THE GEOMETRY OF GENERALIZED LAMÉ EQUATION, I

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Abstract. In this paper, we prove that the spectral curve $\Gamma_{\mathbf{n}}$ of the generalized Lamé equation with the Treibich-Verdier potential

$$y''(z) = \left[\sum_{k=0}^{3} n_k(n_k+1)\wp(z+\tfrac{\omega_k}{2}|\tau) + B\right] y(z), \quad n_k \in \mathbb{Z}_{\geq 0}$$

can be embedded into the symmetric space $\mathrm{Sym}^N E_\tau$ of the N-th copy of the torus E_τ, where $N = \sum n_k$. This embedding induces an addition map $\sigma_{\mathbf{n}}(\cdot|\tau)$ from $\Gamma_{\mathbf{n}}$ onto E_τ. The main result is to prove that the degree of $\sigma_{\mathbf{n}}(\cdot|\tau)$ is equal to $\sum_{k=0}^{3} n_k(n_k+1)/2$. This is the first step toward constructing the pre-modular form associated with this generalized Lamé equation.

Résumé: Dans cet article, nous montrons que la courbe spectrale $\Gamma_{\mathbf{n}}$ du L'équation de Lamé généralisée avec le potentiel de Treibich-Verdier

$$y''(z) = \left[\sum_{k=0}^{3} n_k(n_k+1)\wp(z+\tfrac{\omega_k}{2}|\tau) + B\right] y(z), \quad n_k \in \mathbb{Z}_{\geq 0}$$

peut être plongée dans l'espace symétrique $\mathrm{Sym}^N E_\tau$ de la N'ème copie du tore E_τ, où $N = \sum n_k$. Cette injection induit une application dáddition $\sigma_{\mathbf{n}}(\cdot|\tau)$ à partir de $\Gamma_{\mathbf{n}}$ sur E_τ. Le résultat principal est de prouver que le degré de $\sigma_{\mathbf{n}}(\cdot|\tau)$ égale $\sum_{k=0}^{3} n_k(n_k+1)/2$. C'est la première étape vers la construction de la forme prémodulaire associée à cette équation de Lamé généralisée.

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1. Introduction

Throughout the paper, we let $\tau \in \mathbb{H} = \{\tau \in \mathbb{C} | \operatorname{Im}\tau > 0\}$, $E_\tau = \mathbb{C}/\Lambda_\tau$ be a flat torus with the lattice $\Lambda_\tau = \mathbb{Z} + \mathbb{Z}\tau$, and $\wp(z) = \wp(z|\tau)$ be the Weierstrass elliptic function with periods $\omega_1 = 1$, $\omega_2 = \tau$ and $\omega_3 = 1+\tau$. Let $\zeta(z) = \zeta(z|\tau) := -\int^z \wp(\xi|\tau)d\xi$ be the Weierstrass zeta function with two quasi-periods $\eta_j(\tau)$, $j = 1, 2$:

$$\eta_j(\tau) := 2\zeta(\tfrac{\omega_j}{2}|\tau) = \zeta(z+\omega_j|\tau) - \zeta(z|\tau), \quad j = 1, 2, \tag{1.1}$$

and $\sigma(z) = \sigma(z|\tau)$ be the Weierstrass sigma function defined by $\sigma(z) := \exp \int^z \zeta(\xi)d\xi$. Notice that $\zeta(z)$ is an odd meromorphic function with simple poles at Λ_τ and $\sigma(z)$ is an odd entire function with simple zeros at Λ_τ. For

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$z \in \mathbb{C}$ we denote $[z] := z \pmod{\Lambda_\tau} \in E_\tau$. For a point $[z]$ in E_τ we often write z instead of $[z]$ to simplify notations when no confusion arises.

In this paper, we consider the complex second order ODE:

$$y''(z) = (I_{\mathbf{n}}(z;\tau) + B)y(z), \quad z \in \mathbb{C}, \tag{1.2}$$

where $B \in \mathbb{C}$ and

$$I_{\mathbf{n}}(z;\tau) := \sum_{k=0}^{3} n_k(n_k+1)\wp(z + \tfrac{\omega_k}{2}|\tau) \tag{1.3}$$

with $\omega_0 = 0$ and $n_k \in \mathbb{Z}_{\geq 0}$ for all k. We also denote $I_{\mathbf{n}}(z;B,\tau) := I_{\mathbf{n}}(z;\tau) + B$. The $I_{\mathbf{n}}(z;\tau)$ is called the *Treibich-Verdier potential*. In [24, 25] Treibich and Verdier proved that $I_{\mathbf{n}}(z;\tau)$ is an algebro-geometric solution of the KdV hierarchy equations or equivalently a finite-gap potential. Later Gesztesy and Weikard [9] generalized their result to prove that any Picard potential is an algebro-geometric solution of the KdV hierarchy equations.

We briefly recall the notion of algebro-geometric solutions. Let $y_1(z;B)$, $y_2(z;B)$ be two solutions of (1.2) and $\Phi(z;B) := y_1(z;B)y_2(z;B)$. Then a direct computation shows that Φ satisfies the following third order ODE (called the second symmetric product equation of (1.2)):

$$\Phi'''(z;B) - 4(I_{\mathbf{n}}(z;\tau) + B)\Phi'(z;B) - 2I'_{\mathbf{n}}(z;\tau)\Phi(z;B) = 0. \tag{1.4}$$

Multiplying Φ and integrating (1.4) ,we obtain that

$$\Phi'(z;B)^2 - 2\Phi(z;B)\Phi''(z;B) + 4(I_{\mathbf{n}}(z;\tau) + B)\Phi(z;B)^2 \tag{1.5}$$

is independent of z. Let $Q_{\mathbf{n}}(B) = Q_{\mathbf{n}}(B;\tau)$ denote the above expression of (1.5). Then $I_{\mathbf{n}}(z;\tau)$ is an algebro-geometric solution of the KdV hierarchy equations if $Q_{\mathbf{n}}(B)$ is a polynomial of B for some solution $\Phi(z;B)$; see [9]. In this case, $Q_{\mathbf{n}}(B)$ is known as the *spectral polynomial* and $\Gamma_{\mathbf{n}} = \Gamma_{\mathbf{n}}(\tau) := \{(B,W)|W^2 = Q_{\mathbf{n}}(B;\tau)\}$ is called the *spectral curve* of the Treibich-Verdier potential.

When $\mathbf{n} = (n,0,0,0)$, the potential $n(n+1)\wp(z|\tau)$ is called the Lamé potential and

$$y''(z) = (n(n+1)\wp(z|\tau) + B)y(z), \quad z \in \mathbb{C} \tag{1.6}$$

is called the Lamé equation. The fact that the Lamé potential is a finite-gap potential was first discovered by Ince [12]. We refer the readers to the classic texts [11, 17, 26] and recent works [1, 2, 8, 15, 16] for the Lamé equation. Therefore, (1.2) is called a *generalized Lamé equation (GLE)* in this paper. See [6, 10, 19, 20, 21, 22, 23] and references therein for recent developments of GLE (1.2).

In this paper, we want to study (1.2) from the aspect of monodromy representation. Clearly, (1.2) can be described as a Fuchsian equation defined on the torus E_τ with four regular singularities at $\frac{\omega_k}{2}$'s. The local exponents, i.e. the roots of its indicial equation at $\frac{\omega_k}{2}$ are $-n_k$, n_k+1. Since $n_k \in \mathbb{Z}$, it is well-known (cf. [10]) that $I_{\mathbf{n}}(z)$ is a Picard potential, i.e. any solution of (1.2) is meromorphic in $\mathbb{C}$. By applying $x = \wp(z)$, (1.2) can be transformed to a second order ODE on $\mathbb{CP}^1$ with four regular singular points at e_k's and ∞, where $e_k := \wp(\frac{\omega_k}{2}), k = 1, 2, 3$. This is the well-known Heun equation with four singular points, which has been extensively studied since its isomonodromic deformation gives arise to the famous Painlevé VI equation; see e.g. [13]. For GLE (1.2), we proved in [4] that the corresponding isomonodromic deformation equation is the elliptic form of Painlevé VI equation. In this paper, we also denote GLE (1.2) by $H(\mathbf{n}, B, \tau)$.

The monodromy representation ρ_τ of $H(\mathbf{n}, B, \tau)$ is a group homomorphism from $\pi_1(E_\tau)$ to $SL(2, \mathbb{C})$ because $I_{\mathbf{n}}(z; \tau)$ is a Picard potential. Since $\pi_1(E_\tau)$ is abelian, the monodromy group is always abelian. Thus the monodromy group is comparely easier to compute for $H(\mathbf{n}, B, \tau)$ on E_τ than the Heun equation on $\mathbb{CP}^1$. In terms of any linearly independent solutions $y_1(z)$ and $y_2(z)$, the monodromy group is generated by two matrices M_1, $M_2 \in SL(2, \mathbb{C})$ satisfying

$$(y_1, y_2)(z + \omega_i) = (y_1(z), y_2(z))M_i,\ i = 1, 2, \text{ and } M_1M_2 = M_2M_1. \tag{1.7}$$

By (1.7), M_1 and M_2 can be normalized to satisfy one of the followings.

a) If ρ_τ is completely reducible, then

$$M_1 = \begin{pmatrix} e^{-2\pi is} & 0 \\ 0 & e^{2\pi is} \end{pmatrix}, M_2 = \begin{pmatrix} e^{2\pi ir} & 0 \\ 0 & e^{-2\pi ir} \end{pmatrix}, \quad (r, s) \in \mathbb{C}^2. \tag{1.8}$$

See Section 2, where we will see that $(r, s) \notin \frac{1}{2}\mathbb{Z}^2$.

b) If ρ_τ is not completely reducible, then

$$M_1 = \varepsilon_1 \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, M_2 = \varepsilon_2 \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix}, \ \varepsilon_j = \pm 1,\ C \in \mathbb{C} \cup \{\infty\}. \tag{1.9}$$

When $C = \infty$, the monodromy matrices are understood as

$$M_1 = \varepsilon_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, M_2 = \varepsilon_2 \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \tag{1.10}$$

The aforementioned spectral polynomial $Q_{\mathbf{n}}(B; \tau)$ also plays an important role for the monodromy representation: *ρ_τ is completely reducible if and only if $Q_{\mathbf{n}}(B; \tau) \neq 0$.* Obviously, not all 2×2 matrices of the form (1.8)-(1.9) are monodromy matrices of (1.2). Thus the following questions naturally arise:

(1) If $Q_{\mathbf{n}}(B; \tau) \neq 0$, how to determine the monodromy data (r, s)?
(2) If $Q_{\mathbf{n}}(B; \tau) = 0$, how to determine the monodromy data C?

For the Lamé equation (1.6), in [2, 14, 15] Chai, Wang and the third author have constructed a *pre-modular form* $Z^n_{r,s}(\tau)$ such that the monodromy matrices M_1, M_2 of (1.6) at $\tau = \tau_0$ with some B are given by (1.8) if and only if $Z^n_{r,s}(\tau_0) = 0$. Therefore, the image of M_1, M_2 for ρ_{τ_0} is $\{(r, s) \in \mathbb{C}^2 \setminus \frac{1}{2}\mathbb{Z}^2 | Z^n_{r,s}(\tau_0) = 0\}$. We note that $Z^n_{r,s}(\tau)$ is holomorphic in τ if $(r, s) \in \mathbb{R}^2 \setminus \frac{1}{2}\mathbb{Z}^2$. Moreover, $Z^n_{r,s}(\tau)$ is a modular form of weight $\frac{n(n+1)}{2}$ w.r.t. the principal congruence subgroup $\Gamma(m)$ if (r, s) is a m-torsion point; see [15]. Thus $Z^n_{r,s}(\tau)$ is called a *pre-modular form*.

In this paper and the subsequent one [3], we want to extend the result in [15] to include the Trebich-Verdier potential. Precisely, we will establish the following theorem in [3]:

Theorem 1.1. *There exists a pre-modular form $Z^{\mathbf{n}}_{r,s}(\tau)$ defined in $\tau \in \mathbb{H}$ for any pair $(r,s) \in \mathbb{C}^2 \setminus \frac{1}{2}\mathbb{Z}^2$ such that the followings hold.*

(a) *If $(r,s) = (\frac{k_1}{m}, \frac{k_2}{m})$ with $m \in 2\mathbb{N}_{\geq 2}$, $k_1, k_2 \in \mathbb{Z}_{\geq 0}$ and $\gcd(k_1,k_2,m) = 1$, then $Z^{\mathbf{n}}_{r,s}(\tau)$ is a modular form of weight $\sum_{k=0}^{3} n_k(n_k+1)/2$ with respect to the principal congruence subgroup $\Gamma(m)$.*

(b) *For $(r,s) \in \mathbb{C}^2 \setminus \frac{1}{2}\mathbb{Z}^2$ and $\tau_0 \in \mathbb{H}$ such that $r + s\tau_0 \notin \Lambda_{\tau_0}$, $Z^{\mathbf{n}}_{r,s}(\tau_0) = 0$ if and only if there is $B \in \mathbb{C}$ such that $H(\mathbf{n}, B, \tau_0)$ has its monodromy matrices M_1 and M_2 given by (1.8).*

Following the ideas in [2, 15], the spectral curve $\Gamma_{\mathbf{n}}(\tau)$ can be embedded into $\mathrm{Sym}^N E_\tau := E_\tau^N / S_N$, the symmetric space of N-th copy of E_τ, where $N := \sum_{k=0}^{3} n_k$. Obviously, $\mathrm{Sym}^N E_\tau$ has a natural addition map to E_τ: $\{a_1, \cdots, a_N\} \mapsto \sum_{i=1}^{N} a_i$. Then the composition gives arise to a finite morphism $\sigma_{\mathbf{n}}(\cdot|\tau) : \overline{\Gamma_{\mathbf{n}}(\tau)} \to E_\tau$, still called the *addition map*. The degree of $\sigma_{\mathbf{n}}$ is defined as $\deg \sigma_{\mathbf{n}}(\cdot|\tau) = \#\sigma_{\mathbf{n}}^{-1}(z|\tau)$, $z \in E_\tau$, counted with multiplicity. Our main theorem in this paper is

Theorem 1.2 (=Theorem 6.1)**.** *Let $\tau \in \mathbb{H}$. Then the addition map $\sigma_{\mathbf{n}}(\cdot|\tau) : \overline{\Gamma_{\mathbf{n}}(\tau)} \to E_\tau$ has degree $\sum_{k=0}^{3} n_k(n_k+1)/2$.*

A corollary of Theorem 1.2 is that $\deg \sigma_{\mathbf{n}}(\cdot|\tau)$ (the same as the weight of the pre-modular form in Theorem 1.1) is independent of τ, which is not very obvious at the moment. For the case of the Lamé equation, Theorem 1.2 was proved in [15] by applying *Theorem of the Cube* for morphisms between varieties in algebraic geometry. But this method seems not work in the general case. Our strategy is to study the general class of ODE:

$$y''(z) = I_{\mathbf{n}}(z; p, A, \tau) y(z), \tag{1.11}$$

where the potential $I_{\mathbf{n}}(z; p, A, \tau)$ is given by

(1.12)
$$I_{\mathbf{n}}(z;p,A,\tau) = \begin{bmatrix} \sum_{k=0}^{3} n_k(n_k+1)\wp(z+\frac{\omega_k}{2}|\tau) + \frac{3}{4}(\wp(z+p|\tau) \\ +\wp(z-p|\tau)) + A(\zeta(z+p|\tau) - \zeta(z-p|\tau)) + B \end{bmatrix},$$

with $A \in \mathbb{C}$, $p \in E_\tau \setminus E_\tau[2]$, $E_\tau[2] := \{\frac{\omega_k}{2} | k = 0,1,2,3\} + \Lambda_\tau$ and

$$B = A^2 - \zeta(2p)A - \frac{3}{4}\wp(2p) - \sum_{k=0}^{3} n_k(n_k+1)\wp(p+\tfrac{\omega_k}{2}|\tau). \tag{1.13}$$

The identity (1.13) is to guarantee that all the singular points of (1.11) are apparent, i.e. all solutions of (1.11) are free of logarithmic singularity at any singular point. See [4, 5, 7, 18] for recent developments of (1.11). Like (1.2), we could associate a hyperelliptic curve $\Gamma_{\mathbf{n},p}(\tau) := \{(A,W) | W^2 = Q_{\mathbf{n},p}(A;\tau)\}$ and an addition map $\sigma_{\mathbf{n},p}$ with (1.11). These will be established in Sections 2 and 3. The reason we introduce (1.11) is that as $p \to \omega_k/2, k = 0,1,2,3$, the limiting equation of (1.11) would be (1.2) with $\mathbf{n} = \mathbf{n}_k^{\pm}$, where $\mathbf{n}_k^{\pm}$ is defined by replacing n_k in $\mathbf{n}$ with $n_k \pm 1$. Due to this relation with (1.2), we expect to have the following connection.

Theorem 1.3 (=Theorem 6.6)**.** *For $k \in \{0,1,2,3\}$, there holds*

$$\deg \sigma_{\mathbf{n},p}(\cdot|\tau) = \deg \sigma_{\mathbf{n}_k^+}(\cdot|\tau) + \deg \sigma_{\mathbf{n}_k^-}(\cdot|\tau).$$

The paper is organized as follows. In Section 2, we will give a brief review of the monodromy representation of (1.11). The similar argument also holds for (1.2). We will prove the existence of the embedding of $\Gamma_{\mathbf{n},p}$ and the addition map from $\Gamma_{\mathbf{n},p}$ onto E_τ in Section 3. In Sections 4 and 5, we will study the limiting problem of (1.11) under two case: (i) fix p and $A \to \infty$; (ii) $p \to \frac{\omega_k}{2}$ and $A(p) \to \infty$. This limit problem plays a crucial role in our study of the degree in Section 6, where Theorem 1.3 is proved and then Theorem 1.2 will be obtained by applying Theorem 1.3.

