ON ALGEBRO-GEOMETRIC SIMPLY-PERIODIC SOLUTIONS OF THE KDV HIERARCHY

ZHIJIE CHEN AND CHANG-SHOU LIN

ABSTRACT. In this paper, we show that as $\tau \to \sqrt{-1}\infty$, any zero of the Lamé function converges to either ∞ or a finite point p satisfying $\operatorname{Re} p = \frac{1}{2}$ and $e^{2\pi i p}$ being an algebraic number. Our proof is based on studying a special family of simply-periodic KdV potentials with period 1, i.e. algebro-geometric simply-periodic solutions of the KdV hierarchy. We show that except the pole 0, all other poles of such KdV potentials locate on the line $\operatorname{Re} z = \frac{1}{2}$. We also compute explicitly the eigenvalue set of the corresponding $L^2[0,1]$ eigenvalue problem for such KdV potentials, thus extends Takemura's works [26, 27]. Our main idea is to apply the classification result for simply-periodic KdV potentials by Gesztesy, Unterkofler and Weikard [11] and the Darboux transformation.

1. INTRODUCTION

In this paper, we study two problems related to algebro-geometric solutions of the KdV hierarchy. The first problem is about the asymptotics of the classical Lamé equation

$$y''(z) = [\ell(\ell+1)\wp(z;\tau) - E]y(z), \quad \ell \in \mathbb{N}, \tag{1.1}$$

as $\tau \to i\infty$. Here $\tau \in i\mathbb{R}_{>0}$ and $\wp(z;\tau) := \wp(z;\Lambda_\tau)$ is the Weierstrass $\wp$-function with respect to $\Lambda_\tau = \mathbb{Z} + \mathbb{Z}\tau$. We refer the reader to the classic texts [15, 33] and recent works [3, 13, 20, 21] for more introduction of (1.1).

In the seminal paper [16], Ince proved that the Lamé potential $q(z;\tau) = \ell(\ell+1)\wp(z;\tau)$ is a finite-gap potential. In the literature, a smooth periodic function $q(x)$, $x \in \mathbb{R}$, is called a *finite-gap potential* if the set $\sigma_b(L)$ of $L = -d^2/dx^2 + q(x)$ satisfies

$$\sigma_b(L) = [E_0, E_1] \cup [E_2, E_3] \cup \cdots \cup [E_{2g}, +\infty) \tag{1.2}$$

with $E_0 < E_1 < \cdots < E_{2g}$, where $\sigma_b(L)$ is the spectrum of bounded bands, that is,

$$E \in \sigma_b(L) \Longleftrightarrow \textit{There is a solution of } (L-E)f(x) = 0 \textit{ which is bounded on the whole real axis.} \tag{1.3}$$

It is well-known that a finite-gap potential is an algebro-geometric solution of the KdV hierarchy. A potential $q(z)$ is called *an algebro-geometric solution of the KdV hierarchy* or simply *a KdV potential* if there is a monic odd-order differential operator P such that the commutator $[P, -d^2/dz^2 + q(z)] = 0$;

see Section 2 for a brief review. Under the condition that $q(z)$ is smooth periodic and real-valued on $\mathbb{R}$, it is known (see e.g. [10]) that $q(z)$ is a finite-gap potential if and only if it is a KdV potential.

For the Lamé potential, it follows from [16] that there are

$$E_0(\tau) < E_1(\tau) < \cdots < E_{2\ell}(\tau) \tag{1.4}$$

such that for $q(x) = \ell(\ell+1)\wp(x + \frac{\omega_k}{2}; \tau)$, $k \in \{2,3\}$ where $\omega_2 = \tau$, $\omega_3 = 1+\tau$, the corresponding $\sigma_b(L)$ is of the form (1.2) with $g = \ell$. The polynomial $Q(E;\tau) := \prod_{j=0}^{2\ell}(E - E_j(\tau))$ is called *the spectral polynomial* of the Lamé potential. Fix $0 \le k \le 2\ell$. It is classical (cf. [33]) that there exists a unique $\boldsymbol{a} = \boldsymbol{a}(\tau) := \{a_1, \cdots, a_\ell\} \subset E_\tau \setminus \{0\}$ such that $\boldsymbol{a} \equiv -\boldsymbol{a} \mod \Lambda_\tau$ and the classical *Hermite-Halphen ansatz*

$$y_{\boldsymbol{a}}(z;\tau) := e^{\sum_{j=1}^{\ell} \zeta(a_j;\tau) z} \frac{\prod_{j=1}^{\ell} \sigma(z - a_j; \tau)}{\sigma(z;\tau)^\ell}$$

is a solution of (1.1) with $E = E_k(\tau)$. Here $\zeta(z;\tau)$ and $\sigma(z;\tau)$ are the associated Weierstrass functions of $\wp(z;\tau)$ and $E_\tau = \mathbb{C}/\Lambda_\tau$ is a flat torus. This $y_{\boldsymbol{a}}(z;\tau)$ is known as the *Lamé function* in the literature.

In this paper, we study the asymptotics of the zeros of the Lamé function as $\tau \to i\infty$. Our first result is

Theorem 1.1. *Fix $1 \le k \le \ell$ and consider the Lamé function $y_{\boldsymbol{a}}(z;\tau)$ at $E \in \{E_{2k-1}(\tau), E_{2k}(\tau)\}$ with its zero set*

$$\boldsymbol{a} = \boldsymbol{a}(\tau) = \{a_1(\tau), \cdots, a_\ell(\tau)\} \subset E_\tau \setminus \{0\}, \quad \operatorname{Re} a_j \in [0,1],\ \forall j.$$

Then as $\tau \to i\infty$, the followings hold.

1. *There are k zeros $a_j(\tau)$'s converging to infinity;*
2. *The other $\ell - k$ zeros $a_j(\tau)$'s converge to $\ell - k$ distinct points p_j's which satisfy that* $\operatorname{Re} p_j = \frac{1}{2}$ *and $e^{2\pi i p_j}$ is an algebraic number for any j.*

Our basic idea of proving Theorem 1.1 comes from the well-known fact:

$$\ell(\ell+1) \lim_{\tau\to i\infty} \wp(z;\tau) = \ell(\ell+1)\left(\tfrac{\pi^2}{(\sin \pi z)^2} - \tfrac{\pi^2}{3}\right) =: q_0(z) + e_0,$$

where $e_0 := -\frac{\pi^2}{3}\ell(\ell+1)$ and $q_0(z) := \ell(\ell+1)\frac{\pi^2}{(\sin \pi z)^2}$ is also a KdV potential. From here we will see in Section 6 that $E_{2k-1}(\tau), E_{2k}(\tau) \to e_0 + k^2\pi^2$ as $\tau \to i\infty$. Noting that $q_0(\frac{1}{2} + ix) = \ell(\ell+1)\frac{\pi^2}{(\cosh x)^2}$, we will see in Section 4 that $\operatorname{Im} p_j$'s are the zeros of eigenfunctions of the negative spectrum of the following Schrodinger equation

$$-y''(x) - \frac{\ell(\ell+1)\pi^2}{[\cosh(\pi x)]^2} y(x) = \lambda y(x). \tag{1.5}$$

It is known that the negative part of the spectrum is discrete and finite, which consists of simple eigenvalues

$$\lambda_j = -(\ell - j)^2\pi^2, \quad 0 \le j \le \ell - 1. \tag{1.6}$$

The classical proof of this statement is to transform (1.5) to the hypergeometric equation; see e.g. [18, p.73-74]. In Section 4 of this paper, we will give a new proof of (1.6) via the KdV theory, thus it also works for a large family of simply-periodic KdV potentials of the form

$$\begin{aligned} q(z) =& g(g+1)\widetilde{\mathcal{P}}(z) + m(m+1)\widetilde{\mathcal{P}}(z-\tfrac{1}{2}) \\ &+ \sum_{j=1}^{r} m_j(m_j+1)(\widetilde{\mathcal{P}}(z-p_j)+\widetilde{\mathcal{P}}(z+p_j)), \\ &\text{with} \quad 0 \le m, m_j \le g-1, \end{aligned} \tag{1.7}$$

where we denote

$$\widetilde{\mathcal{P}}(z) := \pi^2[\sin(\pi z)]^{-2} \tag{1.8}$$

for convenience. In this paper, a simply-periodic KdV potential $q(z)$ of the form (1.7) is called *strict* if its associated spectral polynomial is of degree $2g+1$. We will briefly review this notion for all kinds of KdV potentials in Section 2. By applying the classification result of simply-periodic KdV potentials by Gesztesy, Unterkofler and Weikard [11], we will prove in Section 3 that any simply-periodic KdV potential with its basic period 1 and its spectral polynomial of degree $2g+1$ is isospectral to a strict simply-periodic KdV potential $q(z)$ of the form (1.7).

Remark that for a strict KdV potential $q(z)$ given by (1.7), it follows from Theorem 3.2 below from [11] that there are $1 \le n_1 < \cdots < n_g$ satisfying $\gcd(n_1, \cdots, n_g) = 1$ and

$$\sum_{j=1}^{g} n_j = \frac{g(g+1)+m(m+1)}{2} + \sum_{j=1}^{r} m_j(m_j+1)$$

such that the spectral polynomial of $q(z)$ is given by

$$Q_{q,2g+1}(E) = E\prod_{j=1}^{g}(E-n_j^2\pi^2)^2. \tag{1.9}$$

See Sections 3-4 for more details. Here we have the following characterization of the poles p_j's. For an algebraic number α, any other root of its minimial polynomial $f(x) \in \mathbb{Q}[x]$ is called *a conjugate* of α.

Theorem 1.2. *Let $q(z)$ be a strict KdV potential given by (1.7) with its spectral polynomial given by (1.9). Then we have*

1. *for any $1 \le j \le r$, $e^{2\pi i p_j}$ is an algebraic number. Moreover, any conjugate of $e^{2\pi i p_j}$ belongs to $\{e^{\pm 2\pi j p_k} | 1 \le k \le r\}$.*
2. *for any $1 \le k \le g$, any root p of the Baker-Akhiezer function $\psi(P_k, z, z_0)$ of $q(z)$ at $P_k = (n_k^2\pi^2, 0)$ satisfies that $e^{2\pi i p}$ is an algebraic number.*

The notion of the Baker-Akhiezer function in Theorem 1.2 will be reviewed in Section 2.

Remark 1.3. For any KdV potential $q(z)$ of the form (1.7), we take any $b \in \mathbb{R}$ such that $q(z)$ has no poles on $\mathbb{R} + ib$ and denote $\hat{q}(x) = q(x + ib)$. Then the spectrum $\sigma_b(L)$ of bounded bands for $L = -d^2/dx^2 + \hat{q}(x)$ satisfies

$$\sigma_b(L) = [0, +\infty). \tag{1.10}$$

This easily follows from the two facts: (i) By applying the Darboux transformations via the Baker-Akhiezer functions to $q(z)$ finitely many times, we obtain the constant potential $\tilde{q}(z) = 0$; See [11] or Section 3 of this paper. (ii) The Hill's discriminant $\Delta(E)$ (i.e. the trace of the monodromy matrix of $y''(z) = [q(z) - E]y(z)$ with respect to $z \to z + 1$) is *invariant* under the Darboux transformation via the Baker-Akhiezer function. In particular, the Hill's discriminant $\Delta(E)$ of $q(z)$ is the same as the one of the constant potential $\tilde{q}(z) = 0$, which implies (1.10) and

$$\{E \,|\, \Delta(E) = \pm 2\} = \{\pi^2 n^2 \,|\, n \in \mathbb{Z}_{\geq 0}\}.$$

Remark 1.3 shows that the spectrum information obtained from the Hill's discriminant $\Delta(E)$ can not distinguish the potentials. In this paper, we study another type of eigenvalue problem which was already studied by Takemura [26, 27, 29]. Let $q(z)$ be a strict KdV potential given by (1.7) such that $p_j \notin \mathbb{R}$ for all j. We will see in Remark 4.3 that $q(x)$ is real-valued for $x \in [0, 1]$. Clearly $q(x)$ has singularities at $0, 1$ (and $\frac{1}{2}$ if $m \geq 1$) on $[0, 1]$. Consider the following eigenvalue problem which is of particular physical interest:

$$\begin{cases} -\varphi''(x) + q(x)\varphi(x) = \lambda\varphi(x), \quad x \in [0, 1], \\ \varphi(x) \in L^2[0, 1], \text{ i.e. } \int_0^1 |\varphi(x)|^2 dx < \infty. \end{cases} \tag{1.11}$$

Here "physical interest" means that the eigenfunction is contained in an appropriate Hilbert space, which is often the L^2 space (cf. [27]). We will see that $\varphi(x) \in L^2[0, 1]$ is equivalent to $\varphi(0) = \varphi(1) = 0$ (and $\varphi(\frac{1}{2}) = 0$ if $m \geq 1$). Clearly this eigenvalue problem depends on the potential $q(z)$.

Our second problem is to find all the eigenvalues λ's such that (1.11) has a solution. The special case $m_j = 0$ for all j, i.e.

$$\begin{aligned} q(z) =& g(g+1)\widetilde{\mathcal{P}}(z) + m(m+1)\widetilde{\mathcal{P}}(z - \tfrac{1}{2}), \\ =& g(g+1)\pi^2[\sin \pi z]^{-2} + m(m+1)\pi^2[\cos \pi z]^{-2}, \;\; 0 \leq m < g, \end{aligned} \tag{1.12}$$

was already considered by Takemura [26, 27, 29]. Like the case $m = 0$, this $q(z)$ is a limit of the following Treibich-Verdier potential [31] (which is well known as new elliptic KdV potentials besides the Lamé potential)

$$\begin{aligned} & g(g+1)\wp(z;\tau) + m(m+1)\wp(z - \tfrac{1}{2};\tau) \\ & + [g(g+1) + m(m+1)]\tfrac{\pi^2}{3}. \end{aligned} \tag{1.13}$$

The complete set of eigenvalues of (1.11) with $q(z)$ being (1.12) are given by (see e.g. [26, Section 5] for $m \geq 1$ and [29, Section 3] for $m = 0$)

$$\{\pi^2(g + 1 + k)^2 \,|\, k \in \mathbb{Z}_{\geq 0}\} \quad \text{if } m = 0, \tag{1.14}$$

$$\{\pi^2(g+m+2+2k)^2 \,|\, k \in \mathbb{Z}_{\geq 0}\} \quad \text{if } m \geq 1. \tag{1.15}$$

Since $q(z)$ in (1.12) has only two singularities $0, \frac{1}{2} \mod \mathbb{Z}$, the proof in [26, 29] is to transform (1.11) to *hypergeometric equations* via gauge transformations, but this method can not work for general $q(z)$ given by (1.7).

In this paper, we develop a unified approach to solve the eigenvalue problem (1.11) for *a special family of KdV potentials* given by (1.7) including (1.12). Our third result of this paper is as follows, which generalizes (1.14)-(1.15) to a large family of KdV potentials.

Theorem 1.4. *Let $q(z)$ be a strict simply-periodic KdV potential of the form (1.7) with its spectral polynomial given by (1.9). Suppose that*

$$\pm p_j \notin \mathbb{R} \text{ for all } 1 \leq j \leq r \text{ if } r \geq 1. \tag{1.16}$$

Then the following hold.

(1) *If $m = 0$, then the set*

$$\Theta_0 := \left\{ \pi^2 n^2 \middle| \, n \in \mathbb{N} \setminus \{n_1, \cdots, n_g\} \right\}$$

gives all the eigenvalues of the eigenvalue problem (1.11). Furthermore, the eigenfunction $y(x)$ of the eigenvalue $\pi^2 n^2$ satisfies $y(\frac{1}{2}) = 0$ if and only if $n - g$ is even.

