The Orlicz-Petty bodies *

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Abstract

This paper is dedicated to the Orlicz-Petty bodies. We first propose the homogeneous Orlicz affine and geominimal surface areas, and establish their basic properties such as homogeneity, affine invariance and affine isoperimetric inequalities. We also prove that the homogeneous geominimal surface areas are continuous, under certain conditions, on the set of convex bodies in terms of the Hausdorff distance. Our proofs rely on the existence of the Orlicz-Petty bodies and the uniform boundedness of the Orlicz-Petty bodies of a convergent sequence of convex bodies. Similar results for the nonhomogeneous Orlicz geominimal surface areas are proved as well.

2010 Mathematics Subject Classification: 52A20, 53A15.

1 Introduction

The theory of convex geometry was greatly enriched by the combination of two notions: the volume and the linear Orlicz addition of convex bodies [13, 45]. This new theory, usually called the Orlicz-Brunn-Minkowski theory for convex bodies, started from the works of Lutwak, Yang and Zhang [30, 31], and received considerable attention (see e.g., [5, 6, 7, 11, 17, 18, 46, 47, 55, 56]). The linear Orlicz addition of convex bodies was proposed by Gardner, Hug and Weil [13] (independently Xi, Jin and Leng [45]). Let $\varphi_i : [0,\infty) \to [0,\infty)$, $i = 1, 2$, be convex functions such that φ_i is strictly increasing with $\varphi_i(1) = 1$, $\varphi_i(0) = 0$ and $\lim_{t\to\infty} \varphi_i(t) = \infty$. Let S^{n-1} be the unit sphere in $\mathbb{R}^n$ and $h_K : S^{n-1} \to (0,\infty)$ denote the support function of K. For any given $\varepsilon > 0$ and two convex bodies K and L with the origin in their interiors, the linear Orlicz addition $K +_{\varphi,\varepsilon} L$ is determined by its support function $h_{K+_{\varphi,\varepsilon}L}$, the unique solution of

$$\varphi_1\Big(\frac{h_K(u)}{\lambda}\Big) + \varepsilon\varphi_2\Big(\frac{h_L(u)}{\lambda}\Big) = 1 \quad \text{for} \quad u \in S^{n-1}.$$

Denote by $|K +_{\varphi,\varepsilon} L|$ the volume of $K +_{\varphi,\varepsilon} L$. If $(\varphi_1)'_l(1)$, the left derivative of φ_1 at $t = 1$, exists and is positive, then

$$\frac{(\varphi_1)'_l(1)}{n} \cdot \frac{d}{d\varepsilon}|K +_{\varphi,\varepsilon} L|\bigg|_{\varepsilon=0^+} = \frac{1}{n}\int_{S^{n-1}} \varphi_2\bigg(\frac{h_L(u)}{h_K(u)}\bigg) h_K(u) dS_K(u), \tag{1.1}$$

where S_K is the surface area measure of K (see [13, 45] for more details). That is, formula (1.1) provides a geometric interpretation of $V_\phi(K, L)$ for ϕ being convex and strictly increasing. Here, for any continuous function $\phi : (0,\infty) \to (0,\infty)$, $V_\phi(K, L)$ denotes the nonhomogeneous Orlicz L_ϕ mixed volume of K and L:

$$V_\phi(K, L) = \frac{1}{n}\int_{S^{n-1}} \phi\bigg(\frac{h_L(u)}{h_K(u)}\bigg) h_K(u) dS_K(u). \tag{1.2}$$

*Keywords: affine isoperimetric inequalities, affine surface area, geominimal surface area, Orlicz-Brunn-Minkowski theory, Orlicz mixed volume, Petty body.

To the best of our knowledge, there are no geometric interpretations of $V_\phi(K,L)$ for non-convex functions ϕ (even for $\phi(t)=t^p$ with $p<1$) in literature; and such geometric interpretations will be provided in Subsection 5.1 in this paper. Note that formula (1.1) is essential for the Orlicz-Minkowski inequality and many other objects, such as the Orlicz affine and geominimal surface areas [49].

Introduced by Blaschke in 1923 [4], the classical affine surface area was thought to be one of the core concepts in the Brunn-Minkowski theory of convex bodies due to its important applications in, such as, approximation of convex bodies by polytopes [15, 26, 41] and valuation theory [2, 3, 24]. Since the groundbreaking paper by Lutwak [28], considerable progress has been made on the theory of the L_p affine surface areas (see e.g., [21, 25, 33, 34, 36, 40, 42, 43, 44]). Like the classical affine surface area, the L_p affine surface areas play fundamental roles in applications and provide powerful tools in convex geometry. Note that the L_p affine surface areas are affine invariant valuations with homogeneity.

In the Orlicz-Brunn-Minkowski theory for convex bodies, a central task is to find the "right" definitions for the Orlicz affine surface areas. Here, we will discuss two different approaches by Ludwig [23] and the third author [49]. Based on an integral formula, Ludwig proposed the general affine surface areas [23]. Ludwig's definitions work perfectly in studying properties such as valuation [23], the characterization of valuation [18, 23] and the monotonicity under the Steiner symmetrization [47]. In order to define the Orlicz geominimal surface areas, new ideas are needed because geominimal surface areas do not have convenient integral expression like their affine relatives. The third author provided a unified approach to define the Orlicz affine and geominimal surface areas [49] based on the Orlicz L_ϕ mixed volume $V_\phi(\cdot,\cdot)$ defined in formula (1.2). In fact, the approach in [49] is related to an optimization problem for the f-divergence [20] and could be used to define other versions of Orlicz affine and geominimal surface areas [9, 50, 51].

Note that the natural property of "homogeneity" is missing in the Orlicz affine surface areas in [23, 49]. To define the Orlicz affine surface areas with homogeneity is one of the main objects in this paper; and it will be done in Section 3. As an example, we give the definition for $\phi\in\widehat{\Phi}_1$, where $\widehat{\Phi}_1$ is the set of functions $\phi:[0,\infty)\to[0,\infty)$ such that ϕ is strictly increasing with $\phi(0)=0,\phi(1)=1$, $\lim_{t\to\infty}\phi(t)=\infty$ and $\phi(t^{-1/n})$ being strictly convex on $(0,\infty)$. For convex body K and star body L with the origin in their interiors, define $\widehat{V}_\phi(K,L^\circ)$ for $\phi\in\widehat{\Phi}_1$ by

$$\widehat{V}_\phi(K,L^\circ)=\inf_{\lambda>0}\left\{\int_{S^{n-1}}\phi\Big(\frac{n|K|}{\lambda\cdot\rho_L(u)\cdot h_K(u)}\Big)h_K(u)\,dS_K(u)\le n|K|\right\},$$

where ρ_L denotes the radial function of L. We now define $\widehat{\Omega}^{orlicz}_\phi(K)$ for $\phi\in\widehat{\Phi}_1$, the homogeneous Orlicz L_ϕ affine surface area of K, by the infimum of $\widehat{V}_\phi(K,L^\circ)$ where L runs over all star bodies with the origin in their interiors and $|L|=|B_2^n|$ (the volume of the Euclidean unit ball of $\mathbb{R}^n$). In Proposition 3.1, we show that $\widehat{\Omega}^{orlicz}_\phi(K)$ is invariant under the volume preserving linear maps and has homogeneous degree $(n-1)$. Moreover, the following affine isoperimetric inequality is established in Theorem 3.1: *if K has its centroid at the origin and $\phi\in\widehat{\Phi}_1$, then*

$$\frac{\widehat{\Omega}^{orlicz}_\phi(K)}{\widehat{\Omega}^{orlicz}_\phi(B_2^n)}\ \le\ \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-1}{n}},$$

with equality if and only if K is an origin-symmetric ellipsoid. Note that affine isoperimetric inequalities are fundamental in convex geometry; and these inequalities compare affine invariant functionals with the volume (see e.g., [10, 19, 32, 29, 30, 31, 44, 53]).

The Petty body and its L_p extensions for $p>1$ were used to study the continuity of the classical geominimal surface area and its L_p counterparts [28, 37]. To prove the existence and uniqueness of the Orlicz-Petty bodies is one of the main goals of Section 4 in this paper. In order to fulfill these goals, we first define $\widehat{G}_{\phi}^{orlicz}(K)$, the homogeneous Orlicz geominimal surface area of K, by the infimum of $\widehat{V}_{\phi}(K, L^{\circ})$ where L runs over all convex bodies with the origin in their interiors and $|L| = |B_2^n|$. The classical geominimal surface area, which corresponds to $\phi(t) = t$, was introduced by Petty [37] in order to study the affine isoperimetric problems [37, 38]. The classical geominimal surface area and its L_p extensions (corresponding to $\phi(t) = t^p$) for $p > 1$ by Lutwak [28] are continuous on the set of convex bodies in terms of the Hausdorff distance; while their affine relatives are only semicontinuous. The main ingredients to prove the continuity of the L_p geominimal surface area for $p \geq 1$ are the existence of the L_p Petty bodies and the uniform boundedness of the L_p Petty bodies of a convergent sequence of convex bodies (hence, the Blaschke selection theorem can be used). In Section 4, we will prove that $\widehat{G}_{\phi}^{orlicz}(\cdot)$ is also continuous for $\phi \in \widehat{\Phi}_1$. Note that $\phi(t) = t^p \in \widehat{\Phi}_1$ if $p \in (0, \infty)$. Consequently, the L_p geominimal surface area for $p \in (0, 1)$, proposed by the third author in [48], is also continuous. Our approach basically follows the steps in [28, 37]; however, our proof is more delicate and requires much more careful analysis due to the lack of convexity of ϕ (note that in $\widehat{\Phi}_1$, $\phi(t^{-1/n})$ is assumed to be convex, not ϕ itself). In particular, we prove the existence and uniqueness of the Orlicz-Petty bodies in Proposition 4.3. Our main result is Theorem 4.1: *if* $\phi \in \widehat{\Phi}_1$, *then the homogeneous* L_ϕ *Orlicz geominimal surface area is continuous on the set of convex bodies with respect to the Hausdorff distance.* The continuity of nonhomogeneous Orlicz geominimal surface areas [49] will be discussed in Subsection 5.2. The L_p Petty body for $p \in (-1, 0)$ is more involved and will be discussed in Section 6.

2 Background and Notation

We now introduce the basic well-known facts and standard notations needed in this paper. For more details and more concepts in convex geometry, please see [12, 16, 39].

A convex and compact subset $K \subset \mathbb{R}^n$ with nonempty interior is called a convex body in $\mathbb{R}^n$. By $\mathscr{K}$ we mean the set of all convex bodies containing the origin and by $\mathscr{K}_0$ the set of all convex bodies with the origin in their interiors. A convex body K is said to be origin-symmetric if $K = -K$ where $-K = \{x \in \mathbb{R}^n : -x \in K\}$. Let $\mathscr{K}_e$ denote the set of all origin-symmetric convex bodies in $\mathbb{R}^n$. The volume of K is denoted by $|K|$ and the volume radius of K is denoted by $\text{vrad}(K)$. By B_2^n and S^{n-1}, we mean the Euclidean unit ball and the unit sphere in $\mathbb{R}^n$ respectively. The volume of B_2^n will be often written by ω_n and the natural spherical measure on S^{n-1} is written by σ. Consequently, $\text{vrad}(K) = (|K|/\omega_n)^{1/n}$. The standard notation $GL(n)$ stands for the set of all invertible linear transforms on $\mathbb{R}^n$. For $A \in GL(n)$, we use $\det A$ to denote the determinant of A. Let $SL(n) = \{A : \ A \in GL(n) \text{ and } \det A = \pm 1\}$. By A^t and A^{-t} we mean the transpose of A and the inverse of A^t respectively.

Each convex body $K \in \mathscr{K}$ has a continuous support function $h_K : S^{n-1} \to [0, \infty)$ defined by $h_K(u) = \max_{x \in K} \langle x, u \rangle$ for $u \in S^{n-1}$, where $\langle \cdot, \cdot \rangle$ denotes the usual inner product. Note that h_K for $K \in \mathscr{K}$ is nonnegative on S^{n-1}, but it is strictly positive on S^{n-1} if $K \in \mathscr{K}_0$. Moreover, one can define a probability measure $\widetilde{V}_K$ on each $K \in \mathscr{K}$ by

$$d\widetilde{V}_K(u) = \frac{h_K(u)\, dS_K(u)}{n|K|} \quad \text{for} \;\; u \in S^{n-1},$$

where S_K is the surface area measure of K. It is well known that S_K satisfies

$$\int_{S^{n-1}} u\, dS_K(u) = 0 \;\; \text{and} \;\; \int_{S^{n-1}} |\langle u, v \rangle|\, dS_K(u) > 0 \text{ for each } v \in S^{n-1}. \tag{2.3}$$

The first formula of (2.3) asserts that S_K has its centroid at the origin and the second one states that S_K is not concentrated on any great subsphere. Let μ_K denote the usual surface area of ∂K, the boundary of K, and $N_K(x)$ denote a unit outer normal vector of $x \in \partial K$. For each $f \in C(S^{n-1})$, where $C(S^{n-1})$ denotes the set of all continuous functions defined on S^{n-1}, one has

$$\int_{S^{n-1}} f(u)\, dS_K(u) = \int_{\partial K} f(N_K(x))\, d\mu_K(x).$$

The dilation of K is of form $sK = \{sx : x \in K\}$ for $s > 0$. Clearly, $h_{sK}(u) = s \cdot h_K(u)$ for all $u \in S^{n-1}$. Moreover, sK and K share the same probability measure $d\widetilde{V}_K(u)$. Two convex bodies K and L are said to be dilates of each other if $K = sL$ for some constant $s > 0$.

For $u \in S^{n-1}$, let $l_u = \{tu : t \geq 0\}$. We say $L \subset \mathbb{R}^n$ is star-shaped at the origin if, for each $u \in S^{n-1}$, $L \cap l_u$ is a closed line segment containing the origin. One can define the radial function $\rho_L : S^{n-1} \to [0, \infty)$ for L a star-shaped set about the origin by

$$\rho_L(u) = \max\{\lambda \geq 0 : \lambda u \in L\} \qquad \text{for} \quad u \in S^{n-1}.$$

If ρ_L is positive and continuous on S^{n-1}, then L is called a star body about the origin. Denote by $\mathscr{S}_0$ the set of star bodies about the origin in $\mathbb{R}^n$ and clearly $\mathscr{K}_0 \subset \mathscr{S}_0$. The volume of $L \in \mathscr{S}_0$ can be calculated by

$$|L| = \frac{1}{n} \int_{S^{n-1}} \rho_L^n(u)\, d\sigma(u) \quad \text{and} \quad |K^\circ| = \frac{1}{n} \int_{S^{n-1}} \frac{1}{h_K^n(u)}\, d\sigma(u). \tag{2.4}$$

Hereafter, $K^\circ \in \mathscr{K}_0$ is the polar body of $K \in \mathscr{K}_0$; and the support function h_{K° and the radial function ρ_{K° are given by

$$h_{K^\circ}(u) = \frac{1}{\rho_K(u)} \quad \text{and} \quad \rho_{K^\circ}(u) = \frac{1}{h_K(u)}, \quad \text{for all} \quad u \in S^{n-1}.$$

Alternatively, K° can be defined by

$$K^\circ = \{x \in \mathbb{R}^n : \langle x, y\rangle \leq 1 \;\; \text{for all } y \in K\}.$$

The bipolar theorem states that $(K^\circ)^\circ = K$ if $K \in \mathscr{K}_0$.

Let $\mathscr{K}_c \subset \mathscr{K}_0$ be the set of convex bodies with their centroids at origin; that is, $\int_K x\, dx = 0$ if $K \in \mathscr{K}_c$. We say $K \in \mathscr{K}_0$ has the Santaló point at the origin if $K^\circ \in \mathscr{K}_c$. Denote by $\mathscr{K}_s \subset \mathscr{K}_0$ the set of convex bodies with their Santaló points at the origin, and let $\widetilde{\mathscr{K}} = \mathscr{K}_s \cup \mathscr{K}_c$. The set $\widetilde{\mathscr{K}}$ is important in the famous Blaschke-Santaló inequality: for $K \in \widetilde{\mathscr{K}}$, one has

$$|K| \cdot |K^\circ| \leq \omega_n^2$$

with equality if and only if K is an origin-symmetric ellipsoid (i.e., $K = A(B_2^n)$ for some $A \in GL(n)$).

On the set $\mathscr{K}$, we consider the topology generated by the Hausdorff distance $d_H(\cdot, \cdot)$. For $K, K' \in \mathscr{K}$, define $d_H(K, K')$ by

$$d_H(K, K') = \|h_K - h_{K'}\|_\infty = \sup_{u \in S^{n-1}} |h_K(u) - h_{K'}(u)|.$$

A sequence $\{K_i\}_{i\geq 1} \subset \mathscr{K}$ is said to be convergent to a convex body K_0 if $d_H(K_i, K_0) \to 0$ as $i \to \infty$. Note that if $K_i \to K_0$ in the Hausdorff distance, then S_{K_i} is weakly convergent to S_{K_0}. That is, for all $f \in C(S^{n-1})$, one has

$$\lim_{i\to\infty} \int_{S^{n-1}} f(u)\, dS_{K_i}(u) = \int_{S^{n-1}} f(u)\, dS_{K_0}(u).$$

We will use a modified form of the above limit: if $\{f_i\}_{i\geq 1}\subset C(S^{n-1})$ is uniformly convergent to $f_0\in C(S^{n-1})$ and $\{K_i\}_{i\geq 1}\subset \mathscr{K}$ converges to $K_0\in\mathscr{K}$ in the Hausdorff distance, then

$$\lim_{i\to\infty}\int_{S^{n-1}} f_i(u)\ dS_{K_i}(u)=\int_{S^{n-1}} f_0(u)\ dS_{K_0}(u). \tag{2.5}$$

The Blaschke selection theorem is a powerful tool in convex geometry (see e.g., [16, 39]) and will be often used in this paper. It reads: *every bounded sequence of convex bodies has a subsequence that converges to a convex body.*

The following result, proved by Lutwak [28], is essential for our main results.

Lemma 2.1. *Let $\{K_i\}_{i\geq 1}\subset\mathscr{K}_0$ be a convergent sequence with limit K_0, i.e., $K_i\to K_0$ in the Hausdorff distance. If the sequence $\{|K_i^\circ|\}_{i\geq 1}$ is bounded, then $K_0\in\mathscr{K}_0$.*

3 The homogeneous Orlicz affine and geominimal surface areas

This section is dedicated to Orlicz affine and geominimal surface areas with homogeneity. Let $\mathscr{I}$ denote the set of continuous functions $\phi:[0,\infty)\to[0,\infty)$ which are strictly increasing with $\phi(1)=1$, $\phi(0)=0$ and $\phi(\infty)=\lim_{t\to\infty}\phi(t)=\infty$. Similarly, $\mathscr{D}$ denotes the set of continuous functions $\phi:(0,\infty)\to(0,\infty)$ which are strictly decreasing with $\phi(1)=1$, $\phi(0)=\lim_{t\to 0}\phi(t)=\infty$ and $\phi(\infty)=\lim_{t\to\infty}\phi(t)=0$. Note that the conditions on $\phi(0),\phi(1)$ and $\phi(\infty)$ are mainly for convenience; results may still hold for more general strictly increasing or decreasing functions.

The Orlicz L_ϕ mixed volume of convex bodies K and L, $V_\phi(K,L)$, given in formula (1.2) does not have homogeneity in general. In order to define the homogeneous Orlicz affine and geominimal surface areas, a homogeneous Orlicz L_ϕ mixed volume of convex bodies K and L, denoted by $\widehat{V}_\phi(K,L)$, is needed.

