Sharp geometric inequalities for the general p-affine capacity *

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Abstract

In this article, we propose the notion of the general p-affine capacity and prove some basic properties for the general p-affine capacity, such as affine invariance and monotonicity. The newly proposed general p-affine capacity is compared with several classical geometric quantities, e.g., the volume, the p-variational capacity and the p-integral affine surface area. Consequently, several sharp geometric inequalities for the general p-affine capacity are obtained. These inequalities extend and strengthen many well-known (affine) isoperimetric and (affine) isocapacitary inequalities.

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1 Introduction

Many objects of interest and fundamental results in convex geometry are related to the L_p projection bodies [22, 24, 25, 33]. For $p \geq 1$, the L_p projection body of a convex body (i.e., a compact convex subset with nonempty interior) $K \subset \mathbb{R}^n$ containing the origin in its interior is determined by its support function $h_{\Pi_p(K)} : S^{n-1} \to \mathbb{R}$, whose definition is formulated as follows (up to a multiplicative constant): for any $\theta \in S^{n-1}$,

$$h_{\Pi_p(K)}(\theta) = \left(\int_{\partial K} \left(\frac{\theta \cdot \nu_K(x)}{2} \right)^p \cdot |x \cdot \nu_K(x)|^{1-p} \, d\mathscr{H}^{n-1}(x) \right)^{\frac{1}{p}}$$

with ν_K the unit outer normal vector of K at $x \in \partial K$ and $\mathscr{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure of ∂K, the boundary of K (see Section 2 for details on the notations). Define $\Phi_p(K)$, the p-integral affine surface area of K, by

$$\Phi_p(K) = \left(\int_{S^{n-1}} \left[h_{\Pi_p(K)}(u) \right]^{-n} du \right)^{-\frac{p}{n}}$$

where du is the normalized spherical measure on the unit sphere S^{n-1}. Let B_n be the unit Euclidean ball in $\mathbb{R}^n$ and $V(K)$ denote the volume of K. The following L_p affine isoperimetric inequality for the p-integral affine surface area holds [22, 24, 25, 33, 47]: for $p \geq 1$ and for K a convex body with the origin in its interior,

$$\left(\frac{\Phi_p(K)}{\Phi_p(B_n)} \right)^{\frac{1}{n-p}} \geq \left(\frac{V(K)}{V(B_n)} \right)^{\frac{1}{n}} \tag{1.1}$$

with equality if and only if $K = TB_n$ if $p > 1$ and $K = TB_n + x_0$ if $p = 1$ for some invertible linear transform T on $\mathbb{R}^n$ and some $x_0 \in \mathbb{R}^n$. Note that inequality (1.1) is invariant under the

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volume preserving linear transforms and hence is stronger than the well-known L_p isoperimetric inequality [8, 23, 34]:

$$\left(\frac{S_p(K)}{S_p(B_n)}\right)^{\frac{1}{n-p}} \geq \left(\frac{V(K)}{V(B_n)}\right)^{\frac{1}{n}} \tag{1.2}$$

with equality if and only if K is an Euclidean ball in $\mathbb{R}^n$ (if $p > 1$, the center needs to be at the origin). Here $S_p(K)$ is the p-surface area of K and can be formulated by

$$S_p(K) = \int_{\partial K} |x \cdot \nu_K(x)|^{1-p} \, d\mathscr{H}^{n-1}(x). \tag{1.3}$$