(2) *If $m \geq 1$, then the eigenvalue set is given by*

$$\Theta_m := \left\{ \pi^2 \Big(g+m+2-2\sum_{j=1}^{r} m_j + 2k \Big)^2 \middle| \, k \in \mathbb{Z}_{\geq 0}, \right.$$
$$\left. k > \sum_{j=1}^{r} m_j - \tfrac{g+m+2}{2} \right\} \setminus \{n_1^2\pi^2, \cdots, n_g^2\pi^2\}.$$

Remark 1.5. Remark that $q(z)$ given by (1.12) satisfies all assumptions of Theorem 1.4. A more precise statement of Theorem 1.4 will be given in Sections 4-5, where we will give *infinitely many new examples* of potentials $q(z)$ of the form (1.7) satisfying all assumptions of Theorem 1.4, and the exact values of the corresponding $n_1, \cdots, n_g$ will also be given. We believe that Theorem 1.4 should have important applications.

The rest of this paper is organized as follows. As we will see that the notion of strict KdV potentials plays an essential role in our proof. Thus in Section 2, we first review the KdV hierarchy in general setting and then introduce the notion of strict KdV potentials. Some important properties for strict KdV potentials are also collected. We also review the basic theory concerning the Darboux transformation of the KdV potentials from [7, 9] for later usage. In Section 3, we introduce Gesztesy, Unterkofler and Weikard's classification result [11] for simply-periodic KdV potentials and apply it to prove the existence of strict simply-periodic KdV potentials. Theorem 1.2 will be proved as a consequence. In Section 4, we show that the poles of some special strict simply-periodic KdV potentials locate on $\operatorname{Re} z = \frac{1}{2}$. As applications, Theorem 1.4 will be proved in Section 5, and Theorem 1.1 will be proved in Section 6.

2. Preliminaries

2.1. **The KdV hierarchy.** In this section, we review basic facts on the stationary KdV hierarchy following [10, Chapter 1]. Assuming $q(z)$ meromorphic in $\mathbb{C}$, we define $\{f_\ell(z)\}_{\ell\in\mathbb{N}\cup\{0\}}$ recursively by

$$f_0 = 1, \quad f_\ell' = -\tfrac{1}{4}f_{\ell-1}^{(3)} + qf_{\ell-1}' + \tfrac{1}{2}q' f_{\ell-1}, \quad \ell\in\mathbb{N}. \tag{2.1}$$

Explicitly, one finds

$$f_0 = 1, \quad f_1 = \tfrac{1}{2}q + c_1,$$
$$f_2 = -\tfrac{1}{8}(q'' - 3q^2) + c_1\tfrac{1}{2}q + c_2, \quad \text{etc.}$$

Here $\{c_\ell\}_{\ell\in\mathbb{N}}\subset\mathbb{C}$ denote integration constants that naturally arise when solving (2.1). Consider a second-order differential operator of Schrödinger-type

$$L = -\frac{d^2}{dz^2} + q(z)$$

and a $(2g+1)$-order differential operator

$$P_{2g+1} = \sum_{j=0}^{g}\left(f_j\frac{d}{dz} - \tfrac{1}{2}f_j'\right)L^{g-j}, \quad g\in\mathbb{N}\cup\{0\}.$$

By the recursion (2.1), a direct computation leads to ($[\cdot,\cdot]$ the commutator symbol)

$$[L, P_{2g+1}] = -2f_{g+1}', \quad g\in\mathbb{N}\cup\{0\}.$$

In particular, (L, P_{2g+1}) represents the celebrated *Lax pair* [19] of the KdV hierarchy. Varying $g\in\mathbb{N}\cup\{0\}$, the stationary KdV hierarchy is then defined in terms of the vanishing of the commutator of L and P_{2g+1} by

$$\text{s-KdV}_g(q) = [L, P_{2g+1}] = -2f_{g+1}' = 0, \quad g\in\mathbb{N}\cup\{0\}. \tag{2.2}$$

For example,

$$\text{s-KdV}_0(q) = -q' = 0,$$
$$\text{s-KdV}_1(q) = \tfrac{1}{4}q^{(3)} - \tfrac{3}{2}qq' + c_1(-q') = 0, \quad \text{etc.},$$

represent the first few equations of the stationary KdV hierarchy. By definition, the set of solutions of (2.2), with g ranging in $\mathbb{N}\cup\{0\}$ and c_ℓ in $\mathbb{C}$, represents the class of *algebro-geometric KdV solutions*. As in [10, 11], it will be convenient to abbreviate algebro-geometric stationary KdV solutions q simply as *KdV potentials*. It was shown by Segal and Wilson [24, Theorem 6.10] that any KdV potential which is smooth in some real interval can be extended to a meromorphic function on $\mathbb{C}$.

The KdV potentials play an important role for the study of the KdV equation itself and have been widely studied in the literature. We refer the reader to Gesztesy and Holden's text [10] for a complete introduction and detailed references for this topic. There are three kinds of KdV potentials that have received great interest: *rational, simply-periodic, and elliptic KdV potentials*; see e.g. [1, 5, 6, 8, 11, 12, 17, 22, 24, 25, 26, 27, 30, 31, 32] and references therein. Remark that in [8], Etingof and Rains also studied higher order differential operators $L_m = \partial^m + a_2(z)\partial^{m-2} + \cdots + a_m(z)$ with $m \geq 3$ which are *algebraically integrable* (i.e. there is a nonzero differential operator P of order relatively prime to L_m such that $[L_m, P] = 0$). Such kinds of problems were first studied by Burchnall and Chaundy [2].

Next, we introduce a monic polynomial $\Phi_g = \Phi_{q,g}$ of degree g with respect to the spectral parameter $E \in \mathbb{C}$ by

$$\Phi_g(z;E) = \Phi_{q,g}(z;E) := \sum_{\ell=0}^{g} f_{g-\ell}E^{\ell}, \quad g \in \mathbb{N} \cup \{0\}. \tag{2.3}$$

The recursive relation (2.1) and (2.2) together imply that $\Phi_{q,g}$ solves

$$\Phi''' - 4(q-E)\Phi' - 2q'\Phi = 0. \tag{2.4}$$

Consequently,

$$Q_{q,2g+1}(E) := \tfrac{1}{2}\Phi_{q,g}\Phi''_{q,g} - \tfrac{1}{4}\Phi'^2_{q,g} - (q-E)\Phi^2_{q,g} \tag{2.5}$$

is a monic polynomial in E of degree $2g+1$ which is independent of z. Since $[L, P_{2g+1}] = 0$ implies $P^2_{2g+1}\phi = -Q_{q,2g+1}(E)\phi$ for any $L\phi = E\phi$, we conclude that

$$\mathcal{F}_g(L, iP_{2g+1}) := -P^2_{2g+1} - Q_{q,2g+1}(L) = 0, \tag{2.6}$$

a celebrated theorem by Burchnall and Chaundy [2]. Equation (2.6) leads naturally to the hyperelliptic curve $\mathcal{K}_g$ of (arithmetic) genus g:

$$\mathcal{K}_g: \quad \mathcal{F}_g(E, \mathcal{C}) = \mathcal{C}^2 - Q_{q,2g+1}(E) = 0. \tag{2.7}$$

Remark 2.1. (1) As mentioned in [10, Remark 1.5], if $q(z)$ satisfies one stationary KdV equation s-KdV$_g(q) = 0$ for some g, then it also satisfies s-KdV$_p(q) = 0$ for any $p > g$. In fact, (2.2) says that s-KdV$_g(q) = 0$ is equivalent to $f_{g+1} = d_{g+1}$ for some constant $d_{g+1} \in \mathbb{C}$. By replacing f_{g+1} by $f_{g+1} - d_{g+1}$, we may assume $f_{g+1} = 0$. Consequently, (2.1) and (2.2) yield s-KdV$_{g+1}(q) = -2f'_{g+2} = 0$.

(2) In this paper, we say q is *a genus g KdV potential* if g is the smallest integer such that s-KdV$_g(q) = 0$, i.e. s-KdV$_{g-1}(q) \neq 0$ for any choices of integration constants c_k's. For a genus g KdV potential $q(z)$, the corresponding $Q_{q,2g+1}(E)$ is called its *spectral polynomial*.

Now we recall the notion of the Baker-Akhiezer function, the common eigenfunction of L and P_{2g+1}. Let $q(z)$ be a genus g KdV potential and $\mathcal{K}_g$ given in (2.7) be its associate hyperelliptic curve. The one-point compactification of $\mathcal{K}_g$ by joining P_∞, the point at infinity, is still denoted by $\mathcal{K}_g$. A general point $P \in \mathcal{K}_g \setminus P_\infty$ will be denoted by $P = (E, \mathcal{C})$, where $\mathcal{F}_g(E, \mathcal{C}) = \mathcal{C}^2 - Q_{q,2g+1}(E) = 0$. We also define the involution $*$ on $\mathcal{K}_g$ by

$$* : \mathcal{K}_g \to \mathcal{K}_g, \quad P = (E, \mathcal{C}) \mapsto P^* = (E, -\mathcal{C}),\ P^*_\infty = P_\infty.$$

Recalling $\Phi_g(z;E)$ in (2.3), we define the following fundamental meromorphic function $\phi(P,z)$ by

$$\phi(P,z) := \frac{iC(P) + \frac{1}{2}\Phi_g'(z;E)}{\Phi_g(z;E)}, \quad P = (E,C) \in \mathcal{K}_g,\ z \in \mathbb{C}, \tag{2.8}$$

where $C(P)$ denotes the meromorphic function on $\mathcal{K}_g$ obtained upon sloving $C^2 = Q_{2g+1}(E)$ with $P = (E,C)$. Then the stationary Baker-Akhiezer function $\psi(P,z,z_0)$ on $\mathcal{K}_g \setminus \{P_\infty\}$ is defined by

$$\psi(P,z,z_0) := \exp\left(\int_{z_0}^{z} \phi(P,\xi)d\xi\right),\ P = (E,C) \in \mathcal{K}_g \setminus \{P_\infty\},\ z,z_0 \in \mathbb{C}, \tag{2.9}$$

where the integral path is chosen a smooth non-selfintersecting path from z_0 to z which avoids singularities of $\phi(P,z)$. It is known (cf. [9, Lemma 2.1]) that

$$\psi''(P,z,z_0) = [q(z) - E]\psi(P,z,z_0), \tag{2.10}$$

$$\phi(P,z) = \frac{\psi'(P,z,z_0)}{\psi(P,z,z_0)}, \quad \phi'(P,z) = q(z) - E - \phi(P,z)^2, \tag{2.11}$$

$$\psi(P,z,z_0)\psi(P^*,z,z_0) = \frac{\Phi_g(z;E)}{\Phi_g(z_0;E)}, \tag{2.12}$$

$$W(\psi(P,z,z_0),\psi(P^*,z,z_0)) = \frac{2iC(P)}{\Phi_g(z_0;E)}, \tag{2.13}$$

where $' = \frac{d}{dz}$ and $W(f,g) = f'g - fg'$ denotes the Wronskian of f,g. Remark that different choices of z_0 give the same solution of the linear equation (2.10) up to multiplying a constant. The following well-known result shows that the Baker-Akhiezer function $\psi(P,\cdot,z_0)$ is meromorphic in $\mathbb{C}$.

Theorem 2.2. *(cf.* [24] *or* [32, Theorem 1]*) Let $q(z)$ be a KdV potential. Then*

(1) *Any pole z_0 of $q(z)$ is a regular singular point of*

$$y''(z) = [q(z) - E]y''(z). \tag{2.14}$$

The principal part of the Laurent expansion of $q(z)$ near z_0 is given by $\frac{k(k+1)}{(z-z_0)^2}$ for some $k \in \mathbb{N}$. In particular, the residue of $q(z)$ at z_0 is 0*.*

(2) *For any $E \in \mathbb{C}$, all solutions of (2.14) are meromorphic in $\mathbb{C}$.*

Remark that in the case when the hyperelliptic curve associated with $q(z)$ is non-singular, Theorem 2.2 also follows from Its and Matveev [17].

2.2. **Strict KdV potentials.** In this section, we introduce the notion of strict KdV potentials and collect important properties from our another work [4] joint with Kuo.

Let $q(z)$ be a genus $g \geq 1$ KdV potential with $\{p_i\}_{i\in I} \neq \emptyset$ being the set of poles of $q(z)$. Since $q(z)$ is meromorphic, any pole is isolated in $\{p_i\}_{i\in I}$ and the index set I is countable. Fix any pole p_i of $q(z)$. By Theorem 2.2, the Laurent expansion of $q(z)$ at $z = p_i$ is written as

$$q(z) = \sum_{j=0}^{\infty} b_j (z - p_i)^{j-2}, \tag{2.15}$$

with

$$b_0 = n_i(n_i + 1) \text{ for some } n_i \in \mathbb{N}, \quad b_1 = 0. \tag{2.16}$$

Definition 2.3. *We call that a genus $g \geq 1$ KdV potential is a strict KdV potential if the pole set $\{p_i\}_{i\in I} \neq \emptyset$ and the corresponding $I_g := \{j \in I | n_j = g\} \neq \emptyset$.*

The typical example of strict KdV potentials is the Lamé potential $n(n + 1)\wp(z;\Lambda)$, $n \in \mathbb{N}$, which is a genus n strict KdV potential. We can give the following characterization of strict KdV potentials in [4].

Theorem 2.4. [4] *Let $q(z)$ be a genus $g \geq 1$ strict KdV potential with I_g defined as above. By changing variable $z \to z + p_i$ we can always assume $0 \in \{p_i\}_{i\in I_g}$. Then $q(z)$ is even. More precisely, one of the following holds:*

1. *$\{p_i\}_{i\in I_g} = \{0\}$ and $q(z) - \frac{g(g+1)}{z^2}$ is an even function. Furthermore, the leading term $\frac{n(n+1)}{(z-p)^2}$ of $q(z)$ at any pole $p \neq 0$ (if exists) satisfies $n < g$.*
2. *there is $\omega \in \mathbb{C} \setminus \{0\}$ such that $\{p_i\}_{i\in I_g} = \mathbb{Z}\omega$, and*
$$q(z) - g(g + 1)\tfrac{\pi^2}{\omega^2}[\sin(\pi z/\omega)]^{-2}$$
is an even and simply-periodic function with period ω. Furthermore, the leading term $\frac{n(n+1)}{(z-p)^2}$ of $q(z)$ at any pole $p \notin \mathbb{Z}\omega$ (if exists) satisfies $n < g$.
3. *there are ω_1, ω_2 satisfying $\mathrm{Im}(\omega_2/\omega_1) > 0$ such that $\{p_i\}_{i\in I_g} = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 =: \Lambda$, and*
$$\begin{aligned} q(z) =& g(g + 1)\wp(z;\Lambda) + \sum_{k=1}^{3} m_k(m_k + 1)\wp(z - \tfrac{\omega_k}{2};\Lambda) \\ &+ \sum_{j=1}^{N} n_j(n_j + 1)[\wp(z - p_j;\Lambda) + \wp(z + p_j;\Lambda)] + E \end{aligned}$$
is even and elliptic with respect to Λ. Here $E \in \mathbb{C}$, $\omega_3 = \omega_1 + \omega_2$ and $m_k \in [0, g - 1]$ for all $k = 1, 2, 3$; $N \geq 0$ and if $N > 0$, then $2p_j \notin \Lambda$, $1 \leq n_j < g$ for all j and $p_i \not\equiv \pm p_j \mod \Lambda$ for any $i \neq j$.

It is well-known that KdV potentials are not unique with respect to the spectral polynomial. For example, $6\wp(z;\Lambda)$ and $2\wp(z;\Lambda) + 2\wp(z - \frac{\omega_1}{2};\Lambda) +$

$2\wp(z-\frac{\omega_2}{2};\Lambda)$ are both genus 2 KdV potentials and their spectral polynomials are the same; see e.g. [28]. Remark that $6\wp(z;\Lambda)$ is a strict KdV potential while the latter is not. Here we recall the uniqueness result of strict KdV potentials (up to a translation) in a given isospectral set of KdV potentials (i.e. a set consisting of those KdV potentials such that their associate spectral polynomials are the same). This is another important property of strict KdV potentials.

