SHARP NONEXISTENCE RESULTS FOR CURVATURE EQUATIONS WITH FOUR SINGULAR SOURCES ON RECTANGULAR TORI

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ABSTRACT. In this paper, we prove that there are no solutions for the curvature equation

$$\Delta u + e^u = 8\pi n \delta_0 \text{ on } E_\tau, \quad n \in \mathbb{N},$$

where E_τ is a flat rectangular torus and δ_0 is the Dirac measure at the lattice points. This confirms a conjecture in [18] and also improves a result of Eremenko and Gabrielov [11]. The nonexistence is a delicate problem because the equation always has solutions if $8\pi n$ in the RHS is replaced by $2\pi\rho$ with $0 < \rho \notin 4\mathbb{N}$. Geometrically, our result implies that a rectangular torus E_τ admits a metric with curvature $+1$ acquiring a conic singularity at the lattice points with angle $2\pi\alpha$ if and only if α is not an odd integer.

Unexpectedly, our proof of the nonexistence result is to apply the spectral theory of finite-gap potential, or equivalently the algebro-geometric solutions of stationary KdV hierarchy equations. Indeed, our proof can also yield a sharp nonexistence result for the curvature equation with singular sources at three half periods and the lattice points.

1. INTRODUCTION

Throughout the paper, we use the notations $\omega_0 = 0$, $\omega_1 = 1$, $\omega_2 = \tau$, $\omega_3 = 1 + \tau$ and $\Lambda_\tau = \mathbb{Z} + \mathbb{Z}\tau$, where $\tau \in \mathbb{H} = \{\tau | \operatorname{Im} \tau > 0\}$. Define $E_\tau := \mathbb{C}/\Lambda_\tau$ to be a flat torus in the plane and $E_\tau[2] := \{\frac{\omega_k}{2} | 0 \leq k \leq 3\} + \Lambda_\tau$ to be the set consisting of the lattice points and 2-torsion points in E_τ. Consider the following curvature equation with *four singular sources*:

$$\Delta u + e^u = 8\pi \sum_{k=0}^{3} n_k \delta_{\frac{\omega_k}{2}} \quad \text{on } E_\tau, \tag{1.1}$$

where $\delta_{\omega_k/2}$ is the Dirac measure at $\frac{\omega_k}{2}$, and $n_k \in \mathbb{Z}_{\geq 0}$ for all k with $\sum n_k \geq 1$. By changing variable $z \mapsto z + \frac{\omega_k}{2}$, we can always assume $n_0 = \max_k n_k \geq 1$.

Not surprisingly, (1.1) is related to various research areas. In conformal geometry, a solution u to (1.1) leads to a metric $ds^2 = \frac{1}{2}e^u(dx^2 + dy^2)$ with constant Gaussian curvature $+1$ acquiring *conic singularities at* $\frac{\omega_k}{2}$*'s*. It also appears in statistical physics as the equation for the *mean field limit* of the Euler flow in Onsager's vortex model (cf. [1]), hence also called a *mean field equation*. Recently equation (1.1) was shown to be related to the self-dual condensates of the Chern-Simons-Higgs equation in superconductivity. We refer the readers to [3, 10, 11, 20, 22] and references therein for recent developments of related subjects of equation (1.1).

The existence of solutions of equation (1.1) is very challenging from the PDE point of view. In fact, *the solvability of (1.1) essentially depends on the moduli τ in a sophisticated manner*. This phenomena was first discovered by Wang and the second author [17] when they studied the case $n_0 = 1$ and $n_1 = n_2 = n_3 = 0$, i.e.

$$\Delta u + e^u = 8\pi\delta_0 \text{ on } E_\tau. \tag{1.2}$$

For example, they proved that when $\tau \in i\mathbb{R}_{>0}$ (i.e. E_τ is a rectangular torus), equation (1.2) has *no* solution; while for $\tau = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ (i.e. E_τ is a rhombus torus), equation (1.2) has solutions. Later, equation (1.2) was thoroughly investigated in [7, 19].

For the case $n_0 = n \geq 2$ and $n_1 = n_2 = n_3 = 0$, i.e.

$$\Delta u + e^u = 8n\pi\delta_0 \text{ on } E_\tau, \tag{1.3}$$

Chai, Lin, Wang [2] and subsequently Lin, Wang [18] studied it from the viewpoint of algebraic geometry. They developed a theory to connect this PDE problem with hyperelliptic curves and modular forms. Among other things, they proposed the following conjecture.

Conjecture. [16, 18] *When $\tau \in i\mathbb{R}_{>0}$, i.e. E_τ is a rectangular torus, equation (1.3) has no solutions for any $n \geq 2$.*

Geometrically, this conjecture is equivalent to assert that a rectangular torus admits *no* metric with constant curvature $+1$ and a conical singularity with angle $2\pi(1+2n)$, $n \in \mathbb{N}$. Recently in [5], we proved this conjecture for $n = 2$. However, this proof can not work for general $n \geq 3$. One main purpose of this paper is to resolve this conjecture via a completely different idea.

Theorem 1.1. *If $\tau \in i\mathbb{R}_{>0}$, i.e. E_τ is a rectangular torus, then equation (1.3) has no solutions for any $n \geq 1$.*

There are two important consequences of Theorem 1.1. One is related to the pre-modular form $Z_{r,s}^{(n)}(\tau)$ associated with the Lamé equation introduced by [18]. See Section 6. The other is related to the following conjecture.

Conjecture 1.2. *Suppose $\tau \in i\mathbb{R}_{>0}$ and $\rho \in (8\pi(n-1), 8\pi n)$, $n \in \mathbb{N}$. Then the equation*

$$\Delta u + e^u = \rho\delta_0 \quad \text{on } E_\tau$$

possesses exactly n solutions.

Conjecture 1.2 was already proved for $\rho \in (0, 8\pi)$ in [19] and for $\rho = 8\pi(n - \frac{1}{2})$ in [2]. In [9], we will apply Theorem 1.1 to prove Conjecture 1.2 for $|\rho - 8\pi n| \ll 1$ and $|\rho - 8\pi(n-1)| \ll 1$.

In fact, our proof of Theorem 1.1 also works for equation (1.1) with more general n_k's. Our first main result of this paper is the following sharp nonexistence result.

Theorem 1.3. *Let $n_k \in \mathbb{Z}_{\geq 0}$ for all k with $\max_k n_k \geq 1$. If (n_0, n_1, n_2, n_3) satisfies neither*

$$\frac{n_1 + n_2 - n_0 - n_3}{2} \geq 1, \quad n_1 \geq 1, \quad n_2 \geq 1 \tag{1.4}$$

nor

$$\frac{n_1 + n_2 - n_0 - n_3}{2} \leq -1, \quad n_0 \geq 1, \quad n_3 \geq 1, \tag{1.5}$$

then for any $\tau \in i\mathbb{R}_{>0}$, equation (1.1) on E_τ has no even solutions.

Remark 1.4. It was proved in [2] that once (1.3) has a solution, then it has also an even solution. Thus Theorem 1.1 follows directly from Theorem 1.3.

It follows from [11] that our condition on n_k in Theorem 1.3 is *sharp*. In fact, recently Eremenko and Gabrielov [11] studied (1.1) from the viewpoint of geometry, i.e. by studying a related problem concerning spherical quadrilaterals. Among other things, they prove the following result.

Theorem A. [11, Theorem 1.3] *Let $n_k \in \mathbb{Z}_{\geq 0}$ for all k with $\max_k n_k \geq 1$. Then equation (1.1) has an even and symmetric solution $u(z)$, i.e.*

$$u(z) = u(-z) \quad \text{and} \quad u(z) = u(\bar{z}), \tag{1.6}$$

on some rectangular torus E_τ (i.e. for some $\tau \in i\mathbb{R}_{>0}$) if and only if (n_0, n_1, n_2, n_3) satisfies either (1.4) or (1.5).

Theorem A indicates that our condition on n_k in Theorem 1.3 is *sharp*. Furthermore, Theorem 1.3 improves Theorem A because we remove the assumption of symmetry $u(z) = u(\bar{z})$. We emphasize that this improvement is not trivial at all, because our numerical computation shows that there exist $1 < b_1 < b_2 < \sqrt{3}$ such that for any $\tau = ib$ with $b \in (b_1, b_2)$,

$$\Delta u + e^u = 16\pi\delta_0 + 16\pi\delta_{\omega_3/2} \text{ on } E_\tau$$

has *no even and symmetric solutions* but does have *two even solutions*. In view of Theorem 1.1, we suspect that the even assumption $u(z) = u(-z)$ is not necessary either, namely we propose the following conjecture.

Conjecture 1.5. *Equation (1.1) has no solution for any $\tau \in i\mathbb{R}_{>0}$ if and only if (n_0, n_1, n_2, n_3) satisfies neither (1.4) nor (1.5).*

Differently from Eremenko and Gabrielov's geometric approach [11], we prove Theorem 1.3 from the viewpoint of the integrable system in the sense that any solution u can be expressed as some holomorphic data, by which we will connect the curvature equation (1.1) to the following generalized Lamé equation (GLE, a second order linear ODE)

$$y''(z) = I(z)y(z) = \left[\sum_{k=0}^{3} n_k(n_k+1)\wp\left(z + \tfrac{\omega_k}{2}; \tau\right) + E\right] y(z). \tag{1.7}$$

See Section 2 for details. Here $\wp(z) = \wp(z;\tau)$ is the Weierstrass elliptic function with periods $\omega_1 = 1$ and $\omega_2 = \tau$, defined by

$$\wp(z;\tau) := \frac{1}{z^2} + \sum_{\omega\in\Lambda_\tau\setminus\{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right). \tag{1.8}$$

Note that GLE (1.7) becomes the classical Lamé equation when three n_k's vanish, such as $n_1 = n_2 = n_3 = 0$. GLE (1.7) is the elliptic form of Heun's equation and the potential

$$q^{(n_0,n_1,n_2,n_3)}(z) := -\sum_{k=0}^{3} n_k(n_k+1)\wp\left(z + \tfrac{\omega_k}{2};\tau\right) \tag{1.9}$$

is the so-called *Treibich-Verdier potential* ([30]), which is known as an algebro-geometric finite-gap potential associated with the stationary KdV hierarchy. We refer the readers to [14, 24, 25, 26, 27, 28, 30] and references therein for historical reviews and subsequent developments.

In Section 2, we will prove that *once (1.1) has an even solution, then there exists $E \in \mathbb{C}$ such that the monodromy group of GLE (1.7) is conjugate to a subgroup of $SU(2)$, i.e. the monodromy of GLE (1.7) is unitary*. Therefore, to prove Theorem 1.3, it suffices for us to prove the following result, which is also interesting from the viewpoint of the monodromy theory of linear ODEs.

Theorem 1.6. *Let $n_k \in \mathbb{Z}_{\geq 0}$ for all k with $\max_k n_k \geq 1$ and $\tau \in i\mathbb{R}_{>0}$. If (n_0, n_1, n_2, n_3) satisfies neither (1.4) nor (1.5), then the monodromy of GLE (1.7) can not be unitary for any $E \in \mathbb{C}$.*

It is well known (cf. [14, 24]) that a *spectral polynomial* $Q^{(n_0,n_1,n_2,n_3)}(E;\tau)$ is associated for the Treibich-Verdier potential (1.9); see Section 3 for a brief review. In this paper, as applications of Theorem 1.3 and Theorem A, we have the following surprising result.

Theorem 1.7 (=Corollary 4.5)**.** *Let $n_k \in \mathbb{Z}_{\geq 0}$ for all k with $\max_k n_k \geq 1$. Then all the zeros of $Q^{(n_0,n_1,n_2,n_3)}(\cdot;\tau)$ are real and distinct for each $\tau \in i\mathbb{R}_{>0}$ if and only if (n_0, n_1, n_2, n_3) satisfies neither (1.4) nor (1.5).*

This paper is organized as follows. In Section 2, we establish the connection between the curvature equation (1.1) and GLE (1.7). Theorem 1.6 will be proved by applying the spectral theory of Hill's equations with complex-valued potentials [13]; see Section 3, where we will prove a more general result (see Theorem 3.4) which contains Theorem 1.6 as a special case. In Section 4, we apply Theorem 3.4 to prove a more general result for equation (1.1) (see Theorem 4.1) which contains Theorem 1.3 as a special case. In Section 5, we prove the sufficient part of Theorem 1.7. The necessary part will be a consequence of our results as explained in Section 4. In Section 6, we will apply Theorem 1.1 to study pre-modular forms introduced in [18]. In Appendix A, we briefly review the spectral theory of Hill's equation from [13] that are needed in Section 3.

