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Local-in-space estimates near initial time for weak solutions of the Navier-Stokes equations and forward self-similar solutions

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Abstract We show that the classical Cauchy problem for the incompressible 3d Navier-Stokes equations with (-1)-homogeneous initial data has a global scale-invariant solution which is smooth for positive times. Our main technical tools are local-in-space regularity estimates near the initial time, which are of independent interest.

1 Introduction

We consider the classical Cauchy problem for the incompressible Navier-Stokes equation in $R^3 \times (0, \infty)$

$$\left.\begin{aligned} u_t + u\nabla u + \nabla p - \Delta u = 0 \\ \operatorname{div} u = 0 \end{aligned}\right\} \quad \text{in } R^3 \times (0, \infty), \tag{1.1}$$

$$u|_{t=0} = u_0 \quad \text{in } R^3. \tag{1.2}$$

We recall that the problem is invariant under the scaling

$$\begin{aligned} u(x,t) &\to u_\lambda(x,t) = \lambda u\big(\lambda x, \lambda^2 t\big), \\ p(x,t) &\to p_\lambda(x,t) = \lambda^2 p\big(\lambda x, \lambda^2 t\big), \\ u_0(x) &\to u_{0\lambda}(x) = \lambda u_0(\lambda x), \end{aligned} \tag{1.3}$$

where $\lambda > 0$. We say that a solution u is *scale-invariant* if $u_\lambda = u$ and $p_\lambda = p$ for each $\lambda > 0$. Similarly, we say that an initial condition u_0 is scale-invariant,

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if $u_{0\lambda} = u_0$ for each $\lambda > 0$. This is of course the same as requiring that u_0 be (-1)-homogeneous.

One of our goals in this paper is to give a proof of the following result.

Theorem 1.1 *Assume u_0 is scale-invariant and locally Hölder continuous in $R^3 \setminus \{0\}$ with $\operatorname{div} u_0 = 0$ in R^3. Then the Cauchy problem (1.1), (1.2) has at least one scale-invariant solution u which is smooth in $R^3 \times (0, \infty)$ and locally Hölder continuous in $R^3 \times [0, \infty) \setminus \{(0, 0)\}$.*

Previously this result has been known only under suitable smallness conditions on u_0, see for example [5, 15]. For small u_0 one can also prove uniqueness (in suitable function classes). It is quite conceivable that uniqueness may fail for large data. We will comment on this point in more detail below.

The second important theme of our paper can be perhaps called local-in-space regularity estimates near the initial time $t = 0$. It is known that if $u_0 \in L^q$ for $q \geq 3$, then the initial value problem (1.1), (1.2) has a unique local-in-time "mild solution" defined on some time interval $(0, T)$, which is smooth in $R^3 \times (0, T)$ and has in many respects the same regularity as the solution of the heat equation in $R^3 \times [0, T)$ for times close to $t = 0$, see for example [6, 10]. A natural question is under which condition this result can be localized in space: if u_0 is a quite general initial condition for which a generalized suitable weak solution u in the sense of [18] is defined and $u_0|_{B_r} \in L^q(B_r)$ for some $q > 3$, say, can we conclude that u is regular in $B_{\frac{r}{2}} \times [0, t_1)$ for some time $t_1 > 0$? We prove that this is indeed the case under quite general assumptions, which include u_0 which is in L^2_{loc} and $\int_{B_{x,r}} |u_0|^2\, dx \to 0$ for $x \to \infty$. Due to non-local effects of the pressure the solution u in $B_{\frac{r}{2}} \times [0, t_1)$ may not have the same amount of regularity as the solution of the heat equation in this situation, but the non-local effects are limited to the influence of the "harmonic part of the pressure" in a suitable pressure decomposition. We can formulate this type of results somewhat loosely in the following statement.

(S) *Modulo the usual (and quite mild) non-local influences of the pressure, local regularity of the initial data propagates for at least a short time.*

We refer the reader to Sect. 3 for precise statements. Statement (S), in addition to being of independent interest, is one of the main ingredients of our proof of Theorem 1.1.

Results in the direction of (S) can be found already in the classical paper [3]. More recently, related questions about vorticity propagation have been studied in [24]. Our main result concerning (S), Theorem 3.1, takes a somewhat different angle on (S).

Our proof of (S) (see also Theorem 3.1) is based on a combination of techniques from [12, 16–18, 23]. Heuristically, the main point is that one can obtain a sufficient control of the energy flux into “good regions” from the rest of the space, see Sect. 3. Once we know that only small amount of energy can move into the “good region” one can use (a slight modification of) partial regularity schemes in [16, 17] to prove regularity.

To prove Theorem 1.1, we seek the solution $u(x,t)$ in the form

$$u(x,t)=\frac{1}{\sqrt{t}}U\left(\frac{x}{\sqrt{t}}\right). \tag{1.4}$$

The Navier-Stokes equation for u gives

$$-\Delta U-\frac{1}{2}U-\frac{1}{2}x\nabla U+U\nabla U+\nabla P=0, \quad \operatorname{div} U=0, \tag{1.5}$$

in R^3. For a scale-invariant u_0 the problem of finding a scale-invariant solution of the Cauchy problem (1.1), (1.2) is equivalent to the problem of finding a solution of (1.5) with the asymptotics

$$\left|U(x)-u_0(x)\right|=o\left(\frac{1}{|x|}\right), \quad x\to\infty. \tag{1.6}$$

The problem (1.5), (1.6) is reminiscent of the classical Leray’s problem of finding steady-state solution of the Navier-Stokes equation in a bounded domain, with given boundary conditions for U. We will show that one can solve this problem using the Leray-Schauder degree theory, just as in the case of the bounded domain. The non-trivial part is to find the right functional-analytic setup and establish the necessary a-priori estimates. The main difficulty is to find good estimates near ∞. This difficulty will be overcome by applying statement (S) above. Heuristically it is clear that when u is given by (1.4), then estimates of u near $t=0$ are closely related to estimates of U near ∞. In Sect. 4 we will make this more precise.

As in the case of the bounded domains, the Leray-Schauder approach gives existence of the solutions, but not uniqueness. In the case of bounded domains one does not generically expect uniqueness for large data, and this non-uniqueness is in fact expected to be quite typical in the context of the steady Navier-Stokes, once the data is large. Could this also be the case for the problem (1.5), (1.6)? This would lead to non-uniqueness for the Cauchy problem (1.1), (1.2) with scale-invariant u_0, and by a suitable truncation of u_0 at large $|x|$ possibly also to non-uniqueness for the Leray-Hopf solutions of the Cauchy problem for $u_0\in L^2$. We plan to address these issues in future work.

Our paper is organized as follows: in Sect. 2, we prove an ‘ϵ-regularity’ criteria for a generalized Navier-Stokes system; in Sect. 3, we study the local in space near initial time smoothness of Leray solutions; in Sect. 4 we study the asymptotics of forward self-similar solutions to Stokes and Navier-Stokes equations; in Sect. 5 we prove the existence of forward self-similar solutions for large -1 homogeneous initial data.

Notation We use standard notations in our paper. For instance, $B_R(x_0)$ denotes a ball centered at x_0 with radius R in R^3, $B_R := B_R(0)$; for $z_0 = (x_0, t_0)$, $Q(R, z_0) := B_R(x_0) \times (t_0 - R^2, t_0]$, $Q_R := Q(R, (0,0))$; for any f in $\mathcal{O}$, $\fint_{\mathcal{O}} f := \frac{1}{|\mathcal{O}|}\int_{\mathcal{O}} f$. We also use the following standard notations in the literature: for vectors a and v, $a \otimes v$ is a matrix with $(a \otimes v)_{ij} = a_i v_j$; for two matrices a, b, $(a : b) = a_{ij} b_{ij}$ where we assume the usual Einstein summation convention; for a matrix valued function $f = (f_{ij})$, $\operatorname{div} f$ is a vector with $(\operatorname{div} f)_i = (\sum_j \partial_j f_{ij})$; $(u)_{Q(R,z_0)} := \fint_{Q(R,z_0)} u\,dz$, $(u)_r := (u)_{Q_r}$; $(p)_{B_R(x_0)}(t) := \fint_{B_{R(x_0)}} p(x,t)dx$, $(p)_r(t) := (p)_{B_r}(t)$; $Y(u, p, Q(R, z_0)) := (\fint_{Q(R,z_0)} |u - (u)_{Q(R,z_0)}|^3 dz)^{1/3} + R(\fint_{Q(R,z_0)} |p - (p)_{B_R(x_0)}(t)|^{3/2} dz)^{2/3}$; $Y(u, p, Q_R) := Y(u, p, Q(R, (0,0)))$. We use C to denote an absolute and often large positive number, c a positive small absolute number, ϵ the positive small numbers, $C(\alpha, \beta, \dots)$ when the number depends on the parameters $\alpha, \beta, \dots$. $C^{\alpha}_{\text{par}}(\mathcal{O})$ denotes the Hölder space with respect to the parabolic distance when $\mathcal{O}$ is a space time domain. We adopt the convention that nonessential constants can change from line to line. We use u_0 as a divergence free initial data throughout the paper, unless defined otherwise.

2 ϵ-Regularity criteria

Our goal in this section is to prove an ϵ-regularity criteria similar to that of Caffarelli-Kohn-Nirenberg for a generalized Navier-Stokes equation. Our setting is as follows:
Let $\mathcal{O}$ be an open subset of $R^3_x \times R_t$, $a \in L^m_{loc}(\mathcal{O})$ with $m > 5$ (not necessarily an integer), $\operatorname{div} a = 0$. We call a pair of functions (u, p) suitable weak solution to

$$\left.\begin{aligned} \partial_t u - \Delta u + a \cdot \nabla u + \operatorname{div}(a \otimes u) + u \cdot \nabla u + \nabla p = 0 \\ \operatorname{div} u = 0 \end{aligned}\right\} \quad (x,t) \in \mathcal{O}, \tag{2.1}$$

if $u \in L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x(\mathcal{O}')$ for any open subset $\mathcal{O}' \subseteq \overline{\mathcal{O}'} \Subset \mathcal{O}$, $p \in L^{3/2}_{loc}(\mathcal{O})$, such that (u, p) satisfies equations (2.1) in the sense of distributions in $\mathcal{O}$, and

$$\begin{aligned} &\partial_t \frac{|u|^2}{2} - \Delta \frac{|u|^2}{2} + |\nabla u|^2 + \operatorname{div}\left(\frac{|u|^2}{2}(u + a)\right) + u \operatorname{div}(a \otimes u) + \operatorname{div}(up) \\ &\le 0, \end{aligned} \tag{2.2}$$

in the sense of distributions. Recall that a distribution v in $\mathcal{O}$ is called non-negative if $(v, \phi) \geq 0$ for any $\phi \in C_c^\infty(\mathcal{O})$ with $\phi \geq 0$; $u\,\mathrm{div}(a \otimes u)$ is a distribution with

$$\big(u\,\mathrm{div}(a \otimes u), \phi\big) = -\int_{\mathcal{O}} a_i u_j \partial_j u_i \phi(x,t)dxdt - \int_{\mathcal{O}} a_i u_j u_i \partial_j \phi(x,t)dxdt.$$

The terms in (2.2) make sense due to the regularity assumptions and $u \in L^{10/3}_{loc}(\mathcal{O})$ by known multiplicative inequalities.

The main theorem in this section can be stated as the following:

Theorem 2.1 (ϵ-regularity criterion) *Let* (u, p) *be a suitable weak solution to Eq.* (2.1) *in* Q_1 *with* $a \in L^m(Q_1)$, $m > 5$, $\mathrm{div}\, a = 0$. *Then there exists* $\epsilon_0 = \epsilon_0(m) > 0$ *with the following property*: *if*

$$\left(\fint_{Q_1} |u|^3 dxdt\right)^{1/3} + \left(\fint_{Q_1} |p|^{3/2} dxdt\right)^{2/3} + \left(\fint_{Q_1} |a|^m dxdt\right)^{1/m} \leq \epsilon_0, \tag{2.3}$$

then u *is Hölder continuous in* $Q_{1/2}$ *with exponent* $\alpha = \alpha(m) > 0$ *and*

$$\|u\|_{C^{\alpha}_{\mathrm{par}}(Q_{1/2})} \leq C(m, \epsilon_0). \tag{2.4}$$

Remarks The proof of this theorem follows the general line of presentation in [7, 16, 17]. There are some additional complications due to the new terms $a \cdot \nabla u + \mathrm{div}(a \otimes u)$ as we shall see below. As pointed out by a referee, with only minor changes the proof could be carried out for a more general version of (2.1), with $a\nabla u + \mathrm{div}(a \otimes u)$ replaced by $a\nabla u + \mathrm{div}(b \otimes u)$, where b satisfies the same assumptions as a.