Theorem 2.5. [4] *Let $q_1(z), q_2(z)$ be both genus $g \geq 1$ strict KdV potentials. If their corresponding spectral polynomials $Q_{q_1,2g+1}(E) = Q_{q_2,2g+1}(E)$, then $q_1(z) = q_2(z+z_0)$ for some $z_0 \in \mathbb{C}$.*

We emphasize that the above results hold for all kinds of KdV potentials. In [4] Theorems 2.4-2.5 are applied to classify genus 1 KdV potentials. Here they will be applied to simply-periodic KdV potentials in Sections 3-5.

2.3. **The Darboux transformation.** In this section, we recall the important property of the well-known Darboux transformation for the KdV hierarchy from [7, 9].

Theorem 2.6. [7, 9] *Suppose $q(z)$ is a genus g KdV potential with the associate spectral polynomial $Q_{q,2g+1}(E)$. For any $P_0 = (E_0, \mathcal{C}_0) \in \mathcal{K}_g \setminus \{P_\infty\}$, we let $y(z)$ be any solution of*

$$y''(z) = [q(z) - E_0]y(z) \tag{2.17}$$

and define a new potential $\tilde{q}(z)$ via the Darboux transformation

$$\tilde{q}(z) := q(z) - 2\left(\frac{y'(z)}{y(z)}\right)'. \tag{2.18}$$

Then the followings hold.

1. *If $y(z)$ is not a Baker-Akhiezer function of (2.17), then $\tilde{q}(z)$ is a genus $g+1$ KdV potential with the associate spectral polynomial $Q_{\tilde{q},2g+3}(E) = (E-E_0)^2 Q_{q,2g+1}(E)$.*
2. *If $y(z) = \psi(P_0, z, z_0)$ is the Baker-Akhiezer function and E_0 is not a multiple zero of $Q_{q,2g+1}(E)$, then $\tilde{q}(z)$ is a genus g KdV potential isospectral to $q(z)$.*
3. *If $y(z) = \psi(P_0, z, z_0)$ is the Baker-Akhiezer function and E_0 is a multiple zero of $Q_{q,2g+1}(E)$, then $\tilde{q}(z)$ is a genus $g-1$ KdV potential with the associate spectral polynomial $Q_{\tilde{q},2g-1}(E) = (E-E_0)^{-2} Q_{q,2g+1}(E)$.*

The Darboux transformations have many applications; see e.g. [14] and references therein. For example, they can be applied to study algebraically integrable differential operators. Such idea was originally introduced by Burchnall and Chaundy [2] and later used in e.g. [23] and references therein.

Now we study the relation between poles of $q(z)$ and $\tilde{q}(z)$ for later usage. Let $\{p_i\}_{i\in I}$ be the set of poles of $q(z)$ as in Section 2.2. By (2.15)-(2.16), we have that at $z = p_i$

$$q(z) = \frac{n_i(n_i+1)}{(z-p_i)^2} + O(1) \text{ for some } n_i \in \mathbb{N},\ \forall i. \tag{2.19}$$

Proposition 2.7. *Suppose $q(z)$ is a KdV potential with the pole set $\{p_i\}_{i\in I}$ and (2.19). For any $E_0 \in \mathbb{C}$, we let $y(z)$ be any solution of*

$$y''(z) = [q(z) - E_0]y(z)$$

with its zero set $\{\hat{p}_i\}_{i\in I'}$ and $\tilde{q}(z)$ be the new KdV potential defined by (2.18). Then the pole set $\{\tilde{p}_j\}_{j\in J}$ of $\tilde{q}(z)$ is a subset of $\{p_i\}_{i\in I} \cup \{\hat{p}_i\}_{i\in I'}$. More precisely,

(1) *if $\tilde{p}_j = p_{i_0} \in \{p_i\}_{i\in I} \cap \{\hat{p}_i\}_{i\in I'}$, then at $\tilde{p}_j = p_{i_0}$ we have*

$$\tilde{q}(z) = \frac{(n_{i_0}+1)(n_{i_0}+2)}{(z-p_{i_0})^2} + O(1); \tag{2.20}$$

(2) *if $\tilde{p}_j = p_{i_0} \in \{p_i\}_{i\in I} \setminus \{\hat{p}_i\}_{i\in I'}$, then at $\tilde{p}_j = p_{i_0}$ we have*

$$\tilde{q}(z) = \frac{(n_{i_0}-1)n_{i_0}}{(z-p_{i_0})^2} + O(1); \tag{2.21}$$

(3) *if $\tilde{p}_j = \hat{p}_{i_0} \in \{\hat{p}_i\}_{i\in I'} \setminus \{p_i\}_{i\in I}$, then at $\tilde{p}_j = \hat{p}_{i_0}$ we have*

$$\tilde{q}(z) = \frac{2}{(z-\hat{p}_{i_0})^2} + O(1). \tag{2.22}$$

Proof. The first assertion is trivial. Note from (2.19) that the local exponent of any solution $y(z)$ at p_i is either $-n_i$ or $n_i + 1$.

(1) If $\tilde{p}_j = p_{i_0} \in \{p_i\}_{i\in I} \cap \{\hat{p}_i\}_{i\in I'}$, then p_{i_0} is a zero of $y(z)$ with multiplicity $n_{i_0}+1$, i.e. $y(z) = c(z-p_{i_0})^{n_{i_0}+1}(1 + O(z-p_{i_0}))$. Inserting this and (2.19) into (2.18), we immediately obtain (2.20).

(2) If $\tilde{p}_j = p_{i_0} \in \{p_i\}_{i\in I} \setminus \{\hat{p}_i\}_{i\in I'}$, then p_{i_0} is a pole of $y(z)$ with order n_{i_0}, i.e. $y(z) = c(z-p_{i_0})^{-n_{i_0}}(1 + O(z-p_{i_0}))$. This implies (2.21).

(3) If $\tilde{p}_j = \hat{p}_{i_0} \in \{\hat{p}_i\}_{i\in I'} \setminus \{p_i\}_{i\in I}$, then $q(z)$ is holomorphic at $\hat{p}_{i_0}$ and so $\hat{p}_{i_0}$ is a simple zero of $y(z)$. This implies (2.22). □

3. Existence of strict simply-periodic KdV potentials

In this section we will prove Theorem 1.2. Let $\omega \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$. A meromorphic simply-periodic function $q(z)$ with period ω which is bounded as $|\operatorname{Im}(z/\omega)| \to \infty$ has only finite many poles in the period strip

$$\mathcal{S}_\omega := \{z \in \mathbb{C} \mid 0 \le \operatorname{Re}(z/\omega) < 1\}. \tag{3.1}$$

As in [11], we call such functions *bounded near the ends of the period strip $\mathcal{S}_\omega$*.

From now on, we only consider simply-periodic KdV potentials bounded near the ends of the period strip $\mathcal{S}_\omega$, which is known to be of the form

$$q(z) = \sum_{j=1}^{M} m_j(m_j+1)\mathcal{P}(z-p_j) + E_0, \quad m_j \in \mathbb{N}, \tag{3.2}$$
$$= 2\sum_{i=1}^{N} \mathcal{P}(z-p_i) + E_0,$$

where $p_{j_1} \not\equiv p_{j_2} \mod \mathbb{Z}\omega$ for any $1 \le j_1 < j_2 \le M$,

$$N := \frac{1}{2}\sum_{i=1}^{M} m_i(m_i+1),$$

and

$$\mathcal{P}(z) := \tfrac{\pi^2}{\omega^2}([\sin(\pi z/\omega)]^{-2} - \tfrac{1}{3}) = \tfrac{1}{z^2} + O(z^2) \text{ near } 0.$$

See e.g. [11, Theorem 2.5]. Remark that $\mathcal{P}(z)$ is the limit of the Weierstrass $\wp$-function $\wp(z;\Lambda)$ with respect to the lattice $\Lambda := \mathbb{Z}\omega + \mathbb{Z}\omega_2$ as $\frac{\omega_2}{\omega} \to i\infty$.

This section is devoted to studying the existence and further properties of strict simply-periodic KdV potentials. In particular, Theorem 1.2 will be proved as a consequence. Clearly not every simply-periodic KdV potential is necessarily strict. The first result of this section is

Theorem 3.1. *Let $q(z)$ be any genus $g \ge 1$ simply-periodic KdV potential, bounded near the ends of the period strip $\mathcal{S}_\omega$. Then there is a strict genus g simply-periodic KdV potential $\tilde{q}(z)$, bounded near the ends of the period strip $\mathcal{S}_\omega$, such that $\tilde{q}(z)$ is isospectral to $q(z)$, i.e. their associate spectral polynomials $Q_{\tilde{q},2g+1}(E) = Q_{q,2g+1}(E)$.*

We will give two different proofs of Theorem 3.1. To this goal, we need to apply the beautiful classification result of simply-periodic KdV potentials by Gesztesy, Unterkofler and Weikard [11]. As in [11], we define the sets

$$\mathcal{N}_g := \{(n_1, \cdots, n_g) \in \mathbb{N}^g \mid n_1 < n_2 < \cdots < n_g, \gcd(n_1, \cdots, n_g) = 1\}$$

and

$$\mathcal{N} := \bigcup_{g=1}^{\infty} \mathcal{N}_g.$$

For $\underline{n} = (n_1, \cdots, n_g) \in \mathcal{N}$ we denote the number of its components by $\#(\underline{n}) = g$. For $\underline{n} = (n_1, \cdots, n_g) \in \mathcal{N}_g$ and $\underline{v} = (v_1, \cdots, v_g) \in \mathbb{C}^{*g}$, we recall the $g \times g$ matrix $T(\underline{n}, \underline{v}, u)$ defined in [11] by

$$T(\underline{n}, \underline{v}, u) := \Big(n_l^{k-1}[v_l u^{n_l} - (-1)^k]\Big)_{1\le k,l\le g} \tag{3.3}$$

and the corresponding polynomial $\tau_N(\underline{n}, \underline{v}, u)$ of degree $N := \sum_{i=1}^{g} n_i$ in u by

$$\tau_N(\underline{n}, \underline{v}, u) := (-1)^{\lfloor g/2 \rfloor} \frac{\det T(\underline{n}, \underline{v}, u)}{\vartheta(n_1, \cdots, n_g)}. \tag{3.4}$$

Here $\lfloor x \rfloor$ denotes the greatest integer n such that $n \leq x$, and

$$\vartheta(n_1, \cdots, n_g) := \prod_{k<l}(n_l - n_k).$$

Then it was proved in [11, Lemma 3.10] that

$$\tau_N(\underline{n}, \underline{v}, u) = \sum_{k=0}^{N} r_k(\underline{v}) u^k, \tag{3.5}$$

with

$$r_k(\underline{v}) = \sum_{\substack{\sigma_1=0,\cdots,\sigma_g=0 \\ \sigma_1 n_1+\cdots+\sigma_g n_g=k}}^{1} \frac{\vartheta((-1)^{\sigma_1} n_1, \cdots, (-1)^{\sigma_g} n_g)}{\vartheta(n_1, \cdots, n_g)} v_1^{\sigma_g} \cdots v_g^{\sigma_g},\ 0 \leq k \leq N.$$

In particular, $r_0(\underline{v}) = 1$ and $r_N(\underline{v}) = (-1)^{\lfloor g/2 \rfloor} v_1 \cdots v_g$. The following interesting result concerning the classification of simply-periodic KdV potentials was proved in [11].

Theorem 3.2. [11, Theorem 3.14]

(1) *The set*

$$\mathcal{S} = \left\{ q(z) = e_0 - 2[\ln(\tau_N(\underline{n}, \underline{v}, e^{\frac{2\pi i z}{\omega}}))]'' \middle| e_0 \in \mathbb{C}, \underline{n} \in \mathcal{N}_g, \underline{v} \in \mathbb{C}^{*g}, g \in \mathbb{N}_0 \right\} \tag{3.6}$$

is precisely the set of simply-periodic KdV potentials of period ω, bounded near the ends of the period strips $\mathcal{S}_\omega$.

(2) *For any*

$$q(z) \in \mathcal{S}(g, \underline{n}, e_0) := \left\{ q(z) = e_0 - 2[\ln(\tau_N(\underline{n}, \underline{v}, e^{\frac{2\pi i z}{\omega}}))]'' \middle| \underline{v} \in \mathbb{C}^{*g} \right\}, \tag{3.7}$$

$q(z)$ is of genus g and its spectral polynomial is

$$Q_{q,2g+1}(E) = (E - e_0) \prod_{i=1}^{g} (E - e_i)^2, \text{ where } e_i = e_0 + n_i^2 \tfrac{\pi^2}{\omega^2}.$$

Namely, all potentials in $\mathcal{S}(g, \underline{n}, e_0)$ are isospectral.

(3) *By rewriting $q(z)$ in the form of (3.2):*

$$\begin{aligned} q(z) =& e_0 - 2[\ln(\tau_N(\underline{n}, \underline{v}, e^{\frac{2\pi i z}{\omega}}))]'' \\ =& \sum_{j=1}^{M} m_j(m_j+1)\mathcal{P}(z - p_j) + E_0, \quad m_j \in \mathbb{N}, \end{aligned} \tag{3.8}$$

there hold

$$\tau_N(\underline{n}, \underline{v}, u) = r_N(\underline{v}) \prod_{j=1}^{M} \left(u - e^{2\pi i p_j/\omega} \right)^{\frac{m_j(m_j+1)}{2}} \tag{3.9}$$

and

$$\sum_{i=1}^{g} n_i = N = \frac{1}{2} \sum_{j=1}^{M} m_j(m_j+1).$$

Remark 3.3. Note that $\tau_N(\underline{n},\underline{v},u)$ is also well-defined for $\gcd(n_1,\cdots,n_g)\geq 2$. In Theorem 3.2, the condition $\gcd(n_1,\cdots,n_g)=1$ is to guarantee that ω is the basic period of $q(z)$. For $\underline{n}=(n_1,\cdots,n_g)$ with $1\leq n_1<n_2<\cdots<n_g$ and $d:=\gcd(n_1,\cdots,n_g)\geq 2$,

$$q(z):=e_0-2[\ln(\tau_N(\underline{n},\underline{v},e^{\frac{2\pi iz}{\omega}}))]''$$

is still a genus g simply-periodic KdV potential of period ω, bounded near the ends of the period strips $\mathcal{S}_\omega$, but the basic period of $q(z)$ is actually ω/d. Indeed, by defining $\underline{n}':=(n_1',\cdots,n_g')$ with $n_i'=n_i/d$ and $N':=\sum n_i'=N/d$, we have $\gcd(n_1',\cdots,n_g')=1$ and it follows from (3.5) that $\tau_N(\underline{n},\underline{v},u)=\tau_{N'}(\underline{n}',\underline{v},u^d)$, so

$$q(z)=e_0-2[\ln(\tau_N(\underline{n},\underline{v},e^{\frac{2\pi iz}{\omega}}))]''=e_0-2[\ln(\tau_{N'}(\underline{n}',\underline{v},e^{\frac{2\pi iz}{\omega/d}}))]''.$$

Note from Theorem 2.4 that $m_j\leq g$ for all j in (3.8). Recall that $q(z)$ is strict if $m_j=g$ for some j. Now we can give the first proof of Theorem 3.1.

The first proof of Theorem 3.1. Let $\underline{v}^*=(-1,\cdots,-1)$, i.e. $v_j=-1$ for all $1\leq j\leq g$. We will prove that this $\underline{v}^*$ gives rise to a strict KdV potential in $\mathcal{S}(g,\underline{n},e_0)$ for any $g\geq 1,\underline{n}\in\mathcal{N}_g$ and $e_0\in\mathbb{C}$.

