslant 6k_0 - 3m.$$

Therefore, using the inductive hypothesis we get (we use that $n \geqslant -l \geqslant 3k_0 - m - k$):

$$\begin{aligned}\frac{a_{2(k-l)+1}}{(1-\lambda^2_{\frac{1}{2k+1}})^n}e^{2n} &= O(e^{2k_0+4(k_0-m)+2})\cdot O(e^{2n})\cdot(1+O(e^2))\\ &= O(e^{2k_0+4(k_0-m)+2})\cdot O(e^{6k_0-2m-2k})\cdot(1+O(e^2))\\ &= O(e^{12k_0-6m-2k+2})\cdot(1+O(e^2))\\ &= O(e^{2N(k)+2})\cdot(1+O(e^2))\end{aligned}$$

where in the last equality we have used that $m \leqslant k_0$, $k \leqslant 3k_0 - m$ and therefore

$$\begin{aligned}12k_0-6m-2k+2 &= 2(3k_0-m)+2(3k_0-m-k)-2m+2\\ &\geqslant 2(k-m+1)+2\\ &= 2N(k)+2.\end{aligned}$$

Using these estimates, we see that (7.8) becomes:

$$a_{2k+1}+\sum_{j=1}^{N(k)}\frac{\xi_{j,j}(2k+1-2j)e^{2j}}{(1-\lambda^2_{\frac{1}{2k+1}})^j}a_{2(k-j)+1} = O(e^{2N(k)+2}\|\mu\|_{C^{3k_0}}).$$

Using Lemma 3.2 and the inductive hypothesis, we see that for $j < N(k)$ (which implies $m \leqslant k-j < 3k_0 - m$), we have:

$$\begin{aligned}\frac{a_{2(k-j)+1}\,e^{2j}}{(1-\lambda^2_{\frac{1}{2k+1}})^j} &= \frac{a_{2(k-j)+1}\,e^{2j}}{\cos^{2j}(\frac{\pi}{2k+1})}(1+O(e^2))\\ &= \frac{a_{2(k-j)+1}\,e^{2j}}{\cos^{2j}(\frac{\pi}{2k+1})}+e^{2j+2}\,O(e^{2(k-j-m)+2})\\ &= \frac{a_{2(k-j)+1}\,e^{2j}}{\cos^{2j}(\frac{\pi}{2k+1})}+O(e^{2N(k)+2}).\end{aligned}$$

Clearly, for $j = N(k)$

$$\frac{a_{2(k-N(k))+1}\,e^{2N(k)}}{(1-\lambda^2_{\frac{1}{2k+1}})^{N(k)}} = \frac{a_{2(k-N(k))+1}\,e^{2N(k)}}{\cos^{2N(k)}(\frac{\pi}{2k+1})}+O(e^{2N(k)+2}).$$

Hence, (7.8) reduces to

$$a_{2k+1}+\sum_{j=1}^{N(k)}\frac{\xi_{j,j}(2k+1-2j)e^{2j}}{\cos^{2j}(\frac{\pi}{2k+1})}a_{2k+1-2j} = O(e^{2N(k)+2}\|\mu\|_{C^{3k_0}}) \quad (7.9)$$

for $k = k_0, \ldots, 3k_0 - m$.

If $k \geqslant 2k_0$, then $\frac{2}{2k+1} < \frac{1}{2k_0} = \frac{1}{q_0}$. Since Ω is q_0-integrable, then we have also the existence of integrable caustics with rotation numbers $\frac{2}{2k+1}$. Hence, proceeding as above, for $k = 2k_0, \ldots, 3k_0 - m$, we get

$$a_{2k+1} + \sum_{j=1}^{N(k)} \frac{\xi_{j,j}(2k+1-2j)e^{2j}}{\cos^{2j}(\frac{2\pi}{2k+1})} a_{2(k-j)+1} = O(e^{2N(k)+2}\|\mu\|_{C^{3k_0}}) \quad (7.10)$$

for $k = k_0, \ldots, 3k_0 - m$.

Remark 7.5. We obtain $3k_0-2m+2$ linear equations with $3k_0-2m+2$ unknown variables: a_{2m-1}, $a_{2m+1}, \ldots, a_{2(3k_0-m)+1}$. Let us consider the system of linear equations consisting of:

- the k_0 equations corresponding to (7.9) for $k = k_0, \ldots, 2k_0 - 1$;
- the (k_0-m+1) couples of equations corresponding to (7.9) and (7.10) for $k = 2k_0, \ldots, 3k_0 - m$.

Recall that $q_0 = 2k_0$. Denote by $\mathcal{A}^{(odd)}_{q_0,m} \in \mathcal{M}_{3k_0-2m+2}(\mathbb{R})$ the square matrix of the coefficients associated to this system. In particular, the matrix $\mathcal{A}^{(odd)}_{q_0,m}$ has the following structure

$$\mathcal{A}^{(odd)}_{q_0,m} = \left(\begin{array}{c|c|c} * & \mathcal{L} & \mathcal{O} \\ \hline * & * & \mathcal{K} \end{array}\right) \quad (7.11)$$

where

- $\mathcal{L}$ is a lower triangular $k_0 \times k_0$ matrix with 1's on the diagonal;
- $\mathcal{K}$ is a $(k_0 - m + 1) \times 2(k_0 - m + 1)$ matrix of the form

$$\mathcal{K} = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 0 & \ldots & 0 \\ * & 1 & 0 & \ldots & 0 \\ * & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \ldots & 1 \\ * & * & * & \ldots & 1 \end{pmatrix};$$

- $\mathcal{O}$ is a block of zeros of size $k_0 \times (k_0 - m + 1)$;
- observe that each row has a unit on it and the $*$ entries are a multiple of e^{2j}, where $j \in \mathbb{N}$ represents the "distance" from the unit within the row; in particular all $*$ entries are of the form $\xi \cos^{-2j}(w\pi)e^{2j}$, where $\xi \in \mathbb{Q}$, $w \in \{\frac{1}{2k+1}, \frac{2}{2k+1} : k > j\}$.
- Notice that this hierarchical structure of the powers of e in a given row/column, implies a similar hierarchical structure for the rows of its inverse, as we have already pointed out in (4.9), (5.4), (5.6), (6.2), (6.5) and (6.6).

If $\mathcal{A}^{(odd)}_{q_0,m}$ is *non-degenerate*, then solving this system of linear equations, we obtain

$$a_{2k+1} = O(e^{2(k-(m-1))+2}\|\mu\|_{C^{3k_0}}), \quad k = m-1, m, \ldots, k_0.$$

With this new relation, using the arguments in Lemma 7.1, we show that assumption (7.7) still holds by replacing m with $m-1$. This completes the proof of the inductive step.

Iterating the procedure until $m = 1$, we conclude that

$$a_{2k+1} = O(e^2\|\mu\|_{C^{3k_0}}), \quad k = 1, \ldots, k_0 - 1.$$

In the same way, we may show that

$$b_{2k+1} = O(e^2\|\mu\|_{C^{3k_0}}), \quad k = 1, \ldots, k_0 - 1.$$

7.2. **Fourier coefficients of even order Fourier modes.** Let $2 \leqslant m \leqslant k_0$. Denote $N_m = 3k_0 + 3\left\lfloor \frac{k_0-m}{2} \right\rfloor + \nu_{k_0,m}$, where

$$\nu_{k_0,m} := \begin{cases} 1 & \text{if } k_0 - m \text{ is even} \\ 2 & \text{if } k_0 - m \text{ is odd} \end{cases}$$

and $\lfloor \cdot \rfloor$ denotes the floor function. This choice of N_m will be clarified in Remark 7.8.
Assume that for some $1 \leqslant m \leqslant k_0$ we have

$$a_{2k} = \left\{ \begin{array}{ll} O(e^{2(k-m)}\|\mu\|_{C^{5k_0}}), & k = m+1, \ldots, N_m, \\ O(e^{2(N_m-m+1)}\|\mu\|_{C^{5k_0}}), & k = N_m + 1, \ldots, 2N_m - m. \end{array} \right. \tag{7.12}$$

Remark 7.6. The above estimates hold with sharper choices of the norms $\|\cdot\|_{C^k}$ (see (3.8) and Lemma 7.1). However, for the sake of simplicity we have opted for common choice that is suitable for all steps involved in the needed algorithm (see Remark 7.7).

Observe that Lemma 7.1 implies the assumption above for $m = k_0$.

We denote $N'(k) := k - m$.

Let us fix $k \in \{k_0 + 1, \ldots, N_m\}$; observe that for such a choice of k, there exists a caustic with rotation number $\frac{1}{2k}$. Let us now apply (3.8) with $N = N'(k)$:

$$a_{2k} + \sum_{n=1}^{N'(k)} \sum_{|l| \leq n} \frac{\xi_{n,l}(2(k-l))\, e^{2n}}{(1-\lambda_{\frac{1}{2k}}^2)^n} a_{2(k-l)} = O(e^{2N'(k)+2} \|\mu\|_{C^{N'(k)+1}}). \tag{7.13}$$

Remark 7.7. Notice that all estimates involve $\|\mu\|_{C^{k+1}}$ and $\|\mu\|_{C^{N'(k)+1}}$, for some $k \leqslant N_m$ and $m \geqslant 1$; in particular, $N'(k)+1 \leqslant N_m + 1 \leqslant 5k_0$ (as one can easily verify, by choosing $m = 1$ and estimating the corresponding expression both for $k_0 \geqslant 2$ even or odd). Hence, we can choose to bound all terms with respect to $\|\mu\|_{C^{5k_0}}$. Hereafter, in order to simplify the notation, we shall neglect this term and concentrate on the part involving powers of the eccentricity e.

Similarly to what we have done in the odd-order case, we want to show that in (7.13) the only terms in the sum that are not of the same order as the remainder are the ones corresponding to $l = n$.