2. Monodromy representation

Let $\mathbf{n} = (n_0, n_1, n_2, n_3)$, $n_k \in \mathbb{C}$. For any fixed $\tau \in \mathbb{H}$ and $\pm p \notin E_\tau[2] := \{\frac{\omega_k}{2} | k = 0,1,2,3\} + \Lambda_\tau$, we consider the following generalized Lamé equation GLE$(\mathbf{n}, p, A, \tau)$ on the torus E_τ:

$$y''(z) = I_{\mathbf{n}}(z; p, A, \tau) y(z) \quad \text{on } E_\tau, \tag{2.1}$$

where the potential $I_{\mathbf{n}}(z; p, A, \tau)$ is given by

$$I_{\mathbf{n}}(z; p, A, \tau) = \begin{bmatrix} \sum_{k=0}^{3} n_k(n_k+1)\wp(z + \frac{\omega_k}{2}|\tau) + \frac{3}{4}(\wp(z+p|\tau) \\ +\wp(z-p|\tau)) + A(\zeta(z+p|\tau) - \zeta(z-p|\tau)) + B \end{bmatrix}, \tag{2.2}$$

with $A \in \mathbb{C}$ and

$$B = A^2 - \zeta(2p)A - \frac{3}{4}\wp(2p) - \sum_{k=0}^{3} n_k(n_k+1)\wp(p + \tfrac{\omega_k}{2}|\tau). \tag{2.3}$$

GLE$(\mathbf{n}, p, A, \tau)$ is of Fuchsian type with singularities at $S := E_\tau[2] \cup \{\pm[p]\}$.

Let us briefly recall the associated monodromy representation. Let $Y(z;\tau) = (y_1(z;\tau), y_2(z;\tau))$ be a fundamental system of solutions of GLE$(\mathbf{n}, p, A, \tau)$ near a fixed base point $q_0 \notin S$. In general, $Y(z;\tau)$ is multi-valued with respect to z and might have branch points at S. For any loop $\ell \in \pi_1(E_\tau \backslash S, q_0)$, there exists a matrix $\rho_\tau(\ell) \in SL(2,\mathbb{Z})$ such that $\ell^* Y(z;\tau) = Y(z;\tau)\rho_\tau(\ell)$. Here $\ell^* Y(z;\tau)$ denotes the analytic continuation of $Y(z;\tau)$ along the loop ℓ. This induces a group homomorphism

$$\rho_\tau : \pi_1(E_\tau \backslash S, q_0) \to SL(2,\mathbb{C}). \tag{2.4}$$

which is called the monodromy representation of the GLE$(\mathbf{n}, p, A, \tau)$.

The local exponent of GLE$(\mathbf{n}, p, A, \tau)$ at $\frac{\omega_k}{2}$ are $-n_k, n_k+1$. In this paper, we are interested in the case $n_k \in \mathbb{Z}_{\geq 0}$ (hereafter we always assume $n_k \in \mathbb{Z}_{\geq 0}$ for all k), because in this case the local monodromy matrix at each $\frac{\omega_k}{2}$ is I_2 and then the monodromy representation can be reduced to a homomorphism $\rho_\tau : \pi_1(E_\tau\backslash\{\pm[p]\}, q_0) \to SL(2,\mathbb{Z})$. Let $\gamma_\pm \in \pi_1(E_\tau\backslash S, q_0)$ be a simple loop encircling $\pm p$ counterclockwise respectively, and ℓ_j, $j=1,2$, be two fundamental cycles of E_τ connecting q_0 with $q_0+\omega_j$ such that ℓ_j does not intersect with $\ell_p + \Lambda_\tau$ (here ℓ_p is the straight segment connecting $\pm p$) and satisfies

$$\gamma_+\gamma_- = \ell_1\ell_2\ell_1^{-1}\ell_2^{-1} \text{ in } \pi_1\left(E_\tau\backslash\{\pm[p]\}, q_0\right). \tag{2.5}$$

On the other hand, the local exponents at $\pm p$ are $\frac{-1}{2}, \frac{3}{2}$. Since (2.3) implies that $\pm p$ are apparent singularities (i.e. non-logarithmic, see [4]), we have

$$\rho_\tau(\gamma_\pm) = -I_2. \tag{2.6}$$

Denote $\rho_\tau(\ell_j)$ by M_j. Then the monodromy group of GLE$(\mathbf{n}, p, A, \tau)$ is generated by $\{-I_2, M_1, M_2\}$. Together with (2.5)-(2.6), we immediately obtain $M_1M_2 = M_2M_1$, which implies that there is always a solution denoted by $y_1(z) = y_1(z;A)$ being a common eigenfunction, i.e. $\ell_j^* y_1(z;A) = \varepsilon_j y_1(z;A)$, $j=1,2$. Therefore the monodromy representation is always *reducible*.

From the local exponents at $\frac{\omega_k}{2}$ and $\pm p$, it is easy to see that $\frac{\omega_k}{2}$ is not a branch point of $y_1(z)$ but $\pm p$ is a branch point with ramification index 2, i.e. $y_1(\pm p + e^{2\pi i}z) = -y_1(\pm p + z)$ if $|z|>0$ is small. Then $y_1(z)$ can be viewed as a single-valued meromorphic function in $\mathbb{C}\backslash(\ell_p+\Lambda_\tau)$. Therefore, the analytic continuation of $y_1(z)$ along the fundamental cycles ℓ_j is the translation of y_1 by ω_j, $j=1,2$, namely

$$y_1(z+\omega_j;A) = \ell_j^* y_1(z;A) = \varepsilon_j y_1(z;A), \quad j=1,2. \tag{2.7}$$

Since $\mathbb{C}\backslash(\ell_p+\Lambda_\tau)$ is symmetric about 0, $y_1(-z;A)$ is well-defined in $\mathbb{C}\setminus(\ell_p+\Lambda_\tau)$ and also a solution of the same GLE$(\mathbf{n}, p, A, \tau)$. Define

$$y_2(z) = y_2(z;A) := y_1(-z;A) \quad \text{in } \mathbb{C}\backslash(\ell_p+\Lambda_\tau). \tag{2.8}$$

Then by (2.7) we see that $y_2(z;A)$ is also a common eigenfunction, i.e.

$$y_2(z+\omega_j;A) = \ell_j^* y_2(z;A) = \varepsilon_j^{-1} y_2(z;A), \ j=1,2. \tag{2.9}$$

If $\varepsilon_j \neq \pm 1$ for some j, then $y_1(z)$ and $y_2(z)$ are linearly independent. In general, $y_1(z)$ and $y_2(z)$ might be linearly dependent.

Definition 2.1. *GLE$(\mathbf{n}, p, A, \tau)$ is called completely reducible if its monodromy group acting on the 2-dimensional solution space has two linearly independent common eigenfunctions. Otherwise, it is called not completely reducible.*

We will see later that *GLE$(\mathbf{n}, p, A, \tau)$ is completely reducible if and only if the above $y_1(z)$ and $y_2(z) := y_1(-z)$ are linearly independent.*

The branch points $\pm p$ of $y_1(z)$ might cause trouble in analysis. To avoid it, we introduce

$$\Psi_p(z) := \frac{\sigma(z)}{\sqrt{\sigma(z-p)\sigma(z+p)}}. \tag{2.10}$$

By using the transformation law of $\sigma(z)$,

$$\sigma(z+\omega_j) = -e^{\eta_j(z+\frac{1}{2}\omega_j)}\sigma(z), \quad j=1,2, \tag{2.11}$$

we see that $\Psi_p(z)^2$ is an *elliptic function*. We have the following lemma.

Lemma 2.2. *Recall that $\ell_j, j=1,2$ are the two fixed fundamental cycles of E_τ which do not intersect with $\ell_p+\Lambda_\tau$. Then the analytic continuation of $\Psi_p(z)$ along ℓ_j satisfies*

$$\ell_j^*\Psi_p(z) = \Psi_p(z),\ j=1,2. \tag{2.12}$$

Proof. We only need to prove (2.12) in a small neighborhood U of the base point q_0. Since $\ell_j \in \pi_1(E_\tau\backslash\{\pm[p]\}, q_0)$ does not intersect with $\ell_p+\Lambda_\tau$, $\Psi_p(z)$ can be viewed as a single-valued meromorphic function in $\mathbb{C}\backslash(\ell_p+\Lambda_\tau)$, and in this region we have

$$\ell_j^*\Psi_p(z) = \Psi_p(z+\omega_j) = \pm\Psi_p(z),\ z\in U,$$

because $\Psi_p(z)^2 = \frac{\sigma(z)^2}{\sigma(z+p)\sigma(z-p)}$ is an elliptic function. Suppose

$$\Psi_p(z+\omega_j) = -\Psi_p(z),\ z\in U \tag{2.13}$$

holds true. By fixing q_0 and ℓ_j, (2.13) always holds true as $p\to 0$ along ℓ_p. Note from (2.10) that for any $z\in\ell_j$, $\lim_{p\to 0}\Psi_p(z)$ is identical to either 1 or -1, but (2.13) implies that $\lim_{p\to 0}\Psi_p(z)$ for z along ℓ_j contains both 1 and -1, a contradiction. □

Classically, it has been known (cf. [26]) that the second symmetric product equation for any second order ODE play an important role. Let $\tilde{y}_1(z)$, $\tilde{y}_2(z)$ are any two solutions of GLE$(\mathbf{n},p,A,\tau)$ and set $\Phi(z)=\tilde{y}_1(z)\tilde{y}_2(z)$. Then $\Phi(z)$ satisfies the following third order ODE:

$$\Phi'''(z) - 4I_{\mathbf{n}}(z;p,A,\tau)\Phi'(z) - 2I_{\mathbf{n}}'(z;p,A,\tau)\Phi(z) = 0. \tag{2.14}$$

Recall (2.7)-(2.9) that $y_1(z)$ is an common eigenfunction and $y_2(z) = y_1(-z)$. Then $\Phi(z) := y_1(z)y_2(z)$ is a solution of (2.14), and also an *even elliptic function* due to (2.7)-(2.9). The following result was proved by Takemura [18], but we give a proof here for the convenience of readers because it plays a fundamental role in our theory.

Proposition 2.3. [18] *The dimension of the space of even elliptic solutions to (2.14) is* 1.

Proof. It is easy to see that the dimension of the space of even solutions to (2.14) is 2. Suppose the proposition is not true. Since we already know there is at least one even elliptic solution, the dimension of even elliptic solutions to (2.14) is 2, which implies any even solution of (2.14) must be elliptic.

Since the local exponents of GLE$(\mathbf{n},p,A,\tau)$ at 0 are $-n_0, n_0+1$, and $I_{\mathbf{n}}(\cdot;p,A,\tau)$ is even, there are local solutions of the following form at 0:

$$\hat{y}_1(z) = z^{-n_0}\left(1+\sum_{j=1}^{\infty}a_jz^{2j}\right),\quad \hat{y}_2(z) = z^{n_0+1}\left(1+\sum_{j=1}^{\infty}b_jz^{2j}\right).$$

Then $\hat{y}_j(z)^2$, $j=1,2$, are even solutions of (2.14) and hence even elliptic functions by our assumption. Define

$$\tilde{y}_j(z) := \frac{\hat{y}_j(z)}{\Psi_p(z)}, \quad j=1,2, \tag{2.15}$$

where $\Psi_p(z)$ is given by (2.10). Then $\tilde{y}_j(z)$ is a meromorphic function with poles at most at $E_\tau[2]$. Since $\hat{y}_j(z)^2$ and $\Psi_p(z)^2$ are even elliptic, so do $\tilde{y}_j(z)^2$ for $j=1,2$. Assume

$$\begin{pmatrix} \tilde{y}_1(z+\omega_i) \\ \tilde{y}_2(z+\omega_i) \end{pmatrix} = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \begin{pmatrix} \tilde{y}_1(z) \\ \tilde{y}_2(z) \end{pmatrix}, \quad i=1,2$$

for some $\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in SL(2,\mathbb{C})$. Then

$$\tilde{y}_1(z)^2 = \tilde{y}_1(z+\omega_i)^2 = a_i^2\tilde{y}_1(z)^2 + 2a_ib_i\tilde{y}_1(z)\tilde{y}_2(z) + b_i^2\tilde{y}_2(z)^2.$$

So we have $a_i^2=1$, $b_i=0$. Together with $a_id_i - b_ic_i = 1$, we see that $d_i = a_i =: \varepsilon_i \in \{\pm 1\}$. Similarly, we could use $\tilde{y}_2$ to obtain $c_i = 0$. Hence

$$\tilde{y}_k(z+\omega_i) = \varepsilon_i \tilde{y}_k(z) \text{ for } k=1,2 \text{ and } i=1,2.$$

This implies that $\tilde{y}_1(z)\tilde{y}_2(z)$ is odd and elliptic. By the local exponent of $\tilde{y}_1(z)\tilde{y}_2(z)$ at $\frac{\omega_k}{2}$ being one of $\{-2n_k, 1, 2n_k+2\}$ for $k=1,2,3$, we see that $\frac{\omega_k}{2}$ is a simple zero of $\tilde{y}_1(z)\tilde{y}_2(z)$ for $k=1,2,3$. Thus the elliptic function $\tilde{y}_1(z)\tilde{y}_2(z)$ has only a simple pole at $z=0$, a contradiction. This proves the proposition. □

Now we can apply Proposition 2.3 to answer the question proposed earlier.

Proposition 2.4. *Let $y_1(z)$ be a common eigenfunction of the monodromy representation of GLE$(\mathbf{n}, p, A, \tau)$. Then GLE$(\mathbf{n}, p, A, \tau)$ is completely reducible if and only if $y_1(z)$ and $y_2(z) := y_1(-z)$ are linearly independent.*

Proof. Clearly $y_2(z)$ is also a common eigenfunction. So the sufficient part is trivial. For the necessary part, since the monodromy is completely reducible, there exists another common eigenfunction $y_3(z)$ which is linearly independent with $y_1(z)$. Then $y_3(z)y_3(-z)$ is also an even elliptic solution of (2.14). So Proposition 2.3 implies $y_1(z)y_1(-z) = y_3(z)y_3(-z)$ up to a constant, which implies $y_3(z) = cy_1(-z)$ for some constant $c \neq 0$. This proves that $y_1(z)$ and $y_2(z) = y_1(-z)$ are linearly independent. □

Recall (2.7) and (2.9) that $\varepsilon_j, \varepsilon_j^{-1}$ are the eigenvalues of $M_j = \rho_\tau(\ell_j)$. Then Propositions 2.3 and 2.4 have the following consequence.

Corollary 2.5. *If GLE$(\mathbf{n}, p, A, \tau)$ is completely reducible then $(\varepsilon_1, \varepsilon_2) \notin \{\pm(1,1), \pm(1,-1)\}$.*

Suppose GLE$(\mathbf{n}, p, A, \tau)$ is completely reducible and $y_j(z), j=1,2$, are its linearly independent common eigenfunctions in Proposition 2.4. then the monodromy matrices M_j can be written as

$$M_1 = \begin{pmatrix} e^{-2\pi is} & 0 \\ 0 & e^{2\pi is} \end{pmatrix} \text{ and } M_2 = \begin{pmatrix} e^{2\pi ir} & 0 \\ 0 & e^{-2\pi ir} \end{pmatrix}, \tag{2.16}$$

where $(r,s) \in \mathbb{C}^2$ satisfies $(\varepsilon_1, \varepsilon_2) = (e^{-2\pi is}, e^{2\pi ir})$. We call that (r,s) are the monodromy data of GLE$(\mathbf{n}, p, A, \tau)$ if it is completely reducible, and Corollary 2.5 implies

$$(r,s) \notin \tfrac{1}{2}\mathbb{Z}^2. \tag{2.17}$$

Later we will show how to compute (r,s) from the zero set of the common eigenfunction $y_1(z)$.

The above argument shows that (2.14) has a unique even elliptic solution $\Phi(z)$ up to a constant, which is given by $\Phi(z) = y_1(z)y_2(z)$, where $y_1(z)$ is a common eigenfunction and $y_2(z) = y_1(-z)$. Furthermore, GLE$(\mathbf{n}, p, A, \tau)$ is completely reducible if and only if $y_1(z)$ and $y_2(z)$ are linearly independent. This is where the even elliptic solution Φ plays the role which is of fundamental importance for GLE$(\mathbf{n}, p, A, \tau)$. To determine the linear independence, we should consider the Wronskian of $y_1(z)$ and $y_2(z)$: $W := y_1(z)y_2'(z) - y_1'(z)y_2(z)$, which is a constant independent of z.

Lemma 2.6. *Let $\Phi(z) = y_1(z)y_2(z)$ be the even elliptic solution of* (2.14) *and W is the Wronkian of y_1 and y_2. Then*

$$W^2 = \Phi'(z)^2 - 2\Phi''(z)\Phi(z) + 4I_{\mathbf{n}}(z;p,A,\tau)\Phi(z)^2, \tag{2.18}$$

$$y_1(z) = \sqrt{\Phi(z)}\exp\int^z \frac{W}{2\Phi(\xi)}d\xi \tag{2.19}$$

Proof. Since $\Phi(z) = y_1(z)y_2(z)$ and $W = y_1'y_2 - y_1y_2'$, we have

$$\frac{y_1'}{y_1} = \frac{\Phi' + W}{2\Phi}, \quad \frac{y_2'}{y_2} = \frac{\Phi' - W}{2\Phi}$$

which implies (2.19) and

$$\frac{\Phi''}{2\Phi} - \frac{\Phi' + W}{2\Phi^2}\Phi' = \left(\frac{y_1'}{y_1}\right)' = \frac{y_1''}{y_1} - \left(\frac{y_1'}{y_1}\right)^2 = I_{\mathbf{n}} - \left(\frac{\Phi' + W}{2\Phi}\right)^2,$$

$$\frac{\Phi''}{2\Phi} - \frac{\Phi' - W}{2\Phi^2}\Phi' = I_{\mathbf{n}} - \left(\frac{\Phi' - W}{2\Phi}\right)^2.$$

Adding these two formulas together, we easily obtain (2.18). □

To normalize the even elliptic solution $\Phi(z)$, we apply the following result due to Takemura [18].