Step 1. We claim that for any $g\geq 1$ and any $\underline{n}=(n_1,\cdots,n_g)$ with $1\leq n_1<n_2<\cdots<n_g$,

(3.10) *$u=1$ is a root of $\tau_N(\underline{n},\underline{v}^*,u)$ with multiplicity $\frac{g(g+1)}{2}$.*

We prove this claim by induction on g. The case $g=1$ is trivial since

$$\tau_N(\underline{n},\underline{v}^*,u)=1-u^{n_1}.$$

Suppose that (3.10) holds for $g=k-1$ for some $k\geq 2$. We want to prove it for $g=k$ and any $\underline{n}=(n_1,\cdots,n_k)$ with $1\leq n_1<n_2<\cdots<n_k$.

Let $\underline{n}_l=(n_1,\cdots,n_{l-1},n_{l+1},\cdots,n_k)$ and $N_l=N-n_l=\sum_{j=1}^k n_j-n_l$ for any $1\leq l\leq k$. Then $\#(\underline{n}_l)=k-1$ and so our assumption implies

$$(u-1)^{\frac{(k-1)k}{2}}\Big|\tau_{N_l}(\underline{n}_l,\underline{v}^*,u),\quad\forall l,$$

i.e.

$$(u-1)^{\frac{(k-1)k}{2}}\Big|\det(T(\underline{n}_l,\underline{v}^*,u)),\quad\forall l.$$

Case 1. k is odd. Then the k-th row of the matrix $T(\underline{n},\underline{v}^*,u)$ is

$$(n_1^{k-1}(1-u^{n_1}),\cdots,n_k^{k-1}(1-u^{n_k})).$$

Consequently, computing $\det(T(\underline{n},\underline{v}^*,u))$ by expanding along the k-th row leads to

$$\det(T(\underline{n},\underline{v}^*,u))=\sum_{l=1}^k(-1)^{k+l}n_l^{k-1}(1-u^{n_l})\det(T(\underline{n}_l,\underline{v}^*,u))$$

and so

$$(u-1)^{\frac{(k-1)k}{2}+1}\Big|\det(T(\underline{n},\underline{v}^*,u)). \tag{3.11}$$

Let $t\geq\frac{(k-1)k}{2}+1$ be the largest integer such that

$$(u-1)^t\Big|\det(T(\underline{n},\underline{v}^*,u)),\ \text{ i.e. }\ (u-1)^t\Big|\tau_N(\underline{n},\underline{v}^*,u).$$

Since Theorem 3.2 and Remark 3.3 yield that $q(z):=-2[\ln(\tau_N(\underline{n},\underline{v}^*,e^{\frac{2\pi iz}{\omega}}))]''$ is a genus k KdV potential, we see from (3.8)-(3.9) that $t=\frac{m(m+1)}{2}$ for some integer $1\leq m\leq k$ and so $t=\frac{k(k+1)}{2}$. This proves (3.10) for $g=k$.

Case 2. k is even. Then the $(k-1)$-th row of the matrix $T(\underline{n},\underline{v}^*,u)$ is

$$(n_1^{k-2}(1-u^{n_1}),\cdots,n_k^{k-2}(1-u^{n_k}))$$

and the k-th row is

$$-(n_1^{k-1}(1+u^{n_1}),\cdots,n_k^{k-1}(1+u^{n_k})).$$

Note that

$$(u-1)\Big|\left(n_l^{k-1}(1+u^{n_l})+\tfrac{2}{u-1}\times n_l^{k-2}(1-u^{n_l})\right),\ \forall l.$$

To compute $\det(T(\underline{n},\underline{v}^*,u))$, we add $\frac{-2}{u-1}$ multiplying the $(k-1)$-th row to the k-th row first, and then expand along the new k-th row, which again implies (3.11). The rest argument is the same as Case 1. This proves (3.10).

Step 2. We prove the existence of strict KdV potentials.

Let $q(z)$ be any genus $g\geq 1$ simply-periodic KdV potential, bounded near the ends of the period strip $\mathcal{S}_\omega$. Then Theorem 3.2 implies the existence of $\underline{n}\in\mathcal{N}_g$ and $e_0\in\mathbb{C}$ such that $q(z)\in\mathcal{S}(g,\underline{n},e_0)$. Let

$$\tilde{q}(z):=e_0-2[\ln(\tau_N(\underline{n},\underline{v}^*,e^{\frac{2\pi iz}{\omega}}))]''.$$

Then $\tilde{q}(z)$ is also a genus g simply-periodic KdV potential, bounded near the ends of the period strip $\mathcal{S}_\omega$, and is isospectral to $q(z)$. Furthermore, (3.10) and (3.8)-(3.9) imply that

$$\tilde{q}(z)=g(g+1)\mathcal{P}(z)+\sum_{i=1}^{M'}m_i(m_i+1)\mathcal{P}(z-p_i)+E_0, \tag{3.12}$$

where $p_i\notin\mathbb{Z}\omega$ and $m_i\leq g$ for all i if $M'\geq 1$. Thus $\tilde{q}(z)$ is a strict genus g KdV potential. Theorems 2.4 and 2.5 indicate that this $\tilde{q}(z)$ is even and actually unique in $\mathcal{S}(g,\underline{n},e_0)$ up to a translation. Furthermore, Theorem 2.4 implies $m_i<g$ for all i if $M'\geq 1$, because if $m_i=g$ for some i, then Theorem 2.4-(2) shows that the basic period of $\tilde{q}(z)$ is ω/k for some $k\geq 2$, a contradiction to that the basic period of $\tilde{q}(z)$ is ω. In conclusion,

$$\begin{aligned}\tilde{q}(z)=&g(g+1)\mathcal{P}(z)+m(m+1)\mathcal{P}(z-\tfrac{\omega}{2})\\&+\sum_{j=1}^{r}m_j(m_j+1)(\mathcal{P}(z-p_j)+\mathcal{P}(z+p_j))+E_0,\end{aligned} \tag{3.13}$$

where $0 \leq m \leq g-1$ and if $r \geq 1$, then $1 \leq m_j \leq g-1$, $p_j \not\equiv 0, \frac{\omega}{2}, \pm p_{j'}$ mod $\mathbb{Z}\omega$ for any $j \neq j'$. The proof is complete. $\square$

Remark 3.4. Clearly $\tau_N(\underline{n}, \underline{v}^*, u) \in \mathbb{Q}[u]$, i.e. any root of $\tau_N(\underline{n}, \underline{v}^*, u)$ is an algebraic number. Therefore, in contrast to generic KdV potentials in $\mathcal{S}(g, \underline{n}, e_0)$, the location of any pole p_j for the strict KdV potential $\tilde{q}(z)$ in (3.13) can not be arbitrary; it turns out that $e^{2\pi i p_j/\omega}$ is an *algebraic number* and any conjugate of $e^{2\pi i p_j/\omega}$ belongs to $\{e^{\pm 2\pi i p_k/\omega} | 1 \leq k \leq r\}$. We will prove that for a special class of $\underline{n}$'s, all poles p_j's of $\tilde{q}(z)$ locate on the line $\operatorname{Re} z = \frac{1}{2}$; see Theorem 4.2.

Remark 3.5. Fix any $g \geq 1$ and let $\underline{n}_g := (1, 2, \cdots, g)$. Then $\tau_N(\underline{n}_g, \underline{v}^*, u)$ is a polynomial of degree $\frac{g(g+1)}{2}$ in u and so the first proof of Theorem 3.1 gives

$$\tau_N(\underline{n}_g, \underline{v}^*, u) = (1-u)^{g(g+1)/2},$$

i.e.

$$\begin{aligned} e_0 - 2[\ln(\tau_N(\underline{n}_g, \underline{v}^*, e^{\frac{2\pi i z}{\omega}}))]'' &= g(g+1)\tfrac{\pi^2}{\omega^2}[\sin(\pi z/\omega)]^{-2} + e_0 \\ &= g(g+1)\mathcal{P}(z) + g(g+1)\tfrac{\pi^2}{3\omega^2} + e_0. \end{aligned}$$

This is the well-known genus g simply-periodic KdV potential as a limit of the Lamé potential $g(g+1)\wp(z;\Lambda) + g(g+1)\frac{\pi^2}{3\omega^2} + e_0$ with $\Lambda = \mathbb{Z}\omega + \mathbb{Z}\omega_2$ and $\frac{\omega_2}{\omega} \to \sqrt{-1}\infty$. Together with Theorem 3.2, the spectral polynomial is

$$Q_{2g+1}(E) = (E - e_0)\prod_{j=1}^{g}(E - e_0 - j^2\tfrac{\pi^2}{\omega^2})^2.$$

This formula was proved via different ideas in the literature; see e.g. [26, Proposition 4.1].

Consider another typical genus g simply-periodic KdV potential

$$\begin{aligned} q_{g,m}(z) :=& g(g+1)\tfrac{\pi^2}{\omega^2}[\sin\tfrac{\pi z}{\omega}]^{-2} \\ &+ m(m+1)\tfrac{\pi^2}{\omega^2}[\sin(\tfrac{\pi(z-\frac{\omega}{2})}{\omega})]^{-2} + e_0, \ 1 \leq m < g. \end{aligned} \tag{3.14}$$

Like the case $m = 0$, $q_{g,m}(z)$ is a limit of the Treibich-Verdier potential [31]

$$\begin{aligned} & g(g+1)\wp(z;\Lambda) + m(m+1)\wp(z - \tfrac{\omega}{2};\Lambda) \\ &+ [g(g+1) + m(m+1)]\tfrac{\pi^2}{3\omega^2} + e_0. \end{aligned} \tag{3.15}$$

See e.g. [26, 27], where Takemura also computed the spectral polynomial of $q_{g,m}(z)$ as a limit of the corresponding one of (3.15); see [26, Proposition 4.1]. Here we can give a different approach to compute it. By Theorem 3.2, it is suffices to determine what the $\underline{n}$ of (3.14) is. Define $\underline{n}_{g,m} \in \mathcal{N}_g$ by

$$\underline{n}_{g,m} := (1, 2, \cdots, g-m, g-m+2, g-m+4, \cdots, g-m+2m). \tag{3.16}$$

Clearly $\#(\underline{n}_{g,m}) = g$ and the sum of all elements of $\underline{n}_{g,m}$ is $\frac{g(g+1)}{2} + \frac{m(m+1)}{2}$. Recall $\mathcal{S}(g, \underline{n}, e_0)$ and $\tau_N(\underline{n}, \underline{v}, u)$ in Theorem 3.2.

Proposition 3.6. *Let $q_{g,m}(z)$ and $\underline{n}_{g,m}$ be given by (3.14) and (3.16). Then $q_{g,m}(z) \in \mathcal{S}(g, \underline{n}_{g,m}, e_0)$, i.e. the spectral polynomial of $q_{g,m}(z)$ is*

$$Q_{q_{g,m},2g+1}(E) = (E - e_0) \prod_{j=1}^{g-m} (E - e_0 - j^2 \tfrac{\pi^2}{\omega^2})^2 \cdot \prod_{j=1}^{m} (E - e_0 - (g - m + 2j)^2 \tfrac{\pi^2}{\omega^2})^2.$$

In particular,

$$\tau_N(\underline{n}_{g,m}, \underline{v}^*, u) = (1 - u)^{g(g+1)/2}(1 + u)^{m(m+1)/2}. \tag{3.17}$$

To give the second proof of Theorem 3.1 and a new proof of Proposition 3.6, we need to use the Darboux transformation recalled in Section 2.

Let $q(z)$ be a genus g simply-periodic KdV potential, bounded near the ends of the period strip $\mathcal{S}_\omega$. Then it follows from (3.2) that there is $e_0 \in \mathbb{C}$ such that

$$\begin{aligned} q(z) &= e_0 + O(e^{-2\pi|\operatorname{Im}(\frac{z}{\omega})|}) \quad \text{as } \mathcal{S}_\omega \ni z \to \infty, \\ |q^{(k)}(z)| &\backsim e^{-2\pi|\operatorname{Im}(\frac{z}{\omega})|} \to 0 \quad \text{as } \mathcal{S}_\omega \ni z \to \infty, \forall k \geq 1. \end{aligned} \tag{3.18}$$

Consequently, we see from the definition (2.3) of $\Phi_g(z; E)$ that $\Phi_g(z; E)$ is also simply-periodic with period ω, bounded near the ends of the period strip $\mathcal{S}_\omega$ and

$$\lim_{\mathcal{S}_\omega \ni z \to \infty} \Phi_g(z; E) = E^g + d_1 E^{g-1} + \cdot + d_g =: \prod_{i=1}^{g} (E - e_i), \tag{3.19}$$

or more precisely,

$$\Phi_g(z; E) - \prod_{i=1}^{g} (E - e_i) = O(e^{-2\pi|\operatorname{Im}(\frac{z}{\omega})|}) \to 0 \text{ as } \mathcal{S}_\omega \ni z \to \infty. \tag{3.20}$$

These, together with (2.5), easily imply

$$Q_{q,2g+1}(E) = (E - e_0) \prod_{i=1}^{g} (E - e_i)^2. \tag{3.21}$$

Remark that Theorem 3.2 shows that $e_i \neq e_j$ for any $i \neq j$.

Lemma 3.7. *Fix any $E \in \mathbb{C}$. Let $y(z)$ be a solution of*

$$y''(z) = [q(z) - E]y(z), \quad P = (E, \mathcal{C}), \tag{3.22}$$

such that $y(z + \omega) = cy(z)$ for some $c \in \mathbb{C}$. Then the new KdV potential $\tilde{q}(z)$ given by the Darboux transformation

$$\tilde{q}(z) := q(z) - 2 \left(\tfrac{y'(z)}{y(z)} \right)'$$

is also a simply-periodic KdV potential, bounded near the ends of the period strip $\mathcal{S}_\omega$, i.e. $\tilde{q}(z) \in \mathcal{S}$, where $\mathcal{S}$ is given in Theorem 3.2.

Proof. Clearly $y(z+\omega) = cy(z)$ implies that $\tilde{q}(z)$ is simply-periodic with period ω. Note that the fundamental system of solutions of

$$y''(z) = (e_0 - E)y(z)$$

is

$$e^{(e_0-E)^{1/2}z}, e^{-(e_0-E)^{1/2}z} \text{ if } e_0 - E \neq 0; \quad 1, z \text{ if } e_0 - E = 0. \tag{3.23}$$

By (3.18), the leading term of $y(z)$ as $\mathcal{S}_\omega \ni z \to \infty$ is a multiple of some function in (3.23). From here, it is easy to see that $y'(z)/y(z)$ is bounded as $\mathcal{S}_\omega \ni z \to \infty$ and so does $(y'(z)/y(z))'$ by using

$$\left(\tfrac{y'(z)}{y(z)}\right)' = \tfrac{y''(z)}{y(z)} + \left(\tfrac{y'(z)}{y(z)}\right)^2 = q(z) - E + \left(\tfrac{y'(z)}{y(z)}\right)^2.$$

This proves that $\tilde{q}(z)$ is bounded as $\mathcal{S}_\omega \ni z \to \infty$, i.e. $\tilde{q}(z) \in \mathcal{S}$. □

Now consider the Baker-Akhiezer function $\psi(P,z,z_0)$ of $q(z)$, which is a solution of (3.22) by (2.10). Since

$$\frac{\psi'(P,z,z_0)}{\psi(P,z,z_0)} = \phi(P,z) = \frac{i\mathcal{C}(P) + \frac{1}{2}\Phi_g'(z;E)}{\Phi_g(z;E)}$$

is simply-periodic with period ω, we see hat

$$\psi(P, z+\omega, z_0) = c\psi(P,z,z_0) \quad \text{for some } c \in \mathbb{C}.$$

Therefore, Lemma 3.7 shows that the new KdV potential

$$\tilde{q}(z) := q(z) - 2\left(\frac{\psi'(P,z,z_0)}{\psi(P,z,z_0)}\right)' = q(z) - 2\phi'(P,z) \tag{3.24}$$

is also a simply-periodic KdV potential, bounded near the ends of the period strip $\mathcal{S}_\omega$, i.e. $\tilde{q}(z) \in \mathcal{S}$. Remark that for $E \in \{e_i\}_{i=1}^g$, we have $\mathcal{C}(P)^2 = Q_{q,2g+1}(E) = 0$ and so $\psi(P,z,z_0) = \psi(P^*,z,z_0)$. Then (2.12) and (3.19) yield

$$\psi(P,z,z_0) = \sqrt{\frac{\Phi_g(z;E)}{\Phi_g(z_0;E)}} \to 0 \quad \text{as } \mathcal{S}_\omega \ni z \to \infty. \tag{3.25}$$

Our second proof of Theorem 3.1 relies on the following observation.