- Observe that if $0 \leqslant l \leqslant n-1$, then

  $$k - l \geqslant k - (N'(k) - 1) = m + 1$$

  and clearly $k - l \leqslant N_m$. Using the inductive hypothesis, the fact that $0 \leqslant l \leqslant n-1$ and (7.2), we get:

  $$\begin{aligned} \frac{a_{2(k-l)}}{(1-\lambda_{\frac{1}{2k}}^2)^n} e^{2n} &= O(e^{2(k-l-m)}) \cdot O(e^{2l+2}) \cdot (1 + O(e^2)) \\ &= O(e^{2(k-m)+2}) \cdot (1 + O(e^2)) \\ &= O(e^{2N'(k)+2}). \end{aligned}$$

- Let us now consider negative l.
  First observe that if $l = -N'(k)$, then clearly

  $$\frac{a_{2(k+N(k))}}{(1-\lambda_{\frac{1}{2k}}^2)^{N'(k)}} e^{2N'(k)} = O(e^{2N'(k)+2}),$$

  where we have used that $a_{2(k+N'(k))} = O(e^2)$, as it follows applying (3.8) with $N = 0$ (in fact, since $k + N'(k) \leq k_0$, there exists by assumption a caustic of rotation number $\frac{1}{k+N'(k)}$).
  Let us now assume that $k - N_m \leqslant l < 0$, hence $m + 1 \leqslant k - l \leqslant$

N_m. Using the inductive hypothesis we get:

$$\begin{aligned}\frac{a_{2(k-l)}}{(1-\lambda^2_{\frac{1}{2k}})^n}e^{2n} &= O(e^{2(k-l-m)})\cdot O(e^2)\cdot(1+O(e^2))\\ &= O(e^{2(k-m)+2})\cdot(1+O(e^2))\\ &= O(e^{2N'(k)+2}).\end{aligned}$$

On the other hand, if $-n\leqslant l<k-N_m$, then

$$k-l\geqslant N_m+1 \quad \text{and} \quad k-l\leqslant k+N'(k)\leqslant 2N_m-m.$$

Therefore, using the inductive hypothesis we get (we use that $n\geqslant -l\geqslant N_m-k\geqslant 0$):

$$\begin{aligned}\frac{a_{2(k-l)}}{(1-\lambda^2_{\frac{1}{2k}})^n}e^{2n} &= O(e^{2(N_m-m+1)})\cdot O(e^{2n})\cdot(1+O(e^2))\\ &= O(e^{2(N_m-m+1)})\cdot O(e^{2(N_m-k)})\cdot(1+O(e^2))\\ &= O(e^{2(2N_m-m-k)+2})\cdot(1+O(e^2))\\ &= O(e^{2N'(k)+2}).\end{aligned}$$

Using these estimates, we see that (7.13) becomes:

$$a_{2k}+\sum_{j=1}^{N'(k)}\frac{\xi_{j,j}(2k-2j)e^{2j}}{(1-\lambda^2_{\frac{1}{2k}})^j}a_{2(k-j)}=O(e^{2N'(k)+2}\|\mu\|_{C^{5k_0}}).$$

Using Lemma 3.2 and the inductive hypothesis, we see that for $j<N'(k)$ (which implies $m+1\leqslant k-j<N_m$), we have:

$$\begin{aligned}\frac{a_{2(k-j)}\,e^{2j}}{(1-\lambda^2_{\frac{1}{2k}})^j} &= \frac{a_{2(k-j)}\,e^{2j}}{\cos^{2j}(\frac{\pi}{2k})}(1+O(e^2))\\ &= \frac{a_{2(k-j)}\,e^{2j}}{\cos^{2j}(\frac{\pi}{2k})}+e^{2j+2}\,O(e^{2(k-j-m)})\\ &= \frac{a_{2(k-j)+1}\,e^{2j}}{\cos^{2j}(\frac{\pi}{2k})}+O(e^{2N'(k)+2}).\end{aligned}$$

Clearly, for $j=N'(k)$

$$\frac{a_{2(k-N'(k))}\,e^{2N'(k)}}{(1-\lambda^2_{\frac{1}{2k}})^{N'(k)}} = \frac{a_{2(k-N'(k))}\,e^{2N'(k)}}{\cos^{2N'(k)}(\frac{\pi}{2k})}+O(e^{2N'(k)+2}).$$

Hence, (7.13) reduces to

$$a_{2k}+\sum_{j=1}^{N'(k)}\frac{\xi_{j,j}(2k-2j)e^{2j}}{\cos^{2j}(\frac{\pi}{2k})}a_{2(k-j)}=O(e^{2N'(k)+2}\|\mu\|_{C^{5k_0}}), \tag{7.14}$$

for $k = k_0 + 1, \ldots, N_m$.

If $k \geqslant 3k_0 + 1$, since Ω is q_0-integrability (recall that $q_0 = 2k_0$), then we also have the existence of caustics with rotation numbers $\frac{3}{2k}$.
In particular, let $2k \not\equiv 0 \pmod 3$, with $k = 3k_0 + 1, \ldots, N_m$. Proceeding as above, using the existence of a caustic with rotation number $\frac{3}{2k}$, we can conclude that:

$$a_{2k} + \sum_{j=1}^{N'(k)} \frac{\xi_{j,j}(2k-2j)e^{2j}}{\cos^{2j}(\frac{3\pi}{2k})} a_{2k-2j} = O(e^{2N'(k)+2}\|\mu\|_{C^{5k_0}}). \tag{7.15}$$

Remark 7.8. We obtain $N_m - m + 1$ linear equations in $N_m - m + 1$ variables: a_{2k} with $k = m, \ldots, N_m$.
Observe, in fact, The number N_m was chosen in such a way that the number of equations is the same as the number of unknowns. Indeed:

- For $k = k_0 + 1, \ldots, 3k_0$, we obtain $2k_0$ equations.
- For $k = 3k_0 + 1, \ldots, N_m$, we have $N_m - 3k_0$ values of k which contribute with 2 equations when k is not a multiple of 3, and with only one equation otherwise. Hence, each group $\{3j + 1, 3j + 2, 3(j + 1)\}$ produces 5 equations and in our case $j = k_0, \ldots, \lfloor N_m/3 \rfloor$. Let us define $\alpha_m \in \{0, 1, 2\}$ such that $N_m = 3\lfloor N_m/3 \rfloor + \alpha_m$.
- Hence, the number of total equations is

$$\underbrace{2k_0}_{1^{\text{st}}\text{ block}} + \underbrace{5\left(\lfloor N_m/3 \rfloor - k_0\right) + 2\alpha_m}_{2^{\text{nd}}\text{ block}}.$$

- The number of unknowns that we get is $N_m - m + 1$.

In conclusion, we want to choose N_m such that

$$\begin{aligned} & 2k_0 + 5\left(\lfloor N_m/3 \rfloor - k_0\right) + 2\alpha_m = N_m - m + 1 \\ \iff \quad & 5\lfloor N_m/3 \rfloor - 3k_0 + 2\alpha_m = 3\lfloor N_m/3 \rfloor + \alpha_m - m + 1 \\ \iff \quad & 2\lfloor N_m/3 \rfloor = 3k_0 - m + 1 - \alpha_m. \end{aligned} \tag{7.16}$$

We want to solve this equation. We distinguish two cases according to the parity of $3k_0 - m$ (or, equivalently, of $k_0 - m$):

- If $k_0 - m$ is even, (7.16) can have an integral solution only if $\alpha_m = 1$; in this case:

$$\left\lfloor \frac{N_m}{3} \right\rfloor = \frac{3k_0 - m}{2}$$

and

$$N_m = 3\left\lfloor \frac{3k_0 - m}{2} \right\rfloor + 1 = 3k_0 + \left\lfloor \frac{k_0 - m}{2} \right\rfloor + 1.$$

- Similarly, if $k_0 - m$ is odd, (7.16) can have an integral solution only if $\alpha_m = 0$ or 2; in case $\alpha_m = 2$:

$$\left\lfloor \frac{N_m}{3} \right\rfloor = \frac{3k_0 - m - 1}{2} = \left\lfloor \frac{3k_0 - m}{2} \right\rfloor$$

  and

$$N_m = 3\left(\left\lfloor \frac{3k_0 - m}{2} \right\rfloor\right) + 2 = 3k_0 + 3\left\lfloor \frac{k_0 - m}{2} \right\rfloor + 2.$$

  Observe that if we chose $\alpha_m = 0$, then we could get a larger N_m, namely $3k_0 + 3\left\lfloor \frac{k_0-m}{2} \right\rfloor + 3$.

Summarizing, we choose

$$N_m := 3k_0 + 3\left\lfloor \frac{k_0 - m}{2} \right\rfloor + \nu_{k_0,m},$$

where

$$\nu_{k_0,m} = \begin{cases} 1 & \text{if } k_0 - m \text{ is even} \\ 2 & \text{if } k_0 - m \text{ is odd.} \end{cases}$$

Observe that for $m = k_0$, we have exactly $N_{k_0} = 3k_0 + 1$, as needed to recover (7.12) from Lemma 7.1.

Remark 7.9. Recall that $q_0 = 2k_0$. We denote by $\mathcal{A}^{(even)}_{q_0,m} \in \mathcal{M}_{N_m-m+1}(\mathbb{R})$ the square matrix of the coefficients associated to the linear system of equations, consisting of

- the first $2k_0$ equations correspond to (7.14) for $k = k_0{+}1, \ldots, 3k_0$,
- the other $N_m - 2k_0$ rows correspond to the equations (7.14)–(7.15) for $k = 3k_0 + 1, \ldots, N_m$.

In particular, the matrix $\mathcal{A}^{(even)}_{q_0,m}$ has the following structure

$$\mathcal{A}^{(even)}_{q_0,m} = \left(\begin{array}{c|c|c} * & \mathcal{L}' & \mathcal{O}' \\ \hline * & * & \mathcal{K}' \end{array}\right), \tag{7.17}$$

where

- $\mathcal{L}'$ a lower triangular $2k_0 \times 2k_0$ matrix with 1's on the diagonal;

- $\mathcal{K}'$ is a $(N_m - 2k_0) \times (N_m - 3k_0 + m - 1)$ matrix of the form

$$\mathcal{K}' = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ * & 1 & 0 & \dots & 0 \\ * & 1 & 0 & \dots & 0 \\ * & * & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \dots & 1 \\ * & * & * & \dots & 1 \end{pmatrix};$$

- $\mathcal{O}'$ is a block of zeros of size $2k_0 \times (N_m - 3k_0 + m - 1)$;
- observe that each row has a unit on it and the $*$ entries are a multiple of e^{2j}, where $j \in \mathbb{N}$ represents the "distance" from the unit within the row; in particular all $*$ entries are of the form $\xi \cos^{-2j}(w\pi)e^{2j}$, where $\xi \in \mathbb{Q}$, $w \in \{\frac{1}{2k}, \frac{3}{2k} : \ k > j\}$.
- Notice that this hierarchical structure of the powers of e within a given row/colum, implies a similar hierarchical structure for the rows of its inverse, as we have already pointed out before, for example in (4.9), (5.4), (5.6), (6.2), (6.5) and (6.6).