Theorem 2.A. ([18]) *Fix $\tau \in \mathbb{H}$ and $p \notin E_\tau[2]$. Then equation (2.14) has a unique even elliptic solution $\Phi_e(z;A)$ of the form*

$$\Phi_e(z;A) = C_0(A) + \sum_{k=0}^{3}\sum_{j=0}^{n_k-1} b_j^{(k)}(A)\wp(z + \tfrac{\omega_k}{2})^{n_k-j} + \frac{d(A)}{\wp(z) - \wp(p)}, \tag{2.20}$$

such that the coefficients $C_0(A)$, $b_j^{(k)}(A)$ and $d(A)$ are all polynomials in A, and they do not have common zeros, and the leading coefficient of $C_0(A)$ is $\frac{1}{2}$. Moreover,

$$\deg_A C_0(A) > \max\left(\deg_A b_j^{(k)}(A), \deg_A d(A)\right).$$

The proof of Theorem 2.A is not difficult but a little tedious in computation. The expression of (2.20) is due to the fact that $\frac{\omega_k}{2}$ and $\pm p$ might be poles of $\Phi(z)$ with order $-2n_k$ and -1 respectively (depending on A). We substitute (2.20) into GLE$(\mathbf{n}, p, A, \tau)$ and compare A of the two sides at each singularity. Then we could obtain that all the coefficients are polynomials in A after normalization. For details, we refer the readers to [18, 19].

Recalling Proposition 2.3, we apply Lemma 2.6 to the normalized $\Phi_e(z; A)$ of Theorem 2.A. Then we have the main result of this section.

Theorem 2.7. *There is a monic polynomial $Q_{\mathbf{n},p}(A) = Q_{\mathbf{n},p}(A; \tau)$ in A such that the Wronskian of $y_1(z; A)$, $y_2(z; A) = y_1(-z; A)$, where $\Phi_e(z; A) = y_1(z; A)y_2(z; A)$, satisfies*

$$W^2 = Q_{\mathbf{n},p}(A). \tag{2.21}$$

Moreover, $Q_{\mathbf{n},p}(A) \neq 0$ if and only if GLE$(\mathbf{n}, p, A, \tau)$ is completely reducible.

Proof. By (2.2) and (2.3), we could write $I(z; \mathbf{n}, p, A, \tau)$ as

$$I_{\mathbf{n}}(z; p, A, \tau) = A^2 + I_1(z)A + I_2(z).$$

By Theorem 2.A, we have

$$\Phi_e(z) = \frac{1}{2}A^g + \sum_{j=0}^{g-1} \varphi_j(z)A^j, \quad \text{where } \; g := \deg C_0(A),$$

Inserting these into (2.18), we easily obtain

$$W^2 = A^{2g+2} + \sum_{j=0}^{2g+1} q_j(z)A^j, \tag{2.22}$$

where $q_j(z) \equiv c_j$ are independent of z for all j because W is independent of z. Thus, $W^2 = Q_{\mathbf{n},p}(A)$ is a monic polynomial in A of degree $2g + 2$. □

In view of Theorem 2.7, we define the hyperelliptic curve $\Gamma_{\mathbf{n},p} = \Gamma_{\mathbf{n},p}(\tau)$ by

$$\Gamma_{\mathbf{n},p}(\tau) := \{(A, W) | W^2 = Q_{\mathbf{n},p}(A; \tau)\}. \tag{2.23}$$

We remark that the polynomials $Q_{\mathbf{n},p}(A; \tau)$ might have multiple zeros. But we could prove that for each τ, $Q_{\mathbf{n},p}(A; \tau)$ has distinct roots except for finitely many $p \in E_\tau \setminus E_\tau[2]$. See e.g. our subsequent work [3].

Since $\deg_A Q_{\mathbf{n},p}(A; \tau)$ is even, the curve $\Gamma_{\mathbf{n},p}(\tau)$ has two points at infinity which are denoted by $\infty_\pm$. Indeed, for a curve in $\mathbb{C}^2$ defined by $y^2 = \Pi_{i=1}^{2g+2}(x - x_i)$, to study its points at infinity, we let $x = 1/x'$ and $y = y'/x'^{g+1}$.

Then the equation becomes $y'^2 = \prod_{i=1}^{2g+2}(1-x'x_i)$. Thus $(x', y') = (0, \pm 1)$ represents the two points at infinity and they are unramified. Hence

$$\overline{\Gamma_{\mathbf{n},p}(\tau)} = \Gamma_{\mathbf{n},p}(\tau) \cup \{\infty_\pm\} \text{ is smooth at } \infty_\pm. \tag{2.24}$$

Remark 2.8. Given $(A, W) \in \Gamma_{n,p}$, the unique $\Phi_e(z; A)$ is the product of $y_1(z; A)$ and $y_2(z; A) = y_1(-z; A)$. If $W \neq 0$, then $y_1(z; A)$ and $y_2(z; A)$ are linearly independent. We rename $y_1(z; A), y_2(z; A)$ by requiring that the Wronskian of y_1 and y_2 is equal to W (i.e. the Wronskian of y_2 and y_1 is $-W$). Then $y_1(z; A)$ is unique up to a sign, and will be denoted by $y_1(z; A, W)$. In particular, the zero set of $y_1(z; A, W)$ is unique.

3. The embedding of $\Gamma_{\mathbf{n},p}$ in $Sym^N E_\tau$

The main purpose of this section is to define the addition map from $\Gamma_{\mathbf{n},p}$ to E_τ. First we discuss the embedding of $\Gamma_{\mathbf{n},p}$ into $\mathrm{Sym}^N E_\tau$, the symmetric N-th copy product of E_τ, where $N = \sum_{k=0}^3 n_k + 1$ in the sequel.

For any $\mathbf{a} = (a_1, \cdots, a_N) \in \mathbb{C}^N$, we define $y_{\mathbf{a},c}(z) = y_{\mathbf{a},c}(z; p)$ by

$$y_{\mathbf{a},c}(z; p) := \frac{e^{cz} \prod_{i=1}^N \sigma(z - a_i)}{\sqrt{\sigma(z-p)\sigma(z+p)} \prod_{k=0}^3 \sigma(z - \frac{\omega_k}{2})^{n_k}} \text{ for } c \in \mathbb{C}. \tag{3.1}$$

Proposition 3.1. *Fix $p \in E_\tau \setminus E_\tau[2]$ and $A \in \mathbb{C}$. Let $y_1(z; A, W)$ be the common eigenfunction determined in Remark 2.8. Then there always exists $\mathbf{a} \in \mathbb{C}^N$ and $c \in \mathbb{C}$ such that $y_1(z; A, W) = y_{\mathbf{a},c}(z; p)$ up to a constant.*

Proof. Since $y_1(z; A, W)$ is an common eigenfunction, we have

$$\ell_i^* y_1(z; A, W) = \varepsilon_i y_1(z; A, W) \text{ for some } \varepsilon_i \neq 0, i = 1, 2. \tag{3.2}$$

As we discussed in Section 2, $y_1(z; A, W)$ has branch points at $\pm p$. Set $\tilde{y}(z) := y_1(z; A, W)/\Psi_p(z)$, where $\Psi_p(z)$ is defined in (2.10). Then $\tilde{y}(z)$ is a meromorphic function, and it follows from Lemma 2.2 and (3.2) that

$$\tilde{y}(z + \omega_i) = \varepsilon_i \tilde{y}(z), \quad i = 1, 2. \tag{3.3}$$

Conventionally, a meromorphic function satisfying (3.3) is called *an elliptic function of second kind* with periods 1 and τ. Then a classic theorem says that up to a constant, $\tilde{y}(z)$ can be written as

$$\tilde{y}(z) = \frac{e^{cz} \prod_{i=1}^N \sigma(z - a_i)}{\sigma(z) \prod_{k=0}^3 \sigma(z - \frac{\omega_k}{2})^{n_k}}, \tag{3.4}$$

because $\tilde{y}(z)$ have poles at most at 0 with order $-n_0 - 1$ and at $\omega_k/2$ with order $-n_k$, $k = 1, 2, 3$. The proof is complete. □

Remark 3.2. A consequence of Remark 2.8 and Proposition 3.1 is that, up to a constant, the unique even elliptic solution $\Phi_e(z; A)$ in Theorem 2.A can be written as

$$\Phi_e(z; A) = \frac{\prod_{i=1}^N \sigma(z - a_i)\sigma(z + a_i)}{\sigma(z-p)\sigma(z+p) \prod_{k=0}^3 \sigma(z - \frac{\omega_k}{2})^{n_k} \sigma(z + \frac{\omega_k}{2})^{n_k}}. \tag{3.5}$$

On the other hand, if $y_{\mathbf{a},c}(z)$ is a solution of GLE$(\mathbf{n},p,A,\tau)$, it is easy to see that (i) $y_{-\mathbf{a},\tilde{c}}(z) = d_1 y_{\mathbf{a},c}(-z)$ (for some constants $d_1 \neq 0$ and $\tilde{c}$) is also a solution of GLE$(\mathbf{n},p,A,\tau)$, (ii) $y_{\mathbf{a},c}(z)$ is a common eigenfunction of M_j's. Together with Proposition 3.1, we have that (iii) $y(z)$ is a common eigenfunction if and only if $y(z)$ is of the form $y_{\mathbf{a},c}(z)$ up to a constant, and (iv) $y_2(z) := y_1(-z; A, W) = y_{-\mathbf{a},\tilde{c}}(z)$ up to a constant. In particular, Proposition 2.4 implies that GLE$(\mathbf{n},p,A,\tau)$ is completely reducible if and only if $y_{\mathbf{a},c}(z)$ and $y_{-\mathbf{a},\tilde{c}}(z) = d_1 y_{\mathbf{a},c}(-z)$ are linearly independent.

Proposition 3.1 says that $y_1(z; A, W) = d y_{\mathbf{a},c}(z)$ for some $\mathbf{a} = (a_1, \cdots, a_N) \in \mathbb{C}^N$, $c \in \mathbb{C}$ and $d \neq 0$. Obviously $\{a_1, \cdots, a_N\} \setminus (E_\tau[2] \cup \{\pm[p]\})$ is the zero set of $y_1(z; A, W)$ and we expect that c can be uniquely determined by $\mathbf{a}$. On the other hand, we could apply the transformation law (2.11) of σ to obtain the monodromy data (r, s) in (2.16) if GLE$(\mathbf{n},p,A,\tau)$ is completely reducible. These are proved in the following result. Denote $\eta_3 := \eta_1 + \eta_2$.

Theorem 3.3. *Let $p \notin E_\tau[2]$ and $A \in \mathbb{C}$. Suppose GLE$(\mathbf{n},p,A,\tau)$ is completely reducible with solution $y_{\mathbf{a},c}(z)$ given in Proposition 3.1. Then the monodromy data (r,s) in (2.16) and $c = c(\mathbf{a})$ can be determined by $\mathbf{a}$ as follows:*

$$\sum_{i=1}^{N} a_i = r + s\tau + \sum_{k=1}^{3} \frac{n_k \omega_k}{2}, \quad c(\mathbf{a}) = r\eta_1 + s\eta_2, \tag{3.6}$$

$$c(\mathbf{a}) = \frac{1}{2} \sum_{i=1}^{N} \left(\zeta(a_i + \tfrac{\omega_k}{2}) + \zeta(a_i - \tfrac{\omega_k}{2})\right) - \sum_{i=1}^{3} \frac{n_i \eta_i}{2} \text{ if } n_k \neq 0. \tag{3.7}$$

Furthermore, $a_i \notin E_\tau[2]$ for all i and $\mathbf{a} = \{a_1, \cdots, a_N\}$ must satisfy one of the following three alternatives:

(a-i) *$a_i \neq \pm p$ and $a_i \neq \pm a_j$ for any $i \neq j$, i.e. $\pm p$ are simple poles of $\Phi_e(z; A)$. In this case,*

$$c(\mathbf{a}) = \frac{1}{2} \sum_{i=1}^{N} (\zeta(a_i + p) + \zeta(a_i - p)) - \sum_{k=1}^{3} \frac{n_k \eta_k}{2}. \tag{3.8}$$

(a-ii) *$a_{N-1} = a_N = p$, $a_i \neq \pm p$ and $a_i \neq \pm a_j$ for any $i \neq j \leq N-2$, i.e. $\pm p$ are simple zeros of $\Phi_e(z; A)$.*

(a-iii) *$a_{N-1} = a_N = -p$, $a_i \neq \pm p$ and $a_i \neq \pm a_j$ for any $i \neq j \leq N-2$, i.e. $\pm p$ are simple zeros of $\Phi_e(z; A)$.*

Proof. Since GLE$(\mathbf{n},p,A,\tau)$ is completely reducible, Remark 3.2 says that $y_{\mathbf{a},c}(z)$ and $y_{\mathbf{a},c}(-z)$ are linearly independent, which implies

$y_{\mathbf{a},c}(z)$ and $y_{\mathbf{a},c}(-z)$ has a pole of order $-n_k$ at $z = \frac{\omega_k}{2}$ if $n_k > 0$ and non-zero at $z = \frac{\omega_k}{2}$ if $n_k = 0$.

Thus $a_i \notin E_\tau[2]$ for all i.

Recalling $\tilde{y}(z)$ in (3.4) and $\Psi_p(z)$ in (2.10), we have

$$y_{\mathbf{a},c}(z)=\tilde{y}(z)\Psi_p(z). \tag{3.9}$$

By applying the transformation law (2.11) of σ to $\tilde{y}(z)$, we can determine ε_i in (3.2)-(3.3) by

$$\varepsilon_i=\exp\left(c\omega_i-\eta_i\left(\sum_{i=1}^{N}a_i-\sum_{k=1}^{3}\frac{n_k\omega_k}{2}\right)\right),\quad i=1,2.$$

Define $(r,s)\in\mathbb{C}^2$ by

$$-2\pi is:=c-\eta_1\left(\sum_{i=1}^{N}a_i-\sum_{k=1}^{3}\frac{n_k\omega_k}{2}\right) \tag{3.10}$$

$$2\pi ir:=c\tau-\eta_2\left(\sum_{i=1}^{N}a_i-\sum_{k=1}^{3}\frac{n_k\omega_k}{2}\right). \tag{3.11}$$

Then $(\varepsilon_1,\varepsilon_2)=(e^{-2\pi is},e^{2\pi ir})$, and it follows from (3.2) that this (r,s) is precisely the monodromy data in (2.16). By the Legendre relation $\tau\eta_1-\eta_2=2\pi i$, we see that (3.10)-(3.11) are equivalent to (3.6).

If $n_k>0$, then $\frac{\omega_k}{2}$ is a singularity. Since $a_j\notin E_\tau[2]$ for all j, (3.7) follows directly by inserting the expression (3.1) of $y_{\mathbf{a},\mathbf{c}}(z)$ into GLE$(\mathbf{n},p,A,\tau)$ and computing the leading terms at the singularity $\frac{\omega_k}{2}$. We omit the details here.

To prove (i)-(iii), we recall Remark 2.8 that $\Phi_e(z;A)=y_{\mathbf{a},c}(z)y_{\mathbf{a},c}(-z)$. Since $\Phi_e(z;A)$ is even, it has the same local exponent α at p and $-p$, and $\alpha\in\{-1,1,3\}$.

If $\alpha=3$, then the leading term of both $y_{\mathbf{a},c}(z)$ and $y_{\mathbf{a},c}(-z)$ near p is $(z-p)^{\frac{3}{2}}$, which implies that $y_{\mathbf{a},c}(z)$ and $y_{\mathbf{a},c}(-z)$ are linearly dependent, a contradiction.

If $\alpha=-1$, i.e. $\pm p$ are both simple poles of $\Phi_e(z;A)$, then by (3.5) we have $a_j\neq\pm p$ in E_τ for all j. Consequently, $a_1,\cdots,a_N$ are all zeros of $y_{\mathbf{a},c}(z)$ and so are all simple, i.e. $a_i\neq a_j$ for $i\neq j$. Since $y_{\mathbf{a},c}(z)$ and $y_{\mathbf{a},c}(-z)$ can not have common zeros, we also have $a_i\neq -a_j$ for all i,j. This proves (a-i), where (3.8) can be proved similarly by inserting (3.1) into GLE$(\mathbf{n},p,A,\tau)$ and computing the leading terms at singularities $\pm p$.

If $\alpha=1$, i.e. $\pm p$ are both simple zeros of $\Phi_e(z;A)$, one possibility is that the local exponent of $y_{\mathbf{a},c}(z)$ at p is $\frac{3}{2}$ and at $-p$ is $\frac{-1}{2}$. Then (3.1) implies that $-p\notin\{a_1,\cdots,a_N\}$ and there are exactly two elements in $\{a_1,\cdots,a_N\}$ equal to p in E_τ. By reordering $a_1,\cdots,a_N$, (a-ii) is proved. Similarly, for the other possibility that the local exponent of $y_{\mathbf{a},c}(z)$ at p is $\frac{-1}{2}$ and at $-p$ is $\frac{3}{2}$, (a-iii) holds. Remark that if (a-ii) or (a-iii) happens, then $N\geq 2$ and so (3.7) holds for at least one $k\in\{0,1,2,3\}$.

