SPECTRUM OF THE LAMÉ OPERATOR AND APPLICATION, II: WHEN AN ENDPOINT IS A CUSP

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ABSTRACT. This article is the second part of our study of the spectrum $\sigma(L_n;\tau)$ of the Lamé operator

$$L_n = \frac{d^2}{dx^2} - n(n+1)\wp(x+z_0;\tau) \quad \text{in } L^2(\mathbb{R},\mathbb{C}),$$

where $n \in \mathbb{N}$, $\wp(z;\tau)$ is the Weierstrass elliptic function with periods 1 and τ, and $z_0 \in \mathbb{C}$ is chosen such that L_n has no singularities on $\mathbb{R}$. An endpoint of $\sigma(L_n;\tau)$ is called a *cusp* if it is an intersection point of at least three semi-arcs of $\sigma(L_n;\tau)$. We obtain a necessary and sufficient condition for the existence of cusps in terms of monodromy datas and prove that $\sigma(L_n;\tau)$ has at most one cusp for fixed τ. We also consider the case $n = 2$ and study the distribution of τ's such that $\sigma(L_2;\tau)$ has a cusp. For any $\gamma \in \Gamma_0(2)$ and the fundamental domain $\gamma(F_0)$, where $F_0 := \{\tau \in \mathbb{H} |\ 0 \leqslant \operatorname{Re}\tau \leqslant 1, |z - \frac{1}{2}| \geqslant \frac{1}{2}\}$ is the basic fundamental domain of $\Gamma_0(2)$, we prove that there are either 0 or 3 τ's in $\gamma(F_0)$ such that $\sigma(L_2;\tau)$ has a cusp and also completely characterize those γ's.

To prove such results, we will give a complete description of the critical points of the classical modular forms $e_1(\tau), e_2(\tau), e_3(\tau)$, which is of independent interest.

1. INTRODUCTION

This article is the second in a series, initiated in Part I [8], devoted to the geometry of the spectrum $\sigma(L_n) = \sigma(L_n;\tau)$ of the Lamé operator

$$L_n := \frac{d^2}{dx^2} - n(n+1)\wp(x+z_0;\tau), \quad x \in \mathbb{R} \tag{1.1}$$

in $L^2(\mathbb{R},\mathbb{C})$, where $n \in \mathbb{N}$ and $z_0 \in \mathbb{C}$ is chosen such that $\wp(x+z_0;\tau)$ has no singularities on $\mathbb{R}$. Here $\wp(z) = \wp(z;\tau)$ is the Weierstrass $\wp$-function with basic periods $\omega_1 = 1$ and $\omega_2 = \tau \in \mathbb{H} = \{\tau |\operatorname{Im}\tau > 0\}$. Denote also $\omega_3 = 1+\tau$. We take [1, 29] as our general reference on elliptic functions and modular forms.

Since the Lamé potential $n(n+1)\wp(z;\tau)$ is a Picard potential in the sense of Gesztesy and Weikard [17, 18] (i.e. *all solutions* of the Lamé equation

$$y''(z) = [n(n+1)\wp(z;\tau)+E]y(z), \quad z \in \mathbb{C} \tag{1.2}$$

are *meromorphic* in $\mathbb{C}$), the spectrum $\sigma(L_n)$ does not depend on the choice of z_0 and indeed we can even take $z_0 = 0$ where the Lamé potential has singularities on $\mathbb{R}$; see [8, 34]. In particular, the Hill's discriminant $\Delta(E)$

(i.e. $\Delta(E)$ is the trace of the monodromy matrix of (1.2) with respect to $z \to z+1$) is well-defined and is an entire function of $E \in \mathbb{C}$. The spectral theory of the Schrödinger operator $L = \frac{d^2}{dx^2} - q(x)$ with periodic, regular, but complex-valued potentials $q(x)$ has been studied widely in the literature; see e.g. [2, 3, 17, 19, 20, 30] and references therein. In particular, it is known [30] that the spectrum $\sigma(L)$ satisfies

$$\sigma(L) = \Delta^{-1}([-2,2]) = \{E \in \mathbb{C} \mid -2 \leq \Delta(E) \leq 2\}. \tag{1.3}$$

Furthermore, it was proved in [17] that $\sigma(L)$ consists of *one* semi-infinite simple analytic arc tending to $-\infty + \int_{x_0}^{x_0+1} q(x)dx$ (here we assume that 1 is a basic period of $q(x)$) and *finitely many bounded* analytic arcs, provided that $q(x)$ is a Picard potential or equivalently an algebro-geometric finite gap potential in the KdV theory. Indeed, since $q(x)$ is an algebro-geometric finite gap potential, there is a linear differential operator $P_{2g+1} = d^{2g+1}/dx^{2g+1} + \cdots$ with smallest odd order $2g+1$ such that $[P_{2g+1}, L] = 0$. A celebrated theorem of Burchnall and Chaundy [6] implies the existence of the so-called *spectral polynomial* $Q(E)$ of degree $2g+1$ in E associated to $q(x)$ such that

$$P_{2g+1}^2 = Q(\tfrac{d^2}{dx^2} - q(x)).$$

Then [17, Theorem 4.1] says that the *finite endpoints* of $\sigma(L)$ coincide with those zeros of the *spectral polynomial* $Q(E)$ with *odd order*, which implies that the number of the spectral arcs is finite.

Remark 1.1. An endpoint E_0 of a spectral arc of $\sigma(L)$ is a point where the arc can not be analytically continued (cf. [17]). By (1.3) E_0 is an endpoint if and only if

$$d(E_0) := \mathrm{ord}_{E_0}(\Delta(\cdot)^2 - 4)$$

is *odd* (which infers $Q(E_0) = 0$ with odd order; see [17, Theorem 4.1]), and in this case there are $d(E_0)$ arcs meeting at E_0, each of which can not be analytically continued at E_0 because adjacent arcs meet at E_0 with an angle $2\pi/d(E_0)$. If $d(E_0) = 2k \geq 2$ is even, then E_0 is an inner point of k arcs which are all analytic at E_0, so such E_0 is not considered as an endpoint.

For the Lamé case (1.1), the associate spectral polynomial $Q_n(E;\tau)$ is also known as the Lamé polynomial with degree $2n+1$. For example (see e.g. [26, 28, 35])

$$Q_1(E;\tau) = \prod_{k=1}^{3}(E - e_k(\tau)),$$

$$Q_2(E;\tau) = (E^2 - 3g_2(\tau))\prod_{k=1}^{3}(E + 3e_k(\tau)). \tag{1.4}$$

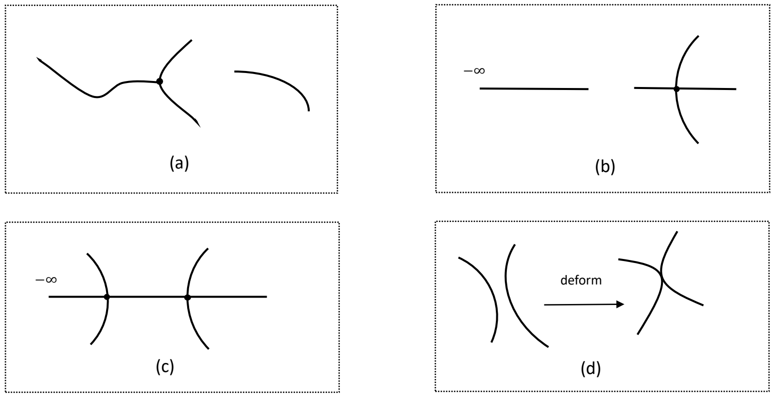

FIGURE 1. The dark point in (a) is a cusp, while the dark point in (b) is an intersection point but not a cusp.

Here $e_k = e_k(\tau) := \wp(\frac{\omega_k}{2};\tau)$ for $k \in \{1,2,3\}$, and g_2 is the well known invariant of the elliptic curve $E_\tau := \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$:

$$\wp'(z;\tau)^2 = 4\prod_{k=1}^{3}(\wp(z;\tau)-e_k(\tau)) = 4\wp(z;\tau)^3 - g_2(\tau)\wp(z;\tau) - g_3(\tau),$$

i.e. $(e_k(\tau),0)$'s are the branch points of the elliptic curve. Thus, $\sigma(L_1)$ has only three finite endpoints $e_k(\tau)$, $k=1,2,3$, and $\sigma(L_2)$ has five finite endpoints $\pm(3g_2(\tau))^{1/2}$ and $-3e_k(\tau)$, $k=1,2,3$.

Definition 1.2. *We call that a finite endpoint E of $\sigma(L_n)$ is a cusp if E is an intersection point of at least three semi-arcs.*

For example, the dark point in (a) of Figure 1 is a cusp. While in (b), although the dark point is an intersection point of different arcs, it is not a cusp because it is not an endpoint.

Our first question is how to characterize a cusp for the Lamé operator? As mentioned before, all solutions of (1.2) are meromorphic, so the monodromy matrices S_j of (1.2) with respect to $z \to z+\omega_j$, $j=1,2$, satisfies $S_1S_2 = S_2S_1$ and hence there are two cases: Case (i) S_1 and S_2 can be diagonized simultaneously, i.e. there is a fundamental system of solutions of (1.2) such that in terms of these solutions,

$$S_1 = \begin{pmatrix} e^{-2\pi i s} & 0 \\ 0 & e^{2\pi i s}\end{pmatrix}, \quad S_2 = \begin{pmatrix} e^{2\pi i r} & 0 \\ 0 & e^{-2\pi i r}\end{pmatrix}, \tag{1.5}$$

where $(r,s) \in \mathbb{C}^2 \setminus \frac{1}{2}\mathbb{Z}^2$; Case (ii) there is a fundamental system of solutions of (1.2) such that in terms of these solutions,

$$S_1 = \varepsilon_1\begin{pmatrix} 1 & 0 \\ 1 & 1\end{pmatrix}, \quad S_2 = \varepsilon_2\begin{pmatrix} 1 & 0 \\ \mathcal{C} & 1\end{pmatrix}, \tag{1.6}$$

where $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$ and $\mathcal{C} \in \mathbb{C} \cup \{\infty\}$. Remark that if $\mathcal{C} = \infty$, then (1.6) should be understood as

$$S_1 = \varepsilon_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S_2 = \varepsilon_2 \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \tag{1.7}$$

A well-known result about the spectral polynomial $Q_n(E;\tau)$ is

(1.8) *Case (ii) occurs if and only if $Q_n(E;\tau) = 0$.*

We will briefly review these facts in Section 2. Our first main result is

Theorem 1.3. *Fix any τ such that the multiplicity of any zero of $Q_n(E;\tau)$ is at most* 2*, then the following hold.*

1. *An endpoint E of $\sigma(L_n;\tau)$ is a cusp if and only if the corresponding monodromy data $\mathcal{C} = \infty$.*
2. *$\sigma(L_n;\tau)$ has at most one cusp.*

Remark 1.4. It is well known (cf. [35]) that except finitely many τ's modulo $SL(2,\mathbb{Z})$, $Q_n(E;\tau)$ has *distinct* zeros. On the other hand, for $n \leq 5$ it can be computed directly via the expression of $Q_n(E;\tau)$ (see [28, Table 3]) that the multiplicity of any zero of $Q_n(E;\tau)$ is at most 2. [28, Proposition 3.2] said that this assertion also holds for all n, but it seems that no proof was provided in [28]. By [28, Proposition 3.2], the assumption that the multiplicity of any zero of $Q_n(E;\tau)$ is at most 2 holds automatically.

Remark 1.5. In Part I [8], we proved the existence of $\tau = \frac{1}{2} + ib$ such that $\sigma(L_2;\tau)$ is of the form (c) in Figure 1, namely $\sigma(L_2;\tau)$ have exactly 2 intersection points but no cusps. Therefore, unlike Theorem 1.3 we can not expect that $\sigma(L_n;\tau)$ has at most one intersection point for all τ if $n \geq 2$.

Generally it is hard to find the explicit formula for C in terms of τ. Here we give two examples. Recall the Weierstrass zeta function $\zeta(z) = \zeta(z;\tau)$ defined by $\zeta'(z) = -\wp(z)$ and $\eta_k(\tau)$ are two quasi-periods of $\zeta(z;\tau)$ defined by

$$\eta_k(\tau) := 2\zeta(\tfrac{\omega_k}{2};\tau) = \zeta(z+\omega_k;\tau) - \zeta(z;\tau), \quad k = 1,2. \tag{1.9}$$

Example 1.6. *Let $n = 1$ and $E = e_k(\tau)$, then*

$$\mathcal{C} = \tau - \frac{2\pi i}{e_k(\tau) + \eta_1(\tau)}, \quad k = 1,2,3.$$

Let $n = 2$ and recall (1.4). If $E = -3e_k(\tau)$, $k = 1,2,3$, then

$$C = \tau - \frac{2\pi i e_k(\tau)}{\frac{g_2(\tau)}{6} + \eta_1(\tau)e_k(\tau) - e_k(\tau)^2}; \tag{1.10}$$

If $E = \pm(3g_2)^{1/2}$, then

$$C = \tau - \frac{2\pi i}{\eta_1(\tau) \pm (g_2(\tau)/12)^{1/2}}. \tag{1.11}$$

See e.g. [12] *for the proof.*

It is interesting to note that the denominators of both (1.10) and (1.11) are related to the derivatives of $e_k(\tau)$ and $\eta_1(\tau)$:

$$e_k'(\tau) = \frac{i}{\pi}\left[\frac{1}{6}g_2(\tau) + \eta_1(\tau)e_k(\tau) - e_k(\tau)^2\right], \tag{1.12}$$

$$\eta_1'(\tau) = \frac{i}{2\pi}[\eta_1(\tau)^2 - g_2(\tau)/12]. \tag{1.13}$$

See e.g. [5] for (1.12)-(1.13). Remark that (1.13) is also one of the famous Ramanujan's formula for the Eisenstein series $E_2(\tau)$. Then Theorem 1.3 for $n = 2$ immediately implies the following

Corollary 1.7. *For any $\tau \in \mathbb{H}$, τ can not be a common critical point of any two of $\eta_1(\tau)$, $e_k(\tau)$, $k = 1, 2, 3$.*

In view of theorem 1.3, we naturally consider the problem of determining the distribution of those τ's such that $\sigma(L_n)$ has a cusp, which has been studied for the case $n = 1$ in [2, 10, 20, 33]. In this article, we give a complete answer for the case $n = 2$. For this case, it follows from Theorem 1.3 and (1.10)-(1.13) that the distribution of those τ's such that $\sigma(L_2)$ has a cusp is equivalent to the distribution of *critical points* of the classical (quasi)-modular forms $\eta_1(\tau)$ and $e_k(\tau)$, $k = 1, 2, 3$. These special functions can be expressed explicitly by theta functions (see e.g. [25, Section 9]), and it is well-known that $e_k(\tau)$ is a modular form of weight 2 with respect to $\Gamma(2)$ while $\eta_1(\tau)$ is not a modular form but only a quasi-modular form on $SL(2,\mathbb{Z})$. The distribution of critical points of (quasi)-modular forms is also an interesting problem from the viewpoint of number theory; see e.g. [4, 23, 31] and references therein. In particular, it was proved in [31] that for each modular form $f(\tau)$ for a subgroup of $SL(2,\mathbb{Z})$, its derivative $f'(\tau)$ has infinitely many inequivalent zeros and all, but a finite number, are simple.

To study the zeros of $e_k'(\tau)$ and $\eta_1'(\tau)$, we consider the congruence subgroup $\Gamma_0(2)$ of $SL(2,\mathbb{Z})$ defined by

$$\Gamma_0(2) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z}) \middle| \, c \equiv 0 \mod 2 \right\},$$

with the "basic" fundamental domain[1]

$$F_0 := \{\tau \in \mathbb{H} \mid 0 \leqslant \operatorname{Re}\tau \leqslant 1 \text{ and } |z - \tfrac{1}{2}| \geqslant \tfrac{1}{2}\}, \tag{1.14}$$

and for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)/\{\pm I_2\}$ (i.e. *consider γ and $-\gamma$ to be the same*),

$$\gamma(F_0) := \left\{ \gamma \cdot \tau := \frac{a\tau + b}{c\tau + d} \middle| \, \tau \in F_0 \right\} = (-\gamma)(F_0)$$

is another fundamental domain of $\Gamma_0(2)$. Note that

$$\mathbb{H} = \bigcup_{\gamma \in \Gamma_0(2)/\{\pm I_2\}} \gamma(F_0). \tag{1.15}$$

Moreover, $\gamma(F_0) = F_0 + m$ for some $m \in \mathbb{Z}$ if and only if $c = 0$.

[1] Of course, the standard definition of F_0 should be $F_0 = \{\tau \in \mathbb{H} \mid 0 \leqslant \operatorname{Re}\tau < 1 \text{ and } |z - \frac{1}{2}| \geqslant \frac{1}{2}\} \setminus \{\tau | \operatorname{Re}\tau > \frac{1}{2}, |z - \frac{1}{2}| = \frac{1}{2}\}$. But it is more convenient for us to use the definition (1.14) in this article.