Before going into the proof of the theorem, we need the following two lemmas to be used below.

Lemma 2.1 *Let* f *be a nonnegative nondecreasing bounded function defined on* $[0, 1]$ *with the following property*:
for any $3/4 \leq s < t < 1$ *and some positive constants* $0 < \theta < 1$, $M > 0$, $\beta > 0$, *we have*

$$f(s) \leq \theta f(t) + \frac{M}{(t-s)^\beta}. \tag{2.5}$$

Then,

$$\sup_{s \in [0,3/4]} f(s) \leq C(\theta, \beta, M), \tag{2.6}$$

for some positive constant depending only on θ, β, M.

Remarks The lemma is well-known, one can find a proof for example in [8].

Our next lemma is an estimate of a generalized Stokes system.

Lemma 2.2 *Let $a \in L^m(Q_1)$, with $\operatorname{div} a = 0$ and $(\int_{Q_1} |a|^m dxdt)^{1/m} \le M$, for some positive $M > 0$, $m > 5$, let $\lambda \in R^n$, $|\lambda| \le M$, $f = (f_{ij}) \in L^m(Q_1)$ with $(\int_{Q_1} |f|^m dxdt)^{1/m} \le M$. Let $u \in L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1(Q_1)$ and $p \in L^{3/2}(Q_1)$ with*

$$\left(\int_{Q_1} |u|^3 dxdt\right)^{1/3} + \left(\int_{Q_1} |p|^{3/2} dxdt\right)^{2/3} \le M. \tag{2.7}$$

Assume (u, p) satisfies

$$\left.\begin{aligned} \partial_t u - \Delta u + a \cdot \nabla u + \lambda \cdot \nabla u + \operatorname{div}(a \otimes u) + \nabla p = \operatorname{div} f \\ \operatorname{div} u = 0 \end{aligned}\right\} \quad \text{in } Q_1, \tag{2.8}$$

in the sense of distributions. Then u is Hölder continuous in $Q_{1/2}$ with exponent $\alpha = \alpha(m) > 0$ and

$$\|u\|_{C^\alpha_{\text{par}}(Q_{1/2})} \le C(m, M). \tag{2.9}$$

Proof The proof is based on standard bootstrapping procedure using elliptic and parabolic estimates. We first show that if $u \in L^q(Q_R)$, $q \ge 3$, then $u \in L^{\tilde{q}}(Q_{R-\delta})$, for each $\tilde{q} \ge 1$ with $\frac{1}{\tilde{q}} \ge \frac{1}{q} - \frac{1}{2}(\frac{1}{5} - \frac{1}{m})$. This improves the regularity of u, and after finitely many such bootstrapping steps we can conclude that $u \in L^q$ (on a slightly smaller ball) for any given q (which can be as large as we wish). Another bootstrapping step then gives that u is in a Hölder space (on a slightly smaller ball). We now sketch some of the details. We can assume $R > 3/4$ and δ is a small positive number. Let us rewrite the equations of (u, p) as

$$\left.\begin{aligned} \partial_t u - \Delta u + \nabla p = \operatorname{div}(f - a \otimes u - u \otimes a - u \otimes \lambda) \\ \operatorname{div} u = 0 \end{aligned}\right\} \quad \text{in } Q_R.$$

By Hölder inequality, we see $h := f - a \otimes u - u \otimes a - u \otimes \lambda \in L^{\frac{mq}{m+q}}(Q_R)$. Taking divergence in the first equation, we obtain

$$\Delta p = \operatorname{div}\operatorname{div}(f - a \otimes u - u \otimes a - u \otimes \lambda) \quad \text{in } Q_R.$$

Set

$$p_1 = \Delta^{-1} \operatorname{div}\operatorname{div}\big((f - a \otimes u - u \otimes a - u \otimes \lambda)\chi_{B_R}\big),$$

and write $p = p_1 + p_2$. Then $\Delta p_2 = 0$ in Q_R. Recall that R is in [3/4, 1]. By elliptic estimates, we get

$$\|p_1\|_{L^{\frac{mq}{m+q}}(Q_R)} \leq C\|h\|_{L^{\frac{mq}{m+q}}(Q_R)}.$$

Since $\frac{mq}{m+q} > 3/2$, we see p_2 verifies estimate

$$\|p_2\|_{L_t^{3/2}C_x^2(Q_{R-\delta/2})} \leq C(\delta, M),$$

with δ being a small positive number. Then,

$$\partial_t u - \Delta u = -\nabla p_1 - \nabla p_2 + \operatorname{div} h \quad \text{in } Q_{R-\delta/2},$$

where p_1, p_2 and h satisfy above estimates. For a smooth cutoff function η with $\eta \equiv 1$ in $Q_{R-3\delta/4}$ and $\eta \equiv 0$ outside $Q_{R-\delta/2}$, set

$$v_1(\cdot, t) = \int_{-\infty}^{t} e^{\Delta(t-s)}\big[-\nabla(p_1\eta)(\cdot, s) + \operatorname{div}(h\eta)(\cdot, s)\big]ds,$$

$$v_2(\cdot, t) = -\int_{-\infty}^{t} e^{\Delta(t-s)}\nabla(p_2\eta)(\cdot, s)ds.$$

Write $u = v_1 + v_2 + v_3$. By estimates of heat equation, we see $\|v_2\|_{L^\infty(Q_{R-\delta})} \leq C(\delta, M)$. As for v_1, by Young's inequality and the properties of heat kernel, we see $v_1 \in L^r(Q_{R-\delta})$ for any $r > 0$ such that

$$\frac{1}{r} > \frac{1}{q} + \frac{1}{m} - \frac{1}{5}.$$

Since v_3 satisfies heat equation in $Q_{R-3\delta/4}$, we see v_3 is smooth in $Q_{R-\delta}$. Thus in summary, we get, $u \in L^q(Q_R)$ implies $u \in L^r(Q_{R-\delta})$ for $r \geq 1$ with $\frac{1}{r} \geq \frac{1}{q} - \frac{1}{2}(\frac{1}{5} - \frac{1}{m})$.

Since $m > 5$, after applying this bootstrapping argument for finitely many times, we can conclude $u \in L^{r_0}(Q_{5/8})$ with r_0 sufficiently large such that

$$|a||u| \in L^{\frac{m+5}{2}}(Q_{5/8}). \tag{2.10}$$

Then we can go back to the decompositions v_1, v_2, and v_3, it is not difficult to verify that all of them are Hölder continuous in $Q_{1/2}$ with exponent $\alpha = \alpha(m)$. If we keep track of the constants in the above process, it's clear we also have the estimates claimed in the lemma. Alternatively, one can use the closed graph theorem with appropriate function spaces to obtain the estimates, we omit the details here. The lemma is proved. □

Now we can return to the proof of Theorem 2.1. We first prove the following 'oscillation lemma', which roughly speaking asserts that if u is of 'small oscillation' in Q_1, then the oscillation is even smaller in Q_θ for $\theta < 1$.

For the next lemma let us recall we use the notation

$$Y\big(u, p, Q(R, z_0)\big) := \left(\fint_{Q(R,z_0)} \left|u - (u)_{Q(R,z_0)}\right|^3 dz\right)^{1/3} + R\left(\fint_{Q(R,z_0)} \left|p - (p)_{B_R(x_0)}(t)\right|^{3/2} dz\right)^{2/3}$$

and

$$Y(u, p, Q_R) := Y\big(u, p, Q\big(R, (0, 0)\big)\big).$$

Lemma 2.3 (Oscillation lemma) *Let* (u, p) *be a suitable weak solution to Eq.* (2.1) *in* Q_1 *with* $a \in L^m(Q_1)$, $m > 5$, $\operatorname{div} a = 0$, $\|a\|_{L^m(Q_1)} \le c$, $|(u)_1| \le M$, *for some small absolute number* $c > 0$, *and some positive number* M. *Then for any* $\theta \in (0, 1/3)$, *there exists an* $\epsilon = \epsilon(\theta, M, m) > 0$, $C_1(M, m) > 0$, *and* $\alpha = \alpha(m) > 0$ *such that if*

$$Y(u, p, Q_1) + \left|(u)_1\right| \left(\fint_{Q_1} |a|^m dx dt\right)^{1/m} < \epsilon,$$

then

$$Y(u, p, Q_\theta) \le C_1(M, m)\theta^\alpha \left(Y(u, p, Q_1) + |(u)_1| \left(\fint_{Q_1} |a|^m dx dt\right)^{1/m}\right).$$

Proof Suppose the lemma is false. Then there exists (u_i, p_i) and a_i with the following properties:

$$\left|(u_i)_1\right| \le M, \quad \|a_i\|_{L^m(Q_1)} \le c, \quad \operatorname{div} a_i = 0,$$

$$Y(u_i, p_i, Q_1) + \left|(u_i)_1\right| \left(\fint_{Q_1} |a_i|^m dx dt\right)^{1/m} = \epsilon_i \to 0+ \quad \text{as } i \to +\infty,$$

$$Y(u_i, p_i, Q_\theta) > C_1(M, m)\theta^\alpha \epsilon_i,$$

and (u_i, p_i) satisfies Eq. (2.1) and inequality (2.2) with a replaced by a_i.

Set

$$v_i = \frac{u_i - (u_i)_1}{\epsilon_i},$$

$$q_i = \frac{p_i - (p_i)_1(t)}{\epsilon_i},$$

$$f_i = \frac{a_i \otimes (u_i)_1}{\epsilon_i}.$$

Then we have $(v_i)_1 = 0$, $\operatorname{div}\operatorname{div} f_i = 0$,

$$\left(\fint_{Q_1} |v_i|^3 dxdt\right)^{1/3} + \left(\fint_{Q_1} |q_i|^{3/2} dxdt\right)^{2/3} + \left(\fint_{Q_1} |f_i|^m dxdt\right)^{1/m} \le 1,$$

and

$$\left(\fint_{Q_\theta} \left|v_i - (v_i)_\theta\right|^3 dxdt\right)^{1/3} + \theta\left(\fint_{Q_\theta} \left|q_i - (q_i)_\theta(t)\right|^{3/2} dxdt\right)^{2/3} > C_1(M, m)\theta^\alpha.$$

Moreover, (v_i, q_i) satisfies:

$$\left.\begin{array}{r} \partial_t v_i - \Delta v_i + \epsilon_i v_i \cdot \nabla v_i + a_i \cdot \nabla v_i \\ + \operatorname{div}(a_i \otimes v_i) + \operatorname{div} f_i + (u_i)_1 \cdot \nabla v_i + \nabla q_i = 0 \\ \operatorname{div} v_i = 0 \end{array}\right\} \tag{2.11}$$

in the sense of distributions in Q_1 and

$$\partial_t \frac{|v_i|^2}{2} - \Delta \frac{|v_i|^2}{2} + |\nabla v_i|^2 + \operatorname{div}\left(\frac{|v_i|^2}{2}(\epsilon_i v_i + (u_i)_1 + a_i)\right) + v_i \operatorname{div}(f_i + a_i \otimes v_i) + \operatorname{div} v_i q_i \le 0, \tag{2.12}$$

in the sense of distributions in Q_1. Here again the terms make sense due to our regularity assumptions and the interpretation of $v_i \operatorname{div}(a_i \otimes v_i + f_i)$ as the one below inequalities (2.2).

Since $v_i \in L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1(Q_1)$ and v_i satisfies Eq. (2.11), we can change the value of v_i on a set of measure zero such that $t \to v_i(\cdot, t)$ is continuous from $(-1, 0)$ to $L_w^2(B_1(0))$, the weak L^2 space.