The proof is complete. $\square$

By the Legendre relation, the matrix $\begin{pmatrix} 1 & \tau \\ \eta_1(\tau) & \eta_2(\tau) \end{pmatrix}$ is always invertible. Hence (3.6)-(3.8) imply the monodromy data (r, s) can be uniquely determined by $a_1, \cdots, a_N$, which contain all the zeros of $y_1(z; A, W)$. The next result is to show that A of GLE$(\mathbf{n}, p, A, \tau)$ can be expressed in terms of $\{a_1, \cdots, a_N\}$ if Case (a-i) happens, i.e. $\pm p \notin \{a_1, \cdots, a_N\}$.

Proposition 3.4. *Let $p \notin E_\tau[2]$ and $A \in \mathbb{C}$. Suppose GLE$(\mathbf{n}, p, A, \tau)$ is completely reducible and Case (a-i) in Theorem 3.3 occurs, then*

$$A = \frac{1}{2}\sum_{i=1}^{N}(\zeta(a_i + p) - \zeta(a_i - p)) - \frac{1}{2}\zeta(2p) - \frac{1}{2}\sum_{k=0}^{3} n_k \left(\zeta(p + \tfrac{\omega_k}{2}) + \zeta(p - \tfrac{\omega_k}{2})\right). \tag{3.12}$$

Again, (3.12) can be obtained directly by inserting the expression (3.1) of $y_{\mathbf{a},\mathbf{c}}(z)$ into GLE$(\mathbf{n}, p, A, \tau)$ and computing the leading terms at singularities $\pm p$. We omit the proof here.

Remark 3.5. Theorem 2.A implies that for fixed τ and $p \notin E_\tau[2]$, there are *only finite many* A's (i.e. zeros of the polynomial $d(A)$) such that Case (a-ii) or (a-iii) occurs. Therefore, if $\mathbf{a}$ is in Case (a-ii) or (a-iii), we could use a sequence of $\mathbf{a}^k$ of Case (a-i) to approximate $\mathbf{a}$. In particular, a_{N-1}^k and a_N^k converge to p for Case (a-ii). The fact that $c(\mathbf{a}^k)$ converges and (3.8) implies the sum $\zeta(a_{N-1}^k - p) + \zeta(a_N^k - p)$ also converges, namely $(a_{N-1}^k - p)^{-1} + (a_N^k - p)^{-1}$ tends to a finite limit as $\mathbf{a}^k \to \mathbf{a}$. Then $c(\mathbf{a})$ and A can be also expressed in terms of $\{a_1, \cdots, a_{N-2}\}$ in Case (a-ii) and (a-iii).

Now we consider that GLE$(\mathbf{n}, p, A_0, \tau)$ is *not completely reducible.* Then $y_{\mathbf{a},c}(z; p)$ in Proposition 3.1 is the only common eigenfunction up to a constant.

Theorem 3.6. *Let $p \notin E_\tau[2]$ and $A \in \mathbb{C}$. Suppose that GLE$(\mathbf{n}, p, A, \tau)$ is not completely reducible with solution $y_{\mathbf{a},c}(z)$ given in Proposition 3.1. Then There exists $(m_1, m_2) \in \frac{1}{2}\mathbb{Z}^2$ such that*

$$\{a_1, \cdots, a_N\} \equiv \{-a_1, \cdots, -a_N\} \ mod \ \Lambda_\tau, \tag{3.13}$$

$$\sum_{j=1}^{N} a_j = m_1 + m_2\tau + \sum_{k=1}^{3} \frac{n_k \omega_k}{2} \in E_\tau[2], \tag{3.14}$$

$$c = m_1\eta_1 + m_2\eta_2. \tag{3.15}$$

Proof. Remark 3.2 shows that $y_{\mathbf{a},c}(-z)$ and $y_{\mathbf{a},c}(z)$ are linearly dependent, so (3.13) follows trivially from the expression (3.1). Besides, (3.2) gives

$$\ell_i^* y_{\mathbf{a},c}(z) = \varepsilon_i y_{\mathbf{a},c}(z), \quad \varepsilon_i \in \{\pm 1\}, \ i = 1, 2,$$

because the linear dependence of $y_{\mathbf{a},c}(-z)$ and $y_{\mathbf{a},c}(z)$ imply $\varepsilon_i = \varepsilon_i^{-1}$. Consequently, (3.14)-(3.15) can be proved by the same way as (3.6), where $(m_1, m_2) \in \frac{1}{2}\mathbb{Z}^2$ follows from $\varepsilon_i \in \{\pm 1\}$ for $i = 1, 2$. □

Now we want to define the embedding of $\Gamma_{n,p}$ into $\mathrm{Sym}^N E_\tau$, where $N = \sum_{i=0}^{3} n_i + 1$ and $\mathrm{Sym}^N E_\tau$ is the N-th symmetric product of E_τ. For any $(A, W) \in \Gamma_{n,p}$, it follows from Remark 2.8 that the solution $y_1(z; A, W)$ is unique up to a sign. By (3.1) and Proposition 3.1, there is $\mathbf{a} = \{a_1, \cdots, a_N\}$ (unique mod Λ_τ) such that $y_1(z; A, W) = y_{\mathbf{a},c}(z)$, i.e. $\mathbf{a}$ gives all the zeros of $y_1(z; A, W)$. Then we define a map $i_{n,p} : \Gamma_{n,p} \to \mathrm{Sym}^N E_\tau$ by

$$i_{\mathbf{n},p}(A, W) := \{[a_1], \cdots, [a_N]\} \in \mathrm{Sym}^N E_\tau, \tag{3.16}$$

where as introduced in Section 1, $[a_i] := a_i \pmod{\Lambda_\tau} \in E_\tau$. The above argument shows that $i_{\mathbf{n},p}$ is well-defined. Furthermore, if $W \neq 0$, then we see from Remark 2.8 that

$$i_{\mathbf{n},p}(A, -W) = \{-[a_1], \cdots, -[a_N]\}. \tag{3.17}$$

Proposition 3.7. *$i_{\mathbf{n},p}$ is an embedding from $\Gamma_{n,p}$ into Sym$^N E_\tau$.*

Proof. Suppose $i_{\mathbf{n},p}(A, W) = i_{\mathbf{n},p}(\tilde{A}, \tilde{W}) = \{[a_1], \cdots, [a_N]\}$. Then (3.5) implies $\Phi_e(z; A) = \Phi_e(z; \tilde{A})$ up to a constant and so $A = \tilde{A}$ by (2.14). Together with Theorem 2.7, we have $W^2 = \tilde{W}^2$. If $W = 0$ then $\tilde{W} = 0$. If $W \neq 0$ then by (3.16) and (3.17) we also have $W = \tilde{W}$. This proves the embedding of $i_{\mathbf{n},p}$. □

By applying $i_{\mathbf{n},p}$, we could define $\sigma_{\mathbf{n},p} : \Gamma_{n,p} \to E_\tau$ by

$$\sigma_{\mathbf{n},p}(A, W) := \sum_{i=1}^{N} [a_i] - \sum_{k=1}^{3} [\tfrac{n_k \omega_k}{2}], \tag{3.18}$$

which is the composition of $i_{\mathbf{n},p}$ and the addition map $\{[a_1], \cdots, [a_N]\} \mapsto \sum_{i=1}^{N} [a_i] - \sum_{k=1}^{3} [\frac{n_k \omega_k}{2}]$. Thus $i_{\mathbf{n},p}$ is also called the *addition map*. Clearly

$$\sigma_{\mathbf{n},p}(A, -W) = -\sum_{i=1}^{N} [a_i] - \sum_{k=1}^{3} [\tfrac{n_k \omega_k}{2}] = -\sigma_{\mathbf{n},p}(A, W).$$

The map $\sigma_{\mathbf{n},p}$ is a holomorphic map (or a finite morphism) from $\Gamma_{\mathbf{n},p}$ to E_τ if the irreducible curve $\Gamma_{\mathbf{n},p}$ is smooth (or if $\Gamma_{n,p}$ is singular). Hence in both cases, the degree $\deg \sigma_{\mathbf{n},p} = \#\sigma_{\mathbf{n},p}^{-1}(z), z \in E_\tau$, is well-defined. How to calculate it is the main purpose of the paper.

Definition 3.8. *We let $Y_{\mathbf{n},p}(\tau)$ be the image of $\Gamma_{\mathbf{n},p}(\tau)$ in Sym$^N E_\tau$ under $i_{\mathbf{n},p}$, and $X_{\mathbf{n},p}(\tau)$ be the image of $\{(A, W) \in \Gamma_{\mathbf{n},p} | W \neq 0\}$ under $i_{\mathbf{n},p}$.*

The next two section will discuss of the limiting of $Y_{\mathbf{n},p}(\tau)$ for two case (i) $A \to \infty$ and $p \notin E_\tau[2]$ is fixed; (ii) $p \to \omega_k/2$.

4. The closure of $Y_{\mathbf{n},p}(\tau)$

The purpose of this section is to prove the following result.

Theorem 4.1. *Let $\mathbf{n}=(n_0,n_1,n_2,n_3)$ where $n_k\in\mathbb{Z}_{\geq 0}$. Then the followings hold.*

(i)
$$\overline{X_{\mathbf{n},p}(\tau)}=\overline{Y_{\mathbf{n},p}(\tau)}=Y_{\mathbf{n},p}(\tau)\cup\{\infty_+(p),\infty_-(p)\},$$
where
$$\infty_\pm(p):=\Big(\overbrace{0,\cdots,0}^{n_0},\overbrace{\tfrac{\omega_1}{2},\cdots,\tfrac{\omega_1}{2}}^{n_1},\overbrace{\tfrac{\omega_2}{2},\cdots,\tfrac{\omega_2}{2}}^{n_2},\overbrace{\tfrac{\omega_3}{2},\cdots,\tfrac{\omega_3}{2}}^{n_3},\pm p\Big). \tag{4.1}$$
Furthermore, $\overline{X_{\mathbf{n},p}(\tau)}$ is smooth at $\infty_\pm(p)$.

(ii) *The map $i_{\mathbf{n},p}:\Gamma_{\mathbf{n},p}\to Y_{\mathbf{n},p}$ has a natural extension to $\bar{\imath}_{\mathbf{n},p}:\overline{\Gamma_{\mathbf{n},p}}\to\overline{Y_{\mathbf{n},p}}$ by $\bar{\imath}_{\mathbf{n},p}(\infty_\pm):=\infty_\pm(p)$, where $\{\infty_\pm\}:=\overline{\Gamma_{\mathbf{n},p}}\backslash\Gamma_{\mathbf{n},p}$ is given in (2.24).*

We will prove a slight generalization of Theorem 4.1. Indeed, we want to study the limiting of $Y_{\mathbf{n},p}(\tau)$ as $p\to p_0\notin E_\tau[2]$. For each p near p_0, we associate a $A(p)\in\mathbb{C}$ and consider GLE$(\mathbf{n},p,A(p),\tau)$. Letting $\mathbf{a}(p)=\{[a_1(p)],\cdots,[a_N(p)]\}=i_{\mathbf{n},p}(A(p),W(p))$ and $c(p):=c(\mathbf{a}(p))=c$ given by Proposition 3.1, we study the limits of $\mathbf{a}(p)$ and $c(p)$ as $p\to p_0$. Since $\mathbf{a}(p)\in\mathrm{Sym}^N E_\tau$, up to a subsequence, we can always assume that $\mathbf{a}(p)\to\mathbf{a}^0=\{[a_1^0],\cdots,[a_N^0]\}$ and
$$a_j(p)\to a_j^0\quad\text{as }p\to p_0,\quad\forall\, j. \tag{4.2}$$

Proposition 4.2. *Let $p\to p_0\notin E_\tau[2]$ and $S:=E_\tau[2]\cup\{\pm[p_0]\}$. Then $I_{\mathbf{n}}(z;p,A(p),\tau)$ converges in $E_\tau\backslash S$ if and only if $A(p)$ converges if and only if the corresponding $B(p)$ converges if and only if $c(p)$ converges.*

Proof. From (2.2), it is easy to see that $I_{\mathbf{n}}(z;p,A(p),\tau)$ converges for $z\in E_\tau\backslash S$ if and only if $A(p)$ converges if and only if $B(p)$ converges. To study its relation with $c(p)$, we recall the solution $y_{\mathbf{a}(p),c(p)}(z;p)$ of GLE$(\mathbf{n},p,A(p),\tau)$ given in Proposition 3.1. Then
$$\begin{aligned}\frac{y'_{\mathbf{a}(p),c(p)}(z;p)}{y_{\mathbf{a}(p),c(p)}(z;p)}=&c(p)+\sum_{j=1}^N\zeta(z-a_j(p))-\sum_{k=0}^3 n_k\zeta(z-\tfrac{\omega_k}{2})\\&-\tfrac12(\zeta(z+p)+\zeta(z-p))\end{aligned}$$
and so
$$\begin{aligned}I_{\mathbf{n}}(z;p,A(p),\tau)&=\frac{y''_{\mathbf{a}(p),c(p)}(z;p)}{y_{\mathbf{a}(p),c(p)}(z;p)}=\left(\frac{y'_{\mathbf{a}(p),c(p)}}{y_{\mathbf{a}(p),c(p)}}\right)'+\left(\frac{y'_{\mathbf{a}(p),c(p)}}{y_{\mathbf{a}(p),c(p)}}\right)^2\\&=c(p)^2+2c(p)D(z;\mathbf{a}(p),p)+D(z;\mathbf{a}(p),p)^2+E(z;\mathbf{a}(p),p),\end{aligned} \tag{4.3}$$

where

$$D(z;\mathbf{a}(p),p):=\left[\begin{array}{c}\sum_{j=1}^{N}\zeta(z-a_j(p))-\sum_{k=0}^{3}n_k\zeta(z-\frac{\omega_k}{2})\\ -\frac{1}{2}(\zeta(z+p)+\zeta(z-p))\end{array}\right] \tag{4.4}$$

and $E(z;\mathbf{a}(p),p)=D'(z;\mathbf{a}(p),p)$ is given by

$$E(z;\mathbf{a}(p),p):=\left[\begin{array}{c}\sum_{k=0}^{3}n_k\wp(z-\frac{\omega_k}{2})-\sum_{j=1}^{N}\wp(z-a_j(p))\\ +\frac{1}{2}(\wp(z+p)+\wp(z-p))\end{array}\right]. \tag{4.5}$$

By (4.2), both $D(z;\mathbf{a}(p),p)$ and $E(z;\mathbf{a}(p),p)$ converge uniformly for $z\in E_\tau\backslash(\{[a_j^0]\}_{j=1}^N\cup\{\pm[p_0]\}\cup E_\tau[2])$. Thus, $I_{\mathbf{n}}(z;p,A(p),\tau)$ converges if and only if $c(p)$ converges. □

Now we discuss the limit $\mathbf{a}^0$ of $\mathbf{a}(p)$ as $p\to p_0\notin E_\tau[2]$. There are two cases: (i) the corresponding $B(p)\to B_0\in\mathbb{C}$; (ii) $B(p)\to\infty$. Case (i) is easy, because $B(p)\to B_0\in\mathbb{C}$ implies $A(p)$ converges to some $A_0\in\mathbb{C}$, i.e. GLE$(\mathbf{n},p,A(p),\tau)$ converges to GLE$(\mathbf{n},p_0,A_0,\tau)$, and so $\lim_{p\to p_0}\mathbf{a}(p)=\mathbf{a}^0\in Y_{\mathbf{n},p_0}(\tau)$. We discuss case (ii) in the following proposition.

Proposition 4.3. *Suppose $\mathbf{a}(p)\to\mathbf{a}^0$ and $B(p)\to\infty$ as $p\to p_0\notin E_\tau[2]$. Then $\mathbf{a}^0\in\{\infty_\pm(p_0)\}$, where $\infty_\pm(p)$ is defined in (4.1).*

Proof. Since $B(p)\to\infty$, by Proposition 4.2, we have both $A(p)$ and $c(p)\to\infty$ as $p\to p_0$. We claim that $c(p)/A(p)\to\beta$ for some $\beta\in\mathbb{C}\setminus\{0\}$ as $p\to p_0$. In fact, by differentiating both expressions (2.1) and (4.3) of $I_{\mathbf{n}}(z;p,A(p),\tau)$, we obtain

$$2c(p)E(z;\mathbf{a}(p),p)=A(p)(\wp(z-p)-\wp(z+p))+\text{ other terms,} \tag{4.6}$$

where other terms remain bounded outside the singularities $\{[a_j^0]\}_{j=1}^N\cup\{\pm[p_0]\}\cup E_\tau[2]$ as $p\to p_0$. Therefore, $c(p)/A(p)\to\beta\neq0$. Since $\mathbf{a}^0=\lim_{p\to p_0}\mathbf{a}(p)$, (4.6) yields $2\beta E(z;\mathbf{a}^0,p_0)=\wp(z-p_0)-\wp(z+p_0)$, and it follows from (4.5) that

$$\begin{aligned}2\beta&\left[\begin{array}{c}\sum_{k=0}^{3}n_k\wp(z-\frac{\omega_k}{2})-\sum_{j=1}^{N}\wp(z-a_j^0)\\ +\frac{1}{2}(\wp(z+p_0)+\wp(z-p_0))\end{array}\right]\\ &=\wp(z-p_0)-\wp(z+p_0).\end{aligned} \tag{4.7}$$

From (4.7), we have two possibilities: One is $\mathbf{a}^0=\infty_+(p_0)$ and $\beta=-1$; the other one is $\mathbf{a}^0=\infty_-(p_0)$ and $\beta=1$. This completes the proof. □

Proof of Theorem 4.1 . Obviously, Theorem 4.1-(i) follows from Propositions 4.2-4.3 easily.