Note that for any

$$E(\tau) \in \{\pm(3g_2(\tau))^{1/2},\ -3e_1(\tau),\ -3e_2(\tau),\ -3e_3(\tau)\}, \tag{1.16}$$

$E'(\tau)$ and $\eta_1'(\tau)$ are not modular forms but only quasi-modular forms. Therefore, the following result is quite unexpected.

Theorem 1.8. *Let $F = \gamma(F_0)$ be any fundamental domain of $\Gamma_0(2)$ with $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)/\{\pm I_2\}$, and $E(\tau)$ be one of the 5 points in (1.16). Then there is at most one $\tau_0 \in F$ such that $E(\tau_0)$ is a cusp of $\sigma(L_2;\tau_0)$. Moreover, the following hold.*

(1) *One of $\pm(3g_2(\tau_0))^{1/2}$ is a cusp for some $\tau_0 \in F$ if and only if $c \neq 0$.*
(2) *$-3e_1(\tau_0)$ is a cusp for some $\tau_0 \in F$ if and only if $c \neq 0$ and $\frac{-d}{c} \in (-\infty, 0) \cup (1, +\infty)$.*
(3) *$-3e_2(\tau_0)$ is a cusp for some $\tau_0 \in F$ if and only if $c \neq 0$ and*
$$\text{either } b \in 2\mathbb{Z},\ \tfrac{-d}{c} < 1 \text{ or } b \in 2\mathbb{Z}+1,\ \tfrac{-d}{c} > 0.$$
(4) *$-3e_3(\tau_0)$ is a cusp for some $\tau_0 \in F$ if and only if $c \neq 0$ and*
$$\text{either } b \in 2\mathbb{Z}+1,\ \tfrac{-d}{c} < 1 \text{ or } b \in 2\mathbb{Z},\ \tfrac{-d}{c} > 0.$$
(5) *If $c = 0$ (i.e. $F = F_0 + m$ with $m \in \mathbb{Z}$), $\sigma(L_2;\tau)$ has no cusps for all $\tau \in F$.*
(6) *If $c \neq 0$, there are exactly 3 τ's in F such that $\sigma(L_2;\tau)$ has a cusp.*

Corollary 1.9. (1) *There exists $b_0 > 0$ such that $\eta_1(\frac{1}{2}+ib)$ is strictly increasing for $0 < b < b_0$ and is strictly decreasing for $b > b_0$.*
(2) *$e_k(\tau)$ has no critical points on the line $\operatorname{Re}\tau = \frac{1}{2}$. In particular, $e_1(\frac{1}{2}+ib)$ is strictly increasing in b.*

Note that the distribution of critical points of $\eta_1(\tau)$ was studied in [11], which implies Theorem 1.8-(1) and Corollary 1.9-(1) as a consequence. Theorem 1.8 (2)-(4) gives the distribution of critical points of $e_k(\tau)$, $k = 1, 2, 3$; see Theorems 4.1-4.3 for more precise statements. The modular form $e_1(\frac{1}{2} + ib)$ is real-valued for $b > 0$, and its $q = e^{2\pi i\tau} = -e^{-2\pi b}$ expansion is given by (see (3.6))

$$e_1(\tfrac{1}{2}+ib) = \frac{2\pi^2}{3} + 16\pi^2 \sum_{k=1}^{\infty}(-1)^k a_k e^{-2\pi bk}, \text{ where } a_k > 0,$$

so Corollary 1.9-(2) is not obvious although it is elementary. We believe it is known in the literature but we could not find any reference related to it until [25] where Corollary 1.9-(2) was proved via the theta functions. Here we will give a new proof of Corollary 1.9-(2) without using the theta functions.

Remark 1.10. This paper mainly deals with the $n = 2$ Lamé potential. When $n \geq 3$, the roots of the Lamé spectral polynomial (i.e. the endpoints of the spectral arcs), as well as the corresponding monodromy data $\mathcal{C}$'s, could become much more complicated algebraic functions of $\eta_1(\tau)$ and $e_k(\tau)$'s, which seem too difficult to study so far. So we think that new approaches are needed to deal with general $n \geq 3$ cases.

We believe that Theorems 1.3 and 1.8 will play important roles while deforming τ. For example, it is well known [22] that for $\tau \in i\mathbb{R}_{>0} \subset F_0$,

$$\sigma(L_2;\tau) = (-\infty, -(3g_2)^{\frac{1}{2}}] \sqcup [-3e_1, -3e_3] \sqcup [-3e_2, (3g_2)^{\frac{1}{2}}]. \tag{1.17}$$

On the other hand, we proved in Part I [8] that for $\tau = \frac{1}{2} + ib \in F_0$ with $b > \frac{\sqrt{3}}{2}$,

$$\sigma(L_2;\tau) = (-\infty, -3e_1] \sqcup [-(3g_2)^{1/2}, (3g_2)^{1/2}] \cup \sigma_2, \tag{1.18}$$

where σ_2 denotes a simple arc symmetric with respect to $\mathbb{R}$ with endpoints $-3e_2$ and $-3e_3$ and

$$\sigma_2 \cap \mathbb{R} = \sigma_2 \cap (-(3g_2)^{1/2}, (3g_2)^{1/2}) = \text{one intersection point};$$

namely the picture of $\sigma(L_2;\tau)$ is of the form (b) in Figure 1. Now consider any continuous loop

$$l : [0,1] \to F_0 \quad \text{with } \ l(0) \in i\mathbb{R}_{>0}, \ \ l(1) = \tfrac{1}{2} + ib, \ b > \tfrac{\sqrt{3}}{2}.$$

Then (1.17)-(1.18) imply that during the deformation of $\sigma(L_2;\tau)$ with τ along the loop $l[0,1]$, there is $t_0 \in (0,1)$ such that different arcs of $\sigma(L_2;l(t))$ have no intersections for $t \in [0,t_0)$, but at least two different arcs of $\sigma(L_2; l(t_0))$ have intersection points, which can *not* be cusps by Theorem 1.8-(5). Therefore, we immediately obtain

Corollary 1.11. *Under the above notations, we let E_0 be any intersection point of two different arcs of $\sigma(L_2;l(t_0))$. Then E_0 is an inner point of both arcs.*

In other words, during the deformation of $\sigma(L_2;\tau)$ with $\tau \in F_0$ varying, the intersection of different arcs can only appear "tangentially" (see (d) of Figure 1), i.e. the intersection can not appear from their endpoints.

On the other hand, during the deformation process, the set of τ's with *real* monodromy data $\mathcal{C}$ is also an important geometric object.

Theorem 1.12. *The following hold.*

1. *For any $C \in (-\infty,0) \cup (1,+\infty)$, there is a unique τ in F_0, denoted it by $\tau_1(C)$, such that Equation (1.10) with $k = 1$ holds.*
2. *For any $C \in (-\infty,0) \cup (0,1)$, there is a unique τ in F_0, denoted it by $\tau_2(C)$, such that Equation (1.10) with $k = 2$ holds.*
3. *For any $C \in (0,1) \cup (1,+\infty)$, there is a unique τ in F_0, denoted it by $\tau_3(C)$, such that Equation (1.10) with $k = 3$ holds.*

Theorem 1.12, which plays a crucial role in the proof of Theorem 1.8, defines 6 curves:

$$\mathcal{C}_{1,-} := \{\tau_1(C) | C \in (-\infty,0)\}, \ \ \mathcal{C}_{1,+} := \{\tau_1(C) | C \in (1,+\infty)\},$$
$$\mathcal{C}_{2,-} := \{\tau_2(C) | C \in (-\infty,0)\}, \ \ \mathcal{C}_{2,+} := \{\tau_2(C) | C \in (0,1)\},$$

$$\mathcal{C}_{3,-} := \{\tau_3(C)|C \in (0,1)\}, \ \mathcal{C}_{3,+} := \{\tau_3(C)|C \in (1,+\infty)\},$$

which will be proved to be smooth curves in Section 5. We will prove in Section 4 that up to Möbius transformations of $\Gamma_0(2)$ action, the critical points of $e_k(\tau)$ locate densely on these curves. Furthermore, the numerical simulation for these curves will be shown in Figure 2 of Section 5.

In addition to these 6 curves, Equation (1.11) also defines 3 smooth curves for $C \in (-\infty,0) \cup (0,1) \cup (1,+\infty)$, which has been proved in [11]. These 9 smooth curves are also related to the degeneracy curve of the multiple Green function $G_2(z_1,z_2;\tau)$ defined by

$$G_2(z_1,z_2;\tau) := G(z_1-z_2;\tau) - 2G(z_1;\tau) - 2G(z_2;\tau), \tag{1.19}$$

where $0 \neq z_1 \neq z_2 \neq 0$, and $G(z;\tau)$ is the Green function of the flat torus E_τ. See Section 5 for a precise statement.

Since the Lamé potential is doubly periodic, we can also consider its spectrum along the $\omega_2 = \tau$ direction. Denote the Hill's discriminant and the spectrum by $\Delta_j(E;\tau)$ and

$$\sigma_j(L_n;\tau) := \{E \in \mathbb{C} \mid -2 \leq \Delta_j(E;\tau) \leq 2\}$$

respectively along the ω_j direction (i.e. $\Delta_j(E;\tau)$ is the trace of the monodromy matrix with respect to $z \to z+\omega_j$), $j=1,2$. Clearly the aforementioned $\sigma(L_n;\tau) = \sigma_1(L_n;\tau)$. Recalling (1.5)-(1.8), we have

$$\Delta_1(E;\tau) = 2\cos 2\pi s, \quad \Delta_2(E;\tau) = 2\cos 2\pi r,$$

and the monodromy of the Lamé equation (1.2) is unitary (i.e. the monodromy is conjugate to a subgroup of $SU(2)$) if and only if $(r,s) \in \mathbb{R}^2 \setminus \frac{1}{2}\mathbb{Z}^2$, or equivalent to

$$E \in \sigma_1(L_n;\tau) \cap \sigma_2(L_n;\tau) \setminus \{E|Q_n(E;\tau)=0\}.$$

See e.g. [7]. Recently Eremenko [15] proved that $\sigma_1(L_n;\tau) \cap \sigma_2(L_n;\tau) \setminus \{E|Q_n(E;\tau)=0\}$ is a finite set. In general, the geometry of $\sigma_1(L_n;\tau)$ and $\sigma_2(L_n;\tau)$ might be quite different, which makes the set $\sigma_1(L_n;\tau) \cap \sigma_2(L_n;\tau)$ very difficult to study. In some situation, it might be more convenient for us to replace $\sigma_2(L_n;\tau)$ with the spectrum along another direction. For example for $\operatorname{Re}\tau = \frac{1}{2}$, it is simpler to consider the spectrum along $2\omega_2 - \omega_1 = 2\tau - 1$ direction:

$$\Delta_3(E;\tau) := 2\cos 2\pi(2r+s), \ \sigma_3(L_n;\tau) := \{E \in \mathbb{C} \mid -2 \leq \Delta_3(E;\tau) \leq 2\},$$

which clearly satisfies

$$\sigma_1(L_n;\tau) \cap \sigma_2(L_n;\tau) = \sigma_1(L_n;\tau) \cap \sigma_3(L_n;\tau).$$

In Part I [8] we used this idea to prove the existence of $b>0$ such that for $\tau = \frac{1}{2} + ib$, the set for $n=2$

$$\sigma_1(L_2;\tau) \cap \sigma_3(L_2;\tau) \setminus \{E|Q_2(E;\tau)=0\}$$

contains at least two points $E, \bar{E}$ with $E \neq \bar{E}$, and consequently, the curvature (or the mean field) equation

$$\triangle u + e^u = 16\pi\delta_0 \quad \text{on } E_\tau, \quad \tau = \tfrac{1}{2} + ib$$

has two even but not-axisymmetric solutions. In general, this class of even but not-axisymmetric solutions is difficult to obtain (see e.g. [16]), and Part I [8] gives the first result on this aspect.

Here we can generalize Theorem 1.3 to $\sigma_j(L_n;\tau)$, $j = 2, 3$.

Theorem 1.13. *Let $j \in \{2, 3\}$ and fix any τ such that the multiplicity of any zero of $Q_n(E;\tau)$ is at most 2. Then $\sigma_j(L_n;\tau)$ has at most one cusp. Furthermore,*

(1) *An endpoint E of $\sigma_2(L_n;\tau)$ is a cusp if and only if the monodromy data $\mathcal{C} = 0$;*
(2) *An endpoint E of $\sigma_3(L_n;\tau)$ is a cusp if and only if the monodromy data $\mathcal{C} = \frac{1}{2}$.*

Theorems 1.8 and 1.13 imply that both $\sigma_1(L_2;\tau)$ and $\sigma_2(L_2;\tau)$ has no cusps for $\tau \in F_0$, while $\sigma_3(L_2;\tau)$ has a cusp for some $\tau \in F_0$.

The rest of this article is organized as follows. In Section 2, we discuss the relation between the monodromy data $\mathcal{C}$ and the cusp, and prove Theorems 1.3 and 1.13. In Section 3, we give the precise definition of the curves $\mathcal{C}_{k,\pm}$'s via a parametrization and prove Theorem 1.12. In Section 4, we apply Theorem 1.12 to give the complete distribution of the critical points of $e_k(\tau)$'s (see Theorems 4.1-4.3), which will imply Theorem 1.8 and Corollary 1.9-(2) as consequences. In Section 5, we establish the relation between the six curves and the multiple Green function G_2 and prove the smoothness of the curves $\mathcal{C}_{k,\pm}$'s.

2. The monodromy data and the cusp

This section is devoted to the proof of Theorems 1.3 and 1.13. First we explain how to determine the monodromy data $\mathcal{C}$ for the Lamé equation

$$y''(z) = [n(n+1)\wp(z;\tau) + E]y(z), \quad z \in \mathbb{C}. \tag{2.1}$$

It is well known (see e.g. [7, 35]) that for any $E \in \mathbb{C}$, there is a unique pair $\pm\boldsymbol{a} = \pm\{a_1, \cdots, a_n\} \subset E_\tau \setminus \{0\}$ such that the Hermite-Halphen ansatz

$$y_{\pm\boldsymbol{a}}(z) := e^{z\sum_{i=1}^n \zeta(\pm a_i)} \frac{\prod_{i=1}^n \sigma(z \mp a_i)}{\sigma(z)^n}$$

solve (2.1) with $E = (2n-1)\sum_{i=1}^n \wp(a_i)$. Here $\sigma(z) := \exp \int^z \zeta(\xi)d\xi$ is the Weierstrass sigma function.

When $Q_n(E;\tau) \neq 0$, $y_{\boldsymbol{a}}(z)$ and $y_{-\boldsymbol{a}}(z)$ are linearly independent, and the monodromy matrices with respect to $(y_{\boldsymbol{a}}(z), y_{-\boldsymbol{a}}(z))$ are give by (1.5).

When $Q_n(E;\tau) = 0$, $y_{\boldsymbol{a}}(z) = (-1)^n y_{-\boldsymbol{a}}(z)$ satisfies

$$y_{\boldsymbol{a}}(z + \omega_k) = \varepsilon_k y_{\boldsymbol{a}}(z), \quad \varepsilon_k = \pm 1, \quad k = 1, 2, \tag{2.2}$$

i.e. $y_{\boldsymbol{a}}(z)^{-2}$ is even elliptic, and there is an odd meromorphic function $\chi(z)$ such that $\chi'(z) = y_{\boldsymbol{a}}(z)^{-2}$ and $\chi(z)$ is quasi-periodic, i.e.

$$\chi_k := \chi(z + \omega_k) - \chi(z), \quad k = 1, 2,$$

are two constants which can not vanish simultaneously. Then a direct computation shows that

$$y_2(z) := y_{\boldsymbol{a}}(z)\chi(z) \tag{2.3}$$

is also a solution of the Lamé equation (2.1), which is linearly independent with $y_{\boldsymbol{a}}(z)$ and satisfies

$$y_2(z + \omega_k) = \varepsilon_k(y_2(z) + \chi_k y_{\boldsymbol{a}}(z)), \quad k = 1, 2. \tag{2.4}$$

Define

$$\mathcal{C} := \tfrac{\chi_2}{\chi_1} \in \mathbb{C} \cup \{\infty\}. \tag{2.5}$$

We claim that this $\mathcal{C}$ is precisely the *monodromy data*. In fact, if $\chi_1 = 0$, then $\chi_2 \neq 0$, $\mathcal{C} = \infty$ and it follows from (2.2)-(2.4) that

$$\begin{pmatrix} \chi_2 y_{\boldsymbol{a}}(z + \omega_1) \\ y_2(z + \omega_1) \end{pmatrix} = \varepsilon_1 \begin{pmatrix} \chi_2 y_{\boldsymbol{a}}(z) \\ y_2(z) \end{pmatrix}, \tag{2.6}$$

$$\begin{pmatrix} \chi_2 y_{\boldsymbol{a}}(z + \omega_2) \\ y_2(z + \omega_2) \end{pmatrix} = \varepsilon_2 \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \chi_2 y_{\boldsymbol{a}}(z) \\ y_2(z) \end{pmatrix},$$

which is precisely (1.7). If $\chi_1 \neq 0$, then $\mathcal{C} \neq \infty$ and (2.2)-(2.3) give

$$\begin{pmatrix} \chi_1 y_{\boldsymbol{a}}(z + \omega_1) \\ y_2(z + \omega_1) \end{pmatrix} = \varepsilon_1 \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \chi_1 y_{\boldsymbol{a}}(z) \\ y_2(z) \end{pmatrix}, \tag{2.7}$$

$$\begin{pmatrix} \chi_1 y_{\boldsymbol{a}}(z + \omega_2) \\ y_2(z + \omega_2) \end{pmatrix} = \varepsilon_2 \begin{pmatrix} 1 & 0 \\ \mathcal{C} & 1 \end{pmatrix} \begin{pmatrix} \chi_1 y_{\boldsymbol{a}}(z) \\ y_2(z) \end{pmatrix}, \tag{2.8}$$

which is precisely (1.6). The above arguments (2.2)-(2.8) can be found in [13], where we also proved the following interesting result.