2. FROM PDE TO ODE WITH UNITARY MONODROMY

The purpose of this section is to establish the connection between the curvature equation (1.1) and GLE (1.7) from the viewpoint of the integrable system. See [2] for a complete discussion for the special case $n_1 = n_2 = n_3 = 0$, i.e. equation (1.3). Our argument of proving the unitary monodromy is different from [2] and works for the general case

$$\Delta u + e^u = 4\pi \sum_{k=1}^{m} \alpha_k \delta_{a_k} \text{ on } E_\tau, \tag{2.1}$$

where a_k's are m different points on E_τ, $\alpha_k > -1$ for all k and $\sum \alpha_k > 0$.

The Liouville theorem says that for any solution $u(z)$ of (2.1), there is a local meromorphic function $f(z)$ away from $\{a_k\}$'s such that

$$u(z) = \log \frac{8|f'(z)|^2}{(1+|f(z)|^2)^2}. \tag{2.2}$$

We remark that the classical Liouville theorem holds only for the case when the domain is simply connected and the equation does not include any singularity. For our present case, see [2] for a proof.

This $f(z)$ is called a developing map. By differentiating (2.2), we have

$$u_{zz} - \frac{1}{2}u_z^2 = \{f; z\} := \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2. \tag{2.3}$$

Conventionally, the RHS of this identity is called the Schwarzian derivative of $f(z)$, denoted by $\{f; z\}$. Note that outside the singularities $\{a_k \,|\, 1 \le k \le m\} + \Lambda_\tau$,

$$\left(u_{zz} - \tfrac{1}{2}u_z^2\right)_{\bar{z}} = \left(u_{z\bar{z}}\right)_z - u_z u_{z\bar{z}} = -\tfrac{1}{4}\left(e^u\right)_z + \tfrac{1}{4}e^u u_z = 0.$$

Furthermore, using the local behavior of $u(z)$ at a_k: $u(z) = 2\alpha_k \ln|z - a_k| + O(1)$, we see that $u_{zz} - \frac{1}{2}u_z^2$ has at most double poles at each a_k. In conclusion, $u_{zz} - \frac{1}{2}u_z^2$ is an *elliptic function* with at most *double poles* at $\{a_k \,|\, 1 \le k \le m\} + \Lambda_\tau$. Denote

$$I(z) := -\tfrac{1}{2}(u_{zz} - \tfrac{1}{2}u_z^2), \tag{2.4}$$

and consider the Fuchsian type linear differential equation

$$y''(z) = I(z)y(z). \tag{2.5}$$

Since $\{f; z\} = -2I(z)$, a classical result says that there exist linearly independent solutions $y_1(z), y_2(z)$ such that

$$f(z) = \frac{y_1(z)}{y_2(z)}. \tag{2.6}$$

On the other hand, the monodromy representation of (2.5) is a group homomorphism $\rho(\cdot) : \pi_1(E_\tau \setminus (\{a_k \,|\, 1 \le k \le m\} + \Lambda_\tau)) \to SL(2, \mathbb{C})$. In general, the monodromy of such linear differential equation could be very complicate. The main result of this section is the following.

Theorem 2.1. *If equation (2.5) comes from a solution $u(z)$ of (2.1) via (2.4), then the monodromy group with respect to the linearly independent solutions $(y_1(z), y_2(z))$ is contained in $SU(2)$, i.e. the monodromy is unitary.*

Proof. Define the Wronskian

$$W = y_1'(z)y_2(z) - y_1(z)y_2'(z).$$

Then W is a nonzero constant. By inserting (2.6) into (2.2), a direct computation leads to

$$2\sqrt{2}We^{-\frac{1}{2}u(z)} = |y_1(z)|^2 + |y_2(z)|^2.$$

Since $u(z)$ is single-valued and doubly periodic, we immediately see that the monodromy group with respect to $(y_1(z), y_2(z))$ is contained in $SU(2)$, namely the monodromy is unitary. □

Now we apply Theorem 2.1 to the curvature equation (1.1). Suppose (1.1) has an *even* solution $u(z)$, i.e. $u(z) = u(-z)$. Then the previous argument shows that $u_{zz} - \frac{1}{2}u_z^2$ is an *even elliptic function* with singularities only at $E_\tau[2] := \{\frac{\omega_k}{2} \,|\, 0 \le k \le 3\} + \Lambda_\tau$. By using the local behaviors of $u(z)$ near $\frac{\omega_k}{2}$: $u(z) = 4n_k \log|z - \frac{\omega_k}{2}| + O(1)$, it is easy to prove that

$$u_{zz} - \frac{1}{2}u_z^2 = -2\left[\sum_{k=0}^{3} n_k(n_k+1)\wp(z + \tfrac{\omega_k}{2};\tau) + E\right] =: -2I(z), \tag{2.7}$$

where E is some constant, because due to the evenness, $u_{zz} - \frac{1}{2}u_z^2$ has no residues at $z \in E_\tau[2]$. Therefore, (2.5) becomes the following GLE

$$y''(z) = I(z)y(z) = \left[\sum_{k=0}^{3} n_k(n_k+1)\wp(z + \tfrac{\omega_k}{2};\tau) + E\right]y(z). \tag{2.8}$$

We say that *GLE (2.8) comes from the curvature equation (1.1) if the potential $I(z)$ is given by an even solution $u(z)$ of (1.1) via (2.7)*. Since $n_k \in \mathbb{Z}_{\ge 0}$, we will see in Section 3 that the local monodromy matrix of GLE (2.8) at $\frac{\omega_k}{2}$ is I_2, so the developing map $f(z) = y_1(z)/y_2(z)$ is single-valued near each $\frac{\omega_k}{2}$ and then can be extended to be an entire meromorphic function in $\mathbb{C}$. Applying Theorem 2.1, we have the following result.

Theorem 2.2. *If the curvature equation (1.1) has an even solution, then there exists $E \in \mathbb{C}$ such that the monodromy representation of GLE (2.8) is unitary.*

Remark 2.3. Actually the converse statement of Theorem 2.2 is also true, i.e. if there exists $E \in \mathbb{C}$ such that the monodromy representation of GLE (2.8) is unitary, then (1.1) has even solutions. Since this statement is not needed in proving the results of this paper and its proof is more delicate and longer than that of Theorem 2.2, we would like to postpone it in a future work.

3. GLE AND FINITE-GAP POTENTIAL

The purpose of this section is to prove a more general result (see Theorem 3.4 below) which contains Theorem 1.6 as a consequence. To state Theorem 3.4, we need to recall some basic facts about the monodromy theory of the generalized Lamé equation (GLE)

$$y''(z) = I(z;E)y(z) = \left[\sum_{k=0}^{3} n_k(n_k+1)\wp\left(z+\tfrac{\omega_k}{2};\tau\right)+E\right]y(z), \tag{3.1}$$

where $n_k \in \mathbb{Z}_{\geq 0}$ for all k and $\max_k n_k \geq 1$. Since the local exponents of GLE (3.1) at $\frac{\omega_k}{2}$ are $-n_k$ and n_k+1 and the potential $I(\cdot;E)$ is even elliptic, it is easy to prove (cf. [14] or [24, Proposition 3.4]) that the local monodromy matrix of GLE (3.1) at $\frac{\omega_k}{2}$ is the unit matrix I_2. Therefore, the monodromy representation of GLE (3.1) is a group homomorphism $\rho(\cdot;E) : \pi_1(E_\tau) \to SL(2,\mathbb{C})$. Let $\ell_j \in \pi_1(E_\tau)$, $j = 1,2$, be the two fundamental cycles of E_τ such that $\ell_1\ell_2\ell_1^{-1}\ell_2^{-1} = Id$. Then

$$\rho(\ell_1;E)\rho(\ell_2;E) = \rho(\ell_2;E)\rho(\ell_1;E), \tag{3.2}$$

where $\rho(\ell_j;E)$ denotes the monodromy matrix of GLE (3.1) with respect to any pair of linearly independent solutions. That is, the monodromy representation of GLE (3.1) is always *abelian* and hence *reducible*. Consequently, there exists a solution $y_1(z) = y_1(z;E)$ of GLE (3.1) such that $y_1(z;E)$ is a common eigenfunction of $\rho(\ell_1;E)$ and $\rho(\ell_2;E)$:

$$y_1(z+1;E) = e^{\pi i\theta_1(E)}y_1(z;E), \quad y_1(z+\tau;E) = e^{\pi i\theta_2(E)}y_1(z;E), \tag{3.3}$$

where $\theta_j(E) \in \mathbb{C}$ are some constants, i.e. $y_1(z;E)$ is elliptic of the second kind. Since $I(-z;E) = I(z;E)$ implies that $y_2(z) = y_2(z;E) := y_1(-z;E)$ is also a solution of the same GLE (3.1) and also a common eigenfunction of $\rho(\ell_1;E)$ and $\rho(\ell_2;E)$:

$$y_2(z+1;E) = e^{-\pi i\theta_1(E)}y_2(z;E), \quad y_2(z+\tau;E) = e^{-\pi i\theta_2(E)}y_2(z;E), \tag{3.4}$$

we conclude that $\Phi(z;E) := y_1(z;E)y_2(z;E) = y_1(z;E)y_1(-z;E)$ is an *even elliptic* solution of the second symmetric product equation of GLE (3.1):

$$\Phi'''(z) - 4I(z;E)\Phi'(z) - 2I'(z;E)\Phi(z) = 0. \tag{3.5}$$

On the other hand, it was proved in [29, Proposition 2.9] that *the dimension of the even elliptic solutions of equation (3.5) is* 1. Together this with [24, Proposition 3.5], we immediately obtain

Lemma 3.1. *Up to a constant multiple, $\Phi(z;E) = y_1(z;E)y_1(-z;E)$ has the following expression:*

$$\Phi(z;E) = c_0(E) + \sum_{k=0}^{3}\sum_{j=0}^{n_k-1} b_j^{(k)}(E)\wp(z+\tfrac{\omega_k}{2};\tau)^{n_k-j}, \tag{3.6}$$

where the coefficients $c_0(E)$ and $b_j^{(k)}(E)$ are polynomials in E, they do not have common divisors, and $c_0(E)$ is monic. Set $g = \deg c_0(E)$, then $\deg b_j^{(k)}(E) < g$ for all k and j.

Define
$$W(E) := y_1'(z;E)y_2(z;E) - y_1(z;E)y_2'(z;E)$$
to be the Wronskian of $y_1(z;E)$ and $y_2(z;E)$. Clearly $W(E)$ is a constant independent of z. It is easy to see that
$$\frac{y_1'(z;E)}{y_1(z;E)} = \frac{\Phi'(z;E)+W(E)}{2\Phi(z;E)}, \quad \frac{y_2'(z;E)}{y_2(z;E)} = \frac{\Phi'(z;E)-W(E)}{2\Phi(z;E)},$$
which implies
$$\begin{aligned}\frac{\Phi''(z;E)}{2\Phi(z;E)} - \frac{\Phi'(z;E)+W(E)}{2\Phi(z;E)^2}\Phi'(z;E) &= \left(\frac{y_1'(z;E)}{y_1(z;E)}\right)' \\ &= I(z;E) - \left(\frac{\Phi'(z;E)+W(E)}{2\Phi(z;E)}\right)^2,\end{aligned}$$
and
$$\frac{\Phi''(z;E)}{2\Phi(z;E)} - \frac{\Phi'(z;E)-W(E)}{2\Phi(z;E)^2}\Phi'(z;E) = I(z;E) - \left(\frac{\Phi'(z;E)-W(E)}{2\Phi(z;E)}\right)^2.$$
From here we immediately obtain
$$\frac{W(E)^2}{4} = I(z;E)\Phi(z;E)^2 + \frac{\Phi'^2}{4} - \frac{\Phi(z;E)\Phi''(z;E)}{2}. \tag{3.7}$$
Remark that (3.7) is well known (cf. [14, 24]) and the fact that the RHS of (3.7) is independent of z can be also seen from equation (3.5). Define
$$\begin{aligned}Q(E) =&Q(E;\tau) = Q^{(n_0,n_1,n_2,n_3)}(E;\tau) \\ :=&I(z;E)\Phi(z;E)^2 + \frac{\Phi'^2}{4} - \frac{\Phi(z;E)\Phi''(z;E)}{2}.\end{aligned} \tag{3.8}$$
Then it follows from the expression of $I(z;E)$ and Lemma 3.1 that $Q(E)$ is *a monic polynomial of degree* $2g+1$. This polynomial $Q(E)$ is known as *the special polynomial of the Treibich-Verdier potential* (cf. [14, 24]) and will play a crucial role in this paper. The number g, i.e. the arithmetic genus of the hyperelliptic curve $F^2 = Q(E)$, was computed in [14, 28]: Let m_k be the rearrangement of n_k such that $m_0 \geq m_1 \geq m_2 \geq m_3$, then
$$g = \begin{cases} m_0, & \text{if } \sum m_k \text{ is even and } m_0+m_3 \geq m_1+m_2; \\ \frac{m_0+m_1+m_2-m_3}{2}, & \text{if } \sum m_k \text{ is even and } m_0+m_3 < m_1+m_2; \\ m_0, & \text{if } \sum m_k \text{ is odd and } m_0 > m_1+m_2+m_3; \\ \frac{m_0+m_1+m_2+m_3+1}{2}, & \text{if } \sum m_k \text{ is odd and } m_0 \leq m_1+m_2+m_3. \end{cases} \tag{3.9}$$
Furthermore, it is known (cf. [14, 28]) that the roots of $Q(\cdot;\tau) = 0$ are *distinct* for generic $\tau \in \mathbb{H}$.