If $\mathcal{A}_{q_0,m}^{(even)}$ is *non-degenerate*, then solving the linear system, we get that

$$a_{2k} = O(e^{2k-2m+2}\|\mu\|_{C^{5k_0}}), \quad k = m, \dots, k_0.$$

Then one can show that replacing m by $m-1$, assumption (7.12) continues to hold. Therefore, iterating the procedure until $m = 1$, we conclude that

$$a_{2k} = O(e^2\|\mu\|_{C^{5k_0}}), \quad k = 2, \dots, k_0.$$

Similarly, one can show that

$$b_{2k} = O(e^2\|\mu\|_{C^{5k_0}}), \quad k = 2, \dots, k_0.$$

To summarize, the discussion in Subsections 7.1 and 7.2 leads to the following statement.

Proposition 7.10. *If all the $q_0 - 2$ matrices in (7.11) and (7.17) are non-degenerate, then there exists $C_{q_0} > 0$ depending only on q_0 such that*

$$|a_k|, |b_k| \leqslant C_{q_0} e^2 \|\mu\|_{C^{3q_0}}, \quad k = 3, \dots, q_0.$$

Remark 7.11. Notice that the algorithm that we have described, can be easily implemented on a computer, hence all of the above non-degeneracy conditions can be explicitely verified, via symbolic computations, for arbitrary q_0; see Sections 4–6 for the cases corresponding to $q_0 = 3, 4, 5$.

8. Deformed Fourier modes

Let Ω be a strictly convex domain and s denote the arc-length parametrisation of $\partial\Omega$ and denote by $|\partial\Omega|$ its length. Let $\rho(s)$ be its radius of curvature at s. Observe that if Ω is C^r, then ρ is C^{r-2}. The Lazutkin parametrisation of the boundary, first introduced in [23], is defined as

$$x(s) = C_\Omega \int_0^s \rho(\sigma)^{-2/3} d\sigma, \quad \text{where} \quad C_\Omega = 2\pi \Big[\int_0^{|\partial\Omega|} \rho(\sigma)^{-2/3} d\sigma \Big]^{-1}.$$

Observe that if $\partial\Omega = \mathcal{E}_e$ is an ellipse, ρ is analytic, thus, the Lazutkin parametrisation is itself an analytic parametrisation of $\mathcal{E}_e$. Let (μ, φ) be the elliptic coordinates associated to the ellipse $\mathcal{E}_e$,

$$\mathcal{E}_e = \{(\mu_0, \varphi) : \ \varphi \in [0, 2\pi)\}.$$

Let $\varphi_L(x)$ denote the change of parametrisation from x to φ. Then we have the following lemma.

Lemma 8.1. *For each $r \in \mathbb{N}$, there exists C_r such that*

$$\|\varphi_L(x) - x\|_{C^r} \leqslant C_r e^2.$$

The proof of this lemma is straightforward. The reader is kindly referred to [22, Appendix A] for some details.

Now let us introduce the change of variables from the action-angle parametrisation θ of $\mathcal{E}_e$, derived from the smooth convex caustic with rotation number $1/q$, to the Lazutkin parametrisation x, *i.e*,

$$x = X_q(\theta) := \varphi_L^{-1}\Big(\varphi_{\lambda_{1/q}}(\theta)\Big).$$

The following lemma is proven in [1, Lemma 11].

Lemma 8.2. *There exists $C(e)$, with $C(e) \to 0$ as $e \to 0$, such that*

$$\|X_q(\cdot) - \mathbb{I}(\cdot)\|_{C^1} \leqslant \frac{C(e)}{q^2}.$$

where $\mathbb{I}$ stands for the identity map.

Let us denote $L^2(\mathbb{T})$ the L^2-space of 2π-periodic functions, with trigonometric basis $\{v_k\}_{k\in\mathbb{Z}}$, where

$$v_0 = 1,\ v_k(x) = \frac{1}{\sqrt{\pi}}\cos kx, \quad v_{-k} = \frac{1}{\sqrt{\pi}}\sin kx, \quad k = 1, 2, \ldots.$$

Consider another set of functions $\{c_k\}_{k\in\mathbb{Z}}$, where

$$c_0(x) = v_0, \quad c_k(x) = v_k(x), \quad c_{-k}(x) = v_{-k}(x) \quad k = 1, \ldots, q_0,$$

and for $k > q_0$,

$$c_k(x) = \frac{\cos\big(kX_k^{-1}(x)\big)}{\sqrt{\pi}\,X_k'\big(X_k^{-1}(x)\big)} \quad , \quad c_{-k}(x) = \frac{\sin\big(kX_k^{-1}(x)\big)}{\sqrt{\pi}\,X_k'\big(X_k^{-1}(x)\big)}.$$

Note here that the functions $c_{\pm k}$ have zero average. From Lemma 8.2, for each $k \geq 1$ we have

$$\|c_k - v_k\|_{C^0} \leqslant \frac{C(e)}{k} \quad , \quad \|c_{-k} - v_{-k}\|_{C^0} \leqslant \frac{C(e)}{k}, \tag{8.1}$$

and

$$C(e) \longrightarrow 0 \quad \text{as} \quad e \to 0,$$

and for e small enough, $\{c_k\}_{k\in\mathbb{Z}}$ form a basis of L^2 (see [1, Proposition 22] and Lemma 8.3 hereafter).

For any integer $r \geqslant 1$, we consider the Sobolev space $H^r(\mathbb{T})$, which is defined as

$$H^r(\mathbb{T}) := \{u \in L^2(\mathbb{T}) :\ u^{(r)} \in L^2(\mathbb{T})\},$$

where $u^{(r)}$ denotes the r-th (weak) derivative of u. Recall that $H^r(\mathbb{T})$ is a Hilbert space with inner product

$$\langle u, v\rangle_r = \Big(\int_{\mathbb{T}} u dx\Big)\Big(\int_{\mathbb{T}} v dx\Big) + \int_{\mathbb{T}} u^{(r)} v^{(r)} dx,$$

and we have

$$\|u\|_r^2 = \sum_{k\in\mathbb{Z}} (|k|^{2r} \wedge 1)\hat{u}_k^2 = \langle u, u\rangle_r,$$

where $a \wedge b = \max\{a, b\}$ and $\hat{u}_k$ are the Fourier coefficient of u, *i.e.*,

$$\hat{u}_k = \int_{\mathbb{T}} u(x) v_k(x) dx, \quad k \in \mathbb{Z}.$$

Notice that the choice of norms is somewhat non-standard and for each $r \geq 1$ we have

$$\|u\|_r \leqslant \|u\|_{r+1}.$$

Denote $\mathcal{V}_k(x)$ be the functions that have zero mean value and

$$\mathcal{V}_k^{(r)}(x) = v_k(x), \quad k \in \mathbb{Z}\setminus\{0\}.$$

Then, we have that the set of functions $\{\mathcal{V}_0 = 1, \mathcal{V}_k,\ k \in \mathbb{Z} \setminus \{0\}\}$ form an orthonormal basis of $H^r(\mathbb{T})$, *i.e.*,

$$\langle \mathcal{V}_k, \mathcal{V}_j \rangle_r = \delta_{k,j}, \quad \forall\, k,\ j \in \mathbb{Z},$$

and for every $u \in H^r(\mathbb{T})$, we have

$$u(x) = \sum_{k \in \mathbb{Z}} u_k \mathcal{V}_k(x),$$

and

$$\|u\|_r^2 = \sum_{k \in \mathbb{Z}} u_k^2,$$

where $u_k = \langle u, \mathcal{V}_k \rangle_r$. Observe that $u_k^2 = (k^{2r} \wedge 1)\hat{u}_k^2$, for $k \in \mathbb{Z}$.

Now we introduce a set of functions

$$\{\mathcal{C}_0 = 1, \mathcal{C}_k,\ \mathcal{C}_{-k}, k \in \mathbb{Z}_+\},$$

where $\mathcal{C}_{\pm k}$, $k \in \mathbb{Z}_+$ are the zero mean value functions on $\mathbb{T}$ such that

$$\mathcal{C}_k^{(r)}(x) = c_k(x), \quad \mathcal{C}_{-k}^{(r)}(x) = c_{-k}(x), \quad k \in \mathbb{Z} \setminus \{0\}.$$

Therefore, we have

$$\mathcal{C}_k = \mathcal{V}_k \quad , \quad \mathcal{C}_{-k} = \mathcal{V}_{-k}, \quad k \in \mathbb{Z}_+,\ k \leqslant q_0,$$

and

$$\begin{aligned} \|\mathcal{V}_k - \mathcal{C}_k\|_r^2 &= \langle \mathcal{V}_k - \mathcal{C}_k, \mathcal{V}_k - \mathcal{C}_k \rangle_r \leqslant \frac{[C(e)]^2}{k^2}, \\ \|\mathcal{V}_{-k} - \mathcal{C}_{-k}\|_r^2 &= \langle \mathcal{V}_{-k} - \mathcal{C}_{-k}, \mathcal{V}_{-k} - \mathcal{C}_{-k} \rangle_r \leqslant \frac{[C(e)]^2}{k^2} \end{aligned} \tag{8.2}$$

for $k \in \mathbb{Z}$, $k > q_0$. Consider the linear operator

$$\mathcal{L} : H^r(\mathbb{T}) \to H^r(\mathbb{T}), \quad u \mapsto \mathcal{L}u = u_0 + \sum_{k \in \mathbb{Z}_+} u_k \mathcal{C}_k(x) + u_{-k}\mathcal{C}_{-k}(x),$$

where $u_k = \langle u, \mathcal{V}_k \rangle_r$.

Define $D(q_0) := \Big[\sum_{|k|>q_0, k \in \mathbb{Z}} \frac{1}{k^2} \Big]^{\frac{1}{2}} < \sqrt{\frac{\pi^2}{3}}$.

Lemma 8.3. *Let $C(e)$ be from Lemma 8.2. Assume e_0 satisfies*

$$C(e_0)D(q_0) < 1.$$

Then, for each $e \in [0, e_0]$ the operator $\mathcal{L}$ is bounded and invertible in the Hilbert space $H^r(\mathbb{T})$. In particular, $\{\mathcal{C}_0,\ \mathcal{C}_k,\ \mathcal{C}_{-k}, k \in \mathbb{Z}_+\}$ form a basis of $H^r(\mathbb{T})$.