Definition 3.1. *For $K,L\in\mathscr{K}_0$ and $\phi\in\mathscr{I}$, define $\widehat{V}_\phi(K,L)$ by*

$$\widehat{V}_\phi(K,L)=\inf_{\lambda>0}\bigg\{\int_{S^{n-1}}\phi\Big(\frac{n|K|\cdot h_L(u)}{\lambda\cdot h_K(u)}\Big)\,d\widetilde{V}_K(u)\leq 1\bigg\}. \tag{3.6}$$

While if $\phi\in\mathscr{D}$, $\widehat{V}_\phi(K,L)$ is defined as above with "≤ 1" replaced by "≥ 1".

Clearly $\widehat{V}_\phi(K,L)>0$ for $K,L\in\mathscr{K}_0$. Definition 3.1 is motivated by formula (10.5) in [13] with a slight modification; namely, an extra term $n|K|$ has been added in the numerator of the variable inside ϕ. This extra term $n|K|$ is added in order to get, as $\phi(1)=1$,

$$\widehat{V}_\phi(K,K)=n|K|. \tag{3.7}$$

Formula (3.6) coincides with formula (10.5) in [13] if $\phi\in\mathscr{I}$ is convex.

The following corollary states the homogeneity of $\widehat{V}_\phi(K,L)$, which has been made to be the same as the classical mixed volume $V_1(K,L)$.

Corollary 3.1. *Let $s,t>0$ be constants. For $K,L\in\mathscr{K}_0$, one has, for $\phi\in\mathscr{I}\cup\mathscr{D}$,*

$$\widehat{V}_\phi(sK,tL)=s^{n-1}t\cdot\widehat{V}_\phi(K,L). \tag{3.8}$$

Proof. For $\phi \in \mathscr{I}$, one has, by letting $\eta = s^{n-1}t\lambda$,

$$\begin{aligned}\widehat{V}_\phi(sK, tL) &= \inf_{\eta>0}\Big\{\int_{S^{n-1}} \phi\Big(\frac{t\cdot n|K|\cdot h_L(u)}{\eta\cdot s^{1-n}\cdot h_K(u)}\Big)\, d\widetilde{V}_K(u) \le 1\Big\} \\ &= s^{n-1}t\cdot \inf_{\lambda>0}\Big\{\int_{S^{n-1}} \phi\Big(\frac{n|K|\cdot h_L(u)}{\lambda\cdot h_K(u)}\Big)\, d\widetilde{V}_K(u) \le 1\Big\}.\end{aligned}$$

That is, $\widehat{V}_\phi(sK, tL) = s^{n-1}t\cdot \widehat{V}_\phi(K, L)$. In particular, if $s = 1$ and $t > 0$, then

$$\widehat{V}_\phi(K, tL) = t\cdot \widehat{V}_\phi(K, L);$$

while if $t = 1$ and $s > 0$, then

$$\widehat{V}_\phi(sK, L) = s^{n-1}\cdot \widehat{V}_\phi(K, L).$$

The case for $\phi \in \mathscr{D}$ follows along the same way. □

Let the function $G : (0, \infty) \to (0, \infty)$ be given by

$$G(\lambda) = \int_{S^{n-1}} \phi\Big(\frac{n|K|\cdot h_L(u)}{\lambda\cdot h_K(u)}\Big)\, d\widetilde{V}_K(u).$$

For $\phi \in \mathscr{I}$, the function G is strictly decreasing on λ with

$$\lim_{\lambda\to 0} G(\lambda) = \lim_{t\to\infty} \phi(t) \quad \text{and} \quad \lim_{\lambda\to\infty} G(\lambda) = \lim_{t\to 0} \phi(t).$$

As an example, we show that $\lim_{\lambda\to 0} G(\lambda) = \lim_{t\to\infty} \phi(t)$. To this end, as $\phi \in \mathscr{I}$ is strictly increasing, we have

$$\begin{aligned}G(\lambda) &= \int_{S^{n-1}} \phi\Big(\frac{n|K|\cdot h_L(u)}{\lambda\cdot h_K(u)}\Big)\, d\widetilde{V}_K(u) \\ &\ge \int_{S^{n-1}} \phi\bigg(\frac{n|K|\cdot \min_{u\in S^{n-1}} h_L(u)}{\lambda\cdot \max_{u\in S^{n-1}} h_K(u)}\bigg)\, d\widetilde{V}_K(u) \\ &= \phi\bigg(\frac{n|K|\cdot \min_{u\in S^{n-1}} h_L(u)}{\lambda\cdot \max_{u\in S^{n-1}} h_K(u)}\bigg).\end{aligned}$$

This yields

$$\lim_{\lambda\to 0} G(\lambda) \ge \lim_{\lambda\to 0} \phi\bigg(\frac{n|K|\cdot \min_{u\in S^{n-1}} h_L(u)}{\lambda\cdot \max_{u\in S^{n-1}} h_K(u)}\bigg) = \lim_{t\to\infty} \phi(t).$$

Similarly, one has

$$\lim_{\lambda\to 0} G(\lambda) \le \lim_{\lambda\to 0} \phi\bigg(\frac{n|K|\cdot \max_{u\in S^{n-1}} h_L(u)}{\lambda\cdot \min_{u\in S^{n-1}} h_K(u)}\bigg) = \lim_{t\to\infty} \phi(t),$$

and the desired result follows. On the other hand, the function G for $\phi \in \mathscr{D}$ is strictly increasing on λ with

$$\lim_{\lambda\to 0} G(\lambda) = \lim_{t\to\infty} \phi(t) \quad \text{and} \quad \lim_{\lambda\to\infty} G(\lambda) = \lim_{t\to 0} \phi(t).$$

Together with $\phi(1) = 1$, we have proved the following corollary.

Corollary 3.2. *Let $\phi \in \mathscr{I}\cup\mathscr{D}$ and $K, L \in \mathscr{K}_0$. Then $\widehat{V}_\phi(K, L) > 0$, and $\lambda_0 = \widehat{V}_\phi(K, L)$ if and only if*

$$G(\lambda_0) = \int_{S^{n-1}} \phi\Big(\frac{n|K|\cdot h_L(u)}{\lambda_0\cdot h_K(u)}\Big)\, d\widetilde{V}_K(u) = 1.$$

For $\phi(t)=t^p$, one writes $\widehat{V}_p(K,L)$ instead of $\widehat{V}_\phi(K,L)$. A simple calculation shows that

$$\begin{aligned}\widehat{V}_p(K,L) &= n|K|\cdot\bigg[\int_{S^{n-1}}\bigg(\frac{h_L(u)}{h_K(u)}\bigg)^p\,d\widetilde{V}_K(u)\bigg]^{1/p}\\ &= (n|K|)^{1-\frac{1}{p}}\cdot\big(nV_p(K,L)\big)^{1/p},\end{aligned}$$

where $V_p(K,L)$ is the L_p mixed volume of K and L for $p\in\mathbb{R}$ [28, 48], i.e.,

$$V_p(K,L)=\frac{1}{n}\int_{S^{n-1}}h_L(u)^p h_K(u)^{1-p}\,dS_K(u).$$

If $\phi\in\mathscr{I}$ is convex, the Orlicz-Minkowski inequality holds [13]: for $K,L\in\mathscr{K}_0$, one has

$$\widehat{V}_\phi(K,L)\geq n\cdot|K|^{\frac{n-1}{n}}|L|^{\frac{1}{n}}. \tag{3.9}$$

If in addition ϕ is strictly convex, equality holds if and only if K and L are dilates to each other. In particular, the classical Minkowski inequality is related to $\phi(t)=t$: for $K,L\in\mathscr{K}_0$, one has

$$V_1(K,L)^n\geq|K|^{n-1}|L|, \tag{3.10}$$

with equality if and only if K and L are homothetic to each other (i.e., there exist a constant $s>0$ and a vector $a\in\mathbb{R}^n$ such that $K=sL+a$).

In order to define the homogeneous Orlicz affine surface areas, we need to define $\widehat{V}_\phi(K,L^\circ)$ for $L\in\mathscr{S}_0$. The definition is similar to Definition 3.1 but with h_{L° replaced by $1/\rho_L$. That is, for $\phi\in\mathscr{I}\cup\mathscr{D}$, $\widehat{V}_\phi(K,L^\circ)$ for $K\in\mathscr{K}_0$ and $L\in\mathscr{S}_0$ is defined by the constant λ_0 such that

$$\int_{S^{n-1}}\phi\Big(\frac{n|K|}{\lambda_0\cdot\rho_L(u)\cdot h_K(u)}\Big)\,d\widetilde{V}_K(u)=1. \tag{3.11}$$

Of course, $\widehat{V}_\phi(K,L^\circ)$ for $K,L\in\mathscr{K}_0$ given by formula (3.11) coincides with the one given by formula (3.6). Note that $\widehat{V}_\phi(K,L^\circ)$ for $K\in\mathscr{K}_0$ and $L\in\mathscr{S}_0$ is also homogeneous as stated in Corollary 3.1.

For function $\phi\in\mathscr{I}\cup\mathscr{D}$, let $F(t)=\phi(t^{-1/n})$ and hence $\phi(t)=F(t^{-n})$. The relations between ϕ and F have been discussed in [49]. For example, a): ϕ and F have opposite monotonicity, that is, if one is strictly decreasing (increasing), then the other one will be strictly increasing (decreasing); b): if one is convex and increasing, then the other one is convex and decreasing. As mentioned in [49], to define Orlicz affine and geominimal surface areas, one needs to consider the convexity and concavity of F instead of the convexity and concavity of ϕ itself. Let

$$\begin{aligned}\widehat{\Phi}_1 &= \big\{\phi:\phi\in\mathscr{I}\text{ and }F\text{ is strictly convex}\big\};\\ \widehat{\Phi}_2 &= \{\phi:\phi\in\mathscr{D}\text{ and }F\text{ is strictly concave}\}.\end{aligned}$$

We often use $\widehat{\Phi}$ for $\widehat{\Phi}_1\cup\widehat{\Phi}_2$. Sample functions in $\widehat{\Phi}$ are: t^p with $p\in(-n,0)\cup(0,\infty)$. Similarly, let

$$\widehat{\Psi}=\{\phi:\phi\in\mathscr{D}\text{ and }F\text{ is strictly convex}\}.$$

Note that if $\phi\in\mathscr{I}$ such that F is strictly concave, then ϕ is a constant. We are not interested in this case. The set $\widehat{\Psi}$ contains functions such as t^p with $p\in(-\infty,-n)$.

Definition 3.2. *Let $K \in \mathscr{K}_0$. The homogeneous Orlicz L_ϕ affine surface area of K, denoted by $\widehat{\Omega}_\phi^{orlicz}(K)$, is defined by*

$$\widehat{\Omega}_\phi^{orlicz}(K) = \inf_{L\in\mathscr{S}_0}\Big\{\widehat{V}_\phi(K, \mathrm{vrad}(L)L^\circ)\Big\} \quad \text{for} \quad \phi\in\widehat{\Phi}; \tag{3.12}$$

$$\widehat{\Omega}_\phi^{orlicz}(K) = \sup_{L\in\mathscr{S}_0}\Big\{\widehat{V}_\phi(K, \mathrm{vrad}(L)L^\circ)\Big\} \quad \text{for} \quad \phi\in\widehat{\Psi}. \tag{3.13}$$

The homogeneous Orlicz L_ϕ geominimal surface area of K, denoted by $\widehat{G}_\phi^{orlicz}(K)$, is defined similarly with $\mathscr{S}_0$ replaced by $\mathscr{K}_0$.

Clearly $\widehat{\Omega}_\phi^{orlicz}(K) \le \widehat{G}_\phi^{orlicz}(K)$ if $\phi\in\widehat{\Phi}$ and $\widehat{\Omega}_\phi^{orlicz}(K) \ge \widehat{G}_\phi^{orlicz}(K)$ if $\phi\in\widehat{\Psi}$. For $\phi(t)=t^p$, one writes $\widehat{\Omega}_p^{orlicz}(K)$ instead of $\widehat{\Omega}_\phi^{orlicz}(K)$. In particular, for $-n\neq p\in\mathbb{R}$,

$$\widehat{\Omega}_p^{orlicz}(K) = (n\omega_n)^{-1/n}\cdot\big(as_p(K)\big)^{\frac{n+p}{np}}\cdot(n|K|)^{1-\frac{1}{p}},$$

where $as_p(K)$ is the L_p affine surface area of K (see e.g., [28, 48]):

$$as_p(K) = \inf_{L\in\mathscr{S}_0}\Big\{nV_p(K, L^\circ)^{\frac{n}{n+p}}\ |L|^{\frac{p}{n+p}}\Big\}, \quad p\ge 0;$$

$$as_p(K) = \sup_{L\in\mathscr{S}_0}\Big\{nV_p(K, L^\circ)^{\frac{n}{n+p}}\ |L|^{\frac{p}{n+p}}\Big\}, \quad -n\neq p< 0.$$

Similarly, for $-n\neq p\in\mathbb{R}$,

$$\widehat{G}_p^{orlicz}(K) = (n\omega_n)^{-1/n}\cdot\big(\tilde{G}_p(K)\big)^{\frac{n+p}{np}}\cdot(n|K|)^{1-\frac{1}{p}},$$

where $\tilde{G}_p(K)$ is the L_p geominimal surface area [28, 48]:

$$\tilde{G}_p(K) = \inf_{L\in\mathscr{K}_0}\Big\{nV_p(K, L^\circ)^{\frac{n}{n+p}}\ |L|^{\frac{p}{n+p}}\Big\}, \quad p\ge 0;$$

$$\tilde{G}_p(K) = \sup_{L\in\mathscr{K}_0}\Big\{nV_p(K, L^\circ)^{\frac{n}{n+p}}\ |L|^{\frac{p}{n+p}}\Big\}, \quad -n\neq p< 0.$$

When $K=B_2^n$, both $\widehat{\Omega}_\phi^{orlicz}(B_2^n)$ and $\widehat{G}_\phi^{orlicz}(B_2^n)$ can be calculated precisely.

Corollary 3.3. *For $\phi\in\widehat{\Phi}\cup\widehat{\Psi}$, one has*

$$\widehat{\Omega}_\phi^{orlicz}(B_2^n) = \widehat{G}_\phi^{orlicz}(B_2^n) = n\omega_n. \tag{3.14}$$

Proof. We only prove $\widehat{\Omega}_\phi^{orlicz}(B_2^n)=n\omega_n$ with $\phi\in\widehat{\Phi}_1$, and the other cases follow along the same lines. As $\phi\in\widehat{\Phi}_1$, one sees that ϕ is strictly increasing and $F(t)=\phi(t^{-1/n})$ is strictly convex. First of all, by formulas (3.7) and (3.12), one has

$$\widehat{\Omega}_\phi^{orlicz}(B_2^n) \le \widehat{V}_\phi(B_2^n, B_2^n) = n\omega_n. \tag{3.15}$$

From Corollary 3.2 and Jensen's inequality, the fact that F is strictly convex yields

$$\begin{aligned} 1 &= \int_{S^{n-1}} \phi\bigg(\frac{n\omega_n\cdot \mathrm{vrad}(L)}{\widehat{V}_\phi(B_2^n, \mathrm{vrad}(L)L^\circ)\cdot\rho_L(u)}\bigg)\cdot\frac{1}{n\omega_n}\,d\sigma(u) \\ &\ge F\bigg(\int_{S^{n-1}} \frac{[\widehat{V}_\phi(B_2^n, \mathrm{vrad}(L)L^\circ)]^n\cdot\rho_L^n(u)}{[n\omega_n\cdot\mathrm{vrad}(L)]^n\cdot n\omega_n}\,d\sigma(u)\bigg) \\ &= \phi\bigg(\frac{n\omega_n}{\widehat{V}_\phi(B_2^n, \mathrm{vrad}(L)L^\circ)}\bigg). \end{aligned}$$

As $\phi(1)=1$ and ϕ is strictly increasing, one gets, for all $L\in\mathscr{S}_0$,

$$\widehat{V}_\phi(B_2^n,\mathrm{vrad}(L)L^\circ)\geq n\omega_n.$$

The desired equality follows by taking the infimum over $L\in\mathscr{S}_0$ and by formula (3.15). $\square$

Proposition 3.1. *Let $K\in\mathscr{K}_0$ and $A\in GL(n)$. For $\phi\in\widehat{\Phi}\cup\widehat{\Psi}$, one has*

$$\widehat{\Omega}_\phi^{orlicz}(AK)=|\det A|^{\frac{n-1}{n}}\cdot\widehat{\Omega}_\phi^{orlicz}(K)\quad\text{and}\quad\widehat{G}_\phi^{orlicz}(AK)=|\det A|^{\frac{n-1}{n}}\cdot\widehat{G}_\phi^{orlicz}(K).$$

Proof. Let $A\in GL(n)$ and $\|\cdot\|$ be the usual Euclidean norm. For $v\in S^{n-1}$, let $u=u(v)=\frac{A^tv}{\|A^tv\|}$. By the definitions of support and radial functions, one can easily check that

$$h_{AK}(v)=\|A^tv\|\cdot h_K(u)\quad\text{and}\quad\rho_{A^{-t}L}(v)\|A^tv\|=\rho_L(u).$$

Consequently, $h_{AK}(v)\rho_{A^{-t}L}(v)=h_K(u)\rho_L(u)$ and

$$\begin{aligned}\widehat{V}_\phi(AK,(A^{-t}L)^\circ)&=\inf_{\lambda>0}\left\{\int_{S^{n-1}}\phi\Big(\frac{n|AK|}{\lambda\cdot h_{AK}(v)\rho_{A^{-t}L}(v)}\Big)\,d\widetilde{V}_{AK}(v)\leq1\right\}\\&=\inf_{\lambda>0}\left\{\int_{S^{n-1}}\phi\Big(\frac{|\det A|\cdot n|K|}{\lambda\cdot h_K(u)\rho_L(u)}\Big)\,d\widetilde{V}_K(u)\leq1\right\}\\&=|\det A|\cdot\inf_{\eta>0}\left\{\int_{S^{n-1}}\phi\Big(\frac{n|K|}{\eta\cdot h_K(u)\rho_L(u)}\Big)\,d\widetilde{V}_K(u)\leq1\right\},\end{aligned}$$

where $\lambda=|\det A|\cdot\eta$. Consequently,

$$\widehat{V}_\phi(AK,(A^{-t}L)^\circ)=|\det A|\cdot\widehat{V}_\phi(K,L^\circ).\tag{3.16}$$

Combining with equation (3.8), one gets, for $\phi\in\widehat{\Phi}$,

$$\begin{aligned}\widehat{V}_\phi(AK,\mathrm{vrad}(A^{-t}L)\cdot(A^{-t}L)^\circ)&=|\det A|\cdot|\det A^t|^{-1/n}\cdot\widehat{V}_\phi(K,\mathrm{vrad}(L)L^\circ)\\&=|\det A|^{\frac{n-1}{n}}\cdot\widehat{V}_\phi(K,\mathrm{vrad}(L)L^\circ).\end{aligned}$$

The desired result follows immediately by taking the infimum over $L\in\mathscr{S}_0$. Other cases follow along the same lines. $\square$

Proposition 3.1 implies that both $\widehat{\Omega}_\phi^{orlicz}(\cdot)$ and $\widehat{G}_\phi^{orlicz}(\cdot)$ are invariant under the volume preserving linear transforms on $\mathbb{R}^n$. That is, for all $A\in SL(n)$ and $K\in\mathscr{K}_0$,

$$\widehat{\Omega}_\phi^{orlicz}(AK)=\widehat{\Omega}_\phi^{orlicz}(K)\quad\text{and}\quad\widehat{G}_\phi^{orlicz}(AK)=\widehat{G}_\phi^{orlicz}(K).$$

In particular, $\widehat{\Omega}_\phi^{orlicz}(\lambda K)=\lambda^{n-1}\cdot\widehat{\Omega}_\phi^{orlicz}(K)$ and $\widehat{G}_\phi^{orlicz}(\lambda K)=\lambda^{n-1}\cdot\widehat{G}_\phi^{orlicz}(K)$ for $\lambda>0$ a constant. This means that both $\widehat{\Omega}_\phi^{orlicz}(\cdot)$ and $\widehat{G}_\phi^{orlicz}(\cdot)$ have homogeneity.