We summarize the above argument in the following

Lemma 3.2. *The Wronskian $W(E)$ of $y_1(z;E), y_2(z;E) = y_1(-z;E)$ satisfies*

$$(W(E)/2)^2 = Q(E),$$

where $Q(E)$ is a monic polynomial of degree $2g+1$, defined by (3.8) with g given by (3.9). Furthermore,

(1) *if $Q(E) \neq 0$, then the monodromy group of GLE (3.1) with respect to $(y_1(z;E), y_2(z;E))$ is generated by*

$$\rho(\ell_1;E) = \begin{pmatrix} e^{\pi i\theta_1(E)} & 0 \\ 0 & e^{-\pi i\theta_1(E)} \end{pmatrix}, \quad \rho(\ell_2;E) = \begin{pmatrix} e^{\pi i\theta_2(E)} & 0 \\ 0 & e^{-\pi i\theta_2(E)} \end{pmatrix}, \tag{3.10}$$

where $(\theta_1(E), \theta_2(E))$ is seen in (3.3)-(3.4). Besides, $(\theta_1(E), \theta_2(E)) \notin \mathbb{Z}^2$.

(2) *if $Q(E) = 0$, then the dimension of the common eigenfunctions is 1 and $(\theta_1(E), \theta_2(E)) \in \mathbb{Z}^2$.*

(3) *the roots of $Q(\cdot;\tau) = 0$ are distinct for generic $\tau \in \mathbb{H}$.*

Proof. It suffices for us to prove the assertions (1)-(2).

(1) Suppose $Q(E) \neq 0$, then $y_1(z;E)$ and $y_2(z;E)$ are linearly independent and hence (3.10) follows from (3.3)-(3.4). Assume by contradiction that $(\theta_1(E), \theta_2(E)) \in \mathbb{Z}^2$, i.e. $e^{\pi i\theta_j(E)} = e^{-\pi i\theta_j(E)} \in \{\pm 1\}$ for $j = 1, 2$. Let $y(z) = y_1(z;E) + y_2(z;E)$ be a solution of GLE (3.1). Then it follows from (3.3)-(3.4) that $y(z)y(-z)$ is an even elliptic solution of (3.5). Again by [29, Proposition 2.9] that the dimension of even elliptic solutions of (3.5) is 1, we conclude $y(z)y(-z) = cy_1(z;E)y_2(z;E)$ for some constant $c \neq 0$, i.e. either $y(z) = c_1y_1(z;E)$ or $y(z) = c_1y_2(z;E)$ with some constant $c_1 \neq 0$, a contradiction with $y(z) = y_1(z;E) + y_2(z;E)$. This proves $(\theta_1(E), \theta_2(E)) \notin \mathbb{Z}^2$.

(2) Suppose $Q(E) = 0$, then $y_1(z;E)$ and $y_2(z;E) = y_1(-z;E)$ are linearly dependent and hence (3.3)-(3.4) imply $e^{\pi i\theta_j(E)} = e^{-\pi i\theta_j(E)}$ for $j = 1, 2$, i.e. $(\theta_1(E), \theta_2(E)) \in \mathbb{Z}^2$. If there exists another common eigenfunction $y(z)$ which is linearly independent with $y_1(z;E)$, then the same argument as (1) shows $y(z)y(-z) = cy_1(z;E)^2$, clearly a contradiction. This proves that the dimension of the common eigenfunctions is 1, i.e. the monodormy matrix $\rho(\ell_1;E)$ and $\rho(\ell_2;E)$ can not be diagonized simultaneously. □

By Lemma 3.2, it is easy to see that the following corollary holds.

Corollary 3.3. *The monodromy representation of GLE (3.1) is unitary, i.e. the monodromy group is contained in $SU(2)$ up to a common conjugation, if and only if $Q(E) \neq 0$ and $(\theta_1(E), \theta_2(E)) \in \mathbb{R}^2 \setminus \mathbb{Z}^2$.*

The main result of this section is as follows, which is interesting from the viewpoint of the monodromy theory of linear ODEs. Theorem 1.6 will be a consequence of this result.

Theorem 3.4. *Let $\tau \in i\mathbb{R}_{>0}$ and $n_k \in \mathbb{Z}_{\geq 0}$ for all k with $\max_k n_k \geq 1$. Suppose that all zeros of $Q^{(n_0,n_1,n_2,n_3)}(\cdot;\tau)$ are real and distinct. Then the monodromy representation of GLE (3.1) can not be unitary for any $E \in \mathbb{C}$.*

The rest of this section is devoted to the proof of Theorem 3.4. Recalling Lemma 3.2, we have

$$\operatorname{tr}\rho(\ell_1;E)=e^{\pi i\theta_1(E)}+e^{-\pi i\theta_1(E)}=2\cos(\pi\theta_1(E))\quad\text{for all } E. \tag{3.11}$$

Remark that $\operatorname{tr}\rho(\ell_1;E)$ is independent of the choice of linearly independent solutions. Define

$$\tilde{\mathcal{S}}=\tilde{\mathcal{S}}^{(n_0,n_1,n_2,n_3)}(\tau):=\{E\in\mathbb{C}\mid -2\leq \operatorname{tr}\rho(\ell_1;E)\leq 2\}. \tag{3.12}$$

Lemma 3.5. *Let $\tau\in i\mathbb{R}_{>0}$. Then the set $\tilde{\mathcal{S}}$ is symmetric with respect to the real line $\mathbb{R}$.*

Proof. It suffices to prove $\bar{E}\in\tilde{\mathcal{S}}$ if $E\in\tilde{\mathcal{S}}$. Suppose $E\in\tilde{\mathcal{S}}$, then $\theta_1(E)\in\mathbb{R}$. Since $\tau\in i\mathbb{R}_{>0}$, it follows from the expression (1.8) of $\wp(z;\tau)$ that $\overline{\wp(z;\tau)}=\wp(\bar{z};\tau)$. Note that $\wp(z+\frac{\overline{\omega_k}}{2};\tau)=\wp(z+\frac{\omega_k}{2};\tau)$. Then it is easy to see that $y(z)$ is a solution of GLE (3.1) if and only if $\tilde{y}(z):=\overline{y(\bar{z})}$ is a solution of GLE

$$\tilde{y}''(z)=I(z;\bar{E})\tilde{y}(z)=\left[\sum_{k=0}^{3}n_k(n_k+1)\wp\left(z+\tfrac{\omega_k}{2};\tau\right)+\bar{E}\right]\tilde{y}(z). \tag{3.13}$$

Let $y_1(z;E)$ be the common eigenfunction of the monodromy matrices of GLE (3.1) such that (3.3) holds, then $\tilde{y}_1(z):=\overline{y_1(\bar{z};E)}$ satisfies

$$\tilde{y}_1(z+1)=e^{-\pi i\theta_1(E)}\tilde{y}_1(z),\quad \tilde{y}_1(z+\tau)=e^{\pi i\overline{\theta_2(E)}}\tilde{y}_1(z),$$

where $\theta_1(E)\in\mathbb{R}$ is used. Therefore, $\tilde{y}_1(z)$ is a common eigenfunction of the monodromy matrices of GLE (3.13) and so

$$\operatorname{tr}\rho(\ell_1;\bar{E})=e^{-\pi i\theta_1(E)}+e^{\pi i\theta_1(E)}=\operatorname{tr}\rho(\ell_1;E)\in[-2,2].$$

This proves $\bar{E}\in\tilde{\mathcal{S}}$. □

The following lemma is the only place of this paper where we need to use some notions and classical results about the spectral theory of Hill's equations that will be recalled in Appendix A.

Lemma 3.6. *Let $\tau\in i\mathbb{R}_{>0}$ and $n_k\in\mathbb{Z}_{\geq 0}$ for all k with $\max_k n_k\geq 1$. Suppose $Q(E;\tau)=Q^{(n_0,n_1,n_2,n_3)}(E;\tau)$ has $2g+1$ real distinct zeros, denoted by $E_{2g}<E_{2g-1}<\cdots<E_1<E_0$. Then*

$$\tilde{\mathcal{S}}^{(n_0,n_1,n_2,n_3)}(\tau)=(-\infty,E_{2g}]\cup[E_{2g-1},E_{2g-2}]\cup\cdots\cup[E_1,E_0]. \tag{3.14}$$

Proof. Though the Treibich-Verdier potential of GLE (3.1) is real-valued for $z\in\mathbb{R}$, it has poles at $\mathbb{Z}$ and $\frac{1}{2}+\mathbb{Z}$. To avoid these singularities on $\mathbb{R}$, we let $z=x+\frac{\tau}{4}$ with $x\in\mathbb{R}$ and $w(x):=y(z)=y(x+\frac{\tau}{4})$ in GLE (3.1). Then $w(x)$ satisfies the following Hill's equation

$$w''(x)-\left[\sum_{k=0}^{3}n_k(n_k+1)\wp\left(x+\tfrac{\tau}{4}+\tfrac{\omega_k}{2};\tau\right)\right]w(x)=Ew(x),\ x\in\mathbb{R}, \tag{3.15}$$

with the potential $q(x) := -\sum_{k=0}^{3} n_k(n_k+1)\wp\left(x+\frac{\tau}{4}+\frac{\omega_k}{2};\tau\right)$ being *continuous* on $\mathbb{R}$ with period $\Omega = 1$ but *not real-valued*. Thus the classic theory of finite-gap potentials can not be applicable to this $q(x)$ either.

Let $w_1(x)$, $w_2(x)$ be any two linearly independent solutions of (3.15). Then so do $w_1(x+\Omega)$, $w_2(x+\Omega)$ and hence there is a monodromy matrix $M(E) \in SL(2,\mathbb{C})$ such that

$$(w_1(x+\Omega), w_2(x+\Omega)) = (w_1(x), w_2(x))M(E).$$

As in Appendix A, we define the *Hill's discriminant* $\Delta(E)$ by

$$\Delta(E) := \mathrm{tr}M(E), \tag{3.16}$$

which is independent of the choice of solutions. This $\Delta(E)$ is an entire function and plays a fundamental role since it encodes all the spectrum information of the associated operator. Indeed, we define

$$\mathcal{S} := \Delta^{-1}([-2,2]) = \{E \in \mathbb{C} \mid -2 \le \Delta(E) \le 2\}$$

to be *the conditional stability set* of the operator $L = \frac{d^2}{dx^2} + q(x)$. Since $q(x)$ is continuous, it was proved in [23] that *this $\mathcal{S}$ coincides with the spectrum $\sigma(H)$ of the associated linear operator H in $L^2(\mathbb{R},\mathbb{C})$* (i.e. H is defined as $Hf = Lf$, $f \in H^{2,2}(\mathbb{R},\mathbb{C})$).