Lemma 3.8. *Let $q(z)$ be a genus $g \geq 2$ simply-periodic KdV potential, bounded near the ends of the period strip $\mathcal{S}_\omega$, with its spectral polynomial $Q_{q,2g+1}(E)$ given by (3.21). If $q(z)$ is strict, then for any $1 \leq k \leq g$,*

$$\tilde{q}(z) := q(z) - 2\left(\frac{\psi'(P,z,z_0)}{\psi(P,z,z_0)}\right)' = q(z) - 2\phi'(P,z), \;\; P = (e_k, 0) \tag{3.26}$$

is a strict genus $g-1$ simply-periodic KdV potential, bounded near the ends of the period strip $\mathcal{S}_\omega$, with its spectral polynomial

$$Q_{\tilde{q},2g-1}(E) = (E - e_0)\prod_{j=1,\neq k}^{g}(E - e_j)^2.$$

Proof. Since e_k is a double root of $Q_{q,2g+1}(E)$, by Theorem 2.6 and Lemma 3.7, we obtain all the desired assertions except that $\tilde{q}(z)$ is strict. Since $q(z)$ is strict, there is a pole p of $q(z)$ such that

$$q(z) = \frac{g(g+1)}{(z-p)^2} + O(1) \quad \text{at } p.$$

Consequently, Proposition 2.7 implies that p is also a pole of $\tilde{q}(z)$ and either

$$\tilde{q}(z) = \frac{(g+1)(g+2)}{(z-p)^2} + O(1) \quad \text{at } p \tag{3.27}$$

(if p is zero of $\psi(P, z, z_0)$ with multiplicity $g+1$) or

$$\tilde{q}(z) = \frac{(g-1)g}{(z-p)^2} + O(1) \quad \text{at } p \tag{3.28}$$

(if p is a pole of $\psi(P, z, z_0)$ with order g). Since $\tilde{q}(z)$ is of genus $g-1$, (3.27) is impossible by Theorem 2.4. So (3.28) holds, which just says that $\tilde{q}(z)$ is a strict genus $g-1$ KdV potential. □

Now we can prove Theorem 1.2.

Proof of Theorem 1.2. Let $q(z)$ be a strict genus g KdV potential given by (1.7). The assertion (1) follows from Remark 3.4. For the assertion (2), we note from Proposition 2.7 that any zero of the Baker-Akhiezer function $\psi(P_k, z, z_0)$ is a pole of the new potential $\tilde{q}(z)$ defined in (3.26). Since Lemma 3.8 shows that $\tilde{q}(z) = (g-1)g\tilde{\mathcal{P}}(z) + \cdots$ is a strict genus $g-1$ KdV potential, the assertion (2) follows from the assertion (1). □

We conclude this section by giving the second proof of Theorem 3.1 and our new proof of Proposition 3.6.

The second proof of Theorem 3.1. Let $q(z)$ be any genus $g \geq 1$ simply-periodic KdV potential, bounded near the ends of the period strip $\mathcal{S}_\omega$. Then Theorem 3.2 implies the existence of $\underline{n} = (n_1, n_2, \cdots, n_g) \in \mathcal{N}_g$ and $e_0 \in \mathbb{C}$ such that $q(z) \in \mathcal{S}(g, \underline{n}, e_0)$, i.e. the spectral polynomial is

$$Q_{q,2g+1}(E) = (E - e_0) \prod_{i=1}^{g} (E - e_i)^2, \text{ where } e_i = e_0 + n_i^2 \tfrac{\pi^2}{\omega^2}.$$

Recall $1 \leq n_1 < \cdots < n_g$ and $\gcd(n_1, \cdots, n_g) = 1$. So $\ell := n_g \geq g$. Consider

$$\begin{aligned} q_0(z) :=& \ell(\ell+1) \tfrac{\pi^2}{\omega^2} [\sin(\pi z/\omega)]^{-2} + e_0 \\ =& \ell(\ell+1)\mathcal{P}(z) + e_0 + \ell(\ell+1) \tfrac{\pi^2}{3\omega^2}. \end{aligned} \tag{3.29}$$

It is known (see e.g. Remark 3.5) that $q_0(z)$ is a strict genus ℓ simply-periodic KdV potential with its spectral polynomial

$$Q_{q_0, 2\ell+1}(E) = (E - e_0) \prod_{j=1}^{\ell} (E - e_0 - j^2 \tfrac{\pi^2}{\omega^2})^2.$$

Clearly $\{n_1, \cdots, n_g\} \subset \{1, \cdots, \ell\}$. If $g = \ell$, then we are done since $q_0(z)$ is isospectral to $q(z)$. So we consider the case $g < \ell$ and denote

$$\{1, \cdots, \ell\} \setminus \{n_1, \cdots, n_g\} = \{n_{g+1}, \cdots, n_\ell\}.$$

Then

$$Q_{q_0,2\ell+1}(E) = Q_{q,2g+1}(E) \times \prod_{h=g+1}^{\ell} (E - e_0 - n_h^2 \tfrac{\pi^2}{\omega^2})^2.$$

Applying Lemma 3.8 to $q_0(z)$ at $P = (e_0 + n_{g+1}^2 \frac{\pi^2}{\omega^2}, 0)$, we obtain that

$$q_1(z) := q_0(z) - 2\phi'(P, z)$$

is a strict genus $\ell - 1$ simply-periodic KdV potential, bounded near the ends of the period strip $\mathcal{S}_\omega$, with its spectral polynomial

$$Q_{q_1,2\ell-1}(E) = (E - e_0) \prod_{j=1, \neq n_{g+1}}^{\ell} (E - e_0 - j^2 \tfrac{\pi^2}{\omega^2})^2.$$

Again we can apply Lemma 3.8 to $q_1(z)$ to eliminating the term $j = n_{g+2}$ (i.e. $(E - e_0 - n_{g+2}^2 \frac{\pi^2}{\omega^2})^2$) from $Q_{q_1,2\ell-1}(E)$. In conclusion, we can apply this process $\ell - g$ times, by eliminating the terms $j = n_{g+1}, \cdots, n_\ell$ one by one from $Q_{q_0,2\ell+1}(E)$, to obtain a strict genus g simply-periodic KdV potential $\tilde{q}(z)$, which is bounded near the ends of the period strip $\mathcal{S}_\omega$ and isospectral to $q(z)$. This completes the proof. □

Proof of Proposition 3.6. Fix any $\ell \geq 3$. We will prove by induction that for any $k \in \mathbb{N}$ satisfying $1 \leq k < \ell/2$,

$$q_{\ell-k,k}(z) \in \mathcal{S}(\ell - k, \underline{n}_{\ell-k,k}, e_0). \tag{3.30}$$

Clearly $q_{g,m}(z) \in \mathcal{S}(g, \underline{n}_{g,m}, e_0)$ follows from (3.30) by letting $\ell = g + m$ and $k = m$.

Note from (3.16) that

$$\underline{n}_{\ell-k,k} = (1, 2, \cdots, \ell - 2k, \ell - 2k + 2, \ell - 2k + 4, \cdots, \ell - 2, \ell). \tag{3.31}$$

In particular,

$$\underline{n}_{\ell-1,1} = (1, 2, \cdots, \ell - 2, \ell).$$

Recalling the genus ℓ potential $q_0(z)$ defined in (3.29), we apply the same argument as the second proof of Theorem 3.1: We apply Lemma 3.8 to obtain that

$$\begin{aligned} q_1(z) :=& q_0(z) - 2\left(\frac{\psi'(P, z, z_0)}{\psi(P, z, z_0)}\right)', \text{ where } P = (e_0 + (\ell - 1)^2 \tfrac{\pi^2}{\omega^2}, 0) \\ =& (\ell - 1)\ell \tfrac{\pi^2}{\omega^2} [\sin \tfrac{\pi z}{\omega}]^{-2} + 2 \sum_{j=1}^{t} \tfrac{\pi^2}{\omega^2} [\sin(\tfrac{\pi(z - p_j)}{\omega})]^{-2} + e_0, \end{aligned} \tag{3.32}$$

is a strict genus $\ell - 1$ simply-periodic KdV potential, bounded near the ends of the period strip $\mathcal{S}_\omega$, with its spectral polynomial

$$Q_{q_1,2\ell-1}(E) = (E - e_0) \prod_{j=1,\neq \ell-1}^{\ell} (E - e_0 - j^2 \tfrac{\pi^2}{\omega^2})^2.$$

Here the second equality of (3.32) follows from Lemma 3.8 and (3.2), and $\{p_j\}_{j=1}^t$ are the zero set of $\psi(P, z, z_0)$ satisfying $p_j \neq 0$ for all j. Remark that (3.18) and (3.21) imply that the constant term e_0 is preserved in (3.32). Clearly $q_1(z) \in \mathcal{S}(\ell-1, \underline{n}_{\ell-1,1}, e_0)$, so Theorem 3.2 implies

$$\frac{(\ell-1)\ell}{2} + t = \sum_{j=1,\neq \ell-1}^{\ell} j = \frac{(\ell-1)\ell}{2} + 1,$$

i.e. $t = 1$. Since Theorem 2.4 says that $q_1(z)$ is even, we conclude that $q_1 = \frac{\omega}{2}$, i.e. $q_1(z) = q_{\ell-1,1}(z)$. This proves (3.30) for $k = 1$.

Suppose (3.30) holds for some $1 \leq k < \frac{\ell}{2} - 1$. We want to prove

$$q_{\ell-k-1,k+1}(z) \in \mathcal{S}(\ell-k-1, \underline{n}_{\ell-k-1,k+1}, e_0). \tag{3.33}$$

By our assumption, for the genus $\ell - k$ KdV potential

$$\begin{aligned} q_{\ell-k,k}(z) =& (\ell-k)(\ell-k+1)\tfrac{\pi^2}{\omega^2}[\sin \tfrac{\pi z}{\omega}]^{-2} \\ &+ k(k+1)\tfrac{\pi^2}{\omega^2}[\sin(\tfrac{\pi(z-\frac{\omega}{2})}{\omega})]^{-2} + e_0, \end{aligned} \tag{3.34}$$

its spectral polynomial is

$$\begin{aligned} Q_{q_{\ell-k,k},2\ell-2k+1}(E) =& (E-e_0)\prod_{j=1}^{\ell-2k}(E - e_0 - j^2\tfrac{\pi^2}{\omega^2})^2 \\ &\cdot \prod_{j=1}^{k}(E - e_0 - (\ell-2k+2j)^2\tfrac{\pi^2}{\omega^2})^2. \end{aligned}$$

Again we apply Lemma 3.8 to $q_{\ell-k,k}(z)$ and obtain that

$$\begin{aligned} &q_{k+1}(z) \\ :=& q_{\ell-k,k}(z) - 2\left(\frac{\psi'(P,z,z_0)}{\psi(P,z,z_0)}\right)' \text{ where } P = (e_0 + (\ell-2k-1)^2\tfrac{\pi^2}{\omega^2}, 0) \\ =& (\ell-k-1)(\ell-k)\tfrac{\pi^2}{\omega^2}[\sin\tfrac{\pi z}{\omega}]^{-2} \\ &+ \tilde{k}(\tilde{k}+1)\tfrac{\pi^2}{\omega^2}[\sin(\tfrac{\pi(z-\frac{\omega}{2})}{\omega})]^{-2} + 2\sum_{j=1}^{t}\tfrac{\pi^2}{\omega^2}[\sin(\tfrac{\pi(z-p_j)}{\omega})]^{-2} + e_0, \end{aligned} \tag{3.35}$$

with

$$p_j \notin \{0, \tfrac{\omega}{2}\} + \mathbb{Z}\omega \quad \forall 1 \leq j \leq t, \tag{3.36}$$

is a strict genus $\ell-k-1$ simply-periodic KdV potential, bounded near the ends of the period strip $\mathcal{S}_\omega$, with its spectral polynomial

$$\begin{aligned} Q_{q_{k+1},2\ell-2k-1}(E) =& \frac{Q_{q_{\ell-k,k},2\ell-2k+1}(E)}{(E-e_0-(\ell-2k-1)^2\frac{\pi^2}{\omega^2})^2} \\ =&(E-e_0)\prod_{j=1}^{\ell-2k-2}(E-e_0-j^2\tfrac{\pi^2}{\omega^2})^2 \\ &\cdot\prod_{j=1}^{k+1}(E-e_0-(\ell-2k-2+2j)^2\tfrac{\pi^2}{\omega^2})^2. \end{aligned}$$

From here and (3.31) that

$$\underline{n}_{\ell-k-1,k+1}=(1,2,\cdots,\ell-2k-2,\ell-2k,\cdots,\ell-2,\ell),$$

we have $q_{k+1}(z)\in\mathcal{S}(\ell-k-1,\underline{n}_{\ell-k-1,k+1},e_0)$, so Theorem 3.2 implies

$$\begin{aligned} \tfrac{(\ell-k-1)(\ell-k)}{2}+\tfrac{\tilde{k}(\tilde{k}+1)}{2}+t=&\sum_{j=1}^{\ell-2k-2}j+\sum_{j=1}^{k+1}(\ell-2k-2+2j) \\ =&\tfrac{(\ell-k-1)(\ell-k)}{2}+\tfrac{(k+1)(k+2)}{2}. \end{aligned}$$

On the other hand, we see from Proposition 2.7 that $\tilde{k}\in\{k-1,k+1\}$. If $\tilde{k}=k-1$, then $t=2k+1$ and Proposition 2.7 shows that $p_i\neq p_j$ for any $i\neq j$. However, Theorem 2.4 says that $q_{k+1}(z)$ is even, i.e.

$$\{p_1,\cdots,p_{2k+1}\}\equiv\{-p_1,\cdots,-p_{2k+1}\}\qquad \mathrm{mod}\ \mathbb{Z}\omega,$$

which contradicts with (3.36). Therefore, we have $\tilde{k}=k+1$, i.e. $t=0$ and so $q_{k+1}(z)=q_{\ell-k-1,k+1}(z)$. This proves (3.33).

In conclusion, we have proved by induction that (3.30) holds for all $1\leq k<\ell/2$. Therefore, $q_{g,m}(z)\in\mathcal{S}(g,\underline{n}_{g,m},e_0)$. In particular, the first proof of Theorem 3.1 implies

$$q_{g,m}(z)=e_0-2[\ln(\tau_N(\underline{n}_{g,m},\underline{v}^*,e^{\frac{2\pi iz}{\omega}}))]'',$$

and hence (3.17) follows from (3.9). This completes the proof. □

4. Spectrum of strict simply-periodic KdV potentials along $\mathrm{Re}\, z=a$

In this section, we always consider the normalized case $\omega=1$ and prove Theorem 4.2, which plays a crucial role in our proof of Theorem 1.1.