Proof. Observe that if $\|\mathcal{L}-\mathbb{I}\|_{H^r\to H^r}<1$, then $\mathcal{L}$ is a bounded invertible operator with a bounded inverse; recall that

$$\|\mathcal{L}-\mathbb{I}\|_{H^r\to H^r} := \sup_{\|u\|_r\leqslant 1} \|[\mathcal{L}-\mathbb{I}](u)\|_r. \tag{8.3}$$

For each $v \in H^r$, we have

$$u=\sum_{k\in\mathbb{Z}} u_k\mathcal{V}_k, \quad u_k=\langle u_k, \mathcal{V}_k\rangle_r, \quad k\in\mathbb{Z}.$$

By the definition of the operator $\mathcal{L}$, we have

$$[\mathcal{L}-\mathbb{I}](u)=\sum_{k\in\mathbb{Z},|k|>q_0} u_k(\mathcal{C}_k-\mathcal{V}_k).$$

By the Cauchy inequality, we have

$$\|[\mathcal{L}-\mathbb{I}](u)\|_r \leqslant \sum_{k\in\mathbb{Z},|k|>q_0} |u_k|\cdot\|\mathcal{C}_k-\mathcal{V}_k\|_r \leqslant \Big[\sum_{k\in\mathbb{Z}} u_k^2\Big]^{\frac{1}{2}}\Big[\sum_{k\in\mathbb{Z}}\|\mathcal{C}_k-\mathcal{V}_k\|_r^2\Big]^{\frac{1}{2}}.$$

By (8.2), we have

$$\Big[\sum_{k\in\mathbb{Z}}\|\mathcal{C}_k-\mathcal{V}_k\|_r^2\Big]^{\frac{1}{2}} \leqslant C(e)\Big[\sum_{|k|>q_0,k\in\mathbb{Z}}\frac{1}{k^2}\Big]^{\frac{1}{2}} < C(e)\sqrt{\frac{\pi^2}{3}}.$$

Therefore, the assertion of the lemma follows from (8.3) and the fact that $\|u\|_r^2=\sum_{k\in\mathbb{Z}} u_k^2$. ☐

Remark 8.4. The basis $\{\mathcal{C}_0,\ \mathcal{C}_k,\ \mathcal{C}_{-k}, k\in\mathbb{Z}_+\}$ of $H^r(\mathbb{T})$ is not necessarily an orthogonal basis.

Corollary 8.5. *There exists $C'(e)>0$, with $C'(e)\to 1$ as $e\to 0$, such that for each $u\in H^r(\mathbb{T})$,*

$$\|u\|_r^2 \leqslant C'(e)\sum_{k\in\mathbb{Z}}\tilde{u}_k^2,$$

where $\tilde{u}_k=\langle u,\mathcal{C}_k\rangle_r$.

Proof. The operator $\mathcal{L}$ is bounded and invertible with a bounded inverse, so does its adjoint operator $\mathcal{L}^*$. Let us denote

$$C'(e)=\|(\mathcal{L}^*)^{-1}\|_{H^r\to H^r}.$$

Hence we have that for each $u\in H^r(\mathbb{T})$,

$$\begin{aligned}\|u\|_r^2 &= \|(\mathcal{L}^*)^{-1}\mathcal{L}^*u\|_r \leqslant C'(e)\|\mathcal{L}^*u\|_r^2\\ &\leqslant C'(e)\sum_{k\in\mathbb{Z}}\langle\mathcal{L}^*u,\mathcal{V}_k\rangle_r^2 = C'(e)\sum_{k\in\mathbb{Z}}\langle u,\mathcal{L}\mathcal{V}_k\rangle_r^2.\end{aligned}$$

Since $\mathcal{L}\mathcal{V}_k = \mathcal{C}_k$, we have

$$\|u\|_{C^r} \leqslant C'(e) \sum_{k\in\mathbb{Z}} \tilde{u}_k^2.$$

The assertion that $C'(e) \to 1$ as $e \to 0$ follows from the fact that $\|\mathcal{L} - \mathbb{I}\|_{H^r \to H^r} \to 0$ as $e \to 0$. $\square$

Corollary 8.6. *Let $u(x) \in H^{r+1}(\mathbb{T})$. Then, there exists $C''(e) > 0$ such that*

$$\big|\langle u, \mathcal{C}_k\rangle_r\big| \leqslant \frac{C''(e)\|u\|_{r+1}}{|k|} \quad \forall\, k \in \mathbb{Z}\setminus\{0\}.$$

Proof. Using (8.2), we have

$$\big|\langle u, \mathcal{V}_k - \mathcal{C}_k\rangle_r\big| \leqslant \|u\|_r \|\mathcal{V}_k - \mathcal{C}_k\|_r \leqslant \frac{C(e)\|u\|_r}{|k|}.$$

Since $u \in H^{r+1}$, we have

$$\big|\langle u, \mathcal{V}_k\rangle_r\big| \leqslant \frac{\|u\|_{r+1}}{|k|}.$$

Therefore we have

$$\big|\langle u, \mathcal{C}_k\rangle_r\big| \leqslant \frac{C''(e)\|u\|_{r+1}}{|k|},$$

where $C''(e) = 1 + C(e)$. $\square$

Consider a domain Ω, whose boundary $\partial\Omega$ is close to the ellipse $\mathcal{E}_e$, written in elliptic coordinates associated to $\mathcal{E}_e$ as

$$\partial\Omega = \mathcal{E}_e + \mu(\varphi),$$

where $\|\mu\|_{C^m} \leqslant M$ with $m > r + 2$ and $\|\mu\|_{C^1}$ is small enough. Let x denote the Lazukin parametrisation of $\mathcal{E}_e$. Define

$$f_\mu(x) = \mu(\varphi_L(x)).$$

Then we have:

Lemma 8.7. *For any integer $r > 0$, there exists $C_r > 0$ independent of φ and μ, such that*

$$(1 - C_r e^2)\|\mu\|_{C^r} \leqslant \|f_\mu\|_{C^r} \leqslant (1 + C_r e^2)\|\mu\|_{C^r}.$$

Moreover, the following holds.

Lemma 8.8. *There exists $C > 0$ such that*

$$|\hat{f}_k - \hat{\mu}_k| \leqslant C e^2 \|\mu\|_{C^1},$$

where $\hat{f}_k$ and $\hat{\mu}_k$ are the Fourier coefficients of the functions f_μ and μ.

The two lemmata above directly follow from Lemma 8.1.

Let us now show the following result, where we assume $e \leqslant e_0$ and $C_r e^2 \leqslant \frac{1}{2}$.

Lemma 8.9. *For any integer $q > q_0$, if the billiard dynamics inside the domain Ω admits an integrable rational caustic with rotation number $1/q$, then*

$$\big|\langle f_\mu, \mathcal{C}_{\pm q}\rangle_r\big| \leqslant C(M) q^7 \|f_\mu\|_{C^1}^{\frac{2(m-r-2)}{m-1}}.$$

Proof. By Lemmata 3.2 and 3.3, from the existence of a smooth convex caustic with rotation number $1/q$, we have that

$$\sum_{k=1}^{q} f_\mu(X_q(\theta + \frac{k}{q}2\pi)) = c_{1/q} + \Upsilon(X_q(\theta)),$$

and denoting $\widetilde{\Upsilon} = \Upsilon(X_q(\theta))$,

$$\|\widetilde{\Upsilon}\|_{C^0} \leqslant q^8 C \|f_\mu\|_{C^1}^2, \quad \text{and} \quad \|\widetilde{\Upsilon}\|_{C^{m-1}} \leqslant q^2 C'(M).$$

By the Sobolev interpolating inequality

$$\|u\|_{C^r} \leqslant C\|u\|_{H^{r+1}} \leqslant C\|u\|_{C^{m-1}}^{\frac{r+1}{m-1}} \|u\|_{C^0}^{\frac{m-r-2}{m-1}},$$

we have

$$\|\widetilde{\Upsilon}\|_{C^r} \leqslant q^8 C'(M) \|f_\mu\|_{C^1}^{\frac{2(m-r-2)}{m-1}}.$$

Notice that

$$\begin{aligned}
&\int_0^{2\pi} D^r \sum_{k=1}^{q} f_\mu(X_q(\theta + \frac{k}{q}2\pi)) \sin q\theta \, d\theta \\
&= \sum_{k=1}^{q} \int_0^{2\pi} D^r f_\mu(X_q(\theta + \frac{k}{q}2\pi)) \sin q\theta \, d\theta \\
&= q \int_0^{2\pi} D^r f_\mu(X_q(\theta)) \sin q\theta \, d\theta,
\end{aligned}$$

here, we denote D^r for the r-th derivative. Then

$$\Big| \int_0^{2\pi} D^r f_\mu(X_q(\theta)) \sin q\theta \, d\theta \Big| \leqslant \frac{\|\widetilde{\Upsilon}\|_{C^r}}{q} \leqslant q^7 C'(M) \|f_\mu\|_{C^1}^{\frac{2(m-r-2)}{m-1}}.$$

Let $x = X_q(\theta)$ and $\theta = X_q^{-1}(x)$, we have

$$d\theta = \frac{1}{X_q'(X_q^{-1}(x))} dx.$$

Therefore,

$$\int_0^{2\pi} D^r f_\mu(X_q(\theta))\sin q\theta\, d\theta = \int_0^{2\pi} D^r f_\mu(x)\sin qX_q^{-1}(x)\frac{1}{X'(X_q^{-1}(x))}dx$$
$$= \sqrt{\pi}\int_0^{2\pi} D^r f_\mu(x) D^r \mathcal{C}_{-q}(x)dx = \sqrt{\pi}\langle f_\mu, \mathcal{C}_{-q}\rangle_r.$$

Hence

$$\left|\langle f_\mu, \mathcal{C}_{-q}\rangle_r\right| \leqslant q^7 C'(M)\|f_\mu\|_{C^1}^{\frac{2(m-r-2)}{m-1}}.$$

Repeating a similar argument, we obtain the corresponding inequality for $\langle f_\mu, \mathcal{C}_q\rangle_r$. □

9. Proof of The Main result

In this section, we prove Theorems 1.1 and 1.2.

Denote

$$n = 3q_0, \quad \text{and} \quad m = 40q_0.$$

Let $\mathcal{E}_e$ be an ellipse with eccentricity $e \in (0, 4e_0/5]$ and the semimajor axis 1, where e_0 is from Lemma 8.3. Consider a C^m-smooth domain Ω, which is a C^n-perturbation of the ellipse $\mathcal{E}_e$, *i.e.*, in the elliptic coordinates associated to $\mathcal{E}_e$,

$$\partial\Omega = \mathcal{E}_e + \mu(\varphi),$$

where

$$\|\mu\|_{C^n} \leqslant \varepsilon, \quad \text{and} \quad \|\mu\|_{C^m} \leqslant M.$$

Here $\varepsilon \leqslant e^{6q_0}$ is a small parameter to be determined below and $M > 0$ is a fixed constant. We make the following assumption:

Assumption A: The domain Ω is q_0-rationally integrable and the non-degeneracy conditions in Proposition 7.10 hold true if $q_0 \geqslant 6$. More exactly, matrices (7.11) *and* (7.17) *are non-degenerate.*

The proof consists of two main steps:

- Find an ellipse $\mathcal{E}''$, close to $\mathcal{E}_e$, which best approximates Ω.
- Show that $\Omega = \mathcal{E}''$.