Observe that

($\star$) *if $(y_1(z), y_2(z))$ is a pair of linearly independent solutions of GLE (3.1), then $(w_1(x), w_2(x)) := (y_1(x+\frac{\tau}{4}), y_2(x+\frac{\tau}{4}))$ is a pair of linearly independent solutions of equation (3.15).*

Thus, the monodromy matrix $\rho(\ell_1; E)$ is also the monodromy matrix of equation (3.15), which gives

$$\Delta(E) = \mathrm{tr}\,\rho(\ell_1; E) = 2\cos(\pi\theta_1(E))$$

and so we obtain the following important identity

$$\sigma(H) = \mathcal{S} = \{E \in \mathbb{C} \mid -2 \le \Delta(E) \le 2\} = \tilde{\mathcal{S}}. \tag{3.17}$$

On the other hand, Lemma 3.2-(1) and ($\star$) imply that if $Q(E) \neq 0$, then $w_j(x) := y_j(x+\frac{\tau}{4}; E)$, $j = 1,2$, are linearly independent Floquet solutions of (3.15). So we can apply Theorem B in Appendix A to (3.15). In particular, ($\star$) infers that the polynomial $R_{2g+1}(E)$ in Theorem B-(ii) is precisely $Q(E)$; see also [14]. Together with our assumption, we obtain

$$R_{2g+1}(E) = Q(E) = \prod_{j=0}^{2g}(E - E_j).$$

Then it follows from (A.5) that (we refer to Appendix A for the definitions of $p_i(E)$ and $\langle q \rangle$ below)

$$d(E_j) := \mathrm{ord}_{E_j}(\Delta(\cdot)^2 - 4) = 1 + 2p_i(E_j) \text{ is } \textit{odd} \text{ for all } j \in [0, 2g]. \tag{3.18}$$

By $\sigma(H) = \{E \mid -2 \le \Delta(E) \le 2\}$, it is easy to prove (see e.g. the proof of Theorem B-(iii) in [13]) that there are $d(E_j)$ semi-arcs of the spectrum $\sigma(H)$

meeting at E_j. On the other hand, Theorem B-(iii) says that: The spectrum $\sigma(H)=\mathcal{S}$ consists of finitely many bounded spectral arcs σ_k, $1\leq k\leq \tilde{g}$ for some $\tilde{g}\leq g$ and *one* semi-infinite arc σ_∞ which tends to $-\infty+\langle q\rangle$, i.e.

$$\sigma(H)=\mathcal{S}=\sigma_\infty\cup\cup_{k=1}^{\tilde{g}}\sigma_k.$$

Furthermore, the set of the finite end points of such arcs is precisely $\{E_j\}_{j=0}^{2g}$ because of (3.18). Together these with the following three facts:

(a) Our assumption gives $E_j\in\mathbb{R}$ and $E_{2g}<E_{2g-1}<\cdots<E_1<E_0$;
(b) Lemma 3.5 and (3.17) imply that the spectrum $\sigma(H)=\mathcal{S}=\tilde{\mathcal{S}}$ is symmetric with respect to the real line $\mathbb{R}$;
(c) A classical result (see e.g. [15, Theorem 2.2]) says that $\mathbb{C}\setminus\sigma(H)$ is path-connected;

we easily conclude that (i) $\sigma(H)\subset\mathbb{R}$, (ii) $d(E_j)=1$ (i.e. $\frac{d}{dE}\Delta(E_j)\neq 0$) for all j and so

$$\tilde{\mathcal{S}}=\sigma(H)=(-\infty,E_{2g}]\cup[E_{2g-1},E_{2g-2}]\cup\cdots\cup[E_1,E_0]. \tag{3.19}$$

Indeed, since (a) says that all finite end points of spectral arcs are on $\mathbb{R}$, the assertion (i) $\sigma(H)\subset\mathbb{R}$ follows immediately from (b)-(c). Consequently, there are at most two semi-arcs of $\sigma(H)$ meeting at each E_j. This, together with (3.18), yields the assertion (ii) $d(E_j)=1$ for all j, namely there is exactly one semi-arc of $\sigma(H)$ ending at E_j, which finally implies (3.19). The proof is complete. □

On the other hand, we have the following important observation.

Lemma 3.7. *Fix $n_k\in\mathbb{Z}_{\geq 0}$ for all k with $\max_k n_k\geq 1$. Let $E_j=E_j(\tau)$, $j=0,1,\cdots,2g$, are all roots of $Q^{(n_0,n_1,n_2,n_3)}(\cdot;\tau)=0$, i.e.*

$$Q^{(n_0,n_1,n_2,n_3)}(E;\tau)=\prod_{j=0}^{2g}(E-E_j(\tau)).$$

Then

$$Q^{(n_0,n_2,n_1,n_3)}(E;\tfrac{-1}{\tau})=\prod_{j=0}^{2g}(E-\tau^2E_j(\tau)). \tag{3.20}$$

Proof. Recall the modular property of $\wp(z;\tau)$:

$$\wp(z;\tfrac{-1}{\tau})=\tau^2\wp(\tau z;\tau),$$

which gives

$$\wp(z+\tfrac{1}{2};\tfrac{-1}{\tau})=\tau^2\wp(\tau z+\tfrac{\tau}{2};\tau),$$
$$\wp(z+\tfrac{-1}{2\tau};\tfrac{-1}{\tau})=\tau^2\wp(\tau z+\tfrac{1}{2};\tau),$$
$$\wp(z+\tfrac{\tau-1}{2\tau};\tfrac{-1}{\tau})=\tau^2\wp(\tau z+\tfrac{1+\tau}{2};\tau).$$

From here, we immediately see that $y(z)$ is a solution of GLE (3.1) if and only if $\tilde{y}(z) := y(\tau z)$ is a solution of GLE

$$\tilde{y}''(z) = \left[\sum_{k=0}^{3} \tilde{n}_k(\tilde{n}_k+1)\wp\left(z+\tfrac{\omega_k}{2};\tfrac{-1}{\tau}\right) + \tau^2 E\right]\tilde{y}(z) \tag{3.21}$$

with $(\tilde{n}_0, \tilde{n}_1, \tilde{n}_2, \tilde{n}_3) = (n_0, n_2, n_1, n_3)$ (of course, we mean $\omega_2 = \frac{-1}{\tau}$ and $\omega_3 = 1 + \frac{-1}{\tau}$ in (3.21)).

Let $y_1(z; E)$ be the common eigenfunction of the monodromy matrices of GLE (3.1) such that (3.3) holds, then $\tilde{y}_1(z; E) := y_1(\tau z; E)$ satisfies

$$\tilde{y}_1(z+1; E) = e^{\pi i \theta_2(E)}\tilde{y}_1(z; E), \quad \tilde{y}_1(z+\tfrac{-1}{\tau}; E) = e^{-\pi i \theta_1(E)}\tilde{y}_1(z; E),$$

i.e. $\tilde{y}_1(z; E)$ is a common eigenfunction of the monodromy matrices of GLE (3.21). When $E = E_j(\tau)$ (resp. $E \notin \{E_j\}_{j=0}^{2g}$), Lemma 3.2 says that $y_1(z; E)$ and $y_1(-z; E)$ are linearly dependent (resp. linearly independent) and so do $\tilde{y}_1(z; E)$ and $\tilde{y}_1(-z; E)$. This yields that $\tau^2 E_j(\tau)$, $j = 0, 1, \cdots, 2g$, are all the roots of

$$Q^{(\tilde{n}_0,\tilde{n}_1,\tilde{n}_2,\tilde{n}_3)}(\cdot; \tfrac{-1}{\tau}) = Q^{(n_0,n_2,n_1,n_3)}(\cdot; \tfrac{-1}{\tau}) = 0.$$

Since (3.9) gives $\deg Q^{(n_0,n_2,n_1,n_3)}(\cdot; \frac{-1}{\tau}) = \deg Q^{(n_0,n_1,n_2,n_3)}(\cdot; \tau) = 2g+1$, and $E_j(\tau)$, $j = 0, 1, \cdots, 2g$, are all distinct for generic $\tau \in \mathbb{H}$, it follows that (3.20) holds for generic $\tau \in \mathbb{H}$ and hence for all $\tau \in \mathbb{H}$ by continuity with respect to τ. □

Now we are in the position to prove Theorem 3.4.

Proof of Theorem 3.4. Let $\tau \in i\mathbb{R}_{>0}$ and suppose $Q^{(n_0,n_1,n_2,n_3)}(\cdot; \tau)$ has $2g+1$ real distinct zeros, denoted by $E_{2g} < E_{2g-1} < \cdots < E_1 < E_0$. Then Lemma 3.6 gives

$$\tilde{\mathcal{S}}^{(n_0,n_1,n_2,n_3)}(\tau) = (-\infty, E_{2g}] \cup [E_{2g-1}, E_{2g-2}] \cup \cdots \cup [E_1, E_0].$$

Assume by contradiction that the monodromy representation of GLE (3.1) is unitary for some $E = \hat{E}$. Then Corollary 3.3 implies

$$Q^{(n_0,n_1,n_2,n_3)}(\hat{E}; \tau) \neq 0 \quad \text{and} \quad (\theta_1(\hat{E}), \theta_2(\hat{E})) \in \mathbb{R}^2 \setminus \mathbb{Z}^2. \tag{3.22}$$

It follows from the definition (3.12) of $\tilde{\mathcal{S}}^{(n_0,n_1,n_2,n_3)}(\tau)$ that $\hat{E} \in \tilde{\mathcal{S}}^{(n_0,n_1,n_2,n_3)}(\tau)$ and $E \neq E_j$ for all j, i.e.

$$\hat{E} \in (-\infty, E_{2g}) \cup (E_{2g-1}, E_{2g-2}) \cup \cdots \cup (E_1, E_0). \tag{3.23}$$

Note that $\frac{-1}{\tau} \in i\mathbb{R}_{>0}$ and

$$\tau^2 E_0 < \tau^2 E_1 < \cdots < \tau^2 E_{2g-1} < \tau^2 E_{2g}.$$

Since Lemma 3.7 shows that $Q^{(n_0,n_2,n_1,n_3)}(\cdot; \frac{-1}{\tau})$ has $2g+1$ real distinct zeros $\{\tau^2 E_j\}_{j=0}^{2g}$, Lemma 3.6 applies for $\frac{-1}{\tau}$ and (n_0, n_2, n_1, n_3) and gives

$$\tilde{\mathcal{S}}^{(n_0,n_2,n_1,n_3)}(\tfrac{-1}{\tau}) = (-\infty, \tau^2 E_0] \cup [\tau^2 E_1, \tau^2 E_2] \cup \cdots \cup [\tau^2 E_{2g-1}, \tau^2 E_{2g}].$$

On the other hand, the proof of Lemma 3.7 shows that $\tilde{y}_1(z) := y_1(\tau z;\hat{E})$ is a common eigenfunction of the monodromy matrices of GLE

$$\tilde{y}''(z) = \left[\sum_{k=0}^{3}\tilde{n}_k(\tilde{n}_k+1)\wp\left(z+\tfrac{\omega_k}{2};\tfrac{-1}{\tau}\right)+\tau^2\hat{E}\right]\tilde{y}(z) \tag{3.24}$$

with $(\tilde{n}_0,\tilde{n}_1,\tilde{n}_2,\tilde{n}_3)=(n_0,n_2,n_1,n_3)$ and

$$\tilde{y}_1(z+1)=e^{\pi i\theta_2(\hat{E})}\tilde{y}_1(z).$$

Consequently, for GLE (3.24) there holds

$$\operatorname{tr}\rho(\ell_1;\tau^2\hat{E})=e^{\pi i\theta_2(\hat{E})}+e^{-\pi i\theta_2(\hat{E})}=2\cos(\pi\theta_2(\hat{E}))\in[-2,2]$$

by (3.22). This implies $\tau^2\hat{E}\in\tilde{\mathcal{S}}^{(n_0,n_2,n_1,n_3)}(\frac{-1}{\tau})$, i.e.

$$\tau^2\hat{E}\in(-\infty,\tau^2E_0]\cup[\tau^2E_1,\tau^2E_2]\cup\cdots\cup[\tau^2E_{2g-1},\tau^2E_{2g}]$$

and hence

$$\hat{E}\in[E_{2g},E_{2g-1}]\cup\cdots\cup[E_2,E_1]\cup[E_0,+\infty),$$

which is a contradiction with (3.23). Therefore, the monodromy representation of GLE (3.1) can not be unitary for any $E\in\mathbb{C}$. □

4. Application to curvature equation and $Q^{(n_0,n_1,n_2,n_3)}$

The purpose of this section is apply the previous results to prove Theorems 1.3, 1.6 and 1.7.