From inequality (2.12) we obtain,

$$\begin{aligned} &\int_{B_1(0)} \frac{|v_i|^2}{2}(x,t)\phi(x,t)dx + \int_{-1}^t \int_{B_1(0)} |\nabla v_i|^2 \phi(x,s) dxds \\ &\le \int_{-1}^t \int_{B_1(0)} \frac{|v_i|^2}{2}(\partial_t \phi + \Delta\phi) dxds \\ &\quad + \int_{-1}^t \int_{B_1(0)} \frac{|v_i|^2}{2}\left[(u_i)_1 + a_i + \epsilon_i v_i\right]\nabla\phi dxds \\ &\quad + \int_{-1}^t \int_{B_1(0)} \left[(f_i + a_i \otimes v_i) : (\nabla v_i \phi + v_i \otimes \nabla\phi)\right] dxds \end{aligned}$$

$$+ \int_{-1}^{t}\int_{B_1(0)} q_i v_i \nabla\phi dxds,$$

for any $\phi \geq 0$ with $\phi \in C_c^\infty(B_1(0)\times(-1,t])$.

Let us define

$$E_i(r) = \text{ess} \sup_{-r^2<t\leq 0} \int_{B_r} \frac{|v_i|^2}{2}(x,t)dx + \int_{-r^2}^{0}\int_{B_r} |\nabla v_i|^2(x,s)dxds, \quad (2.13)$$

for $0 < r < 1$. By known multiplicative inequalities we have

$$\|v_i\|^2_{L^{10/3}(Q_r)} \leq CE_i(r). \quad (2.14)$$

Then for any $1/2 < r_1 < r_2 \leq 1$, if we choose nonnegative test function ϕ with support in Q_{r_2} appropriately, we obtain the following estimates, with the help of the above local energy estimates, Hölder inequality, and the estimates on v_i, q_i, a_i:

$$\begin{aligned}
E(r_1) &\leq \frac{C}{(r_2-r_1)^2} + \frac{C}{r_2-r_1}\int_{Q_{r_2}} \frac{|v_i|^2}{2}\big(M + |a_i| + \epsilon_i|v_i|\big)dxds \\
&\quad + \int_{Q_{r_2}} |f_i||\nabla v_i| + |a_i||v_i||\nabla v_i|dxdt \\
&\quad + \frac{C}{r_2-r_1}\int_{Q_{r_2}} |f_i||v_i| + |a_i||v_i|^2 + |q_i||v_i|dxdt \\
&\leq \frac{C(M)}{(r_2-r_1)^2} + \left(\int_{Q_{r_2}} |f_i|^2 dxdt\right)^{1/2} E(r_2)^{1/2} \\
&\quad + \|a_i\|_{L^5(Q_{r_2})}\|v_i\|_{L^{10/3}(Q_{r_2})}\|\nabla v\|_{L^2(Q_{r_2})} \\
&\leq \frac{C(M)}{(r_2-r_1)^2} + CE(r_2)^{1/2} + C\|a\|_{L^m(Q_1)}E(r_2) \\
&\leq \frac{C(M)}{(r_2-r_1)^2} + \big(C\|a_i\|_{L^m(Q_1)} + 1/2\big)E(r_2).
\end{aligned}$$

Note that we have $\|a_i\|_{L^m(Q_1)} \leq c$ with c small. So if we choose c such that $Cc < 1/2$, then we can apply Lemma 2.1 and conclude that $E(3/4) \leq C(M,m)$. That is, v_i are uniformly bounded in $L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1(Q_{3/4})$. Thus by known embedding theorems and the fact that v_i satisfies Eq. (2.11) (which provide crucial regularity in t), we can choose a subsequence of v_i (which we still denote as v_i), such that for some $\lambda \in R^3$, $v \in L^3(Q_{3/4})$, $q \in L^{3/2}(Q_{3/4})$ and a, $f \in L^m(Q_{3/4})$ with $\operatorname{div} a = 0$, we have

$v_i(\cdot,t) \rightharpoonup v(\cdot,t)$ weakly in $L^2(B_{3/4})$ for every $t \in (-(\frac{3}{4})^2, 0)$,
$v_i \to v$ strongly in $L^3(Q_{3/4})$,
$q_i \rightharpoonup q$ weakly in $L^{3/2}(Q_{3/4})$,
$(u_i)_1 \to \lambda$, $a_i \rightharpoonup a$ weakly in $L^m(Q_{3/4})$,
$f_i \rightharpoonup f$ in $L^m(Q_{3/4})$.

Moreover, we have

$$\left(\int_{Q_{3/4}} |v|^3 dxdt\right)^{1/3} + \left(\int_{Q_{3/4}} |q|^{3/2} dxdt\right)^{2/3} + \left(\int_{Q_{3/4}} \big(|f|+|a|\big)^m dxdt\right)^{1/m} \le C,$$

and $|\lambda| \le M$.

From Eq. (2.11) for (v_i, q_i), we see

$$\left.\begin{aligned} \partial_t v - \Delta v + \lambda\cdot\nabla v + \operatorname{div}(a\otimes v + v\otimes a + f) + \nabla q = 0 \\ \operatorname{div} v = 0 \end{aligned}\right\} \quad \text{in } Q_{3/4}. \tag{2.15}$$

By Lemma 2.2 on generalized Stokes system, we see for some $\alpha = \alpha(m) > 0$, v is Hölder continuous in $Q_{1/2}$ with exponent α, with respect to parabolic distance. More precisely,

$$\big|v(x_1,t_1) - v(x_2,t_2)\big| \le C(M,m)\big(|x_1-x_2| + |t_1-t_2|^{1/2}\big)^{\alpha}.$$

Since $v_i \to v$ strongly in $L^3(Q_{3/4})$, we see

$$\left(\fint_{Q_\theta} \big|v_i - (v_i)_\theta\big|^3 dxdt\right)^{1/3} \le C(M,m)\theta^{\alpha},$$

for i sufficiently large.

Note that from Eq. (2.11) we have

$$-\Delta q_i = \operatorname{div}\operatorname{div}(\epsilon_i v_i\otimes v_i + v_i\otimes a_i + a_i\otimes v_i).$$

Let $q_i = q_i^1 + q_i^2$, where

$$q_i^1 = (-\Delta)^{-1}\operatorname{div}\operatorname{div}\big((\epsilon_i v_i\otimes v_i + v_i\otimes a_i + a_i\otimes v_i)\chi_{B_{3/4}}\big),$$

with $\chi_{B_{3/4}}$ being the characteristic function on $B_{3/4}$. Since v_i strongly converges to v in $L^3(Q_{4/3})$, we have $q_i^1 - \tilde{q}_i$ strongly converges to 0 in

$L^{3/2}(Q_{4/3})$, where

$$\tilde{q}_i = (-\Delta)^{-1}\operatorname{div}\operatorname{div}\big((v\otimes a_i + a_i\otimes v)\chi_{B_{3/4}}\big).$$

Since v is bounded, we obtain by estimates of Riesz operators $\tilde{q}_i \in L^m(Q_{1/2})$. Thus

$$\begin{aligned}
&\theta\left(\fint_{Q_\theta} |\tilde{q}_i|^{3/2}dxdt\right)^{3/2} \\
&\quad\le \theta\left(\fint_{Q_\theta} |\tilde{q}_i|^m dxdt\right)^{1/m} \le C(M,m)\theta^{1-5/m}.
\end{aligned}$$

Therefore, for i sufficiently large, we have

$$\theta\left(\fint_{Q_\theta} |q_i^1|^{3/2}dxdt\right)^{3/2} \le C(M,m)\theta^{1-5/m}.$$

By definition, $\Delta q_i^2 = 0$ in $Q_{3/4}$ and $(\fint_{Q_{3/4}} |q_i^2|^{3/2}dxdt)^{2/3} \le C$. Thus by elliptic estimates, we obtain,

$$\begin{aligned}
&\theta\left(\fint_{Q_\theta} \big|q_i^2 - \big(q_i^2\big)_\theta(t)\big|^{3/2}dxdt\right)^{2/3} \\
&\quad\le C\theta\left(\theta^{3/2}\fint_{-\theta^2}^0 \big\|\nabla q_i^2(\cdot,t)\big\|^{3/2}_{L^\infty(B_{5/12})}dt\right)^{2/3} \\
&\quad\le C\theta\left(\theta^{-1/2}\int_{-\theta^2}^0\int_{B_{1/2}(0)} |q|^{3/2}dxdt\right)^{2/3} \\
&\quad\le C\theta^{2/3}.
\end{aligned}$$

Therefore, summarizing the above, we see

$$\theta\left(\fint_{Q_\theta} \big|q_i - (q_i)_\theta(t)\big|^{3/2}dxdt\right)^{2/3} \le C(M,m)\theta^{\min\{2/3,1-5/m\}},$$

for i sufficiently large. This, together with the estimates on v_i, shows

$$Y(v_i, q_i, Q_\theta) \le C(M,m)\theta^\alpha,$$

for i sufficiently large, if we choose $\alpha(m)$ sufficiently small. This contradicts $Y(v_i, q_i, Q_\theta) \ge C_1(M,m)\theta^\alpha$ if we choose $C_1(M,m) > 2C(M,m)$. Thus the lemma is proved. □

The above lemma admits the following iterations.

Lemma 2.4 (Iteration of the oscillation lemma) *Let* (u, p), M, $\epsilon(\theta, M, m)$, $C_1(M, m)$, $\alpha(m)$, c *and* a *be as in the above lemma, with* $|(u)_{Q_1}| \le M/2$. *Let* $\beta = \alpha/2$. *Choose* $\theta \in (0, 1/3)$ *such that* $C_1(M, m)\theta^{\alpha-\beta} < 1$, *and* $\theta < c_1$ *with* $c_1 = c_1(M, m)$ *being some small number. Then there exists* $\epsilon_*(\theta, M, m)$ *sufficiently small, such that if*

$$Y(u, p, Q_1) + M\left(\fint_{Q_1} |a|^m dxdt\right)^{1/m} < \epsilon_*, \tag{2.16}$$

then for any $k = 1, 2, \ldots,$ *we have*

$$\left|(u)_{Q_{\theta^{k-1}}}\right| \le M, \tag{2.17}$$

$$Y(u, p, Q_{\theta^{k-1}}) + \left|(u)_{\theta^{k-1}}\right|\left(\fint_{Q_{\theta^{k-1}}} |a|^m dxdt\right)^{1/m} \times \theta^{k-1} < \epsilon_* \le \epsilon(\theta, M, m), \tag{2.18}$$

$$Y(u, p, Q_{\theta^k}) \le \theta^\beta\left(Y(u, p, Q_{\theta^{k-1}}) + \left|(u)_{\theta^{k-1}}\right|\left(\fint_{Q_{\theta^{k-1}}} |a|^m dxdt\right)^{1/m} \theta^{k-1}\right). \tag{2.19}$$

Proof We prove the lemma by induction.

For $k = 1$, the conclusion follows from Lemma 2.3, if we choose ϵ_* such that $\epsilon_* < \epsilon(\theta, M, p)$. Suppose the conclusion is true for $k \le k_0$, $k_0 \ge 1$, we show it remains true for $k = k_0 + 1$.