It is well known that inequality (1.2) can be strengthened by the isocapacitary inequality related to the p-variational capacity. For a compact set $K \subset \mathbb{R}^n$, its p-variational capacity, denoted by $C_p(K)$, can be formulated by (see e.g. [6, 29, 30])

$$C_p(K) = \inf\Big\{ \int_{\mathbb{R}^n} |\nabla f|^p \, dx : \quad f \in C_c^\infty \text{ and } f \geq 1 \text{ on } K\Big\},$$

where ∇f denotes the gradient of f and C_c^∞ is the set of smooth functions with compact supports in $\mathbb{R}^n$. The p-variational capacity is an important geometric invariant which has close connection with the p-Laplacian partial differential equation and has important applications in many areas, e.g., analysis, mathematical physics and partial differential equations (see e.g., [6, 29, 30] and references therein). In particular, the Brunn-Minkowski type inequalities and the Hadamard variational formulas for the p-variational capacity have been established in, e.g., [1, 2, 4, 5, 17, 18, 19, 49]. The following inequality for the p-variational capacity holds [22, 29]: for $p \in [1, n)$ and for K being a Lipschitz star body with the origin in its interior,

$$\left(\frac{S_p(K)}{S_p(B_n)}\right)^{\frac{1}{n-p}} \geq \left(\frac{C_p(K)}{C_p(B_n)}\right)^{\frac{1}{n-p}} \geq \left(\frac{V(K)}{V(B_n)}\right)^{\frac{1}{n}}. \tag{1.4}$$

The p-variational capacity behaves rather similar to the p-surface area and is lack of the affine invariance. Very recently, Xiao [42, 43] introduced an affine relative of the p-variational capacity and named it as the p-affine capacity. This new notion is denoted by $C_{p,0}(K)$ in this article and its definition is equivalent to, as proved in Section 3, the following: for $p \in [1, n)$ and for K a compact set in $\mathbb{R}^n$,

$$C_{p,0}(K) = \inf\Big\{\mathscr{H}_p(f) : \ f \in C_c^\infty \text{ and } f \geq 1 \ \text{ on } \ K\Big\},$$

where $\mathscr{H}_p(f)$ is the p-affine energy of f:

$$\mathscr{H}_p(f) = \bigg(\int_{S^{n-1}} \Big(\int_{\mathbb{R}^n} \frac{|\langle u \cdot \nabla f\rangle|^p}{2} dx\Big)^{-\frac{n}{p}} du\bigg)^{-\frac{p}{n}}.$$

The following affine isocapacity inequality was also established in [43, Theorems 3.2 and 3.5] and [42, Theorems 1.3' and 1.4']: for $p \in [1, n)$ and for K an origin-symmetric convex body, one has

$$\left(\frac{\Phi_p(K)}{\Phi_p(B_n)}\right)^{\frac{1}{n-p}} \geq \left(\frac{C_{p,0}(K)}{C_{p,0}(B_n)}\right)^{\frac{1}{n-p}} \geq \left(\frac{V(K)}{V(B_n)}\right)^{\frac{1}{n}}. \tag{1.5}$$

The second inequality of (1.5) indeed also holds for any compact set $K \subset \mathbb{R}^n$. Again inequality (1.5) is invariant under the volume preserving linear transforms and hence is stronger than inequality (1.4). Moreover, inequality (1.5) can be viewed as the affine relative of inequality (1.4). See e.g., [38, 44, 45] for more works related to affine capacities. We would like to mention that the p-affine energy is the key ingredient in many fundamental analytical inequalities, see e.g., [3, 15, 26, 31, 35, 36, 41, 46].

It is our goal in this article to study a concept more general than the p-affine capacity and to establish stronger sharp geometric inequalities. The motivation is a result from recent studies, such as, the general L_p affine isoperimetric inequalities and asymmetric affine L_p Sobolev inequalities by Haberl and Schuster [12, 13], asymmetric affine Pólya-Szegö principle by Haberl, Schuster and Xiao [14] and Minkowski valuations by Ludwig [20]. The key in [12] is to replace $h_{\Pi_p(K)}$ by its asymmetric counterpart $h_{\Pi_{p,\tau}(K)}: S^{n-1} \to \mathbb{R}$: for any $p \geq 1$, for any $\tau \in [-1, 1]$ and for K a convex body with the origin in its interior,

$$\left[h_{\Pi_{p,\tau}(K)}(\theta)\right]^p = \int_{\partial K} \left[\varphi_\tau(\theta \cdot \nu_K(x))\right]^p \cdot |x \cdot \nu_K(x)|^{1-p} \, d\mathscr{H}^{n-1}(x)$$