By Theorems 3.4 and 2.2, we immediately obtain the following general result which contains Theorem 1.3 as a consequence.

Theorem 4.1. *Let $\tau\in i\mathbb{R}_{>0}$ and $n_k\in\mathbb{Z}_{\geq 0}$ for all k with $\max_k n_k\geq 1$. Suppose that all zeros of $Q^{(n_0,n_1,n_2,n_3)}(\cdot;\tau)$ are real and distinct. Then the curvature equation (1.1) on this E_τ has no even solutions.*

Remark 4.2. The converse statement of Theorem 4.1 does not necessarily hold. Here is an example. Define $e_k=e_k(\tau):=\wp(\frac{\omega_k}{2};\tau)$, $k=1,2,3$. It is well known that

$$e_1(\tau)>e_3(\tau)>e_2(\tau)\quad\text{for }\tau\in i\mathbb{R}_{>0},$$

$$e_1(\tau)\in\mathbb{R},\quad e_2(\tau)=\overline{e_3(\tau)}\notin\mathbb{R}\quad\text{for }\tau\in\tfrac{1}{2}+i\mathbb{R}_{>0}.$$

Now for $\tau\in i\mathbb{R}_{>0}$, we have $\frac{1+\tau}{2}\in\frac{1}{2}+i\mathbb{R}_{>0}$ and it is easy to compute that

$$Q^{(1,0,0,1)}(E;\tau)=(E-E_0(\tau))(E-E_1(\tau))(E-E_2(\tau))$$

with $E_0(\tau)=e_1(\frac{1+\tau}{2})-2e_3(\tau)\in\mathbb{R}$, $E_1(\tau)=\overline{E_2(\tau)}=e_2(\frac{1+\tau}{2})-2e_3(\tau)\notin\mathbb{R}$, namely $Q^{(1,0,0,1)}(E;\tau)$ always has two roots in $\mathbb{C}\setminus\mathbb{R}$ for any $\tau\in i\mathbb{R}_{>0}$. On the other hand, it was proved in [4, Theorem 1.1] (see also [11, 12]) that there exist $0<b_0<1<b_1<\sqrt{3}$ such that

$$\Delta u+e^u=8\pi\delta_0+8\pi\delta_{\omega_3/2}\quad\text{on }E_\tau,\ \tau=ib,\ b>0$$

has no even solutions if and only if $b\in[b_0,b_1]$.

Remark 4.2 indicates that the zeros of $Q^{(n_0,n_1,n_2,n_3)}(\cdot;\tau)$ are not necessarily real distinct for $\tau \in i\mathbb{R}_{>0}$ without further conditions on n_k's. Naturally we ask: *What are the n_k's such that $Q^{(n_0,n_1,n_2,n_3)}(\cdot;\tau)$ has real distinct zeros*? We have the following result on this subject.

Theorem 4.3. *Let $n_k \in \mathbb{Z}_{\geq 0}$ for all k with* $\max_k n_k \geq 1$. *If neither*

$$\frac{n_1+n_2-n_0-n_3}{2} \geq 1, \quad n_1 \geq 1, \quad n_2 \geq 1 \tag{4.1}$$

nor

$$\frac{n_1+n_2-n_0-n_3}{2} \leq -1, \quad n_0 \geq 1, \quad n_3 \geq 1 \tag{4.2}$$

hold, then for any $\tau \in i\mathbb{R}_{>0}$, the zeros of $Q^{(n_0,n_1,n_2,n_3)}(\cdot;\tau)$ are real and distinct.

The proof of Theorem 4.3 is long and will be postponed in Section 5. We will see from Corollary 4.5 that our condition on n_k in Theorem 4.3 is sharp. Now we can prove Theorems 1.3 and 1.6 by applying Theorem 4.3.

Proof of Theorem 1.6. Theorem 1.6 follows from Theorems 3.4 and 4.3. □

Proof of Theorem 1.3. Theorem 1.3 follows from Theorems 4.1 and 4.3. □

Together with Eremenko and Gabrielov's result Theorem A and our Theorem 4.1, we immediately obtain

Theorem 4.4. *Let $n_k \in \mathbb{Z}_{\geq 0}$ for all k with* $\max_k n_k \geq 1$. *Suppose either*

$$\frac{n_1+n_2-n_0-n_3}{2} \geq 1, \quad n_1 \geq 1, \quad n_2 \geq 1 \tag{4.3}$$

or

$$\frac{n_1+n_2-n_0-n_3}{2} \leq -1, \quad n_0 \geq 1, \quad n_3 \geq 1. \tag{4.4}$$

Then there exists $\tau \in i\mathbb{R}_{>0}$ such that $Q^{(n_0,n_1,n_2,n_3)}(\cdot;\tau)$ has either multiple zeros or complex zeros.

Theorem 4.4 indicates that our condition on n_k in Theorem 4.3 is sharp, namely Theorem 1.7 holds.

Corollary 4.5 (=Theorem 1.7)**.** *Let $n_k \in \mathbb{Z}_{\geq 0}$ for all k with* $\max_k n_k \geq 1$. *Then all the zeros of $Q^{(n_0,n_1,n_2,n_3)}(\cdot;\tau)$ are real and distinct for each $\tau \in i\mathbb{R}_{>0}$ if and only if (n_0, n_1, n_2, n_3) satisfies neither (4.3) nor (4.4).*

In general, it is very difficult to prove such an optimal algebraic result for the spectral polynomial $Q^{(n_0,n_1,n_2,n_3)}(\cdot;\tau)$ of the Treibich-Verdier potential (1.9). Corollary 4.5 is a beautiful application of Eremenko and Gabrielov's result (Theorem A, via geometric approach) and our result (Theorem 4.1, via analytic approach).

5. Real distinct roots of $Q^{(n_0,n_1,n_2,n_3)}$

The purpose of this section is to prove Theorem 4.3. As pointed out by Corollary 4.5, our condition on n_k in Theorem 4.3 is optimal. A non-optimal version of Theorem 4.3 was proved in [6, Theorem 1.1].

5.1. As in [6], first we need to investigate polynomial solutions of

$$\frac{d^2y}{dx^2}+\left(\frac{\gamma_1}{x-t_1}+\frac{\gamma_2}{x-t_2}+\frac{\gamma_3}{x-t_3}\right)\frac{dy}{dx}+\frac{\alpha\beta(x-t_3)-q}{\prod_{j=1}^3(x-t_j)}y=0, \tag{5.1}$$

where

$$t_1\neq t_2\neq t_3\neq t_1,\quad \gamma_3\notin -\mathbb{Z}_{\geq 0}, \tag{5.2}$$

$$\alpha=-N \text{ with } N\in\mathbb{Z}_{\geq 0},\quad \alpha+\beta+1=\gamma_1+\gamma_2+\gamma_3. \tag{5.3}$$

It is a Fuchsian equation on $\mathbb{CP}^1$ with four regular singularities $\{t_1,t_2,t_3,\infty\}$, with the exponents being $0,1-\gamma_j$ at t_j and α,β at ∞. Set

$$y=\sum_{m=0}^{\infty}c_m(x-t_3)^m,\quad \text{where } c_0=1, \tag{5.4}$$

and substitute it to the differential equation (5.1) multiplied by $\prod_{j=1}^3(x-t_j)$. Then the coefficients satisfy the following recursive relations:

$$(t_1-t_3)(t_2-t_3)\gamma_3c_1=qc_0=q, \tag{5.5}$$

$$\begin{aligned}(t_1-t_3)(t_2-t_3)(m+1)(m+\gamma_3)c_{m+1}&=-(m-1+\alpha)(m-1+\beta)c_{m-1}\\ &+[m\{(m-1+\gamma_3)(t_1+t_2-2t_3)+(t_2-t_3)\gamma_1+(t_1-t_3)\gamma_2\}+q]c_m.\end{aligned}$$

Consequently, it is easy to see that c_r *is a polynomial in* q *of degree* r *and we denote it by* $c_r(q)$.

Let q_0 be a solution to the equation $c_{N+1}(q)=0$, where N is given by (5.3). Then it follows from (5.5) for $m=N+1$ that $c_{N+2}(q_0)=0$. By applying (5.5) for $m=N+2,N+3,\ldots$, we have $c_m(q_0)=0$ for $m\geq N+3$. Hence, if $c_{N+1}(q_0)=0$, then (5.1) have a non-zero polynomial solution. More precisely, we obtain the following proposition.

Proposition 5.1. *Suppose (5.2)-(5.3) hold. If* q *is a solution to the equation* $c_{N+1}(q)=0$*, then the differential equation (5.1) have a non-zero polynomial solution of degree no more than* N.

Now we restrict to the case that all the parameters are *real*. Then $c_r(q)$ is a polynomial of q with real coefficients.

Theorem 5.2. [6] *Let* t_j,γ_j *are all real. Assume that* $\alpha=-N$ *with* $N\in\mathbb{Z}_{\geq 0}$, $\beta=\gamma_1+\gamma_2+\gamma_3+N-1>0$, $\gamma_3>0$ *and* $(t_1-t_3)(t_2-t_3)<0$. *Then the equation* $c_{N+1}(q)=0$ *has all its roots real and unequal.*

The above theorem was proved in [6] by applying the standard method of Sturm sequence. In this paper, we prove the following result.

Theorem 5.3. *Let t_j, γ_j are all real. Assume that there are integers $n_0 \geq n_3 \geq 1$ such that*

$$(t_1 - t_3)(t_2 - t_3) < 0, \quad -\alpha = N = n_0 + n_3, \tag{5.6}$$

$$\gamma_3 = \beta = \gamma_1 + \gamma_2 + \gamma_3 - 1 - \alpha = \tfrac{1}{2} - n_3 < 0. \tag{5.7}$$

Then the equation $c_{N+1}(q) = 0$ has all its roots real and unequal.

Proof. Under our assumptions (5.6)-(5.7), we have

$$-(m - 1 + \alpha) > 0, \quad \forall\, m \in [1, N],$$

$$m + \gamma_3 = m + \beta = m + \tfrac{1}{2} - n_3 \begin{cases} < 0 & \text{if } m \leq n_3 - 1, \\ > 0 & \text{if } m \geq n_3. \end{cases}$$

Together with the recursive formula (5.5), we easily obtain the following properties:

(P1) Up to a positive constant, the leading term in $c_m(q)$ is

$$\begin{cases} q^m & \text{if } 1 \leq m \leq n_3, \\ (-1)^{m-n_3} q^m & \text{if } n_3 \leq m \leq N+1. \end{cases}$$

(P2) If $c_m(q) = 0$ and $c_{m-1}(q) \neq 0$ for $q \in \mathbb{R}$, then

$$c_{m+1}(q) c_{m-1}(q) \begin{cases} < 0 & \text{if } 1 \leq m \leq N, m \neq n_3, \\ > 0 & \text{if } m = n_3. \end{cases}$$

Therefore, $c_m(q)$ is not a Sturm sequence. However, we can still show that the polynomial $c_m(q)$ $(1 \leq m \leq N+1)$ has m real distinct roots $s_i^{(m)}$ $(i = 1, \ldots, m)$ such that

$$s_1^{(m)} < s_1^{(m-1)} < s_2^{(m)} < s_2^{(m-1)} < \cdots < s_{m-1}^{(m)} < s_{m-1}^{(m-1)} < s_m^{(m)}$$

by induction on m. The case $m = 1$ is trivial. Let $1 \leq k \leq N$ and assume that the statement is true for $m \leq k$. From the assumption of the induction,

$$s_1^{(k)} < s_1^{(k-1)} < s_2^{(k)} < s_2^{(k-1)} < \cdots < s_{k-1}^{(k)} < s_{k-1}^{(k-1)} < s_k^{(k)}. \tag{5.8}$$

Case 1. We consider $k \leq n_3 - 1$.