By induction

$$\left|(u)_{Q_{\theta^{k-1}}}\right| \le M,$$

$$Y(u, p, Q_{\theta^{k-1}}) + \left|(u)_{\theta^{k-1}}\right|\left(\fint_{Q_{\theta^{k-1}}} |a|^m dxdt\right)^{1/m} \theta^{k-1} < \epsilon_*,$$

$$Y(u, p, Q_{\theta^k}) \le \theta^\beta\left(Y(u, p, Q_{\theta^{k-1}}) + \left|(u)_{\theta^{k-1}}\right|\left(\fint_{Q_{\theta^{k-1}}} |a|^m dxdt\right)^{1/m} \theta^{k-1}\right) \le \theta^\beta \epsilon_*,$$

for all $k \le k_0$. Thus,

$$Y(u, p, Q_{\theta^k}) \le \theta^\beta\left(Y(u, p, Q_{\theta^{k-1}}) + \theta^{k-1} M\left(\fint_{Q_{\theta^{k-1}}} |a|^m dxdt\right)^{1/m}\right)$$

$$\le \theta^{\beta}\left(Y(u,p,Q_{\theta^{k-1}}) + \theta^{(k-1)(1-5/m)} M\left(\fint_{Q_1} |a|^m dxdt\right)^{1/m}\right)$$
$$\le \theta^{\beta} Y(u,p,Q_{\theta^{k-1}}) + \theta^{k\beta_1}\epsilon_*$$

for all $k \le k_0$, with $\beta_1 = \min\{\beta, 1-5/m\}$. Simple calculations with a repeated use of the above inequalities show:

$$Y(u,p,Q_{\theta^k}) \le \theta^{k\beta} Y(u,p,Q_1) + k\theta^{k\beta_1}\epsilon_*, \quad \forall k \le k_0.$$

Thus,

$$\left|(u)_{Q_{\theta^{k_0}}}\right| \le \sum_{k=1}^{k_0} \left|(u)_{Q_{\theta^k}} - (u)_{Q_{\theta^{k-1}}}\right| + \left|(u)_{Q_1}\right|$$
$$\le \sum_{k=1}^{k_0}\left(\fint_{Q_{\theta^k}} \left|u - (u)_{Q_{\theta^{k-1}}}\right|^3 dxdt\right)^{1/3} + \left|(u)_{Q_1}\right|$$
$$\le \theta^{-5/3}\sum_{k=1}^{k_0} Y(u,p,Q_{\theta^{k-1}}) + \left|(u)_{Q_1}\right|$$
$$\le \theta^{-5/3}\sum_{k=1}^{k_0}\left(\theta^{(k-1)\beta}\epsilon_* + \epsilon_*(k-1)\theta^{(k-1)\beta_1}\right) + M/2$$
$$\le \theta^{-5/3}\left(1-\theta^{\beta}\right)^{-1}\epsilon_* + \theta^{-5/3}\epsilon_* C(\beta_1,\theta) + M/2.$$

If we choose $\epsilon_* = \epsilon_*(\theta, M, m)$ to be sufficiently small, we see

$$\left|(u)_{Q_{\theta^{k_0}}}\right| \le M.$$

Moreover,

$$Y(u,p,Q_{\theta^{k_0}}) + \theta^{k_0}\left|(u)_{\theta^{k_0}}\right|\left(\fint_{Q_{\theta^{k_0}}} |a|^m dxdt\right)^{1/m}$$
$$\le \theta^{\beta}\epsilon_* + \theta^{(1-5/m)k_0}\epsilon_* < \epsilon_*,$$

if we choose $\theta < c(M,m)$ to be sufficiently small. Set

$$u(x,t) = \frac{1}{\theta^{k_0}} v\left(\frac{x}{\theta^{k_0}}, \frac{t}{\theta^{2k_0}}\right),$$
$$p(x,t) = \frac{1}{\theta^{2k_0}} q\left(\frac{x}{\theta^{k_0}}, \frac{t}{\theta^{2k_0}}\right), \quad \text{and}$$

$$a(x,t)=\frac{1}{\theta^{k_0}}b\left(\frac{x}{\theta^{k_0}},\frac{t}{\theta^{2k_0}}\right).$$

One can verify that (v,q) is a suitable weak solution to Eq. (2.1) with a replaced by b in Q_1. Moreover,

$$Y(v,q,Q_1)+\left|(v)_{Q_1}\right|\left(\fint_{Q_1}|b|^m dxdt\right)^{1/m}$$

$$=\theta^{k_0}\left(Y(u,p,Q_{\theta^{k_0}})+\theta^{k_0}\left|(u)_{Q_{\theta^{k_0}}}\right|\left(\fint_{Q_{\theta^{k_0}}}|a|^m dxdt\right)^{1/m}\right)<\epsilon_*,$$

$$\left(\int_{Q_1}|b|^m dxdt\right)^{1/m}\le\theta^{k_0-\frac{5k_0}{m}}\left(\int_{Q_1}|a|^m dxdt\right)^{1/m}<c.$$

Thus, by Lemma 2.3, we obtain,

$$Y(v,q,Q_\theta)\le\theta^\beta\left(Y(v,q,Q_1)+\left|(v)_{Q_1}\right|\left(\fint_{Q_1}|b|^m dxdt\right)^{1/m}\right), \tag{2.20}$$

that is,

$$Y(u,p,Q_{\theta^{k_0+1}})\le\theta^\beta\left(Y(u,p,Q_{\theta^{k_0}})+\left|(u)_{\theta^{k_0}}\right|\left(\fint_{Q_{\theta^{k_0}}}|a|^m dxdt\right)^{1/m}\theta^{k_0}\right). \tag{2.21}$$

The lemma is then proved. □

By translation and dilation, we obtain the following corollary.

Corollary 2.1 *Let (u,p) be a suitable weak solution to Eq. (2.1) in $Q(R,z_0)$, with $a\in L^m(Q(R,z_0))$, $\operatorname{div} a=0$, $|(u)_{Q(R,z_0)}|R<M/2$, θ, β are as in the above. Then there exists $\epsilon_*=\epsilon_*(\theta,M,m)$ such that*

$$RY\big(u,p,Q(R,z_0)\big)+RM\left(\fint_{Q(R,z_0)}|a|^m dxdt\right)^{1/m}<\epsilon_*$$

implies, for $k\ge1$:

$$R\left|(u)_{Q(\theta^{k-1}R,z_0)}\right|\le M,\quad \textit{and}$$

$$Y\big(u,p,Q(\theta^k R,z_0)\big)$$

$$\le\theta^\beta\Big(Y\big(u,p,Q(\theta^{k-1}R,z_0)\big)$$

$$+ R\theta^{k-1}\left|(u)_{Q_{\theta^{k-1}R}}\right|\left(\fint_{Q(\theta^{k-1}R,z_0)}|a|^m dxdt\right)^{1/m}\Bigg).$$

Proof of Theorem 2.1 It is clear if we choose ϵ_0 sufficiently small, we can apply Corollary 2.1 in $Q(1/2, z_0)$ for any $z_0 \in Q_{1/2}$. Note that $|(u)_{Q_{\theta^k R}}|$ is bounded and $m > 5$. Thus we can conclude

$$Y\big(u, p, Q(z_0, Q_{\theta^k})\big) \le C(\theta, M, m)\theta^{k\alpha},$$

for some $\alpha = \alpha(m)$, where we can choose $M < 1$, $\theta = \theta(M, m) = \theta(m)$. (There is a slight abuse of notation, in particular, this α is smaller than those appearing in the oscillation lemma.) In particular,

$$\left(\fint_{Q(\theta^k,z_0)}\left|u - (u)_{Q(\theta^k,z_0)}\right|^3 dxdt\right)^{1/3} \le C(\theta, M, m)\theta^{k\alpha},$$

for all $z_0 \in Q_{1/2}$ and $k \ge 1$. By Campanato's lemma, we conclude u is Hölder continuous in $Q_{1/2}$. The theorem is proved. □

In applications, it is cumbersome to have the "smallness condition" on a. We can remove this condition and get the following theorem.

Theorem 2.2 (Improved ϵ-regularity criteria) *Let (u, p) be a suitable weak solution to Eq. (2.1) in Q_1, with $a \in L^m(Q_1)$, $\operatorname{div} a = 0$, $\|a\|_{L^m(Q_1)} \le M$, for some $M > 0$ and $m > 5$. Then there exists $\epsilon_1 = \epsilon_1(m, M) > 0$ with the following properties: if*

$$\left(\fint_{Q_1}|u|^3 dxdt\right)^{1/3} + \left(\fint_{Q_1}|p|^{3/2} dxdt\right)^{2/3} \le \epsilon_1,$$

then u is Hölder continuous in $Q_{1/2}$ with exponent $\alpha = \alpha(m) > 0$ and

$$\|u\|_{C^\alpha_{\text{par}}(Q_{1/2})} \le C(m, \epsilon_1, M) = C(m, M). \tag{2.22}$$

Proof Choose $0 < R_0 < 1/2$, a small positive number to be determined below. For any $z_0 = (x_0, t_0) \in Q_{1/2}$, we would like to apply a scaled version of Theorem 2.1 for (u, p) in $Q(R_0, z_0)$. Set

$$u(x,t) = \frac{1}{R_0}v\left(\frac{x-x_0}{R_0}, \frac{t-t_0}{R_0^2}\right),$$

$$p(x,t) = \frac{1}{R_0^2}q\left(\frac{x-x_0}{R_0}, \frac{t-t_0}{R_0^2}\right),$$

$$a(x,t) = \frac{1}{R_0} b\left(\frac{x - x_0}{R_0}, \frac{t - t_0}{R_0^2}\right).$$

We see that (v, q) is a suitable weak solution to Eq. (2.1) with a replaced by b in Q_1. Moreover,

$$\|b\|_{L^m(Q_1)} \le R_0^{1-5/m} \|a\|_{L^m(Q(R_0,z_0))} \le C R_0^{1-5/m} M,$$

and

$$\left(\fint_{Q_1} |v|^3 dxdt\right)^{1/3} + \left(\fint_{Q_1} |q|^{3/2} dxdt\right)^{2/3}$$
$$= R_0 \left(\fint_{Q(R_0,z_0)} |u|^3 dxdt\right)^{1/3} + \left(\fint_{Q(R_0,z_0)} |p|^{3/2} dxdt\right)^{2/3} R_0^2$$
$$\le C\big(R_0 R_0^{-5/3} + R_0^2 R_0^{-10/3}\big)\epsilon_1 \le C R_0^{-4/3} \epsilon_1.$$

Thus,

$$\left(\fint_{Q_1} |v|^3 dxdt\right)^{1/3} + \left(\fint_{Q_1} |q|^{3/2} dxdt\right)^{2/3} + \left(\fint_{Q_1} |b|^m dxdt\right)^{1/m}$$
$$\le R_0^{1-5/m} M + C R_0^{-4/3} \epsilon_1.$$

Thus, if we choose R_0 such that $R_0^{1-5/m} M < \epsilon_0/2$, fix R_0, $R_0 = R_0(M, m)$ and choose ϵ_1 such that $C R_0^{-4/3} \epsilon_1 < \frac{\epsilon_0}{2}$. Then we can apply Theorem 2.1 to (v, q) and conclude v is Hölder continuous in $Q_{1/2}$. Scale back and collect all constants, the theorem is then proved. □

3 Local in space near initial time smoothness of Leray solutions

In this section, we use the 'ϵ-regularity' theorem proved in the last section to study the local in space near initial time smoothness of the so called Leray solutions. Our setting is as follows.

Let $u_0 \in L^2_{loc}(R^3)$ with $\operatorname{div} u_0 = 0$ and $\sup_{x_0 \in R^3} \int_{B_1(x_0)} |u_0|^2 dx < \infty$. We recall the definition of Leray solutions in [18], see also [12].

Definition 3.1 (Leray solution) A vector field $u \in L^2_{loc}(R^3 \times [0, \infty))$ is called a Leray solution to Navier-Stokes equations with initial data u_0 if it satisfies:

(i) $\operatorname{ess\,sup}_{0\le t<R^2} \sup_{x_0\in R^3} \int_{B_R(x_0)} \frac{|u|^2}{2}(x,t)dx + \sup_{x_0\in R^3} \int_0^{R^2} \times \int_{B_R(x_0)} |\nabla u|^2 dxdt < \infty$, and

$$\lim_{|x_0|\to\infty} \int_0^{R^2} \int_{B_R(x_0)} |u|^2(x,t)dxdt = 0,$$

for any $R < \infty$.

(ii) for some distribution p in $R^3 \times (0,\infty)$, (u,p) verifies Navier Stokes equations

$$\left.\begin{aligned} \partial_t u - \Delta u + u\cdot\nabla u + \nabla p = 0 \\ \operatorname{div} u = 0 \end{aligned}\right\} \quad \text{in } R^3 \times (0,\infty), \tag{3.1}$$

in the sense of distributions and for any compact set $K \subseteq R^3$, $\lim_{t\to 0+} \|u(\cdot,t) - u_0\|_{L^2(K)} = 0$.

(iii) u is suitable in the sense of Caffarelli-Kohn-Nirenberg, more precisely, the following local energy inequality holds:

$$\int_0^\infty \int_{R^3} |\nabla u|^2 \phi(x,t)dxdt \le \int_0^\infty \int_{R^3} \frac{|u|^2}{2}(\partial_t\phi + \Delta\phi) + \frac{|u|^2}{2} u\cdot\nabla\phi + pu\cdot\nabla\phi dxdt \tag{3.2}$$

for any smooth $\phi \ge 0$ with $\operatorname{supp}\phi \Subset R^3 \times (0,\infty)$. The set of all Leray solutions starting from u_0 will be denoted as $\mathcal{N}(u_0)$.