An immediate consequence of formula (3.14) and Proposition 3.1 is: for $\phi\in\widehat{\Phi}\cup\widehat{\Psi}$ and for the ellipsoid $\mathscr{E}=AB_2^n$ with $A\in GL(n)$,

$$\widehat{\Omega}_\phi^{orlicz}(\mathscr{E})=\widehat{G}_\phi^{orlicz}(\mathscr{E})=|\det A|^{\frac{n-1}{n}}\cdot n\omega_n.$$

We can prove the following affine isoperimetric inequalities for the homogeneous Orlicz L_ϕ affine and geominimal surface areas.

Theorem 3.1. *Let $K \in \widetilde{\mathscr{K}}$ be a convex body with its centroid or Santaló point at the origin.*
(i) For $\phi \in \widehat{\Phi}$, one has

$$\frac{\widehat{\Omega}_\phi^{orlicz}(K)}{\widehat{\Omega}_\phi^{orlicz}(B_2^n)} \le \frac{\widehat{G}_\phi^{orlicz}(K)}{\widehat{G}_\phi^{orlicz}(B_2^n)} \le \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-1}{n}}$$

with equality if and only if K is an origin-symmetric ellipsoid.
(ii) For $\phi \in \widehat{\Psi}$, there is a universal constant $c > 0$ such that

$$\frac{\widehat{\Omega}_\phi^{orlicz}(K)}{\widehat{\Omega}_\phi^{orlicz}(B_2^n)} \ge \frac{\widehat{G}_\phi^{orlicz}(K)}{\widehat{G}_\phi^{orlicz}(B_2^n)} \ge c \cdot \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-1}{n}}.$$

Remark. Theorem 3.1 asserts that among all convex bodies in $\widetilde{\mathscr{K}}$ with fixed volume, the homogeneous Orlicz L_ϕ affine and geominimal surface areas for $\phi \in \widehat{\Phi}$ attain their maximum at origin-symmetric ellipsoids. The L_p affine isoperimetric inequalities for the L_p affine and geominimal surface areas are special cases of Theorem 3.1 with $\phi(t) = t^p$ (see e.g., [28, 37, 38, 44, 48]).

Proof. Formulas (3.7) and (3.8) together with Definition 3.2 imply that for all $\phi \in \widehat{\Phi}$ and $K \in \mathscr{K}_0$,

$$\widehat{\Omega}_\phi^{orlicz}(K) \le \widehat{G}_\phi^{orlicz}(K) \le \widehat{V}_\phi(K, \mathrm{vrad}(K^\circ)K) = n|K| \cdot \mathrm{vrad}(K^\circ). \tag{3.17}$$

If $K \in \widetilde{\mathscr{K}}$, the Blaschke-Santaló inequality further implies, for all $\phi \in \widehat{\Phi}$,

$$\widehat{\Omega}_\phi^{orlicz}(K) \le \widehat{G}_\phi^{orlicz}(K) \le n|K|^{\frac{n-1}{n}} \cdot \omega_n^{1/n}$$

with equality if and only if K is an origin-symmetric ellipsoid (i.e., those make the equality hold in the Blaschke-Santaló inequality). Dividing both sides by $\widehat{\Omega}_\phi^{orlicz}(B_2^n) = n\omega_n$, one gets the desired inequality in (i).

Similarly, for all $\phi \in \widehat{\Psi}$ and for all $K \in \mathscr{K}_0$,

$$\widehat{\Omega}_\phi^{orlicz}(K) \ge \widehat{G}_\phi^{orlicz}(K) \ge n|K| \cdot \mathrm{vrad}(K^\circ). \tag{3.18}$$

Dividing both sides by $\widehat{\Omega}_\phi^{orlicz}(B_2^n) = n\omega_n$, one gets

$$\frac{\widehat{\Omega}_\phi^{orlicz}(K)}{\widehat{\Omega}_\phi^{orlicz}(B_2^n)} \ge \frac{\widehat{G}_\phi^{orlicz}(K)}{\widehat{G}_\phi^{orlicz}(B_2^n)} \ge c \cdot \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-1}{n}},$$

where the inverse Santaló inequality [8] has been used: there is a universal constant $c > 0$ such that for all $K \in \widetilde{\mathscr{K}}$,

$$|K| \cdot |K^\circ| \ge c^n \omega_n^2. \tag{3.19}$$

See [22, 35] for estimates of the constant c. □

The following Santaló type inequalities follow immediately from Theorem 3.1 and the Blaschke-Santaló inequality.

Theorem 3.2. *Let $K \in \widetilde{\mathscr{K}}$ be a convex body with its centroid or Santaló point at the origin.*
(i) For $\phi \in \widehat{\Phi}$, one has

$$\frac{\widehat{\Omega}_\phi^{orlicz}(K) \cdot \widehat{\Omega}_\phi^{orlicz}(K^\circ)}{[\widehat{\Omega}_\phi^{orlicz}(B_2^n)]^2} \le \frac{\widehat{G}_\phi^{orlicz}(K) \cdot \widehat{G}_\phi^{orlicz}(K^\circ)}{[\widehat{G}_\phi^{orlicz}(B_2^n)]^2} \le 1.$$

Equality holds if and only if K is an origin-symmetric ellipsoid.
(ii) For $\phi \in \widehat{\Psi}$, there is a universal constant $c > 0$ such that

$$\frac{\widehat{\Omega}_\phi^{orlicz}(K) \cdot \widehat{\Omega}_\phi^{orlicz}(K^\circ)}{[\widehat{\Omega}_\phi^{orlicz}(B_2^n)]^2} \ge \frac{\widehat{G}_\phi^{orlicz}(K) \cdot \widehat{G}_\phi^{orlicz}(K^\circ)}{[\widehat{G}_\phi^{orlicz}(B_2^n)]^2} \ge c^{n+1}.$$

A finer calculation could lead to stronger arguments than Theorem 3.1, where the conditions on the centroid or the Santaló point of K can be removed. That is, $\widetilde{\mathscr{K}}$ in Theorem 3.1 can be replaced by $\mathscr{K}_0$. See similar results in [47, 48, 49, 53].

Corollary 3.4. *Let $K \in \mathscr{K}_0$. If either $\phi \in \widehat{\Phi}_1$ is concave or $\phi \in \widehat{\Phi}_2$ is convex, then*

$$\frac{\widehat{\Omega}_\phi^{orlicz}(K)}{\widehat{\Omega}_\phi^{orlicz}(B_2^n)} \le \frac{\widehat{G}_\phi^{orlicz}(K)}{\widehat{G}_\phi^{orlicz}(B_2^n)} \le \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-1}{n}}.$$

In addition, if either $\phi \in \widehat{\Phi}_1$ is strictly concave or $\phi \in \widehat{\Phi}_2$ is strictly convex, equality holds if and only if K is an origin-symmetric ellipsoid.

To prove this corollary, one needs the following cyclic inequality. For convenience, let $H = \phi \circ \psi^{-1}$, where ψ^{-1}, the inverse of ψ, always exists if $\psi \in \widehat{\Phi} \cup \widehat{\Psi}$.

Theorem 3.3. *Let $K \in \mathscr{K}_0$. Assume one of the following conditions holds: a) $\phi \in \widehat{\Phi}$ and $\psi \in \widehat{\Psi}$; b) H is convex with $\phi \in \widehat{\Phi}_2$ and $\psi \in \widehat{\Phi}_1$; c) H is concave with $\phi, \psi \in \widehat{\Phi}_1$; d) H is convex with either $\phi, \psi \in \widehat{\Phi}_2$ or $\phi, \psi \in \widehat{\Psi}$. Then*

$$\widehat{\Omega}_\phi^{orlicz}(K) \le \widehat{\Omega}_\psi^{orlicz}(K) \quad \text{and} \quad \widehat{G}_\phi^{orlicz}(K) \le \widehat{G}_\psi^{orlicz}(K).$$

Proof. The case for condition a) follows immediately from formulas (3.17) and (3.18). We only prove the case for condition b), and the other cases follow along the same fashion. Assume that condition b) holds and then H is convex. Corollary 3.2 and Jensen's inequality imply that

$$\begin{aligned} 1 &= \int_{S^{n-1}} \phi\Big(\frac{n|K|}{\widehat{V}_\phi(K, L^\circ) \cdot \rho_L(u) \cdot h_K(u)}\Big)\, d\widetilde{V}_K(u) \\ &= \int_{S^{n-1}} H\bigg(\psi\Big(\frac{n|K|}{\widehat{V}_\phi(K, L^\circ) \cdot \rho_L(u) \cdot h_K(u)}\Big)\bigg)\, d\widetilde{V}_K(u) \\ &\ge H\bigg(\int_{S^{n-1}} \psi\Big(\frac{n|K|}{\widehat{V}_\phi(K, L^\circ) \cdot \rho_L(u) \cdot h_K(u)}\Big)\, d\widetilde{V}_K(u)\bigg). \end{aligned}$$

Together with Corollary 3.2 and the facts that H is decreasing and $H(1) = 1$, one has

$$\int_{S^{n-1}} \psi\Big(\frac{n|K|}{\widehat{V}_\psi(K, L^\circ) \cdot \rho_L(u) \cdot h_K(u)}\Big)\, d\widetilde{V}_K(u) \le \int_{S^{n-1}} \psi\Big(\frac{n|K|}{\widehat{V}_\phi(K, L^\circ) \cdot \rho_L(u) \cdot h_K(u)}\Big)\, d\widetilde{V}_K(u). \quad (3.20)$$

Note that $\psi \in \mathscr{I}$ (increasing). It follows from formula (3.20) that $\widehat{V}_\phi(K, L^\circ) \le \widehat{V}_\psi(K, L^\circ)$. Together with Corollary 3.1 and Definition 3.2, one gets the desired result. □

Proof of Corollary 3.4. Let $\psi(t) = t$ and $\phi \in \widehat{\Phi}_2$ be convex. Then $H = \phi$ satisfies condition b) in Theorem 3.3 and thus $\widehat{\Omega}_{\phi}^{orlicz}(K) \leq \widehat{\Omega}_{1}^{orlicz}(K)$. Note that $\widehat{\Omega}_{1}^{orlicz}(K)$ is essentially the classical geominimal surface area and is translation invariant. That is, for any $z_0 \in \mathbb{R}^n$, $\widehat{\Omega}_{1}^{orlicz}(K-z_0) = \widehat{\Omega}_{1}^{orlicz}(K)$. In particular, one selects z_0 to be the point in $\mathbb{R}^n$ such that $K-z_0 \in \widetilde{\mathscr{K}}$ (i.e., z_0 is either the centroid or the Santaló point of K). Theorem 3.1 implies that

$$\frac{\widehat{\Omega}_{\phi}^{orlicz}(K)}{\widehat{\Omega}_{\phi}^{orlicz}(B_2^n)} \leq \frac{\widehat{\Omega}_{1}^{orlicz}(K-z_0)}{\widehat{\Omega}_{1}^{orlicz}(B_2^n)} \leq \left(\frac{|K-z_0|}{|B_2^n|}\right)^{\frac{n-1}{n}} = \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-1}{n}}.$$

To characterize the equality, due to the homogeneity of $\widehat{\Omega}_{\phi}^{orlicz}(\cdot)$, it is enough to prove that if ϕ is in addition strictly convex, $\widehat{\Omega}_{\phi}^{orlicz}(K) = \widehat{\Omega}_{\phi}^{orlicz}(B_2^n)$ if and only if K is an origin-symmetric ellipsoid with $|K| = \omega_n$. First of all, if K is an origin-symmetric ellipsoid with $|K| = \omega_n$, then $\widehat{\Omega}_{\phi}^{orlicz}(K) = \widehat{\Omega}_{\phi}^{orlicz}(B_2^n)$ follows from Corollary 3.3 and Proposition 3.1. On the other hand, by Theorem 3.2, $\widehat{\Omega}_{\phi}^{orlicz}(K) = \widehat{\Omega}_{\phi}^{orlicz}(B_2^n)$ holds only if $K - z_0$ is an origin-symmetric ellipsoid with $|K| = \omega_n$. By Proposition 3.1, it is enough to claim $K = B_2^n + z_0$ with $z_0 = 0$. Corollary 3.3 and Definition 3.2 yield

$$n\omega_n = \widehat{\Omega}_{\phi}^{orlicz}(B_2^n) = \widehat{\Omega}_{\phi}^{orlicz}(K) = \widehat{\Omega}_{\phi}^{orlicz}(B_2^n + z_0) \leq \widehat{V}_{\phi}(B_2^n + z_0, B_2^n).$$

Note that $\phi \in \widehat{\Phi}$ is convex and decreasing. Combining with Corollary 3.2, one has

$$\begin{aligned} 1 &= \int_{S^{n-1}} \phi\left(\frac{n\omega_n}{\widehat{V}_{\phi}(B_2^n + z_0, B_2^n) \cdot h_{B_2^n+z_0}(u)}\right) \cdot \frac{h_{B_2^n+z_0}(u)}{n\omega_n} \cdot d\sigma(u) \\ &\geq \phi\left(\int_{S^{n-1}} \frac{d\sigma(u)}{\widehat{V}_{\phi}(B_2^n + z_0, B_2^n)}\right) \geq 1. \end{aligned}$$

As ϕ is strictly convex, equality holds if and only if $h_{B_2^n+z_0}(u)$ is a constant on S^{n-1}. This yields $z_0 = 0$ as desired.

The case for $\phi \in \widehat{\Phi}_1$ being concave (with characterization for equality) follows along the same lines. □

4 The Orlicz-Petty bodies and the continuity of the homogeneous Orlicz geominimal surface areas

This section concentrates on the continuity of the homogeneous Orlicz geominimal surface areas. In Subsection 4.1, we first show that the homogeneous Orlicz geominimal surface areas are semicontinuous on $\mathscr{K}_0$ with respect to the Hausdorff distance. The existence and uniqueness of the Orlicz-Petty bodies, under certain conditions, will be proved in Subsection 4.2. Our main result on the continuity will be given in Subsection 4.3.

4.1 Semicontinuity of the homogeneous Orlicz geominimal surface areas

Let us first establish the semicontinuity of the homogeneous Orlicz geominimal surface areas. Recall that for $\phi \in \widehat{\Phi}$ and for $K \in \mathscr{K}_0$,

$$\widehat{G}_{\phi}^{orlicz}(K) = \inf_{L \in \mathscr{K}_0} \{\widehat{V}_{\phi}(K, \mathrm{vrad}(L)L^{\circ})\}.$$

It is often more convenient, by the bipolar theorem (i.e., $(L^\circ)^\circ = L$ for $L \in \mathscr{K}_0$) and Corollary 3.1, to formulate $\widehat{G}_\phi^{orlicz}(K)$ for $\phi \in \widehat{\Phi}$ by

$$\widehat{G}_\phi^{orlicz}(K) = \inf\{\widehat{V}_\phi(K,L): \ L \in \mathscr{K}_0 \quad \text{with} \quad |L^\circ| = \omega_n\}. \tag{4.21}$$

Similarly, for $\phi \in \widehat{\Psi}$,

$$\widehat{G}_\phi^{orlicz}(K) = \sup\{\widehat{V}_\phi(K,L): \ L \in \mathscr{K}_0 \quad \text{with} \quad |L^\circ| = \omega_n\}. \tag{4.22}$$

Denote by r_K and R_K the inner and outer radii of convex body $K \in \mathscr{K}_0$, respectively. That is,

$$r_K = \min\{h_K(u): \ u \in S^{n-1}\} \quad \text{and} \quad R_K = \max\{h_K(u): \ u \in S^{n-1}\}.$$

Lemma 4.1. *Let $K, L \in \mathscr{K}_0$. For $\phi \in \mathscr{I} \cup \mathscr{D}$, one has*

$$\frac{n\omega_n \cdot r_K^n \cdot r_L}{R_K} \le \widehat{V}_\phi(K,L) \le \frac{n\omega_n \cdot R_K^n \cdot R_L}{r_K}.$$

Proof. For $\phi \in \mathscr{I}$, let $\lambda = \widehat{V}_\phi(K,L)$. By Corollary 3.2 and the fact that ϕ is increasing on $(0,\infty)$, one has

$$\begin{aligned} 1 &= \int_{S^{n-1}} \phi\left(\frac{n|K| \cdot h_L(u)}{\lambda \cdot h_K(u)}\right) d\widetilde{V}_K(u) \\ &\le \int_{S^{n-1}} \phi\left(\frac{n|K| \cdot R_L}{\lambda \cdot r_K}\right) d\widetilde{V}_K(u) \\ &\le \phi\left(\frac{n\omega_n \cdot R_K^n \cdot R_L}{\lambda \cdot r_K}\right). \end{aligned}$$

Moreover, as $\phi(1) = 1$, one gets

$$\widehat{V}_\phi(K,L) = \lambda \le \frac{n\omega_n \cdot R_K^n \cdot R_L}{r_K}.$$

For the lower bound,

$$\begin{aligned} 1 &= \int_{S^{n-1}} \phi\left(\frac{n|K| \cdot h_L(u)}{\lambda \cdot h_K(u)}\right) d\widetilde{V}_K(u) \\ &\ge \int_{S^{n-1}} \phi\left(\frac{n|K| \cdot r_L}{\lambda \cdot R_K}\right) d\widetilde{V}_K(u) \\ &\ge \phi\left(\frac{n\omega_n \cdot r_K^n \cdot r_L}{\lambda \cdot R_K}\right). \end{aligned}$$

As ϕ is increasing on $(0,\infty)$ and $\phi(1) = 1$, one gets

$$\widehat{V}_\phi(K,L) \ge \frac{n\omega_n \cdot r_K^n \cdot r_L}{R_K}.$$

The case for $\phi \in \mathscr{D}$ follows along the same lines. □

We will often need the following result.