Then **(P1)** implies

$$\lim_{q \to -\infty} c_{k-1}(q) = (-1)^{k-1} \infty, \ \lim_{q \to +\infty} c_{k-1}(q) = +\infty. \tag{5.9}$$

Since $s_j^{(k-1)}$, $1 \leq j \leq k-1$, are all the roots of c_{k-1}, it follows from (5.8) and (5.9) that

$$c_{k-1}(s_i^{(k)}) \sim (-1)^{k-i}, \quad \forall i \in [1, k]. \tag{5.10}$$

Here $c \sim (-1)^j$ means $c = (-1)^j \tilde{c}$ for some $\tilde{c} > 0$. Then we see from **(P2)** that

$$c_{k+1}(s_i^{(k)}) \sim (-1)^{k+1-i}, \quad \forall i \in [1, k].$$

On the other hand, **(P1)** implies

$$\lim_{q\to-\infty} c_{k+1}(q) = (-1)^{k+1}\infty, \ \lim_{q\to+\infty} c_{k+1}(q) = +\infty.$$

From here, it follows from the intermediate value theorem that the polynomial $c_{k+1}(q)$ has $k+1$ real distinct roots $s_i^{(k+1)}$ $(1 \le i \le k+1)$ such that

$$s_1^{(k+1)} < s_1^{(k)} < s_2^{(k+1)} < s_2^{(k)} < \cdots < s_k^{(k+1)} < s_k^{(k)} < s_{k+1}^{(k+1)}. \tag{5.11}$$

Case 2. We consider $k = n_3$.

Then (5.10) still holds, and so **(P2)** gives

$$c_{k+1}(s_i^{(k)}) \sim (-1)^{k-i}, \quad \forall i \in [1,k],$$

which is different from Case 1! However, **(P1)** says that the leading term of $c_{k+1} = c_{n_3+1}$ is $-q^{k+1}$ (up to a positive constant), which implies

$$\lim_{q\to-\infty} c_{k+1}(q) = (-1)^{k}\infty, \ \lim_{q\to+\infty} c_{k+1}(q) = -\infty.$$

This is also different from Case 1! Thanks to these two facts, we see again that $c_{k+1}(q)$ has $k+1$ real distinct roots $s_i^{(k+1)}$ $(1 \le i \le k+1)$ such that (5.11) holds.

Case 3. We consider $n_3 + 1 \le k \le N$.

Then **(P1)** says that the leading term of c_{k-1} is $(-1)^{k-1-n_3}q^{k-1}$, which implies

$$\lim_{q\to-\infty} c_{k-1}(q) = (-1)^{-n_3}\infty, \ \lim_{q\to+\infty} c_{k-1}(q) = (-1)^{k-1-n_3}\infty.$$

From here and (5.8), we obtain

$$c_{k-1}(s_i^{(k)}) \sim (-1)^{i-1-n_3}, \quad \forall i \in [1,k],$$

and so **(P2)** implies

$$c_{k+1}(s_i^{(k)}) \sim (-1)^{i-n_3}, \quad \forall i \in [1,k].$$

Recall **(P1)** that the leading term of c_{k+1} is $(-1)^{k+1-n_3}q^{k+1}$, which gives

$$\lim_{q\to-\infty} c_{k+1}(q) = (-1)^{-n_3}\infty, \ \lim_{q\to+\infty} c_{k+1}(q) = (-1)^{k+1-n_3}\infty.$$

Therefore, we conclude again that $c_{k+1}(q)$ has $k+1$ real distinct roots $s_i^{(k+1)}$ $(1 \le i \le k+1)$ such that (5.11) holds.

This proves that $c_{N+1}(q) = 0$ has all its roots real and unequal. The proof is complete. □

5.2. Recalling GLE (3.1), we let $y(z)$ be a solution of GLE,

$$\left(\frac{d^2}{dz^2} - \sum_{k=0}^{3} n_k(n_k+1)\wp(z+\tfrac{\omega_k}{2};\tau) - E\right)y(z) = 0. \tag{5.12}$$

Set $x = \wp(z)$ and recall e_k defined in Remark 4.2. Applying the formula

$$\wp(z+\tfrac{\omega_i}{2}) = e_i + \frac{(e_i - e_{i'})(e_i - e_{i''})}{\wp(z) - e_i}, \text{ where } \{i, i', i''\} = \{1,2,3\},$$

it is easy to see that equation (5.12) is equivalent to

$$\begin{aligned}\Bigg\{\frac{d^2}{dx^2} + \frac{1}{2}\left(\sum_{i=1}^{3}\frac{1}{x-e_i}\right)\frac{d}{dx} - \frac{1}{4\prod_{j=1}^{3}(x-e_j)}\Bigg(\tilde{C}+ \\ n_0(n_0+1)x + \sum_{i=1}^{3} n_i(n_i+1)\frac{(e_i-e_{i'})(e_i-e_{i''})}{x-e_i}\Bigg)\Bigg\}\tilde{f}(x) = 0,\end{aligned} \tag{5.13}$$

where $\tilde{f}(\wp(z)) = y(z)$ and $\tilde{C} = E + \sum_{i=1}^{3} n_i(n_i+1)e_i$. Note that $e_1 + e_2 + e_3 = 0$. It is easy to see that the Riemann scheme of equation (5.13) is

$$\left\{\begin{matrix} e_1 & e_2 & e_3 & \infty \\ \frac{-n_1}{2} & \frac{-n_2}{2} & \frac{-n_3}{2} & \frac{-n_0}{2} \\ \frac{n_1+1}{2} & \frac{n_2+1}{2} & \frac{n_3+1}{2} & \frac{n_0+1}{2} \end{matrix}\right\}.$$

Let $\tilde{\alpha}_i \in \{-n_i/2, (n_i+1)/2\}$ for each $i \in \{0,1,2,3\}$ such that $N := -\sum \tilde{\alpha}_i \in \mathbb{Z}_{\geq 0}$. Set

$$\Phi^{(\tilde{\alpha}_1,\tilde{\alpha}_2,\tilde{\alpha}_3)}(x) = \prod_{j=1}^{3}(x-e_j)^{\tilde{\alpha}_j} \quad \text{and} \quad \tilde{f}(x) = \Phi^{(\tilde{\alpha}_1,\tilde{\alpha}_2,\tilde{\alpha}_3)}(x)f(x).$$

Then $\tilde{f}(x)$ solves equation (5.13) if and only if $f(x)$ satisfies

$$\begin{aligned}\frac{d^2f(x)}{dx^2} + \sum_{i=1}^{3}\frac{2\tilde{\alpha}_i+\frac{1}{2}}{x-e_i}\frac{df(x)}{dx} + \Bigg(\frac{(\sum_{i=1}^{3}\tilde{\alpha}_i - \frac{n_0}{2})(\sum_{i=1}^{3}\tilde{\alpha}_i + \frac{n_0+1}{2})x}{(x-e_1)(x-e_2)(x-e_3)} \\ - \frac{\frac{E}{4} + e_1(\tilde{\alpha}_2+\tilde{\alpha}_3)^2 + e_2(\tilde{\alpha}_1+\tilde{\alpha}_3)^2 + e_3(\tilde{\alpha}_1+\tilde{\alpha}_2)^2}{(x-e_1)(x-e_2)(x-e_3)}\Bigg)f(x) = 0.\end{aligned} \tag{5.14}$$

This equation is in the form of equation (5.1) by setting

$$\alpha = \tilde{\alpha}_0 + \tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3 = -N, \quad N \in \mathbb{Z}_{\geq 0},$$
$$\beta = -\tilde{\alpha}_0 + \tfrac{1}{2} + \tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3,$$
$$\gamma_i = 2\tilde{\alpha}_i + \tfrac{1}{2}, \quad t_i = e_i, \quad i = 1,2,3,$$
$$q = \tfrac{E}{4} + e_1(\tilde{\alpha}_2+\tilde{\alpha}_3)^2 + e_2(\tilde{\alpha}_1+\tilde{\alpha}_3)^2 + e_3(\tilde{\alpha}_1+\tilde{\alpha}_2)^2 - e_3\alpha\beta.$$

It is well known that $e_j = e_j(\tau) \in \mathbb{R}$ and $e_1 > e_3 > e_2$ for $\tau \in i\mathbb{R}_{>0}$, i.e.

$$(e_1 - e_3)(e_2 - e_3) < 0 \quad \text{if } \tau \in i\mathbb{R}_{>0}. \tag{5.15}$$

Write $f(x) = \sum_{r=0}^{\infty} c_r(x-e_3)^r$ with $c_0 = 1$, then c_r is a polynomial in q and equivalently in E of degree r. We denote it by $c_r(E)$. Then it follows from

Proposition 5.1 that if $c_{N+1}(E)=0$, then the differential equation (5.13) has a "polynomial" solution $\tilde{f}(x)=\Phi^{(\tilde{\alpha}_1,\tilde{\alpha}_2,\tilde{\alpha}_3)}(x)f(x)$ in the sense that $f(x)$ is a polynomial of degree no more than N.

Let $P_{\tilde{\alpha}_0,\tilde{\alpha}_1,\tilde{\alpha}_2,\tilde{\alpha}_3}(E)$ be *the monic polynomial obtained by normalising* $c_{N+1}(E)$. Then

$$\deg P_{\tilde{\alpha}_0,\tilde{\alpha}_1,\tilde{\alpha}_2,\tilde{\alpha}_3}(E)=N+1=-\tilde{\alpha}_0-\tilde{\alpha}_1-\tilde{\alpha}_2-\tilde{\alpha}_3+1.$$

Recall that $n_k\in\mathbb{Z}_{\geq 0}$ for all k. We recall the following important result from [24], which establishes the precise relation between the spectral polynomial $Q^{(n_0,n_1,n_2,n_3)}(E)$ and the aforementioned polynomial $P_{\tilde{\alpha}_0,\tilde{\alpha}_1,\tilde{\alpha}_2,\tilde{\alpha}_3}(E)$. For our purpose, we only consider the case that $\sum_k n_k$ is even, then $Q(E)=Q^{(n_0,n_1,n_2,n_3)}(E)$ is written as $Q(E)=P^{(0)}(E)P^{(1)}(E)P^{(2)}(E)P^{(3)}(E)$, where

$$P^{(0)}(E)=P_{-n_0/2,-n_1/2,-n_2/2,-n_3/2}(E),$$

$$P^{(1)}(E)=\begin{cases}P_{-n_0/2,-n_1/2,(n_2+1)/2,(n_3+1)/2}(E), & n_0+n_1\geq n_2+n_3+2,\\ 1, & n_0+n_1=n_2+n_3,\\ P_{(n_0+1)/2,(n_1+1)/2,-n_2/2,-n_3/2}(E), & n_0+n_1\leq n_2+n_3-2,\end{cases}$$

$$P^{(2)}(E)=\begin{cases}P_{-n_0/2,(n_1+1)/2,-n_2/2,(n_3+1)/2}(E), & n_0+n_2\geq n_1+n_3+2,\\ 1, & n_0+n_2=n_1+n_3,\\ P_{(n_0+1)/2,-n_1/2,(n_2+1)/2,-n_3/2}(E), & n_0+n_2\leq n_1+n_3-2,\end{cases}$$

$$P^{(3)}(E)=\begin{cases}P_{-n_0/2,(n_1+1)/2,(n_2+1)/2,-n_3/2}(E), & n_0+n_3\geq n_1+n_2+2,\\ 1, & n_0+n_3=n_1+n_2,\\ P_{(n_0+1)/2,-n_1/2,-n_2/2,(n_3+1)/2}(E), & n_0+n_3\leq n_1+n_2-2.\end{cases}$$

Furthermore, it was shown in [24, Theorem 3.2] that the equations $P^{(i)}(E)=0$ and $P^{(j)}(E)=0$ $(i\neq j)$ *do not have common solutions*.

Recall the following result proved in [6].

Theorem 5.4. [6] *Suppose* $n_0,n_1,n_2,n_3\in\mathbb{Z}_{\geq 0}$ *with* $\max_k n_k\geq 1$. *If* $n_3=0$, $n_0\geq n_1+n_2-1$ *and* $\tau\in i\mathbb{R}_{>0}$, *then the zeros of* $Q^{(n_0,n_1,n_2,n_3)}(E)$ *are all real and distinct.*

Here we prove the following analogous result for new cases.

Theorem 5.5. *Suppose* $n_0,n_1,n_2,n_3\in\mathbb{Z}_{\geq 0}$ *satisfying* $\max_k n_k\geq 1$ *and*

$$n_0+n_3=n_1+n_2. \tag{5.16}$$

Then for $\tau\in i\mathbb{R}_{>0}$, *the zeros of* $Q^{(n_0,n_1,n_2,n_3)}(E)$ *are all real and distinct.*

Proof. By changing variable $z\mapsto z+\frac{\omega_k}{2}$ in GLE (3.1), we have

$$\begin{aligned}Q^{(n_0,n_1,n_2,n_3)}(E)&=Q^{(n_1,n_0,n_3,n_2)}(E)\\ &=Q^{(n_2,n_3,n_0,n_1)}(E)=Q^{(n_3,n_2,n_1,n_0)}(E).\end{aligned}\tag{5.17}$$

Therefore, we may always assume $n_0=\max n_k$ and then (5.16) implies $n_3=\min n_k$. If $n_3=0$, then this theorem follows from Theorem 5.4. Therefore, we only consider $n_3\geq 1$, i.e.

$$n_0=\max n_k,\quad n_3=\min n_k\geq 1. \tag{5.18}$$

Note that $\sum n_k$ is even. We only need to show that the zeros of each polynomial $P^{(j)}(E)$, $j \in \{0,1,2,3\}$, are all real and distinct.