Remarks For general existence results concerning Leray solutions, see [4, 14, 18]. In the case the initial data is in $L^2(R^3)$, the notion of Leray-Hopf weak solutions is often used (see for example [16]). The difference is that Leray-Hopf weak solutions belong to $L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1(R^3 \times [0,\infty))$. It is clear that our definition includes such solutions. Note that we impose a decay condition on u in (i). This condition allows us to calculate p in the following way: $\forall (x,t) \in B_r(x_0) \times (0,t_*) \subseteq R^3 \times (0,\infty)$, take a smooth cutoff function ϕ with $\phi|_{B_{2r}(x_0)} = 1$, then there exists a function $p(t)$ depending only on x_0, r, t, ϕ (we suppress the dependence on x_0, r, ϕ in our notation) such that for $(x,t) \in B_r(x_0) \times (0,t_*)$

$$\begin{aligned} p(x,t) &= -\Delta^{-1} \operatorname{div}\operatorname{div}(u\otimes u\phi) \\ &\quad - \int_{R^3} \big(k(x-y) - k(x_0-y)\big) u\otimes u(y,t)\big(1-\phi(y)\big)dy + p(t) \end{aligned} \tag{3.3}$$

where $k(x)$ is the kernel of $\Delta^{-1}\operatorname{div}\operatorname{div}$.

The right hand side is well defined since u satisfies the estimates in (i) and

$$\left|k(x-y)-k(x_0-y)\right| = O\left(\frac{1}{|x_0-y|^4}\right) \quad \text{as } |y| \to \infty. \tag{3.4}$$

The situation is similar to extending the domain of singular integrals to bounded functions, see for example [18] and [22].

For Leray solution $u \in \mathcal{N}(u_0)$, we have the following a priori estimates, first proved in [18], see also a simpler proof in [12]. These estimates have played an important role in [12, 21], see also [23].

Lemma 3.1 (A priori estimate for Leray solutions)
Let $\alpha = \sup_{x_0\in R^3} \int_{B_R(x_0)} \frac{|u_0|^2}{2}(x)dx < \infty$ *for some* $R > 0$ *and let* u *be a Leray solution with initial data* u_0. *Then there exists some small absolute number* $c > 0$ *such that for* λ *satisfying* $0 < \lambda \le c\min\{\alpha^{-2}R^2, 1\}$, *we have*

$$\begin{aligned} &\operatorname*{ess\,sup}_{0\le t\le\lambda R^2} \sup_{x_0\in R^3} \int_{B_R(x_0)} \frac{|u|^2}{2}(x,t)dx \\ &\quad + \sup_{x_0\in R^3} \int_0^{\lambda R^2} \int_{B_R(x_0)} |\nabla u|^2(x,t)dxdt \le C\alpha. \end{aligned} \tag{3.5}$$

Remarks Note that from the formula (3.3) and the a priori estimate of u, we get the following estimate for p which will be useful:

$$\sup_{x_0\in R^3} \int_0^{\lambda R^2} \int_{B_R(x_0)} \left|p - p(t)\right|^{3/2} dxdt \le C\alpha^{3/2} R^{1/2}. \tag{3.6}$$

Strictly speaking, one should really write $p_{x_0,R}(t)$ rather than just $p(t)$ in the last estimate. In other words, for a given x_0, t and R we need to choose an appropriate constant $p(t) = p_{x_0,R}(t)$ to satisfy the inequality. The reason is that the decay assumption on u is too weak to imply decay of p, and the pressure may have a large mean value for large $|x_0|$. However, the quantity entering the equation is ∇p, which does not change in a given open set if we change p in that set by a constant. The estimates show that under our assumptions p can be controlled up to such constants. The convention that $p(t)$ can depend on the corresponding x_0 and R is used throughout the paper.

Now we can prove our first important result.

Theorem 3.1 *Let* $u_0 \in L^2_{loc}(R^3)$ *with* $\sup_{x_0\in R^3} \int_{B_1(x_0)} |u_0|^2(x)dx \le \alpha < \infty$. *Suppose* u_0 *is in* $L^m(B_2(0))$ *with* $\|u_0\|_{L^m(B_2(0))} \le M < \infty$ *and* $m > 3$. *Let* *us decompose*[1] $u_0 = u_0^1 + u_0^2$ *with* $\operatorname{div} u_0^1 = 0$, $u_0^1|_{B_{4/3}} = u_0$, $\operatorname{supp} u_0^1 \Subset B_2(0)$ *and* $\|u_0^1\|_{L^m(R^3)} \le C(M,m)$. *Let a be the locally in time defined mild solution to Navier-Stokes equations with initial data* u_0^1. *Then there exists a positive* $T = T(\alpha, m, M) > 0$, *such that any Leray solution* $u \in \mathcal{N}(u_0)$ *satisfies*: $u - a \in C^{\gamma}_{\text{par}}(\overline{B_{1/2}} \times [0,T])$, *and* $\|u-a\|_{C^{\gamma}_{\text{par}}(\overline{B_{1/2}} \times [0,T])} \le C(M,m,\alpha)$, *for some* $\gamma = \gamma(m) \in (0,1)$.

Remark We can certainly choose $T(M,m) > 0$ such that a is defined on $R^3 \times [0, T(M,m)]$. The point of the theorem is that regularity of solution to Navier-Stokes equations depends locally on initial data, as least when Hölder continuity is concerned.

Proof By assumption a solves the Cauchy problem for Navier-Stokes equations with initial data u_0^1 in $R^3 \times [0, T_1]$, where $T_1 = T_1(M,m)$, namely:

$$\left.\begin{aligned} \partial_t a - \Delta a + a\cdot\nabla a + \nabla \tilde{p} = 0 \\ \operatorname{div} a = 0 \end{aligned}\right\} \quad \text{in } R^3 \times (0,T_1), \quad \text{and} \tag{3.7}$$

$$a(\cdot, 0) = u_0^1. \tag{3.8}$$

It is well-known how to construct the so called mild solution to Navier-Stokes equations, see for example [13, 15, 19]. In our case, it is even simpler, since $u_0^1 \in L^m$ with $m > 3$ is subcritical with respect to the natural scaling of the equation. We can follow the arguments in the Appendix of [7], and obtain $a \in L^{\frac{5m}{3}}(R^3 \times (0,T_1))$ with $\|a\|_{L^{\frac{5m}{3}}(R^3\times(0,T_1))} \le CM$. Note that $\frac{5m}{3} > 5$ since $m > 3$. Moreover, by the estimates on a and by treating the nonlinear term as perturbation, we can recover a local energy estimate for a:

$$\operatorname*{ess\,sup}_{0<t<T_1} \int_{B_1(x_0)} \frac{|a|^2}{2}(x,t)dx + \int_0^{T_1}\int_{B_1(x_0)} |\nabla a|^2(x,t)dxdt \le C(M,m),$$

for any $x_0 \in R^3$. Write $u = a + v$, we can verify that v satisfies:

$$\left.\begin{aligned} \partial_t v - \Delta v + v\cdot\nabla v + a\cdot\nabla v + \operatorname{div}(a\otimes v) + \nabla q = 0 \\ \operatorname{div} v = 0 \end{aligned}\right\}, \tag{3.9}$$

[1]Such decomposition is well-known. One can for example first localize u_0 using a smooth cutoff function, and then use Bogovskii's lemma to deal with the divergence-free condition. See for example [1, 9].

in the sense of distributions in $R^3 \times (0, T_1)$, here $q = p - \tilde{p}$ with p being the associated pressure for u; and the local energy inequality

$$\partial_t \frac{|v|^2}{2} - \Delta \frac{|v|^2}{2} + |\nabla v|^2 + \operatorname{div}\left(\frac{|v|^2}{2}(v + a)\right) + v \operatorname{div}(a \otimes v) + \operatorname{div}(vq) \leq 0,$$

in the sense of distributions in $R^3 \times (0, T_1)$;

$$\lim_{t \to 0+} \|v(\cdot, t) - u_0^2\|_{L^2(B_1(x_0))} = 0, \quad \text{for any } x_0 \in R^3.$$

Note also that $u_0^2|_{B_{4/3}} \equiv 0$. Since (u, p) satisfies the a priori estimates in Lemma 3.1 (and the remarks below it), $(a, \tilde{p})$ is regular, we obtain the following estimates for (v, q) in $B_2(0) \times [0, T_2)$, $T_2 = T_2(\alpha, M, m)$:

$$\operatorname*{ess\,sup}_{0<t<T_2} \frac{1}{2} \int_{B_2(0)} |v|^2(x, t) dx + \int_0^{T_2} \int_{B_2(0)} |\nabla v|^2(x, s) dx ds$$
$$+ \left(\int_0^{T_2} \int_{B_2(0)} |q|^{3/2} dx ds\right)^{2/3} \leq C(\alpha, m, M).$$

From the local energy inequality for v, and $\lim_{t \to 0+} \|v(\cdot, t)\|_{L^2(B_{4/3}(0))} = 0$, we obtain

$$\frac{1}{2} \int_{B_{4/3}} |v|^2(x, t)\phi(x) dx + \int_0^t \int_{B_{4/3}} |\nabla v|^2(x, t)\phi(x) dx ds$$
$$\leq \int_0^t \int_{B_{4/3}} \frac{|v|^2}{2} \Delta\phi dx ds + \int_0^t \int_{B_{4/3}} \frac{|v|^2}{2}(v + a)\nabla\phi dx ds$$
$$+ \int_0^t \int_{B_{4/3}} \left[a \otimes v : (\nabla v \phi + v \otimes \nabla\phi)\right] + qv \cdot \nabla\phi dx ds,$$

where $\phi \in C_c^\infty(B_{4/3})$, $\phi|_{B_1} \equiv 1$, $\phi \geq 0$.

By multiplicative inequalities, we know

$$\left(\int_0^{T_2} \int_{B_2(0)} |v|^{10/3} dx dt\right)^{3/10} \leq C(\alpha, m, M).$$

Thus from the above, we see by Schwartz inequality:

$$\frac{1}{2} \int_{B_1(0)} |v|^2(x, t) dx + \int_0^t \int_{B_1(0)} |\nabla v|^2(x, s) dx ds$$
$$\leq C(\alpha, m, M) t^{\min\{1/30, \frac{m-3}{5m}\}},$$

for $t < T_2$. From

$$\Delta q = -\operatorname{div}\operatorname{div}(v \otimes v + a \otimes v + v \otimes a),$$

we can see $q \in L^{5/3}_{loc}$. Thus

$$\left(\int_0^t \int_{B_1(0)} |q|^{3/2} dx ds\right)^{2/3} \leq C(\alpha, m, M) t^{1/15}.$$

The importance of these estimates lies in the fact that they provide crucial "quantitative" information on the decay in time as $t \to 0+$. Now for t_0 fixed, whose precise value is to be determined later, extend v, q to $B_1(0) \times (-1 + t_0, t_0]$ by setting $v = 0$, $q = 0$ for $(x, t) \in B_1 \times (-1 + t_0, 0]$. Extend a to $B_1(0) \times (-1 + t_0, t_0]$ by setting $a(t, x) = 0$ for $t < 0$. The extended function (v, q) is a suitable weak solution to the generalized Navier-Stokes equations (2.1) with the extended a in $B_1(0) \times [-1 + t_0, t_0]$. Note here that

$$\lim_{t \to 0+} \|v(\cdot, t)\|_{L^2(B_1(0))} = 0$$

plays a crucial role: it guarantees that $\partial_t v$ and $\partial_t \frac{|v|^2}{2}$ will not cause any problem across $t = 0$. Then clearly if we choose $t_0 = t_0(\alpha, m, M)$ sufficiently small, we can apply Theorem 2.2 and conclude v is Hölder continuous in $B_{1/2} \times [0, t_0]$, with $\|v\|_{C^\gamma_{\text{par}}(B_{1/2} \times [0, t_0])} \leq C(\alpha, m, M)$, for some $\gamma = \gamma(m)$. The theorem is proved. □

For applications below, we state the following simple (and certainly well-known) lemma for heat equation without proof.

Lemma 3.2 *We have the following estimates*:

1. *If* $u_0 \in C^\beta(R^3)$ *for some* $\beta \in (0, 1)$, *then* $e^{\Delta t} u_0(x) \in C^\beta_{\text{par}}(R^3 \times [0, 1])$, *with*

$$\left\|e^{\Delta t} u_0(x)\right\|_{C^\beta_{\text{par}}(R^3 \times [0,1])} \leq C \|u_0\|_{C^\beta(R^3)}. \tag{3.10}$$

2. *If* $f \in L^\infty(R^3 \times [0, 1])$, *then* $\int_0^t \nabla e^{\Delta(t-s)} f(\cdot, s) ds \in C^\beta_{\text{par}}(R^3 \times [0, 1])$ *for any* $\beta \in (0, 1)$, *and*

$$\left\|\int_0^t \nabla e^{\Delta(t-s)} f(\cdot, s) ds\right\|_{C^\beta_{\text{par}}(R^3 \times [0,1])} \leq C(\beta) \|f\|_{L^\infty(R^3 \times [0,1])}. \tag{3.11}$$

The above theorem implies the following result.