(ii). The above argument shows that $\lim_{A\to\infty}i_{\mathbf{n},p}(A,\pm W)=\infty_\pm(p)$ (by renaming $W,-W$ if necessary). As pointed out in (2.24), $\overline{\Gamma_{\mathbf{n},p}}\backslash\Gamma_{\mathbf{n},p}$ consists of two points $\infty_\pm:=\lim_{A\to\infty}(A,\pm W)$. Thus the embedding $i_{\mathbf{n},p}$ has a natural extension to $\bar{\imath}_{\mathbf{n},p}:\overline{\Gamma_{\mathbf{n},p}}\to\overline{Y_{\mathbf{n},p}}$ by defining $\bar{\imath}_{\mathbf{n},p}(\infty_\pm):=\infty_\pm(p)$. □

We have the following corollary.

Corollary 4.4. *The map $\sigma_{\mathbf{n},p}$ can be extended to be a finite morphism (still denoted by $\sigma_{\mathbf{n},p}$) from the irreducible curve $\overline{\Gamma_{\mathbf{n},p}}$ to E_τ such that*

$$\lim_{(A,\pm W)\to\infty_\pm} \sigma_{\mathbf{n},p}(A,\pm W) = \pm[p].$$

5. The limiting of $Y_{\mathbf{n},p}(\tau)$ as $p\to\frac{\omega_k}{2}$

5.1. **The counterpart of the above theory for $H(\mathbf{n},B,\tau)$.** Before stating the main result of this section, we recall the following generalized Lamé equation (denoted by $H(\mathbf{n},B,\tau)$)

$$y''(z) = I_{\mathbf{n}}(z;B,\tau)y(z), \quad z\in\mathbb{C}. \tag{5.1}$$

where

$$I_{\mathbf{n}}(z;B,\tau) := \sum_{k=0}^{3} n_k(n_k+1)\wp(z+\tfrac{\omega_k}{2}|\tau)+B, \tag{5.2}$$

and $n_k\in\mathbb{Z}_{\geq 0}$ for all k.

There is a counterpart of the theory about $\text{GLE}(\mathbf{n},p,A,\tau)$ established in previous sections for $H(\mathbf{n},B,\tau)$, and the proof is simpler due to the absence of singularities $\pm p\notin E_\tau[2]$. Therefore, we only write down the conclusions without any details of the proofs. Remark that part of the statements listed below can be found in [10, 19, 21, 22] and references therein. In this section, we denote $\hat{N} := \sum_k n_k$.

(i) Any solution of $H(\mathbf{n},B,\tau)$ is meromorphic in $\mathbb{C}$. The corresponding second symmetric product equation (1.4) has a unique even elliptic solution $\hat{\Phi}_e(z;B)$ expressed by

$$\hat{\Phi}_e(z;B) = \hat{C}_0(B) + \sum_{k=0}^{3}\sum_{j=0}^{n_k-1} \hat{b}_j^{(k)}(B)\wp(z+\tfrac{\omega_k}{2})^{n_k-j} \tag{5.3}$$

where $\hat{C}_0(B), \hat{b}_j^{(k)}(B)$ are all polynomials in B with $\deg\hat{C}_0 > \max_{j,k}\deg\hat{b}_j^{(k)}$ and the leading coefficient of $\hat{C}_0(B)$ being $\frac{1}{2}$. Moreover, $\hat{\Phi}_e(z;B) = \hat{y}_1(z;B)$ $\hat{y}_1(-z;B)$, where $\hat{y}_1(z;B)$ is a common eigenfunction of the monodromy matrices of $H(\mathbf{n},B,\tau)$ and up to a constant, can be written as

$$\hat{y}_1(z;B) = y_{\mathbf{a}}(z) := \frac{e^{c(\mathbf{a})z}\prod_{i=1}^{\hat{N}}\sigma(z-a_i)}{\prod_{k=0}^{3}\sigma(z-\frac{\omega_k}{2})^{n_k}}. \tag{5.4}$$

(ii) Let $\hat{W}$ be the Wroskian of $\hat{y}_1(z;B)$ and $\hat{y}_1(-z;B)$ (and denote $\hat{y}_1(z;B)$ by $\hat{y}_1(z;B,\hat{W})$), then $\hat{W}^2 = Q_{\mathbf{n}}(B;\tau)$, where

$$Q_{\mathbf{n}}(B;\tau) := \hat{\Phi}_e'(z;B)^2 - 2\hat{\Phi}_e(z;B)\hat{\Phi}_e''(z;B) + 4I_{\mathbf{n}}(z;B,\tau)\hat{\Phi}_e(z;B)^2$$

is a monic polynomial in B with *odd degree* and independent of z. Define the hyperelliptic curve $\Gamma_{\mathbf{n}}(\tau)$ by

$$\Gamma_{\mathbf{n}}(\tau) := \{(B,\hat{W})\,|\,\hat{W}^2 = Q_{\mathbf{n}}(B;\tau)\}.$$

Then the map $i_{\mathbf{n}}:\Gamma_{\mathbf{n}}(\tau)\to \mathrm{Sym}^{\hat{N}}E_\tau$ defined by

$$i_{\mathbf{n}}(B,\hat{W}):=\{[a_1],\cdots,[a_{\hat{N}}]\}$$

is an embedding, where $\{[a_1],\cdots,[a_{\hat{N}}]\}$ is uniquely determined by $\hat{y}_1(z;B,\hat{W})$ via (5.4). Let $Y_{\mathbf{n}}(\tau)$ be the image of $\Gamma_{\mathbf{n}}(\tau)$ in $\mathrm{Sym}^{\hat{N}}E_\tau$ under $i_{\mathbf{n}}$, and $X_{\mathbf{n}}(\tau)$ be the image of $\{(B,\hat{W})\in\Gamma_{\mathbf{n}}|\hat{W}\neq 0\}$ under $i_{\mathbf{n}}$.

(iii) $\overline{X_{\mathbf{n}}(\tau)}=\overline{Y_{\mathbf{n}}(\tau)}=Y_{\mathbf{n}}(\tau)\cup\{\infty_0\}$ where

$$\infty_0:=\Big(\overbrace{0,\cdot\cdot\cdot,0}^{n_0},\overbrace{\frac{\omega_1}{2},\cdot\cdot\cdot,\frac{\omega_1}{2}}^{n_1},\overbrace{\frac{\omega_2}{2},\cdot\cdot\cdot,\frac{\omega_2}{2}}^{n_2},\overbrace{\frac{\omega_3}{2},\cdot\cdot\cdot,\frac{\omega_3}{2}}^{n_3}\Big). \tag{5.5}$$

Furthermore, the map $i_{\mathbf{n}}:\Gamma_{\mathbf{n}}\to Y_{\mathbf{n}}$ has a natural extension to $\bar{\imath}_{\mathbf{n}}:\overline{\Gamma_{\mathbf{n}}}\to\overline{Y_{\mathbf{n}}(\tau)}$ by $\bar{\imath}_{\mathbf{n}}(\infty):=\infty_0$ where $\{\infty\}:=\overline{\Gamma_{\mathbf{n}}}\backslash\Gamma_{\mathbf{n}}$.

(iv) The embedding $i_{\mathbf{n}}$ also induces the addition map $\sigma_{\mathbf{n}}:\Gamma_{\mathbf{n}}\to E_\tau$ by

$$\sigma_{\mathbf{n}}(B,\hat{W}):=\sum_{i=1}^{\hat{N}}[a_i]-\sum_{k=1}^{3}[\tfrac{n_k\omega_k}{2}].$$

Remark that the map $-\sigma_{\mathbf{n}}$ was already defined in [22, Section 4] and called a covering map there; while for the Lamé case $n_1=n_2=n_3=0$, $\sigma_{\mathbf{n}}$ was called an addition map in [15]. As before, $\sigma_{\mathbf{n}}$ can be extended as a finite morphism from the irreducible curve $\overline{\Gamma_{\mathbf{n}}(\tau)}$ to E_τ such that

$$\lim_{(B,\hat{W})\to\infty}\sigma_{\mathbf{n}}(B,\hat{W})=0. \tag{5.6}$$

5.2. **The limit of $Y_{\mathbf{n},p}$ as $p\to\frac{\omega_k}{2}$.** For any $\mathbf{n}=(n_0,n_1,n_2,n_3)$, we define $\mathbf{n}_k^{\pm}:=(n_{k,0}^{\pm},n_{k,1}^{\pm},n_{k,2}^{\pm},n_{k,3}^{\pm})$, where $n_{k,i}^{\pm}=n_i$ for $i\neq k$ and $n_{k,k}^{\pm}=n_k\pm1$. Recall $N=\sum_k n_k+1$. In the following, we always identify a point $\mathbf{a}=\{[a_1],\cdot\cdot\cdot,[a_{N-2}]\}\in\overline{Y_{\mathbf{n}_k^-}(\tau)}$ by $\mathbf{a}=\{[a_1],\cdot\cdot\cdot,[a_{N-2}],[\frac{\omega_k}{2}],[\frac{\omega_k}{2}]\}$.

The main result of this section is the following theorem:

Theorem 5.1. *Let $\mathbf{n}=(n_0,n_1,n_2,n_3)$ with $n_k\geq 0$ for all k. Then*

$$\lim_{p\to\frac{\omega_k}{2}}\overline{Y_{\mathbf{n},p}(\tau)}=\overline{Y_{\mathbf{n}_k^+}(\tau)}\cup\overline{Y_{\mathbf{n}_k^-}(\tau)}.$$

In the following, we will give the complete proof for the case $p\to 0$. For other cases the proof is similar. Let p be a sequence of points in E_τ and a sequence $(A(p),W(p))\in\Gamma_{\mathbf{n},p}$ with the corresponding $\{[a_1(p)],\cdot\cdot\cdot,[a_N(p)]\}\in Y_{\mathbf{n},p}$ satisfying $([a_1(p)],\cdot\cdot\cdot,[a_N(p)])\to([a_1^0],\cdot\cdot\cdot,[a_N^0])$. We assume that up to a subsequence if necessary, $A(p)\to A_0\in\mathbb{C}\cup\{\infty\}$ as $p\to 0$. Then the same proof as Proposition 4.2 implies

$$\begin{aligned}&I_{\mathbf{n}}(z;p,A(p),\tau)\text{ converges if and only if }c(\mathbf{a}(p))\text{ converges}\\&\text{if and only if }B(p)\text{ converges as }p\to 0.\end{aligned} \tag{5.7}$$

Indeed, we have a more precise statement. Recall $e_k:=\wp(\frac{\omega_k}{2})$ for $k=1,2,3$.

Proposition 5.2. *$B(p)$ is convergent as $p \to 0$ if and only if*

$$A(p) = A^{\pm}(p) := \alpha_0^{\pm} p^{-1} + \alpha_1^{\pm} p + o(p) \tag{5.8}$$

with

$$\alpha_0^{+} := -(\tfrac{1}{4} + n_0), \quad \alpha_0^{-} := \tfrac{3}{4} + n_0 \tag{5.9}$$

and some $\alpha_1^{\pm} \in \mathbb{C}$. Furthermore, if (5.8) holds, then $I_{\mathbf{n}}(z; p, A^{\pm}(p), \tau)$ converges to $I_{\mathbf{n}_0^{\pm}}(z; B^{\pm}, \tau)$, where $B^{\pm}$ and $\alpha_1^{\pm}$ are related by

$$B^{\pm} = \mp(2n_0 + 1)\alpha_1^{\pm} - \sum_{k=1}^{3} n_k(n_k + 1)e_k. \tag{5.10}$$

Proof. Recall (2.3) and it yields

$$\begin{aligned} B(p) &= A(p)^2 - \zeta(2p)A(p) - \frac{3}{4}\wp(2p) - \sum_{k=0}^{3} n_k(n_k+1)\wp(p + \tfrac{\omega_k}{2}) \\ &= \frac{1}{p^2}\left((A(p)p)^2 - \frac{A(p)p}{2} - \frac{3}{16} - n_0(n_0+1)\right) \\ &\quad - \sum_{k=1}^{3} n_k(n_k+1)e_k + o(1) \\ &= \frac{(A(p)p - \alpha_0^{+})(A(p)p - \alpha_0^{-})}{p^2} - \sum_{k=1}^{3} n_k(n_k+1)e_k + o(1), \end{aligned} \tag{5.11}$$

where $\alpha_0^{\pm}$ is defined in (5.9). From here, we easily see that $B(p)$ converges as $p \to 0$ if and only if $A(p)$ satisfies (5.8).

Now suppose (5.8) holds, then (5.11) implies $B(p) \to B^{\pm}$ as $p \to 0$. Furthermore,

$$\begin{aligned} &n_0(n_0+1)\wp(z) + \tfrac{3}{4}(\wp(z+p) + \wp(z-p)) + A(p)(\zeta(z+p) - \zeta(z-p)) \\ &= n_0(n_0+1)\wp(z) + \tfrac{3}{2}\wp(z) - 2A(p)p\wp(z) + o(1) \\ &\to [n_0(n_0+1) + \tfrac{3}{2} - 2\alpha_0^{\pm}]\wp(z) = n_{0,0}^{\pm}(n_{0,0}^{\pm} + 1)\wp(z), \end{aligned}$$

where $n_{0,0}^{\pm} = n_0 \pm 1$. Thus $I_{\mathbf{n}}(z; p, A^{\pm}(p), \tau)$ converges to $I_{\mathbf{n}_0^{\pm}}(z; B^{\pm}, \tau)$. □

Now, we discuss the limit $\mathbf{a}^0$ of $\mathbf{a}(p)$ as $p \to 0$. We divide two cases to discuss: (i) $B(p) \to B_0 \in \mathbb{C}$, and (ii) $B(p) \to \infty$.

Proposition 5.3. *Suppose $B(p) \to B_0 \in \mathbb{C}$ as $p \to 0$. Then either*

(i) *$\mathbf{a}^0 \in Y_{\mathbf{n}^+}(\tau)$, where $\mathbf{n}^+ = \mathbf{n}_0^+ = (n_0 + 1, n_1, n_2, n_3)$; or*

(ii) *$\mathbf{a}^0 = \{[a_1^0], \cdots, [a_{N-2}^0], 0, 0\}$ and $\{[a_1^0], \cdots, [a_{N-2}^0]\} \in Y_{\mathbf{n}^-}(\tau)$, where $\mathbf{n}^- = \mathbf{n}_0^- = (n_0 - 1, n_1, n_2, n_3)$.*

Proof. By Proposition 5.2, we see that $I_{\mathbf{n}}(z; p, A(p), \tau)$ converges to either (i) $I_{\mathbf{n}^+}(z; B^+, \tau)$ or (ii) $I_{\mathbf{n}^-}(z; B^-, \tau)$.

For the case (i), because $c(\mathbf{a}(p)) \to c_0$ for some $c_0 \in \mathbb{C}$ by (5.7), we see that

$$y_{\mathbf{a}(p),c(\mathbf{a}(p))}(z;p) \to y_{\mathbf{a}^0}(z) := \frac{e^{c_0 z}\prod_{j=1}^{N}\sigma(z-a_j^0)}{\sigma(z)^{n_0+1}\prod_{k=1}^{3}\sigma(z-\frac{\omega_k}{2})^{n_k}}, \tag{5.12}$$

and $y_{\mathbf{a}^0}(z)$ is a solution to $H(\mathbf{n}^+, B^+, \tau)$. This implies $\mathbf{a}^0 \in Y_{\mathbf{n}^+}(\tau)$.

For the case (ii), we also have (5.12) but $y_{\mathbf{a}^0}(z)$ is a solution to $H(\mathbf{n}^-, B^-, \tau)$. It follows from (5.4) that at least two of $\{[a_1^0], \cdots, [a_N^0]\}$ must be 0. After a rearrangement of the index, we have might assume

$$\mathbf{a}^0 = \{[a_1^0], \cdots, [a_{N-2}^0], 0, 0\}.$$

Then we have $\{[a_1^0], \cdots, [a_{N-2}^0]\} \in Y_{\mathbf{n}^-}(\tau)$. □

Remark 5.4. In the following, we always identify a point $\{[a_1^0], \cdots, [a_{N-2}^0]\} \in Y_{\mathbf{n}^-}(\tau)$ with the point $\{[a_1^0], \cdots, [a_{N-2}^0], 0, 0\}$ in $\mathrm{Sym}^N E_\tau$.

Next, we consider the case $I_{\mathbf{n}}(z; p, A(p), \tau)$ diverges, i.e. $B(p) \to \infty$ as $p \to 0$.

Proposition 5.5. *Suppose $B(p) \to \infty$ as $p \to 0$. Then*

$$\mathbf{a}(p) \to \mathbf{a}^0 = \infty(0)$$

where

$$\infty(0) := \Big(\overbrace{0, \cdots, 0}^{n_0}, \overbrace{\frac{\omega_1}{2}, \cdots, \frac{\omega_1}{2}}^{n_1}, \overbrace{\frac{\omega_2}{2}, \cdots, \frac{\omega_2}{2}}^{n_2}, \overbrace{\frac{\omega_3}{2}, \cdots, \frac{\omega_3}{2}}^{n_3}, 0\Big). \tag{5.13}$$

Proof. Since $B(p) \to \infty$, we see from (5.7) that $I_{\mathbf{n}}(z; p, A(p), \tau)$ diverges and $c(\mathbf{a}(p)) \to \infty$ as $p \to 0$. By differentiating both expressions (2.1) and (4.3) of $I_{\mathbf{n}}(z; p, A(p), \tau)$ and considering $p \to 0$, we obtain

$$\begin{aligned} &2c(\mathbf{a}(p))E(z;\mathbf{a}(p),p) + 2D(z;\mathbf{a}(p),p)E(z;\mathbf{a}(p),p) + E'(z;\mathbf{a}(p),p) \\ &= \sum_{k=0}^{3} n_k(n_k+1)\wp'(z+\tfrac{\omega_k}{2}) + \frac{3}{4}(\wp'(z+p)+\wp'(z-p)) \\ &\quad + A(p)[-2\wp'(z)p + O(p^3)]. \end{aligned} \tag{5.14}$$

Then we divide our discussion into two cases.