Step 1 Denote $\mathbb{E}_\varepsilon = \mathbb{E}_\varepsilon(\mathcal{E}_e)$ the set of ellipses whose C^0-Hausdorff distance to $\mathcal{E}_e$ is not greater than 2ε, *i.e.*,

$$\mathbb{E}_\varepsilon := \{\mathcal{E}' \subset \mathbb{R}^2 : \ \text{dist}_H(\mathcal{E}',\ \mathcal{E}_e) \leqslant 2\varepsilon\}.$$

Clearly, $\mathbb{E}_\varepsilon$ is a compact set in any C^r-topology (it is completely determined by 5 parameters). We could choose ε small enough so that the eccentricities of all the ellipses in $\mathbb{E}_\varepsilon$ are between $4e/5$ and $5e/4$.

For each $\mathcal{E}' \in \mathbb{E}_\varepsilon$, we can write the domain Ω in the elliptic coordinate frame associated to $\mathcal{E}'$, as

$$\partial\Omega = \mathcal{E}' + \mu_{\mathcal{E}'}(\varphi).$$

Choosing smaller ε if necessary, assuming $\|\mu_{\mathcal{E}'}\|_{C^m} \leqslant 2M$, $\forall \mathcal{E}' \in \mathbb{E}_\varepsilon$, from Lemma A.1, we know that $\|\mu_{\mathcal{E}'}\|_{C^n}$ changes continuously with respect to $\mathcal{E}'$.

The proof is by contradiction. Assume that the statement of the theorem was not true – namely, $\partial\Omega$ was not an ellipse – since $\mathbb{E}_\varepsilon$ is compact, then we choose $\mathcal{E}'' \in \mathbb{E}$ such that

$$\|\mu_{\mathcal{E}''}\|_{C^n} = \min\{\mathcal{E}' \in \mathbb{E} : \ \|\mu_{\mathcal{E}'}\|_{C^n}\} > 0.$$

We also have that

$$\|\mu_{\mathcal{E}''}\|_{C^m} \leqslant 2M \quad \text{ and } \quad \|\mu_{\mathcal{E}''}\|_{C^n} \leqslant \|\mu_{\mathcal{E}'}\|_{C^n}.$$

Step 2. Prove the following

Lemma 9.1. *There exists an ellipse $\bar{\mathcal{E}} \in \mathbb{E}_\varepsilon$ such that in the elliptic coordinate frame associated to $\bar{\mathcal{E}}$*

$$\|\mu_{\bar{\mathcal{E}}}\|_{C^n} < \frac{1}{2}\|\mu_{\mathcal{E}''}\|_{C^n},$$

Notice that this contradicts minimality of $\|\mu_{\mathcal{E}''}\|_{C^n} > 0$ among all $\mathcal{E}' \in \mathbb{E}_\varepsilon$.

Proof. By Lemma B.1, there exists an ellipse $\bar{\mathcal{E}} \in \mathbb{E}_\varepsilon$ such that in the elliptic coordinate frame associated to $\bar{\mathcal{E}}$, the domain Ω reads as

$$\partial\Omega = \bar{\mathcal{E}} + \mu_{\bar{\mathcal{E}}}(\varphi),$$

with

$$\|\mu_{\bar{\mathcal{E}}}\|_{C^m} \leqslant 2M, \quad \|\mu_{\bar{\mathcal{E}}}\|_{C^n} \leqslant 2\|\mu_{\mathcal{E}''}\|_{C^n}.$$

and the first five Fourier coefficients of $\mu_{\bar{\mathcal{E}}}$ satisfy (9.1). Write $\mu_{\bar{\mathcal{E}}}$ as Fourier series, *i.e.*,

$$\mu_{\bar{\mathcal{E}}}(\varphi) := \sum_{k=0}^{+\infty} a_k \cos k\varphi + b_k \sin k\varphi.$$

We split the perturbation into four parts:

(1) (Elliptic motions) $|k| \leqslant 2$;
(2) (Low-order modes) $2 < |k| \leqslant q_0$;
(3) (Intermediate-order modes) $q_0 < |k| < N := \|\mu_{\mathcal{E}''}\|_{C^n}^{-1/15}$;
(4) (High-order modes) $|k| \geqslant N$.

Each of these regimes requires different type of estimates.

- *Elliptic motions:* $|k|\leqslant 2$.

By Lemma B.1, there exists $C>0$ such that

$$|a_k| \leqslant Ce^2\|\mu_{\mathcal{E}''}\|_{C^1}, \quad |b_k| \leqslant Ce^2\|\mu_{\mathcal{E}''}\|_{C^1}, \quad |k| \leqslant 2. \tag{9.1}$$

- *Low-order modes:* $2<|k|\leqslant q_0$.

With Assumption A, from Proposition 7.10 we have that there exist $C_{q_0}>0$ depending only on q_0 such that

$$|a_k|,\ |b_k| \leqslant C_{q_0}e^2\|\mu_{\bar{\mathcal{E}}}\|_{C^n} \leqslant 2C_{q_0}e^2\|\mu_{\mathcal{E}''}\|_{C^n}, \quad 3\leqslant k\leqslant q_0. \tag{9.2}$$

Denote x the Lazutkin parametrisation of the ellipse $\bar{\mathcal{E}}$. Define

$$F(x) := \mu_{\bar{\mathcal{E}}}(\varphi(x)).$$

By Lemma 8.7, we have

$$\|\mu_{\bar{\mathcal{E}}}\|_{C^n} \leqslant (1-C_ne^2)^{-1}\|F\|_{C^n},$$

and

$$\|F\|_{C^n} \leqslant (1+C_ne^2)\|\mu_{\bar{\mathcal{E}}}\|_{C^n} \leqslant 2(1+C_ne^2)\|\mu_{\mathcal{E}''}\|_{C^n}.$$

We consider the Hilbert space $H^{n+1}(\mathbb{T})$ and define the basis $\{\mathcal{C}_k, k\in\mathbb{Z}\}$ for $H^{n+1}(\mathbb{T})$ as those in Section 8. Denote

$$\alpha_k^F = \langle F, \mathcal{C}_k\rangle_{n+1} \quad k\in\mathbb{Z}.$$

Then, due to Lemma 8.8, (9.1) and (9.2), we have that there exists $\bar{C}_{q_0}>0$ such that

$$|\alpha_k^F| \leqslant (1\wedge|k|^{n+1})\bar{C}_{q_0}e^2\|\mu_{\mathcal{E}''}\|_{C^n} \quad |k|\leqslant q_0,$$

Therefore, we have

$$\sum_{k=-q_0}^{q_0}(\alpha_k^F)^2 \leqslant 3\bar{C}_{q_0}q_0^{2n+3}e^4\|\mu_{\mathcal{E}''}\|_{C^n}^2.$$

- *Intermediate-order modes:* $q_0<|k|<N := \|\mu_{\mathcal{E}''}\|_{C^n}^{-1/15}$.

By Lemma 8.9, for $q_0<|k|<N$ we have that

$$|\alpha_k^F| \leqslant C(M)|k|^7\|F\|_{C^1}^{\frac{2(m-n-3)}{m-1}} \leqslant 4C(M)|k|^7\|\mu_{\mathcal{E}''}\|_{C^1}^{\frac{2(m-n-3)}{m-1}}.$$

So we have,

$$\sum_{q_0<|k|<N}(\alpha_k^F)^2 \leqslant C(M)N^{15}\|\mu_{\mathcal{E}''}\|_{C^1}^{\frac{4(m-n-3)}{m-1}}.$$

- *High-order modes:* $|k|\geq N$.

For $|k| \geqslant N$, due to Lemma 8.6, we have

$$|\alpha_k^F| \leqslant \frac{\|F\|_{C^{n+2}}}{|k|} \leqslant \frac{C\|\mu_{\mathcal{E}''}\|_{C^{n+2}}}{|k|}.$$

So we have

$$\sum_{|k|\geqslant N} (\alpha_k^F)^2 \leqslant \sum_{|k|\geqslant N} C\frac{\|\mu_{\mathcal{E}''}\|^2_{C^{n+2}}}{k^2} \leqslant \frac{C}{N}\|\mu_{\mathcal{E}''}\|^2_{C^{n+2}}.$$

Using Sobolev interpolation inequality, we have

$$\|\mu_{\mathcal{E}''}\|_{C^{n+2}} \leqslant C\|\mu_{\mathcal{E}''}\|_{H^{n+3}} \leqslant C\|\mu_{\mathcal{E}''}\|_{C^m}^{\frac{3}{m-n}}\|\mu_{\mathcal{E}''}\|_{C^n}^{\frac{m-n-3}{m-n}} \leqslant C(M)\|\mu_{\mathcal{E}''}\|_{C^n}^{\frac{m-n-3}{m-n}}.$$

Since $m = 40q_0$ and $n = 3q_0$, we have

$$\frac{2(m-n-3)}{m-n} \geqslant \frac{72}{37} \quad \text{and} \quad \frac{4(m-n-3)}{m-1} \geqslant \frac{18}{5}.$$

Choose $N = \|\mu_{\mathcal{E}''}\|_{C^n}^{-1/15}$. Then, we obtain

$$\frac{1}{N}C\|\mu_{\mathcal{E}''}\|^2_{C^{n+2}} \leqslant CM\|\mu_{\mathcal{E}''}\|_{C^n}^{72/37+1/15} = C(M)\|\mu_{\mathcal{E}''}\|_{C^n}^{2+7/555},$$

and

$$C(M)N^{15}\|\mu_{\mathcal{E}''}\|_{C^n}^{\frac{4(m-n-3)}{m-1}} \leqslant C(M)\|\mu_{\mathcal{E}''}\|_{C^n}^{13/5}.$$

Then, due to Corollary 8.5, we conclude

$$\|F\|^2_{H^{n+1}} \leqslant C'(e)\big(3\bar{C}_{q_0}q_0^{2n+3}e^4\|\mu_{\mathcal{E}''}\|^2_{C^n} + C(M)\|\mu_{\mathcal{E}''}\|_{C^n}^{2+7/555}\big).$$