Theorem 3.2 (Local Hölder regularity of Leray solutions) *Let* $u_0 \in L^2_{loc}(R^3)$ *with* $\sup_{x_0\in R^3}\int_{B_1(x_0)}|u|^2(x)dx \le \alpha < \infty$. *Suppose* u_0 *is in* $C^\gamma(B_2(0))$ *with* $\|u_0\|_{C^\gamma(B_2(0))} \le M < \infty$. *Then there exists a positive* $T = T(\alpha,\gamma,M) > 0$, *such that any Leray solution* $u \in \mathcal{N}(u_0)$ *satisfies*:

$$u \in C^\gamma_{\mathrm{par}}\big(\overline{B_{1/4}} \times [0,T]\big), \quad \textit{and} \quad \|u\|_{C^\gamma_{\mathrm{par}}(\overline{B_{1/4}}\times[0,T])} \le C(M,\alpha,\gamma). \tag{3.12}$$

Proof Let us decompose $u_0 = u_0^1 + u_0^2$ with $\operatorname{div} u_0^1 = 0$, $u_0^1|_{B_{4/3}(0)} = u_0$, $\operatorname{supp} u_0^1 \Subset B_2(0)$ and $\|u_0^1\|_{C^\gamma(R^3)} \le CM$. Let a be the mild solution to Navier-Stokes equations with initial data u_0^1 in $R^3 \times (0, T(M))$. Then Theorem 3.1 implies that $u - a$ is Hölder continuous with some exponent $\beta \in (0,\gamma)$ in $B_{1/2} \times [0,T]$ with some $T = T(\alpha,\gamma,M) \in (0, T(M))$. Since the initial data u_0^1 for a is in C^γ, it is not difficult to show that $a \in C^\gamma_{\mathrm{par}}(R^3 \times (0,T))$. Thus u is Hölder continuous with exponent β in $B_{1/2} \times [0, T(M)]$. By using a routine boostraping argument, one can improve the exponent to γ. Since this argument will be used one more time below, we sketch some of the details here for the reader's convenience. Note that u is Hölder continuous in $\overline{B_{1/2}} \times [0,T]$, thus from the representation formula (3.3) for p and estimates for Riesz transform, we know p is bounded in $\overline{B_{7/16}} \times [0,T]$ modulo some function $p(t)$. Now rewrite the equation for u as

$$\partial_t u - \Delta u = -\operatorname{div}(u \otimes u) - \nabla p. \tag{3.13}$$

Choose a smooth cutoff function η with $\eta \equiv 1$ on $\overline{B_{3/8}}$ and $\eta \equiv 0$ outside $B_{7/16}$. Write

$$u_1(\cdot,t) = \int_0^t e^{\Delta(t-s)}\big[-\operatorname{div}(u\otimes u\eta) - \nabla(p\eta)\big](\cdot,s)ds,$$
$$u_2(\cdot,t) = e^{\Delta t}(u_0\eta).$$

Let $u = u_1 + u_2 + u_3$. By Lemma 3.2 we see u_1 and u_2 are Hölder continuous with exponent β. Note that u_3 satisfies

$$\partial_t u_3 - \Delta u_3 = 0 \quad \text{in } B_{3/8} \times [0,T],$$

and $u_3(\cdot,0)|_{B_{3/8}} = 0$. Thus u_3 is smooth in $\overline{B_{1/4}} \times [0,T]$. In summary u is Hölder continuous in $\overline{B_{1/4}} \times [0,T]$ with exponent β. Then the theorem is proved. □

4 Estimates of forward self-similar solutions to Navier-Stokes and Stokes equations

In this section, we start to study forward self-similar solutions to Navier-Stokes equations and a related nonhomogeneous Stokes system. Our setting is as follows.

Let u be a Leray solution with initial data u_0. Suppose $\lambda u_0(\lambda x) = u_0(x)$, $\lambda u(\lambda x, \lambda^2 t) = u(x,t)$ for any $\lambda > 0$. We also assume $u_0|_{\partial B_1(0)} \in C^\infty(\partial B_1(0))$. Then it is easy to see

$$\left|\nabla^\alpha u_0(x)\right| \le \frac{C(\alpha, u_0)}{|x|^{1+|\alpha|}}, \quad \forall |\alpha| \ge 0.$$

Our first main result in this section is the following theorem.

Theorem 4.1 (A-priori estimate for forward self-similar solutions) *Let divergence free initial data u_0 be scale-invariant, $u \in \mathcal{N}(u_0)$ be scale-invariant. Then $U(\cdot) := u(\cdot, 1)$, the solution profile at time $t = 1$, belongs to $C^\infty(R^3)$ and*

$$\left|\partial^\alpha\left(U(x) - e^\Delta u_0(x)\right)\right| \le \frac{C(\alpha, u_0)}{(1+|x|)^{3+|\alpha|}}, \quad \forall |\alpha| \ge 0. \tag{4.1}$$

Remarks Here and below, constants $C(u_0, \dots), T(u_0, \dots) \dots$ only depend on the magnitude of u_0 and its finitely many derivatives on the unit sphere. Similar estimates with more precise asymptotics have been proved in [2] when the initial data is small in appropriate senses.

Proof Apply Lemma 3.1 with $R = 1$, we see (set $M := \|u_0\|_{C(\partial B_1)}$)

$$\sup_{0<t<T_1} \frac{1}{2}\int_{B_1(0)} \left|u(x,t)\right|^2 dx + \int_0^{T_1}\int_{B_1(0)} \left|\nabla u(x,t)\right|^2 dxdt \le C(M),$$
$$T_1 = T_1(M). \tag{4.2}$$

Since $u(x,t) = \frac{1}{\sqrt{t}} u(\frac{x}{\sqrt{t}}, 1) = \frac{1}{\sqrt{t}} U(\frac{x}{\sqrt{t}})$, we have for a fixed $t_* < T_1$ which is to be determined later

$$C(M) \ge 1/2 \int_{B_1(0)} \left|u(x,t_*)\right|^2 dx + \int_{t_*/2}^{t_*}\int_{B_1(0)} \left|\nabla u(x,t)\right|^2 dxdt$$
$$\ge \frac{\sqrt{t_*}}{2}\int_{B_{\frac{1}{\sqrt{t_*}}}(0)} \left|u(x,1)\right|^2 dx + \frac{\sqrt{t_*}}{8}\int_{B_{\frac{1}{\sqrt{t_*}}}(0)} \left|\nabla u(x,1)\right|^2 dx \tag{4.3}$$

$$\geq \frac{\sqrt{t_*}}{2} \int_{B_{\frac{1}{\sqrt{t_*}}}(0)} |U(x)|^2 dx + \frac{\sqrt{t_*}}{8} \int_{B_{\frac{1}{\sqrt{t_*}}}(0)} |\nabla U(x)|^2 dx. \quad (4.4)$$

On the other hand, for $\forall x_0$, $|x_0| = 8$, since $u_0 \in C^\infty(B_4(x_0))$, we can apply Theorem 3.1 and some simple boostraping arguments to show the following: there exists $T_2 = T_2(M) > 0$ such that $\forall \alpha$,

$$\left\| \partial_t \partial_x^\alpha u \right\|_{L^\infty(B_{1/8}(x_0)\times[0,T_2])} \leq C(\alpha, u_0),$$

this is true for any $u \in \mathcal{N}(u_0)$.

Since $\forall \lambda > 0$, $\lambda u(\lambda x, \lambda^2 t)$ is also a Leray solution with initial data u_0, we obtain

$$\left| \lambda^{|\alpha|+1} \partial^\alpha u\big(\lambda x_0, \lambda^2 t\big) - \partial^\alpha u_0(x_0) \right| \leq C(\alpha, u_0) t,$$

for any $\lambda > 0$, $|\alpha| \geq 0$, $t \leq T_2(u_0)$.

Take $\lambda = \frac{1}{\sqrt{t}}$, we obtain $|(\frac{1}{\sqrt{t}})^{|\alpha|+1} \partial^\alpha u(\frac{x_0}{\sqrt{t}}, 1) - \partial^\alpha u(x_0)| \leq C(\alpha, u_0) t$.

Setting $y = \frac{x_0}{\sqrt{t}}$, and using the homogeneity of $\partial^\alpha u_0$, we get

$$\left| \partial^\alpha U(y) - \partial^\alpha u_0(y) \right| \leq \frac{C(\alpha, u_0)}{|y|^{|\alpha|+3}}, \quad \forall |y| > \frac{8}{\sqrt{T_2}}. \quad (4.5)$$

Now choose t_* sufficiently small, $t_* = t_*(M)$, we see from inequality (4.3):

$$\int_{B_{\frac{16}{\sqrt{T_2}}}} \left(|U(y)|^2 + |\nabla U(y)|^2 \right) dy \leq C(M).$$

Since $u(x,t)$ satisfies Navier-Stokes equations, it is easy to verify U satisfies

$$\left.\begin{aligned} -\Delta U - \tfrac{x}{2} \cdot \nabla U - \tfrac{U}{2} + U \cdot \nabla U + \nabla P = 0 \\ \operatorname{div} U = 0 \end{aligned}\right\} \quad \text{in } R^3. \quad (4.6)$$

Thus elliptic estimates give

$$\left\| U(\cdot) \right\|_{C^k(B_{\frac{9}{\sqrt{T_2}}})} \leq C(k, M).$$

These estimates, combined with the properties of heat equation, finish the proof. □

For later use, let us study a nonhomogeneous Stokes system with singular forcing. Our result is the following lemma.

Lemma 4.1 (Decay for the linear singularly forced Stokes system) *Let* $f \in C(R^3)$, *suppose* $v \in L_t^\infty L_x^\gamma(R^3 \times (0,T))$ *for any* $T < \infty$, *and some* $\gamma > 1$, *suppose* v *satisfies*

$$\left.\begin{array}{c}\partial_t v - \Delta v + \nabla p = t^{-3/2} f(\frac{x}{\sqrt{t}}) \\ \operatorname{div} v = 0\end{array}\right\} \quad \text{in } R^3 \times (0,\infty), \tag{4.7}$$

for some distribution p, *and* $\lim_{t\to 0+} \|v(\cdot,t)\|_{L^\gamma(R^3)} = 0$. *Then*

(i) *if* $\tilde{v}$ *also satisfies the above conditions, then* $v = \tilde{v}$.
(ii) *if* f *satisfies* $M := \sup_{x\in R^3}(1+|x|)^3|f(x)| < \infty$, *then*

$$v(\cdot,t) = \int_0^t e^{\Delta(t-s)} P \frac{1}{s^{3/2}} f\left(\frac{\cdot}{\sqrt{s}}\right) ds, \tag{4.8}$$

where P *is the Helmholtz projection operator. Let* $V(x) = v(x,1)$, *then* $\|V\|_{C^{1,\alpha}(B_R)} \le C(\alpha,R)M$ *for* $\alpha \in (0,1)$ *and*

$$\sup_{x\in R^3} \left((1+|x|)^2|V(x)| + (1+|x|)^3|\nabla V(x)|\right) \le CM. \tag{4.9}$$

(iii) *if* f *satisfies* $M := \sup_{x\in R^3}(1+|x|)^4|f(x)| < \infty$, *then* v *is given by formula* (4.8). *Let* $V(x) = v(x,1)$, *then* $\|V\|_{C^{1,\alpha}(B_R)} \le C(\alpha,R)M$ *for* $\alpha \in (0,1)$ *and*

$$\sup_{x\in R^3} \left((1+|x|)^3|V(x)| + (1+|x|)^4|\nabla V(x)|\right) \le CM. \tag{4.10}$$

Proof The uniqueness is easy. We only need to show that if $f = 0$ and for some $\gamma_1, \gamma_2 > 1$,

$$v \in L_t^\infty\left(L_x^{\gamma_1} + L_x^{\gamma_2}\right)\left(R^3 \times (0,T)\right) \quad \text{for any } T > 0 \quad \text{and,}$$
$$\lim_{t\to 0+} \left\|v(t,\cdot)\right\|_{(L_x^{\gamma_1}+L_x^{\gamma_2})(R^3)} = 0,$$

then $v = 0$. Set $\omega = \operatorname{curl} v$, then $\partial_t \omega - \Delta\omega = 0$ in $R^3 \times (0,\infty)$. Since

$$\lim_{t\to 0+} \left\|v(\cdot,t)\right\|_{(L^{\gamma_1}+L^{\gamma_2})(R^3)} = 0,$$

we can extend ω to $R^3 \times R$ by setting $\omega = 0$ for $v < 0$, and the extended function, which we still denote as ω, satisfies $\partial_t\omega - \Delta\omega = 0$ in $R^3 \times R$. Here again there is no problem showing that the equation is satisfied across $t = 0$ since ω decays to 0 as $t \to 0+$. One can for example first mollify ω in x and the mollified function is smooth in both x and t. For the mollified function the claim is clear, then we can pass to the limit to show our claim. Since we have bounds for ω in some negative Sobolev space and $\omega = 0$ for $t < 0$, we conclude $\omega \equiv 0$. Thus $\Delta v = 0$ in $R^3 \times (0, \infty)$. Therefore $v \equiv 0$.