Proposition 2.1. [13, Proposition 3.2] *Let E_1, E_2 be two zeros of $Q_n(E;\tau)$ such that $E_1 \neq E_2$, and $\mathcal{C}_j$ be the monodromy data of the corresponding Lamé equation (2.1) with $E = E_j$. Then $\mathcal{C}_1 \neq \mathcal{C}_2$.*

Now we are in the position to prove Theorems 1.3 and 1.13.

Proof of Theorem 1.3 and 1.13. Fix any τ such that the multiplicity of any zero of $Q_n(E;\tau)$ is at most 2. Denote $\tilde{\omega}_1 := \omega_1 = 1$, $\tilde{\omega}_2 := \omega_2 = \tau$ and $\tilde{\omega}_3 := 2\tau - 1$ in this proof. Recall Section 1 that $\sigma_k(L_n;\tau)$ denotes the spectrum of the Lamé operator along the $\tilde{\omega}_k$ direction and

$$\sigma_k(L_n;\tau) = \{E \in \mathbb{C} \mid -2 \leq \Delta_k(E;\tau) \leq 2\}, \quad k = 1, 2, 3, \tag{2.9}$$

where

(2.10) $\Delta_k(E;\tau)$ *is the trace of the monodromy matrix of the Lamé equation (2.1) with respect to* $z \to z + \tilde{\omega}_k$.

As introduced in Section 1, it was proved in [17, Theorem 4.1] that E_0 is a finite endpoint of $\sigma_k(L_n;\tau)$ if and only if E_0 be a zero of the spectral polynomial (or the Lamé polynomial) $Q_n(E;\tau)$ with *odd* order. Let E_0 be any finite endpoint of $\sigma_k(L_n;\tau)$, then our assumption implies $\mathrm{ord}_{E_0}Q_n(\cdot;\tau)=1$. By (2.2) and (2.4) we have $\Delta_k(E_0;\tau)=\pm 2$. Define

$$d_k(E_0):=\mathrm{ord}_{E_0}(\Delta_k(\cdot;\tau)^2-4)$$

as in [17], where it was proved that

$$d_k(E_0)=\mathrm{ord}_{E_0}Q_n(\cdot;\tau)+2p_k(E_0)=1+2p_k(E_0).$$

Here $p_k(E_0)\in\mathbb{Z}_{\geq 0}$ denotes the immovable part of E_0 as a Dirichlet eigenvalue along $z\to z+\tilde{\omega}_k$ and can be characterized by that $p_k(E_0)\geq 1$ if and only if all solutions of the Lamé equation (2.1) are (anti)periodic with respect to $z\to z+\tilde{\omega}_k$, i.e.

$$p_k(E_0)\geq 1 \Leftrightarrow \textit{ the monodromy matrix of (2.1) with respect to } z\to z+\tilde{\omega}_k \textit{ is } \pm I_2; \tag{2.11}$$

see [17, Proposition 3.1]. Indeed, if E_0 is an immovable Dirichlet eigenvalue (i.e. $p_k(E_0)\geq 1$), then [17, Proposition 3.1-(i)] says that all solutions of the Lamé equation (2.1) are (anti)periodic with respect to $z\to z+\tilde{\omega}_k$. Conversely, if all solutions are (anti)periodic with respect to $z\to z+\tilde{\omega}_k$, then so does the special solution $s(E_0,z,z_0)$ (resp. $c(E_0,z,z_0)$) satisfying the Dirichlet condition $s(E_0,z_0,z_0)=0$ and $s'(E_0,z_0,z_0)=1$ (resp. the Neumann condition $c(E_0,z_0,z_0)=1$ and $c'(E_0,z_0,z_0)=0$). This implies $s(E_0,z_0+\tilde{\omega}_k,z_0)=0$ and $c'(E_0,z_0+\tilde{\omega}_k,z_0)=0$, so E_0 is also both a Dirichlet and a Neumann eigenvalue with respect to the interval $[z_0,z_0+\tilde{\omega}_k]$. Then it follows from [17, Proposition 3.1-(ii)] that E_0 is an immovable Dirichlet eigenvalue (i.e. $p_k(E_0)\geq 1$). This proves (2.11).

By (2.2) and (2.4) we have

$$y_{\boldsymbol{a}}(z+\tilde{\omega}_k)=\varepsilon_k y_{\boldsymbol{a}}(z),\;\; y_2(z+\tilde{\omega}_k)=\varepsilon_k(y_2(z)+\chi_k y_{\boldsymbol{a}}(z)),\;\; k=1,2,$$
$$y_{\boldsymbol{a}}(z+\tilde{\omega}_3)=\varepsilon_1\varepsilon_2^2 y_{\boldsymbol{a}}(z),$$
$$y_2(z+\tilde{\omega}_3)=\varepsilon_1\varepsilon_2^2(y_2(z)+(2\chi_2-\chi_1)y_{\boldsymbol{a}}(z)),$$

so the monodromy matrix of (2.1) with respect to $z\to z+\tilde{\omega}_k$ is $\pm I_2$ if and only if $\chi_1=0$ (i.e. $\mathcal{C}=\infty$) for $k=1$; $\chi_2=0$ (i.e. $\mathcal{C}=0$) for $k=2$; $2\chi_2-\chi_1=0$ (i.e. $\mathcal{C}=1/2$) for $k=3$. In conclusion,

$$p_1(E_0)\geq 1\Leftrightarrow \mathcal{C}=\infty,\quad p_2(E_0)\geq 1\Leftrightarrow \mathcal{C}=0,$$
$$p_3(E_0)\geq 1\Leftrightarrow \mathcal{C}=1/2.$$

On the other hand, it is easy to see from (2.9) and $\Delta_k(E_0;\tau)=\pm 2$ that there are exactly $d_k(E_0)$'s semi-arcs of $\sigma_k(L_n;\tau)$ meeting at E_0. So E_0 is a cusp of $\sigma_k(L_n;\tau)$ if and only if $d_k(E_0)\geq 3$, if and only if $p_k(E_0)\geq 1$, if and only if $\mathcal{C}=\infty$ for $k=1$, $\mathcal{C}=0$ for $k=2$ and $\mathcal{C}=1/2$ for $k=3$.

Finally, Proposition 2.1 says that for any fixed τ, among the monodromy datas $\mathcal{C}$'s corresponding to the finite endpoints E_0's of $\sigma_k(L_n;\tau)$, there is at most one $\mathcal{C}$ being ∞ for $k=1$ (resp. being 0 for $k=2$ and being $1/2$ for $k=3$), which yields that $\sigma_k(L_n;\tau)$ has at most one cusp.

The proof is complete. $\square$

3. Parametrization of the curves

The purpose of this section is to give the precise definition and parametrization of the curves $\mathcal{C}_{k,-}$ and $\mathcal{C}_{k,+}$, $k=1,2,3$, and prove Theorem 1.12.

Given $C\in\mathbb{R}$, we define the holomorphic functions $f_{k,C}(\tau)$ on $\mathbb{H}$ by

$$f_{k,C}(\tau) := 3e_k(\tau)(C\eta_1(\tau)-\eta_2(\tau)) + (\tfrac{g_2(\tau)}{2}-3e_k(\tau)^2)(C-\tau). \tag{3.1}$$

By the Legendre relation $\tau\eta_1-\eta_2=2\pi i$ we see that (1.10) is equivalent to $f_{k,C}(\tau)=0$. Recall the fundamental domain F_0 of $\Gamma_0(2)$:

$$F_0=\{\tau\in\mathbb{H}\mid 0\le \operatorname{Re}\tau\le 1,\ \ |\tau-\tfrac{1}{2}|\ge\tfrac{1}{2}\}.$$

Clearly $\bar{F}_0=F_0\cup\{0,1\}$ and so $F_0=\bar{F}_0\cap\mathbb{H}$. Denote $\mathring{F}_0=F_0\setminus\partial F_0$ to be the set of interior points of F_0. The main results of this section are as follows, which imply Theorem 1.12.

Theorem 3.1. *Let $k=1$.*

(1) *(=Theorem 1.12-(1)) For $C\in(-\infty,0)\cup(1,+\infty)$, $f_{1,C}(\tau)$ has a unique zero $\tau_1(C)$ in F_0. Furthermore, $\tau_1(C)\in\mathring{F}_0$ and is simple.*
(2) *For $C\in[0,1]$, $f_{1,C}(\tau)$ has no zeros in F_0.*

Theorem 3.2. *Let $k=2$.*

(1) *(=Theorem 1.12-(2)) For $C\in(-\infty,0)\cup(0,1)$, $f_{2,C}(\tau)$ has a unique zero $\tau_2(C)$ in F_0. Furthermore, $\tau_2(C)\in\mathring{F}_0$ and is simple.*
(2) *For $C\in\{0\}\cup[1,+\infty)$, $f_{2,C}(\tau)$ has no zeros in F_0.*

Theorem 3.3. *Let $k=3$.*

(1) *(=Theorem 1.12-(3)) For $C\in(0,1)\cup(1,+\infty)$, $f_{3,C}(\tau)$ has a unique zero $\tau_3(C)$ in F_0. Furthermore, $\tau_3(C)\in\mathring{F}_0$ and is simple.*
(2) *For $C\in(-\infty,0)\cup\{1\}$, $f_{3,C}(\tau)$ has no zeros in F_0.*

Remark 3.4. By Theorems 3.1-3.3 and the implicit function theorem, we see that $(-\infty,0)\cup(1,+\infty)\ni C\mapsto\tau_1(C)$ is a smooth function and so do for $\tau_k(C)$, $k=2,3$. Define six curves via the parametrization in terms of C:

$$\mathcal{C}_{1,-}:=\{\tau_1(C)|C\in(-\infty,0)\},\ \ \mathcal{C}_{1,+}:=\{\tau_1(C)|C\in(1,+\infty)\},$$
$$\mathcal{C}_{2,-}:=\{\tau_2(C)|C\in(-\infty,0)\},\ \ \mathcal{C}_{2,+}:=\{\tau_2(C)|C\in(0,1)\},$$
$$\mathcal{C}_{3,-}:=\{\tau_3(C)|C\in(0,1)\},\ \ \mathcal{C}_{3,+}:=\{\tau_3(C)|C\in(1,+\infty)\}.$$

We will prove in Sections 4-5 that these are precisely the smooth curves related to the critical points of $e_k(\tau)$'s.

To prove Theorems 3.1-3.3, first we need to recall the modular property. Given any $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in SL(2,\mathbb{Z})$, it is well known that

$$g_2(\tfrac{a\tau+b}{c\tau+d})=(c\tau+d)^4g_2(\tau), \tag{3.2}$$

$$\wp\left(\tfrac{z}{c\tau+d};\tfrac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^2\,\wp(z;\tau),\tag{3.3}$$

$$\zeta\left(\tfrac{z}{c\tau+d};\tfrac{a\tau+b}{c\tau+d}\right)=(c\tau+d)\,\zeta(z;\tau),$$

i.e. $g_2(\tau)$ is a modular form of weight 4 with respect to $SL(2,\mathbb{Z})$. From here and (1.9) we can obtain

$$\begin{pmatrix}\eta_2(\frac{a\tau+b}{c\tau+d})\\ \eta_1(\frac{a\tau+b}{c\tau+d})\end{pmatrix}=(c\tau+d)\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\eta_2(\tau)\\ \eta_1(\tau)\end{pmatrix}.\tag{3.4}$$

Thus each $\eta_j(\tau)$ is not a modular forms, but $(\eta_1(\tau),\eta_2(\tau))$ is what is now called a "vector-valued modular form". Again by (3.3), we can easily derive

$$e_1(\tfrac{a\tau+b}{c\tau+d})=\begin{cases}(c\tau+d)^2e_1(\tau), & \text{if } c \text{ even and } d \text{ odd}\\ (c\tau+d)^2e_2(\tau), & \text{if } c \text{ odd and } d \text{ even}\\ (c\tau+d)^2e_3(\tau), & \text{if } c \text{ odd and } d \text{ odd,}\end{cases}$$

$$e_2(\tfrac{a\tau+b}{c\tau+d})=\begin{cases}(c\tau+d)^2e_1(\tau), & \text{if } a \text{ even and } b \text{ odd}\\ (c\tau+d)^2e_2(\tau), & \text{if } a \text{ odd and } b \text{ even}\\ (c\tau+d)^2e_3(\tau), & \text{if } a \text{ odd and } b \text{ odd,}\end{cases}$$

$$e_3(\tfrac{a\tau+b}{c\tau+d})=\begin{cases}(c\tau+d)^2e_1(\tau), & \text{if } a+c \text{ even and } b+d \text{ odd}\\ (c\tau+d)^2e_2(\tau), & \text{if } a+c \text{ odd and } b+d \text{ even}\\ (c\tau+d)^2e_3(\tau), & \text{if } a+c \text{ odd and } b+d \text{ odd.}\end{cases}\tag{3.5}$$

In particular, each $e_k(\tau)$ is a modular form of weight 2 on $\Gamma(2)$. In the rest of this article, we will freely use the formulas (3.2)-(3.5). Denote $q:=e^{2\pi i\tau}$. We recall the following q-expansions for later usage:

$$e_1(\tau)=\frac{2\pi^2}{3}+16\pi^2\sum_{k=1}^{\infty}a_kq^k,\ \text{where } a_k=\sum_{d|k,d\text{ is odd}}d,\tag{3.6}$$

$$e_2(\tau)=-\frac{\pi^2}{3}-8\pi^2\sum_{k=1}^{\infty}a_kq^{\frac{k}{2}},\tag{3.7}$$

$$e_3(\tau)=-\frac{\pi^2}{3}-8\pi^2\sum_{k=1}^{\infty}(-1)^ka_kq^{\frac{k}{2}},\tag{3.8}$$

$$\eta_1(\tau)=\frac{\pi^2}{3}-8\pi^2\sum_{k=1}^{\infty}\sigma_1(k)q^k,\ \text{ where } \sigma_1(k)=\sum_{1\le d|k}d,\tag{3.9}$$

$$g_2(\tau)=\frac{4}{3}\pi^4+320\pi^4\sum_{k=1}^{\infty}\sigma_3(k)q^k,\ \text{ where } \sigma_3(k)=\sum_{1\le d|k}d^3.\tag{3.10}$$

See e.g. [32, p.70] for (3.6)-(3.9) and [24, p.44] for (3.10).

Lemma 3.5. *For any $k \in \{1,2,3\}$ and $C \in \mathbb{R}\backslash\{0,1\}$, $f_{k,C}(\tau) \neq 0$ for $\tau \in \partial F_0 \cap \mathbb{H}$.*

Proof. Fix $k \in \{1,2,3\}$ and $C \in \mathbb{R}\backslash\{0,1\}$. Suppose $f_{k,C}(\tau) = 0$ for some $\tau \in \partial F_0 \cap \mathbb{H}$. Clearly (3.1) and the Legendre relation $\tau\eta_1 - \eta_2 = 2\pi i$ imply

$$C = \tau - \frac{2\pi i e_k}{e_k\eta_1 + \frac{g_2}{6} - e_k^2}. \tag{3.11}$$

Case 1. $\tau \in i\mathbb{R}_{>0}$.

Then it is known that $\eta_1, g_2, e_k \in \mathbb{R}$, which can be seen from the q-expansions. So (3.11) gives $C \in i\mathbb{R} \cup \{\infty\}$, a contradiction with $C \in \mathbb{R} \setminus \{0\}$.

Case 2. $|\tau - \frac{1}{2}| = \frac{1}{2}$.

Then $\tau' = \frac{\tau}{1-\tau} \in i\mathbb{R}_{>0}$. Define $C' := \frac{C}{1-C} \in \mathbb{R} \setminus \{0\}$. By applying $g_2(\tau') = (1-\tau)^4 g_2(\tau)$,

$$\eta_2(\tau') = (1-\tau)\eta_2(\tau),\ \eta_1(\tau') = (1-\tau)(\eta_1(\tau) - \eta_2(\tau)), \tag{3.12}$$

$$e_1(\tau') = (1-\tau)^2 e_3(\tau),\ \ e_2(\tau') = (1-\tau)^2 e_2(\tau),$$

$$e_3(\tau') = (1-\tau)^2 e_1(\tau), \tag{3.13}$$

a straightforward computation leads to

$$f_{1,C'}(\tau') = \frac{(1-\tau)^3}{1-C} f_{3,C}(\tau),\ \ f_{2,C'}(\tau') = \frac{(1-\tau)^3}{1-C} f_{2,C}(\tau),$$

$$f_{3,C'}(\tau') = \frac{(1-\tau)^3}{1-C} f_{1,C}(\tau).$$

Then we obtain a contradiction as Case 1.