Recall the potential $q_0(z)$ defined by (3.29) with $e_0=0$:

$$q_0(z):=\ell(\ell+1)\pi^2[\sin(\pi z)]^{-2}=\ell(\ell+1)\widetilde{\mathcal{P}}(z), \tag{4.1}$$

where $\widetilde{\mathcal{P}}(z)$ is defined in (1.8). Comparing to $\mathcal{P}(z)$, the advantage of $\widetilde{\mathcal{P}}(z)$ is that $\lim_{\operatorname{Im} z\to\infty}\widetilde{\mathcal{P}}(z)=0$. The spectral polynomial of $q_0(z)$ is

$$Q_{q_0,2\ell+1}(E)=E\prod_{j=1}^{\ell}(E-j^2\pi^2)^2.$$

Lemma 4.1. *Fix any $1\leq k\leq \ell$ and consider the Baker-Akhiezer function $\psi(P_k,z,z_0)$ of $q_0(z)$ at $P_k:=(k^2\pi^2,0)$. Then $\psi(P_k,z,z_0)$ has exactly $\ell-k$ zeros $p_1,\cdots,p_{\ell-k}$ in $\mathcal{S}_1$.*

Proof. Fix $1\leq k\leq\ell$ and consider

$$q_k(z):=q_0(z)-2\left(\frac{\psi'(P_k,z,z_0)}{\psi(P_k,z,z_0)}\right)', \tag{4.2}$$

which is a strict genus $\ell-1$ simply-periodic KdV potential, bounded near the ends of the period strip

$$\mathcal{S}_1=\{z\in\mathbb{C}\mid 0\leq\operatorname{Re}z<1\},$$

with its spectral polynomial

$$Q_{q_k,2\ell-1}(E)=E\prod_{j=1,\neq k}^{\ell}(E-j^2\pi^2)^2. \tag{4.3}$$

As in the proof of Theorem 3.6, we have

$$q_k(z)=(\ell-1)\ell\widetilde{\mathcal{P}}(z)+2\sum_{j=1}^{t}\widetilde{\mathcal{P}}(z-p_j), \tag{4.4}$$

and $p_j\in\mathcal{S}_1\setminus\{0\}$ is a zero of the Baker-Akhiezer function $\psi(P_k,z,z_0)$ by the definition (4.2) of $q_k(z)$. Since any zero of $\psi(P_k,z,z_0)$ is simple, we have

$$p_i\neq p_j,\quad \forall i\neq j,\quad \text{if } t>0.$$

To determine t, we denote $\underline{n}_k=(1,\cdots,k-1,k+1,\cdots,\ell)$. Then Theorem 3.2 and (4.3) imply $q_k(z)\in\mathcal{S}(\ell-1,\underline{n}_k,0)$ and so

$$t=\sum_{j=1,\neq k}^{\ell}j-\frac{(\ell-1)\ell}{2}=\ell-k.$$

Indeed, since $q_k(z)$ is strict, it follows from the first proof of Theorem 3.1 that

$$q_k(z)=-2[\ln(\tau_{N_k}(\underline{n}_k,\underline{v}^*,e^{2\pi iz})]'',\ \text{where } N_k=\tfrac{\ell(\ell+1)}{2}-k,$$

which leads to

$$\tau_{N_k}(\underline{n}_k,\underline{v}^*,u)=r_{N_k}(\underline{v}^*)(u-1)^{\frac{(\ell-1)\ell}{2}}\prod_{j=1}^{\ell-k}(u-e^{2\pi ip_j}).$$

In conclusion, $\psi(P_k,z,z_0)$ has exactly $\ell-k$ zeros $p_1,\cdots,p_{\ell-k}$ in $\mathcal{S}_1$. □

The following result determines the location of these zeros, which indicates that all zeros of the polynomial $\tau_{N_k}(\underline{n}_k, \underline{v}^*, u)$ except 1 are *simple and negative*. This result plays a crucial role in our proof of Theorem 1.1.

Theorem 4.2. *Under the above notations, all the zeros $p_1, \cdots, p_{\ell-k}$ of the Baker–Akhiezer function $\psi(P_k, z, z_0)$ lie on the line $\operatorname{Re} z = \frac{1}{2}$. In particular, by renaming $p_1, \cdots, p_{\ell-k}$ if necessary, the following holds.*

(1) *If $t = \ell - k$ is even, then*

$$q_k(z) = (\ell-1)\ell\widetilde{\mathcal{P}}(z) + 2\sum_{j=1}^{\frac{\ell-k}{2}}(\widetilde{\mathcal{P}}(z-p_j) + \widetilde{\mathcal{P}}(z+p_j)), \tag{4.5}$$

where $p_j = \frac{1}{2} + i \operatorname{Im} p_j$ with $\operatorname{Im} p_j \neq 0$ for all j.

(2) *If $t = \ell - k$ is odd, then*

$$q_k(z) = (\ell-1)\ell\widetilde{\mathcal{P}}(z) + 2\widetilde{\mathcal{P}}(z-\tfrac{1}{2}) + 2\sum_{j=1}^{\frac{\ell-k-1}{2}}(\widetilde{\mathcal{P}}(z-p_j) + \widetilde{\mathcal{P}}(z+p_j)), \tag{4.6}$$

where $p_j = \frac{1}{2} + i \operatorname{Im} p_j$ with $\operatorname{Im} p_j \neq 0$ for all j.

Proof. Let $z = \frac{1}{2} + ix$ with $x \in \mathbb{R}$. Then

$$q_0(\tfrac{1}{2}+ix) = \ell(\ell+1)\pi^2[\sin(\pi(\tfrac{1}{2}+ix))]^{-2} = \frac{\ell(\ell+1)\pi^2}{[\cosh(\pi x)]^2}.$$

Consider the spectrum of the following Schrödinger equation

$$-y''(x) - \frac{\ell(\ell+1)\pi^2}{[\cosh(\pi x)]^2}y(x) = \lambda y(x). \tag{4.7}$$

Clearly the positive part of the spectrum is continuous because the potential converges to 0 as $x \to \infty$; while we will prove in Proposition 4.4 that the negative part of the spectrum is discrete and finite, which consists of simple eigenvalues

$$\lambda_j = -(\ell-j)^2\pi^2, \quad 0 \le j \le \ell-1. \tag{4.8}$$

Recalling that

$$\psi''(P_{\ell-j}, z, z_0) = [q_0(z) - (\ell-j)^2\pi^2]\psi(P_{\ell-j}, z, z_0), \quad \forall 0 \le j \le \ell-1,$$

we define

$$\eta_j(x) := \psi(P_{\ell-j}, \tfrac{1}{2}+ix, z_0), \quad x \in \mathbb{R}, \quad \forall 0 \le j \le \ell-1.$$

then we have: (1) $\eta_j(x)$ has at most j zeros on $\mathbb{R}$ because $\psi(P_{\ell-j}, z, z_0)$ has exactly j zeros in $\mathcal{S}_1$; (2) $\lim_{x\to\infty}\eta_j(x) = 0$ by (3.25); (3) $\eta_j(x)$ solves (4.7) with $\lambda = \lambda_j = -(\ell-j)^2\pi^2$. In particular, (2)-(3) imply that $\eta_j(x)$ is precisely the eigenfunction of (4.7) with respect to the simple eigenvalue λ_j. Since $\lambda_0 < \lambda_1 < \cdots < \lambda_{\ell-1}$, it is standard to conclude from (1) and the Sturm comparison principle that $\eta_j(x)$ has precisely j zeros on $\mathbb{R}$ for any $0 \le j \le \ell-1$, i.e. all the zeros $p_1, \cdots, p_j$ of the Baker–Akhiezer function

$\psi(P_{\ell-j}, z, z_0)$ lie on the line $\operatorname{Re} z = \frac{1}{2}$. The proof is complete by noting from Theorem 2.4 that $q_k(z)$ is even since it is strict. □

Now we want to prove (4.8) for general strict simply-periodic KdV potentials. Given any $g \geq 2$ and $\underline{n} = (n_1, \cdots, n_g) \in \mathcal{N}_g$, we consider

$$q(z) = q_{g,\underline{n}}(z) := -2[\ln(\tau_N(\underline{n}, \underline{v}^*, e^{2\pi i z}))]''.$$

It follows from the first proof of Theorem 3.1 that $q(z)$ is a strict genus g simply-periodic KdV potential with basic period $\omega = 1$, and has the following form

$$\begin{aligned} q_{g,\underline{n}}(z) =& g(g+1)\widetilde{\mathcal{P}}(z) + m(m+1)\widetilde{\mathcal{P}}(z - \tfrac{1}{2}) \\ &+ \sum_{j=1}^{r} m_j(m_j+1)(\widetilde{\mathcal{P}}(z - p_j) + \widetilde{\mathcal{P}}(z + p_j)), \end{aligned} \tag{4.9}$$

with

$$N = \sum_{j=1}^{g} n_j = \frac{g(g+1) + m(m+1)}{2} + \sum_{j=1}^{r} m_j(m_j+1), \tag{4.10}$$

where $0 \leq m \leq g-1$ and if $r \geq 1$, we have $1 \leq m_j \leq g-1$, $p_j \not\equiv 0, \frac{1}{2}, \pm p_{j'}$ mod $\mathbb{Z}$ for any $j \neq j'$. Here we used (3.18) to see that there is no constant term in (4.9). Besides, the spectral polynomial is

$$Q_{q_{g,\underline{n}}, 2g+1}(E) = E \prod_{j=1}^{g} (E - n_j^2 \pi^2)^2. \tag{4.11}$$

Remark 4.3. Since

$$\begin{aligned} \tau_N(\underline{n}, \underline{v}^*, u) =& (1-u)^{\frac{g(g+1)}{2}} (1+u)^{\frac{m(m+1)}{2}} \\ &\cdot \prod_{j=1}^{r} [(e^{2\pi i p_j} - u)(e^{-2\pi i p_j} - u)]^{\frac{m_j(m_j+1)}{2}} \in \mathbb{Q}[u], \end{aligned}$$

so $e^{\mp 2\pi i \bar{p}_j} = \overline{e^{\pm 2\pi i p_j}}$ are also roots of $\tau_N(\underline{n}, \underline{v}^*, u)$ with multiplicity $\frac{m_j(m_j+1)}{2}$. That is if $\bar{p}_j \not\equiv \pm p_j$ mod $\mathbb{Z}$, then there is $k \neq j$ such that $\bar{p}_j \equiv \pm p_k$ mod $\mathbb{Z}$. From here and (4.9) we conclude that $\overline{q_{g,\underline{n}}(z)} = q_{g,\underline{n}}(\bar{z})$ and so

$$q_{g,\underline{n}}(z) \in \mathbb{R} \cup \infty \quad \text{for } z = x \in \mathbb{R}.$$

Proposition 4.4. *Under the above notations, we fix any $a \in \mathbb{R}$ such that $q_{g,\underline{n}}(z)$ has no poles on the line $a + i\mathbb{R}$. Then the negative part of the spectrum of the following Schrödinger equation*

$$-y''(x) - q_{g,\underline{n}}(a + ix)y(x) = \lambda y(x), \tag{4.12}$$

is discrete and finite, which consists of simple eigenvalues

$$\lambda_j = -n_{g-j}^2 \pi^2, \quad 0 \leq j \leq g-1.$$

Proof. **Step 1.** Suppose $\lambda < 0$ belongs to the negative part of the spectrum of (4.12) with the corresponding eigenfunction $\varphi(x) \in L^\infty(\mathbb{R}, \mathbb{C})$ (Indeed $\varphi(x) \in L^2(\mathbb{R}, \mathbb{C})$; see Step 2). Then there is a solution $y(z)$ of

$$y''(z) = [q_{g,\underline{n}}(z) + \lambda]y(z), \quad z \in \mathbb{C} \tag{4.13}$$

such that $\varphi(x) = y(a + ix)$. We claim $\lambda = -n_j^2\pi^2$ for some $1 \le j \le g$.

Suppose by contradiction that $\lambda \neq -n_j^2\pi^2$ for any $1 \le j \le g$, i.e. $Q_{q_{g,\underline{n}},2g+1}(-\lambda) \neq 0$. Denote $P = (-\lambda, \mathcal{C})$. Then the Baker-Akhiezer functions $\psi(P, z, z_0)$ and $\psi(P^*, z, z_0)$ of (4.13) are linearly independent, so

$$y(z) = c_1\psi(P, z, z_0) + c_2\psi(P^*, z, z_0) \text{ for some } (c_1, c_2) \neq (0, 0). \tag{4.14}$$

Without loss of generality, we may assume $c_1 \neq 0$.

Since $q(z)$ is even, so does $\Phi_{q,g}(z)$. It follows that

$$\psi(P^*, z, z_0) = \psi(P, -z, z_0) \textit{ up to a multiplying constant}. \tag{4.15}$$

Since $\lim_{\mathrm{Im}\, z\to\infty} q(z) = 0$, as in Lemma 3.7 we see that the leading term of $\psi(P, z, z_0)$ as $\mathrm{Im}\, z \to \pm\infty$ is a multiple of one of $e^{i\sqrt{-\lambda}z}$ and $e^{-i\sqrt{-\lambda}z}$. By $\lambda < 0$, (4.15) and the linear independence of $\psi(P, z, z_0)$ and $\psi(P^*, z, z_0)$, it is easy to see that

$$\lim_{\mathrm{Im}\, z\to+\infty} \psi(P, z, z_0) = \lim_{\mathrm{Im}\, z\to-\infty} \psi(P^*, z, z_0) = A,$$
$$\lim_{\mathrm{Im}\, z\to-\infty} \psi(P, z, z_0) = \lim_{\mathrm{Im}\, z\to+\infty} \psi(P^*, z, z_0) = B,$$

with $\{A, B\} = \{0, \infty\}$. Without loss of generality we may assume $A = \infty$ and $B = 0$. Then (4.14) with $c_1 \neq 0$ implies that $\varphi(x) = y(a + ix) \to \infty$ as $x \to +\infty$, a contradiction with $\varphi(x) \in L^\infty(\mathbb{R}, \mathbb{C})$. This proves $\lambda = -n_j^2\pi^2$ for some $1 \le j \le g$.

Step 2. Suppose $\lambda = -n_j^2\pi^2$ for some $1 \le j \le g$. Letting $\varphi(x) = \psi(P, a + ix, z_0)$ with $P = (-n_j^2\pi^2, 0)$, we easily conclude from (3.20), (3.21) and (3.25) that $\lim_{x\to\infty} \varphi(x) = 0$ and indeed $\varphi(x) \in L^\infty(\mathbb{R}, \mathbb{C}) \cap L^2(\mathbb{R}, \mathbb{C})$. This shows that $\lambda = -n_j^2\pi^2$ belongs to the negative part of the spectrum of (4.12). This completes the proof. □

5. Spectrum of strict simply-periodic KdV potentials along $[0, 1]$

In this section, we study the eigenvalue problem for the potentials (4.5)-(4.6) in Theorem 4.2 and prove Theorem 1.4. Consider the eigenvalue problem

$$\begin{cases} -\varphi''(x) + q(x)\varphi(x) = \lambda\varphi(x), \quad x \in [0, 1], \\ \varphi(x) \in L^2[0, 1], \text{ i.e. } \int_0^1 |\varphi(x)|^2 dx < \infty. \end{cases} \tag{5.1}$$

where $q(x + 1) = q(x)$ and λ is the eigenvalue. As mentioned in Section 1, when $q(z)$ is given by (3.14) with $\omega = 1$:

$$q(z) = g(g + 1)\widetilde{\mathcal{P}}(z) + m(m + 1)\widetilde{\mathcal{P}}(z - \tfrac{1}{2}), \quad 0 \le m < g, \tag{5.2}$$

the set

$$\{\pi^2(g+1+k)^2 \,|\, k \in \mathbb{Z}_{\geq 0}\} \quad \text{if } m=0,$$

$$\{\pi^2(g+m+2+2k)^2 \,|\, k \in \mathbb{Z}_{\geq 0}\} \quad \text{if } m \geq 1,$$

gives all the eigenvalues of (5.1), but the proof can not work for the potentials (4.5)-(4.6) in Theorem 4.2.