Case 1. Suppose $A(p)p \to \beta \in \mathbb{C}$ up to a subsequence. Then the RHS of (5.14) is uniformly convergent outside the singularties $E_\tau[2]$. Since $c(\mathbf{a}(p)) \to \infty$, we have

$$E(z;\mathbf{a}(p),p) \to 0 \text{ uniformly outside } E_\tau[2].$$

So we see from (4.5) that

$$\sum_{k=0}^{3} n_k \wp(z - \tfrac{\omega_k}{2}) - \sum_{j=1}^{N} \wp(z - a_j^0) + \wp(z) = 0. \tag{5.15}$$

This implies that there are (n_0+1) of a_j^0 equal to 0 and n_k of a_j^0 equal to $\frac{\omega_k}{2}$ for $k\in\{1,2,3\}$, namely $\mathbf{a}^0=\infty(0)$.

Case 2. Suppose $A(p)p\to\infty$. By dividing $A(p)p$ on both sides of (5.14), we see that

$$\begin{aligned}\frac{2c(\mathbf{a}(p))}{A(p)p}E(z;\mathbf{a}(p),p)=&\frac{-2D(z;\mathbf{a}(p),p)E(z;\mathbf{a}(p),p)-E'(z;\mathbf{a}(p),p)}{A(p)p}\\ &+\frac{1}{A(p)p}\left[\begin{array}{c}\sum_{k=0}^3 n_k(n_k+1)\wp'(z+\frac{\omega_k}{2})\\ +\frac{3}{4}(\wp'(z+p)+\wp'(z-p))\end{array}\right]-2\wp'(z)+O(p^2).\end{aligned}\tag{5.16}$$

Since the RHS of (5.16) converges uniformly to $-2\wp'(z)$ outside $E_\tau[2]$, we have either $\frac{c(\mathbf{a}(p))}{A(p)p}\to\beta\in\mathbb{C}$ or $\frac{c(\mathbf{a}(p))}{A(p)p}\to\infty$. Suppose $\frac{c(\mathbf{a}(p))}{A(p)p}\to\beta\in\mathbb{C}$. Letting $p\to 0$, it follows from (5.16) and (4.5) that

$$\beta\Bigg[\sum_{k=0}^3 n_k\wp(z-\tfrac{\omega_k}{2})-\sum_{j=1}^N\wp(z-a_j^0)+\wp(z)\Bigg]=-\wp'(z).\tag{5.17}$$

But the RHS of (5.17) has a pole of order 3 at $z=0$ while the LHS does not, a contradiction. Thus, $\frac{c(\mathbf{a}(p))}{A(p)p}\to\infty$ and then we obtain (5.15) again, which gives $\mathbf{a}^0=\infty(0)$. This completes the proof. □

Corollary 5.6. *Let $(A(p),W(p))\in\Gamma_{\mathbf{n},p}$. Assume that $\sigma_{\mathbf{n},p}(A(p),W(p))\to[u_0]\neq 0$ as $p\to 0$. Then $B(p)$ are uniformly bounded as $p\to 0$. In particular, $I_{\mathbf{n}}(z;p,A(p),\tau)$ converges up to a subsequence of $p\to 0$.*

Proof. Suppose by contradiction that up to a subsequence of $p\to 0$, $B(p)\to\infty$. Let $\mathbf{a}(p)=i_{\mathbf{n},p}(A(p),W(p))$ be the corresponding point in $Y_{\mathbf{n},p}(\tau)$. Then Proposition 5.5 implies $\mathbf{a}(p)\to\infty(0)$ and so $[u_0]=0$, a contradiction. Thus $B(p)$ are uniformly bounded. The proof is complete by applying (5.7). □

Now we are ready to prove Theorem 5.1.

Proof of Theorem 5.1. We prove the case for $p\to 0$ and the other three cases are similar. Let $p\to 0$. By Propositions 5.3 and 5.5, for any $\mathbf{a}(p)\in\overline{Y_{\mathbf{n},p}(\tau)}$, $\mathbf{a}(p)\to\mathbf{a}^0\in\overline{Y_{\mathbf{n}^+}(\tau)}\cup\overline{Y_{\mathbf{n}^-}(\tau)}$ as $p\to 0$. Conversely, we want to show that any point $\mathbf{a}\in\overline{Y_{\mathbf{n}^+}(\tau)}\cup\overline{Y_{\mathbf{n}^-}(\tau)}$ could be a limit point of some $\mathbf{a}(p)\in\overline{Y_{\mathbf{n},p}(\tau)}$ as $p\to 0$.

First we note that $\infty(0)$ is the limit of $\infty_\pm(p)$ as $p\to 0$.

Given any $\mathbf{a}=\{[a_1],\cdots,[a_N]\}\in Y_{\mathbf{n}^+}(\tau)$ and let $i_{\mathbf{n}^+}(B,W)=\mathbf{a}$ for some B. For p close to 0, we set $A(p):=-(\frac{1}{4}+n_0)p^{-1}+\alpha_1^+p$, where α_1^+ is given by (5.10):

$$\alpha_1^+:=-\frac{B+\sum_{k=1}^3 n_k(n_k+1)e_k}{2n_0+1},$$

and $B(p)$ by (2.3). Let $(A(p),\pm W(p))\in\Gamma_{\mathbf{n},p}$ and $\pm\mathbf{a}(p)=i_{\mathbf{n},p}(A(p),\pm W(p))$. By Propositions 5.2-5.3, $I_{\mathbf{n}}(z;p,A(p),\tau)$ converges to $I_{\mathbf{n}^+}(z;B,\tau)$ and so $\{\pm\mathbf{a}(p)\}$ converges to $\{\pm\mathbf{a}\}$. By a similar argument, we could prove that any point of $Y_{\mathbf{n}^-}(\tau)$ can be approximated by $Y_{\mathbf{n},p}(\tau)$ as $p \to 0$. This completes the proof. $\square$

6. The degree of the addition map

In previous sections, we have defined the addition map $\sigma_{\mathbf{n},p}(\cdot|\tau)$ and $\sigma_{\mathbf{n}}(\cdot|\tau)$ from $\overline{\Gamma_{\mathbf{n},p}(\tau)}$ and $\overline{\Gamma_{\mathbf{n}}(\tau)}$ onto E_τ respectively. Since $\overline{\Gamma_{\mathbf{n},p}(\tau)}$ and $\overline{\Gamma_{\mathbf{n}}(\tau)}$ are irreducible algebraic curves, it is an elementary fact that the degrees of $\sigma_{\mathbf{n},p}(\cdot|\tau)$ and $\sigma_{\mathbf{n}}(\cdot|\tau)$ are well-defined. The purpose of this section is to prove the following result.

Theorem 6.1. *Let* $\mathbf{n} = (n_0, n_1, n_2, n_3)$ *with* $n_k \in \mathbb{Z}_{\geq 0}$ *and* $p \in E_\tau \backslash E_\tau[2]$. *Then*

$$\deg \sigma_{\mathbf{n}} = \sum_{k=0}^{3} \frac{n_k(n_k+1)}{2}, \tag{6.1}$$

and

$$\deg \sigma_{\mathbf{n},p} = \sum_{k=0}^{3} n_k(n_k+1) + 1. \tag{6.2}$$

To compute the degree of the map $\sigma_{\mathbf{n},p}$, we consider $\wp(\sigma_{\mathbf{n},p}(A,W)|\tau)$. Since $\sigma_{\mathbf{n},p}(A,-W) = -\sigma_{\mathbf{n},p}(A,W)$, $\wp(\sigma_{\mathbf{n},p}(A,W)|\tau)$ depends on A only and is a meromorphic function of $A \in \mathbb{C}$. Together with the facts that (i) $\wp(\sigma_{\mathbf{n},p}(\cdot,W)|\tau)$ has finitely many poles A's and (ii) $\sigma_{\mathbf{n},p}(A,W) \to \pm[p]$ by Corollary 4.4 and so

$$\wp(\sigma_{\mathbf{n},p}(A,W)|\tau) \to \wp(p|\tau) \text{ as } A \to \infty \text{ for any } \tau \text{ and } p \notin E_\tau[2], \tag{6.3}$$

we conclude that there are *coprime polynomials* $P_j(A,p;\tau) \in \mathbb{C}[A]$, $j = 1,2$, such that

$$\wp(\sigma_{\mathbf{n},p}(A,W)|\tau) = \frac{P_1(A,p;\tau)}{P_2(A,p;\tau)}. \tag{6.4}$$

Indeed, a more delicate argument shows $P_j(A,p;\tau) \in \mathbb{Q}[e_k(\tau), \wp(p|\tau), \wp'(p|\tau)][A]$; see e.g. [18, Proposition 3.1]. Here $e_k(\tau) := \wp(\frac{\omega_k}{2}|\tau)$, $k = 1,2,3$. Write

$$P_1(A,p;\tau) = \sum_{k=0}^{m_1} a_k(p;\tau)A^k \ \text{ and } P_2(A,p;\tau) = \sum_{k=0}^{m_2} b_k(p;\tau)A^k,$$

where $a_k(p;\tau), b_k(p;\tau) \in \mathbb{Q}[e_k(\tau), \wp(p|\tau), \wp'(p|\tau)]$ with $a_{m_1}(p;\tau) \not\equiv 0$ and $b_{m_2}(p;\tau) \not\equiv 0$.

Lemma 6.2. *Let* $\tau \in \mathbb{H}$ *and* $p \notin E_\tau[2]$. *Then under the above notations, the followings hold.*

(1) $m_1 = m_2 =: m$ *and*

$$\frac{a_m(p;\tau)}{b_m(p;\tau)} = \wp(p|\tau). \tag{6.5}$$

(2) *If* $b_m(p;\tau)\neq 0$, *then*

$$\deg_A P_1(A,p;\tau)\leq \deg_A P_2(A,p;\tau)=\deg\sigma_{\mathbf{n},p}=m. \tag{6.6}$$

Furthermore,

$$\deg_A P_1(A,p;\tau)=\deg_A P_2(A,p;\tau)\ \Leftrightarrow\ \wp(p|\tau)\neq 0. \tag{6.7}$$

Proof. (1) Take any τ and $p\notin E_\tau[2]$ such that $\wp(p|\tau)\neq 0$, $a_{m_1}(p;\tau)\neq 0$ and $b_{m_2}(p;\tau)\neq 0$. Then it follows from (6.3) and (6.4) that $m_1=m_2$ (denote it by m) and (6.5) holds. Since both sides of (6.5) are meromorphic in (τ,p), we conclude that (6.5) holds for all τ and $p\notin E_\tau[2]$.

(2) Fix any τ and $p\notin E_\tau[2]$ such that $b_m(p;\tau)\neq 0$. Then $\deg_A P_2=m\geq \deg_A P_1$ and (6.7) follows from (6.5). It suffices to prove $\deg\sigma_{\mathbf{n},p}=m$. Denote $\tilde m:=\deg\sigma_{\mathbf{n},p}$. Take any $\sigma_0\notin E_\tau[2]\cup\{\pm[p]\}$ such that

$$\sigma_{\mathbf{n},p}^{-1}(\sigma_0)=\{(A_1,W_1),\cdots,(A_{\tilde m},W_{\tilde m})\} \tag{6.8}$$

consists of $\tilde m$ distinct elements in $\Gamma_{\mathbf{n},p}(\tau)$ and the polynomial

$$P_1(\cdot,p;\tau)-\wp(\sigma_0|\tau)P_2(\cdot,p;\tau) \text{ has only simple zeros.} \tag{6.9}$$

If $W_i=0$ for some i, then it follows from Theorem 3.6 that $\sigma_0=\sigma_{\mathbf{n},p}(A_i,W_i)\in E_\tau[2]$, a contradiction. Hence $W_i\neq 0$ for all i. If $A_i=A_j$ for some $i\neq j$, then $W_i=-W_j$. By using

$$\sigma_0=\sigma_{\mathbf{n},p}(A_j,W_j)=\sigma_{\mathbf{n},p}(A_i,-W_i)=-\sigma_{\mathbf{n},p}(A_i,W_i)=-\sigma_0 \text{ in } E_\tau,$$

we have $\sigma_0\in E_\tau[2]$, a contradiction again. Thus these A_i's are $\tilde m$ distinct roots of (6.9), namely

$$\deg_A(P_1(A,p;\tau)-\wp(\sigma_0|\tau)P_2(A,p;\tau))\geq \tilde m. \tag{6.10}$$

Assume by contradiction that $\deg_A(P_1-\wp(\sigma_0|\tau)P_2)>\tilde m$. By (6.9), the polynomial $P_1-\wp(\sigma_0|\tau)P_2$ has another zero $A_{\tilde m+1}\notin\{A_1,\cdots,A_{\tilde m}\}$. Then the points $(A_{\tilde m+1},\pm W_{\tilde m+1})\in\Gamma_{\mathbf{n},p}$ satisfy

$$\wp(\sigma_{\mathbf{n},p}(A_{\tilde m+1},\pm W_{\tilde m+1})|\tau)=\frac{P_1(A_{\tilde m+1},p;\tau)}{P_2(A_{\tilde m+1},p;\tau)}=\wp(\sigma_0|\tau),$$

i.e. either $\sigma_{\mathbf{n},p}(A_{\tilde m+1},W_{\tilde m+1})=\sigma_0$ or $\sigma_{\mathbf{n},p}(A_{\tilde m+1},-W_{\tilde m+1})=\sigma_0$, which is a contradiction with (6.8) and $A_{\tilde m+1}\notin\{A_1,\cdots,A_{\tilde m}\}$. This proves

$$\deg_A(P_1(A,p;\tau)-\wp(\sigma_0|\tau)P_2(A,p;\tau))=\tilde m$$

hold for almost $\sigma_0\notin E_\tau[2]\cup\{\pm[p]\}$, which implies

$$m=\max\{\deg_A P_1,\deg_A P_2\}=\tilde m=\deg\sigma_{\mathbf{n},p}.$$

The proof is complete. □

Lemma 6.3. *Let* $\tau_0\in\mathbb{H}$ *and* $p_0\notin E_{\tau_0}[2]$. *If at least one of* $\{a_m(p_0;\tau_0),\cdots,a_0(p_0;\tau_0),\ b_m(p_0;\tau_0),\ \cdots,\ b_0(p_0;\tau_0)\}$ *is not zero, then* $b_m(p_0;\tau_0)\neq 0$.

Proof. Assume by contradiction that $b_m(p_0;\tau_0)=0$. By our assumption and Lemma 6.2-(1), we can take $\sigma_0\notin E_{\tau_0}[2]\cup\{\pm[p_0]\}$ such that

$$P(A,p_0;\tau_0):=P_1(A,p_0;\tau_0)-\wp(\sigma_0|\tau_0)P_2(A,p_0;\tau_0)$$

is a nonzero polynomial with $\deg P(A,p_0;\tau_0)\le m-1$. Let $\tau_\ell,p_\ell\notin E_{\tau_\ell}[2]$ such that $(p_\ell,\tau_\ell)\to(p_0,\tau_0)$ as $\ell\to\infty$, $b_m(p_\ell;\tau_\ell)\neq 0$ and $\sigma_0\notin E_{\tau_\ell}[2]\cup\{\pm[p_\ell]\}$ for all ℓ. Then Lemma 6.2-(2) gives $\deg\sigma_{\mathbf{n},p_\ell}(\cdot|\tau_\ell)=m$, so

$$\sigma^{-1}_{\mathbf{n},p_\ell}(\sigma_0|\tau_\ell)=\{(A_{\ell,1},W_{\ell,1}),\cdots,(A_{\ell,m},W_{\ell,m})\}$$

and $A_{\ell,i}$, $i=1,\cdots,m$, are all the zeros of

$$P(A,p_\ell;\tau_\ell):=P_1(A,p_\ell;\tau_\ell)-\wp(\sigma_0|\tau_\ell)P_2(A,p_\ell;\tau_\ell).$$

Since $\deg P(A,p_0;\tau_0)\le m-1$, there is some i such that $A_{\ell,i}\to\infty$ as $\ell\to\infty$. Then Proposition 4.3 implies that the corresponding $(a_1(A_{\ell,i},p_\ell,\tau_\ell),\cdots,a_N(A_{\ell,i},p_\ell,\tau_\ell))$ of $(A_{\ell,i},W_{\ell,i})$ converges to $\infty_\pm(p_0)$, which yields

$$\sigma_0=\lim_{\ell\to\infty}\sum_{j=1}^{N}a_j(A_{\ell,i},p_\ell,\tau_\ell)-\sum_{k=1}^{3}\frac{n_k\omega_k}{2}=\pm p_0,$$

a contradiction to the choice of σ_0. This proves $b_m(p_0;\tau_0)\neq 0$. □

Lemma 6.4. *$\deg\sigma_{\mathbf{n},p}(\cdot|\tau)=m$ is independent of $\tau\in\mathbb{H}$ and $p\notin E_\tau[2]$.*

Proof. By Lemma 6.2-(2), it suffices to prove that $\deg\sigma_{\mathbf{n},p_0}(\cdot|\tau_0)=m$ for any τ_0, $p_0\notin E_{\tau_0}[2]$ satisfying $b_m(p_0;\tau_0)=0$.