By Sobolev embedding theorem, we have

$$\begin{aligned}\|\mu_{\bar{\mathcal{E}}}\|^2_{C^n} &\leqslant \frac{1}{(1-C_ne^2)^{-2}}\|F\|^2_{C^n} \leqslant \frac{C'_n}{(1-C_ne^2)^{-2}}\|F\|^2_{n+1}\\ &\leqslant \frac{3C'_nC'(e)\bar{C}_{q_0}}{(1-C_ne^2)^{-2}}q_0^{2n+3}e^4\|\mu_{\mathcal{E}''}\|^2_{C^n} + C(M)\|\mu_{\mathcal{E}''}\|_{C^n}^{2+7/555}.\end{aligned}$$

Therefore if

$$\frac{3C'_nC'(e)\bar{C}_{q_0}}{(1-C_ne^2)^{-2}}q_0^{2n+3}e^4 < \frac{1}{16} \tag{9.3}$$

and ε small enough, then we get

$$\|\mu_{\bar{\mathcal{E}}}\|_{C^n} < \frac{1}{2}\|\mu_{\mathcal{E}''}\|_{C^n},$$

which contradicts the minimality of $\|\mu_{\mathcal{E}''}\|_{C^n}$. So $\partial\Omega$ must be an ellipse.
□

Appendix A. Elliptic Polar Coordinates

Consider an ellipse

$$\mathcal{E} = \left\{(x,y) \in \mathbb{R}^2 : \ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right\}, \quad a > b > 0.$$

Associated to $\mathcal{E}$, there exists an elliptic coordinate frame (μ, φ) given by the relations

$$x = c\,\cosh(\mu_0 + \mu)\,\cos\varphi \quad , \quad y = c\,\sinh(\mu_0 + \mu)\,\sin\varphi,$$

where $c = \sqrt{a^2 - b^2}$ is the semifocal distance of $\mathcal{E}$, e denotes its eccentricity and $\mu_0 = \cosh^{-1}(e^{-1})$. Observe that $\mathcal{E}$ in this elliptic coordinates is represented by

$$\{(\mu_0, \varphi) : \ \varphi \in [0, 2\pi)\},$$

Then, any (small) smooth perturbation Ω of the ellipse $\mathcal{E}$ can be written in elliptic coordinates as

$$\partial\Omega = \{(\mu_0 + \mu(\varphi), \varphi) : \ \varphi \in [0, 2\pi)\}.$$

where $\mu(\varphi)$ is a 2π-periodic smooth functions. We will denote

$$\partial\Omega = \mathcal{E} + \mu(\varphi).$$

Lemma A.1. [22, Lemma 35] *Let $\mathcal{E}_{e_0,c} = \mathcal{E}(0,0,c,\mu_0,0)$ be an ellipse of eccentricity $e_0 = 1/\cosh\mu_0$ and semi-focal distance c, and suppose that Ω is a perturbation of $\mathcal{E}_{e_0,c}$, which can be written (in the elliptic coordinate frame (μ,φ) associated to $\mathcal{E}_{e_0,c}$) as $\Omega = \mathcal{E}_{e_0,c} + \mu_\Omega(\varphi)$. Consider another ellipse $\overline{\mathcal{E}}$ sufficiently close to $\mathcal{E}_{e_0,c}$, which can be written (in elliptic coordinates frame associated to $\mathcal{E}_{e_0,c}$) as*

$$\overline{\mathcal{E}} = \mathcal{E}_{e_0,c} + \mu_{\overline{\mathcal{E}}}.$$

If $\overline{\mathcal{E}}$ is sufficiently close to $\mathcal{E}_{e_0,c}$, we can write (in the elliptic coordinate frame $(\overline{\mu},\overline{\varphi})$ associated to $\overline{\mathcal{E}}$) $\Omega = \overline{\mathcal{E}} + \overline{\mu}_\Omega(\overline{\varphi})$, for some function $\overline{\mu}_\Omega$. Then, there exists $C = C(e_0, c, n)$ such that

$$\|\mu_\Omega(\varphi) - (\mu_{\overline{\mathcal{E}}}(\varphi) + \overline{\mu}_\Omega(\varphi))\|_{C^n} \leqslant C\|\mu_{\overline{\mathcal{E}}}\|_{C^n}\|\|\mu_\Omega - \mu_{\overline{\mathcal{E}}}\|_{C^n}. \tag{A.1}$$

In particular, for any $C' > 1$, if $\overline{\mathcal{E}}$ is sufficiently close to $\mathcal{E}_{e_0,c}$ then we have

$$\frac{1}{C'}\|\mu_\Omega - \mu_{\overline{\mathcal{E}}}\|_{C^n} \leqslant \|\overline{\mu}_\Omega\|_{C^n} \leqslant C'\|\mu_\Omega - \mu_{\overline{\mathcal{E}}}\|_{C^n}. \tag{A.2}$$

Remark A.2. Lemma 35 in [22] is stated for C^1-norm. The same arguments also work for C^n-norm, $n = 1, \ldots, m$.

Appendix B. Elliptic Motions in elliptic coordinates

In this section we consider a special class of perturbations of the ellipse $\mathcal{E}_e$ (see also [22, Appendix B]). These perturbations written in the corresponding elliptic coordinates are of the form

$$\partial\Omega = \mathcal{E}_e + \tilde{\mu}(\varphi),$$

with

$$\tilde{\mu}(\varphi) = a_0 + a_1 \cos\varphi + a_{-1}\sin\varphi + a_2\cos 2\varphi + a_{-2}\sin 2\varphi.$$

We show that for this type of perturbations, there exists an ellipse $\bar{\mathcal{E}}$, represented in elliptic coordinates as $\bar{\mathcal{E}} = \mathcal{E}_e + \bar{\mu}(\varphi)$ such that

$$\tilde{\mu}(\varphi) - \bar{\mu}(\varphi) = O(e^2\tilde{\mu}).$$

Let us consider a domain $\mathcal{D} \subset \mathbb{R}^2$ close to $\mathcal{E}_e$,

$$\partial\mathcal{D}: \quad \begin{cases} x = c\ \cosh\big(\mu_0 + \mu(\varphi)\big)\cos\varphi, \\ y = c\ \sinh\big(\mu_0 + \mu(\varphi)\big)\sin\varphi, \end{cases} \qquad \varphi \in [0, 2\pi],$$

where $\mu(\varphi)$ is a smooth 2π-periodic function and we assume $\|\mu\|_{C^1}$ is small enough.

Let us define

$$\begin{aligned} r_\mu(\varphi) &:= (c\cosh(\mu_0 + \mu(\varphi))\cos\varphi)^2 + (c\sinh(\mu_0+\mu(\varphi))\sin\varphi)^2 \\ &= (a\cos\varphi + a\sqrt{1-e^2}\mu(\varphi)\cos\varphi + O(\mu^2))^2 \\ &\quad + (a\sqrt{1-e^2}\sin\varphi + a\mu(\varphi)\sin\varphi + O(\mu^2))^2 \\ &= a^2\cos^2\varphi + 2a^2\sqrt{1-e^2}\mu(\varphi)\cos^2\varphi + O(\mu^2) \\ &\quad + a^2(1-e^2)\sin^2\varphi + 2a^2\sqrt{1-e^2}\mu(\varphi)\sin^2\varphi + O(\mu^2) \\ &= a^2(1-e^2\sin^2\varphi) + 2a^2\sqrt{1-e^2}\mu(\varphi) + O(\mu^2). \end{aligned}$$

Here we have used Taylor's expansion and the fact that

$$c\cosh\mu_0 = a, \quad c\sinh\mu_0 = b = a\sqrt{1-e^2}.$$

B.1. **Homotheties.** For any $\lambda \in \mathbb{R}$, denote the homothety of the ellipse $\mathcal{E}_e$ by

$$\mathcal{E}[\lambda, \mathcal{E}_e] := \exp[\lambda]\mathcal{E}_e.$$

Let $\mu_\lambda(\varphi)$ be the function representing $\mathcal{E}[\lambda, \mathcal{E}_e]$ in the elliptic coordinates of $\mathcal{E}_e$. Then we have

$$\begin{pmatrix} c\ \cosh\big(\mu_0 + \mu_\lambda(\varphi)\big)\cos\varphi \\ c\ \sinh\big(\mu_0 + \mu_\lambda(\varphi)\big)\sin\varphi \end{pmatrix} = \exp[\lambda]\begin{pmatrix} c\ \cosh(\mu_0)\cos\big(\varphi_\lambda(\varphi)\big) \\ c\ \sinh(\mu_0)\sin\big(\varphi_\lambda(\varphi)\big) \end{pmatrix},$$

where $\|\varphi_\lambda(\varphi) - \varphi\|_{C^n} \leqslant C_n\lambda$. For $|\lambda|$ small enough, using Taylor's expansion, denoting $\Delta\varphi_\lambda = \varphi_\lambda - \varphi$, we have

$$\begin{aligned}
r_\lambda(\varphi) :&= (\exp[\lambda]a\cos(\varphi_\lambda))^2 + (\exp[\lambda]a\sqrt{1-e^2}\sin\varphi_\lambda)^2 \\
&= a^2\big(\cos\varphi - \Delta\varphi_\lambda\sin\varphi + \lambda\cos\varphi + O(\lambda^2)\big)^2 \\
&\quad + a^2(1-e^2)\big(\sin\varphi + \Delta\varphi_\lambda\cos\varphi + \lambda\sin\varphi + O(\lambda^2)\big)^2 \\
&= a^2(\cos^2\varphi - 2\Delta\varphi_\lambda\sin\varphi\cos\varphi + 2\lambda\cos^2\varphi + O(\lambda^2)) \\
&\quad + a^2(1-e^2)(\sin^2\varphi + 2\Delta\varphi_\lambda\sin\varphi\cos\varphi + 2\lambda\sin^2\varphi + O(\lambda^2)) \\
&= a^2[1 - e^2\sin^2\varphi + 2\lambda - 2\lambda e^2\sin^2\varphi - e^2\Delta\varphi_\lambda\sin 2\varphi + O(\lambda^2)].
\end{aligned}$$

From $r_{\mu_\lambda}(\varphi) = r_\lambda(\varphi)$, we have

$$\mu_\lambda(\varphi) = \frac{\lambda}{\sqrt{1-e^2}} - \frac{2\lambda e^2\sin^2\varphi - e^2\Delta\varphi\sin 2\varphi}{2\sqrt{1-e^2}} + O(\lambda^2) = \lambda + O(e^2\lambda).$$