Case 3. $\tau \in 1 + i\mathbb{R}_{>0}$.

Then $\tau' = \tau - 1 \in i\mathbb{R}_{>0}$. Define $C' := C - 1 \in \mathbb{R} \setminus \{0\}$. By using

$$g_2(\tau') = g_2(\tau),\ \eta_1(\tau') = \eta_1(\tau),\ \eta_2(\tau') = \eta_2(\tau) - \eta_1(\tau),$$

$$e_1(\tau') = e_1(\tau),\ e_2(\tau') = e_3(\tau),\ e_3(\tau') = e_2(\tau),$$

we easily obtain

$$f_{1,C'}(\tau') = f_{1,C}(\tau),\ f_{2,C'}(\tau') = f_{3,C}(\tau),\ f_{3,C'}(\tau') = f_{2,C}(\tau).$$

Again we obtain a contradiction as Case 1. □

To continue our proof, we need to introduce a pre-modular form $Z_{r,s}^{(2)}(\tau)$ from [14, 26]. For any $(r,s) \in \mathbb{R}^2 \setminus \mathbb{Z}^2$, we define

$$\begin{aligned} Z_{r,s}(\tau) :=& \zeta(r+s\tau;\tau) - r\eta_1(\tau) - s\eta_2(\tau) \\ =& \zeta(r+s\tau;\tau) - (r+s\tau)\eta_1(\tau) + 2\pi i s, \end{aligned} \tag{3.14}$$

$$Z_{r,s}^{(2)}(\tau) := Z_{r,s}(\tau)^3 - 3\wp(r+s\tau;\tau)Z_{r,s}(\tau) - \wp'(r+s\tau;\tau). \tag{3.15}$$

Here we use $\tau\eta_1 - \eta_2 = 2\pi i$ in (3.14). Since $\zeta(z;\tau)$ has simple poles at the lattice $\Lambda_\tau := \mathbb{Z} + \mathbb{Z}\tau$, we have $Z_{r,s}(\tau), \wp, \wp' \equiv \infty$ and so $Z^{(2)}_{r,s}(\tau)$ is not well-defined provided $(r,s) \in \mathbb{Z}^2$. If $(r,s) \in \frac{1}{2}\mathbb{Z}^2 \backslash \mathbb{Z}^2$, where

$$\tfrac{1}{2}\mathbb{Z}^2 := \{(\tfrac{m}{2}, \tfrac{n}{2}) \,|\, m, n \in \mathbb{Z}\},$$

then (1.9) and the oddness of $\zeta(z;\tau)$ imply $Z_{r,s}(\tau) \equiv 0$ and so $Z^{(2)}_{r,s}(\tau) \equiv 0$, where $\wp'(\frac{\omega_k}{2}) = 0$ is used. Thus, we only consider $(r,s) \in \mathbb{R}^2 \backslash \frac{1}{2}\mathbb{Z}^2$. Then both $Z_{r,s}(\tau)$ and $Z^{(2)}_{r,s}(\tau)$ are holomorphic in $\mathbb{H}$, and it is easy to see that the following properties hold:

(i) $Z_{r,s}(\tau) = \pm Z_{m\pm r, n\pm s}(\tau)$ and $Z^{(2)}_{r,s}(\tau) = \pm Z^{(2)}_{m\pm r, n\pm s}(\tau)$ for any $(m,n) \in \mathbb{Z}^2$.

(ii) $Z_{r',s'}(\tau') = (c\tau + d) Z_{r,s}(\tau)$ and $Z^{(2)}_{r',s'}(\tau') = (c\tau+d)^3 Z^{(2)}_{r,s}(\tau)$ for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$, where $\tau' = \gamma \cdot \tau := \frac{a\tau+b}{c\tau+d}$ and $(s', r') = (s, r) \cdot \gamma^{-1}$.

Remark that $Z_{r,s}(\tau)$ was first introduced by Hecke [21], who already proved the first identity of property (ii). In particular, when $(r,s) \in Q_N$ for some $N \in \mathbb{N}_{\geq 3}$, where

$$Q_N := \left\{ \left(\tfrac{k_1}{N}, \tfrac{k_2}{N}\right) \middle| \gcd(k_1, k_2, N) = 1,\ 0 \leq k_1, k_2 \leq N-1 \right\}, \tag{3.16}$$

and $\gamma \in \Gamma(N) := \{\gamma \in SL(2,\mathbb{Z}) | \gamma \equiv I_2 \bmod N\}$, then $(r', s') \equiv (r, s) \bmod \mathbb{Z}^2$ and so

$$Z_{r,s}\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d) Z_{r,s}(\tau), \quad Z^{(2)}_{r,s}\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^3 Z^{(2)}_{r,s}(\tau),$$

namely $Z_{r,s}(\tau)$ and $Z^{(2)}_{r,s}(\tau)$ are *modular forms* of weight 1 and 3, respectively, with respect to $\Gamma(N)$. Due to this reason, $Z_{r,s}(\tau)$ and $Z^{(3)}_{r,s}(\tau)$ are called *pre-modular forms* in [26].

To study zeros of $Z^{(2)}_{r,s}(\tau)$, we can restrict τ in the fundamental domain F_0 of $\Gamma_0(2)$ by property (ii), and only need to consider $(r,s) \in [0,1] \times [0, \frac{1}{2}] \backslash \frac{1}{2}\mathbb{Z}^2$ by property (i). Define four open triangles:

$$\begin{aligned} \triangle_0 &:= \{(r,s) \mid 0 < r, s < \tfrac{1}{2},\ r+s > \tfrac{1}{2}\}, \\ \triangle_1 &:= \{(r,s) \mid \tfrac{1}{2} < r < 1,\ 0 < s < \tfrac{1}{2},\ r+s > 1\}, \\ \triangle_2 &:= \{(r,s) \mid \tfrac{1}{2} < r < 1,\ 0 < s < \tfrac{1}{2},\ r+s < 1\}, \\ \triangle_3 &:= \{(r,s) \mid r > 0,\ s > 0,\ r+s < \tfrac{1}{2}\}. \end{aligned} \tag{3.17}$$

Clearly $[0,1] \times [0, \frac{1}{2}] = \cup_{k=0}^{3} \overline{\triangle_k}$. We proved the following result in [11].

Theorem A. [11] *Let $(r,s) \in [0,1] \times [0, \frac{1}{2}] \backslash \frac{1}{2}\mathbb{Z}^2$. Then $Z^{(2)}_{r,s}(\cdot)$ has a zero τ in F_0 if and only if $(r,s) \in \triangle_1 \cup \triangle_2 \cup \triangle_3$. Furthermore, for any $(r,s) \in \triangle_1 \cup \triangle_2 \cup \triangle_3$, the zero $\tau \in F_0$ is unique, simple and satisfies $\tau \in \mathring{F}_0$.*

We will see that Theorem A plays a key role in our proof of Theorems 3.1-3.3. First we need to establish the precise relation between $Z^{(2)}_{r,s}(\tau)$ and

$f_{k,C}(\tau)$. This is *the key point* of our whole idea. Fix any $C \in \mathbb{R}$, and for $s \in (0, \frac{1}{4(1+|C|)^2})$ we define

$$F_{1,C,s}(\tau) := \frac{-1}{s} Z^{(2)}_{\frac{1}{2}-Cs,s}(\tau),$$
$$F_{2,C,s}(\tau) := \frac{1}{s} Z^{(2)}_{Cs,\frac{1}{2}-s}(\tau),$$
$$F_{3,C,s}(\tau) := \frac{1}{s} Z^{(2)}_{\frac{1}{2}+Cs,\frac{1}{2}-s}(\tau).$$

Lemma 3.6. *Let $k \in \{1,2,3\}$. Then as $s \to 0$, $F_{k,C,s}(\tau)$ converges to $f_{k,C}(\tau)$ uniformly in any compact subset of $F_0 = \bar{F}_0 \cap \mathbb{H}$.*

Proof. Denote $u = s(\tau - C)$ for convenience. Then $u \to 0$ as $s \to 0$. Let $\tau \in K$ where K is any compact subset of F_0. Then $s = O(|u|)$ and $|u| = O(s)$.

First we consider the case $k = 1$. Recall $\eta_1(\tau) = 2\zeta(\frac{1}{2};\tau)$. Then it follows from the Taylor expansions that

$$\zeta\left(\tfrac{1}{2}+u;\tau\right) = \frac{1}{2}\eta_1 - e_1 u - \frac{\wp''(\frac{1}{2};\tau)}{6}u^3 + O(|u|^5),$$
$$\wp\left(\tfrac{1}{2}+u;\tau\right) = e_1 + \frac{\wp''(\frac{1}{2};\tau)}{2}u^2 + O(|u|^4),$$
$$\wp'\left(\tfrac{1}{2}+u;\tau\right) = \wp''(\tfrac{1}{2};\tau)u + O(|u|^3),$$

hold uniformly for $\tau \in K$ as $s \to 0$. From here and (3.14), we see that

$$Z_{\frac{1}{2}-Cs,s}(\tau) = -(e_1+\eta_1)u + 2\pi i s + O(|u|^3) = O(|u|),$$

and so (note $\wp''(\frac{1}{2};\tau) = 6e_1^2 - \frac{1}{2}g_2$)

$$\begin{aligned} Z^{(2)}_{\frac{1}{2}-Cs,s}(\tau) &= Z_{\frac{1}{2}-Cs,s}(\tau)^3 - 3\wp\left(\tfrac{1}{2}+u;\tau\right) Z_{\frac{1}{2}-Cs,s}(\tau) - \wp'\left(\tfrac{1}{2}+u;\tau\right) \\ &= 3e_1\eta_1 u - 6\pi i e_1 s + (\tfrac{1}{2}g_2 - 3e_1^2)u + O(|u|^3) \end{aligned} \tag{3.18}$$

uniformly for $\tau \in K$ as $s \to 0$. Consequently, we derive from $u = s(\tau - C)$ and $\eta_2 = \tau\eta_1 - 2\pi i$ that

$$\begin{aligned} F_{1,C,s}(\tau) &= \frac{-1}{s} Z^{(2)}_{\frac{1}{2}-Cs,s}(\tau) \\ &= 3e_1(C\eta_1 - \eta_2) + (\tfrac{g_2}{2} - 3e_1^2)(C-\tau) + O(s^2) \to f_{1,C}(\tau) \end{aligned}$$

uniformly for $\tau \in K$ as $s \to 0$. Therefore, the case $k = 1$ is proved.

The case $k = 2,3$ can be proved in a similar way; we just need to show

$$\begin{aligned} Z^{(2)}_{Cs,\frac{1}{2}-s}(\tau) &= Z_{Cs,\frac{1}{2}-s}(\tau)^3 - 3\wp\left(\tfrac{\tau}{2}-u;\tau\right) Z_{Cs,\frac{1}{2}-s}(\tau) - \wp'\left(\tfrac{\tau}{2}-u;\tau\right) \\ &= -3e_2\eta_1 u + 6\pi i e_2 s - (\tfrac{1}{2}g_2 - 3e_2^2)u + O(|u|^3), \end{aligned} \tag{3.19}$$

$$\begin{aligned} & Z^{(2)}_{\frac{1}{2}+Cs,\frac{1}{2}-s}(\tau) \\ =& Z_{\frac{1}{2}+Cs,\frac{1}{2}-s}(\tau)^3 - 3\wp\left(\tfrac{1+\tau}{2}-u;\tau\right) Z_{\frac{1}{2}+Cs,\frac{1}{2}-s}(\tau) - \wp'\left(\tfrac{1+\tau}{2}-u;\tau\right) \end{aligned}$$

$$= -3e_3\eta_1 u + 6\pi i e_3 s - (\tfrac{1}{2}g_2 - 3e_3^2)u + O(|u|^3).$$

The details are omitted here. The proof is complete. $\square$

Lemma 3.7. *Let $s > 0$. Then as $s \to 0$, any zero $\tau(s) \in \{\tau \in \mathbb{H} | \operatorname{Re}\tau \in [-1,1]\}$ of $Z^{(2)}_{\frac{1}{2}-Cs,s}(\tau)$ (resp. $Z^{(2)}_{Cs,\frac{1}{2}-s}(\tau)$, $Z^{(2)}_{\frac{1}{2}+Cs,\frac{1}{2}-s}(\tau)$), if exist, is uniformly bounded.*

Proof. Suppose by contradiction that up to a subsequence, $Z^{(2)}_{\frac{1}{2}-Cs,s}(\tau)$ (resp. $Z^{(2)}_{Cs,\frac{1}{2}-s}(\tau)$, $Z^{(2)}_{\frac{1}{2}+Cs,\frac{1}{2}-s}(\tau)$) has a zero $\tau(s) \in \{\tau \in \mathbb{H} | \operatorname{Re}\tau \in [-1,1]\}$ such that $\tau(s) \to \infty$ as $s \to 0$. Write $\tau = \tau(s) = a(s) + ib(s)$, then $a(s) \in [-1,1]$ and $b(s) \to +\infty$. Denote $q = e^{2\pi i\tau}$ as before. We recall the q-expansions (cf. [24, p.46] for $\wp$ and [10, (5.3)] for $Z_{r,s}$): for $|q| < |e^{2\pi iz}| < |q|^{-1}$,

$$\frac{\wp(z;\tau)}{-4\pi^2} = \frac{1}{12} + \frac{e^{2\pi iz}}{(1-e^{2\pi iz})^2} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} nq^{nm}(e^{2\pi inz} + e^{-2\pi inz} - 2), \tag{3.20}$$

$$\begin{aligned}\frac{\wp'(z;\tau)}{-4\pi^2} =& \frac{2\pi i e^{2\pi iz}}{(1-e^{2\pi iz})^2} + \frac{4\pi i e^{4\pi iz}}{(1-e^{2\pi iz})^3} \\ &+ 2\pi i \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} n^2 q^{nm}(e^{2\pi inz} - e^{-2\pi inz}),\end{aligned}$$

$$\begin{aligned}Z_{r,s}(\tau) =& 2\pi is - \pi i\frac{1+e^{2\pi iz}}{1-e^{2\pi iz}} \\ &- 2\pi i \sum_{n=1}^{\infty}\left(\frac{e^{2\pi iz}q^n}{1-e^{2\pi iz}q^n} - \frac{e^{-2\pi iz}q^n}{1-e^{-2\pi iz}q^n}\right),\end{aligned} \tag{3.21}$$

where $z = r + s\tau$ in (3.21).

First we consider $Z^{(2)}_{\frac{1}{2}-Cs,s}(\tau)$. Let $z = \frac{1}{2} - Cs + s\tau = \frac{1}{2} + s(a(s) - C + ib(s))$ and denote $x = e^{2\pi iz}$. Clearly $|x| = e^{-2\pi sb(s)} \in (0,1)$ and

$$|x^{-1}q| = e^{-2\pi(1-s)b(s)} \to 0, \;\; x \not\to 1 \;\text{ as }\; s \to 0.$$

Case 1-1. Up to a subsequence $|x^{-1}q| = o(s|1-x|^2) = o(s)$.

Then applying the q-expansions (3.20)-(3.21) leads to

$$\frac{\wp(z;\tau)}{-4\pi^2} = \frac{1}{12} + \frac{x}{(1-x)^2} + o(s),$$

$$\frac{\wp'(z;\tau)}{-4\pi^2} = \frac{2\pi ix}{(1-x)^2} + \frac{4\pi ix^2}{(1-x)^3} + o(s),$$

$$Z_{\frac{1}{2}-Cs,s}(\tau) = -\pi i\frac{1+x}{1-x} + 2\pi is + o(s).$$

Inserting these into (3.15), a straightforward computation gives

$$0 = Z^{(2)}_{\frac{1}{2}-Cs,s}(\tau(s)) = -4\pi^3 is + o(s),$$

which is a contradiction.

Case 1-2. Up to a subsequence $|x^{-1}q| \geq ds$ for some constant $d > 0$.

Then $b(s) \leq \ln \frac{1}{s}$ for $s > 0$ small, which implies $u := s(\tau - C) \to 0$ and

$$s = o(|u|), \ \ u^2 = o(s).$$

Recall (3.6), (3.9) and (3.10). Since $b(s) \to +\infty$ implies that

$$g_2(\tau) = \frac{4}{3}\pi^4 + O(|q|), \ \ \eta_1(\tau) = \frac{1}{3}\pi^2 + O(|q|)$$

$$e_1(\tau) = \frac{2}{3}\pi^2 + O(|q|)$$

are uniformly bounded, so (3.18) still holds, namely

$$\begin{aligned} 0 &= Z^{(2)}_{\frac{1}{2}-Cs,s}(\tau(s)) \\ &= -6\pi i e_1 s + (3e_1\eta_1 + \tfrac{1}{2}g_2 - 3e_1^2)u + O(|u|^3). \end{aligned}$$

Since

$$3e_1\eta_1 + \tfrac{1}{2}g_2 - 3e_1^2 = O(|q|) = O\left(|\tau - C|^{-2}\right),$$

we finally obtain

$$\begin{aligned} 0 &= -6\pi i e_1 s + (3e_1\eta_1 + \tfrac{1}{2}g_2 - 3e_1^2)u + O(|u|^3) \\ &= -4\pi^3 i s + o(s), \end{aligned}$$

which is a contradiction. Therefore, the case $Z^{(2)}_{\frac{1}{2}-Cs,s}(\tau)$ is proved.