for $\theta \in S^{n-1}$, where

$$\left[\varphi_\tau(t)\right]^p = \Big(\frac{1+\tau}{2}\Big) t_+^p + \Big(\frac{1-\tau}{2}\Big) t_-^p \tag{1.6}$$

with $t_+ = \max\{0, t\}$ and $t_- = \max\{0, -t\}$ for any $t \in \mathbb{R}$. We point out that this extension is a key step from the L_p Brunn-Minkowski theory of convex bodies to the Orlicz theory and its dual (see e.g., [9, 10, 21, 27, 28, 40, 48]). Similarly, the key in [13, 14] is to replace the p-affine energy function $\mathscr{H}_p(f)$ by its asymmetric counterpart: for any $p \in [1, n)$, for any $\tau \in [-1, 1]$ and for any $f \in C_c^\infty$,

$$\mathscr{H}_{p,\tau}(f) = \left(\int_{S^{n-1}} \left(\int_{\mathbb{R}^n} \left[\varphi_\tau(\nabla_u f)\right]^p dx\right)^{-\frac{n}{p}} du\right)^{-\frac{p}{n}}.$$

When $\tau = 0$, $\mathscr{H}_{p,\tau}(f)$ goes back to the p-affine energy $\mathscr{H}_p(f)$. It is worth to mention that to deal with $\mathscr{H}_{p,\tau}(f)$ is much more challenging than $\mathscr{H}_p(f)$, mainly because the L_p convexifications of level sets of a smooth function f in the latter case always contain the origin in their interiors but in the former may not contain the origin in their interiors. These asymmetric extensions have also been widely used to study affine Sobolev type inequalities, the affine Pólya-Szegö principle as well as many other affine isoperimetric inequalities, see e.g., [31, 32, 37, 39].

In Section 3, we provide several equivalent definitions for the general p-affine capacity, which will be denoted by $C_{p,\tau}(\cdot)$. One of them reads: for any $p \in [1, n)$, for any $\tau \in [-1, 1]$ and for any compact set $K \subset \mathbb{R}^n$,

$$C_{p,\tau}(K) = \inf\Big\{\mathscr{H}_{p,\tau}(f): \ f \in C_c^\infty \text{ and } f \geq 1 \text{ on } K\Big\}.$$

Basic properties for the general p-affine capacity, such as, monotonicity, affine invariance, translation invariance, homogeneity and the continuity from above, are established in Section 4. Similarly, the general p-integral affine surface area of a Lipschitz star body K is defined in Subsection 5.3 by: for any $p \in [1, n)$ and for any $\tau \in [-1, 1]$,

$$\Phi_{p,\tau}(K) = \left(\int_{S^{n-1}} \left[h_{\Pi_{p,\tau}(K)}(u)\right]^{-n} du\right)^{-\frac{p}{n}}.$$

Note that when $\tau = 0$, then $\Phi_{p,0}(K) = \Phi_p(K)$. The sharp geometric inequalities for the general p-affine capacity are established in Section 5. Roughly speaking, for K a convex body containing the origin in its interior, these sharp geometric inequalities can be summarized as follows: for all $p \in [1,n)$ and for all $0 \le \tau \le \eta \le 1$, then

$$\begin{array}{ccccccc} \left(\frac{V(K)}{V(B_n)}\right)^{\frac{1}{n}} & \le & \left(\frac{C_{p,\eta}(K)}{C_{p,\eta}(B_n)}\right)^{\frac{1}{n-p}} & \le & \left(\frac{\Phi_{p,\eta}(K)}{\Phi_{p,\eta}(B_n)}\right)^{\frac{1}{n-p}} & \le & \left(\frac{S_p(K)}{S_p(B_n)}\right)^{\frac{1}{n-p}} \\ & & \wedge\!| & & \wedge\!| & & \\ \left(\frac{V(K)}{V(B_n)}\right)^{\frac{1}{n}} & \le & \left(\frac{C_{p,\tau}(K)}{C_{p,\tau}(B_n)}\right)^{\frac{1}{n-p}} & \le & \left(\frac{\Phi_{p,\tau}(K)}{\Phi_{p,\tau}(B_n)}\right)^{\frac{1}{n-p}} & \le & \left(\frac{S_p(K)}{S_p(B_n)}\right)^{\frac{1}{n-p}}. \end{array} \tag{1.7}$$