B.2. **Translations.** For any $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$, denote the translation of $\mathcal{E}_e$ along the vector α by

$$\mathcal{T}[\alpha, \mathcal{E}_e] := \mathcal{E}_e + \alpha.$$

We look for the function $\mu_\alpha(\varphi)$ that defines $\mathcal{T}[\alpha, \mathcal{E}_e]$ in the elliptic coordinates of the ellipse $\mathcal{E}_e$. Then, we have

$$\begin{pmatrix} c\ \cosh\big(\mu_0 + \mu_\alpha(\varphi)\big)\cos\varphi \\ c\ \sinh\big(\mu_0 + \mu_\alpha(\varphi)\big)\sin\varphi \end{pmatrix} = \begin{pmatrix} c\ \cosh(\mu_0)\cos\big(\varphi_\alpha(\varphi)\big) + \alpha_1 \\ c\ \sinh(\mu_0)\sin\big(\varphi_\alpha(\varphi)\big) + \alpha_2 \end{pmatrix},$$

where

$$\|\varphi_\alpha - \varphi\|_{C^n} \leqslant C_n|\alpha|.$$

For $|\alpha|$ small enough, using Taylor's expansion and denoting

$$\Delta\varphi_\alpha = \varphi_\alpha - \varphi,$$

we have

$$\begin{aligned}
r_\alpha(\varphi) :&= (a\cos\varphi_\alpha + \alpha_1)^2 + (a\sqrt{1-e^2}\sin\varphi_\alpha + \alpha_2)^2 \\
&= (a\cos\varphi - a\Delta\varphi_\alpha\sin\varphi + O(|\alpha|^2) + \alpha_1)^2 \\
&\quad + (b\sin\varphi + b\Delta\varphi_\alpha\cos\varphi + O(|\alpha|^2) + \alpha_2)^2 \\
&= a^2\cos^2\varphi - 2a^2\Delta\varphi_\alpha\cos\varphi\sin\varphi + 2\alpha_1 a\cos\varphi + O(|\alpha|^2) \\
&\quad + b^2\sin^2\varphi + 2b^2\Delta\varphi_\alpha\cos\varphi\sin\varphi + 2b\alpha_2\sin\varphi + O(|\alpha|^2) \\
&= a^2 - a^2e^2\sin^2\varphi + 2\alpha_1 a\cos\varphi + 2\alpha_2 b\sin\varphi - a^2e^2\Delta\varphi_\alpha\sin 2\varphi + O(|\alpha|^2).
\end{aligned}$$

Since $r_{\mu_\alpha} = r_\alpha$, we obtain

$$\begin{aligned}\mu_\alpha(\varphi) &= \frac{\alpha_1 \cos\varphi + \alpha_2\sqrt{1-e^2}\sin\varphi - \frac{1}{2}ae^2\sin 2\varphi + O(|\alpha|^2)}{a\sqrt{1-e^2}} \\ &= \frac{\alpha_1}{a}\cos\varphi + \frac{\alpha_2}{a}\sin\varphi + O(e^2|\alpha|).\end{aligned}$$

B.3. **Hyperbolic rotations.** For any $\beta = (\beta_1, \beta_2)$, let us denote by $\mathcal{H}[\beta, \mathcal{E}_e]$ the ellipse obtained by applying to $\mathcal{E}_e$ the hyperbolic rotation generated by the linear map

$$\mathcal{H}[\beta] = \exp\begin{pmatrix} \beta_1 & \beta_2 \\ \beta_2 & -\beta_1 \end{pmatrix} = \begin{pmatrix} 1+\beta_1 & \beta_2 \\ \beta_2 & 1-\beta_1 \end{pmatrix} + O(|\beta|^2).$$

Let $\mu_\beta(\varphi)$ be the function that defines $\mathcal{H}[\beta, \mathcal{E}_e]$ in the elliptic coordinates of $\mathcal{E}_e$. Then we have

$$\begin{aligned}&\begin{pmatrix} c\ \cosh\big(\mu_0 + \mu_\beta(\varphi)\big)\cos\varphi \\ c\ \sinh\big(\mu_0 + \mu_\beta(\varphi)\big)\sin\varphi \end{pmatrix} \\ &= \mathcal{H}[\beta]\begin{pmatrix} c\ \cosh(\mu_0)\cos\big(\varphi_\beta(\varphi)\big) \\ c\ \sinh(\mu_0)\sin\big(\varphi_\beta(\varphi)\big) \end{pmatrix} \\ &= \begin{pmatrix} (1+\beta_1)a\cos\varphi_\beta + \beta_2 b\sin\varphi_\beta \\ \beta_2 a\cos\varphi_\beta + (1-\beta_1)b\sin\varphi_\beta \end{pmatrix} + O(|\beta|^2),\end{aligned}$$

where $\|\varphi_\beta - \varphi\|_{C^n} \leqslant C_n|\beta|$. For $|\beta|$ small enough, using Taylor's expansion and denoting $\Delta\varphi_\beta = \varphi_\beta - \varphi$, we have

$$\begin{aligned}r_\beta(\varphi) :&= [(1+\beta_1)a\cos\varphi_\beta + \beta_2 b\sin\varphi_\beta]^2 \\ &\quad + [\beta_2 a\cos\varphi_\beta + (1-\beta_1)b\sin\varphi_\beta]^2 + O(|\beta|^2) \\ &= [a\cos\varphi - a\Delta\varphi_\beta\sin\varphi + \beta_1 a\cos\varphi + \beta_2 b\sin\varphi + O(|\beta|^2)]^2 \\ &\quad + [\beta_2 a\cos\varphi + b\sin\varphi + b\Delta\varphi_\beta\cos\varphi - b\beta_1\sin\varphi + O(|\beta|^2)]^2 \\ &= a^2\cos^2\varphi - a^2\Delta\varphi_\beta\sin 2\varphi + 2\beta_1 a^2\cos^2\varphi + \beta_2 ab\sin 2\varphi \\ &\quad + b^2\sin^2\varphi + ab\beta_2\sin 2\varphi + b^2\Delta\varphi_\beta\sin 2\varphi - 2b^2\beta_1\sin^2\varphi + O(|\beta|^2) \\ &= a^2 - a^2e^2\sin^2\varphi + 2ab\beta_2\sin 2\varphi + 2a^2\beta_1\cos 2\varphi \\ &\quad + 2a^2e^2\sin^2\varphi + a^2e^2\Delta_\beta\sin 2\varphi + O(|\beta|^2).\end{aligned}$$

From $r_{\mu_\beta}(\varphi) = r_\beta(\varphi)$, we have

$$\begin{aligned}\mu_\beta &= \frac{2ab\beta_2\sin 2\varphi + 2a^2\beta_1\cos 2\varphi + 2a^2e^2\sin^2\varphi + a^2e^2\Delta_\beta\sin 2\varphi + O(|\beta|^2)}{2a^2\sqrt{1-e^2}} \\ &= \beta_1\cos 2\varphi + \beta_2\sin 2\varphi + O(e^2|\beta|).\end{aligned}$$

To sum up, combining with Lemma A.1, we obtain the following result.

Lemma B.1. *Let $\mathcal{E}_e$ be an ellipse of eccentricity $e \in (0, \frac{1}{2})$, and Ω be a small perturbation of $\mathcal{E}_e$, which written in the elliptic coordinates associated to $\mathcal{E}$ as*

$$\mu(\varphi) = a_0 + a_1 \cos\varphi + a_{-1}\sin\varphi + a_2\cos 2\varphi + a_{-2}\sin 2\varphi.$$

Assume $\|\mu\|_{C^n}$ small enough for some $n \geqslant 2$, then there exists C_n, independent of the eccentricity e and μ, and an ellipse $\bar{\mathcal{E}}$,

$$\bar{\mathcal{E}} = \mathcal{E}_e + \bar{\mu}(\varphi),$$

such that

$$\|\mu - \bar{\mu}\|_{C^n} \leqslant C_n e^2 \|\mu\|_{C^n}.$$

Appendix C. Expansion with respect to e

The action-angle parametrisation θ of the elliptic coordinate φ corresponding to the caustic C_λ, expanded up to order $O(e^{2N+2})$, $J \in \mathbb{N}$ is as follows:

$$\varphi(\theta, \lambda, e) = \theta + \sum_{j=1}^{N} \varphi_j(\theta) \frac{a^j e^{2j}}{(a^2 - \lambda^2)^j} + O(e^{2N+2}), \tag{C.1}$$

where the functions $\varphi_N(\theta)$ are of the form

$$\varphi_j(\theta) = \sum_{l=1}^{j} \beta_{j,l} \sin 2l\theta.$$

We give below the explicit formulae for $\varphi_N(\theta)$ for $j = 1, \ldots, 6$.

$$\varphi_1(\theta) = \frac{1}{8}\sin 2\theta,$$

$$\varphi_2(\theta) = \frac{1}{256}(16\sin 2\theta + \sin 4\theta),$$

$$\varphi_3(\theta) = \frac{83\sin 2\theta}{2048} + \frac{\sin 4\theta}{256} + \frac{\sin 6\theta}{6144},$$

$$\varphi_4(\theta) = \frac{121\sin 2\theta}{4096} + \frac{29\sin 4\theta}{8192} + \frac{\sin 6\theta}{4096} + \frac{\sin 8\theta}{131072},$$

$$\varphi_5(\theta) = \frac{12071\sin 2\theta}{524288} + \frac{13\sin 4\theta}{4096} + \frac{37\sin 6\theta}{131072} + \frac{\sin 8\theta}{65536} + \frac{\sin 10\theta}{2621440},$$

$$\varphi_6(\theta) = \frac{19651\sin 2\theta}{1048576} + \frac{47955\sin 4\theta}{16777216} + \frac{235\sin 6\theta}{786432} + \frac{45\sin 8\theta}{2097152} + \frac{\sin 10\theta}{1048576} + \frac{\sin 12\theta}{50331648}.$$