Next we consider $Z^{(2)}_{Cs,\frac{1}{2}-s}(\tau)$. The following argument is slightly different from above.

Let $z = \frac{\tau}{2} + Cs - s\tau = Cs + (\frac{1}{2} - s)(a(s) + ib(s))$ and denote $x = e^{2\pi i z}$. Clearly

$$|x^{-1}q| \leq |x| = e^{-2\pi(\frac{1}{2}-s)b(s)} \to 0 \ \text{ as } \ s \to 0.$$

Case 2-1. Up to a subsequence $|x| = o(s)$. Then $|x^{-1}q| = o(s)$.

Applying the q-expansions (3.20)-(3.21) leads to

$$\wp(z;\tau) = -\frac{\pi^2}{3} + o(s), \quad \wp'(z;\tau) = o(s),$$

$$Z_{Cs,\frac{1}{2}-s}(\tau) = -\pi i \frac{1+x}{1-x} + 2\pi i(\tfrac{1}{2} - s) + o(s) = -2\pi i s + o(s).$$

Consequently, a straightforward computation gives

$$0 = Z^{(2)}_{Cs,\frac{1}{2}-s}(\tau(s)) = -2\pi^3 i s + o(s),$$

which is a contradiction.

Case 2-2. Up to a subsequence $|x| \geq ds$ for some constant $d > 0$.

Then $b(s) \leq \ln \frac{1}{s}$ for $s > 0$ small, $u := s(\tau - C) \to 0$, $s = o(|u|)$ and $u^2 = o(s)$. Since $b(s) \to +\infty$ implies that $g_2(\tau)$, $\eta_1(\tau)$ and

$$e_2(\tau) = -\frac{1}{3}\pi^2 + O(|q|^{1/2})$$

are uniformly bounded, so (3.19) still holds, namely

$$\begin{aligned}0 &= Z^{(2)}_{Cs,\frac{1}{2}-s}(\tau(s)) \\ &= 6\pi i e_2 s - (\tfrac{1}{2}g_2 - 3e_2\eta_1 - 3e_2^2)u + O(|u|^3).\end{aligned}$$

Since

$$\tfrac{1}{2}g_2 - 3e_2\eta_1 - 3e_2^2 = O(|q|^{1/2}) = O\left(|\tau - C|^{-2}\right),$$

we finally obtain

$$\begin{aligned}0 &= 6\pi i e_2 s - (\tfrac{1}{2}g_2 - 3e_2\eta_1 - 3e_2^2)u + O(|u|^3) \\ &= -2\pi^3 i s + o(s),\end{aligned}$$

which is a contradiction. Therefore, the case $Z^{(2)}_{Cs,\frac{1}{2}-s}(\tau)$ is proved.

Finally, the third case $Z^{(2)}_{\frac{1}{2}+Cs,\frac{1}{2}-s}(\tau)$ can be proved by the same way as $Z^{(2)}_{Cs,\frac{1}{2}-s}(\tau)$, so we omit the details. The proof is complete. $\square$

Now we consider $f_{1,C}(\tau)$ first. Recall $\triangle_k$ defined in (3.17). For $C \in (-\infty, 0) \cup (1, \infty)$, it follows from $s \in (0, \frac{1}{4(1+|C|)^2})$ that $(\frac{1}{2} - Cs, s) \in \triangle_2 \cup \triangle_3$, so Theorem A implies that $Z^{(2)}_{\frac{1}{2}-Cs,s}(\tau)$ has a unique zero $\tau(s) \in F_0$. By the definition of F_0, we easily see that

$$\frac{-1}{\tau(s)}, \frac{\tau(s)}{1-\tau(s)} \in \{\tau \in \mathbb{H} | \operatorname{Re}\tau \in [-1, 1]\}.$$

Lemma 3.8. *Let $C \in (-\infty, 0) \cup (1, \infty)$. Then as $s \to 0$, the unique zero $\tau(s) \in F_0$ of $Z^{(2)}_{\frac{1}{2}-Cs,s}(\tau)$ can not converge to any of $\{0, 1, \infty\}$.*

Proof. Lemma 3.7 shows that $\tau(s) \not\to \infty$. To prove $\tau(s) \not\to 0$, we use the aforementioned modular property (ii) of $Z^{(2)}_{r,s}(\tau)$:

$$Z^{(2)}_{r',s'}(\tau') = (c\tau + d)^3 Z^{(2)}_{r,s}(\tau),$$

whenever

$$\tau' = \frac{a\tau + b}{c\tau + d} \text{ and } (s', r') = (s, r)\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}).$$

We also use property (i):

$$Z^{(2)}_{m\pm r, n\pm s}(\tau) = \pm Z^{(2)}_{r,s}(\tau), \ \forall m, n \in \mathbb{Z}. \tag{3.22}$$

Letting $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, we obtain

$$Z^{(2)}_{s,-r}(\tfrac{-1}{\tau}) = \tau^3 Z^{(2)}_{r,s}(\tau). \tag{3.23}$$

Assume $C \in (-\infty, 0)$. By defining

$$\tilde{C} := \frac{-1}{C}, \quad \tilde{s} := -Cs,$$

we have $\tilde{s} \in (0, \frac{1}{4(1+|\tilde{C}|)^2})$ for s small and (note (3.22))

$$\tau^3 Z^{(2)}_{\frac{1}{2}-Cs,s}(\tau) = Z^{(2)}_{s,Cs-\frac{1}{2}}(\tfrac{-1}{\tau}) = Z^{(2)}_{s,Cs+\frac{1}{2}}(\tfrac{-1}{\tau}) = Z^{(2)}_{\tilde{C}\tilde{s},\frac{1}{2}-\tilde{s}}(\tfrac{-1}{\tau}).$$

Therefore, $Z^{(2)}_{\tilde{C}\tilde{s},\frac{1}{2}-\tilde{s}}(\tau)$ has zero $\frac{-1}{\tau(s)} \in \{\tau \in \mathbb{H} | \operatorname{Re}\tau \in [-1,1]\}$, and then Lemma 3.7 implies $\frac{-1}{\tau(s)} \not\to \infty$, i.e. $\tau(s) \not\to 0$ as $s \to 0$. The case $C \in (1,\infty)$ can be proved similarly. This proves $\tau(s) \not\to 0$ as $s \to 0$.

To prove $\tau(s) \not\to 1$, we let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)$ and obtain

$$Z^{(2)}_{r,r+s}(\tfrac{\tau}{1-\tau}) = (1-\tau)^3 Z^{(2)}_{r,s}(\tau). \tag{3.24}$$

Let $C \in (-\infty, 0)$. By defining

$$\tilde{C} := \frac{C}{1-C}, \quad \tilde{s} := (1-C)s,$$

we have $\tilde{s} \in (0, \frac{1}{4(1+|\tilde{C}|)^2})$ for s small and (note (3.22))

$$\begin{aligned}(1-\tau)^3 Z^{(2)}_{\frac{1}{2}-Cs,s}(\tau) &= Z^{(2)}_{\frac{1}{2}-Cs,\frac{1}{2}+(1-C)s}(\tfrac{\tau}{1-\tau}) \\ &= -Z^{(2)}_{\frac{1}{2}+Cs,\frac{1}{2}-(1-C)s}(\tfrac{\tau}{1-\tau}) = -Z^{(2)}_{\frac{1}{2}+\tilde{C}\tilde{s},\frac{1}{2}-\tilde{s}}(\tfrac{\tau}{1-\tau}).\end{aligned}$$

Again Lemma 3.7 implies $\frac{\tau(s)}{1-\tau(s)} \not\to \infty$, i.e. $\tau(s) \not\to 1$ as $s \to 0$. The case $C \in (1,+\infty)$ can be proved similarly. This proves $\tau(s) \not\to 1$ as $s \to 0$. $\square$

Now we are in a position to prove Theorems 3.1-3.3 for $C \in \mathbb{R}\backslash\{0,1\}$.

Proof of Theorem 3.1 for $C \in \mathbb{R}\backslash\{0,1\}$. Recall $s \in (0, \frac{1}{4(1+|C|)^2})$. First we consider $C \in (-\infty,0) \cup (1,\infty)$. Then, as pointed out before, $Z^{(2)}_{\frac{1}{2}-Cs,s}(\tau)$ has a unique zero $\tau(s) \in F_0$. By Lemma 3.8, up to a subsequence we have

$$\tau_1(C) := \lim_{s\to 0} \tau(s) \in \bar{F}_0 \cap \mathbb{H} = F_0. \tag{3.25}$$

Recalling

$$F_{1,C,s}(\tau) := \frac{-1}{s} Z^{(2)}_{\frac{1}{2}-Cs,s}(\tau),$$

we have $F_{1,C,s}(\tau(s)) = 0$. Then Lemma 3.6 implies $f_{1,C}(\tau_1(C)) = 0$, i.e. $f_{1,C}(\tau)$ has a zero $\tau_1(C) \in F_0$. Applying Lemma 3.5, we have $\tau_1(C) \in \mathring{F}_0$. Suppose $f_{1,C}(\tau)$ has another zero $\tilde{\tau} \neq \tau_1(C)$ in $\mathring{F}_0$. Since $F_{1,C,s}(\tau)$ and $f_{1,C}(\tau)$ are all holomorphic functions, it follows from Lemma 3.6 and Rouché's theorem that $F_{1,C,s}(\tau)$ has a zero $\tilde{\tau}(s)$ satisfying $\tilde{\tau}(s) \to \tilde{\tau}$ as $s \to 0$, namely

$Z^{(2)}_{\frac{1}{2}-Cs,s}(\tau)$ has two different zeros in F_0 for $s>0$ small, a contradiction with Theorem A. Thus $\tau_1(C)$ is the unique zero of $f_{1,C}(\tau)$ in F_0. Since Theorem A says that $\tau(s)$ is a simple zero of $F_{1,C,s}(\tau)$, the same argument also implies that $\tau_1(C)$ is simple. Besides, This also indicates that (3.25) actually holds for any $s\to 0$ (i.e. not only for a subsequence).

Now we consider $C\in(0,1)$. Suppose that $f_{1,C}(\tau)$ has a zero in $\mathring{F}_0$. Then the same argument implies that $Z^{(2)}_{\frac{1}{2}-Cs,s}(\tau)$ has a zero in $\mathring{F}_0$ for $s>0$ small. However, since $(\frac{1}{2}-Cs,s)\in\triangle_0$, we obtain a contradiction with Theorem A. Thus $f_{1,C}(\tau)$ has no zeros in F_0 for $C\in(0,1)$.

The remaining case $C\in\{0,1\}$ will be postponed in Section 4. The proof is complete. $\square$

Proof of Theorems 3.2 and 3.3 for $C\in\mathbb{R}\backslash\{0,1\}$. Instead of applying the same idea as Theorem 3.1, we give a different proof. Let $\tau'=\frac{1}{1-\tau}$ and $C'=\frac{1}{1-C}$, then it easy to prove that

$$\tau'\in F_0 \Longleftrightarrow \tau\in F_0 \text{ and } C'\in\mathbb{R}\backslash\{0,1\}\Longleftrightarrow C\in\mathbb{R}\backslash\{0,1\}. \tag{3.26}$$

Since $g_2(\tau')=(1-\tau)^4 g_2(\tau)$,

$$\begin{gathered}\eta_2(\tau')=(1-\tau)\eta_1(\tau),\ \eta_1(\tau')=(1-\tau)(\eta_1(\tau)-\eta_2(\tau)),\\ e_2(\tau')=(1-\tau)^2 e_1(\tau),\ \ e_3(\tau')=(1-\tau)^2 e_2(\tau),\\ e_1(\tau')=(1-\tau)^2 e_3(\tau),\end{gathered} \tag{3.27}$$

we easily derive from the definition (3.1) of $f_{k,C}$ that

$$f_{2,C'}(\tau')=\frac{(1-\tau)^3}{1-C}f_{1,C}(\tau), \tag{3.28}$$

$$f_{3,C'}(\tau')=\frac{(1-\tau)^3}{1-C}f_{2,C}(\tau), \tag{3.29}$$

$$f_{1,C'}(\tau')=\frac{(1-\tau)^3}{1-C}f_{3,C}(\tau). \tag{3.30}$$

Applying (3.30) and Theorem 3.1 for $f_{1,C'}(\cdot)$ with $C'\in\mathbb{R}\backslash\{0,1\}$, we immediately obtain: For $C\in(0,1)\cup(1,+\infty)$, $f_{3,C}(\tau)$ has a unique zero in F_0 which is simple; we denote this unique zero by $\tau_3(C)$, then $\tau_3(C)\in\mathring{F}_0$ and

$$\tau_1\left(\frac{1}{1-C}\right)=\frac{1}{1-\tau_3(C)}. \tag{3.31}$$

For $C\in(-\infty,0)$, $f_{3,C}(\tau)$ has no zeros in F_0. This proves Theorem 3.3 for $C\in\mathbb{R}\backslash\{0,1\}$.

Applying (3.29) and Theorem 3.3 for $f_{3,C'}(\cdot)$ with $C'\in\mathbb{R}\backslash\{0,1\}$, we immediately obtain: For $C\in(-\infty,0)\cup(0,1)$, $f_{2,C}(\tau)$ has a unique zero in F_0

which is simple; we denote this unique zero by $\tau_2(C)$, then $\tau_2(C) \in \mathring{F}_0$ and

$$\tau_3\left(\frac{1}{1-C}\right) = \frac{1}{1-\tau_2(C)}. \tag{3.32}$$

For $C \in (1, +\infty)$, $f_{2,C}(\tau)$ has no zeros in F_0. This proves Theorem 3.2 for $C \in \mathbb{R}\backslash\{0,1\}$.

Remark that (3.28) implies

$$\tau_2\left(\frac{1}{1-C}\right) = \frac{1}{1-\tau_1(C)}. \tag{3.33}$$

Again, the remaining case $C \in \{0,1\}$ will be postponed in Section 4. $\square$

As in Theorems 3.1-3.3, we always denote by $\tau_k(C)$ the unique zero of $f_{k,C}(\tau)$ in F_0. Remark 3.4 implies that $\tau_k(C)$ is a *smooth* function of $C \in (-\infty, 0) \cup (1, +\infty)$ if $k = 1$, $C \in (-\infty, 0) \cup (0, 1)$ if $k = 2$ and $C \in (0, 1) \cup (1, +\infty)$ if $k = 3$. We conclude this section by studying the asymptotic behavior of $\tau_k(C)$.

Lemma 3.9. *Let $k \in \{1, 2, 3\}$ and write $\tau_k(C) = a_k(C) + ib_k(C)$ with $a_k(C)$, $b_k(C) \in \mathbb{R}$. Then*

$$b_1(C) \to +\infty, \;\; a_1(C) \to \begin{cases} 1/4 & \text{if } C \to -\infty, \\ 3/4 & \text{if } C \to +\infty, \end{cases} \tag{3.34}$$

$$b_2(C) \to +\infty, \;\; a_2(C) \uparrow 1/2 \;\; \text{if } C \to -\infty, \tag{3.35}$$

$$b_3(C) \to +\infty, \;\; a_3(C) \downarrow 1/2 \;\; \text{if } C \to +\infty. \tag{3.36}$$

Consequently, for all $k \in \{1, 2, 3\}$, there hold $\tau_k(C) \to \infty$ as $C \to \infty$ and

$$\tau_k(C) \to 0 \text{ as } C \to 0, \;\; \tau_k(C) \to 1 \text{ as } C \to 1. \tag{3.37}$$

Proof. The proofs of (3.34)-(3.36) are similar, so we only give the proof of (3.35). Define meromorphic functions on F_0:

$$\phi_k(\tau) := \tau - \frac{2\pi i e_k(\tau)}{e_k(\tau)\eta_1(\tau) + \frac{g_2(\tau)}{6} - e_k(\tau)^2}. \tag{3.38}$$

Write $\tau = a + bi$ and $q = e^{2\pi i \tau}$ as before. Recalling the q-expansions (3.6)-(3.10) that

$$e_2(\tau) = -\frac{1}{3}\pi^2 - 8\pi^2(q^{\frac{1}{2}} + q + 4q^{\frac{3}{2}}) + O(|q|^2),$$

$$\eta_1(\tau) = \frac{1}{3}\pi^2 - 8\pi^2(q + 3q^2) + O(|q|^3),$$

$$g_2(\tau) = \frac{4}{3}\pi^4 + 320\pi^4(q + 9q^2) + O(|q|^3),$$

we easily obtain

$$\begin{aligned}\phi_2(\tau) &= \tau - \frac{i}{12\pi}q^{-\frac{1}{2}} - \frac{11i}{6\pi} + O(|q|^{\frac{1}{2}}) \\ &= a - \frac{\sin \pi a}{12\pi}e^{\pi b} + i\left(b - \frac{\cos \pi a}{12\pi}e^{\pi b} - \frac{11}{6\pi}\right) + O(|q|^{\frac{1}{2}}).\end{aligned}$$

Therefore, when $C \to -\infty$, it is easy to prove the existence of $\tilde{\tau}_2(C) = \tilde{a}_2(C) + i\tilde{b}_2(C) \in \mathring{F}_0$ such that $C = \phi_2(\tilde{\tau}_2(C))$ and

$$\tilde{b}_2(C) \to +\infty,\ \tilde{a}_2(C) \uparrow 1/2 \text{ if } C \to -\infty.$$

Since $C = \phi_2(\tilde{\tau}_2(C))$ implies $f_{2,C}(\tilde{\tau}_2(C)) = 0$ and $\tau_2(C)$ is the unique zero of $f_{2,C}$ in F_0, we conclude $\tau_2(C) = \tilde{\tau}_2(C)$. This proves (3.35).