Inequality (1.5) turns out to be a special (and indeed the maximal) case of the above chain of inequalities. Hence, (1.7) extends and strengthens many well-known (affine) isoperimetric and (affine) isocapacitary inequalities, such as, [12, Theorem 1] by Haberl and Schuster, [22, inequality (13)] by Ludwig, Xiao and Zhang, and [43, Theorems 3.2 and 3.5] by Xiao. Moreover, we also prove that, for any $p \in [1,n)$ and for any $\tau \in [-1,1]$,

$$\left(\frac{S_p(K)}{S_p(B_n)}\right)^{\frac{1}{n-p}} \ge \left(\frac{C_p(K)}{C_p(B_n)}\right)^{\frac{1}{n-p}} \ge \left(\frac{C_{p,\tau}(K)}{C_{p,\tau}(B_n)}\right)^{\frac{1}{n-p}} \ge \left(\frac{V(K)}{V(B_n)}\right)^{\frac{1}{n}}, \tag{1.8}$$

which extends and strengthens, e.g., inequality (1.4), [22, (12)] by Ludwig, Xiao and Zhang, and [43, Remark 2.7] by Xiao. Note that inequalities (1.7) and (1.8) work for more general compact sets than convex bodies, and we will explain the details in Section 5.

2 Background and Notations

A compact set $M \subset \mathbb{R}^n$ is said to be a star body (with respect to the origin o) if the line segment jointing o and x, for all $x \in M$, is contained in M. For each star body M, one can define the radial function ρ_M of M as follows: for all $x \in \mathbb{R}^n \setminus \{o\}$,

$$\rho_M(x) = \max\{\lambda \ge 0 : \ \lambda x \in M\}.$$

The star body M is said to be a Lipschitz star body if the boundary of M is Lipschitz.

A compact convex subset in $\mathbb{R}^n$ with nonempty interior is called a convex body. By $\mathscr{K}_0$, we mean the set of all convex bodies with the origin o in their interiors. Each $K \subset \mathscr{K}_0$ is (uniquely) associated with two continuous functions defined on the unit sphere S^{n-1}: the radial function ρ_K and the support function h_K. Hereafter, for $u \in S^{n-1}$,

$$h_K(u) \ = \ \max\{y \cdot u : \ y \in K\},$$

where $x \cdot y$ is the standard inner product of x and y in $\mathbb{R}^n$. The support function $h_K : S^{n-1} \to (0,\infty)$ of a convex body $K \in \mathscr{K}_0$ can be extended to $\mathbb{R}^n \setminus \{o\}$ as follows: $h_K(x) = r h_K(u)$ for any $x \in \mathbb{R}^n \setminus \{o\}$ with $x = ru$. It can be easily checked that the extended function $h_K : \mathbb{R}^n \setminus \{o\} \to (0,\infty)$ is sublinear, i.e., h_K has the positive homogeneity of degree 1 and satisfies

$$h_K(x+y) \le h_K(x) + h_K(y)$$

for all $x, y \in \mathbb{R}^n \setminus \{o\}$. On the other hand, if a function $h : \mathbb{R}^n \setminus \{o\} \to (0,\infty)$ is sublinear, then h is the support function of a convex body $K \in \mathscr{K}_0$ [34]. For each $K \in \mathscr{K}_0$, its polar body K° is

$$K^\circ = \{x \in \mathbb{R}^n : \ x \cdot y \le 1 \ \text{ for all } \ y \in K\}.$$