Lemma 4.2. *Let $\varphi : I \to \mathbb{R}$ be a uniformly continuous function on an interval $I \subset \mathbb{R}$. Let $\{f_i\}_{i\geq 0}$ be a sequence of functions such that $f_i : E \to I$ for all $i \geq 0$ and $f_i \to f_0$ uniformly on E as $i \to \infty$. Then $\varphi(f_i) \to \varphi(f_0)$ uniformly on E as $i \to \infty$.*

Proof. For any $\epsilon > 0$. As φ is uniformly continuous, there exists $\delta(\epsilon) > 0$ such that $|\varphi(x)-\varphi(y)| < \epsilon$ for all $x, y \in I$ with $|x - y| < \delta(\epsilon)$. On the other hand, as $f_i \to f_0$ uniformly on E, there exists an integer $N_0(\epsilon) := N(\delta(\epsilon)) > 0$ such that $|f_i(z) - f_0(z)| < \delta(\epsilon)$ for all $i > N_0(\epsilon)$ and all $z \in E$. Hence, $|\varphi(f_i(z)) - \varphi(f_0(z))| < \epsilon$ for all $i > N_0(\epsilon)$ and all $z \in E$. That is, $\varphi(f_i) \to \varphi(f_0)$ uniformly on E. □

Proposition 4.1. *Let $\{K_i\}_{i\geq 1}$ and $\{L_i\}_{i\geq 1}$ be two sequences of convex bodies in $\mathscr{K}_0$ such that $K_i \to K \in \mathscr{K}_0$ and $L_i \to L \in \mathscr{K}_0$. For $\phi \in \mathscr{I} \cup \mathscr{D}$, one has $\widehat{V}_\phi(K_i, L_i) \to \widehat{V}_\phi(K, L)$.*

Proof. As $K_i \to K \in \mathscr{K}_0$, one can find constants $c_K, C_K > 0$, such that, for all $i \geq 1$,

$$c_K B_2^n \subset K_i, K \subset C_K B_2^n. \tag{4.23}$$

Similarly, one can find constants $c_L, C_L > 0$, such that, for all $i \geq 1$,

$$c_L B_2^n \subset L_i, L \subset C_L B_2^n. \tag{4.24}$$

For simplicity, let $\lambda_i = \widehat{V}_\phi(K_i, L_i)$. Lemma 4.1 yields, for all $i \geq 1$,

$$\frac{n\omega_n \cdot c_K^n \cdot c_L}{C_K} \leq \lambda_i \leq \frac{n\omega_n \cdot C_K^n \cdot C_L}{c_K}, \tag{4.25}$$

and thus the sequence $\{\lambda_i\}_{i\geq 1}$ is bounded from both sides. Let f_i and f be given by

$$f_i(u) = \frac{n|K_i| \cdot h_{L_i}(u)}{\lambda_i \cdot h_{K_i}(u)} \quad \text{and} \quad f(u) = \frac{n|K| \cdot h_L(u)}{\lambda_0 \cdot h_K(u)} \quad \text{for} \quad u \in S^{n-1}.$$

On the one hand, suppose that $\{\lambda_{i_k}\}_{k\geq 1}$ is a convergent subsequence of $\{\lambda_i\}_{i\geq 1}$ with limit λ_0. That is, $\lim_{k\to\infty} \lambda_{i_k} = \lambda_0$ and then $0 < \lambda_0 < \infty$. Note that $K_i \to K \in \mathscr{K}_0$ yields $h_{K_i} \to h_K$ uniformly on S^{n-1}. Similarly, $h_{L_i} \to h_L$ uniformly on S^{n-1}. Together with (4.23) and (4.24), one sees that $f_{i_k} \to f$ uniformly on S^{n-1}. Moreover, the ranges of f_{i_k}, f are all in the interval

$$I = \left[\frac{c_L}{C_L} \cdot \left(\frac{c_K}{C_K}\right)^{n+1}, \ \frac{C_L}{c_L} \cdot \left(\frac{C_K}{c_K}\right)^{n+1}\right].$$

Note that the interval $I \subsetneq (0, \infty)$ is a compact set. Hence $\phi \in \mathscr{I} \cup \mathscr{D}$ restricted on I is uniformly continuous. Lemma 4.2 implies that $\phi(f_{i_k}) \to \phi(f)$ uniformly on S^{n-1}. Moreover, as both $\{\phi(f_{i_k})\}_{k\geq 1}$ and $\{h_{K_i}\}_{i\geq 1}$ are uniformly bounded on S^{n-1}, one sees that $\phi(f_{i_k})h_{K_{i_k}} \to \phi(f)h_K$ uniformly on S^{n-1}. Formula (2.5) then yields

$$\begin{aligned} 1 &= \lim_{k\to\infty} \int_{S^{n-1}} \phi\left(\frac{n|K_{i_k}| \cdot h_{L_{i_k}}(u)}{\lambda_{i_k} \cdot h_{K_{i_k}}(u)}\right) d\widetilde{V}_{K_{i_k}}(u) \\ &= \lim_{k\to\infty} \int_{S^{n-1}} \frac{\phi\left(f_{i_k}(u)\right) h_{K_{i_k}}(u)}{n|K_{i_k}|}\, dS_{K_{i_k}}(u) \\ &= \int_{S^{n-1}} \frac{\phi\left(f(u)\right) h_K(u)}{n|K|}\, dS_K(u) \\ &= \int_{S^{n-1}} \phi\left(\frac{n|K| \cdot h_L(u)}{\lambda_0 \cdot h_K(u)}\right) d\widetilde{V}_K(u). \end{aligned}$$

Therefore $\lambda_0 = \widehat{V}_\phi(K, L)$ and $\lim_{k\to\infty} \widehat{V}_\phi(K_{i_k}, L_{i_k}) = \widehat{V}_\phi(K, L)$. We have proved that if a subsequence of $\{\widehat{V}_\phi(K_i, L_i)\}_{i\geq 1}$ is convergent, then its limit must be $\widehat{V}_\phi(K, L)$.

To conclude Proposition 4.1, it is enough to claim that the sequence $\{\widehat{V}_\phi(K_i, L_i)\}_{i\geq 1}$ is indeed convergent. Suppose that $\{\widehat{V}_\phi(K_i, L_i)\}_{i\geq 1}$ is not convergent. One has two convergent subsequences whose limits exist by (4.25) and are different. This contradicts with the arguments in the previous paragraph, and hence the sequence $\{\widehat{V}_\phi(K_i, L_i)\}_{i\geq 1}$ is convergent. $\square$

The following result states that the homogeneous Orlicz geominimal surface areas are semicontinuous. For the homogeneous Orlicz affine surface areas, similar semicontinuous arguments also hold.

Proposition 4.2. *For $\phi \in \widehat{\Phi}$, the functional $\widehat{G}_\phi^{orlicz}(\cdot)$ is upper semicontinuous on $\mathscr{K}_0$ with respect to the Hausdorff distance. That is, for any convergent sequence $\{K_i\}_{i\geq 1} \subset \mathscr{K}_0$ whose limit is $K_0 \in \mathscr{K}_0$, then*

$$\widehat{G}_\phi^{orlicz}(K_0) \geq \limsup_{i\to\infty} \widehat{G}_\phi^{orlicz}(K_i).$$

While for $\phi \in \widehat{\Psi}$, the functional $\widehat{G}_\phi^{orlicz}(\cdot)$ is lower semicontinuous on $\mathscr{K}_0$: for any $K_i \to K_0$, then

$$\widehat{G}_\phi^{orlicz}(K_0) \leq \liminf_{i\to\infty} \widehat{G}_\phi^{orlicz}(K_i).$$

Proof. Let $\phi \in \widehat{\Phi}$. For any given $\epsilon > 0$, by formula (4.21), there exists a convex body $L_\epsilon \in \mathscr{K}_0$, such that $|L_\epsilon^\circ| = \omega_n$ and

$$\widehat{G}_\phi^{orlicz}(K_0) + \epsilon > \widehat{V}_\phi(K_0, L_\epsilon) \geq \widehat{G}_\phi^{orlicz}(K_0).$$

By Proposition 4.1, one has

$$\widehat{G}_\phi^{orlicz}(K_0) + \epsilon > \widehat{V}_\phi(K_0, L_\epsilon) = \lim_{i\to\infty} \widehat{V}_\phi(K_i, L_\epsilon) = \limsup_{i\to\infty} \widehat{V}_\phi(K_i, L_\epsilon) \geq \limsup_{i\to\infty} \widehat{G}_\phi^{orlicz}(K_i).$$

The desired result follows by letting $\epsilon \to 0$. The case for $\phi \in \widehat{\Psi}$ can be proved along the same lines. $\square$

4.2 The Orlicz-Petty bodies: existence and basic properties

In this subsection, we will prove the existence of the Orlicz-Petty bodies under the condition $\phi \in \widehat{\Phi}_1$. The following lemma is needed for our goal. Denote by $a_+ = \frac{a+|a|}{2}$ for $a \in \mathbb{R}$. Clearly $a_+ = \max\{a, 0\}$.

Lemma 4.3. *Let $K \in \mathscr{K}_0$ and $\phi \in \widehat{\Phi}_1$. For fixed $v \in S^{n-1}$, define $G_v : (0,\infty) \to (0,\infty)$ by*

$$G_v(\eta) = \int_{S^{n-1}} \phi\left(\frac{n|K| \cdot \langle u, v\rangle_+}{\eta \cdot h_K(u)}\right) d\widetilde{V}_K(u).$$

Then G_v is strictly decreasing, and

$$\lim_{\eta\to 0} G_v(\eta) = \lim_{t\to\infty} \phi(t) = \infty \quad \text{and} \quad \lim_{\eta\to\infty} G_v(\eta) = \lim_{t\to 0} \phi(t) = 0.$$

Proof. Since $K \in \mathscr{K}_0$, (2.3) implies that there exists a constant $c_1 > 0$ such that for all $v \in S^{n-1}$,

$$\int_{S^{n-1}} \langle u, v\rangle_+ dS_K(u) \geq c_1.$$

For any given $v \in S^{n-1}$, let $\Sigma_j(v) = \{u \in S^{n-1} : \langle u, v\rangle_+ > \frac{1}{j}\}$ for all integers $j \geq 1$. It is obvious that $\Sigma_j(v) \subset \Sigma_{j+1}(v)$ for all $j \geq 1$ and $\cup_{j=1}^{\infty}\Sigma_j(v) = \{u \in S^{n-1} : \langle u, v\rangle_+ > 0\}$. Hence,

$$\lim_{j\to\infty} \int_{\Sigma_j(v)} \langle u, v\rangle_+ \, dS_K(u) = \int_{\cup_{j=1}^{\infty}\Sigma_j(v)} \langle u, v\rangle_+ \, dS_K(u) = \int_{S^{n-1}} \langle u, v\rangle_+ \, dS_K(u) \geq c_1.$$

Then, there exists an integer $j_0 \geq 1$ (depending on $v \in S^{n-1}$) such that

$$\frac{c_1}{2} \leq \int_{\Sigma_{j_0}(v)} \langle u, v\rangle_+ \, dS_K(u) \leq \int_{\Sigma_{j_0}(v)} dS_K(u). \tag{4.26}$$

Assume that $\phi \in \widehat{\Phi}_1$ and then ϕ is strictly increasing. Let $0 < \eta_1 < \eta_2 < \infty$. For all $u \in \Sigma_{j_0}(v)$, one has

$$\phi\left(\frac{n|K| \cdot \langle u, v\rangle_+}{\eta_2 \cdot h_K(u)}\right) < \phi\left(\frac{n|K| \cdot \langle u, v\rangle_+}{\eta_1 \cdot h_K(u)}\right),$$

and by (4.26),

$$\int_{\Sigma_{j_0}(v)} \phi\left(\frac{n|K| \cdot \langle u, v\rangle_+}{\eta_2 \cdot h_K(u)}\right) d\widetilde{V}_K(u) < \int_{\Sigma_{j_0}(v)} \phi\left(\frac{n|K| \cdot \langle u, v\rangle_+}{\eta_1 \cdot h_K(u)}\right) d\widetilde{V}_K(u).$$

The desired monotone argument (i.e., G_v is strictly decreasing) follows immediately from

$$G_v(\eta) = \int_{\Sigma_{j_0}(v)} \phi\left(\frac{n|K| \cdot \langle u, v\rangle_+}{\eta \cdot h_K(u)}\right) d\widetilde{V}_K(u) + \int_{S^{n-1}\setminus\Sigma_{j_0}(v)} \phi\left(\frac{n|K| \cdot \langle u, v\rangle_+}{\eta \cdot h_K(u)}\right) d\widetilde{V}_K(u).$$

Now let us prove that

$$\lim_{\eta\to 0} G_v(\eta) = \lim_{t\to\infty} \phi(t) = \infty \quad \text{and} \quad \lim_{\eta\to\infty} G_v(\eta) = \lim_{t\to 0} \phi(t) = 0.$$

To this end, as $\phi \in \widehat{\Phi}_1$ is increasing,

$$G_v(\eta) = \int_{S^{n-1}} \phi\left(\frac{n|K| \cdot \langle u, v\rangle_+}{\eta \cdot h_K(u)}\right) d\widetilde{V}_K(u) \leq \int_{S^{n-1}} \phi\left(\frac{n|K|}{\eta \cdot r_K}\right) d\widetilde{V}_K(u) = \phi\left(\frac{n|K|}{\eta \cdot r_K}\right).$$

By letting $t = \frac{n|K|}{\eta \cdot r_K}$, one has $0 \leq \lim_{\eta\to\infty} G_v(\eta) \leq \lim_{t\to 0} \phi(t) = 0$ and thus $\lim_{\eta\to\infty} G_v(\eta) = 0$. On the other hand,

$$\begin{aligned}
G_v(\eta) &= \int_{S^{n-1}} \phi\left(\frac{n|K| \cdot \langle u, v\rangle_+}{\eta \cdot h_K(u)}\right) d\widetilde{V}_K(u) \\
&\geq \int_{\Sigma_{j_0}(v)} \phi\left(\frac{n|K| \cdot \langle u, v\rangle_+}{\eta \cdot h_K(u)}\right) d\widetilde{V}_K(u) \\
&\geq \int_{\Sigma_{j_0}(v)} \phi\left(\frac{n|K|}{\eta \cdot j_0 \cdot R_K}\right) \cdot \frac{r_K}{n|K|} \cdot dS_K(u) \\
&\geq \phi\left(\frac{n|K|}{\eta \cdot j_0 \cdot R_K}\right) \cdot \frac{r_K}{n|K|} \cdot \frac{c_1}{2}. \qquad (4.27)
\end{aligned}$$

The desired result $\lim_{\eta\to 0} G_v(\eta) = \lim_{t\to\infty} \phi(t) = \infty$ follows by taking $\eta \to 0$. □

A direct consequence of Lemma 4.3 is that if $\phi \in \widehat{\Phi}_1$ and $v \in S^{n-1}$, then there is a unique $\eta_0 \in (0, \infty)$ such that

$$G_v(\eta_0) = \int_{S^{n-1}} \phi\left(\frac{n|K| \cdot \langle u, v\rangle_+}{\eta_0 \cdot h_K(u)}\right) d\widetilde{V}_K(u) = 1.$$

Such a unique η_0 can be defined as the homogeneous Orlicz L_ϕ mixed volume of $K \in \mathscr{K}_0$ and the line segment $[0, v] = \{tv : t \in [0, 1]\}$, namely, $\eta_0 = \widehat{V}_\phi(K, [0, v])$ and

$$\int_{S^{n-1}} \phi\left(\frac{n|K| \cdot \langle u, v\rangle_+}{\widehat{V}_\phi(K, [0, v]) \cdot h_K(u)}\right) d\widetilde{V}_K(u) = 1. \tag{4.28}$$

Proposition 4.3. *Let $K \in \mathscr{K}_0$ and $\phi \in \widehat{\Phi}_1$. There exists a convex body $M \in \mathscr{K}_0$ such that*

$$\widehat{G}^{orlicz}_\phi(K) = \widehat{V}_\phi(K, M) \quad \text{and} \quad |M^\circ| = \omega_n.$$

If in addition ϕ is convex, such a convex body M is unique.

Proof. Formula (4.21) implies that for $\phi \in \widehat{\Phi}_1$, there exists a sequence $\{M_i\}_{i\geq 1} \subset \mathscr{K}_0$ such that $\widehat{V}_\phi(K, M_i) \to \widehat{G}^{orlicz}_\phi(K)$ as $i \to \infty$, $|M_i^\circ| = \omega_n$ and $\widehat{V}_\phi(K, M_i) \leq 2\widehat{V}_\phi(K, B_2^n)$ for all $i \geq 1$. For each fixed $i \geq 1$, let

$$R_i = \rho_{M_i}(u_i) = \max\{\rho_{M_i}(u) : u \in S^{n-1}\}.$$

This yields $\{\lambda u_i : 0 \leq \lambda \leq R_i\} \subset M_i$ and hence for all $u \in S^{n-1}$,

$$h_{M_i}(u) \geq R_i \cdot \frac{|\langle u, u_i\rangle| + \langle u, u_i\rangle}{2} = R_i \cdot \langle u, u_i\rangle_+.$$

Let $\phi \in \widehat{\Phi}_1$ and $\eta_i = \widehat{V}_\phi(K, [0, u_i]) \in (0, \infty)$ for $i \geq 1$. Recall that formula (4.28) states

$$1 = \int_{S^{n-1}} \phi\left(\frac{n|K| \cdot \langle u, u_i\rangle_+}{\eta_i \cdot h_K(u)}\right) d\widetilde{V}_K(u).$$

By Corollary 3.2 and the fact that ϕ is increasing, we have

$$\begin{aligned} 1 &= \int_{S^{n-1}} \phi\left(\frac{n|K| \cdot h_{M_i}(u)}{\widehat{V}_\phi(K, M_i) \cdot h_K(u)}\right) d\widetilde{V}_K(u) \\ &\geq \int_{S^{n-1}} \phi\left(\frac{n|K| \cdot R_i \cdot \langle u, u_i\rangle_+}{2\widehat{V}_\phi(K, B_2^n) \cdot h_K(u)}\right) d\widetilde{V}_K(u). \end{aligned}$$

This further leads to, for all $i \geq 1$,

$$R_i \leq \frac{2\widehat{V}_\phi(K, B_2^n)}{\eta_i}.$$

Next, we prove that $\inf_{i\geq 1} \eta_i > 0$. We will use the method of contradiction and assume that $\inf_{i\geq 1} \eta_i = 0$. Consequently, there is a subsequence of $\{\eta_i\}_{i\geq 1}$ (still denoted by $\{\eta_i\}_{i\geq 1}$), such that, $\eta_i \to 0$ as $i \to \infty$. Due to the compactness of S^{n-1}, one can also have a convergent subsequence of $\{u_i\}_{i\geq 1}$ (again denoted by $\{u_i\}_{i\geq 1}$) whose limit is $v \in S^{n-1}$. In summary, we have two sequences $\{u_i\}_{i\geq 1}$ and $\{\eta_i\}_{i\geq 1}$ such that $u_i \to v$ and $\eta_i \to 0$ as $i \to \infty$. It is easily checked that

$\langle u, u_i\rangle_+ \to \langle u, v\rangle_+$ uniformly on S^{n-1} by the triangle inequality. For any given $\varepsilon > 0$, Corollary 3.2, Fatou's lemma and formula (4.26) imply

$$\begin{aligned}
1 &= \lim_{i\to\infty}\int_{S^{n-1}} \phi\left(\frac{n|K|\cdot\langle u, u_i\rangle_+}{\eta_i\cdot h_K(u)}\right) d\widetilde{V}_K(u)\\
&\geq \liminf_{i\to\infty}\int_{S^{n-1}} \phi\left(\frac{n|K|\cdot\langle u, u_i\rangle_+}{(\eta_i+\varepsilon)\cdot h_K(u)}\right) d\widetilde{V}_K(u)\\
&\geq \int_{S^{n-1}} \liminf_{i\to\infty}\phi\left(\frac{n|K|\cdot\langle u, u_i\rangle_+}{(\eta_i+\varepsilon)\cdot h_K(u)}\right) d\widetilde{V}_K(u)\\
&= \int_{S^{n-1}} \phi\left(\frac{n|K|\cdot\langle u, v\rangle_+}{\varepsilon\cdot h_K(u)}\right) d\widetilde{V}_K(u)\\
&= G_v(\varepsilon).
\end{aligned}$$

It follows from Lemma 4.3 that $\lim_{\varepsilon\to 0^+} G_v(\varepsilon) = \infty$, which leads to a contradiction (i.e., $1 \geq \infty$). Therefore, $\inf_{i\geq 1}\eta_i > 0$ and

$$\sup_{i\geq 1} R_i \leq \frac{2\widehat{V}_\phi(K, B_2^n)}{\inf_{i\geq 1}\eta_i} < \infty.$$

This concludes that the sequence $\{M_i\}_{i\geq 1} \subset \mathscr{K}_0$ is uniformly bounded.