Clearly (3.34)-(3.36) imply $\tau_k(C) \to \infty$ as $C \to \infty$ for all k. Consequently, (3.37) follows directly from (3.31)-(3.33): (3.33) gives $\tau_2(C) \to 0$ as $C \to 0$ and then (3.32) implies $\tau_3(C) \to 1$ as $C \to 1$; (3.32) gives $\tau_3(C) \to 0$ as $C \to 0$ and then (3.31) implies $\tau_1(C) \to 1$ as $C \to 1$; (3.31) gives $\tau_1(C) \to 0$ as $C \to 0$ and then (3.33) implies $\tau_2(C) \to 1$ as $C \to 1$.

The proof is complete. $\square$

4. CRITICAL POINTS OF $e_k(\tau)$'S

This section is devoted to the complete distribution of critical points of $e_k(\tau)$ for $k \in \{1, 2, 3\}$. Our main results of this section are as follows, and Theorem 1.8, Corollary 1.9-(2) will be direct consequences.

Theorem 4.1 (Critical points of $e_1(\tau)$)**.** *Recall Theorem 3.1 that $\tau_1(C)$ is the unique zero of $f_{1,C}(\tau)$ in F_0 for $C \in (-\infty, 0) \cup (1, +\infty)$. Then*

(1) *For any $m \in \mathbb{Z}$, there holds $e_1'(\tau) \neq 0$ in $F_0 + m$. Consequently, $e_1'(\tau) \neq 0$ whenever $\operatorname{Im}\tau \geq \frac{1}{2}$.*

(2) *Given $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)/\{\pm I_2\}$ with $c \neq 0$ and $\frac{-d}{c} \in (0, 1)$. Then $e_1'(\tau) \neq 0$ in the fundamental domain $\gamma(F_0)$ of $\Gamma_0(2)$. In particular, $e_1'(\tau) \neq 0$ along the line $\{\tau \in \mathbb{H} | \operatorname{Re}\tau = \frac{1}{2}\}$.*

(3) *Given $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)/\{\pm I_2\}$ with $c \neq 0$ and $\frac{-d}{c} \in (-\infty, 0) \cup (1, +\infty)$. Then $\frac{a\tau_1(-d/c)+b}{c\tau_1(-d/c)+d}$ is the unique zero of $e_1'(\tau)$ in $\gamma(F_0)$. In particular,*

$$\Theta_1 := \left\{ \frac{a\tau_1(\frac{-d}{c}) + b}{c\tau_1(\frac{-d}{c}) + d} \;\middle|\; \begin{array}{c} \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)/\{\pm I_2\} \textit{ with } c \neq 0 \\ \textit{and } \frac{-d}{c} \in (-\infty, 0) \cup (1, +\infty) \end{array} \right\} \tag{4.1}$$

gives all the zeros of $e_1'(\tau)$ in $\mathbb{H}$.

Theorem 4.2 (Critical points of $e_2(\tau)$)**.** *Recall Theorems 3.2-3.3 that $\tau_2(C)$ (resp. $\tau_3(C)$) is the unique zero of $f_{2,C}(\tau)$ (resp. $f_{3,C}(\tau)$) in F_0 for $C \in (-\infty, 0) \cup (0, 1)$ (resp. $C \in (0, 1) \cup (1, +\infty)$). Then*

(1) *For any $m \in \mathbb{Z}$, there holds $e_2'(\tau) \neq 0$ in $F_0 + m$. Consequently, $e_2'(\tau) \neq 0$ whenever $\operatorname{Im}\tau \geq \frac{1}{2}$.*

(2) *Let* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)/\{\pm I_2\}$ *with* $c \neq 0$ *such that*

$$\text{either } b \in 2\mathbb{Z},\ \tfrac{-d}{c} > 1 \text{ or } b \in 2\mathbb{Z}+1,\ \tfrac{-d}{c} < 0.$$

Then $e_2'(\tau) \neq 0$ *in the fundamental domain* $\gamma(F_0)$ *of* $\Gamma_0(2)$.

(3) *Given* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)/\{\pm I_2\}$ *with* $c \neq 0$. *If* $b \in 2\mathbb{Z}$ *and* $\frac{-d}{c} < 1$, *then* $\frac{a\tau_2(-d/c)+b}{c\tau_2(-d/c)+d}$ *is the unique zero of* $e_2'(\tau)$ *in* $\gamma(F_0)$; *If* $b \in 2\mathbb{Z}+1$ *and* $\frac{-d}{c} > 0$, *then* $\frac{a\tau_3(-d/c)+b}{c\tau_3(-d/c)+d}$ *is the unique zero of* $e_2'(\tau)$ *in* $\gamma(F_0)$. *In particular,*

$$\Theta_2 := \left\{ \frac{a\tau_2(\frac{-d}{c})+b}{c\tau_2(\frac{-d}{c})+d} \,\middle|\, \begin{array}{r} \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)/\{\pm I_2\} \text{ with } c \neq 0, \\ b \in 2\mathbb{Z} \text{ and } \frac{-d}{c} < 1 \end{array} \right\}$$
$$\cup \left\{ \frac{a\tau_3(\frac{-d}{c})+b}{c\tau_3(\frac{-d}{c})+d} \,\middle|\, \begin{array}{r} \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)/\{\pm I_2\} \text{ with } c \neq 0, \\ b \in 2\mathbb{Z}+1 \text{ and } \frac{-d}{c} > 0 \end{array} \right\} \tag{4.2}$$

gives all the zeros of $e_2'(\tau)$ *in* $\mathbb{H}$.

Theorem 4.3 (Critical points of $e_3(\tau)$)**.**

(1) *For any* $m \in \mathbb{Z}$, *there holds* $e_3'(\tau) \neq 0$ *in* $F_0 + m$. *Consequently,* $e_3'(\tau) \neq 0$ *whenever* $\operatorname{Im}\tau \geq \frac{1}{2}$.

(2) *Let* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)/\{\pm I_2\}$ *with* $c \neq 0$ *such that*

$$\text{either } b \in 2\mathbb{Z}+1,\ \tfrac{-d}{c} > 1 \text{ or } b \in 2\mathbb{Z},\ \tfrac{-d}{c} < 0.$$

Then $e_3'(\tau) \neq 0$ *in the fundamental domain* $\gamma(F_0)$ *of* $\Gamma_0(2)$.

(3) *Given* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)/\{\pm I_2\}$ *with* $c \neq 0$. *If* $b \in 2\mathbb{Z}+1$ *and* $\frac{-d}{c} < 1$, *then* $\frac{a\tau_2(-d/c)+b}{c\tau_2(-d/c)+d}$ *is the unique zero of* $e_3'(\tau)$ *in* $\gamma(F_0)$; *If* $b \in 2\mathbb{Z}$ *and* $\frac{-d}{c} > 0$, *then* $\frac{a\tau_3(-d/c)+b}{c\tau_3(-d/c)+d}$ *is the unique zero of* $e_3'(\tau)$ *in* $\gamma(F_0)$. *In particular,*

$$\Theta_3 := \left\{ \frac{a\tau_2(\frac{-d}{c})+b}{c\tau_2(\frac{-d}{c})+d} \,\middle|\, \begin{array}{r} \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)/\{\pm I_2\} \text{ with } c \neq 0, \\ b \in 2\mathbb{Z}+1 \text{ and } \frac{-d}{c} < 1 \end{array} \right\}$$
$$\cup \left\{ \frac{a\tau_3(\frac{-d}{c})+b}{c\tau_3(\frac{-d}{c})+d} \,\middle|\, \begin{array}{r} \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)/\{\pm I_2\} \text{ with } c \neq 0, \\ b \in 2\mathbb{Z} \text{ and } \frac{-d}{c} > 0 \end{array} \right\} \tag{4.3}$$

gives all the zeros of $e_3'(\tau)$ *in* $\mathbb{H}$.

Remark 4.4. Given $\gamma_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \in \Gamma_0(2)/\{\pm I_2\}$ with $c_j \neq 0$ such that $\gamma_1 \neq \pm\gamma_2$, we have $\gamma_1(\mathring{F}_0) \cap \gamma_2(\mathring{F}_0) = \emptyset$ (note that $\gamma_1(\partial F_0) \cap \gamma_2(\partial F_0) \neq \emptyset$ may happen), i.e. the critical points of $e_k(\tau)$ in $\gamma_1(F_0)$ and $\gamma_2(F_0)$ are different. Therefore, the map from

$$\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)/\{\pm I_2\} | c \neq 0, \tfrac{-d}{c} \in (-\infty, 0) \cup (1, +\infty)\}$$

to Θ_1 is one-to-one. Analogous results hold for Θ_k, $k = 2,3$. The above theorems completely determine all the critical points of the classical modular forms $e_k(\tau)$'s. To the best of our knowledge, such results have not appeared in the extensive literature and are new. We believe that these fundamental results will have important applications. Here we apply them to prove Theorem 1.8.

Proof of Theorem 1.8. (1). By (1.11), (1.13) and Theorem 1.3, we see that one of $\pm(3g_2)^{1/2}$ is a cusp of $\sigma(L_2;\tau)$ if and only if the corresponding monodromy data $\mathcal{C} = \infty$, if and only if $\eta_1'(\tau) = 0$. Since we proved in [11] that $\eta_1(\tau)$ has no critical points in $F = \gamma(F_0)$ if $c = 0$, and has a unique critical point in F if $c \neq 0$, we obtain Theorem 1.8-(1).

(2)-(4). Similarly it follows from (1.10), (1.12) and Theorem 1.3 that $-3e_k$ is a cusp of $\sigma(L_2;\tau)$ if and only if $e_k'(\tau) = 0$. Consequently, Theorem 1.8 (2)-(4) follow from Theorems 4.1-4.3.

(5)-(6). Since for any $\gamma \in \Gamma_0(2)$, each of $\{\eta_1(\tau), e_1(\tau), e_2(\tau), e_3(\tau)\}$ has at most one critical point in $F = \gamma(F_0)$, it follows that for each $E(\tau) \in \{\pm(3g_2)^{1/2}, -3e_1, -3e_2, -3e_3\}$, there is at most one $\tau_0 \in F$ such that $E(\tau_0)$ is a cusp of $\sigma(L_2;\tau_0)$. Moreover, if $c = 0$, none of $\{\eta_1(\tau), e_1(\tau), e_2(\tau), e_3(\tau)\}$ has critical points in $F = \gamma(F_0)$, so Theorem 1.8-(5) follows from (1)-(4). If $c \neq 0$, it is easy to see that exactly 3 of $\{\eta_1(\tau), e_1(\tau), e_2(\tau), e_3(\tau)\}$ have critical points simultaneously in this $F = \gamma(F_0)$, so there are exactly 3 τ's in F such that $\sigma(L_2;\tau)$ has a cusp.

The proof is complete. □

New proof of Corollary 1.9-(2). For $\tau = \frac{1}{2} + ib$ with $b > 0$, Theorem 4.1-(2) shows $e_1'(\tau) \neq 0$. By (3.6)-(3.8) we have

$$e_1(\tau) \in \mathbb{R}, \quad e_2(\tau) = \overline{e_3(\tau)} \notin \mathbb{R}.$$

From here and $e_1 + e_2 + e_3 = 0$, we also obtain $e_k'(\tau) \neq 0$ for $k = 2,3$. Finally, it is easy to see from (3.6) that $e_1(\frac{1}{2} + ib)$ is strictly increasing for large b and hence for all $b > 0$. □

To prove Theorems 4.1-4.3, we need the following simple observation.

Lemma 4.5. *Let $\tau = ib$ with $b > 0$. Then*

$$\tfrac{d}{db}e_1(\tau) < 0, \ \ \tfrac{d}{db}e_2(\tau) > 0, \ \ \tfrac{d}{db}e_3(\tau) < 0.$$

Proof. Recall the q-expansions (3.6)-(3.8) of $e_k(\tau)$'s. Let $\tau = ib$, i.e. $q = e^{-2\pi b}$. Since a_j's are positive, clearly

$$\tfrac{d}{db}e_1(\tau) = -32\pi^3 \sum_{k=1}^{\infty} k a_k q^k < 0, \tag{4.4}$$

$$\tfrac{d}{db}e_2(\tau) = 8\pi^3 \sum_{k=1}^{\infty} k a_k q^{\frac{k}{2}} > 0. \tag{4.5}$$

Since $e_3(\tau) = -e_1(\tau) - e_2(\tau)$, we have

$$\tfrac{d}{db}e_3(\tau) = -\tfrac{d}{db}e_2(\tau) - \tfrac{d}{db}e_1(\tau).$$

For $b \geq 1$, it follows from (4.4)-(4.5) and $q = e^{-2\pi b}$ that

$$\tfrac{d}{db}e_2(\tau) > -\tfrac{1}{4}e^{\pi b}\tfrac{d}{db}e_1(\tau)$$

and so

$$\tfrac{d}{db}e_3(\tau) < -(1-4e^{-\pi b})\tfrac{d}{db}e_2(\tau) < 0 \text{ for } b \geq 1.$$

On the other hand,

$$e_3(-1/\tau) = \tau^2 e_3(\tau).$$

This gives $e_3(i) = 0$, so $e_3(\tau) < 0$ for $\tau = ib$ with $b > 1$. Furthermore, letting $\tau = ib$ with $b \in (0,1)$ leads to $e_3(ib) = -b^{-2}e_3(ib^{-1})$, so

$$\tfrac{d}{db}e_3(ib) = b^{-4}\left[2be_3(ib^{-1}) + \left(\tfrac{d}{db}e_3\right)(ib^{-1})\right] < 0 \text{ for } b \in (0,1).$$

This completes the proof. $\square$

Recalling (1.12), we define holomorphic functions

$$G_k(\tau) := \tfrac{1}{2}g_2(\tau) + 3\eta_1(\tau)e_k(\tau) - 3e_k(\tau)^2 = -3\pi i e_k'(\tau). \tag{4.6}$$

Then we only need to study zeros of $G_k(\tau)$.

Lemma 4.6. *For $k \in \{1,2,3\}$ and any $m \in \mathbb{Z}$, $G_k(\tau) \neq 0$ for $\tau \in F_0 + m$.*

Proof. Fix $k \in \{1,2,3\}$. First we claim that

$$G_k(\tau) \neq 0 \text{ for } \tau \in \partial F_0 \cap \mathbb{H}. \tag{4.7}$$

For $\tau \in i\mathbb{R}_{>0}$, $G_k(\tau) \neq 0$ follows from Lemma 4.5. By $\eta_1(\tau+1) = \eta_1(\tau)$, $g_2(\tau+1) = g_2(\tau)$, $e_1(\tau+1) = e_1(\tau)$, $e_2(\tau+1) = e_3(\tau)$ and $e_3(\tau+1) = e_2(\tau)$, we obtain

$$G_1(\tau+1) = G_1(\tau),\ G_2(\tau+1) = G_3(\tau),\ G_3(\tau+1) = G_2(\tau). \tag{4.8}$$

Thus $G_k(\tau) \neq 0$ for $\tau \in i\mathbb{R}_{>0} + 1$.

If $|\tau - \frac{1}{2}| = \frac{1}{2}$, then $\tau' = \frac{\tau}{1-\tau} \in i\mathbb{R}_{>0}$. By using (3.12)-(3.13), we see from (3.1) with $C = -1$ that

$$\begin{aligned} f_{k,-1}(\tau') &= -3e_k(\tau')(\eta_1(\tau') + \eta_2(\tau')) - (\tfrac{g_2(\tau')}{2} - 3e_k(\tau')^2)(1+\tau') \\ &= \begin{cases} -(1-\tau)^3 G_3(\tau) & \text{if } k = 1, \\ -(1-\tau)^3 G_2(\tau) & \text{if } k = 2, \\ -(1-\tau)^3 G_1(\tau) & \text{if } k = 3. \end{cases} \end{aligned}$$

Since Lemma 3.5 shows $f_{k,-1}(\tau') \neq 0$ for all k, we obtain $G_k(\tau) \neq 0$. This proves (4.7).