In this paper, we develop a unified approach to solve the eigenvalue problem (5.1) for a family of potentials including (5.2) and (4.5)-(4.6). Under the above notations, we define

$$\Xi := \{q_{g,\underline{n}}(z) | g \geq 2, \underline{n} \in \mathcal{N}_g, p_j \notin \mathbb{R} \text{ for all } j \text{ in (4.9) if } r \geq 1\}. \tag{5.3}$$

That is for any $q_{g,\underline{n}}(z) \in \Xi$, $q_{g,\underline{n}}(x)$ has at most singularities $0, 1, \frac{1}{2}$ on $[0,1]$. Clearly this is equivalent to that except ± 1, any other zero $u_j = e^{\pm 2\pi i p_j}$ of $\tau_N(\underline{n}, \underline{v}^*, u)$ satisfies $|u_j| \neq 1$.

Remark 5.1. Clearly the potentials given in (5.2) and (4.5)-(4.6) belong to Ξ, so $\Xi \neq \emptyset$. Given small $g \geq 2$ and $\underline{n} = (n_1, \cdots, n_g) \in \mathcal{N}_g$, since all zeros of $\tau_N(\underline{n}, \underline{v}^*, u)$ can be computed (via *mathematica* for instance), it is easy to check whether $q_{g,\underline{n}}(z) \in \Xi$ or not. Here is a new example. We let $g=3$ and $\underline{n} = (1,4,5)$, i.e. $N = 10$. Then a direct computation leads to

$$\begin{aligned}\tau_{10}(\underline{n}, \underline{v}^*, u) &= (1-u)^6(1 + \tfrac{7}{2}u + 6u^2 + \tfrac{7}{2}u^3 + u^4) \\ &= (1-u)^6(u-u_0)(u-\tfrac{1}{u_0})(u-\overline{u_0})(u-\tfrac{1}{\overline{u_0}}),\end{aligned}$$

where $u_0 + \frac{1}{u_0} = -\frac{7}{4} \pm i\frac{\sqrt{15}}{4} \notin [-2,2]$, i.e. $|u_0| \neq 1$. So this $q_{g,\underline{n}}(z) \in \Xi$. Our following theorem shows that the eigenvalue set of (5.1) with $q(x) = q_{g,\underline{n}}(x)$ is precisely $\{\pi^2 n^2 \,|\, n \in \mathbb{N} \setminus \{1,4,5\}\}$.

Theorem 5.2 (=Theorem 1.4)**.** *Let $g \geq 2, \underline{n} \in \mathcal{N}_g$ such that $q_{g,\underline{n}}(z)$ given by (4.9) satisfies $q_{g,\underline{n}}(z) \in \Xi$, where Ξ is defined in (5.3).*

(1) *If $m = 0$, then the set*

$$\Theta_0 := \left\{ \pi^2 n^2 \middle| \, n \in \mathbb{N} \setminus \{n_1, \cdots, n_g\} \right\}$$

gives all the eigenvalues of the eigenvalue problem (5.1) with $q(x) = q_{g,\underline{n}}(x)$. Furthermore, the eigenfunction $y(x)$ of the eigenvalue $\pi^2 n^2$ satisfies $y(\frac{1}{2}) = 0$ if and only if $n - g$ is even.

(2) *If $m \geq 1$, then the eigenvalue set is given by*

$$\Theta_m := \left\{ \pi^2 \Big(g+m+2-2\sum_{j=1}^{r} m_j + 2k\Big)^2 \middle| \, k \in \mathbb{Z}_{\geq 0}, \; k > \sum_{j=1}^{r} m_j - \tfrac{g+m+2}{2} \right\} \setminus \{n_1^2\pi^2, \cdots, n_g^2\pi^2\}.$$

Proof. Recall (4.11) that the spectral polynomial of $q_{g,\underline{n}}(z)$ is

$$Q_{q_{g,\underline{n}},2g+1}(E) = E\prod_{j=1}^{g}(E-n_j^2\pi^2)^2.$$

Since $q_{g,\underline{n}}(z) \in \Xi$, $q_{g,\underline{n}}(x)$ has singularities precisely at $0,1$ (and also at $\frac{1}{2}$ if $m \geq 1$) on $[0,1]$.

Step 1. We assume that E_0 is an eigenvalue of the eigenvalue problem (5.1) with $q(x) = q_{g,\underline{n}}(x)$. We want to prove $E_0 \in \Theta_m$.

Let $\varphi(x) \in L^2[0,1]$ be the corresponding eigenfunction. Then there is a solution $y(z)$ of

$$y''(z) = [q_{g,\underline{n}}(z) - E_0]y(z), \quad z \in \mathbb{C} \tag{5.4}$$

such that $y(x) = \varphi(x)$ for $x \in [0,1]$, i.e. $y(x) \in L^2[0,1]$. Since the local exponent of $y(z)$ at $0,1$ is either $-g$ or $g+1$, and the local exponent of $y(z)$ at $1/2$ is either $-m$ or $m+1$, we see from $y(x) \in L^2[0,1]$ that

(5.5) *the local exponent of $y(z)$ at $0,1$ must be $g+1$; and*
if $m \geq 1$, then the local exponent of $y(z)$ at $1/2$ must be $m+1$.

From here and $y(\cdot + 1)$ is also a solution of (5.4) with the local exponent $g+1$ at 0, we have

$$y(z+1) = cy(z) \quad \text{for some constant } c. \tag{5.6}$$

Then it follows from Lemma 3.7 that

$$\tilde{q}(z) := q_{g,\underline{n}}(z) - 2\left(\tfrac{y'(z)}{y(z)}\right)'$$

is also a simply-periodic KdV potential with period 1, bounded near the ends of the period strip $\mathcal{S}_1$, i.e. $\tilde{q}(z) \in \mathcal{S}$. Furthermore, we see from Proposition 2.7 and (5.5) that $\tilde{q}(z)$ contains the term $(g+1)(g+2)\widetilde{\mathcal{P}}(z)$, so its genus is at least $g+1$. Together with Theorem 2.6, we conclude that $\tilde{q}(z)$ is a strict genus $g+1$ simply-periodic KdV potential with its spectral polynomial

$$Q_{\tilde{q},2g+3}(E) = (E-E_0)^2 E\prod_{j=1}^{g}(E-n_j^2\pi^2)^2. \tag{5.7}$$

Then Theorem 3.2 says that there is $n_{g+1} \in \mathbb{N} \setminus \{n_1, \cdots, n_g\}$ such that $E_0 = \pi^2 n_{g+1}^2$. More precisely, by letting $\underline{n}^* = (\tilde{n}_1, \cdots, \tilde{n}_{g+1}) \in \mathcal{N}_{g+1}$ such that

$$\{\tilde{n}_1, \cdots, \tilde{n}_{g+1}\} = \{n_1, \cdots, n_g, n_{g+1}\}, \quad \tilde{n}_k < \tilde{n}_{k+1}\ \forall k,$$

we have that $\tilde{q}(z) = e_0 - 2[\ln(\tau_{N+n_{g+1}}(\underline{n}^*, \underline{v}^*, e^{2\pi i z}))]''$ is the unique strict KdV potential (up to translation) in $\mathcal{S}(g+1, \underline{n}^*, e_0)$ for some e_0. Furthermore, Theorem 3.2 and (5.7) yield $e_0 = 0$, so

$$\tilde{q}(z) = -2[\ln(\tau_{N+n_{g+1}}(\underline{n}^*, \underline{v}^*, e^{2\pi i z}))]'' = q_{g+1,\underline{n}^*}(z).$$

Remark that for $m = 0$, we already proved $E_0 = \pi^2 n_{g+1}^2 \in \Theta_0$.

Now we consider the case $m \geq 1$. Together with (4.9), (5.5) and Proposition 2.7, we have

$$\begin{aligned}\tilde{q}(z) =&(g+1)(g+2)\widetilde{\mathcal{P}}(z) + (m+1)(m+2)\widetilde{\mathcal{P}}(z-\tfrac{1}{2})\\ &+\sum_{j=1}^{r}\tilde{m}_j(\tilde{m}_j+1)(\widetilde{\mathcal{P}}(z-p_j)+\widetilde{\mathcal{P}}(z+p_j))\\ &+2\sum_{j=1}^{t}(\widetilde{\mathcal{P}}(z-\tilde{p}_j)+\widetilde{\mathcal{P}}(z+\tilde{p}_j)),\end{aligned}$$

where $\tilde{m}_j \in \{m_j-1, m_j+1\}$ by Proposition 2.7, and $\tilde{p}_j \not\equiv 0, \frac{1}{2}, \pm p_i$ for any j, i if $t > 0$. Consequently

$$\begin{aligned}\sum_{j=1}^{g+1} n_j &= \frac{g(g+1)+m(m+1)}{2} + \sum_{j=1}^{r} m_j(m_j+1) + n_{g+1}\\ &= \frac{(g+1)(g+2)+(m+1)(m+2)}{2} + \sum_{j=1}^{r}\tilde{m}_j(\tilde{m}_j+1) + 2t,\end{aligned}$$

i.e.

$$n_{g+1} = g+m+2+2t+\sum_{j=1}^{r}[\tilde{m}_j(\tilde{m}_j+1) - m_j(m_j+1)].$$

Since $n_{g+1} > 0$, $t \geq 0$ and

$$\tilde{m}_j(\tilde{m}_j+1) - m_j(m_j+1) = \begin{cases} -2m_j & \text{if } \tilde{m}_j = m_j - 1,\\ 2(m_j+1) & \text{if } \tilde{m}_j = m_j+1,\end{cases}$$

we see that $n_{g+1} = g+m+2-2\sum_{j=1}^{r} m_j + 2k$ for some $k \in \mathbb{Z}_{\geq 0}$ and $k > \sum_{j=1}^{r} m_j - \frac{g+m+2}{2}$. This proves $E_0 = \pi^2 n_{g+1}^2 \in \Theta_m$.

Step 2. Take any $n_{g+1} \in \mathbb{N} \setminus \{n_1, \cdots, n_g\}$ such that $E_0 := \pi^2 n_{g+1}^2 \in \Theta_m$. We want to show that E_0 is an eigenvalue of the eigenvalue problem (5.1) with $q(x) = q_{g,\underline{n}}(x)$.

Define $\underline{n}^* \in \mathcal{N}_{g+1}$ as in Step 1 and consider the strict genus $g+1$ potential

$$q_{g+1,\underline{n}^*}(z) := -2[\ln(\tau_{N+n_{g+1}}(\underline{n}^*, \underline{v}^*, e^{2\pi i z}))]'',$$

which contains the term $(g+1)(g+2)\widetilde{\mathcal{P}}(z)$ (see e.g. (4.9) with $g, \underline{n}$ replaced by $g+1, \underline{n}^*$). Its spectral polynomial

$$Q_{q_{g+1,\underline{n}^*},2g+3}(E) = E\prod_{j=1}^{g+1}(E - n_j^2\pi^2)^2.$$

Now consider the Baker-Akhiezer function $\psi(P_0, z, z_0)$ of $q_{g+1,\underline{n}^*}(z)$ with $P_0 := (n_{g+1}^2\pi^2, 0) = (E_0, 0)$. By Lemma 3.8, we see that

$$q_0(z) := q_{g+1,\underline{n}^*}(z) - 2\left(\frac{\psi'(P_0,z,z_0)}{\psi(P_0,z,z_0)}\right)' \tag{5.8}$$

is a strict genus g simply-periodic KdV potential containing the term $g(g+1)\widetilde{\mathcal{P}}(z)$, with its spectral polynomial

$$Q_{q_0,2g+1}(E) = E\prod_{j=1}^{g}(E-n_j^2\pi^2)^2 = Q_{q_{g,\underline{n}},2g+1}(E).$$

Then Theorem 2.5 shows that $q_0(z) = q_{g,\underline{n}}(z)$. Note from the proof of Lemma 3.8 that $0,1$ are both poles of $\psi(P_0,z,z_0)$ with order $g+1$. Define

$$y(z) := \frac{1}{\psi(P_0,z,z_0)}.$$

Then

(5.9) $\qquad 0,1$ *are both zeros of* $y(z)$ *with multiplicity* $g+1$.

Furthermore, $\psi''(P_0,z,z_0) = [q_{g+1,\underline{n}^*}(z)-E_0]\psi(P_0,z,z_0)$ implies that $y(z)$ is a solution of (5.4) and

$$q_{g+1,\underline{n}^*}(z) = q_{g,\underline{n}}(z) - 2\left(\tfrac{y'(z)}{y(z)}\right)'. \tag{5.10}$$

By the expression (4.9) of $q_{g,\underline{n}}(z)$ and Proposition 2.7, we deduce from (5.10) that

$$\begin{aligned} q_{g+1,\underline{n}^*}(z) =&(g+1)(g+2)\widetilde{\mathcal{P}}(z) + \tilde{m}(\tilde{m}+1)\widetilde{\mathcal{P}}(z-\tfrac{1}{2}) \\ &+\sum_{j=1}^{r}\tilde{m}_j(\tilde{m}_j+1)(\widetilde{\mathcal{P}}(z-p_j)+\widetilde{\mathcal{P}}(z+p_j)) \\ &+2\sum_{j=1}^{t}(\widetilde{\mathcal{P}}(z-\tilde{p}_j)+\widetilde{\mathcal{P}}(z+\tilde{p}_j)), \end{aligned}$$

where $\tilde{m}\in\{m-1,m+1\}$, $\tilde{m}_j\in\{m_j-1,m_j+1\}$, and $\tilde{p}_j\not\equiv 0,\frac{1}{2},\pm p_i$ for any j,i if $t>0$.

First we consider $m=0$. Then we see from (5.9) that $y(x)\in L^2[0,1]$ and so E_0 is an eigenvalue with eigenfunction $y(x)$. Note $\tilde{m}\in\{-1,1\}$ or equivalently $\tilde{m}\in\{0,1\}$. Clearly $y(1/2)=0$ if and only if $1/2$ is a pole of $\psi(P_0,z,z_0)$ if and only if $\tilde{m}=1$, i.e.

$$\begin{aligned} &\frac{g(g+1)}{2}+\sum_{j=1}^{r}m_j(m_j+1)+n_{g+1} \\ =&\sum_{j=1}^{g+1}n_j = \frac{(g+1)(g+2)}{2}+1+\sum_{j=1}^{r}\tilde{m}_j(\tilde{m}_j+1)+2t. \end{aligned}$$

So $n_{g+1}-g$ is even. Similarly, $y(1/2)\neq 0$ if and only if $\tilde{m}=0$, which easily implies that $n_{g+1}-g$ is odd. This proves the assertion (1).

Now we consider the case $m \geq 1$. Then $n_{g+1} = g + m + 2 - 2\sum_{j=1}^{r} m_j + 2k > 0$ for some $k \in \mathbb{Z}_{\geq 0}$. Suppose $\tilde{m} = m - 1$, then

$$\frac{g(g+1) + m(m+1)}{2} + \sum_{j=1}^{r} m_j(m_j+1) + g + m + 2 - 2\sum_{j=1}^{r} m_j + 2k$$

$$= \sum_{j=1}^{g+1} n_j = \frac{(g+1)(g+2) + (m-1)m}{2} + \sum_{j=1}^{r} \tilde{m}_j(\tilde{m}_j+1) + 2t,$$

i.e.

$$1 = 2t - 2k - 2m + \sum_{j=1}^{r} [\tilde{m}_j(\tilde{m}_j+1) - m_j(m_j+1) + 2m_j],$$

a contradiction. Thus $\tilde{m} = m + 1$, which implies that $\frac{1}{2}$ is a zero of $y(z)$ with multiplicity $m+1$. Together with (5.9), we conclude that $y(x) \in L^2[0,1]$ and so E_0 is an eigenvalue with eigenfunction $y(x)$.