The Blaschke selection theorem yields that there exists a convergent subsequence of $\{M_i\}_{i\geq 1}$ (still denoted by $\{M_i\}_{i\geq 1}$) and a convex body $M \in \mathscr{K}$ such that $M_i \to M$ as $i \to \infty$. Since $|M_i^\circ| = \omega_n$ for all $i \geq 1$, Lemma 2.1 implies $M \in \mathscr{K}_0$. Moreover, $|M^\circ| = \omega_n$ because $|M_i^\circ| = \omega_n$ for all $i \geq 1$ and $M_i \to M$ (hence, $M_i^\circ \to M^\circ$). It follows from Proposition 4.1 that

$$\widehat{V}_\phi(K, M_i) \to \widehat{V}_\phi(K, M) \quad \text{and} \quad |M^\circ| = \omega_n.$$

By the uniqueness of the limit, one gets

$$\widehat{G}_\phi^{orlicz}(K) = \widehat{V}_\phi(K, M) \quad \text{and} \quad |M^\circ| = \omega_n.$$

This concludes the existence of the Orlicz-Petty bodies.

If $\phi \in \widehat{\Phi}_1$ is also convex, the uniqueness of M can be proved as follows. Suppose that $M_1, M_2 \in \mathscr{K}_0$ such that $|M_1^\circ| = |M_2^\circ| = \omega_n$ and

$$\widehat{V}_\phi(K, M_1) = \inf_{L\in\mathscr{K}_0}\{\widehat{V}_\phi(K, \mathrm{vrad}(L^\circ)L)\} = \widehat{V}_\phi(K, M_2).$$

Define $M \in \mathscr{K}_0$ by $M = \frac{M_1+M_2}{2}$. That is, $h_M = \frac{h_{M_1}+h_{M_2}}{2}$. By formula (2.4), it can be checked that $|M^\circ| \leq \omega_n$ (hence $\mathrm{vrad}(M^\circ) \leq 1$) with equality if and only if $M_1 = M_2$. In fact, the function t^{-n} is strictly convex, and hence

$$\begin{aligned}
|M^\circ| &= \frac{1}{n}\int_{S^{n-1}} h_M(u)^{-n}\, d\sigma(u)\\
&= \frac{1}{n}\int_{S^{n-1}} \left(\frac{h_{M_1}(u)+h_{M_2}(u)}{2}\right)^{-n} d\sigma(u)\\
&\leq \frac{1}{n}\int_{S^{n-1}} \frac{h_{M_1}(u)^{-n}+h_{M_2}(u)^{-n}}{2}\, d\sigma(u)\\
&= \frac{|M_1^\circ|+|M_2^\circ|}{2} = \omega_n, \qquad (4.29)
\end{aligned}$$

with equality if and only if $h_{M_1} = h_{M_2}$ on S^{n-1}, i.e., $M_1 = M_2$.

For convenience, let $\lambda = \widehat{V}_\phi(K, M_1) = \widehat{V}_\phi(K, M_2)$. The fact that ϕ is convex imply

$$\begin{aligned}\int_{S^{n-1}} \phi\left(\frac{n|K|\cdot h_M(u)}{\lambda\cdot h_K(u)}\right) d\widetilde{V}_K(u) &= \int_{S^{n-1}} \phi\left(\frac{n|K|\cdot (h_{M_1}(u)+h_{M_2}(u))}{2\cdot\lambda\cdot h_K(u)}\right) d\widetilde{V}_K(u)\\ &\le \frac{1}{2}\int_{S^{n-1}} \left[\phi\left(\frac{n|K|\cdot h_{M_1}(u)}{\lambda\cdot h_K(u)}\right)+\phi\left(\frac{n|K|\cdot h_{M_2}(u)}{\lambda\cdot h_K(u)}\right)\right] d\widetilde{V}_K(u)\\ &= 1.\end{aligned}$$

Hence, $\widehat{V}_\phi(K, M) \le \lambda$ which follows from the facts that ϕ is strictly increasing and

$$\int_{S^{n-1}} \phi\left(\frac{n|K|\cdot h_M(u)}{\lambda\cdot h_K(u)}\right) d\widetilde{V}_K(u) \le 1 = \int_{S^{n-1}} \phi\left(\frac{n|K|\cdot h_M(u)}{\widehat{V}_\phi(K,M)\cdot h_K(u)}\right) d\widetilde{V}_K(u).$$

Assume that $M_1 \ne M_2$, then $\mathrm{vrad}(M^\circ) < 1$. Note that $\widehat{V}_\phi(K, M) > 0$. Together with Corollary 3.1, one can check that

$$\widehat{V}_\phi(K, \mathrm{vrad}(M^\circ)M) < \widehat{V}_\phi(K, M) \le \widehat{V}_\phi(K, M_1).$$

This contradicts with the minimality of M_1. Therefore, $M_1 = M_2$ and the uniqueness follows. □

Definition 4.1. *Let $K \in \mathscr{K}_0$ and $\phi \in \widehat{\Phi}_1$. A convex body M is said to be an L_ϕ Orlicz-Petty body of K, if $M \in \mathscr{K}_0$ satisfies*

$$\widehat{G}^{orlicz}_\phi(K) = \widehat{V}_\phi(K, M) \quad \text{and} \quad |M^\circ| = \omega_n.$$

Denote by $\widehat{T}_\phi K$ the set of all L_ϕ Orlicz-Petty bodies of K.

Clearly, if $\phi \in \widehat{\Phi}_1$, the set $\widehat{T}_\phi K$ is nonempty and may contain more than one convex body. If in addition $\phi \in \widehat{\Phi}_1$ is convex, $\widehat{T}_\phi K$ must contain only one convex body; and in this case $\widehat{T}_\phi K$ is called the L_ϕ Orlicz-Petty body of K. Moreover, the set $\widehat{T}_\phi K$ is $SL(n)$-invariant. In fact, for $A \in SL(n)$ and all $M \in \widehat{T}_\phi K$, by Proposition 3.1 and formula (3.16), one sees

$$\widehat{G}^{orlicz}_\phi(AK) = \widehat{G}^{orlicz}_\phi(K) = \widehat{V}_\phi(K, M) = \widehat{V}_\phi(AK, AM).$$

It follows from $|(AM)^\circ| = \omega_n$ that $AM \in \widehat{T}_\phi(AK)$ and thus $A(\widehat{T}_\phi K) \subset \widehat{T}_\phi(AK)$. Replacing K by AK and A by its inverse, one also gets $\widehat{T}_\phi(AK) \subset A(\widehat{T}_\phi K)$ and thus $\widehat{T}_\phi(AK) = A(\widehat{T}_\phi K)$.

On the other hand, $\widehat{T}_\phi(\lambda K) = \widehat{T}_\phi K$ for all $\lambda > 0$. To this end, for $M \in \widehat{T}_\phi K$, one has $|M^\circ| = \omega_n$ and $\widehat{G}^{orlicz}_\phi(K) = \widehat{V}_\phi(K, M)$. This leads to, by Corollary 3.1 and Proposition 3.1,

$$\widehat{G}^{orlicz}_\phi(\lambda K) = \lambda^{n-1}\widehat{G}^{orlicz}_\phi(K) = \lambda^{n-1}\widehat{V}_\phi(K, M) = \widehat{V}_\phi(\lambda K, M).$$

Thus, $M \in \widehat{T}_\phi(\lambda K)$ and then $\widehat{T}_\phi K \subset \widehat{T}_\phi(\lambda K)$. Similarly, $\widehat{T}_\phi(\lambda K) \subset \widehat{T}_\phi K$ and thus $\widehat{T}_\phi(\lambda K) = \widehat{T}_\phi K$.

When $\phi \in \widehat{\Phi}_1$ is convex, the L_ϕ Orlicz-Petty body $\widehat{T}_\phi K$ satisfies the following inequality: for all $K \in \mathscr{K}_0$, one has

$$|\widehat{T}_\phi K| \cdot |(\widehat{T}_\phi K)^\circ| \le |K| \cdot |K^\circ|. \tag{4.30}$$

In fact, it follows from (3.17) that for $K \in \mathscr{K}_0$, $\widehat{G}_\phi^{orlicz}(K) \le n|K| \cdot \mathrm{vrad}(K^\circ)$. Definition 4.1 and the Orlicz-Minkowski inequality (3.9) imply that

$$\widehat{G}_\phi^{orlicz}(K) = \widehat{V}_\phi(K, \widehat{T}_\phi K) \ge n \cdot |K|^{\frac{n-1}{n}} |\widehat{T}_\phi K|^{\frac{1}{n}}.$$

The desired inequality (4.30) is then a simple consequence of the combination of the two inequalities above and $|(\widehat{T}_\phi K)^\circ| = \omega_n$. Note that it is an open problem (i.e., the famous Mahler conjecture) to find the minimum of $|K| \cdot |K^\circ|$ among all convex bodies $K \in \widetilde{\mathscr{K}}$. The inverse Santaló inequality (3.19) provides an isomorphic solution to the Mahler conjecture. We think that the L_ϕ Orlicz-Petty body $\widehat{T}_\phi K$ and inequality (4.30) may be useful in attacking the Mahler conjecture.

The following proposition states that an L_ϕ Orlicz-Petty body of a polytope is again a polytope.

Proposition 4.4. *Let $K \in \mathscr{K}_0$ be a polytope and $\phi \in \widehat{\Phi}_1$. If $M \in \widehat{T}_\phi K$, then M is a polytope with faces parallel to those of K.*

Proof. Let K be a polytope whose surface area measure S_K is concentrated on a finite set $\{u_1, \cdots, u_m\} \subset S^{n-1}$. Let $M \in \widehat{T}_\phi K$ be an L_ϕ Orlicz-Petty body of K. Denote by P the polytope whose faces are parallel to those of K and P circumscribes M.

Note that S_K is concentrated on $\{u_1, \cdots, u_m\}$ and $h_P(u_i) = h_M(u_i)$ for all $1 \le i \le m$. Let $\lambda = \widehat{V}_\phi(K, P)$. Then

$$\begin{aligned} 1 &= \int_{S^{n-1}} \phi\left(\frac{n|K| \cdot h_P(u)}{\lambda \cdot h_K(u)}\right) d\widetilde{V}_K(u) \\ &= \frac{1}{n|K|} \cdot \sum_{i=1}^m \phi\left(\frac{n|K| \cdot h_P(u_i)}{\lambda \cdot h_K(u_i)}\right) h_K(u_i) S_K(u_i) \\ &= \frac{1}{n|K|} \cdot \sum_{i=1}^m \phi\left(\frac{n|K| \cdot h_M(u_i)}{\lambda \cdot h_K(u_i)}\right) h_K(u_i) S_K(u_i) \\ &= \int_{S^{n-1}} \phi\left(\frac{n|K| \cdot h_M(u)}{\lambda \cdot h_K(u)}\right) d\widetilde{V}_K(u). \end{aligned}$$

Consequently, $\lambda = \widehat{V}_\phi(K, P) = \widehat{V}_\phi(K, M)$.

As P circumscribes M, then $P^\circ \subset M^\circ$ and $|P^\circ| \le |M^\circ| = \omega_n$ with equality if and only if $M = P$. Formula (3.12) and Corollary 3.1 yield that for $\phi \in \widehat{\Phi}_1$,

$$\widehat{G}_\phi^{orlicz}(K) \le \widehat{V}_\phi(K, \mathrm{vrad}(P^\circ) P) \le \widehat{V}_\phi(K, M) = \widehat{G}_\phi^{orlicz}(K).$$

This requires in particular $|P^\circ| = |M^\circ| = \omega_n$. Hence $M = P$ is a polytope whose faces are parallel to those of K. □

Proposition 4.5. *Let $K \in \mathscr{K}_0$ and $r_K, R_K > 0$ be such that $r_K B_2^n \subset K \subset R_K B_2^n$. For $\phi \in \widehat{\Phi}_1$ and $M \in \widehat{T}_\phi K$, there exists an integer $j_0 > 1$ such that, for all $u \in S^{n-1}$,*

$$h_M(u) \le \frac{j_0 \cdot R_K^{n+1}}{r_K^{n+1}} \cdot \phi^{-1}\left(\frac{2n\omega_n \cdot R_K^n}{c_1 \cdot r_K}\right),$$

where $c_1 > 0$ is the constant in (4.26).

Proof. Let $M \in \widehat{T}_\phi K$. First of all, the minimality of M gives that

$$\widehat{V}_\phi(K, M) \le \widehat{V}_\phi(K, B_2^n) \le \frac{n\omega_n \cdot R_K^n}{r_K},$$

where the second inequality follows from Lemma 4.1. Let $\lambda = \widehat{V}_\phi(K, M)$ and $R(M) = \rho_M(v) = \max\{\rho_M(u) : u \in S^{n-1}\}$. A calculation similar to (4.27) leads to

$$\begin{aligned} 1 &= \int_{S^{n-1}} \phi\left(\frac{n|K| \cdot h_M(u)}{\lambda \cdot h_K(u)}\right) d\widetilde{V}_K(u) \\ &\ge \int_{S^{n-1}} \phi\left(\frac{n|K| \cdot R(M) \cdot \langle u, v\rangle_+}{\lambda \cdot R_K}\right) \frac{r_K}{n|K|} dS_K(u) \\ &\ge \int_{\Sigma_{j_0}(v)} \phi\left(\frac{n|K| \cdot R(M)}{\lambda \cdot j_0 \cdot R_K}\right) \frac{r_K}{n|K|} dS_K(u) \\ &\ge \phi\left(\frac{n|K| \cdot R(M)}{\lambda \cdot j_0 \cdot R_K}\right) \frac{r_K \cdot c_1}{2n|K|}. \end{aligned}$$

By the facts that $\phi(1) = 1$ and ϕ is increasing, one has

$$R(M) \le \frac{\lambda \cdot j_0 \cdot R_K}{n|K|} \cdot \phi^{-1}\left(\frac{2n|K|}{c_1 \cdot r_K}\right) \le \frac{j_0 \cdot R_K^{n+1}}{r_K^{n+1}} \cdot \phi^{-1}\left(\frac{2n\omega_n \cdot R_K^n}{c_1 \cdot r_K}\right).$$

This completes the proof. □

4.3 Continuity of the homogeneous Orlicz geominimal surface areas

This subsection is dedicated to prove the continuity of the homogeneous Orlicz geominimal surface areas under the condition $\phi \in \widehat{\Phi}_1$. The following uniform boundedness argument is needed.

Lemma 4.4. *Let $\{K_\alpha\}_{\alpha \in \Lambda} \subset \mathscr{K}_0$ be a family of convex bodies satisfying the uniformly bounded property: there exist constants $r, R > 0$ such that $rB_2^n \subset K_\alpha \subset RB_2^n$ for all $\alpha \in \Lambda$. For $\phi \in \widehat{\Phi}_1$ and for any $M_\alpha \in \widehat{T}_\phi(K_\alpha)$, there exist constants $r', R' > 0$ such that*

$$r'B_2^n \subset M_\alpha \subset R'B_2^n \quad \text{for all} \quad \alpha \in \Lambda.$$

Proof. We only need to prove the case that $\{K_\alpha\}_{\alpha \in \Lambda}$ contains infinite many different convex bodies, as otherwise the argument is trivial.

Let $M_\alpha \in \widehat{T}_\phi(K_\alpha)$. First, we prove the existence of R' by contradiction. To this end, we assume that there is no constant R' such that $M_\alpha \subset R'B_2^n$ for all $\alpha \in \Lambda$. In other words, there is a sequence of $\{M_\alpha\}_{\alpha \in \Lambda}$, denoted by $\{M_i\}_{i \ge 1}$, such that $R(M_i) \to \infty$. Hereafter, for all $i \ge 1$,

$$R(M_i) = \rho_{M_i}(u_i) = \max\{\rho_{M_i}(u) : u \in S^{n-1}\}.$$

Similar to the proof of Proposition 4.3, one can find a subsequence, which will not be relabeled, such that, $u_i \to v \in S^{n-1}$ (due to the compactness of S^{n-1}), $R(M_i) \to \infty$ and $K_i \to K$ (by the Blaschke selection theorem due to the uniform boundedness of $\{K_\alpha\}_{\alpha \in \Lambda}$) as $i \to \infty$.

It follows from Proposition 4.1 that $\widehat{V}_\phi(K_i, B_2^n) \to \widehat{V}_\phi(K, B_2^n)$ as $i \to \infty$. This implies the boundedness of the sequence $\{\widehat{V}_\phi(K_i, B_2^n)\}_{i \ge 1}$ and hence

$$\lambda_i = \frac{\widehat{V}_\phi(K_i, B_2^n)}{R(M_i)} \to 0 \quad \text{as} \quad i \to \infty.$$

Let $\varepsilon > 0$ be given. The triangle inequality yields the uniform convergence of $\langle u, u_i\rangle_+ \to \langle u, v\rangle_+$ on S^{n-1} as $i \to \infty$. Moreover, as $K_i \to K$, one sees $rB_2^n \subset K \subset RB_2^n$ and

$$0 \le \frac{n|K_i| \cdot \langle u, u_i\rangle_+}{(\lambda_i + \varepsilon) \cdot h_{K_i}(u)} \le \frac{n\omega_n \cdot R^n}{\varepsilon \cdot r} \quad \text{and} \quad 0 \le \frac{n|K| \cdot \langle u, v\rangle_+}{\varepsilon \cdot h_K(u)} \le \frac{n\omega_n \cdot R^n}{\varepsilon \cdot r}.$$

A simple calculation yields that

$$\frac{n|K_i| \cdot \langle u, u_i\rangle_+}{(\lambda_i + \varepsilon) \cdot h_{K_i}(u)} \to \frac{n|K| \cdot \langle u, v\rangle_+}{\varepsilon \cdot h_K(u)} \qquad \text{uniformly on} \quad S^{n-1} \quad \text{as} \quad i \to \infty.$$

Let $I = [0, n\omega_n R^n \varepsilon^{-1} r^{-1}]$ and then $\phi \in \widehat{\Phi}_1$ is uniformly continuous on I. By Lemma 4.2, one gets

$$\phi\left(\frac{n|K_i| \cdot \langle u, u_i\rangle_+}{(\lambda_i + \varepsilon) \cdot h_{K_i}(u)}\right) \to \phi\left(\frac{n|K| \cdot \langle u, v\rangle_+}{\varepsilon \cdot h_K(u)}\right) \quad \text{uniformly on} \quad S^{n-1} \quad \text{as } i \to \infty. \tag{4.31}$$

Note that $\phi \in \widehat{\Phi}_1$ is increasing. By Corollary 3.2, (2.5), (4.26) and (4.31), a calculation similar to (4.27) leads to, for any given $\varepsilon > 0$,

$$\begin{aligned}
1 &= \lim_{i\to\infty} \int_{S^{n-1}} \phi\left(\frac{n|K_i| \cdot h_{M_i}(u)}{\widehat{V}_\phi(K_i, M_i) \cdot h_{K_i}(u)}\right) d\widetilde{V}_{K_i}(u) \\
&\ge \lim_{i\to\infty} \int_{S^{n-1}} \phi\left(\frac{n|K_i| \cdot R(M_i) \cdot \langle u, u_i\rangle_+}{\widehat{V}_\phi(K_i, B_2^n) \cdot h_{K_i}(u)}\right) d\widetilde{V}_{K_i}(u) \\
&\ge \lim_{i\to\infty} \int_{S^{n-1}} \phi\left(\frac{n|K_i| \cdot \langle u, u_i\rangle_+}{(\lambda_i + \varepsilon) \cdot h_{K_i}(u)}\right) d\widetilde{V}_{K_i}(u) \\
&= \int_{S^{n-1}} \phi\left(\frac{n|K| \cdot \langle u, v\rangle_+}{\varepsilon \cdot h_K(u)}\right) d\widetilde{V}_K(u) \\
&= G_v(\varepsilon).
\end{aligned}$$

It follows from Lemma 4.3 that $\lim_{\varepsilon\to 0^+} G_v(\varepsilon) = \infty$, which leads to a contradiction (i.e., $1 \ge \infty$). Thus $R(M_i) \to \infty$ is impossible. This concludes the existence of R' such that $M_\alpha \subset R'B_2^n$ for all $\alpha \in \Lambda$. In other words, $\{M_\alpha\}_{\alpha\in\Lambda} \subset \mathscr{K}_0$ is uniformly bounded.