Suppose by contradiction that $G_k(\tau)$ has a zero τ_0 in $\mathring{F}_0$. Recalling $\phi_k(\tau)$ in (3.38), it follows that $\phi_k(\tau)$ is meromorphic at τ_0 with τ_0 being a pole and so maps a small neighborhood $U \subset \mathring{F}_0$ of τ_0 onto a neighborhood V of ∞. Take $C > 1$ large enough such that $C \in V$. Then there exists $\tilde{\tau}_k(C) \in U$ such that $C = \phi_k(\tilde{\tau}_k(C))$, which is equivalent to $f_{k,C}(\tilde{\tau}_k(C)) = 0$. If $k = 2$, we already obtain a contradiction with Theorem 3.2. If $k \in \{1,3\}$, by Theorems 3.1, 3.3 and Lemma 3.9, we immediately obtain $\tilde{\tau}_k(C) = \tau_k(C) \to \infty$ as $C \to +\infty$, which contradicts with $\tilde{\tau}_k(C) \in U$.

Therefore, we have proved $G_k(\tau) \neq 0$ for $\tau \in F_0$. Together with (4.8), we conclude that for any $m \in \mathbb{Z}$, $G_k(\tau) \neq 0$ for $\tau \in F_0 + m$. The proof is complete. $\square$

Proof of Theorem 4.1. The assertion (1) is just Lemma 4.6.

Given $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)/\{\pm I_2\}$ with $c \neq 0$. Write $\tau' = \gamma \cdot \tau = \frac{a\tau+b}{c\tau+d}$ with $\tau \in F_0$. Since $c \in 2\mathbb{Z}$ and $d \in 2\mathbb{Z}+1$, we have

$$e_1(\tau') = (c\tau+d)^2 e_1(\tau). \tag{4.9}$$

This, together with

$$\eta_1(\tau') = (c\tau+d)(c\eta_2(\tau) + d\eta_1(\tau)), \;\; g_2(\tau') = (c\tau+d)^4 g_2(\tau),$$

and (3.1), leads to

$$\begin{aligned} G_1(\tau') &= c(c\tau+d)^3 \left[3e_1(\tau)(\tfrac{d}{c}\eta_1 + \eta_2) + (\tfrac{g_2(\tau)}{2} - 3e_1^2)(\tfrac{d}{c} + \tau)\right] \\ &= -c(c\tau+d)^3 f_{1,\frac{-d}{c}}(\tau). \end{aligned}$$

Clearly $\frac{-d}{c} \in \mathbb{Q}\backslash\mathbb{Z}$.

(2). If $\frac{-d}{c} \in (0,1)$, then Theorem 3.1 shows that $f_{1,\frac{-d}{c}}(\tau)$ has no zeros in F_0. Thus, G_1 has no zeros in $\gamma(F_0)$. In particular, letting $\gamma = \left(\begin{smallmatrix} 1 & -1 \\ 2 & -1 \end{smallmatrix}\right)$ implies that G_1 has no zeros in

$$\gamma(F_0) = \left\{\tau \in \mathbb{H} \,|\, |\tau - \tfrac{1}{2}| \leq \tfrac{1}{2},\ |\tau - \tfrac{1}{4}| \geq \tfrac{1}{4},\ |\tau - \tfrac{3}{4}| \geq \tfrac{1}{4}\right\},$$

so G_1 has no zeros along the line $\{\tau \in \mathbb{H} | \operatorname{Re}\tau = \frac{1}{2}\}$. This proves (2).

(3). If $\frac{-d}{c} \in (-\infty,0) \cup (1,+\infty)$, then Theorem 3.1 shows that $\tau_1(\frac{-d}{c})$ is the unique zero of $f_{1,\frac{-d}{c}}(\tau)$ in F_0. Consequently,

$$\gamma \cdot \tau_1(\tfrac{-d}{c}) = \frac{a\tau_1(\frac{-d}{c}) + b}{c\tau_1(\frac{-d}{c}) + d} \in \gamma(\mathring{F}_0)$$

is the unique zero of G_1 in $\gamma(F_0)$. Together with (1.15), we conclude that the set Θ_1 defined in (4.1) gives all the zeros of G_1 and so e_1'. This proves (3). The proof is complete. $\square$

Proof of Theorem 4.2. The proof is similar to that of Theorem 4.1. Again (1) is just Lemma 4.6. Given $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)/\{\pm I_2\}$ with $c \neq 0$. Note $a, d \in 2\mathbb{Z}+1$. Write $\tau' = \gamma \cdot \tau = \frac{a\tau+b}{c\tau+d}$ with $\tau \in F_0$ as before. The different thing from (4.9) is that

$$e_2(\tau') = \begin{cases} (c\tau+d)^2 e_2(\tau) & \text{if } b \in 2\mathbb{Z}, \\ (c\tau+d)^2 e_3(\tau) & \text{if } b \in 2\mathbb{Z}+1, \end{cases}$$

and so

$$G_2(\tau') = \begin{cases} -c(c\tau+d)^3 f_{2,\frac{-d}{c}}(\tau) & \text{if } b \in 2\mathbb{Z}, \\ -c(c\tau+d)^3 f_{3,\frac{-d}{c}}(\tau) & \text{if } b \in 2\mathbb{Z}+1. \end{cases}$$

(2). If $b \in 2\mathbb{Z}$ and $\frac{-d}{c} > 1$, then Theorem 3.2 shows that $f_{2,\frac{-d}{c}}(\tau)$ has no zeros in F_0. Thus, G_2 has no zeros in $\gamma(F_0)$. If $b \in 2\mathbb{Z}+1$ and $\frac{-d}{c} < 0$, then Theorem 3.3 shows that $f_{3,\frac{-d}{c}}(\tau)$ has no zeros in F_0. Thus, G_2 has no zeros in $\gamma(F_0)$. This proves (2).

(3). If $b \in 2\mathbb{Z}$ and $\frac{-d}{c} < 1$, since $\frac{-d}{c} \neq 0$, Theorem 3.2 shows that $\tau_2(\frac{-d}{c})$ is the unique zero of $f_{2,\frac{-d}{c}}(\tau)$ in F_0. Consequently,

$$\gamma \cdot \tau_2(\tfrac{-d}{c}) = \frac{a\tau_2(\frac{-d}{c}) + b}{c\tau_2(\frac{-d}{c}) + d}$$

is the unique zero of G_2 in $\gamma(F_0)$. If $b \in 2\mathbb{Z}+1$ and $\frac{-d}{c} > 0$, since $\frac{-d}{c} \neq 1$, Theorem 3.3 shows that $\tau_3(\frac{-d}{c})$ is the unique zero of $f_{3,\frac{-d}{c}}(\tau)$ in F_0, which implies that

$$\gamma \cdot \tau_3(\tfrac{-d}{c}) = \frac{a\tau_3(\frac{-d}{c}) + b}{c\tau_3(\frac{-d}{c}) + d}$$

is the unique zero of G_2 in $\gamma(F_0)$. In particular, the set Θ_2 defined in (4.2) gives all the zeros of G_2 and so e_2'. This proves (3). □

Proof of Theorem 4.3. The proof is similar to that of Theorem 4.2; we omit the details here. □

Conversely, we can finish the proof of Theorems 3.1-3.3 by considering the remaining case $C \in \{0, 1\}$.

Completion of the proof of Theorems 3.1-3.3. Fix $k \in \{1,2,3\}$. First we consider $C = 0$, i.e.

$$f_{k,0}(\tau) = -3e_k(\tau)\eta_2(\tau) - \tau(\tfrac{g_2(\tau)}{2} - 3e_k(\tau)^2).$$

Suppose $f_{k,0}(\tau) = 0$ for some $\tau \in F_0$. Then $\tau' := \frac{\tau-1}{\tau} \in F_0$. By $\eta_1(\tau') = \tau\eta_2(\tau)$, $g_2(\tau') = \tau^4 g_2(\tau)$ and

$$e_1(\tau') = \tau^2 e_2(\tau),\ e_2(\tau') = \tau^2 e_3(\tau),\ e_3(\tau') = \tau^2 e_1(\tau),$$

we obtain

$$\begin{aligned} G_j(\tau') &= 3\eta_1(\tau')e_j(\tau') + \tfrac{1}{2}g_2(\tau') - 3e_j(\tau')^2 \\ &= \begin{cases} -\tau^3 f_{2,0}(\tau) & \text{if } j = 1, \\ -\tau^3 f_{3,0}(\tau) & \text{if } j = 2, \\ -\tau^3 f_{1,0}(\tau) & \text{if } j = 3, \end{cases} \end{aligned}$$

i.e. $G_j(\tau') = 0$ for some j, a contradiction with Lemma 4.6.

Now we consider $C = 1$, i.e.

$$f_{k,1}(\tau) := 3e_k(\tau)(\eta_1(\tau) - \eta_2(\tau)) + (\tfrac{g_2(\tau)}{2} - 3e_k(\tau)^2)(1-\tau).$$

Suppose $f_{k,1}(\tau) = 0$ for some $\tau \in F_0$. Then $\tau' := \frac{1}{1-\tau} \in F_0$. By

$$\eta_1(\tau') = (1-\tau)(\eta_1(\tau) - \eta_2(\tau)), \;\; g_2(\tau') = (1-\tau)^4 g_2(\tau),$$
$$e_1(\tau') = (1-\tau)^2 e_3(\tau), \;\; e_2(\tau') = (1-\tau)^2 e_1(\tau),$$
$$e_3(\tau') = (1-\tau)^2 e_2(\tau),$$

we obtain

$$G_j(\tau') = \begin{cases} (1-\tau)^3 f_{3,0}(\tau) & \text{if } j = 1, \\ (1-\tau)^3 f_{1,0}(\tau) & \text{if } j = 2, \\ (1-\tau)^3 f_{2,0}(\tau) & \text{if } j = 3, \end{cases}$$

i.e. $G_j(\tau') = 0$ for some j, again a contradiction with Lemma 4.6.

This proves $f_{k,C}(\tau) \neq 0$ in F_0 provided $C \in \{0, 1\}$. □

By Theorems 4.1-4.3, we can transform every critical point of $e_k(\tau)$ via the Möbius transformation of $\Gamma_0(2)$ action to locate it in F_0. Denote *the collection of such corresponding points in F_0 by $\mathcal{C}_k$*, which consists of countably many points. Recalling the six curves $\mathcal{C}_{k,-}, \mathcal{C}_{k,+}$, $k = 1, 2, 3$, defined in Remark 3.4, we can describe the geometry of $\mathcal{C}_k$ as follows.

Theorem 4.7. *The six curves $\mathcal{C}_{k,-}, \mathcal{C}_{k,+}$, $k = 1, 2, 3$, are all smooth curves and*

$$\mathcal{C}_1 \subset \mathcal{C}_{1,+} \cup \mathcal{C}_{1,-} = \overline{\mathcal{C}_1} \cap F_0, \tag{4.10}$$

$$\mathcal{C}_k \subset \mathcal{C}_{2,+} \cup \mathcal{C}_{2,-} \cup \mathcal{C}_{3,+} \cup \mathcal{C}_{3,-} = \overline{\mathcal{C}_k} \cap F_0, \quad k = 2, 3.$$

In other words, all critical points of $e_1(\tau)$ (resp. $e_2(\tau)$, $e_3(\tau)$) can be mapped via the Möbius transformations of $\Gamma_0(2)$ to locate densely on the union of 2 (resp. 4) smooth curves.

Proof. Recalling the definition of $\mathcal{C}_k$, it follows from (4.1), (4.2) and (4.3) that

$$\mathcal{C}_1 = \left\{ \tau_1(\tfrac{-d}{c}) \;\middle|\; \begin{array}{c} \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)/\{\pm I_2\},\ c \neq 0 \\ \text{and } \frac{-d}{c} \in (-\infty, 0) \cup (1, +\infty) \end{array} \right\} \subset \mathcal{C}_{1,+} \cup \mathcal{C}_{1,-},$$

$$\mathcal{C}_2 = \left\{ \tau_2(\tfrac{-d}{c}) \;\middle|\; \begin{array}{r} \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)/\{\pm I_2\},\ c \neq 0, \\ b \in 2\mathbb{Z} \text{ and } \frac{-d}{c} < 1 \end{array} \right\}$$
$$\cup \left\{ \tau_3(\tfrac{-d}{c}) \;\middle|\; \begin{array}{r} \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)/\{\pm I_2\},\ c \neq 0, \\ b \in 2\mathbb{Z}+1 \text{ and } \frac{-d}{c} > 0 \end{array} \right\} \subset \bigcup_{k=2,3} \mathcal{C}_{k,+} \cup \mathcal{C}_{k,-},$$

$$\mathcal{C}_3 = \left\{ \tau_2(\tfrac{-d}{c}) \;\middle|\; \begin{array}{r} \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)/\{\pm I_2\},\ c \neq 0, \\ b \in 2\mathbb{Z}+1 \text{ and } \frac{-d}{c} < 1 \end{array} \right\}$$
$$\cup \left\{ \tau_3(\tfrac{-d}{c}) \;\middle|\; \begin{array}{r} \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)/\{\pm I_2\},\ c \neq 0, \\ b \in 2\mathbb{Z} \text{ and } \frac{-d}{c} > 0 \end{array} \right\} \subset \bigcup_{k=2,3} \mathcal{C}_{k,+} \cup \mathcal{C}_{k,-}.$$

It remains to prove the denseness: $\mathcal{C}_{1,+}\cup\mathcal{C}_{1,-}\subset\overline{\mathcal{C}_1}$ and

$$\bigcup_{k=2,3}\mathcal{C}_{k,+}\cup\mathcal{C}_{k,-}\subset\overline{\mathcal{C}_k},\quad k=2,3.$$

Since Remark 3.4 says that $\tau_k(C)$ is continuous as a function of C, it suffices to prove that each of

$$Q_0:=\{\tfrac{-d}{c}\mid d\in\mathbb{Z},\,c\in2\mathbb{Z}\setminus\{0\},\,(c,d)=1\},$$

$$Q_1:=\left\{\tfrac{-d}{c}\,\middle|\,\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_0(2)/\{\pm I_2\}\text{ with }c\neq0,b\in2\mathbb{Z}+1\right\}$$

and

$$Q_2:=\left\{\tfrac{-d}{c}\,\middle|\,\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_0(2)/\{\pm I_2\}\text{ with }c\neq0,b\in2\mathbb{Z}\right\}$$

is dense in $\mathbb{R}$. The assertion $\overline{Q_0}=\mathbb{R}$ is trivial because $\mathbb{Q}\subset\overline{Q_0}$. So we only need to prove $Q_1=Q_2=Q_0$. Clearly $Q_1,Q_2\subset Q_0$. Conversely, take any $\frac{m}{n}\in Q_0$. Then $n\in2\mathbb{Z}\setminus\{0\}$, m is odd and $(m,n)=1$. So there exists $a,b\in\mathbb{Z}$ such that $am+bn=1$, which gives

$$\begin{pmatrix}a&b\\-n&m\end{pmatrix}\in\Gamma_0(2)\quad\text{and}\quad\begin{pmatrix}a-n&b+m\\-n&m\end{pmatrix}\in\Gamma_0(2).$$

Therefore, $\frac{m}{n}\in Q_1$ and $\frac{m}{n}\in Q_2$. In conclusion, $Q_0=Q_1=Q_2$.

Finally, the smoothness of the six curves will be proved in Section 5. The proof is complete. □

We conclude this section by studying basic properties of these curves.

Theorem 4.8.

(i) *For each $k\in\{1,2,3\}$, the function $C\mapsto\tau_k(C)$ is one-to-one, where $C\in(-\infty,0)\cup(1,+\infty)$ if $k=1$, $C\in(-\infty,0)\cup(0,1)$ if $k=2$, $C\in(0,1)\cup(1,+\infty)$ if $k=3$. Equivalently, neither $\mathcal{C}_{k,-}$ nor $\mathcal{C}_{k,+}$ has self-intersection, and $\mathcal{C}_{k,-}$ has no intersection with $\mathcal{C}_{k,+}$.*

(ii) $\partial\mathcal{C}_{2,+}=\partial\mathcal{C}_{3,-}=\{0,1\}$ *and*

$$\partial\mathcal{C}_{1,-}=\{0,\tfrac14+i\infty\},\ \ \partial\mathcal{C}_{1,+}=\{1,\tfrac34+i\infty\},$$

$$\partial\mathcal{C}_{2,-}=\{0,\tfrac12+i\infty\},\ \ \partial\mathcal{C}_{3,+}=\{1,\tfrac12+i\infty\}.$$

(iii) *The curve $\mathcal{C}_{1,-}$ is symmetric with $\mathcal{C}_{1,+}$ with respect to the line* $\operatorname{Re}\tau=\frac12$*; $\mathcal{C}_{2,-}$ is symmetric with $\mathcal{C}_{3,+}$ with respect to the line* $\operatorname{Re}\tau=\frac12$*; $\mathcal{C}_{2,+}$ is symmetric with $\mathcal{C}_{3,-}$ with respect to the line* $\operatorname{Re}\tau=\frac12$

(iv) *The curve $\mathcal{C}_{1,-}$ (resp. $\mathcal{C}_{1,+}$) has no intersection with the line* $\operatorname{Re}\tau=\frac12$.