It is easily checked that

$$\rho_{K^\circ} = \frac{1}{h_K} \quad \text{ and } \ h_{K^\circ} = \frac{1}{\rho_K}. \tag{2.9}$$

The standard notation $\mathscr{H}^k$ is for the k-dimensional Hausdorff measure. In the case of $k = n$, we use $V(\cdot)$ to denote the volume instead of $\mathscr{H}^n$. In particular, the volume of the unit Euclidean ball B_n, denoted by ω_n for simplicity, has the following expression:

$$\omega_n = \frac{\pi^{n/2}}{\Gamma(1+n/2)},$$

where $\Gamma(\cdot)$ is the Gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt.$$

The Beta function $B(\cdot,\cdot)$ is closely related to the Gamma function, and it has the form

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt.$$

It is easily checked that

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

It is convention to use $d\sigma$ for the spherical measure of S^{n-1}. In later context, the normalized spherical measure du is often used, i.e.,

$$du = \frac{d\sigma}{n\omega_n} \quad \text{and} \quad \int_{S^{n-1}} du = 1.$$

The volume of each $K \in \mathscr{K}_0$ can be calculated by

$$V(K) = \frac{1}{n}\int_{S^{n-1}} \rho_K^n(u)\, d\sigma(u) \quad \text{or} \quad V(K) = \frac{1}{n}\int_{S^{n-1}} h_K(u)\, dS(K,u), \tag{2.10}$$

where $S(K,\cdot)$ is the classical surface area measure of $K \in \mathscr{K}_0$ defined on S^{n-1}. Denote by $C(S^{n-1})$ the set of continuous functions on S^{n-1}. The classical surface area measure $S(K,\cdot)$ has the following analytic interpretation: for all $f \in C(S^{n-1})$,

$$\int_{S^{n-1}} f(u)\, dS(K,u) = \int_{\partial K} f(\nu_K(x))\, d\mathscr{H}^{n-1}(x), \tag{2.11}$$

where $\nu_K(x)$ is an outer unit normal vector at $x \in \partial K$, the boundary of K. For each $K \in \mathscr{K}_0$, $\nu_K(x)$ exists almost everywhere on ∂K with respect to $\mathscr{H}^{n-1}$ [34].

A smooth function is a real valued function $f : \mathbb{R}^n \to \mathbb{R}$ which is infinitely continuously differentiable. Denote by C^∞ the set of smooth functions with continuous derivatives of all orders, and by C_c^∞ (or $C_c^\infty(\mathbb{R}^n)$) the set of functions in C^∞ with compact support in $\mathbb{R}^n$. The gradient of $f \in C_c^\infty$ is denoted by ∇f. For $1 \le p < \infty$ and $f \in C_c^\infty$, consider the norm

$$\|f\|_{1,p} = \|f\|_p + \|\nabla f\|_p = \left(\int_{\mathbb{R}^n} |f|^p\, dx\right)^{1/p} + \left(\int_{\mathbb{R}^n} |\nabla f|^p\, dx\right)^{1/p}.$$

We also use $\|f\|_\infty$ to denote the maximal value (or supremum) of $|f|$. The closure of C_c^∞ under the norm $\|\cdot\|_{1,p}$ is denoted by $W_0^{1,p}$. Note that the Sobolev space $W_0^{1,p}$ is a Banach space and each $f \in W_0^{1,p}$ is a real valued L_p function on $\mathbb{R}^n$ with weak L_p partial derivative (see e.g. [6] for more details about the Sobolev space). Hereafter, when $f \in W_0^{1,p}$ is not smooth enough, ∇f means the weak partial gradient. By $\nabla_z f$ we mean the inner product of z and ∇f, namely

$\nabla_z f = z \cdot \nabla f$. When $u \in S^{n-1}$, $\nabla_u f$ is just the directional derivative of f along the direction u. Clearly $\nabla_z f$ is linear about $z \in \mathbb{R}^n$.