Next, we show the existence of $r' > 0$ such that $r'B_2^n \subset M_\alpha$ for all $\alpha \in \Lambda$. Assume that there is no such a constant $r' > 0$. In other words, there is a sequence $\{M_j\}_{j\ge 1}$ such that $w_j \to w \in S^{n-1}$ (due to the compactness of S^{n-1}) and $r_j \to 0$ as $j \to \infty$, where

$$r_j = h_{M_j}(w_j) = \min\{h_{M_j}(u) : u \in S^{n-1}\}.$$

Note that the sequence $\{M_j\}_{j\ge 1} \subset \mathscr{K}_0$ is uniformly bounded (as proved above). The Blaschke selection theorem, Lemma 2.1 and $|M_j^\circ| = \omega_n$ for all $j \ge 1$ imply that there exists a subsequence of $\{M_j\}_{j\ge 1}$, which will not be relabeled, and a convex body $M \in \mathscr{K}_0$, such that, $M_j \to M$ as $j \to \infty$. That is,

$$\lim_{j\to\infty} \sup_{u\in S^{n-1}} |h_{M_j}(u) - h_M(u)| = 0.$$

This further implies, as $w_j \to w$,

$$h_M(w) = \lim_{j\to\infty} h_{M_j}(w_j) = \lim_{j\to\infty} r_j = 0.$$

This contradicts with the positivity of the support function of M. Hence, there is a constant $r' > 0$ such that $r'B_2^n \subset M_\alpha$ for all $\alpha \in \Lambda$. □

Now let us prove our main result which states that the homogeneous Orlicz geominimal surface areas are continuous on $\mathscr{K}_0$ with respect to the Hausdorff distance.

Theorem 4.1. *For $\phi \in \widehat{\Phi}_1$, the functional $\widehat{G}_{\phi}^{orilcz}(\cdot)$ on $\mathscr{K}_0$ is continuous with respect to the Hausdorff distance. In particular, the L_p geominimal surface surface area for $p \in (0,\infty)$ is continuous on $\mathscr{K}_0$ with respect to the Hausdorff distance.*

Proof. The upper semicontinuity has been proved in Proposition 4.2. To get the continuity, it is enough to prove that the homogeneous Orlicz geominimal surface areas are lower semicontinuous on $\mathscr{K}_0$. To this end, let $\{K_i\}_{i\geq 1} \subset \mathscr{K}_0$ be a convergent sequence whose limit is $K_0 \in \mathscr{K}_0$. Let $M_i \in \widehat{T}_{\phi}(K_i)$ for $i \geq 1$. Clearly, $\{K_i\}_{i\geq 0}$ satisfies the uniformly bounded condition in Lemma 4.4, which implies the uniform boundedness of the sequence $\{M_i\}_{i\geq 1}$.

Let $l = \liminf_{i\to\infty} \widehat{G}_{\phi}^{orlicz}(K_i)$. Consequently, one can find a subsequence $\{K_{i_k}\}_{k\geq 1}$ such that $l = \lim_{k\to\infty} \widehat{G}_{\phi}^{orlicz}(K_{i_k})$. By the Blaschke selection theorem and Lemma 2.1, there exists a subsequence of $\{M_{i_k}\}_{k\geq 1}$ (still denoted by $\{M_{i_k}\}_{k\geq 1}$) and a body $M \in \mathscr{K}_0$, such that, $M_{i_k} \to M$ as $k \to \infty$ and $|M^\circ| = \omega_n$. Proposition 4.1 then yields

$$\widehat{G}_{\phi}^{orlicz}(K_{i_k}) = \widehat{V}_{\phi}(K_{i_k}, M_{i_k}) \to \widehat{V}_{\phi}(K_0, M) \quad \text{as} \quad k \to \infty.$$

It follows from (4.21) that

$$\widehat{G}_{\phi}^{orlicz}(K_0) \leq \widehat{V}_{\phi}(K_0, M) = \lim_{k\to\infty} \widehat{G}_{\phi}^{orlicz}(K_{i_k}) = \liminf_{i\to\infty} \widehat{G}_{\phi}^{orlicz}(K_i).$$

This completes the proof. □

Proposition 4.3 states that if $\phi \in \widehat{\Phi}_1$ is convex, the L_ϕ Orlicz-Petty body is unique. In this case, $\widehat{T}_{\phi}K$ contains only one element. Consequently, $\widehat{T}_{\phi} : \mathscr{K}_0 \to \mathscr{K}_0$ defines an operator. The following result states that the operator $\widehat{T}_{\phi}$ is continuous.

Proposition 4.6. *Let $\phi \in \widehat{\Phi}_1$ be convex. Then $\widehat{T}_{\phi} : \mathscr{K}_0 \mapsto \mathscr{K}_0$ is continuous with respect to the Hausdorff distance.*

Proof. It is enough to prove that $\{\widehat{T}_{\phi}K_i\}_{i\geq 1} \subset \mathscr{K}_0$ is convergent to $\widehat{T}_{\phi}K_0 \in \mathscr{K}_0$ for every convergent sequence $\{K_i\}_{i\geq 1} \subset \mathscr{K}_0$ with limit $K_0 \in \mathscr{K}_0$, in particular, every subsequence of $\{\widehat{T}_{\phi}K_i\}_{i\geq 1}$ has a convergent subsequence whose limit is $\widehat{T}_{\phi}K_0$.

Let $\{K_{i_k}\}_{k\geq 1}$ be any subsequence of $\{K_i\}_{i\geq 1}$. Of course, $K_{i_k} \to K_0$ as $k \to \infty$ and $\{\widehat{T}_{\phi}K_{i_k}\}_{k\geq 1}$ is uniformly bounded by Lemma 4.4. Following the Blaschke selection theorem, one can find a subsequence of $\{\widehat{T}_{\phi}K_{i_k}\}_{k\geq 1}$, which will not be relabeled, and $M \in \mathscr{K}_0$ such that $\widehat{T}_{\phi}K_{i_k} \to M$ as $k \to \infty$ and $|M^\circ| = \omega_n$. By Proposition 4.1, one has

$$\widehat{G}_{\phi}^{orlicz}(K_{i_k}) = \widehat{V}_{\phi}(K_{i_k}, \widehat{T}_{\phi}K_{i_k}) \to \widehat{V}_{\phi}(K_0, M) \quad \text{as} \quad k \to \infty.$$

By Theorem 4.1, one has

$$\widehat{G}_{\phi}^{orlicz}(K_{i_k}) \to \widehat{G}_{\phi}^{orlicz}(K_0) = \widehat{V}_{\phi}(K_0, \widehat{T}_{\phi}K_0) \quad \text{as} \quad k \to \infty.$$

Hence, $\widehat{V}_{\phi}(K_0, \widehat{T}_{\phi}K_0) = \widehat{V}_{\phi}(K_0, M)$ and then $\widehat{T}_{\phi}K_0 = M$ by the uniqueness of the L_ϕ Orlicz-Petty body for $\phi \in \widehat{\Phi}_1$ being convex. □

5 The nonhomogeneous Orlicz geominimal surface areas

In this section, we will briefly discuss the continuity of the nonhomogeneous Orlicz geominimal surface areas defined in [49]. In particular, we prove the existence, uniqueness and affine invariance for the L_φ Orlicz-Petty bodies in Subsection 5.2. In Subsection 5.1, we provide a geometric interpretation for the nonhomogeneous Orlicz L_φ mixed volume with $\varphi \in \mathscr{I} \cup \mathscr{D}$ (in particular, for $\varphi(t) = t^p$ with $p < 1$).

5.1 The geometric interpretation for the nonhomogeneous Orlicz L_φ mixed volume

For any continuous function $\varphi : (0, \infty) \to (0, \infty)$, $V_\varphi(K, L)$ denotes the nonhomogeneous Orlicz L_φ mixed volume of K and L. It has the following integral expression:

$$V_\varphi(K, L) = \frac{1}{n} \int_{S^{n-1}} \varphi\left(\frac{h_L(u)}{h_K(u)}\right) h_K(u) dS_K(u). \tag{5.32}$$

We can use the following examples to see that $V_\varphi(\cdot, \cdot)$ is not homogeneous:

$$V_\varphi(rB_2^n, B_2^n) = \varphi(1/r) \cdot r^n \cdot \omega_n \quad \text{and} \quad V_\varphi(B_2^n, rB_2^n) = \varphi(r) \cdot \omega_n.$$

The geometric interpretation of $V_\varphi(\cdot, \cdot)$ for convex $\varphi \in \mathscr{I}$ was given in [13, 45]. However, there are no geometric interpretations of $V_\varphi(\cdot, \cdot)$ for non-convex functions φ (even if $\varphi(t) = t^p$ for $p < 1$). In this subsection, we will provide such a geometric interpretation for all $\varphi \in \mathscr{I} \cup \mathscr{D}$.

Denote by $C^+(S^{n-1})$ the set of all positive continuous functions on S^{n-1}. Define K_f, the Aleksandrov body associated with $f \in C^+(S^{n-1})$, by

$$K_f = \cap_{u \in S^{n-1}} H^-(u, f(u)),$$

where $H^-(u, \alpha)$ is the half space with normal vector u and constant $\alpha > 0$:

$$H^-(u, \alpha) = \{x \in \mathbb{R}^n : \langle x, u \rangle \le \alpha\}.$$

This implies that

$$K_f = \{x \in \mathbb{R}^n : \langle x, u \rangle \le f(u) \quad \text{for all} \quad u \in S^{n-1}\}.$$

Equivalently, K_f is the (unique) maximal element (with respect to set inclusion) of the set

$$\{K \in \mathscr{K}_0 : h_K(u) \le f(u) \quad \text{for all} \quad u \in S^{n-1}\}.$$

When $f = h_L$ for some convex body $L \in \mathscr{K}_0$, one sees $K_f = L$.

For $K \in \mathscr{K}_0$ and $f \in C^+(S^{n-1})$, the L_1 mixed volume of K and f, denoted by $V_1(K, f)$, can be formulated by

$$V_1(K, f) = \frac{1}{n} \int_{S^{n-1}} f(u) dS_K(u).$$

When f is the support function of a convex body L, then $V_1(K, f)$ is just the usual L_1 mixed volume of K and L (i.e., $\varphi(t) = t$ in formula (5.32)). In particular, $V_1(K, h_K) = |K|$ for all $K \in \mathscr{K}_0$. Lemma 3.1 in [27] states that

$$|K_f| = V_1(K_f, f). \tag{5.33}$$

In order to prove the geometric interpretation for $V_\varphi(\cdot, \cdot)$, the linear Orlicz addition of functions [20] is needed. A special case is given below.

Definition 5.1. *Assume that either $\varphi_1, \varphi_2 \in \mathscr{I}$ or $\varphi_1, \varphi_2 \in \mathscr{D}$. For $\varepsilon > 0$, define $p_1 +_{\varphi,\varepsilon} p_2$, the linear Orlicz addition of positive functions p_1, p_2 (on whatever common domain), by*

$$\varphi_1\left(\frac{p_1(x)}{(p_1 +_{\varphi,\varepsilon} p_2)(x)}\right) + \varepsilon\varphi_2\left(\frac{p_2(x)}{(p_1 +_{\varphi,\varepsilon} p_2)(x)}\right) = 1.$$

For our context, $p_1 = h_K$ and $p_2 = h_L$ where $K, L \in \mathscr{K}_0$ are two convex bodies. Namely we let $f_\varepsilon = h_K +_{\varphi,\varepsilon} h_L$ and then for any $u \in S^{n-1}$,

$$\varphi_1\left(\frac{h_K(u)}{f_\varepsilon(u)}\right) + \varepsilon\varphi_2\left(\frac{h_L(u)}{f_\varepsilon(u)}\right) = 1. \tag{5.34}$$

When $\varphi_1, \varphi_2 \in \mathscr{I}$ are convex functions, $f_\varepsilon = h_K +_{\varphi,\varepsilon} h_L$ is the support function of a convex body (see [13, 45]). Clearly, $f_\varepsilon \in C^+(S^{n-1})$ determines an Aleksandrov body K_{f_ε}, which will be written as K_ε for simplicity. Moreover, $h_K \leq f_\varepsilon$ if $\varphi_1, \varphi_2 \in \mathscr{I}$ and $h_K \geq f_\varepsilon$ if $\varphi_1, \varphi_2 \in \mathscr{D}$.

Let $(\varphi_1)'_l(1)$ and $(\varphi_1)'_r(1)$ stand for the left and the right derivatives of φ_1 at $t = 1$, respectively, if they exist. From the proof of Theorem 9 in [20], one sees that $f_\varepsilon \to h_K$ uniformly on S^{n-1} as $\varepsilon \to 0^+$. Following similar arguments in [13, 14, 20, 54], we can prove the following result.

Lemma 5.1. *Let $K, L \in \mathscr{K}_0$ and $\varphi_1, \varphi_2 \in \mathscr{I}$ be such that $(\varphi_1)'_l(1)$ exists and is positive. Then*

$$(\varphi_1)'_l(1) \lim_{\varepsilon\to 0^+} \frac{f_\varepsilon(u) - h_K(u)}{\varepsilon} \quad = \quad h_K(u) \cdot \varphi_2\left(\frac{h_L(u)}{h_K(u)}\right) \quad \text{uniformly on } \ S^{n-1}. \tag{5.35}$$

For $\varphi_1, \varphi_2 \in \mathscr{D}$, (5.35) holds with $(\varphi_1)'_l(1)$ replaced by $(\varphi_1)'_r(1)$.

Proof. Let $\varphi_1, \varphi_2 \in \mathscr{I}$. Note that $f_\varepsilon \downarrow h_K$ uniformly on S^{n-1} as $\varepsilon \downarrow 0^+$. Then, for all $u \in S^{n-1}$,

$$\begin{aligned}
(\varphi_1)'_l(1) &= \lim_{\varepsilon\to 0^+} f_\varepsilon(u) \cdot \frac{1 - \varphi_1\left(\frac{h_K(u)}{f_\varepsilon(u)}\right)}{f_\varepsilon(u) - h_K(u)} \\
&= \lim_{\varepsilon\to 0^+} f_\varepsilon(u) \cdot \varphi_2\left(\frac{h_L(u)}{f_\varepsilon(u)}\right) \cdot \frac{\varepsilon}{f_\varepsilon(u) - h_K(u)} \\
&= h_K(u) \cdot \varphi_2\left(\frac{h_L(u)}{h_K(u)}\right) \cdot \lim_{\varepsilon\to 0^+} \frac{\varepsilon}{f_\varepsilon(u) - h_K(u)}.
\end{aligned}$$

Rewrite the above limit as follows:

$$(\varphi_1)'_l(1) \cdot \lim_{\varepsilon\to 0^+} \frac{f_\varepsilon(u) - h_K(u)}{\varepsilon} = \lim_{\varepsilon\to 0^+} f_\varepsilon(u) \cdot \lim_{\varepsilon\to 0^+} \varphi_2\left(\frac{h_L(u)}{f_\varepsilon(u)}\right) = h_K(u) \cdot \varphi_2\left(\frac{h_L(u)}{h_K(u)}\right).$$

Moreover, the convergence is uniform because both $\{f_\varepsilon(u)\}_{\varepsilon>0}$ and $\left\{\varphi_2\left(\frac{h_L(u)}{f_\varepsilon(u)}\right)\right\}_{\varepsilon>0}$ are uniformly convergent and uniformly bounded on S^{n-1}.

If $\varphi_1, \varphi_2 \in \mathscr{D}$ such that $(\varphi_1)'_r(1)$ exists and is nonzero, the proof goes along the same manner. □

The geometric interpretation for the nonhomogeneous Orlicz L_φ mixed volume with $\varphi \in \mathscr{I} \cup \mathscr{D}$ is given in the following theorem. The following result by Aleksandrov [1] is needed: if the sequence $\{f_i\}_{i\geq 1} \subset C^+(S^{n-1})$ converges to $f \in C^+(S^{n-1})$ uniformly on S^{n-1}, then the sequence of $\{K_{f_i}\}_{i\geq 1}$, the Aleksandrov bodies associated to f_i, converges to K_f with respect to the Hausdorff distance.

Theorem 5.1. *Let $K, L \in \mathscr{K}_0$ and $\varphi_1, \varphi_2 \in \mathscr{I}$ be such that $(\varphi_1)'_l(1)$ exists and is positive. Then,*

$$V_{\varphi_2}(K,L) = \frac{(\varphi_1)'_l(1)}{n} \lim_{\varepsilon\to 0^+} \frac{|K_\varepsilon| - |K|}{\varepsilon}. \tag{5.36}$$

For $\varphi_1, \varphi_2 \in \mathscr{D}$, (5.36) holds with $(\varphi_1)'_l(1)$ replaced by $(\varphi_1)'_r(1)$.