Let us now prove part (ii) and part (iii). By the uniqueness result, we only need to prove the claimed estimates. Denote the kernel of Pe^{Δ} by $k(x)$, then $k(\cdot) \in L^{1+\epsilon}(R^3)$ for any $\epsilon > 0$. By Young's inequality it is easy to get

$$\begin{aligned} &\left\| \int_0^t e^{\Delta(t-s)} P s^{-3/2} f\left(\frac{\cdot}{\sqrt{s}}\right) ds \right\|_{L_x^{\frac{1+\epsilon}{1-\epsilon}}} \\ &\quad \le C(\epsilon) \int_0^t \left\| (t-s)^{-3/2} k\left(\frac{\cdot}{\sqrt{t-s}}\right) \right\|_{L_x^{1+\epsilon}} \left\| s^{-3/2} f\left(\frac{\cdot}{\sqrt{s}}\right) \right\|_{L_x^{1+\epsilon}} ds \\ &\quad \le C(\epsilon) M t^{1-\frac{3\epsilon}{1+\epsilon}}. \end{aligned}$$

Thus,

$$v(\cdot, t) = \int_0^t e^{\Delta(t-s)} P \frac{1}{s^{3/2}} f\left(\frac{\cdot}{\sqrt{s}}\right) ds.$$

Now let us prove the decay estimates of V. The proof is a direct consequence of the following inequality (which can be proved by simple calculations) with $\alpha, \beta = 3, 4$ and $R := |x| > 8$:

$$\int_0^1 \int_{R^3} \frac{1}{(|x-y| + \sqrt{1-t})^\alpha} \frac{1}{(|y| + \sqrt{t})^\beta} dy dt \le \begin{cases} R^{-3} \log R & \text{if } \alpha = \beta = 3, \\ R^{-\alpha-\beta+4} & \text{otherwise.} \end{cases} \tag{4.11}$$

For part (ii), we have

$$\begin{aligned} \left| V(x) \right| &\le \int_0^1 \int_{R^3} \frac{1}{(|x-y| + \sqrt{1-t})^3} \frac{1}{(|y| + \sqrt{t})^3} dy dt \le |x|^{-3} \log |x|, \\ \left| \nabla V(x) \right| &\le \int_0^1 \int_{R^3} \frac{1}{(|x-y| + \sqrt{1-t})^4} \frac{1}{(|y| + \sqrt{t})^3} dy dt \le |x|^{-3}, \end{aligned}$$

for $|x| > 8$. For part (iii), we have

$$\begin{aligned} \left| V(x) \right| &\le \int_0^1 \int_{R^3} \frac{1}{(|x-y| + \sqrt{1-t})^3} \frac{1}{(|y| + \sqrt{t})^4} dy dt \le |x|^{-3}, \\ \left| \nabla V(x) \right| &\le \int_0^1 \int_{R^3} \frac{1}{(|x-y| + \sqrt{1-t})^4} \frac{1}{(|y| + \sqrt{t})^4} dy dt \le |x|^{-4}, \end{aligned}$$

for $|x| > 8$. Thus the decay estimates are proved. Since V also satisfies an elliptic equation:

$$\left.\begin{aligned} -\Delta V - \tfrac{x}{2}\cdot\nabla V - \tfrac{V}{2} + \nabla P = f \\ \operatorname{div} V = 0 \end{aligned}\right\} \quad \text{in } R^3, \tag{4.12}$$

the estimates in $B_C(0)$ is simple. □

5 Existence of forward self-similar solution for large initial data

In this section we prove the existence of the scale-invariant solutions as stated in Theorem 1.1. Let us briefly recall the general strategy of the proof, as outlined in the introduction. For any scale-invariant initial data u_0, we introduce a parameter $\mu \in [0, 1]$. When μ is small, the existence and uniqueness of a scale-invariant solution u_μ with initial data μu_0 is known. Theorem 4.1 provides us with a priori estimates sufficient for applying the Leray-Schauder degree theory as μ is varied in $[0, 1]$. (The precise definitions of the relevant operators and function spaces are introduced below.) The degree is continuous in μ and, as usual in this situation, we conclude that the degree is identically 1 for all $\mu \in [0, 1]$. This implies the existence of a scale-invariant solution for initial data μu_0 with $\mu \in [0, 1]$ and taking $\mu = 1$ we obtain the result. The asymptotics for large x obtained in Theorem 4.1 is crucial for our argument. It ensures that although we work on the whole space, our operators defined below remain compact (in suitable spaces) and behave in a way which is quite similar to the situation for bounded domains.

Theorem 5.1 *Let $u_0 \in C^\infty(R^3 \setminus \{0\})$ satisfy $\lambda u_0(\lambda x) = u_0(x)$ for all $\lambda > 0$, $\operatorname{div} u_0 = 0$. Then there exists $u \in C^\infty(R^3 \times (0,\infty))$, with $\lambda u(\lambda x, \lambda^2 t) = u(x,t)$ for all $\lambda > 0$, and $u \in \mathcal{N}(u_0)$, that is, u satisfies*

$$\left.\begin{aligned} \partial_t u - \Delta u + u\cdot\nabla u + \nabla p = 0 \\ \operatorname{div} u = 0 \end{aligned}\right\} \quad \text{in } R^3 \times (0,\infty) \quad \text{for some } p. \tag{5.1}$$

Moreover, let $U(x) = u(x, 1)$, then

$$\left|\partial^\alpha\big(U(x) - e^\Delta u_0(x)\big)\right| \le \frac{C(\alpha, u_0)}{(1+|x|)^{3+|\alpha|}}.$$

Proof By Theorem 4.1, it suffices to show there exists $u \in \mathcal{N}(u_0)$ with the scaling

$$\lambda u\big(\lambda x, \lambda^2 t\big) = u(x,t) \quad \text{for all } \lambda > 0. \tag{5.2}$$

Denote

$$X = \Big\{ V \in C^1\big(R^3\big) : \operatorname{div} V = 0, \\ \sup_{x \in R^3} \Big(\big(1+|x|\big)^2 \big|V(x)\big| + \big(1+|x|\big)^3 \big|\nabla V(x)\big| \Big) < \infty \Big\}. \tag{5.3}$$

For any $V \in X$, we define a natural norm

$$\|V\|_X = \sup_{x \in R^3} \Big(\big(1+|x|\big)^2 \big|V(x)\big| + \big(1+|x|\big)^3 \big|\nabla V(x)\big| \Big). \tag{5.4}$$

Set $U_0 = e^{\Delta} u_0$. Introduce a parameter $\mu \in [0,1]$, set $U_{0\mu} = \mu U_0$. We will follow Leray's method to prove the existence of $u_\mu \in \mathcal{N}(\mu u_0)$ with $\lambda u_\mu(\lambda x, \lambda^2 t) = u_\mu(x,t)$ for all $\lambda > 0$ and $\mu \in [0,1]$. Due to the scaling invariance of $u_\mu(x,t)$, we are essentially seeking the profile function $U_\mu(x) = u_\mu(x,1)$. For ease of notation, we will suppress the explicit indication of the dependence of u and U on μ below. Our goal is to obtain $U(x)$ satisfying

$$\left.\begin{aligned} -\Delta U + U\cdot\nabla U - \tfrac{U}{2} - \tfrac{x}{2}\cdot\nabla U + \nabla P = 0 \\ \operatorname{div} U = 0 \end{aligned}\right\} \quad \text{in } R^3, \tag{5.5}$$

and $|U(x) - U_{0\mu}(x)| = o(\frac{1}{|x|})$ as $|x| \to \infty$. We will seek U in the form

$$U = U_{0\mu} + V, \quad \text{where } V \in X. \tag{5.6}$$

It is clear $u(x,t) = \frac{1}{\sqrt{t}} U(\frac{x}{\sqrt{t}}) \in \mathcal{N}(\mu u_0)$ if and only if $U(x)$ satisfies the above elliptic system and $U(x) = U_{0\mu} + V$ for some $V \in X$, by Theorem 4.1. Thus we have reduced the problem to finding $V \in X$, with

$$\left.\begin{aligned} -\Delta V + V\cdot\nabla V + U_{0\mu}\cdot\nabla V \\ + V\cdot\nabla U_{0\mu} - \tfrac{V}{2} - \tfrac{x}{2}\cdot\nabla V + \nabla P = -U_{0\mu}\cdot\nabla U_{0\mu} \\ \operatorname{div} V = 0 \end{aligned}\right\}, \tag{5.7}$$

in R^3. We rewrite the above as:

$$-\Delta V - \frac{V}{2} - \frac{x}{2}\cdot\nabla V + \nabla P = -V\cdot\nabla V - U_{0\mu}\cdot\nabla V - V\cdot\nabla U_{0\mu} - U_{0\mu}\cdot\nabla U_{0\mu}. \tag{5.8}$$

Since $V \in X$, V satisfies the above equation if and only if $v(x,t) := \frac{1}{\sqrt{t}} V(\frac{x}{\sqrt{t}})$ satisfies

$$\left.\begin{aligned} \partial_t v - \Delta v + \nabla p = t^{-3/2} F(\tfrac{x}{\sqrt{t}}) \\ \operatorname{div} v = 0 \\ v(\cdot,0) = 0 \end{aligned}\right\}, \tag{5.9}$$

where

$$F=-V\cdot\nabla V-U_{0\mu}\cdot\nabla V-V\cdot\nabla U_{0\mu}-U_{0\mu}\cdot\nabla U_{0\mu} \tag{5.10}$$

has the decay properties in Lemma 4.1. Thus for such F, Eq. (5.9) is uniquely solvable, we denote the solution profile at time 1 as $\mathcal{G}(F)\in X$. This enables us to consider the following equivalent formulation,

$$\text{find}\quad V\in X\quad \text{with } V=\mathcal{G}(-V\cdot\nabla V-U_{0\mu}\cdot\nabla V-V\cdot\nabla U_{0\mu}-U_{0\mu}\cdot\nabla U_{0\mu}). \tag{5.11}$$

Let $K: X\times[0,1]\to X$ be defined as:

$$\begin{aligned}&\forall V\in X,\quad \mu\in[0,1],\\ &\quad K(V,\mu):=\mathcal{G}(U_{0\mu}\nabla U_{0\mu})+\mathcal{G}(U_{0\mu}\nabla V+V\nabla U_{0\mu}+V\nabla V).\end{aligned} \tag{5.12}$$

Note that $\mathcal{G}(U_{0\mu}\nabla U_{0\mu})=\mu^2\mathcal{G}(U_0\nabla U_0)$ has a one-dimensional range. We claim that the second term is compact. To verify the claim, let us consider a bounded sequence $V^{(j)}$ in X together with $\mu_j\in[0,1]$ and set $G^{(j)}=\mathcal{G}(U_{0\mu_j}\nabla V^{(j)}+V^{(j)}\nabla U_{0\mu_j}+V^{(j)}\nabla V^{(j)})$. We note that the fields $U_{0\mu_j}\nabla V^{(j)}$, $V^{(j)}\nabla U_{0\mu_j}$, and $V^{(j)}\nabla V^{(j)}$ all have decay $(1+|x|)^{-4}$ or better uniformly in j as $|x|\to\infty$, and hence we can apply Lemma 4.1, (iii) to show that for any given $\varepsilon>0$ there exists $R>0$ such that $\sup_{|x|\geq R}[(1+|x|)^2|G^{(j)}|+(1+|x|)^3|\nabla G^{(j)}|]\leq\varepsilon$ uniformly in j. In the region $|x|\leq R$ the functions $G^{(j)}$ will be bounded in $C^{1,\alpha}$ uniformly in j due to classical elliptic estimates for the linear Stokes operator and the claim follows easily. We conclude that $K: X\times[0,1]\to X$ is compact. We can also easily verify that K is continuous in $X\times[0,1]$ by similar arguments based on the estimates in Lemma 4.1. Therefore to solve the problem:

$$\text{find}\quad V\in X,\quad \text{such that } V+K(V,\mu)=0,\quad \text{where } \mu\in[0,1],$$

we can apply Leray's method, see for example [20]. We need the following conditions to be verified:

1. Solvability for μ small. This is already done, for example in [5, 11], note that it also follows from a simple implicit function theorem in our formulation. In the language of Leray Schauder degree theory, we can verify $d(I+K(\cdot,\mu),B_M(0),0)=1$ for μ small and some fixed $M>0$.
2. A priori estimate for solutions. This is done, in Theorem 4.1.
3. Compactness and continuity of K. This follows from the estimates of $\mathcal{G}$.