Case 1. At least one of the coefficients of $P_i(A,p;\tau_0)$ is not identical 0 in p.

Then we may divide $P_i(A,p;\tau_0)$ by a common factor $(p-p_0)^\ell$ such that all the coefficients of the new $\tilde{P}_i(A,p;\tau_0):=P_i(A,p;\tau_0)/(p-p_0)^\ell$ are holomorphic at p_0 and at least one of them do not vanish at p_0. Then Lemma 6.3 implies $\tilde{b}_m(p_0;\tau_0)\neq 0$, where $\tilde{b}_m(p;\tau):=b_m(p;\tau)/(p-p_0)^\ell$, and so the same proof as Lemma 6.2-(2) shows that $\deg\sigma_{\mathbf{n},p_0}(\cdot|\tau_0)=m$.

Case 2. All the coefficients of $P_i(A,p;\tau_0)$ are identical 0 in p.

Then we may divide $P_i(A,p;\tau)$ by a factor $(\tau-\tau_0)^\ell$ such that Case 1 holds, which again implies that $\deg\sigma_{\mathbf{n},p_0}(\cdot|\tau_0)=m$. The proof is complete. □

For the addition map $\sigma_{\mathbf{n}}(\cdot|\tau)$ from $H(\mathbf{n},B,\tau)$, we have the similar result: There are coprime polynomials $\hat{P}_j(B;\tau)\in\mathbb{Q}[e_k(\tau)][B]$ such that

$$\wp(\sigma_{\mathbf{n}}(B,\hat{W})|\tau)=\frac{\hat{P}_1(B;\tau)}{\hat{P}_2(B;\tau)}. \tag{6.11}$$

Since (5.6) says $\sigma_{\mathbf{n}}(B,\hat{W})\to 0$ as $B\to\infty$, we have $\deg_B\hat{P}_1(B;\tau)>\deg_B\hat{P}_2(B;\tau)$. Takemura [22, Proposition 3.2] proved that

$$\deg_B\hat{P}_1(B;\tau)=\deg_B\hat{P}_2(B;\tau)+1 \text{ for any } \tau\in\mathbb{H}, \tag{6.12}$$

$$\wp(\sigma_{\mathbf{n}}(B,\hat{W})|\tau)=\frac{4B}{[\sum_{k=0}^{3}n_k(n_k+1)]^2}+O(1) \text{ as } B\to\infty \text{ for fixed } \tau. \tag{6.13}$$

Write

$$\hat{P}_1(B;\tau)=\sum_{k=0}^{\hat{m}_1}\hat{a}_k(\tau)B^k \quad\text{and}\quad \hat{P}_2(B;\tau)=\sum_{k=0}^{\hat{m}_2}\hat{b}_k(\tau)B^k,$$

where $\hat{a}_k(\tau),\hat{b}_k(\tau)\in\mathbb{Q}[e_k(\tau)]$ with $\hat{a}_{\hat{m}_1}(\tau)\not\equiv 0$ and $\hat{b}_{\hat{m}_2}(\tau)\not\equiv 0$. Then we have

Lemma 6.5. *Under the above notations, the followings hold.*

(1) $\hat{m}_2+1=\hat{m}_1=:\hat{m}$ *and*

$$\frac{\hat{a}_{\hat{m}}(\tau)}{\hat{b}_{\hat{m}-1}(\tau)}=\frac{4}{[\sum_{k=0}^{3}n_k(n_k+1)]^2}=:C(\mathbf{n}). \tag{6.14}$$

(2) $\deg\sigma_{\mathbf{n}}(\cdot|\tau)=\hat{m}$ *is independent of* $\tau\in\mathbb{H}$. *Furthermore,*

$$\deg_B\hat{P}_1(B;\tau)=\hat{m}=\deg\sigma_{\mathbf{n}}(\cdot|\tau)\ \textit{if}\ \hat{a}_{\hat{m}}(\tau)\neq 0.$$

Proof. The proof is similar to those of Lemmas 6.2-6.4 with minor modifications; we omit the details here. □

Recall Theorem 5.1 that

$$\lim_{p\to\frac{\omega_k}{2}}\overline{Y_{\mathbf{n},p}(\tau)}=\overline{Y_{\mathbf{n}_k^+}(\tau)}\cup\overline{Y_{\mathbf{n}_k^-}(\tau)}.$$

Thus it is reasonably expected that the degree of $\sigma_{\mathbf{n},p}$ should be the sum of the degree of $\sigma_{\mathbf{n}_k^+}$ and $\sigma_{\mathbf{n}_k^-}$. The following result confirms this conjecture, which plays a crucial role in the proof of Theorem 6.1.

Theorem 6.6. *Let* $\mathbf{n}=(n_0,n_1,n_2,n_3)$ *where* $n_k\in\mathbb{Z}_{\geq 0}$ *and* $p\in E_\tau\backslash E_\tau[2]$. *Then for each* $k=0,1,2,3$, *we have*

$$\deg\sigma_{\mathbf{n},p}=\deg\sigma_{\mathbf{n}_k^+}+\deg\sigma_{\mathbf{n}_k^-}.$$

Before giving the proof of Theorem 6.6, we would like to apply it to prove Theorem 6.1.

Proof of Theorem 6.1. It was proved by Wang and the third author [15] that

$$\deg\sigma_{(n_0,0,0,0)}=\frac{n_0(n_0+1)}{2},\quad n_0\in\mathbb{Z}_{\geq 0}. \tag{6.15}$$

Note that $(0,0,0,0)$ and $(-1,0,0,0)$ gives the same GLE, so (6.15) also holds for $n_0=-1$. Then by Theorem 6.6, we have

$$\begin{aligned}\deg\sigma_{(n_0,0,0,0),p}&=\deg\sigma_{(n_0+1,0,0,0)}+\deg\sigma_{(n_0-1,0,0,0)}\\&=\frac{(n_0+1)(n_0+2)}{2}+\frac{(n_0-1)n_0}{2}\\&=n_0(n_0+1)+1.\end{aligned} \tag{6.16}$$

Suppose that

$$\deg\sigma_{(n_0,k,0,0)}=\frac{n_0(n_0+1)}{2}+\frac{k(k+1)}{2}$$

for all $0 \leq k \leq n_1 - 1$. We claim that

$$\deg \sigma_{(n_0,n_1,0,0)} = \frac{n_0(n_0+1)}{2} + \frac{n_1(n_1+1)}{2}. \tag{6.17}$$

We note that the following identities hold

$$\begin{aligned}&\deg \sigma_{(n_0,n_1,0,0)} + \deg \sigma_{(n_0,n_1-2,0,0)} \\ &= \deg \sigma_{(n_0,n_1-1,0,0),p} \\ &= \deg \sigma_{(n_0-1,n_1-1,0,0)} + \deg \sigma_{(n_0+1,n_1-1,0,0)},\end{aligned}$$

where the first identity follows from Theorem 6.6 with $k = 1$ and the second one follows from the $k = 0$. Hence

$$\begin{aligned}&\deg \sigma_{(n_0,n_1,0,0)} \\ &= \deg \sigma_{(n_0-1,n_1-1,0,0)} + \deg \sigma_{(n_0+1,n_1-1,0,0)} - \deg \sigma_{(n_0,n_1-2,0,0)} \\ &= \frac{(n_0-1)n_0}{2} + \frac{(n_1-1)n_1}{2} + \frac{(n_0+1)(n_0+2)}{2} + \frac{(n_1-1)n_1}{2} \\ &\quad - \frac{n_0(n_0+1)}{2} - \frac{(n_1-2)(n_1-1)}{2} \\ &= \frac{n_0(n_0+1)}{2} + \frac{n_1(n_1+1)}{2}.\end{aligned} \tag{6.18}$$

This proves (6.17).

Now we claim that

$$\deg \sigma_{(n_0,n_1,0,0),p} = n_0(n_0+1) + n_1(n_1+1) + 1. \tag{6.19}$$

Indeed, a direct consequence of (6.17) and Theorem 6.6 imply that

$$\begin{aligned}&\deg \sigma_{(n_0,n_1,0,0),p} \\ &= \deg \sigma_{(n_0+1,n_1,0,0)} + \deg \sigma_{(n_0-1,n_1,0,0)} \\ &= \frac{(n_0+1)(n_0+2)}{2} + \frac{n_1(n_1+1)}{2} + \frac{(n_0-1)n_0}{2} + \frac{n_1(n_1+1)}{2} \\ &= n_0(n_0+1) + n_1(n_1+1) + 1.\end{aligned} \tag{6.20}$$

This proves (6.19).

By the same argument as (6.18) and (6.20), we could derive the formulas (6.1)-(6.2) for any $\mathbf{n} = (n_0, n_1, n_2, n_3)$ via induction. □

The rest of this section is devoted to the proof of Theorem 6.6 for $k = 0$ (the other cases $k \in \{1, 2, 3\}$ can be proved in an analogous way and we omit the details). Denote $\mathbf{n}_0^{\pm} = \mathbf{n}^{\pm}$ and

$$\deg \sigma_{\mathbf{n},p} = m, \;\; \deg \sigma_{\mathbf{n}^+} = m^+, \;\; \deg \sigma_{\mathbf{n}^-} = m^-.$$

Our goal is to prove $m = m^+ + m^-$. Recalling Lemmas 6.4 and 6.5, in the sequel we fix τ (so we will omit the notation τ freely) such that

$$\hat{a}_{m^{\pm}}(\tau) \neq 0 \text{ and } b_m(\cdot;\tau) \not\equiv 0.$$

Then $\wp(\sigma_{\mathbf{n}^{\pm}}(B,\hat{W}))$ (see (6.11) and (6.14)) can be rewritten as

$$\wp(\sigma_{\mathbf{n}^{\pm}}(B,\hat{W})) = C(\mathbf{n}^{\pm})\frac{\prod_{i=1}^{m^{\pm}}(B-\hat{B}_i^{0\pm})}{\prod_{i=1}^{m^{\pm}-1}(B-\hat{B}_i^{\infty\pm})}, \tag{6.21}$$

where $\hat{B}_i^{0\pm}$'s and $\hat{B}_i^{\infty\pm}$'s are all the zeros of $\hat{P}_1(B)$ and $\hat{P}_2(B)$ (corresponding to $\mathbf{n}^{\pm}$) respectively. On the other hand, since $b_m(\cdot;\tau)\not\equiv 0$, we take *any sequence* $p\to 0$ such that $b_m(p)\neq 0$ and $\wp(p)\neq 0$, then $\wp(\sigma_{\mathbf{n},p}(A,W))$ (see (6.4) and (6.5)) can be rewritten as

$$\wp(\sigma_{\mathbf{n},p}(A,W)) = \wp(p)\frac{\prod_{i=1}^{m}(A-A_i^0(p))}{\prod_{j=1}^{m}(A-A_j^{\infty}(p))}, \tag{6.22}$$

where $A_i^0(p)$'s and $A_j^{\infty}(p)$'s are all the zeros of $P_1(A,p)$ and $P_2(A,p)$ respectively.

To prove $m=m^{+}+m^{-}$, we need to study the asymptotics of $A_i^0(p)$'s and $A_j^{\infty}(p)$'s as $p\to 0$. In the sequel, *when we say as $p\to 0$, we often mean up to a subsequence of $p\to 0$ if necessary.*

By Corollary 5.6 and Proposition 5.2, $A_i^0(p)$ satisfies the asymptotic formula (5.8) as $p\to 0$. We denote $A_i^0(p)$ by $A_i^{0^+}(p)$ if

$$A_i^{0^+}(p) = -(\tfrac{1}{4}+n_0)p^{-1}+\alpha_i^{0^+}p+o(p) \text{ for some } \alpha_i^{0^+}\in\mathbb{C}, \tag{6.23}$$

and by $A_i^{0^-}(p)$ if

$$A_i^{0^-}(p) = (\tfrac{3}{4}+n_0)p^{-1}+\alpha_i^{0^-}p+o(p) \text{ for some } \alpha_i^{0^-}\in\mathbb{C}. \tag{6.24}$$

We assume that there are $A_1^{0^+}(p),\cdots,A_{m_1}^{0^+}(p)$ and $A_1^{0^-}(p),\cdots,A_{m_2}^{0^-}(p)$ (counted with multiplicity), so $m=m_1+m_2$. Let $B_i^{0^{\pm}}(p)$ be the coefficient of GLE($\mathbf{n}$,p, $A_i^{0^{\pm}}(p)$). Then by (5.10), we have

$$B_i^{0^{\pm}}(p)\to B_i^{0^{\pm}} \text{ as } p\to 0,$$

where

$$B_i^{0^+} := -(2n_0+1)\alpha_i^{0^+}-\sum_{k=1}^{3}n_k(n_k+1)e_k, \tag{6.25}$$

$$B_i^{0^-} := (2n_0+1)\alpha_i^{0^-}-\sum_{k=1}^{3}n_k(n_k+1)e_k. \tag{6.26}$$

Also by Proposition 5.2, GLE($\mathbf{n}$,p, $A_i^{0^{\pm}}(p)$) $\to$H($\mathbf{n}^{\pm}$, $B_i^{0^{\pm}}$) as $p\to 0$. Since (6.22) says $\wp(\sigma_{\mathbf{n},p}(A_i^{0^{\pm}}(p),W_i^{0^{\pm}}(p)))=0$ for each p, we have

$$\wp(\sigma_{\mathbf{n}^{\pm}}(B_i^{0^{\pm}},W_i^{0^{\pm}})) = \lim_{p\to 0}\wp\left(\sigma_{\mathbf{n},p}(A_i^{0^{\pm}}(p),W_i^{0^{\pm}}(p))\right)=0.$$

Recalling $\{\hat{B}_1^{0^+},\cdots,\hat{B}_{m^+}^{0^+}\}$ and $\{\hat{B}_1^{0^-},\cdots,\hat{B}_{m^-}^{0^-}\}$ in (6.21), our above argument yields the following corollary.

Corollary 6.7. *$B_i^{0^+} \in \{\hat{B}_1^{0^+}, \cdots, \hat{B}_{m^+}^{0^+}\}$ for each $i = 1, 2, \cdots, m_1$ and $B_j^{0^-} \in \{\hat{B}_1^{0^-}, \cdots, \hat{B}_{m^-}^{0^-}\}$ for each $j = 1, 2, \cdots, m_2$.*

Later we will prove $m_1 = m^+$ and $m_2 = m^-$. Indeed, our proof will imply $\{B_1^{0^+}, \cdots, B_{m_1}^{0^+}\} = \{\hat{B}_1^{0^+}, \cdots, \hat{B}_{m^+}^{0^+}\}$ and $\{B_1^{0^-}, \cdots, B_{m_2}^{0^-}\} = \{\hat{B}_1^{0^-}, \cdots, \hat{B}_{m^-}^{0^-}\}$. See (6.51) below.

Next we consider the asymptotics of $A_i^\infty(p)$'s as $p \to 0$. Since (6.22) says

$$\sigma_{\mathbf{n},p}(A_i^\infty(p), W_i^\infty(p)) = 0 \quad \text{for each } p, \tag{6.27}$$

it might happen that the corresponding GLE$(\mathbf{n}, p, A_i^\infty(p))$ does not converge as $p \to 0$. If GLE$(\mathbf{n}, p, A_i^\infty(p))$ converges, again $A_i^\infty(p)$ satisfies the asymptotic formula (5.8) as $p \to 0$, and we denote it by $A_i^{\infty^+}(p)$ if

$$A_i^{\infty^+}(p) = -(\tfrac{1}{4} + n_0)p^{-1} + \alpha_i^{\infty^+} p + o(p) \text{ for some } \alpha_i^{\infty^+} \in \mathbb{C}, \tag{6.28}$$

and by $A_i^{\infty^-}(p)$ if

$$A_i^{\infty^-}(p) = (\tfrac{3}{4} + n_0)p^{-1} + \alpha_i^{\infty^-} p + o(p) \text{ for some } \alpha_i^{\infty^-} \in \mathbb{C}. \tag{6.29}$$

If GLE$(\mathbf{n}, p, A_i^\infty(p))$ does not converge, then there are three alternatives of $A_i^\infty(p)$ up to a subsequence, according to Proposition 5.2:

(D-i) $A_i^\infty(p) = A_i^{\infty_1}(p)$ where

$$A_i^{\infty_1}(p) = -(\tfrac{1}{4} + n_0)p^{-1} + \alpha_i^{\infty_1} + o(1),\ \alpha_i^{\infty_1} \neq 0,$$

(D-ii) $A_i^\infty(p) = A_i^{\infty_2}(p)$ where

$$A_i^{\infty_2}(p) = (\tfrac{3}{4} + n_0)p^{-1} + \alpha_i^{\infty_2} + o(1),\ \alpha_i^{\infty_2} \neq 0,$$

(D-iii) $A_i^\infty(p) = A_i^{\infty_3}(p)$ such that

$$A_i^{\infty_3}(p)p \to \beta \in \mathbb{C} \cup \{\infty\} \backslash \{-(\tfrac{1}{4} + n_0), (\tfrac{3}{4} + n_0)\}.$$

The key step is to prove the following result.

Lemma 6.8. *Recall (6.27) that $\sigma_{\mathbf{n},p}^{-1}(0) = \{(A_i^\infty(p), W_i^\infty(p))\}_{i=1}^m$. Then there are $m_1 - 1$ of $\{GLE(\mathbf{n}, p, A_i^\infty(p))\}_{i=1}^m$ converge to $H(\mathbf{n}^+, B)$ for some B, and $m_2 - 1$ of them converge to $H(\mathbf{n}^-, B)$ for some B, and the rest two (say $A_{m-1}^\infty(p)$ and $A_m^\infty(p)$) do not converge. Moreover, $A_{m-1}^\infty(p)$ and $A_m^\infty(p)$ belong to the case **(D-iii)** with $A_i^\infty(p)p \to \beta_i \in \mathbb{C} \backslash \{-(\frac{1}{4} + n_0), (\frac{3}{4} + n_0)\}$, $i = m-1, m$ as $p \to 0$.*

Proof. **Step 1**. We prove that there are some i's such that GLE$(\mathbf{n}, p, A_i^\infty(p))$ does not converge as $p \to 0$.