Proof. The uniform convergence of f_ε on S^{n-1} implies that K_ε converges to K in the Hausdorff distance as $\varepsilon \to 0^+$. In particular $|K_\varepsilon| \to |K|$ as $\varepsilon \to 0^+$ and S_{K_ε} converges to S_K weakly on S^{n-1}. It follows from (2.5), (5.33), the Minkowski inequality (3.10) and Lemma 5.1 that

$$\begin{aligned}
\liminf_{\varepsilon\to 0^+} |K_\varepsilon|^{\frac{n-1}{n}} \cdot \frac{|K_\varepsilon|^{\frac{1}{n}} - |K|^{\frac{1}{n}}}{\varepsilon} &\geq \liminf_{\varepsilon\to 0^+} \frac{|K_\varepsilon| - V_1(K_\varepsilon, K)}{\varepsilon} \\
&= \liminf_{\varepsilon\to 0^+} \frac{V_1(K_\varepsilon, f_\varepsilon) - V_1(K_\varepsilon, h_K)}{\varepsilon} \\
&= \lim_{\varepsilon\to 0^+} \frac{1}{n} \int_{S^{n-1}} \frac{f_\varepsilon(u) - h_K(u)}{\varepsilon} dS_{K_\varepsilon}(u) \\
&= \frac{1}{(\varphi_1)'_l(1)} V_{\varphi_2}(K,L).
\end{aligned}$$

Similarly, due to $h_{K_\varepsilon} \leq f_\varepsilon$,

$$\begin{aligned}
|K|^{\frac{n-1}{n}} \cdot \limsup_{\varepsilon\to 0^+} \frac{|K_\varepsilon|^{\frac{1}{n}} - |K|^{\frac{1}{n}}}{\varepsilon} &\leq \limsup_{\varepsilon\to 0^+} \frac{V_1(K, K_\varepsilon) - |K|}{\varepsilon} \\
&\leq \limsup_{\varepsilon\to 0^+} \frac{V_1(K, f_\varepsilon) - V_1(K, h_K)}{\varepsilon} \\
&= \lim_{\varepsilon\to 0^+} \frac{1}{n} \int_{S^{n-1}} \frac{f_\varepsilon(u) - h_K(u)}{\varepsilon} dS_K(u) \\
&= \frac{1}{(\varphi_1)'_l(1)} V_{\varphi_2}(K,L).
\end{aligned}$$

Combing the inequalities above, one has

$$(\varphi_1)'_l(1) \cdot |K|^{\frac{n-1}{n}} \cdot \lim_{\varepsilon\to 0^+} \frac{|K_\varepsilon|^{\frac{1}{n}} - |K|^{\frac{1}{n}}}{\varepsilon} = V_{\varphi_2}(K,L).$$

Let $g(\varepsilon) = |K_\varepsilon|^{\frac{1}{n}}$ and $g(0) = |K|^{\frac{1}{n}}$. Then

$$\begin{aligned}
\frac{(\varphi_1)'_l(1)}{n} \cdot \lim_{\varepsilon\to 0^+} \frac{|K_\varepsilon| - |K|}{\varepsilon} &= \frac{(\varphi_1)'_l(1)}{n} \cdot \lim_{\varepsilon\to 0^+} \frac{g(\varepsilon)^n - g(0)^n}{\varepsilon} \\
&= (\varphi_1)'_l(1) \cdot g(0)^{n-1} \lim_{\varepsilon\to 0^+} \frac{g(\varepsilon) - g(0)}{\varepsilon} \\
&= V_{\varphi_2}(K,L).
\end{aligned}$$

The result for $\varphi_1, \varphi_2 \in \mathscr{D}$ follows along the same lines. □

Let $\varphi_1(t) = \varphi_2(t) = t^p$ for $0 \neq p \in \mathbb{R}$. Then formula (5.34) gives the L_p addition of h_K and h_L:

$$f_{p,\varepsilon}(u) = \left[h_K(u)^p + \varepsilon h_L(u)^p\right]^{1/p} \quad \text{for } u \in S^{n-1}.$$

Then the L_p mixed volume of K and L [27, 48] is the first order variation at $\varepsilon = 0$ of the volume of $K_{f_{p,\varepsilon}}$, the Aleksandrov body associated to $f_{p,\varepsilon}$:

$$V_p(K,L) = \frac{p}{n} \cdot \lim_{\varepsilon\to 0^+} \frac{|K_{f_{p,\varepsilon}}| - |K|}{\varepsilon}.$$

5.2 The Orlicz-Petty bodies and the continuity of nonhomogeneous Orlicz geominimal surface areas

In this subsection, we establish the continuity of the nonhomogeneous Orlicz geominimal surface areas, whose proof is similar to that in Section 4. For completeness, we still include the proof with emphasis on the modification.

The nonhomogeneous Orlicz geominimal surface areas can be defined as follows.

Definition 5.2. *Let $K \in \mathscr{K}_0$ be a convex body with the origin in its interior.*
(i) For $\varphi \in \widehat{\Phi}_1 \cup \widehat{\Psi}$, define the nonhomogeneous Orlicz L_φ geominimal surface area of K by

$$G_\varphi^{orlicz}(K) = \inf_{L \in \mathscr{K}_0} \{nV_\varphi(K, \text{vrad}(L^\circ)L)\} = \inf\{nV_\varphi(K,L) : L \in \mathscr{K}_0 \text{ with } |L^\circ| = \omega_n\}.$$

(ii) For $\varphi \in \widehat{\Phi}_2$, define the nonhomogeneous Orlicz L_φ geominimal surface area of K by

$$G_\varphi^{orlicz}(K) = \sup_{L \in \mathscr{K}_0} \{nV_\varphi(K, \text{vrad}(L^\circ)L)\} = \sup\{nV_\varphi(K,L) : L \in \mathscr{K}_0 \text{ with } |L^\circ| = \omega_n\}.$$

Note that the nonhomogeneous Orlicz L_φ geominimal surface area can be defined for more general functions than $\varphi \in \mathscr{I} \cup \mathscr{D}$ (see more details in [49]). However, from Section 4, one sees that the monotonicity of φ is crucial to establish continuity of Orlicz geominimal surface areas. Hence, in this section, we only consider $\varphi \in \widehat{\Phi} \cup \widehat{\Psi}$. We can use the following example to see that $G_\varphi^{orlicz}(\cdot)$ is not homogeneous (see Corollary 3.1 in [49]):

$$G_\varphi^{orlicz}(rB_2^n) = \varphi(1/r) \cdot r^n \cdot n\omega_n.$$

Proposition 5.1. *Let $\{K_i\}_{i\geq 1} \subset \mathscr{K}_0$ and $\{L_i\}_{i\geq 1} \subset \mathscr{K}_0$ be such that $K_i \to K \in \mathscr{K}_0$ and $L_i \to L \in \mathscr{K}_0$ as $i \to \infty$. For $\varphi \in \widehat{\Phi} \cup \widehat{\Psi}$, one has $V_\varphi(K_i, L_i) \to V_\varphi(K, L)$ as $i \to \infty$.*

Proof. As $K_i \to K \in \mathscr{K}_0$ and $L_i \to L \in \mathscr{K}_0$, one can find constants $r, R > 0$ such that these bodies contain rB_2^n and are contained in RB_2^n. Moreover, $h_{K_i} \to h_K$ and $h_{L_i} \to h_L$ uniformly on S^{n-1}. Together with Lemma 4.2 (where we can let $I = [r/R, R/r]$), one has

$$\varphi\left(\frac{h_{L_i}(u)}{h_{K_i}(u)}\right) h_{K_i}(u) \to \varphi\left(\frac{h_L(u)}{h_K(u)}\right) h_K(u) \quad \text{uniformly on} \quad S^{n-1}.$$

Formula (2.5) then implies

$$\int_{S^{n-1}} \varphi\left(\frac{h_{L_i}(u)}{h_{K_i}(u)}\right) h_{K_i}(u) dS_{K_i}(u) \to \int_{S^{n-1}} \varphi\left(\frac{h_L(u)}{h_K(u)}\right) h_K(u) dS_K(u).$$

This completes the proof. □

Similar to Proposition 4.2, the nonhomogeneous Orlicz L_φ geominimal surface area is upper (lower, respectively) semicontinuous on $\mathscr{K}_0$ with respect to the Hausdorff distance for $\varphi \in \widehat{\Phi}_1 \cup \widehat{\Psi}$ (for $\varphi \in \widehat{\Phi}_2$, respectively).

The following proposition states that the Orlicz-Petty bodies exist. See [52] for special results when $\varphi \in \mathscr{I}$ is convex (in this case, $\varphi \in \widehat{\Phi}_1$).

Proposition 5.2. *Let $K \in \mathscr{K}_0$ and $\varphi \in \widehat{\Phi}_1$. There exists a convex body $M \in \mathscr{K}_0$ such that*

$$G_\varphi^{orlicz}(K) = nV_\varphi(K, M) \quad \text{and} \quad |M^\circ| = \omega_n.$$

If in addition φ is convex, such a convex body is unique.

Proof. Let $\varphi \in \widehat{\Phi}_1$. It follows from the definition of $G_\varphi^{orlicz}(K)$ that there exists a sequence $\{M_i\}_{i\geq 1} \subset \mathscr{K}_0$ such that $nV_\varphi(K, M_i) \to G_\varphi^{orlicz}(K)$, $|M_i^\circ| = \omega_n$ and $2V_\varphi(K, B_2^n) \geq V_\varphi(K, M_i)$ for all $i \geq 1$. Let $R_i = \rho_{M_i}(u_i) = \max\{\rho_{M_i}(u) : u \in S^{n-1}\}$ and assume that $\sup_{i\geq 1} R_i = \infty$. Without loss of generality, let $R_i \to \infty$ and $u_i \to v$ (due to the compactness of S^{n-1}) as $i \to \infty$. As before, $h_{M_i}(u) \geq R_i \cdot \langle u, u_i\rangle_+$ for all $u \in S^{n-1}$.

Let $D > 0$ be given. By Definition 5.2, Fatou's lemma, continuity of φ, (4.26) and the fact that φ is increasing, one has

$$\begin{aligned} 2V_\varphi(K, B_2^n) &\geq \lim_{i\to\infty} \frac{1}{n} \int_{S^{n-1}} \varphi\left(\frac{h_{M_i}(u)}{h_K(u)}\right) h_K(u) dS_K(u) \\ &\geq \liminf_{i\to\infty} \frac{1}{n} \int_{S^{n-1}} \varphi\left(\frac{R_i \cdot \langle u, u_i\rangle_+}{R_K}\right) r_K dS_K(u) \\ &\geq \liminf_{i\to\infty} \frac{1}{n} \int_{S^{n-1}} \varphi\left(\frac{D \cdot \langle u, u_i\rangle_+}{R_K}\right) r_K dS_K(u) \\ &\geq \frac{1}{n} \int_{S^{n-1}} \varphi\left(\frac{D \cdot \langle u, v\rangle_+}{R_K}\right) r_K dS_K(u) \\ &\geq \varphi\left(\frac{D}{j_0 \cdot R_K}\right) \cdot \frac{r_K}{n} \cdot \frac{c_1}{2}. \end{aligned}$$

A contradiction (i.e., $2V_\varphi(K, B_2^n) > \infty$) is obtained by letting $D \to \infty$ and the fact that $\lim_{t\to\infty} \phi(t) = \infty$ (as $\varphi \in \widehat{\Phi}_1$ is increasing and unbounded). That is, $\{M_i\}_{i\geq 1}$ is uniformly bounded, and a convergent subsequence of $\{M_i\}_{i\geq 1}$, which will not be relabeled, can be found due to the Blaschke selection theorem. Let M be the limit of $\{M_i\}_{i\geq 1}$ and then $M \in \mathscr{K}_0$ due to Lemma 2.1. Moreover, $|M_i^\circ| = \omega_n$ for all $i \geq 1$ implies $|M^\circ| = \omega_n$. It follows from Proposition 5.1 that M is the desired body such that $G_\varphi^{orlicz}(K) = nV_\varphi(K, M)$ and $|M^\circ| = \omega_n$.

For uniqueness, let $M_1, M_2 \in \mathscr{K}_0$ be such that $|M_1^\circ| = |M_2^\circ| = \omega_n$ and

$$V_\varphi(K, M_1) = \inf_{L\in\mathscr{K}_0} \{V_\varphi(K, \mathrm{vrad}(L^\circ)L)\} = V_\varphi(K, M_2).$$

Let $M = \frac{M_1+M_2}{2}$. Then $\mathrm{vrad}(M^\circ) \leq 1$ with equality if and only if $M_1 = M_2$ (see inequality (4.29)). The fact that φ is convex yields that $V_\varphi(K, M) \leq V_\varphi(K, M_1)$. Therefore, if $M_1 \neq M_2$ (hence $\mathrm{vrad}(M^\circ) < 1$), the fact that φ is strictly increasing implies that

$$nV_\varphi(K, \mathrm{vrad}(M^\circ)M) < nV_\varphi(K, M) \leq nV_\varphi(K, M_1) = nV_\varphi(K, \mathrm{vrad}(M_1^\circ)M_1).$$

This contradicts with the minimality of M_1 and hence the uniqueness follows. □

Definition 5.3. *Let $K \in \mathscr{K}_0$ and $\varphi \in \widehat{\Phi}_1$. A convex body $M \in \mathscr{K}_0$ is said to be an L_φ Orlicz-Petty body of K, if $M \in \mathscr{K}_0$ satisfies*

$$G_\varphi^{orlicz}(K) = nV_\varphi(K, M) \quad \text{and} \quad |M^\circ| = \omega_n.$$

Denote by $T_\varphi K$ the set of all L_φ Orlicz-Petty bodies of K.

Let $\varphi \in \widehat{\Phi}_1$. The set $T_\varphi K$ has many properties same as those for $\widehat{T}_\phi K$. For instance, $T_\varphi K$ is $SL(n)$-invariant: $T_\varphi(AK) = A(T_\varphi K)$ for all $A \in SL(n)$. Moreover, if K is a polytope, then any convex body in $T_\varphi K$ must be a polytope with faces parallel to those of K. If in addition φ is convex, $|T_\varphi K| \cdot |(T_\varphi K)^\circ| \leq |K| \cdot |K^\circ|$.

The continuity of the nonhomogeneous Orlicz L_φ geominimal surface areas is proved in the following theorem. See [52] for special results when $\varphi \in \mathscr{I}$ is convex (in this case, $\varphi \in \widehat{\Phi}_1$).

Theorem 5.2. *If $\varphi \in \widehat{\Phi}_1$, then the functional $G_\varphi^{orilcz}(\cdot)$ on $\mathscr{K}_0$ is continuous with respect to the Hausdorff distance.*

Proof. Let $\varphi \in \widehat{\Phi}_1$. The upper semicontinuity has been stated after Proposition 5.1. To conclude the continuity, it is enough to prove the lower semicontinuity.

To this end, we need the following statement: if $K_i \to K$ as $i \to \infty$ with $K_i, K \in \mathscr{K}_0$ for all $i \geq 1$, there exists a constant $R' > 0$ such that $M_i \subset R'B_2^n$ for all (given) $M_i \in T_\varphi K_i$, $i \geq 1$. The proof basically follows the idea in Lemma 4.4. In fact, assume that there is no constant R' such that $M_i \subset R'B_2^n$ for $i \geq 1$. Let $R_i = \rho_{M_i}(u_i) = \max\{\rho_{M_i}(u) : u \in S^{n-1}\}$. It follows from the Blaschke selection theorem and the compactness of S^{n-1} that there is a subsequence of $\{K_i\}_{i\geq 1}$, which will not be relabeled, such that, $R_i \to \infty$ and $u_i \to v$ as $i \to \infty$. For any given $\varepsilon > 0$, one has

$$
\begin{aligned}
V_\varphi(K, B_2^n) &= \lim_{i\to\infty} V_\varphi(K_i, B_2^n) \\
&\geq \lim_{i\to\infty} \frac{1}{n}\int_{S^{n-1}} \varphi\left(\frac{h_{M_i}(u)}{h_{K_i}(u)}\right) h_{K_i}(u)dS_{K_i}(u) \\
&\geq \lim_{i\to\infty} \frac{1}{n}\int_{S^{n-1}} \varphi\left(\frac{\langle u, u_i\rangle_+}{(R_i^{-1}+\varepsilon)\cdot R}\right) rdS_{K_i}(u) \\
&= \frac{1}{n}\int_{S^{n-1}} \varphi\left(\frac{\langle u, v\rangle_+}{\varepsilon\cdot R}\right) rdS_K(u) \\
&\geq \frac{1}{n}\int_{\Sigma_{j_0}(v)} \varphi\left(\frac{1}{\varepsilon\cdot j_0\cdot R}\right) rdS_K(u) \\
&= \varphi\left(\frac{1}{\varepsilon\cdot j_0\cdot R}\right)\cdot\frac{r}{n}\cdot\frac{c_1}{2}
\end{aligned}
$$

where $r, R > 0$ are constants such that $rB_2^n \subset K_i, K \subset RB_2^n$ for all $i \geq 1$. A contradiction (i.e., $V_\varphi(K, B_2^n) \geq \infty$) is obtained by taking $\varepsilon \to 0^+$ and the fact that $\lim_{t\to\infty}\varphi(t) = \infty$.

Now let us prove the lower semicontinuity of $G_\varphi^{orilcz}(\cdot)$ and the continuity then follows. Let $l = \liminf_{i\to\infty} G_\varphi^{orlicz}(K_i)$. There is a subsequence of $\{K_i\}_{i\geq1}$, say $\{K_{i_k}\}_{k\geq1}$, such that, $l = \lim_{k\to\infty} G_\varphi^{orlicz}(K_{i_k})$. From the arguments in the previous paragraph, one sees that $\{M_{i_k}\}_{k\geq1}$ is uniformly bounded. The Blaschke selection theorem and Lemma 2.1 imply that there exists a subsequence of $\{M_{i_k}\}_{k\geq1}$, which will not be relabeled, and a convex body $M \in \mathscr{K}_0$ such that $M_{i_k} \to M$ as $k \to \infty$ and $|M^\circ| = \omega_n$. Proposition 5.1 yields

$$G_\varphi^{orlicz}(K_{i_k}) = nV_\varphi(K_{i_k}, M_{i_k}) \to nV_\varphi(K, M) \geq G_\varphi^{orlicz}(K) \quad \text{as} \quad k\to\infty.$$

Hence, $\liminf_{i\to\infty} G_\varphi^{orlicz}(K_i) \geq G_\varphi^{orlicz}(K)$ and this completes the proof. □

Similar to Proposition 4.6, we can prove that if $\varphi \in \widehat{\Phi}_1$ is convex, then $T_\varphi : \mathscr{K}_0 \mapsto \mathscr{K}_0$ is continuous with respect to the Hausdorff distance.

6 The Orlicz geominimal surface areas with respect to $\mathscr{K}_e$ and the related Orlicz-Petty bodies

In Sections 4 and 5, we prove the existence of the Orlicz-Petty bodies and the continuity for the Orlicz geominimal surface areas under the condition $\phi \in \widehat{\Phi}_1$. For $\phi \in \widehat{\Phi}_2 \cup \widehat{\Psi}$, our method fails. In fact, when $\phi \in \widehat{\Phi}_2$, we can prove the following result.

Proposition 6.1. *Let $\phi, \varphi \in \widehat{\Phi}_2$ and $K \in \mathscr{K}_0$ be a polytope. Then*

$$\widehat{G}_{\phi}^{orlicz}(K) = 0 \quad \text{and} \quad G_{\varphi}^{orlicz}(K) = \infty.$$

Proof. Let $\phi \in \widehat{\Phi}_2$ and $K \in \mathscr{K}_0$ be a polytope. Then the surface area measure of K is concentrated on finite directions, say $\{u_1, \cdots, u_m\} \subset S^{n-1}$. As $\widehat{G}_{\phi}^{orlicz}(K)$ is $SL(n)$ invariant, we can assume that, without loss of generality, $S_K(u_1) > 0$ and $u_1 = e_1$ with $\{e_1, \cdots, e_n\}$ the canonical orthonormal basis of $\mathbb{R}^n$.