Since $P^{(0)}(E) = P_{-n_0/2,-n_1/2,-n_2/2,-n_3/2}(E)$, we have

$$\alpha = -\tfrac{n_0}{2} - \tfrac{n_1}{2} - \tfrac{n_2}{2} - \tfrac{n_3}{2} = -n_0 - n_3, \quad N = -\alpha = n_0 + n_3,$$
$$\gamma_3 = \tfrac{1}{2} - n_3 < 0,$$
$$\begin{aligned}\beta &= -\tilde{\alpha}_0 + \tfrac{1}{2} + \tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3 = \tfrac{n_0}{2} + \tfrac{1}{2} - \tfrac{n_1}{2} - \tfrac{n_2}{2} - \tfrac{n_3}{2} \\ &= \tfrac{1}{2} - n_3 = \gamma_3 < 0.\end{aligned}$$

Then we can apply Theorem 5.3 to see that the zeros of $P^{(0)}(E)$ are all real and distinct.

Clearly (5.16) and (5.18) imply $n_0 + n_1 \geq n_2 + n_3$. Since $P^{(1)}(E) = 1$ for $n_0 + n_1 = n_2 + n_3$, we only need to consider $n_0 + n_1 \geq n_2 + n_3 + 2$ and so

$$P^{(1)}(E) = P_{-n_0/2,-n_1/2,(n_2+1)/2,(n_3+1)/2}(E).$$

Then

$$\alpha = -\tfrac{n_0}{2} - \tfrac{n_1}{2} + \tfrac{n_2+1}{2} + \tfrac{n_3+1}{2} = n_2 + 1 - n_0 \leq 0, \quad N = -\alpha \geq 0,$$
$$\gamma_3 = \tfrac{3}{2} + n_3 > 0,$$
$$\begin{aligned}\beta &= -\tilde{\alpha}_0 + \tfrac{1}{2} + \tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3 = \tfrac{n_0}{2} + \tfrac{1}{2} - \tfrac{n_1}{2} + \tfrac{n_2+1}{2} + \tfrac{n_3+1}{2} \\ &= \tfrac{3}{2} + n_2 > 0.\end{aligned}$$

Thus by Theorem 5.2, we see that the zeros of $P^{(1)}(E)$ are all real and distinct. The same argument shows that the zeros of $P^{(2)}(E)$ are all real and distinct. Finally, we note that $P^{(3)}(E) = 1$.

In conclusion, the zeros of $Q^{(n_0,n_1,n_2,n_3)}(E)$ are all real and distinct. □

Theorem 5.6. *Suppose $n_0, n_1, n_2, n_3 \in \mathbb{Z}_{\geq 0}$ satisfying*

$$n_0 + n_3 = n_1 + n_2 \pm 1. \tag{5.19}$$

Then for $\tau \in i\mathbb{R}_{>0}$, the zeros of $Q^{(n_0,n_1,n_2,n_3)}(E)$ are all real and distinct.

Proof. By (5.17) we only need to consider the case

$$n_0 + n_3 = n_1 + n_2 + 1 \text{ and } n_0 \geq n_3. \tag{5.20}$$

Again by Theorem 5.4, we only need to consider $n_3 \geq 1$. Since $\sum n_k$ is odd, we define

$$\begin{aligned}l_0 &= (n_0 + n_1 + n_2 + n_3 + 1)/2 = n_0 + n_3 > 0,\\ l_1 &= (n_0 + n_1 - n_2 - n_3 - 1)/2 = n_1 - n_3,\\ l_2 &= (n_0 - n_1 + n_2 - n_3 - 1)/2 = n_2 - n_3,\\ l_3 &= (n_0 - n_1 - n_2 + n_3 - 1)/2 = 0.\end{aligned}$$

Then it was proved in [28, Section 4] (see also [6, Section 3]) that

$$Q^{(n_0,n_1,n_2,n_3)}(E) = Q^{(l_0,l_1,l_2,l_3)}(E).$$

Note that if $l_1 < 0$ and $l_2 < 0$, then $n_1 \leq n_3 - 1$ and $n_2 \leq n_3 - 1$, which contradict with our assumption (5.20). Thus there are three cases.

Case 1. $l_1 \geq 0$ and $l_2 \geq 0$.

Then

$$l_0 - l_1 - l_2 + 1 = 2n_3 + 2 > 0.$$

Thus, we can apply Theorem 5.4 to $Q^{(l_0,l_1,l_2,l_3)}(E)$ and obtain that the zeros of $Q^{(n_0,n_1,n_2,n_3)}(E)$ are all real and distinct.

Case 2. $l_1 \geq 0$ and $l_2 < 0$.

Then $-1 - l_2 \geq 0$. Since

$$\frac{d^2}{dz^2} - \sum_{k=0}^{3} l_k(l_k+1)\wp(z + \tfrac{\omega_k}{2})$$

is *invariant* by replacing l_2 to $-l_2 - 1$, we obtain

$$Q^{(n_0,n_1,n_2,n_3)}(E) = Q^{(l_0,l_1,l_2,l_3)}(E) = Q^{(l_0,l_1,-l_2-1,l_3)}(E).$$

Since

$$l_0 - l_1 - (-l_2 - 1) + 1 = 2n_2 + 3 > 0,$$

we can apply Theorem 5.4 to $Q^{(l_0,l_1,-l_2-1,l_3)}(E)$ and obtain that the zeros of $Q^{(n_0,n_1,n_2,n_3)}(E)$ are all real and distinct.

Case 3. $l_1 < 0$ and $l_2 \geq 0$. The proof is the same as Case 2.

In conclusion, the zeros of $Q^{(n_0,n_1,n_2,n_3)}(E)$ are all real and distinct. The proof is complete. □

We are in the position to prove Theorem 4.3.

Proof of Theorem 4.3. Since neither (4.1) nor (4.2) hold, we have one of the followings hold:

$$n_1 + n_2 - n_0 - n_3 \in \{0, 1, -1\}, \tag{5.21}$$

$$n_1 + n_2 - n_0 - n_3 \geq 2, \quad \text{either } n_1 = 0 \text{ or } n_2 = 0, \tag{5.22}$$

$$n_0 + n_3 - n_1 - n_2 \geq 2, \quad \text{either } n_0 = 0 \text{ or } n_3 = 0. \tag{5.23}$$

If (5.21) holds, the assertion follow from Theorems 5.5 and 5.6. If (5.23) holds, by (5.17) we may assume $n_3 = 0$ and so the assertion follows from Theorem 5.4. Finally, we see from (5.17) that the case (5.22) is equivalent to the case (5.23). The proof is complete. □

6. Application to pre-modular form

In this section, we apply Theorem 1.1 to the pre-modular form $Z_{r,s}^{(n)}(\tau)$ introduced in [18]. As pointed out in the introduction, the solvability of the curvature equation (1.2), i.e.

$$\Delta u + e^u = 8n\pi\delta_0 \quad \text{on } E_\tau, \tag{6.1}$$

depends essentially on the moduli τ of the flat torus E_τ and is intricate from the PDE point of view. To settle this challenging problem, Chai, Wang and the second author studied it from the viewpoint of algebraic geometry. They developed a theory to connect this PDE problem with the Lamé equation (i.e. GLE (3.1) with $n_0 = n$ and $n_k = 0$ for $k \in \{1, 2, 3\}$)

$$y''(z) = [n(n+1)\wp(z;\tau) + E]y(z)$$

and pre-modular forms. In particular, Wang and the second author [18] proved the following important result.

Theorem 6.1. [18] *There exists a pre-modular form $Z^{(n)}_{r,s}(\cdot)$ of weight $\frac{n(n+1)}{2}$ such that (6.1) on E_τ has solutions if and only if $Z^{(n)}_{r,s}(\tau) = 0$ for some $(r, s) \in \mathbb{R}^2 \setminus \frac{1}{2}\mathbb{Z}^2$.*

The pre-modular form $Z^{(n)}_{r,s}(\tau)$ is holomorphic in τ for each $(r, s) \in \mathbb{R}^2 \setminus \frac{1}{2}\mathbb{Z}^2$, and possess the following properties (see [18]):

(i) $Z^{(n)}_{r,s}(\tau) = (-1)^{n(n+1)/2} Z^{(n)}_{m\pm r, n\pm s}(\tau)$ for any $(m, n) \in \mathbb{Z}^2$.
(ii) For any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{Z})$, we define $\tau' = \gamma \cdot \tau := \frac{a\tau+b}{c\tau+d}$ and $(s', r') := (s, r) \cdot \gamma^{-1}$. Then

$$Z^{(n)}_{r',s'}(\tau') = (c\tau + d)^{\frac{n(n+1)}{2}} Z^{(n)}_{r,s}(\tau).$$

In particular, when $(r, s) \in Q_N$ is a N-torsion point for some $N \in \mathbb{N}_{\geq 3}$, where

$$Q_N := \left\{ \left(\tfrac{k_1}{N}, \tfrac{k_2}{N}\right) \middle| \gcd(k_1, k_2, N) = 1,\ 0 \leq k_1, k_2 \leq N-1 \right\}, \tag{6.2}$$

and $\gamma \in \Gamma(N) := \{\gamma \in SL(2, \mathbb{Z}) | \gamma \equiv I_2 \bmod N\}$, then $(r', s') \equiv (r, s) \bmod \mathbb{Z}^2$ and so

$$Z^{(n)}_{r,s}\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau + d)^{\frac{n(n+1)}{2}} Z^{(n)}_{r,s}(\tau).$$

Thus $Z^{(n)}_{r,s}(\tau)$ is *a modular form* of weight $\frac{n(n+1)}{2}$ with respect to the principal congruence subgroup $\Gamma(N)$. Due to this reason, $Z^{(n)}_{r,s}(\tau)$ are called *pre-modular forms* in this paper as in [18].

For $n \leq 4$, the explicit expression of $Z^{(n)}_{r,s}(\tau)$ is known; see [18]. Let $\zeta(z;\tau) := -\int^z \wp(\xi;\tau)d\xi$ be the Weierstrass zeta function, which is odd and has two quasi-periods $\eta_k(\tau) := 2\zeta(\frac{\omega_k}{2};\tau)$, $k = 1, 2$:

$$\eta_1(\tau) = \zeta(z + 1;\tau) - \zeta(z;\tau),\ \ \eta_2(\tau) = \zeta(z + \tau;\tau) - \zeta(z;\tau).$$

Define

$$Z = Z_{r,s}(\tau) := \zeta(r + s\tau;\tau) - r\eta_1(\tau) - s\eta_2(\tau).$$

Then it is known [18] that (write $\wp = \wp(r + s\tau;\tau)$ and $\wp' = \wp'(r + s\tau;\tau)$ for convenience): $Z^{(1)}_{r,s}(\tau) = Z_{r,s}(\tau)$,

$$Z^{(2)}_{r,s}(\tau) = Z^3 - 3\wp Z - \wp',$$

$$\begin{aligned} Z_{r,s}^{(3)}(\tau) =& Z^6 - 15\wp Z^4 - 20\wp' Z^3 + \left(\tfrac{27}{4}g_2 - 45\wp^2\right) Z^2 \\ & - 12\wp\wp' Z - \tfrac{5}{4}(\wp')^2. \end{aligned}$$

$$\begin{aligned} Z_{r,s}^{(4)}(\tau) =& Z^{10} - 45\wp Z^8 - 120\wp' Z^7 + (\tfrac{399}{4}g_2 - 630\wp^2) Z^6 - 504\wp\wp' Z^5 \\ & - \tfrac{15}{4}(280\wp^3 - 49g_2\wp - 115g_3)Z^4 + 15(11g_2 - 24\wp^2)\wp' Z^3 \\ & - \tfrac{9}{4}(140\wp^4 - 245g_2\wp^2 + 190g_3\wp + 21g_2^2)Z^2 \\ & - (40\wp^3 - 163g_2\wp + 125g_3)\wp' Z + \tfrac{3}{4}(25g_2 - 3\wp^2)(\wp')^2. \end{aligned}$$

For general n, the expression of $Z_{r,s}^{(n)}(\tau)$ is too complicate to be written down.