(v) *The union $\mathcal{C}_{2,-}\cup\mathcal{C}_{2,+}$ (resp. $\mathcal{C}_{3,+}\cup\mathcal{C}_{3,-}$) has no intersection with $\{\tau=\frac12+ib\,|\,b\geq\sqrt3/2\}$.*

Proof. (i) Recall the *holomorphic* functions $\phi_k(\tau)$ defined on F_0 in (3.38) (note from Lemma 4.6 that the denominator of $\phi_k(\tau)$ does not vanish on F_0). Since $f_{k,C}(\tau_k(C))=0$ is equivalent to $C=\phi_k(\tau_k(C))$, letting $k=1$ we see that $\tau_1:(-\infty,0)\cup(1,+\infty)\to\mathcal{C}_{1,-}\cup\mathcal{C}_{1,+}$ is bijective. That is, each of $\mathcal{C}_{1,-}$, $\mathcal{C}_{1,+}$

has no self-intersection, and $\mathcal{C}_{1,-}$ has no intersection with $\mathcal{C}_{1,+}$. The other two cases $k \in \{2,3\}$ are similar.

(ii) This assertion is just Lemma 3.9.

(iii) By the q-expansions (3.6)-(3.10), we have $e_1(1-\bar{\tau}) = \overline{e_1(\tau)}$, $e_2(1-\bar{\tau}) = \overline{e_3(\tau)}$, $e_3(1-\bar{\tau}) = \overline{e_2(\tau)}$, $\eta_1(1-\bar{\tau}) = \overline{\eta_1(\tau)}$, $\eta_2(1-\bar{\tau}) = \overline{\eta_1(\tau)} - \overline{\eta_2(\tau)}$ and $g_2(1-\bar{\tau}) = \overline{g_2(\tau)}$. Since $C \in \mathbb{R}\backslash\{0,1\}$, we easily see from the expression (3.1) of $f_{k,C}(\tau)$ that

$$f_{1,1-C}(1-\bar{\tau}) = -\overline{f_{1,C}(\tau)},$$
$$f_{2,1-C}(1-\bar{\tau}) = -\overline{f_{3,C}(\tau)}, \quad f_{3,1-C}(1-\bar{\tau}) = -\overline{f_{2,C}(\tau)}.$$

Therefore, it follows from Theorems 3.1-3.3 that

$$\tau_1(1-C) = 1-\overline{\tau_1(C)}, \tag{4.11}$$
$$\tau_2(1-C) = 1-\overline{\tau_3(C)}, \quad \tau_3(1-C) = 1-\overline{\tau_2(C)}.$$

This proves (iii).

(iv) Suppose $\operatorname{Re}\tau_1(C) = \frac{1}{2}$ for some $C \in (-\infty,0)\cup(1,+\infty)$. Then $f_{1,C}(\tau_1(C)) = 0$ gives $C = \phi_1(\tau_1(C))$, i.e.

$$C = \tau_1(C) - \frac{2\pi i e_1(\tau_1(C))}{e_1(\tau_1(C))\eta_1(\tau_1(C)) + \frac{g_2(\tau_1(C))}{6} - e_1(\tau_1(C))^2}.$$

Since $\operatorname{Re}\tau_1(C) = \frac{1}{2}$ imply $g_2(\tau_1(C)), e_1(\tau_1(C)), \eta_1(\tau_1(C)) \in \mathbb{R}$, we obtain $C = \operatorname{Re}\tau_1(C) = \frac{1}{2}$, a contradiction. This proves the assertion (iv).

(v) Assume by contradiction that $\mathcal{C}_{2,-}\cup\mathcal{C}_{2,+}$ contains a point $\frac{1}{2}+ib_0$ with $b_0 \geq \sqrt{3}/2$. Then $\frac{1}{2}+ib_0 = \tau_2(C)$ for some $C \in (-\infty,0)\cup(0,1)$. Again we have

$$C = \tau_2(C) - \frac{2\pi i e_2}{e_2\eta_1 + \frac{g_2}{6} - e_2^2}(\tau_2(C)) = \frac{1}{2} + ib_0 + \frac{2e_2(\tau_2(C))}{e_2'(\tau_2(C))}, \tag{4.12}$$

where we use (1.12) to obtain the second equality.

On the other hand, for $\tau = \frac{1}{2}+ib$, it follows from $e_2 = \overline{e_3} \notin \mathbb{R}$ and $e_1 + e_2 + e_3 = 0$ that $e_2 = -\frac{1}{2}e_1 + i\operatorname{Im} e_2$. Furthermore, (3.7) implies $\operatorname{Im} e_2(\frac{1}{2} + ib) < 0$ for b large and so for all $b > 0$. These, together with (1.12), easily imply

$$\operatorname{Re} e_2'(\tau) = \tfrac{-1}{\pi}(\eta_1+e_1)\operatorname{Im} e_2 =: A > 0,$$
$$\operatorname{Im} e_2'(\tau) = \tfrac{1}{2}e_{1,b} := \tfrac{1}{2}\tfrac{d}{db}e_1(\tfrac{1}{2}+ib),$$

and so

$$\operatorname{Im}\frac{2e_2(\tau)}{e_2'(\tau)} = \frac{2A\operatorname{Im} e_2 + \frac{1}{2}e_1e_{1,b}}{A^2 + \frac{1}{4}e_{1,b}^2}.$$

Inserting this into (4.12) leads to

$$b_0 + \frac{2A\operatorname{Im} e_2 + \frac{1}{2}e_1e_{1,b}}{A^2+\frac{1}{4}e_{1,b}^2}(\tfrac{1}{2}+ib_0) = 0,$$

which is equivalent to

$$\frac{A}{\pi}|\operatorname{Im} e_2|(\eta_1 + e_1 - \tfrac{2\pi}{b_0}) + \tfrac{1}{4}e_{1,b}^2 + \frac{e_1 e_{1,b}}{2b_0} = 0 \text{ at } \tau = \tfrac{1}{2} + ib_0. \tag{4.13}$$

Note from Corollary 1.9-(2) that $e_{1,b}(\frac{1}{2}+ib) > 0$ for all b. Since $b_0 \geq \frac{\sqrt{3}}{2}$, we have $e_1(\frac{1}{2}+ib_0) > e_1(\frac{1}{2}+i\frac{1}{2}) = 0$ and it was proved in [25, Lemma 6.1] that $\eta_1(\frac{1}{2}+ib_0) + e_1(\frac{1}{2}+ib_0) - \frac{2\pi}{b_0} > 0$. Therefore, every term in the LHS of (4.13) is positive, clearly a contradiction. This proves (v).

The proof is complete. □

Theorem 4.8 (ii) and (v) indicates that the curve $\mathcal{C}_{2,+}$ (resp. $\mathcal{C}_{3,-}$) must intersect with $\{\tau = \frac{1}{2} + ib \,|\, b \in (0, \sqrt{3}/2)\}$. As suggested by Figure 2 below, we suspect that $\mathcal{C}_{2,+}$ (resp. $\mathcal{C}_{3,-}$) has a unique intersection point with the line $\operatorname{Re}\tau = \frac{1}{2}$, and the curve $\mathcal{C}_{2,-}$ (resp. $\mathcal{C}_{3,+}$) has no intersection with the line $\operatorname{Re}\tau = \frac{1}{2}$. These problems seem challenging because of $e_k(\frac{1}{2}+ib) \notin \mathbb{R}$ for $k \in \{2,3\}$ and remain open.

5. Geometric interpretation and smoothness

In this final section, we give the geometric meaning of the six curves from the multiple Green function $G_2(z_1, z_2; \tau)$. Let $G(z) = G(z;\tau)$ be the Green function on the torus $E_\tau = \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$:

$$-\Delta G(z;\tau) = \delta_0 - \frac{1}{|E_\tau|} \text{ on } E_\tau, \quad \int_{E_\tau} G(z;\tau) = 0,$$

where δ_0 is the Dirac measure at 0 and $|E_\tau|$ is the area of the torus E_τ. See [25] for a detailed study of $G(z;\tau)$. Define the multiple Green function G_2 by

$$G_2(z_1, z_2; \tau) := G(z_1 - z_2; \tau) - 2G(z_1;\tau) - 2G(z_2;\tau), \tag{5.1}$$

where $0 \neq z_1 \neq z_2 \neq 0$. A critical point (a_1, a_2) of G_2 satisfies

$$2\nabla G(a_1;\tau) = \nabla G(a_1 - a_2;\tau), \;\; 2\nabla G(a_2;\tau) = \nabla G(a_2 - a_1;\tau).$$

Clearly if (a_1, a_2) is a critical point then so does (a_2, a_1), and we consider such two critical points to be *the same one*. A critical point (a_1, a_2) is called a *trivial critical point* if

$$\{a_1, a_2\} = \{-a_1, -a_2\} \text{ in } E_\tau.$$

It is known [27] that G_2 has only five trivial critical points $\{(\frac{1}{2}\omega_i, \frac{1}{2}\omega_j) | i,j \in \{1,2,3\}, i \neq j\}$ and $\{(q_\pm, -q_\pm) | \wp(q_\pm;\tau) = \pm\sqrt{g_2(\tau)/12}\}$. Geometrically, we want to determine those τ's such that one of trivial critical points is *degenerate* (i.e. the Hessian of G_2 at this critical point *vanishes*), because bifurcation phenomena should happen and so *non-trivial* critical points of G_2 should appear near such τ's. Define the *degeneracy curves* of G_2 in F_0 related to $(\frac{1}{2}\omega_i, \frac{1}{2}\omega_j)$:

$$\mathcal{C}_{i,j} := \left\{\tau \in F_0 | \det D^2 G_2(\tfrac{1}{2}\omega_i, \tfrac{1}{2}\omega_j; \tau) = 0\right\}. \tag{5.2}$$

It was calculated in [27, Example 4.2] that the Hessian at $(\frac{1}{2}\omega_i, \frac{1}{2}\omega_j)$ is given by

$$\det D^2 G_2(\tfrac{1}{2}\omega_i, \tfrac{1}{2}\omega_j; \tau) = \frac{4|G_k(\tau)|^2}{(2\pi)^4 \operatorname{Im}\tau} \operatorname{Im}\phi_k(\tau), \ \ \{i,j,k\} = \{1,2,3\} \tag{5.3}$$

(see also [9, (1.7)-(1.8)] for exactly these expressions), where $\phi_k(\tau)$ and $G_k(\tau)$ are functions defined in (3.38) and (4.6), respectively.

Theorem 5.1 (Geometric meaning of the six curves)**.** *For the six curves $\mathcal{C}_{k,\pm}$'s related to critical points of $e_k(\tau)$'s and the degeneracy curves $\mathcal{C}_{i,j}$'s related to critical points of G_2, there holds $\mathcal{C}_{i,j} = \mathcal{C}_{k,-} \cup \mathcal{C}_{k,+}$ for $\{i,j,k\} = \{1,2,3\}$. In particular, $(\frac{1}{2}\omega_i, \frac{1}{2}\omega_j)$ is a non-degenerate critical point of $G_2(\cdot;\tau)$ for $\tau = \frac{1}{2} + ib$ with $b \geq \sqrt{3}/2$.*

Proof of Theorem 5.1 and the smoothness of curves. Lemma 4.6 shows $G_k(\tau) \neq 0$ for all $\tau \in F_0$, then it follows from (5.2)-(5.3) that

$$\begin{aligned} \mathcal{C}_{i,j} &= \{\tau \in F_0 | \operatorname{Im}\phi_k(\tau) = 0\} \\ &= \{\tau \in F_0 | \phi_k(\tau) = C \text{ for some } C \in \mathbb{R}\} \\ &= \{\tau \in F_0 | f_{k,C}(\tau) = 0 \text{ for some } C \in \mathbb{R}\} \\ &= \begin{cases} \{\tau_1(C) | C \in (-\infty,0) \cup (1,+\infty)\} & \text{if } k=1 \\ \{\tau_2(C) | C \in (-\infty,0) \cup (0,1)\} & \text{if } k=2 \\ \{\tau_3(C) | C \in (0,1) \cup (1,+\infty)\} & \text{if } k=3 \end{cases} \\ &= \mathcal{C}_{k,-} \cup \mathcal{C}_{k,+}, \end{aligned} \tag{5.4}$$

where we have used Theorems 3.1-3.3. Together with Theorem 4.8, we see that $(\frac{1}{2}\omega_i, \frac{1}{2}\omega_j)$ is a non-degenerate critical point of $G_2(\cdot;\tau)$ for $\tau = \frac{1}{2} + ib$ with $b \geq \sqrt{3}/2$. This proves Theorem 5.1.

Finally, the smoothness of these curves was proved in [9, Theorem 1.3]. In fact, writing $\tau = a + bi$ with $a, b \in \mathbb{R}$, then

$$\frac{\partial \operatorname{Im}\phi_k}{\partial a} = \operatorname{Im}\phi_k', \quad \frac{\partial \operatorname{Im}\phi_k}{\partial b} = \operatorname{Re}\phi_k'.$$

Since [9, Theorem 3.1] proves

$$\phi_k'(\tau) \neq 0 \quad \forall \tau \in F_0, \ \ k = 1,2,3,$$

we easily see from Theorem 4.8-(i) and $\mathcal{C}_{k,-} \cup \mathcal{C}_{k,+} = \{\tau \in F_0 | \operatorname{Im}\phi_k(\tau) = 0\}$ that the six curves $\mathcal{C}_{k,-}, \mathcal{C}_{k,+}$, $k = 1,2,3$, are all smooth curves in F_0. $\square$

The numerical simulation for the degeneracy curves of G_2 and hence the six smooth curves is shown in Figure 2, which is copied from C. L. Wang. The other three curves $\mathcal{C}_+, \mathcal{C}_0, \mathcal{C}_-$ appearing in Figure 2 are those degeneracy curves of G_2 at the other two trivial critical points $\{(q_\pm, -q_\pm) | \wp(q_\pm;\tau) = \pm\sqrt{g_2(\tau)/12}\}$. We proved in [11, Theorem 5.1] that under the Möbius transformations of $\Gamma_0(2)$ action, all critical points of the weight 2 Eisenstein series $E_2(\tau)$ are mapped to locate densely on these three curves. Clearly

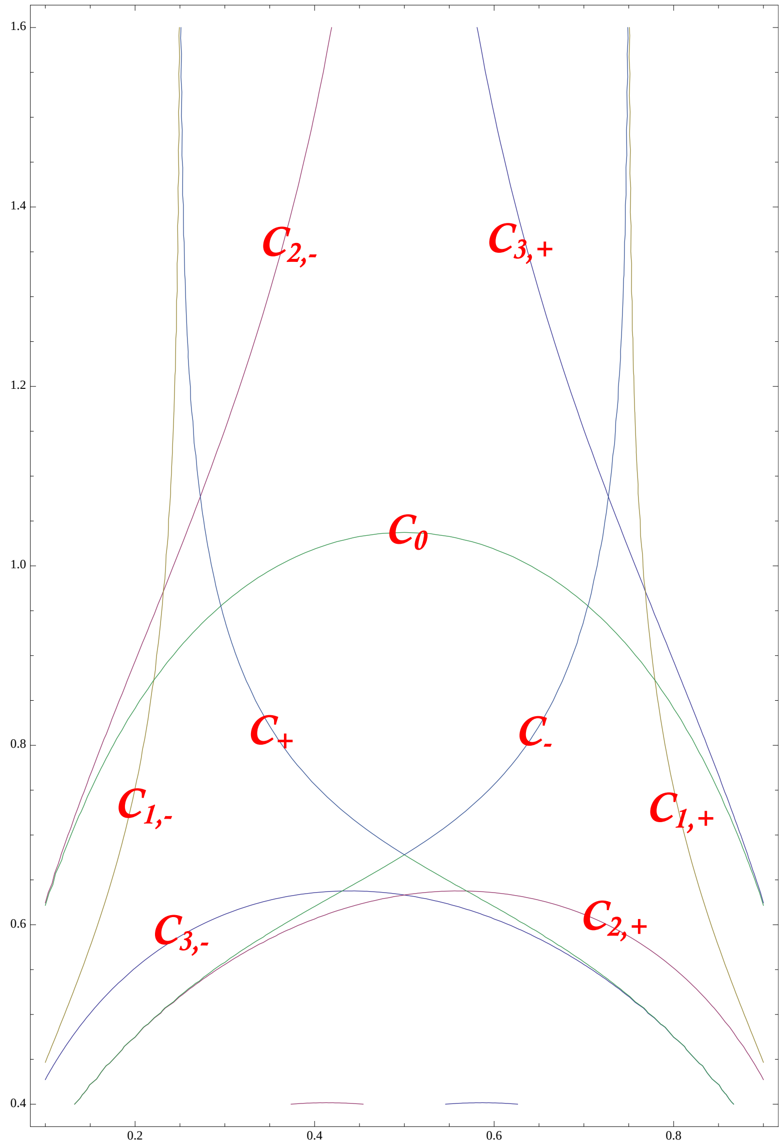

FIGURE 2. The 9 smooth curves in F_0.

Theorem 5.1 and [11, Theorem 5.1] together prove rigorously why Figure 2 contains exactly nine smooth curves. We will study the critical points of the weight 4 and 6 Eisenstein series in a future work.

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