Let $\epsilon > 0$ and $A_\epsilon = \operatorname{diag}(\epsilon, b_2, \cdots, b_n)$ with constants $b_2, \cdots, b_n > 0$ such that $b_2 \cdots b_n = 1/\epsilon$. Clearly $\det A_\epsilon = 1$ and then $A_\epsilon \in SL(n)$. Let $L_\epsilon = A_\epsilon K \in \mathscr{K}_0$ and $\lambda_\epsilon = \widehat{V}_\phi(K, L_\epsilon)$. Then, $h_{L_\epsilon}(e_1) = \epsilon \cdot h_K(e_1)$ for all $\epsilon > 0$ and

$$\begin{aligned}
1 &= \int_{S^{n-1}} \phi\left(\frac{n|K| \cdot h_{L_\epsilon}(u)}{\lambda_\epsilon \cdot h_K(u)}\right) d\widetilde{V}_K(u) \\
&= \frac{1}{n|K|} \cdot \sum_{i=1}^{m} \phi\left(\frac{n|K| \cdot h_{L_\epsilon}(u_i)}{\lambda_\epsilon \cdot h_K(u_i)}\right) h_K(u_i) S_K(u_i) \\
&\geq \frac{1}{n|K|} \cdot \phi\left(\frac{n|K| \cdot h_{L_\epsilon}(e_1)}{\lambda_\epsilon \cdot h_K(e_1)}\right) h_K(e_1) S_K(e_1) \\
&= \frac{1}{n|K|} \cdot \phi\left(\frac{n|K| \cdot \epsilon}{\lambda_\epsilon}\right) h_K(e_1) S_K(e_1).
\end{aligned}$$

Assume that $\inf_{\epsilon>0} \lambda_\epsilon > 0$. There exists a constant $c > 0$ such that $\lambda_\epsilon > c$ for all $\epsilon > 0$. The above inequality and the fact that $\phi \in \widehat{\Phi}_2$ is decreasing imply

$$1 \geq \frac{1}{n|K|} \cdot \phi\left(\frac{n|K| \cdot \epsilon}{\lambda_\epsilon}\right) h_K(e_1) S_K(e_1) \geq \frac{1}{n|K|} \cdot \phi\left(\frac{n|K| \cdot \epsilon}{c}\right) h_K(e_1) S_K(e_1).$$

Recall that $\lim_{t\to 0} \phi(t) = \infty$ as $\phi \in \widehat{\Phi}_2 \subset \mathscr{D}$. A contradiction (i.e., $1 \geq \infty$) is obtained if we let $\epsilon \to 0^+$. This means that

$$\inf_{\epsilon>0} \lambda_\epsilon = \inf_{\epsilon>0} \widehat{V}_\phi(K, L_\epsilon) = 0.$$

On the other hand, $\operatorname{vrad}(L_\epsilon^\circ) = \operatorname{vrad}(K^\circ)$ for all $\epsilon > 0$. This yields that

$$0 \leq \widehat{G}_{\phi}^{orlicz}(K) = \inf_{L \in \mathscr{K}_0} \{\widehat{V}_\phi(K, \operatorname{vrad}(L^\circ) L)\} \leq \inf_{\epsilon>0} \{\widehat{V}_\phi(K, \operatorname{vrad}(L_\epsilon^\circ) L_\epsilon)\} = 0.$$

For the nonhomogeneous Orlicz L_φ geominimal surface area, the proof follows along the same lines. In fact, for all $\epsilon > 0$,

$$\begin{aligned}
V_\varphi(K, \operatorname{vrad}(L_\epsilon^\circ) L_\epsilon) &= \frac{1}{n} \int_{S^{n-1}} \varphi\left(\frac{\operatorname{vrad}(K^\circ) h_{L_\epsilon}(u)}{h_K(u)}\right) h_K(u)\, dS_K(u) \\
&\geq \frac{1}{n} \cdot \varphi(\operatorname{vrad}(K^\circ) \cdot \epsilon) \cdot h_K(e_1) \cdot S_K(e_1).
\end{aligned}$$

and the desired result follows

$$G_{\varphi}^{orlicz}(K) = \sup_{L \in \mathscr{K}_0} \{n V_\varphi(K, \operatorname{vrad}(L^\circ) L)\} \geq \sup_{\epsilon>0} \{n V_\varphi(K, \operatorname{vrad}(L_\epsilon^\circ) L_\epsilon)\} = \infty.$$

This completes the proof. $\square$

An immediate consequence of Proposition 6.1 is that for $\phi \in \widehat{\Phi}_2$, the homogeneous Orlicz L_ϕ geominimal surface area is not continuous but only upper semicontinuous on $\mathscr{K}_0$ with respect to the Hausdorff distance. To this end, let $K = B_2^n$. One can find a sequence of polytopes $\{P_i\}_{i\geq 1}$ such that $P_i \to B_2^n$ as $i \to \infty$ with respect to the Hausdorff distance. However, one cannot expect to have $\widehat{G}_\phi^{orlicz}(P_i) \to \widehat{G}_\phi^{orlicz}(B_2^n)$ as $i \to \infty$, since $\widehat{G}_\phi^{orlicz}(P_i) = 0$ for all $i \geq 1$ and $\widehat{G}_\phi^{orlicz}(B_2^n) = n\omega_n > 0$. Moreover, if $\phi \in \widehat{\Phi}_2$ and K is a polytope, the Orlicz-Petty bodies for K do not exist (i.e., $\widehat{T}_\phi K = \emptyset$). This is because $\widehat{G}_\phi^{orlicz}(K) = 0$, but $\widehat{V}_\phi(K, M) > 0$ for $M \in \widehat{T}_\phi K \subset \mathscr{K}_0$ if $\widehat{T}_\phi K \neq \emptyset$. Similarly, the nonhomogeneous Orlicz L_φ geominimal surface area is not continuous but only lower semicontinuous on $\mathscr{K}_0$ with respect to the Hausdorff distance as $G_\varphi^{orlicz}(P_i) = \infty$ for all $i \geq 1$. Moreover, if $\varphi \in \widehat{\Phi}_2$ and K is a polytope, the Orlicz-Petty bodies for K do not exist.

Our method to prove the existence of the Orlicz-Petty bodies in Sections 4 and 5 heavily relies on the value of the Orlicz mixed volumes of K and line segements $[0, v] = \{tv : t \in [0, 1]\}$ for $v \in S^{n-1}$ (for instance $\widehat{V}_\phi(K, [0, v])$ in Section 4). However, $\widehat{V}_\phi(K, [0, v])$ are always 0 for all $v \in S^{n-1}$ if $\phi \in \mathscr{D}$. It seems impossible to prove the existence of the Orlicz-Petty bodies for $\phi \in \mathscr{D}$ and for general (even with enough smoothness) convex bodies $K \in \mathscr{K}_0$.

When $\phi(t) = t^p$ for $p \in (-1, 0)$, one can calculate that, for all $v \in S^{n-1}$ (see e.g., [53]),

$$\int_{S^{n-1}} |\langle u, v\rangle|^p d\sigma(u) = C_{n,p}, \tag{6.37}$$

where $C_{n,p} > 0$ is a finite constant depending on n and p. Note that the integrand includes $|\langle u, v\rangle|$ rather than $\langle u, v\rangle_+$. This suggests that our method in Sections 4 and 5 may still work for smooth enough $K \in \mathscr{K}_0$ and a modified Orlicz geominimal surface area.

Our modified Orlicz geominimal surface area is given by the following definition. Recall that $\mathscr{K}_e$ is the set of all origin-symmetric convex bodies.

Definition 6.1. *Let $K \in \mathscr{K}_0$ and $\phi \in \widehat{\Phi}$. The homogeneous Orlicz L_ϕ geominimal surface area of K with respect to $\mathscr{K}_e$ is defined by*

$$\widehat{G}_\phi^{orlicz}(K, \mathscr{K}_e) = \inf\{\widehat{V}_\phi(K, L) : \ L \in \mathscr{K}_e \quad \text{with} \quad |L^\circ| = \omega_n\}. \tag{6.38}$$

While if $\phi \in \widehat{\Psi}$, $\widehat{G}_\phi^{orlicz}(\cdot, \mathscr{K}_e)$ can be defined similarly with "$\inf$" replaced by "$\sup$".

Properties for $\widehat{G}_\phi^{orlicz}(\cdot, \mathscr{K}_e)$, such as affine invriance, homogeneity, affine isoperimetric inequalities (requiring $K \in \mathscr{K}_e$), and continuity if $\phi \in \widehat{\Phi}_1$, are the same as those for $\widehat{G}_\phi^{orlicz}(\cdot)$ proved in Sections 3 and 4. The details are left for readers.

In the rest of this section, we will prove the existence of the Orlicz-Petty bodies and the "continuity" of $\widehat{G}_\phi^{orlicz}(\cdot, \mathscr{K}_e)$ for certain $\phi \in \widehat{\Phi}_2$. We will work on convex bodies $K \in C_+^2$. A convex body K is said to be in C_+^2 if K has C^2 boundary and positive curvature function f_K. Hereafter, the curvature function of K is the function $f_K : S^{n-1} \to (0, \infty)$ such that

$$f_K(u) = \frac{dS_K(u)}{d\sigma(u)} \quad \text{for} \quad u \in S^{n-1}.$$

Let $\phi \in \widehat{\Phi}_2$ be such that for all $x \in \mathbb{R}^n$,

$$\int_{S^{n-1}} \phi(|\langle u, x\rangle|)\, d\sigma(u) < \infty \quad \text{and} \quad \lim_{\|x\| \to \infty} \int_{S^{n-1}} \phi(|\langle u, x\rangle|)\, d\sigma(u) = 0. \tag{6.39}$$

Note that $\phi(t) = t^p$ for $p \in (-1, 0)$ satisfies the condition (6.39) due to formula (6.37). Moreover, (6.39) is equivalent to, for all $s > 0$,

$$\int_{S^{n-1}} \phi(s \cdot |\langle u, e_1 \rangle|)\, d\sigma(u) < \infty \quad \text{and} \quad \lim_{s\to\infty} \int_{S^{n-1}} \phi(s \cdot |\langle u, e_1 \rangle|)\, d\sigma(u) = 0.$$

Proposition 6.2. *Let $K \in C^2_+$ and $\phi \in \widehat{\Phi}_2$ satisfy (6.39). Then there exists $M \in \mathscr{K}_e$ such that*

$$\widehat{G}^{orlicz}_{\phi}(K, \mathscr{K}_e) = \widehat{V}_{\phi}(K, M) \quad \text{and} \quad |M^{\circ}| = \omega_n.$$

Proof. Let $K \in C^2_+$. Its curvature function f_K is continuous on S^{n-1} and hence has maximum which will be denoted by $F_K < \infty$. By (6.38), for $\phi \in \widehat{\Phi}_2$, there exists a sequence $\{M_i\}_{i\geq 1} \subset \mathscr{K}_e$ such that $\widehat{V}_{\phi}(K, M_i) \to \widehat{G}^{orlicz}_{\phi}(K, \mathscr{K}_e)$ as $i \to \infty$, $2\widehat{V}_{\phi}(K, B^n_2) \geq \widehat{V}_{\phi}(K, M_i)$ and $|M^{\circ}_i| = \omega_n$ for all $i \geq 1$. Again let $R_i = \rho_{M_i}(u_i) = \max\{\rho_{M_i}(u) : u \in S^{n-1}\}$. Then $h_{M_i}(u) \geq R_i \cdot |\langle u, u_i \rangle|$ for all $u \in S^{n-1}$ and all $i \geq 1$. Corollary 3.2, together with (6.39) and the fact that $\phi \in \widehat{\Phi}_2$ is decreasing, implies that, for all $i \geq 1$,

$$\begin{aligned} 1 &= \int_{S^{n-1}} \phi\left(\frac{n|K| \cdot h_{M_i}(u)}{\widehat{V}_{\phi}(K, M_i) \cdot h_K(u)}\right) d\widetilde{V}_K(u) \\ &\leq \int_{S^{n-1}} \phi\left(\frac{n|K| \cdot R_i \cdot |\langle u, u_i \rangle|}{2\widehat{V}_{\phi}(K, B^n_2) \cdot h_K(u)}\right) \cdot \frac{h_K(u) f_K(u)}{n|K|}\, d\sigma(u) \\ &\leq \int_{S^{n-1}} \phi\left(\frac{n|K| \cdot R_i \cdot |\langle u, u_i \rangle|}{2\widehat{V}_{\phi}(K, B^n_2) \cdot R_K}\right) \cdot \frac{R_K F_K}{n|K|}\, d\sigma(u) < \infty. \end{aligned}$$

Assume that $\sup_{i\geq 1} R_i = \infty$. Without loss of generality, let $\lim_{i\geq 1} R_i = \infty$ and

$$x_i = \frac{n|K| \cdot R_i \cdot u_i}{2\widehat{V}_{\phi}(K, B^n_2) \cdot R_K}.$$

Then $\lim_{i\to\infty} \|x_i\| = \infty$. It follows from (6.39) that

$$1 \leq \frac{R_K F_K}{n|K|} \cdot \lim_{i\to\infty} \int_{S^{n-1}} \phi\left(|\langle u, x_i \rangle|\right) d\sigma(u) = 0.$$

This is a contradiction and hence $\sup_{i\geq 1} R_i < \infty$. In other words, the sequence $\{M_i\}_{i\geq 1}$ is uniformly bounded. By the Blaschke selection theorem, there exists a convergent subsequence of $\{M_i\}_{i\geq 1}$ (still denoted by $\{M_i\}_{i\geq 1}$) and a convex body $M \in \mathscr{K}$ such that $M_i \to M$ as $i \to \infty$. As $|M^{\circ}_i| = \omega_n$ for all $i \geq 1$, Lemma 2.1 gives $M \in \mathscr{K}_e$ and $|M^{\circ}| = \omega_n$. Proposition 4.1 concludes that M is the desired body. □

Definition 6.2. *Let $K \in C^2_+$ and $\phi \in \widehat{\Phi}_2$ satisfy (6.39). A convex body $M \in \mathscr{K}_e$ is said to be an L_{ϕ} Orlicz-Petty body of K with respect to $\mathscr{K}_e$, if $M \in \mathscr{K}_e$ satisfies*

$$\widehat{G}^{orlicz}_{\phi}(K, \mathscr{K}_e) = \widehat{V}_{\phi}(K, M) \quad \text{and} \quad |M^{\circ}| = \omega_n.$$

Denote by $\widehat{T}_{\phi}(K, \mathscr{K}_e)$ the set of all such bodies.

Theorem 6.1. *Let $\phi \in \widehat{\Phi}_2$ satisfy (6.39). Assume that $\{K_i\}_{i\geq 0} \subset C^2_+$ such that $K_i \to K_0$ as $i \to \infty$ and $\{f_{K_i}\}_{i\geq 1}$ is uniformly bounded on S^{n-1}. Then*

$$\lim_{i\to\infty} \widehat{G}^{orlicz}_{\phi}(K_i, \mathscr{K}_e) = \widehat{G}^{orlicz}_{\phi}(K_0, \mathscr{K}_e).$$

Proof. As $K_i \to K_0$, there exist $r, R > 0$ such that $rB_2^n \subset K_i \subset RB_2^n$ for all $i \geq 0$. We claim that there is a finite constant $R' > 0$ such that $M_i \subset R'B_2^n$ for all (given) $M_i \in \widehat{T}_\phi(K_i, \mathscr{K}_e)$, $i \geq 1$. Suppose that there is no such finite constant. Without loss of generality, assume that $\lim_{i\to\infty} R_i = \infty$ and $u_i \to v$ (due to the compactness of S^{n-1}) as $i \to \infty$, where again

$$R_i = \rho_{M_i}(u_i) = \max\{\rho_{M_i}(u) : u \in S^{n-1}\}.$$

As before, $h_{M_i}(u) \geq R_i \cdot |\langle u, u_i\rangle|$ for all $u \in S^{n-1}$ and $i \geq 1$. Corollary 3.2, together with (6.39) and the fact that $\phi \in \widehat{\Phi}_2$ is decreasing, implies that, for all $i \geq 1$,

$$\begin{aligned} 1 &= \int_{S^{n-1}} \phi\left(\frac{n|K_i| \cdot h_{M_i}(u)}{\widehat{V}_\phi(K_i, M_i) \cdot h_{K_i}(u)}\right) d\widetilde{V}_{K_i}(u) \\ &\leq \int_{S^{n-1}} \phi\left(\frac{n|K_i| \cdot R_i \cdot |\langle u, u_i\rangle|}{\widehat{V}_\phi(K_i, B_2^n) \cdot h_{K_i}(u)}\right) \cdot \frac{h_{K_i}(u) f_{K_i}(u)}{n|K_i|}\, d\sigma(u) \\ &\leq \int_{S^{n-1}} \phi\left(\frac{r^{n+1} \cdot R_i \cdot |\langle u, u_i\rangle|}{R^{n+1}}\right) \cdot \frac{R \cdot F_0}{n\omega_n \cdot r^n}\, d\sigma(u), \end{aligned}$$

where the last inequality follows from Lemma 4.1 and F_0 is the uniform bound of $\{f_{K_i}\}_{i\geq 1}$ on S^{n-1} (i.e., $F_0 = \sup_{i\geq 1} \sup_{u\in S^{n-1}} f_{K_i}(u)$). As in the proof of Proposition 6.2, one gets

$$1 \leq \lim_{i\to\infty} \int_{S^{n-1}} \phi\left(\frac{r^{n+1} \cdot R_i \cdot |\langle u, u_i\rangle|}{R^{n+1}}\right) \cdot \frac{R \cdot F_0}{n\omega_n \cdot r^n}\, d\sigma(u) = 0,$$

which is a contradiction. Hence there is a finite constant $R' > 0$ such that $M_i \subset R'B_2^n$ for all (given) $M_i \in \widehat{T}_\phi(K_i, \mathscr{K}_e)$, $i \geq 1$. In other words, $\{M_i\}_{i\geq 1}$ is uniformly bounded.

Let $l = \liminf_{i\to\infty} \widehat{G}_\phi^{orlicz}(K_i, \mathscr{K}_e)$. Clearly, one can find a subsequence $\{K_{i_k}\}_{k\geq 1}$ such that $l = \lim_{k\to\infty} \widehat{G}_\phi^{orlicz}(K_{i_k}, \mathscr{K}_e)$. By the Blaschke selection theorem and Lemma 2.1, there exists a subsequence of $\{M_{i_k}\}_{k\geq 1}$ (still denoted by $\{M_{i_k}\}_{k\geq 1}$) and a body $M \in \mathscr{K}_e$, such that, $M_{i_k} \to M$ as $k \to \infty$ and $|M^\circ| = \omega_n$. Proposition 4.1 then yields

$$\widehat{G}_\phi^{orlicz}(K_{i_k}, \mathscr{K}_e) = \widehat{V}_\phi(K_{i_k}, M_{i_k}) \to \widehat{V}_\phi(K_0, M) \quad \text{as} \quad k \to \infty.$$

By (6.38), one has

$$\widehat{G}_\phi^{orlicz}(K_0, \mathscr{K}_e) \leq \widehat{V}_\phi(K_0, M) = \lim_{k\to\infty} \widehat{G}_\phi^{orlicz}(K_{i_k}, \mathscr{K}_e) = \liminf_{i\to\infty} \widehat{G}_\phi^{orlicz}(K_i, \mathscr{K}_e).$$

On the other hand, for any given $\epsilon > 0$, by (6.38) and Proposition 4.1, there exists a convex body $L_\epsilon \in \mathscr{K}_e$ such that $|L_\epsilon^\circ| = \omega_n$ and

$$\widehat{G}_\phi^{orlicz}(K_0, \mathscr{K}_e) + \epsilon > \widehat{V}_\phi(K_0, L_\epsilon) = \limsup_{i\to\infty} \widehat{V}_\phi(K_i, L_\epsilon) \geq \limsup_{i\to\infty} \widehat{G}_\phi^{orlicz}(K_i, \mathscr{K}_e).$$

By letting $\epsilon \to 0$, one gets $\widehat{G}_\phi^{orlicz}(K_0, \mathscr{K}_e) \geq \limsup_{i\to\infty} \widehat{G}_\phi^{orlicz}(K_i, \mathscr{K}_e)$ and the desired limit follows. □

Let $K \in \mathscr{K}_0$ and $\varphi \in \widehat{\Phi}_1 \cup \widehat{\Psi}$. The nonhomogeneous Orlicz L_φ geominimal surface area of K with respect to $\mathscr{K}_e$ can be defined by

$$G_\varphi^{orlicz}(K, \mathscr{K}_e) = \inf\{nV_\varphi(K, L) : \ L \in \mathscr{K}_e \quad \text{with} \quad |L^\circ| = \omega_n\}.$$

While if $\varphi \in \widehat{\Phi}_2$, $G_{\varphi}^{orlicz}(\cdot, \mathscr{K}_e)$ can be defined similarly with " inf" replaced by " sup". Analogous results to Proposition 6.2 and Theorem 6.1 can be proved for $G_{\varphi}^{orlicz}(\cdot, \mathscr{K}_e)$ if $\varphi \in \widehat{\Phi}_2$ satisfies (6.39). We leave the details for readers.

Acknowledgments. The first author is supported by AARMS, NSERC, NSFC (No. 11501185) and the Doctor Starting Foundation of Hubei University for Nationalities (No. MY2014B001). The third author is supported by a NSERC grant.

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