Lemma C.1. *Let*

$$\mu(\varphi) = a_0 + \sum_{k=1}^{+\infty} a_k \cos k\varphi + b_k \sin k\varphi,$$

and $\mu(\varphi) \in C^m(\mathbb{T})$. *Then for* $N \leqslant m-1$, *the expansion of the function* $\mu(\varphi(\theta, \lambda, e))$ *with respect to* e *up to order* $O(e^{2N+2})$ *is*

$$\mu(\varphi(\theta, \lambda, e)) = \mu(\theta) + \sum_{j=1}^{N} P_j(\theta) \frac{a^j e^{2j}}{(a^2 - \lambda^2)^j} + O(e^{2N+2} \|\mu\|_{C^{N+1}}) \quad \text{(C.2)}$$

where the functions $P_j(\theta)$ *are of the form*

$$P_j(\theta) = \sum_{k=1}^{+\infty} \sum_{l=-j}^{j} \xi_{j,l}(k)(a_k \cos(k+2l)\theta + b_k \sin(k+2l)\theta).$$

The coefficients $\xi_{j,l}(k)$ *can be explicitely computed and, for small* j *and* l, *they are presented below.*

The functions $P_j(\theta)$, $j = 1, \ldots, 6$ are explicitly given by

$$P_1(\theta) = \mu'(\theta)\varphi_1(\theta) = \sum_{k=1}^{+\infty} \sum_{l=-1}^{1} \xi_{1,l}(k)(a_k \cos(k+2l)\theta + b_k \sin(k+2l)\theta),$$

where

$$\xi_{1,-1}(k) = \frac{-k}{16}, \quad \xi_{1,0}(k) = 0, \quad \text{and} \quad \xi_{1,1}(k) = \frac{k}{16}.$$

$$\begin{aligned} P_2(\theta) &= \mu'(\theta)\varphi_2(\theta) + \frac{1}{2}\mu''(\theta)(\varphi_1(\theta))^2 \\ &= \sum_{k=1}^{+\infty} \sum_{l=-2}^{2} \xi_{2,l}(k)(a_k \cos(k+2l)\theta + b_k \sin(k+2l)\theta), \end{aligned}$$

where

$$\xi_{2,-2}(k) = \frac{-k+k^2}{512}, \quad \xi_{2,-1}(k) = \frac{-16k}{512}, \quad \xi_{2,0} = \frac{-2k^2}{512},$$

$$\xi_{2,1}(k) = \frac{16k}{512}, \quad \xi_{2,2}(k) = \frac{k^2+k}{512}.$$

$$\begin{aligned} P_3(\theta) =& \mu'(\theta)\varphi_3(\theta) + \frac{2}{2}\mu''(\theta)\varphi_1(\theta)\varphi_2(\theta) + \frac{1}{6}\mu'''(\theta)(\varphi_1)^3 \\ =& \sum_{k=1}^{+\infty} \sum_{l=-3}^{3} \xi_{3,l}(k)(a_k \cos(k+2l)\theta + b_k \sin(k+2l)\theta), \end{aligned}$$

where

$$\xi_{3,-3}(k) = \frac{-k}{12288} + \frac{k^2}{8192} - \frac{k^3}{24576},$$

$$\xi_{3,-2}(k) = \frac{-k}{512} + \frac{k^2}{512}, \quad \xi_{3,-1}(k) = \frac{-83k}{4096} - \frac{k^2}{8192} + \frac{k^3}{8192},$$

$$\xi_{3,0}(k) = \frac{-k^2}{256}, \quad \xi_{3,1}(k) = \frac{83k}{4096} - \frac{k^2}{8192} - \frac{k^3}{8192}, \quad \xi_{3,2}(k) = \frac{k+k^2}{512},$$

$$\xi_{3,3}(k) = \frac{k}{12288} + \frac{k^2}{8192} + \frac{k^3}{24576}.$$

$$\begin{aligned} P_4(\theta) &= \mu'(\theta)\varphi_4(\theta) + \frac{1}{2}\mu''(\theta)[(\varphi_2(\theta))^2 + 2\varphi_1(\theta)\varphi_3(\theta)] \\ &\quad + \frac{1}{6}\mu'''(\theta)3(\varphi_1(\theta))^2\varphi_2(\theta) + \frac{1}{24}\mu^{(4)}(\theta)(\varphi_1(\theta))^4 \\ &= \sum_{k=1}^{+\infty}\sum_{l=-4}^{4}\xi_{4,l}(k)(a_k\cos(k+2l)\theta + b_k\sin(k+2l)\theta), \end{aligned}$$

where $\xi_{4,j}(k)$, $j = -4, \dots, 4$ are polynomials in k of at most order 4, and

$$\xi_{4,4}(k) = \frac{k}{262144} + \frac{11k^2}{1572864} + \frac{k^3}{262144} + \frac{k^4}{1572864}.$$

$$\begin{aligned} P_5(\theta) &= \mu'(\theta)\varphi_5(\theta) + \frac{2}{2}\mu''(\theta)[\varphi_2(\theta)\varphi_2(\theta) + \varphi_1(\theta)(\varphi_2(\theta))^2] \\ &\quad + \frac{3}{6}\mu'''(\theta)[\varphi_1(\theta)(\varphi_2(\theta))^2 + (\varphi_1(\theta))^2\varphi_3(\theta)] \\ &\quad + \frac{4}{24}\mu^{(4)}(\varphi_1(\theta))^3\varphi_2(\theta) + \frac{1}{120}\mu^{(5)}(\theta)(\varphi_1(\theta))^5 \\ &= \sum_{k=1}^{+\infty}\sum_{l=-5}^{5}\xi_{5,l}(k)(a_k\cos(k+2l) + b_k\sin(k+2l)\theta), \end{aligned}$$

where $\xi_{5,j}$, $j = -5, \dots, 5$ are polynomials in k of at most order 5, and

$$\xi_{5,5}(k) = \frac{k}{5242880} + \frac{5k^2 + k^4}{12582912} + \frac{7k^3}{25165824} + \frac{k^5}{125829120}.$$

$$
\begin{aligned}
P_6(\theta) &= \mu'(\theta)\varphi_6(\theta) + \frac{1}{2}\mu''(\theta)[2\varphi_1(\theta)\varphi_5(\theta) + 2\varphi_2(\theta)\varphi_4(\theta) + (\varphi_3(\theta))^2] \\
&\quad + \frac{1}{6}\mu'''(\theta)[3(\varphi_1(\theta))^2\varphi_4(\theta) + 3\varphi_1(\theta)\varphi_2(\theta)\varphi_3(\theta) + (\varphi_2(\theta))^3] \\
&\quad + \frac{1}{24}\mu^{(4)}(\theta)[4(\varphi_1(\theta))^3\varphi_3(\theta) + 6(\varphi_1(\theta))^2(\varphi_2(\theta))^2] \\
&\quad + \frac{1}{120}\mu^{(5)}(\theta)[5(\varphi_1(\theta))^4\varphi_2(\theta)] + \frac{1}{720}\mu^{(6)}(\theta)[(\varphi_1(\theta))^6] \\
&= \sum_{k=1}^{+\infty}\sum_{l=-6}^{6} \xi_{6,l}(k)(a_k\cos(k+2l) + b_k\sin(k+2l)\theta),
\end{aligned}
$$

where $\xi_{6,j}(k)$ are polynomials in k of at most order 6, and

$$
\xi_{6,6}(k) = \frac{k}{100663296} + \frac{137k^2}{6039797760} + \frac{11k^3 + k^5}{805306368} + \frac{17k^4}{2415919104} + \frac{k^6}{12079595520}.
$$

Appendix D. The inverse and adjugate of a matrix

We recall the definition of the adjugate of a matrix and its relation to the inverse of a square matrix in this section.

Let A be a $n \times n$ matrix with real entries. The adjugate $\mathrm{adj}(A)$ of A is the transpose of the cofactor matrix C of A,

$$
\mathrm{adj}(A) = C^T.
$$

The cofactor matrix of A is the $n \times n$ matrix C whose (i, j)-entry is the (i, j)-cofactor of A,

$$
C_{ij} = (-1)^{i+j} M_{ij},
$$

where $M_{i,j}$ is the determinant of the $(n-1) \times (n-1)$ matrix that results from deleting the row i and the column j of A. Therefore, the adjugate of matrix A is the $n \times n$ matrix adj(A) whose (i, j)-entry is the (j, i)-cofactor of A,

$$
\mathrm{adj}(A)_{ij} = C_{ji} = (-1)^{j+i} M_{ji}.
$$

Theorem D.1. *For a square matrix $A = (a_{ij})$,*

1. *$det(A) = \sum_{i=1}^{n} a_{ij} C_{ij}$, for $j = 1, \ldots, n$.*
2. *A is invertible if and only if $det(A) \neq 0$. Moreover, the inverse has the form*

$$
A^{-1} = \frac{1}{\det(A)}\, \mathrm{adj}(A).
$$

Now consider the coefficient matrix in (6.4), which has the form

$$A = (a_{ij}) = \begin{pmatrix} A_{11}e^4 & A_{12}e^2 & 1 & 0 & 0 & 0 \\ A_{21}e^6 & A_{22}e^4 & A_{23}e^2 & 1 & 0 & 0 \\ A_{31}e^8 & A_{32}e^6 & A_{33}e^4 & A_{34}e^2 & 1 & 0 \\ A_{41}e^8 & A_{42}e^6 & A_{43}e^4 & A_{44}e^2 & 1 & 0 \\ A_{51}e^{10} & A_{52}e^8 & A_{53}e^6 & A_{54}e^4 & A_{55}e^2 & 1 \\ A_{61}e^{10} & A_{62}e^8 & A_{63}e^6 & A_{64}e^4 & A_{65}e^2 & 1 \end{pmatrix}$$

Direct calculations show that

$$\det(A) = \sum_{\sigma \in S_6} \operatorname{sgn}(\sigma) \prod_{i=1}^{6} a_{i\sigma_i} = \sum (\cdots) e^{16} = \mathcal{A} e^{16},$$

where S_6 is the set formed by all the permutation of $\{1, \ldots, 6\}$. One key feature here is that the nonzero quantities in the summation are all exactly of order e^{16}. Using part (1) of Theorem D.1, we obtain that

$$C_{11} = c_1 e^{12},\ C_{12} = c_2 e^{10},\ C_{13} = c_3 e^8,\ C_{14} = c_4 e^8,\ C_{15} = c_5 e^6,\ C_{16} = c_6 e^6.$$

Then using part (2) of Theorem D.1, if $\det A \neq 0$, then the first row of the inverse A^{-1} has the form

$$(O(e^{-4}), O(e^{-6}), O(e^{-8}), O(e^{-8}), O(e^{-10}), O(e^{-10})).$$

In the same way, we obtain that the second row of A^{-1} is of the form

$$(O(e^{-2}), O(e^{-4}), O(e^{-6}), O(e^{-6}), O(e^{-8}), O(e^{-8})).$$

All the matrices appear in Sections 4–7 could be treated similarly.

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