Suppose GLE$(\mathbf{n}, p, A_i^\infty(p))$ are convergent for all i. Suppose k_1 of them converge to H$(\mathbf{n}^+, B)$ and k_2 of them converge to H$(\mathbf{n}^-, B)$. Then we rewrite (6.22) as

$$\wp\left(\sigma_{\mathbf{n},p}\left(A, W\right)\right) = \wp(p) \frac{\prod_{i=1}^{m_1}(A - A_i^{0^+}(p)) \cdot \prod_{i=1}^{m_2}(A - A_i^{0^-}(p))}{\prod_{i=1}^{k_1}\left(A - A_i^{\infty^+}(p)\right) \cdot \prod_{i=1}^{k_2}\left(A - A_i^{\infty^-}(p)\right)}. \tag{6.30}$$

Fix $\sigma_0 \in E_\tau \backslash E_\tau[2]$ such that $\wp(\sigma_0) \notin \{0, \infty\}$. The same proof of Theorem 5.1 implies that there exist $\sigma_p^{\pm} \in E_\tau \backslash E_\tau[2]$ and

$$(A^{\pm}(p), W^{\pm}(p)) \in \sigma_{\mathbf{n},p}^{-1}(\sigma_p^{\pm}) \text{ with } \lim_{p\to 0} \sigma_p^{\pm} = \sigma_0, \tag{6.31}$$

such that $\mathrm{GLE}(\mathbf{n}, p, A^{\pm}(p))$ converges to $\mathrm{H}(\mathbf{n}^{\pm}, B^{\pm})$ for some $B^{\pm}$, namely

$$A^{+}(p) = -\left(\tfrac{1}{4} + n_0\right) p^{-1} + \alpha^{+} p + o(p) \text{ for some } \alpha^{+} \in \mathbb{C}, \tag{6.32}$$

$$A^{-}(p) = \left(\tfrac{3}{4} + n_0\right) p^{-1} + \alpha^{-} p + o(p) \text{ for some } \alpha^{-} \in \mathbb{C}. \tag{6.33}$$

Recalling (6.23)-(6.24) and (6.28)-(6.29), we remark that

$$\alpha^{+} \notin \{\alpha_i^{0^+}\}_i \cup \{\alpha_j^{\infty^+}\}_j, \quad \alpha^{-} \notin \{\alpha_i^{0^-}\}_i \cup \{\alpha_j^{\infty^-}\}_j, \tag{6.34}$$

namely $A^{\pm}(p) - A_i^{0^\pm}(p) = (\alpha^{\pm} - \alpha_i^{0^\pm})p + o(p)$ with $\alpha^{\pm} - \alpha_i^{0^\pm} \neq 0$ for any i and so do $A^{\pm}(p) - A_i^{\infty^\pm}(p)$. For example, if $\alpha^{+} = \alpha_i^{0^+}$ for some i, then $\mathrm{GLE}(\mathbf{n}, p, A_i^{0^+}(p))$ also converges to $\mathrm{H}(\mathbf{n}^{+}, B^{+})$, the same limit as that of $\mathrm{GLE}(\mathbf{n}, p, A^{+}(p))$. This yields from (6.31) that

$$\begin{aligned} 0 &= \wp\left(\sigma_{\mathbf{n},p}\left(A_i^{0^+}(p), W_i^{0^+}(p)\right)\right) = \wp\left(\sigma_{\mathbf{n}^+}\left(B^{+}, W^{+}\right)\right) \\ &= \lim_{p\to 0} \wp\left(\sigma_{\mathbf{n},p}\left(A^{+}(p), W^{+}(p)\right)\right) = \lim_{p\to 0} \wp(\sigma_p^{+}) = \wp(\sigma_0) \notin \{0, \infty\}, \end{aligned}$$

a contradiction.

Now by inserting $A = A^{+}(p)$ into (6.30), we easily see from (6.23)-(6.24), (6.28)-(6.29) and (6.32) that

$$\wp(\sigma_0) + o(1) = \wp(\sigma_p^{+}) = \frac{p^{m_1 - m_2 - 2} \cdot \mathcal{O}(1)}{p^{k_1 - k_2} \cdot \mathcal{O}(1)} \text{ as } p \to 0. \tag{6.35}$$

Here different from the notation $O(1)$, we use the notation $\mathcal{O}(1)$ to denote various quantities depending on p which is uniformly bounded away from 0 and ∞ as $p \to 0$. However, inserting $A = A^{-}(p)$ into (6.30) leads to

$$\wp(\sigma_0) + o(1) = \wp(\sigma_p^{-}) = \frac{p^{m_2 - m_1 - 2} \cdot \mathcal{O}(1)}{p^{k_2 - k_1} \cdot \mathcal{O}(1)}, \tag{6.36}$$

which contradicts with (6.35). This proves Step 1, namely there must exist

$$A_i^{\infty}(p) \in \mathbf{A}^{\infty} := \{A_1^{\infty}(p), \ \cdots \ , A_m^{\infty}(p)\}$$

such that $\mathrm{GLE}(\mathbf{n}, p, A_i^{\infty}(p))$ does not converge. Suppose there are ℓ_1, ℓ_2 and ℓ_3 $A_i^{\infty}(p)$'s satisfying (D-i), (D-ii) and (D-iii) respectively. Then we rewrite (6.22) as

$$\wp\left(\sigma_{\mathbf{n},p}\left(A, W\right)\right) = \frac{\wp(p) \prod_{i=1}^{m_1}(A - A_i^{0^+}(p)) \cdot \prod_{i=1}^{m_2}(A - A_i^{0^-}(p))}{\prod_{i=1}^{k_1}(A - A_i^{\infty^+}(p)) \prod_{i=1}^{k_2}(A - A_i^{\infty^-}(p)) \prod_{j=1}^{3} \prod_{i=1}^{\ell_j}(A - A_i^{\infty_j}(p))}. \tag{6.37}$$

Step 2. We prove that $(\ell_1, \ell_2, \ell_3) \in \{(0,0,2), (0,2,1), (2,0,1)\}$.

By the asymptotics (6.32)-(6.33) and (D-i)-(D-iii), we have

$$\begin{cases} A^+(p) - A_i^{\infty_1}(p) = -\alpha_i^{\infty_1} + o(1),\ \alpha_i^{\infty_1} \neq 0, \\ A^+(p) - A_i^{\infty_2}(p) = -(1+2n_0)p^{-1} + O(1), \end{cases} \tag{6.38}$$

$$\begin{cases} A^-(p) - A_i^{\infty_1}(p) = (1+2n_0)p^{-1} + O(1), \\ A^+(p) - A_i^{\infty_2}(p) = -\alpha_i^{\infty_2} + o(1),\ \alpha_i^{\infty_2} \neq 0, \end{cases} \tag{6.39}$$

and

$$\prod_{i=1}^{\ell_3} \frac{A^+(p) - A_i^{\infty_3}(p)}{A^-(p) - A_i^{\infty_3}(p)} = \mathcal{O}(1). \tag{6.40}$$

Again by inserting $A = A^\pm(p)$ into (6.37) and using (6.31), we have the following identity

$$\begin{aligned} \wp(\sigma_0) &= \frac{p^{m_1-m_2-2}\cdot\mathcal{O}(1)}{p^{k_1-k_2-\ell_2}\prod_{i=1}^{\ell_3}(A^+(p) - A_i^{\infty_3}(p))} + o(1) \\ &= \frac{p^{m_2-m_1-2}\cdot\mathcal{O}(1)}{p^{k_2-k_1-\ell_1}\prod_{i=1}^{\ell_3}(A^-(p) - A_i^{\infty_3}(p))} + o(1). \end{aligned} \tag{6.41}$$

Together with (6.40), we obtain

$$2\left(m_1 - m_2\right) = 2\left(k_1 - k_2\right) + \left(\ell_1 - \ell_2\right), \tag{6.42}$$

which implies $\ell_1 - \ell_2$ is even. Since $\wp(\sigma_0) \neq 0$, (6.41) and $|A^+(p) - A_i^{\infty_3}(p)| \geq |p|^{-1}\cdot\mathcal{O}(1)$ also yield

$$\begin{aligned} |p|^{-\ell_3} &\leq \mathcal{O}(1)\cdot\prod_{i=1}^{\ell_3}|A^+(p) - A_i^{\infty_3}(p)| \\ &= \mathcal{O}(1)\cdot|p|^{m_1-m_2-2-(k_1-k_2-\ell_2)} = \mathcal{O}(1)\cdot|p|^{\frac{\ell_1+\ell_2}{2}-2}, \end{aligned} \tag{6.43}$$

which implies $\frac{\ell_1+\ell_2}{2} - 2 \leq -\ell_3$, namely

$$0 \leq \ell_1 + \ell_2 + 2\ell_3 \leq 4. \tag{6.44}$$

From here and that $\ell_1 - \ell_2$ is even, we see that

$$\begin{aligned} (\ell_1, \ell_2, \ell_3) = {} & (0,0,1), (0,0,2), (0,2,0), (0,2,1), \\ & (2,0,0), (2,0,1), (1,1,0), (1,1,1), (1,3,0), (3,1,0). \end{aligned}$$

On the other hand, we have

$$m_1 + m_2 = m = k_1 + k_2 + \ell_1 + \ell_2 + \ell_3. \tag{6.45}$$

From here and (6.42), we can only have

$$(\ell_1, \ell_2, \ell_3) = (0,0,2), (0,2,1), (2,0,1), (1,1,0).$$

The case $(\ell_1, \ell_2, \ell_3) = (1,1,0)$ is impossible by (6.41). This proves Step 2.

Step 3. We prove that $(\ell_1, \ell_2, \ell_3) \neq (0,2,1), (2,0,1)$ and so $(\ell_1, \ell_2, \ell_3) = (0,0,2)$.

Suppose $(\ell_1,\ell_2,\ell_3)=(0,2,1)$. Then it follows from (6.42) and (6.45) that $k_1=m_1-1$, $k_2=m_2-2$. Inserting these into (6.41) leads to $A^{\pm}(p)-A_1^{\infty_3}(p)=\mathcal{O}(1)\cdot p^{-1}$, so

$$A_1^{\infty_3}(p)p\to\beta_1\in\mathbb{C}\backslash\{-(\tfrac{1}{4}+n_0),\tfrac{3}{4}+n_0\}. \tag{6.46}$$

To get a contradiction, we take any B and consider $A(p)$ such that $\mathrm{GLE}(\mathbf{n},p,A(p))\to\mathrm{H}(\mathbf{n}^-,B)$ as $p\to 0$. The existence of such $A(p)$ is proved in Theorem 5.1, which has the following asymptotics

$$A(p)=(\tfrac{3}{4}+n_0)p^{-1}+\alpha_1^-p+o(p) \tag{6.47}$$

where

$$\alpha_1^-:=\frac{B+\sum_{k=1}^{3}n_k(n_k+1)e_k}{2n_0+1}. \tag{6.48}$$

Furthermore, the proof of Theorem 5.1 also implies $\wp(\sigma_{\mathbf{n},p}(A(p),W(p)))\to\wp(\sigma_{\mathbf{n}^-}(B,\hat{W}))$. By (6.37) and $(\ell_1,\ell_2,\ell_3)=(0,2,1)$ we have

$$\wp(\sigma_{\mathbf{n},p}(A(p),W(p)))=$$
$$\frac{\wp(p)\prod_{i=1}^{m_1}(A(p)-A_i^{0^+}(p))\prod_{i=1}^{m_2}(A(p)-A_i^{0^-}(p))(A(p)-A_1^{\infty_3}(p))^{-1}}{\prod_{i=1}^{m_1-1}(A(p)-A_i^{\infty^+}(p))\prod_{i=1}^{m_2-2}(A(p)-A_i^{\infty^-}(p))\prod_{i=1}^{2}(A(p)-A_i^{\infty_2}(p))}.$$

Inserting the asymptotics of $A(p)$, (6.23)-(6.24), (6.28)-(6.29), (D-2) and (6.46) to the above formula and letting $p\to 0$, we easily obtain

$$\wp(\sigma_{\mathbf{n}^-}(B,\hat{W}))=\frac{\prod_{i=1}^{m_2}(B-B_i^{0^-})}{(1+2n_0)(\frac{3}{4}+n_0-\beta_1)\alpha_1^{\infty_2}\alpha_2^{\infty_2}\prod_{i=1}^{m_2-2}(B-B_i^{\infty^-})}, \tag{6.49}$$

where $B_i^{0^-}=\lim_{p\to 0}B_i^{0^-}(p)$ is given in (6.26) and similarly for $B_i^{\infty^-}=\lim_{p\to 0}B_i^{\infty^-}(p)$. However, (6.49) contradicts with (6.11) and (6.12). This proves $(\ell_1,\ell_2,\ell_3)\neq(0,2,1)$. Similarly we can prove $(\ell_1,\ell_2,\ell_3)\neq(2,0,1)$ and so $(\ell_1,\ell_2,\ell_3)=(0,0,2)$.

Step 4. We complete the proof.

Since $(\ell_1,\ell_2,\ell_3)=(0,0,2)$, then (6.42) and (6.45) imply $k_1=m_1-1$ and $k_2=m_2-1$. Inserting these into (6.41) leads to $\prod_{i=1}^{2}(A^{\pm}(p)-A_i^{\infty_3}(p))=\mathcal{O}(1)\cdot p^{-2}$ and so $A_i^{\infty_3}(p)p\to\beta_i\in\mathbb{C}\backslash\{-(\frac{1}{4}+n_0),(\frac{3}{4}+n_0)\}$. The proof is complete. □

We are in the position to prove Theorem 6.6.

Proof of Theorem 6.6. We only need to prove this theorem for the case $k=0$. For the other cases $k\in\{1,2,3\}$, since $p\to\frac{\omega_k}{2}$ is equivalent to $p-\frac{\omega_k}{2}\to 0$, the desired assertion follows from the case $p\to 0$ by changing variable $z\to z-\frac{\omega_k}{2}$ in $\mathrm{GLE}(\mathbf{n},p,A)$ and $\mathrm{H}(\mathbf{n}^{\pm},B)$.

Thus we consider $k=0$. As pointed out before, we only need to prove $m^+ = m_1$ and $m^- = m_2$. Since by Lemma 6.8, we can rewrite (6.22) as

$$\begin{aligned}(6.50)\quad &\wp(\sigma_{\mathbf{n},p}(A,W))\\ =&\frac{\wp(p)\prod_{i=1}^{m_1}(A-A_i^{0^+}(p))\cdot\prod_{i=1}^{m_2}(A-A_i^{0^-}(p))}{\prod_{i=1}^{m_1-1}(A-A_i^{\infty^+}(p))\prod_{i=1}^{m_2-1}(A-A_i^{\infty^-}(p))\prod_{i=m-1}^{m}(A-A_i^{\infty}(p))},\end{aligned}$$

and then repeat the argument of Step 3 in Lemma 6.8: Take any B and consider $A(p)$ satisfying (6.47)-(6.48) such that $\mathrm{GLE}(\mathbf{n},p,A(p))\to \mathrm{H}(\mathbf{n}^-,B)$ as $p\to 0$. The same proof as that of Theorem 5.1 implies $\wp(\sigma_{\mathbf{n},p}(A(p),W(p)))\to \wp(\sigma_{\mathbf{n}^-}(B,\hat{W}))$. Then by inserting the asymptotics (6.47)-(6.48) of $A(p)$, (6.23)-(6.24), (6.28)-(6.29), and **(D-iii)** in Lemma 6.8 to the formula (6.50), we easily obtain for $p\to 0$ that

$$\wp(\sigma_{\mathbf{n}^-}(B,\hat{W}))=\hat{C}(\mathbf{n}^-)\frac{\prod_{i=1}^{m_2}(B-B_i^{0^-})}{\prod_{i=1}^{m_2-1}(B-B_i^{\infty^-})},\tag{6.51}$$

where $\hat{C}(\mathbf{n}^-)$ is a nonzero constant. Note that (6.51) holds for all B. Comparing (6.51) with (6.21), we conclude $m_2=m^-=\deg\sigma_{\mathbf{n}^-}$. Similarly, by fixing any B and considering $A(p)$ satisfying

$$A(p)=-(\tfrac{1}{4}+n_0)p^{-1}+\alpha_1^+p+o(p)$$

where

$$\alpha_1^+:=-\frac{B+\sum_{k=1}^{3}n_k(n_k+1)e_k}{2n_0+1},$$

such that $\mathrm{GLE}(\mathbf{n},p,A(p))\to \mathrm{H}(\mathbf{n}^+,B)$ as $p\to 0$, we can prove

$$\wp(\sigma_{\mathbf{n}^+}(B,\hat{W}))=\hat{C}(\mathbf{n}^+)\frac{\prod_{i=1}^{m_1}(B-B_i^{0^+})}{\prod_{i=1}^{m_1-1}(B-B_i^{\infty^+})},\quad \hat{C}(\mathbf{n}^+)\neq 0,$$

so $m_1=m^+=\deg\sigma_{\mathbf{n}^+}$. In conclusion,

$$\deg\sigma_{\mathbf{n}^+}+\deg\sigma_{\mathbf{n}^-}=m_1+m_2=\deg\sigma_{\mathbf{n},p}.$$

This completes the proof. □

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