For a subset $E \subset \mathbb{R}^n$, $\mathbf{1}_E$ denotes the indicator function of E, that is, $\mathbf{1}_E(x) = 1$ if $x \in E$ and otherwise 0. Let $|x| = \sqrt{x \cdot x}$ be the Euclidean norm of $x \in \mathbb{R}^n$. The distance from a point $x \in \mathbb{R}^n$ to a subset $E \subset \mathbb{R}^n$, denoted by $\mathrm{dist}(x, E)$, is defined by

$$\mathrm{dist}(x, E) = \inf\{|x - y| : \ y \in E\}.$$

Note that if $x \in \bar{E}$, the closure of E, then $\mathrm{dist}(x, E) = 0$.

For any real number $t > 0$, define the level set $[f]_t$ of $f \in C_c^\infty$ by

$$[f]_t = \{x \in \mathbb{R}^n : |f(x)| \geq t\}. \tag{2.12}$$

For all $t \in (0, \|f\|_\infty)$, $[f]_t$ is a compact set. The Sard's theorem implies that, for almost every $t \in (0, \|f\|_\infty)$, the smooth $(n-1)$ submanifold

$$\partial[f]_t = \{x \in \mathbb{R}^n : |f(x)| = t\}$$

has nonzero normal vector $\nabla f(x)$ for all $x \in \partial[f]_t$. Denoted by $\nu(x) = -\nabla f(x)/|\nabla f(x)|$ and

$$\{\nu(x) : x \in \partial[f]_t\} = S^{n-1}.$$

An often used formula in our proofs is the well-known Federer's coarea formula (see [7], p.289): suppose that Ω is an open set in $\mathbb{R}$ and $f : \mathbb{R}^n \to \mathbb{R}$ is a Lipschitz function, then

$$\int_{f^{-1}(\Omega)\bigcap\{|\nabla f|>0\}} g(x)\ dx = \int_\Omega \int_{f^{-1}(t)} \frac{g(x)}{|\nabla f(x)|}\ d\mathscr{H}^{n-1}(x)\ dt, \tag{2.13}$$

for any measurable function $g : \mathbb{R}^n \to [0, \infty)$.

Denote by $\mathbb{R}^*$ the subset of $\mathbb{R}$ that contains nonnegative real numbers. Let $\varphi_\tau : \mathbb{R} \to \mathbb{R}^*$ be the function given by formula (1.6), that is, for $\tau \in [-1, 1]$ and $t \in \mathbb{R}$,

$$\varphi_\tau(t) = \Big(\frac{1+\tau}{2}\Big)^{1/p} t_+ + \Big(\frac{1-\tau}{2}\Big)^{1/p} t_-. \tag{2.14}$$

It is easily checked that φ_τ has positive homogeneous of degree 1 and subadditive, i.e.

$$\varphi_\tau(\lambda t) = \lambda\varphi_\tau(t) \quad \text{for} \quad \lambda \geq 0 \quad \text{and} \quad \varphi_\tau(t_1 + t_2) \leq \varphi_\tau(t_1) + \varphi_\tau(t_1). \tag{2.15}$$

Special cases, which are commonly used, are $\varphi_0(t) = 2^{-1/p}|t|$, $\varphi_1(t) = t_+$ and $\varphi_{-1}(t) = t_-$. We would like to mention that the function $\psi_\eta : \mathbb{R} \to \mathbb{R}^*$ for each $\eta \in [-1, 1]$ given by

$$\psi_\eta(t) = |t| + \eta t$$

is also commonly used in convex geometry (see e.g. [12, 20]). However, if we let

$$\tau = \frac{(1+\eta)^p - (1-\eta)^p}{(1+\eta)^p + (1-\eta)^p},$$

then $\psi_\eta^p = \big((1+\eta)^p + (1-\eta)^p\big) \cdot \varphi_\tau^p$. In later context, the theory for the general p-affine capacity will be developed only based on φ_τ because it is more convenient to prove the convexity or concavity of the general p-affine capacity with φ_τ.