The proof is complete. □

Remark 5.3. The above proof shows that $1/y(z)$ is a Baker-Akhiezer function of $q_{g+1,\underline{n}^*}(z)$, but we see from Theorem 2.6 that the eigenfunction $y(z)$ itself is not a Baker-Akhiezer function of $q_{g,\underline{n}}(z)$ (otherwise the genus of $q_{g+1,\underline{n}^*}(z)$ is at most g, a contradiction). Consequently, we must have $c = \pm 1$ in (5.6), i.e. $y(z+1) = \pm y(z)$, and so solutions of (5.4) are all periodic or all anti-periodic.

6. Application to Lamé functions

In this section, we apply Theorem 4.2 to the classical Lamé equation

$$y''(z) = [\ell(\ell+1)\wp(z;\tau) - E]y(z), \quad \ell \in \mathbb{N} \tag{6.1}$$

and give the proof of Theorem 1.1. Here $\wp(z;\tau) := \wp(z;\Lambda_\tau)$ with $\Lambda_\tau = \mathbb{Z} + \mathbb{Z}\tau$ and $\tau \in i\mathbb{R}_{>0}$. We recall some classical results (see e.g. [3, 13, 33]).

(i). The spectral polynomial of $q(z;\tau) = \ell(\ell+1)\wp(z;\tau)$ is given by

$$Q_{q,2\ell+1}(E;\tau) = \prod_{j=0}^{2\ell}(E - E_j(\tau)),$$

with $E_0(\tau) < E_1(\tau) < \cdots < E_{2\ell}(\tau)$.

(ii). Let $E_\tau = \mathbb{C}/\Lambda_\tau$ be a flat torus. For any $E \in \mathbb{C}$, there exists a unique pair $\pm\boldsymbol{a} = \pm\boldsymbol{a}(E,\tau) := \pm\{a_1, \cdots, a_\ell\} \subset E_\tau \setminus \{0\}$ satisfying $a_i \not\equiv a_j \mod \Lambda_\tau$ for $i \neq j$ such that the classical *Hermite-Halphen ansatz*

$$y_{\pm\boldsymbol{a}}(z;\tau) := e^{\pm\sum_{j=1}^{\ell}\zeta(a_j;\tau)z}\frac{\prod_{j=1}^{\ell}\sigma(z \mp a_j;\tau)}{\sigma(z;\tau)^\ell}$$

are solutions of (6.1) with

$$E = -(2\ell-1)\sum_{j=1}^{\ell}\wp(a_j;\tau).$$

Here $\zeta(z;\tau)$ and $\sigma(z;\tau)$ are the associated Weierstrass functions of $\wp(z;\tau)$. Indeed, $y_{\pm \boldsymbol{a}}(z;\tau)$ are the *Baker-Akhiezer functions* $\psi(P,z,z_0)$, $\psi(P^*,z,z_0)$ of (6.1) up to multiplying constants. In particular,

(6.2) *$\boldsymbol{a}$, $-\boldsymbol{a}$ are the zero sets of $\psi(P,z,z_0)$, $\psi(P^*,z,z_0)$ respectively.*

(iii). For $E \notin \{E_j(\tau)\}_{j=0}^{2\ell}$, we have $\boldsymbol{a} \cap -\boldsymbol{a} = \emptyset$. Consequently, $y_{\boldsymbol{a}}(z;\tau)$ and $y_{-\boldsymbol{a}}(z;\tau)$ are linearly independent.

(iv). For $E \in \{E_j(\tau)\}_{j=0}^{2\ell}$, we have $\boldsymbol{a} = -\boldsymbol{a}$, i.e. $y_{\boldsymbol{a}}(z;\tau) = (-1)^n y_{-\boldsymbol{a}}(z;\tau)$ and $\psi(P,z,z_0) = \psi(P^*,z,z_0)$. In this case, $y_{\boldsymbol{a}}(z;\tau)$ is known as the *Lamé function* in the literature.

In this section, we study the asymptotics of the zeros of the Lamé function as $\tau \to i\infty$. Recalling the simply-periodic KdV potential $q_0(z)$ given in (4.1), it is well known that

$$\begin{aligned}\lim_{\tau\to i\infty} q(z;\tau) &= \ell(\ell+1)\lim_{\tau\to i\infty}\wp(z;\tau) = \ell(\ell+1)\left(\tfrac{\pi^2}{(\sin\pi z)^2} - \tfrac{\pi^2}{3}\right)\\ &= q_0(z) + e_0 =: q_1(z),\end{aligned}$$

where $e_0 := -\frac{\pi^2}{3}\ell(\ell+1)$. Clearly the spectral polynomial of $q_1(z)$ is

$$Q_{q_1,2\ell+1}(E) = Q_{q_0,2\ell+1}(E-e_0) = (E-e_0)\prod_{j=1}^{\ell}(E-e_0-j^2\pi^2)^2.$$

Consider the corresponding $\Phi_{q,\ell}(z;E)$ of $q(z;\tau)$ defined in (2.3), which solves

$$\Phi''' - 4(q(z;\tau)-E)\Phi' - 2q'(z;\tau)\Phi = 0.$$

It was proved in [3, Theorem 7.3 (i)] that

$$\Phi_{q,\ell}(z;E) = E^\ell + \sum_{j=0}^{\ell-1} f_{\ell-j}(q)E^j \in \mathbb{Q}[g_2(\tau), g_3(\tau), \wp(z;\tau)][E],$$

where $g_2(\tau), g_3(\tau)$ are well-known invariants of the elliptic curve E_τ, which satisfy

$$\lim_{\tau\to i\infty} g_2(\tau) = \tfrac{4}{3}\pi^4, \quad \lim_{\tau\to i\infty} g_3(\tau) = \tfrac{8}{27}\pi^6.$$

Hence,

$$\Phi_1(z;E) := \lim_{\tau\to i\infty}\Phi_{q,\ell}(z;E) = E^\ell + \cdots \tag{6.3}$$

is a well-defined monic polynomial of degree ℓ in E and solves

$$\Phi''' - 4(q_1(z)-E)\Phi' - 2q_1'(z)\Phi = 0,$$

namely $\Phi_1(z;E)$ is the corresponding $\Phi_{q_1,\ell}(z;E)$ of $q_1(z)$ defined in (2.3) and so

$$\lim_{\tau\to i\infty} Q_{q,2\ell+1}(E;\tau) = \lim_{\tau\to i\infty}\prod_{j=0}^{2\ell}(E-E_j(\tau))$$

$$= Q_{q_1,2\ell+1}(E) = (E-e_0)\prod_{j=1}^{\ell}(E-e_0-j^2\pi^2)^2.$$

That is, $E_0(\tau)\to e_0$ and

$$\lim_{\tau\to i\infty} E_{2k-1}(\tau) = \lim_{\tau\to i\infty} E_{2k}(\tau) = e_0 + k^2\pi^2, \quad 1\leq k\leq \ell. \tag{6.4}$$

Fix $1\leq k\leq \ell$ and consider the Baker-Akhiezer function $\psi(P_k(\tau), z, z_0)$ of the Lamé potential $q(z;\tau)$ at $P_k(\tau)\in\{(E_{2k-1}(\tau),0),(E_{2k}(\tau),0)\}$. Then (6.4) implies that

$$\lim_{\tau\to i\infty}\Phi_{q,\ell}(z;E_{2k-1}(\tau)) = \lim_{\tau\to i\infty}\Phi_{q,\ell}(z;E_{2k}(\tau)) = \Phi_{q_1,\ell}(z;e_0+k^2\pi^2)$$

and so

$$\lim_{\tau\to i\infty}\psi(P_k(\tau),z,z_0) = \psi(P_k,z,z_0),$$

where $\psi(P_k,z,z_0)$ is the Baker-Akhiezer function of $q_1(z)$ at $P_k = (e_0 + k^2\pi^2, 0)$ or equivalently, the Baker-Akhiezer function of $q_0(z)$ at $(k^2\pi^2, 0)$. These, together with (6.2), Theorem 4.2 and Remark 3.4, immediately imply Theorem 1.1, i.e. the following result.

Theorem 6.1 (=Theorem 1.1)**.** *Fix $1\leq k\leq\ell$ and consider the Lamé function $y_{\boldsymbol{a}}(z;\tau)$ at $E\in\{E_{2k-1}(\tau),E_{2k}(\tau)\}$ with its zero set*

$$\boldsymbol{a} = \boldsymbol{a}(E,\tau) = \{a_1(\tau),\cdots,a_\ell(\tau)\}\subset E_\tau\setminus\{0\}, \quad \operatorname{Re} a_j\in[0,1],\ \forall j.$$

Then as $\tau\to i\infty$, there are k zeros $a_j(\tau)$'s converging to infinity, and the other $\ell-k$ zeros $a_j(\tau)$'s converges to $\ell-k$ distinct points p_j's which satisfy $\operatorname{Re} p_j = \frac{1}{2}$ and $e^{2\pi i p_j}$ is an algebraic number.

Acknowledgements The authors wish to thank a referee very much for valuable comments and pointing out some references to us. The authors also thank Ting-Jung Kuo for valuable discussions. The research of Z. Chen was supported by Independent Research Program at Tsinghua University (Grant No. 2019Z07L02016) and NSFC (No. 11701312).

REFERENCES

[1] H. Airault, H. McKean and J. Moser; *Rational and elliptic solutions of the Korteweg-deVries equation and a related many-body problem*. Comm. Pure Appl. Math. **30** (1977), 95-148.
[2] J. Burchnall and T. Chaundy; *Commutative ordinary differential operators*. Proc. Lond. Math. Soc. **21** (1923), 420-440.
[3] C.L. Chai, C.S. Lin and C.L. Wang; *Mean field equations, Hyperelliptic curves, and Modular forms: I*. Camb. J. Math. **3** (2015), 127-274.
[4] Z. Chen, T. J. Kuo and C. S. Lin; *On algebro-geometric solutions of the KdV hierarchy*. In preparation.
[5] Z. Chen and C. S. Lin; *Sharp nonexistence results for curvature equations with four singular sources on rectangular tori*. Amer. J. Math. to appear.
[6] B. A. Dubrovin; *Periodic problems for the Korteweg-de Vries equation in the class of finite band potentials*. Funct. Anal. Appl. **9** (1975), 215-223.

[7] F. Ehlers and H. Knörrer; *An algebro-geometric interpertation of the Bäcklund transformation for the Korteweg-de Vries equation*. Comment Math. Helv. **57** (1982), 1-10.
[8] P. Etingof and E. Rains; *On Algebraically Integrable Differential Operators on an Elliptic Curve*. SIGMA Symm. Integ. Geom. Methods Appl. **7** (2011), Paper 062, 19pp.
[9] F. Gesztesy and H. Holden; *Darboux-type transformations and hyperelliptic curves*. J. reine angew. Math. **527** (2000), 151-183.
[10] F. Gesztesy and H. Holden; *Soliton equations and their algebro-geometric solutions. Vol. I. $(1+1)$-dimensional continuous models.* Cambridge Studies in Advanced Mathematics, vol. 79, Cambridge University Press, Cambridge, 2003. xii+505 pp.
[11] F. Gesztesy, K. Unterkofler and R. Weikard; *An explicit characterization of Calogero-Moser systems*. Trans. Am. Math. Soc. **358** (2006), 603-656.
[12] F. Gesztesy and R. Weikard; *Picard potentials and Hill's equation on a torus*. Acta Math. **176** (1996), 73-107.
[13] F. Gesztesy and R. Weikard; *Lamé potentials and the stationary (m)KdV hierarchy*. Math. Nachr. **176** (1995), 73-91.
[14] V. B. Matveev and M. A. Salle; *Darboux Transformations and Solitons*. Springer, Berlin 1991.
[15] W. Magnus and S. Winkler; *Hill's equation*, Reprint of the 1979 second edition, Dover Publications, Inc., Mineola, NY, 2004.
[16] E. Ince; *Further investigations into the periodic Lamé functions*. Proc. Roy. Soc. Edinburgh **60** (1940), 83-99.
[17] A. R. Its and V. B. Matveev; *Schrödinger operators with finite-gap spectrum and N-soliton solutions of the Korteweg-de Vries equation*. Theoret. Math. Phys. **23** (1975), 343-355.
[18] L. D. Landau and E. M. Lifshitz; *Quantum mechanics: non-relativistic theory*. Vol. 3. Pergamon Press, 1991.
[19] P. Lax; *Integrals of nonlinear equations of evolution and solitary waves*. Comm. Pure Appl. Math. **21** (1968), 467-490.
[20] C.S. Lin and C.L. Wang; *Mean field equations, Hyperelliptic curves, and Modular forms: II*. J. Éc. polytech. Math. **4** (2017), 557-593.
[21] R. Maier; *Lamé polynomail, hyperelliptic reductions and Lamé band structure.* Philos. Trans. R. Soc. A **366** (2008), 1115-1153.
[22] S. P. Novikov; *The periodic problem for the Korteweg-de Vries equation*. Funct. Anal. Appl. **8** (1974), 236-246.
[23] E. Previato; *Burchnall-Chaundy bundles*. Algebraic geometry (Catania, 1993/Barcelona, 1994), 377-383, Lecture Notes in Pure and Appl. Math. 200, Dekker, New York, 1998.
[24] G. Segal and G. Wilson; *Loop groups and equations of KdV type*. Publ. Math. IHES. **61** (1985), 5-65.
[25] A. O. Smirnov; *Finite-gap elliptic solutions of the KdV equation*. Acta Appl. Math. **36** (1994), 125-166.
[26] K. Takemura; *The Heun equation and the Calogero-Moser-Sutherland system I: the Bethe Ansatz method*. Comm. Math. Phys. **235** (2003), 467-494.
[27] K. Takemura; *The Heun equation and the Calogero-Moser-Sutherland system II: perturbation and algebraic solution*. Elec. J. Differ. Equ. **2004** (2004), no. 15, 1-30.
[28] K. Takemura; *The Heun equation and the Calogero-Moser-Sutherland system IV: the Hermite-Krichever Ansatz*. Comm. Math. Phys. **258** (2005), 367–403.
[29] K. Takemura; *Analytic continuation of eigenvalues of the Lamé operator*. J. Differ. Equ. **228** (2006), 1-16.
[30] A. Treibich; *Hyperelliptic tangential covers, and finite-gap potentials*. Russ. Math. Surv. **56** (2001), 1107-1151.
[31] A. Treibich and J. L. Verdier; *Revetements exceptionnels et sommes de 4 nombres triangulaires*. Duke Math. J. **68** (1992), 217-236.
[32] R. Weikard; *On rational and periodic solutions of stationary KdV equations*. Doc. Math. J. DMV **4** (1999), 109-126.

[33] E. T. Whittaker and G.N. Watson; *A course of modern analysis*, *4th edition*. Cambridge University Press, 1927.

DEPARTMENT OF MATHEMATICAL SCIENCES, YAU MATHEMATICAL SCIENCES CENTER, TSINGHUA UNIVERSITY, BEIJING, 100084, CHINA
E-mail address: zjchen2016@tsinghua.edu.cn

TAIDA INSTITUTE FOR MATHEMATICAL SCIENCES (TIMS), CENTER FOR ADVANCED STUDY IN THEORETICAL SCIENCES (CASTS), NATIONAL TAIWAN UNIVERSITY, TAIPEI 10617, TAIWAN
E-mail address: cslin@math.ntu.edu.tw