Define

$$F_0 := \{\tau \in \mathbb{H} \mid 0 \leqslant \text{ Re } \tau \leqslant 1 \text{ and } |\tau - \tfrac{1}{2}| \geqslant \tfrac{1}{2}\},$$

which is a fundamental domain of

$$\Gamma_0(2) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \middle| c \equiv 0 \mod 2 \right\}.$$

Here, as an application of Theorems 1.1 and 6.1, we have the following result.

Theorem 6.2. *Let* $(r, s) \in \mathbb{R}^2 \backslash \frac{1}{2}\mathbb{Z}^2$. *Then* $Z_{r,s}^{(n)}(\tau) \neq 0$ *for any* $\tau \in \partial F_0 \cap \mathbb{H}$.

Proof. It does not seem that this assertion could be obtained directly from the expressions of $Z_{r,s}^{(n)}(\tau)$ even for $n \leq 4$. Indeed, this lemma is a consequence of our PDE result.

Given $\tau \in \partial F_0 \cap \mathbb{H}$. If $\tau \in i\mathbb{R}_{>0}$, then Theorems 1.1 and 6.1 together imply $Z_{r,s}^{(n)}(\tau) \neq 0$ for any $(r, s) \in \mathbb{R}^2 \backslash \frac{1}{2}\mathbb{Z}^2$. If $\tau \in i\mathbb{R}_{>0} + 1$, then by applying $\gamma = \left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)$ in property (ii), we have that $\tau - 1 \in i\mathbb{R}_{>0}$ and

$$Z_{r,s}^{(n)}(\tau) = Z_{r+s,s}^{(n)}(\tau - 1) \neq 0 \text{ for any } (r, s) \in \mathbb{R}^2 \backslash \tfrac{1}{2}\mathbb{Z}^2.$$

If $|\tau - \frac{1}{2}| = \frac{1}{2}$, then again by applying $\gamma = \left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)$ in property (ii) we see that $\frac{\tau}{1-\tau} \in i\mathbb{R}_{>0}$ and

$$(1 - \tau)^{\frac{n(n+1)}{2}} Z_{r,s}^{(n)}(\tau) = Z_{r,r+s}^{(n)}(\tfrac{\tau}{1-\tau}) \neq 0 \text{ for any } (r, s) \in \mathbb{R}^2 \backslash \tfrac{1}{2}\mathbb{Z}^2.$$

This completes the proof. □

Theorem 6.2 has important applications to studying the zero structure of $Z_{r,s}^{(n)}(\tau)$. By property (ii), we can restrict τ in the fundamental domain F_0 of $\Gamma_0(2)$, and by (i), we only need to consider $(r, s) \in [0, 1] \times [0, \frac{1}{2}] \backslash \frac{1}{2}\mathbb{Z}^2$. Define four open triangles:

$$\begin{aligned} \triangle_0 &:= \{(r, s) \mid 0 < r, s < \tfrac{1}{2}, \ r + s > \tfrac{1}{2}\}, \\ \triangle_1 &:= \{(r, s) \mid \tfrac{1}{2} < r < 1, \ 0 < s < \tfrac{1}{2}, \ r + s > 1\}, \\ \triangle_2 &:= \{(r, s) \mid \tfrac{1}{2} < r < 1, \ 0 < s < \tfrac{1}{2}, \ r + s < 1\}, \end{aligned}$$

$$\triangle_3 := \{(r,s) \mid r > 0,\ s > 0,\ r+s < \tfrac{1}{2}\}.$$

In [7, 8], Theorem 6.2 was applied to prove the following results.

Theorem 6.3. [7] *Let $(r,s) \in [0,1] \times [0,\frac{1}{2}] \backslash \frac{1}{2}\mathbb{Z}^2$. Then $Z_{r,s}(\tau) = 0$ has a solution τ in F_0 if and only if $(r,s) \in \triangle_0$. Furthermore, for any $(r,s) \in \triangle_0$, the zero $\tau \in F_0$ is unqiue and satisfies $\tau \in \mathring{F}_0 = F_0 \setminus \partial F_0$.*

Theorem 6.4. [8] *Let $(r,s) \in [0,1] \times [0,\frac{1}{2}] \backslash \frac{1}{2}\mathbb{Z}^2$. Then $Z^{(2)}_{r,s}(\tau) = 0$ has a solution τ in F_0 if and only if $(r,s) \in \triangle_1 \cup \triangle_2 \cup \triangle_3$. Furthermore, for any $(r,s) \in \triangle_1 \cup \triangle_2 \cup \triangle_3$, the zero $\tau \in F_0$ is unqiue and satisfies $\tau \in \mathring{F}_0$.*

Among their applications back to the curvature equation (6.1), such results have other interesting applications. For example, Theorem 6.4 can be used to completely determine the critical points of the Eisenstein series $E_2(\tau)$ of weight 2; see [8]. We will study the zero structure of $Z^{(n)}_{r,s}(\tau)$ for $n \in \{3,4\}$ via Theorem 6.2 in future.

APPENDIX **A.** SPECTRAL THEORY AND FINITE-GAP POTENTIAL

In this appendix, we recall the spectral theory for Hill's equation with complex-valued potentials [13], which will be applied in Lemma 3.6.

Let $q(x)$ is a complex-valued continuous nonconstant periodic function of period Ω on $\mathbb{R}$. Consider the following Hill's equation

$$y''(x) + q(x)y(x) = Ey(x), \quad x \in \mathbb{R}. \tag{A.1}$$

This equation has received an enormous amount of consideration due to its ubiquity in applications as well as its structural richness; see e.g. [13, 15] and references therein for historical reviews.

Let $y_1(x)$ and $y_2(x)$ be any two linearly independent solutions of equation (A.1). Then so do $y_1(x+\Omega)$ and $y_2(x+\Omega)$ and hence there exists a monodromy matrix $M(E) \in SL(2,\mathbb{C})$ such that

$$(y_1(x+\Omega), y_2(x+\Omega)) = (y_1(x), y_2(x))M(E).$$

A solution of Hill's equation (A.1) is called *a Floquet solution* if it is a eigenfunction of the monodromy matrix $M(E)$. Define the *Hill's discriminant* $\Delta(E)$ by

$$\Delta(E) := \mathrm{tr} M(E), \tag{A.2}$$

which is clearly an invariant of (A.1), i.e. does not depend on the choice of linearly independent solutions. This $\Delta(E)$ is an entire function and plays a fundamental role since it encodes all the spectrum information of the associated operator; see e.g. [15] and references therein. Indeed, we define

$$\mathcal{S} := \Delta^{-1}([-2,2]) = \{E \in \mathbb{C} \mid -2 \leq \Delta(E) \leq 2\} \tag{A.3}$$

to be *the conditional stability set* of the operator $L = \frac{d^2}{dx^2} + q(x)$. Since $q(x)$ is assumed to be continuous, $\mathcal{S}$ can be characterized as

$$\mathcal{S} = \{E \in \mathbb{C} \mid Ly = Ey \text{ has a bounded solution on } \mathbb{R}\}.$$

Then it was proved in [23] that *this $\mathcal{S}$ coincides with the spectrum $\sigma(H)$ of the associated linear operator H in $L^2(\mathbb{R},\mathbb{C})$* (i.e. H is defined as $Hf = Lf$, $f \in H^{2,2}(\mathbb{R},\mathbb{C})$).

On the other hand, we define

$$d(E) := \mathrm{ord}_E(\Delta(\cdot)^2 - 4).$$

Then it is well known (cf. [21, Section 2.3]) that $d(E)$ equals *the algebraic multiplicity of (anti)periodic eigenvalues*. Let $c(E, x, x_0)$ and $s(E, x, x_0)$ be the special fundamental system of solutions of (A.1) satisfying by the initial values

$$c(E, x_0, x_0) = s'(E, x_0, x_0) = 1,\ c'(E, x_0, x_0) = s(E, x_0, x_0) = 0.$$

Then we have

$$\Delta(E) = c(E, x_0 + \Omega, x_0) + s'(E, x_0 + \Omega, x_0).$$

Define

$$p(E, x_0) := \mathrm{ord}_E s(\cdot, x_0 + \Omega, x_0),$$

$$p_i(E) := \min\{p(E, x_0) : x_0 \in \mathbb{R}\}.$$

It is known that $p(E, x_0)$ is the algebraic multiplicity of a Dirichlet eigenvalue on the interval $[x_0, x_0 + \Omega]$, and $p_i(E)$ denotes the immovable part of $p(E, x_0)$ (cf. [13]). It was proved in [13, Theorem 3.2] that $d(E) - 2p_i(E) \geq 0$. Define

$$D(E) := E^{p_i(0)} \prod_{\lambda \in \mathbb{C}\setminus\{0\}} \left(1 - \tfrac{E}{\lambda}\right)^{p_i(\lambda)}. \tag{A.4}$$

Let us recall the following important result proved in [13].

Theorem B. [13, Theorem 4.1] *Assume that $q(x)$ is a complex-valued continuous nonconstant periodic function of period Ω on $\mathbb{R}$ and that equation (A.1) has two linearly independent Floquet solutions for all $E \in \mathbb{C} \setminus \{E_j\}_{j=1}^{\tilde{m}}$ for some $\tilde{m} \in \mathbb{Z}_{\geq 0}$ and precisely one Floquet solution for each $E = E_j$. Then*

(i) $d(E) - 2p_i(E) > 0$ on a finite set $\{E_j\}_{j=1}^{m}$ including $\{E_j\}_{j=1}^{\tilde{m}}$, $m \geq \tilde{m}$, and $d(E) - 2p_i(E) = 0$ elsewhere. The Wronskian of two nontrivial Floquet solutions which are linearly independent on some punctured disk $0 < |E - \lambda| < \varepsilon$ tends to zero as $E \to \lambda$ if and only if $\lambda \in \{E_j\}_{j=1}^{m}$.

(ii) $\sum_{j=1}^{m}(d(E_j) - 2p_i(E_j)) = 2g + 1$ for some $g \in \mathbb{Z}_{\geq 0}$ and $q(x)$ is an algebro-geometric finite gap potential associated with the compact (possibly singular) hyperelliptic curve obtained upon one-point compactification of the curve

$$F^2 = R_{2g+1}(E) := \prod_{j=1}^{m}(E - E_j)^{d(E_j) - 2p_i(E_j)} = C\frac{\Delta(E)^2 - 4}{D(E)^2}. \tag{A.5}$$

Here $D(E)$ is seen in (A.4) and C is some nonzero constant.

(iii) the spectrum $\sigma(H) = \mathcal{S}$ consists of finitely many bounded spectral arcs σ_k, $1 \leq k \leq \tilde{g}$ for some $\tilde{g} \leq g$ and one semi-infinite arc σ_∞ which tends to $-\infty + \langle q \rangle$, with $\langle q \rangle = \frac{1}{\Omega} \int_{x_0}^{x_0+\Omega} q(x)dx$, i.e.

$$\sigma(H) = \mathcal{S} = \sigma_\infty \cup \cup_{k=1}^{\tilde{g}} \sigma_k.$$

Furthermore, the finite end points of such arcs must be those $E \in \{E_j\}_{j=1}^m$ with $d(E)$ odd.

Remark that if we assume in addition that $q(x)$ is *real-valued* in Theorem B, then it is well-known (cf. [13, 15]) that $R_{2g+1}(E)$ has $2g+1$ distinct real zeros, denoted by $E_{2g} < E_{2g-1} < \cdots < E_1 < E_0$, and

$$\sigma(H) = \mathcal{S} = (-\infty, E_{2g}] \cup [E_{2g-1}, E_{2g-2}] \cup \cdots \cup [E_1, E_0], \tag{A.6}$$

namely the spectrum has the so-called *finite-gap property*, and so $q(x)$ is a so-called *finite-gap potential*. In Section 3, we show that even if the continuous function $q(x)$ is *not real-valued*, the finite-gap property (A.6) might still hold in some special situations; see Lemma 3.6.

Acknowledgements The authors thanks the anonymous referee very much for careful reading and valuable comments. The research of the first author was supported by NSFC (No. 11701312).

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