Thus we can apply Leray's method, and conclude that for each $\mu \in [0, 1]$, there exists a solution $V \in X$ to $V + K(V, \mu) = 0$. Take $\mu = 1$, the theorem is proved. □

With the existence theorem for smooth (away from 0) -1 homogeneous initial data, we can obtain existence results for not so smooth initial data. We illustrate the method with Hölder continuous (away from 0) initial data, although more general initial data can be considered.

Theorem 5.2 *Let $u_0 \in C^{\alpha}_{\text{loc}}(R^3 \setminus \{0\})$ with $\alpha \in (0, 1)$, $\lambda u_0(\lambda x) = u_0(x)$ for all $\lambda > 0$, and $\operatorname{div} u_0 = 0$ in R^3. Denote $M = \|u_0\|_{C^{\alpha}(\partial B_1)}$. Then there exists $u \in \mathcal{N}(u_0)$, and u satisfies $u(x, t) = \lambda u(\lambda x, \lambda^2 t)$ for all $\lambda > 0$. Moreover, let $U(x) = u(x, 1)$. Then $U \in C^{\infty}(R^3)$ with*

$$\left|U(x) - e^{\Delta} u_0(x)\right| \leq \frac{C(M)}{(1 + |x|)^{1+\alpha}}. \tag{5.13}$$

Proof Let us choose $u_0^{\epsilon} \in C^{\infty}(R^3 \setminus \{0\})$ with $\lambda u_0^{\epsilon}(\lambda x) = u_0^{\epsilon}(x)$ for all $\lambda > 0$, $\operatorname{div} u_0^{\epsilon} = 0$ in R^3, $\|u_0^{\epsilon}\|_{C^{\alpha}(\partial B_1(0))} \leq CM$, and $\|u_0^{\epsilon} - u_0\|_{C(\partial B_1)} \to 0$ as $\epsilon \to 0+$. We can construct such u_0^{ϵ} by first mollifying u_0 on the unit sphere and then using the scaling invariance and applying Helmholtz projection operator to form u_0^{ϵ}. We only note that the scaling invariance is preserved by the Helmholtz projection. By Theorem 5.1, we can find $u^{\epsilon} \in \mathcal{N}(u_0^{\epsilon})$ with $\lambda u^{\epsilon}(\lambda x, \lambda^2 t) = u^{\epsilon}(x, t)$, for all $\lambda > 0$. Let $U^{\epsilon}(x) = u^{\epsilon}(x, 1)$, then $u^{\epsilon}(x, t) = \frac{1}{\sqrt{t}} U^{\epsilon}(\frac{x}{\sqrt{t}})$. For any $x_0 \in R^3$ with $|x_0| = 8$, since $u_0^{\epsilon} \in C^{\alpha}(B_4(x_0))$ with $\|u_0^{\epsilon}\|_{C^{\alpha}(B_4(x_0))} \leq C(M)$, by Theorem 3.2, there exists $T(M) > 0$, such that $u^{\epsilon} \in C^{\alpha}_{\text{par}}(B_{1/2} \times [0, T(M)])$ and $\|u^{\epsilon}\|_{C^{\alpha}_{\text{par}}(B_{1/2} \times [0, T(M)])} \leq C(M)$. Thus,

$$\left|\frac{1}{\sqrt{t}} U^{\epsilon}\left(\frac{x_0}{\sqrt{t}}\right) - u_0^{\epsilon}(x_0)\right| \leq C(M) t^{\alpha/2}, \quad \text{for } t < T(M). \tag{5.14}$$

By the homogeneity of u_0^{ϵ}, we get

$$\left|U^{\epsilon}\left(\frac{x_0}{\sqrt{t}}\right) - u_0^{\epsilon}\left(\frac{x_0}{\sqrt{t}}\right)\right| \leq C(M) t^{1/2+\alpha/2}, \quad \text{for } t < T(M). \tag{5.15}$$

Notice that $|x_0| = 8$ is arbitrary, we get

$$\left|U^{\epsilon}(x) - u_0^{\epsilon}(x)\right| \leq \frac{C(M)}{|x|^{1+\alpha}} \quad \text{for } |x| > C_1(M). \tag{5.16}$$

Moreover, by following the same arguments in the proof of Theorem 4.1, we can obtain

$$\left\|U^{\epsilon}\right\|_{C^k(B_R(0))} \leq C(k, M, R) \quad \text{for } \forall R > 0. \tag{5.17}$$

By combining the above estimates and using elementary properties of heat equation, we get

$$\left|U^\epsilon(x) - e^\Delta u^\epsilon(x)\right| \le \frac{C(M)}{(1+|x|)^{1+\alpha}}, \quad \text{for } x \in R^3. \tag{5.18}$$

Note also that since u^ϵ satisfies the Navier-Stokes equations for $t > 0$, U^ϵ satisfies

$$\left.\begin{aligned} -\Delta U^\epsilon + U^\epsilon \cdot \nabla U^\epsilon - \tfrac{x}{2} \cdot \nabla U^\epsilon - \tfrac{U^\epsilon}{2} + \nabla P^\epsilon = 0 \\ \operatorname{div} U^\epsilon = 0 \end{aligned}\right\} \quad \text{in } R^3.$$

By the estimates on U^ϵ, we can pass to a subsequence $\epsilon_i \to 0+$, such that $U^{\epsilon_i} \to U$ in $C^2(B_R(0))$ for all $R > 0$. Thus U satisfies

$$\left.\begin{aligned} -\Delta U + U \cdot \nabla U - \tfrac{x}{2} \cdot \nabla U - \tfrac{U}{2} + \nabla P = 0 \\ \operatorname{div} U = 0 \end{aligned}\right\} \quad \text{in } R^3, \tag{5.19}$$

and

$$\left|U(x) - e^\Delta u_0(x)\right| \le \frac{C(M)}{(1+|x|)^{1+|\alpha|}} \quad \text{for all } x \in R^3. \tag{5.20}$$

Setting $u(x,t) = \frac{1}{\sqrt{t}} U(\frac{x}{\sqrt{t}})$, we can easily verify that u satisfies all the conditions in our theorem. □

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References

1. Bogovskii, M.E.: Solutions of some problems of vector analysis, associated with the operators div and grad. (Russian) Theory of cubature formulas and the application of functional analysis to problems of mathematical physics, pp. 5–40, 149, Trudy Sem. S. L. Soboleva, No. 1, 1980, Akad. Nauk SSSR Sibirsk. Otdel., Inst. Mat., Novosibirsk, 1980. (Reviewer: G. Wildenhain), 26B12 (35C05 35Q99 46E35)
2. Brandolese, L.: Fine properties of self-similar solutions of the Navier-Stokes equations. Arch. Ration. Mech. Anal. **192**(3), 375–401 (2009). (English summary)
3. Caffarelli, L., Kohn, R.-V., Nirenberg, L.: Partial regularity of suitable weak solutions of the Navier-Stokes equations. Commun. Pure Appl. Math. **XXXV**, 771–831 (1982)
4. Calderon, C.P.: Existence of weak solutions for the Navier-Stokes equations with initial data in L^p. Trans. Am. Math. Soc. **318**(1), 179–200 (1990)
5. Cannone, M., Planchon, F.: Self-similar solutions for Navier-Stokes equations in R^3. Commun. Partial Differ. Equ. **21**(1–2), 179–193 (1996)
6. Dong, H., Du, D.: On the local smoothness of solutions of the Navier-Stokes equations. J. Math. Fluid Mech. **9**(2), 139–152 (2007)

7. Escauriaza, L., Seregin, G., Šverák, V.: $L_{3,\infty}$ solutions of Navier-Stokes equations and backward uniqueness. Usp. Mat. Nauk **58**(2(350)), 3–44 (2003). Translation in Russ. Math. Surv. **58**(2), 211–250 (2003)
8. Evans, L.: Quasiconvexity and partial regularity in the calculus of variations. Arch. Ration. Mech. Anal. **95**(3), 227–252 (1986)
9. Geiert, M., Heck, H., Hieber, M.: On the equation $\operatorname{div} u = g$ and Bogovski's operator in Sobolev spaces of negative order. In: Partial Differential Equations and Functional Analysis. Oper. Theory Adv. Appl., vol. 168, pp. 113–121. Birkhäuser, Basel (2006). (English summary)
10. Germain, P., Pavlovic, N., Staffilani, N.: Regularity of solutions to the Navier-Stokes equations evolving from small data in BMO^{-1}. Int. Math. Res. Not. No. 21 (2007). 35 pp
11. Giga, Y., Miyakawa, T.: Navier-Stokes flow in R^3 with measures as initial vorticity and Morrey spaces. Commun. Partial Differ. Equ. **14**(5), 577–618 (1989)
12. Jia, H., Šverák, V.: Minimal L^3-initial data for potential Navier-Stokes singularities. arXiv:1201.1592v1
13. Kato, T.: Strong L^p-solutions of the Navier-Stokes equation in R^m, with applications to weak solutions. Math. Z. **187**(4), 471–480 (1984)
14. Kikuchi, N., Seregin, G.: Weak Solutions to the Cauchy Problem for the Navier-Stokes Equations Satisfying the Local Energy Inequality. AMS Translations, Series 2 vol. 220, pp. 141–164 (2007)
15. Koch, H., Tataru, D.: Well-posedness for the Navier-Stokes equations. Adv. Math. **157**(1), 22–35 (2001)
16. Ladyzhenskaya, O.A., Seregin, G.A.: On partial regularity of suitable weak solutions to the three-dimensional Navier-Stokes equations. J. Math. Fluid Mech. **1**(4), 356–387 (1999)
17. Lin, F.-H.: A new proof of the Caffarelli-Korn-Nirenberg theorem. Commun. Pure Appl. Math. **51**(3), 241–257 (1998)
18. Lemarié-Rieusset, P.G.: Recent Developments in the Navier-Stokes Problem. Chapman & Hall/CRC Research Notes in Mathematics, vol. 431. Chapman & Hall/CRC, Boca Raton (2002)
19. Leray, J.: Sur le mouvement d'un liquide visqueux emplissant l'espace. Acta Math. **63**, 193–248 (1934)
20. Mawhin, J.: Leray-Schauder degree: a half century of extensions and applications. Topol. Methods Nonlinear Anal. **14**, 195–228 (1999)
21. Rudin, W., Šverák, V.: Minimal initial data for potential Navier-Stokes singularities. J. Funct. Anal. 260(3), 879–891
22. Stein, E.: Harmonic Analysis: Real-Variable Methods, Orthogonality and Oscillatory Integrals. Princeton Mathematical Series, Vol. 43: Monographs in Harmonic Analysis, III, Princeton University Press, Princeton (1993). xiv+695 pp
23. Seregin, G.: A certain necessary condition for potential blow up for Navier-Stokes equations. Commun. Math. Phys. **312**(3), 833–845 (2012)
24. Tao, T.: Localization and compactness properties of the Navier-Stokes global regularity problem. arXiv:1108.1165v3