We shall need the following result (see, e.g., [11, Lemma 1.3.1 (ii)]), which is crucial in the computation of involved integral on S^{n-1}.

Lemma 2.1. *If $v \in S^{n-1}$ and Φ is a bounded Lebesgue integrable function on $[-1,1]$, then $\Phi(u\cdot v)$, considered as a function of $u \in S^{n-1}$, is integrable with respect to the normalized spherical measure du. Moreover,*

$$\int_{S^{n-1}} \Phi(u\cdot v)\,du = \frac{(n-1)\omega_{n-1}}{n\omega_n}\int_{-1}^{1}\Phi(t)(1-t^2)^{\frac{n-3}{2}}\,dt.$$

It can be easily checked that for $p>0$

$$\begin{aligned}\int_{-1}^{1} t_+^p(1-t^2)^{\frac{n-3}{2}}\,dt &= \int_{-1}^{1} t_-^p(1-t^2)^{\frac{n-3}{2}}\,dt\\ &= \int_{0}^{1} t^p(1-t^2)^{\frac{n-3}{2}}\,dt\\ &= \frac{1}{2}\cdot\int_{0}^{1} t^{\frac{p+1}{2}-1}(1-t)^{\frac{n-1}{2}-1}\,dt\\ &= \frac{1}{2}\cdot B\Big(\frac{p+1}{2},\frac{n-1}{2}\Big).\end{aligned}$$

In particular, if $\Phi=\varphi_\tau^p$, it follows from (1.6) and Lemma 2.1 that, for $p>0$ and for any $u\in S^{n-1}$,

$$\begin{aligned}\int_{S^{n-1}}[\varphi_\tau(u\cdot v)]^p\,du &= \frac{(n-1)\omega_{n-1}}{n\omega_n}\int_{-1}^{1}\Big[\Big(\frac{1+\tau}{2}\Big)t_+^p+\Big(\frac{1-\tau}{2}\Big)t_-^p\Big](1-t^2)^{\frac{n-3}{2}}\,dt\\ &= \frac{(n-1)\omega_{n-1}}{2n\omega_n}\cdot B\Big(\frac{p+1}{2},\frac{n-1}{2}\Big)\qquad (:=A(n,p)).\end{aligned}\tag{2.16}$$

3 The general p-affine capacity

In this section, the general p-affine capacity is proposed and several equivalent formulas for the general p-affine capacity are provided. Throughout, the general p-affine capacity of a compact set $K\subset\mathbb{R}^n$ will be denoted by $C_{p,\tau}(K)$. For convenience, let

$$\mathscr{E}(K)=\{f: f\in W_0^{1,p},\ f\ge \mathbf{1}_K\}.$$

For each $f\in W_0^{1,p}$, let $\nabla_u^+ f(x)=\max\{\nabla_u f(x),0\}$, $\nabla_u^- f(x)=\max\{-\nabla_u f(x),0\}$, and

$$\begin{aligned}\mathscr{H}_{p,\tau}(f) &= \Big(\int_{S^{n-1}}\|\varphi_\tau(\nabla_u f)\|_p^{-n}\,du\Big)^{-\frac{p}{n}}\\ &= \Bigg(\int_{S^{n-1}}\Big(\int_{\mathbb{R}^n}\Big[\Big(\frac{1+\tau}{2}\Big)(\nabla_u^+ f(x))^p+\Big(\frac{1-\tau}{2}\Big)(\nabla_u^- f(x))^p\Big]\,dx\Big)^{-\frac{n}{p}}\,du\Bigg)^{-\frac{p}{n}}.\end{aligned}\tag{3.17}$$

Definition 3.1. *Let K be a compact subset in $\mathbb{R}^n$ and the function φ_τ be as in (2.14). For $1\le p<n$, define the general p-affine capacity of K by*

$$C_{p,\tau}(K) = \inf_{f\in\mathscr{E}(K)}\mathscr{H}_{p,\tau}(f).$$

Remark. For any compact set $K\subset\mathbb{R}^n$ and for any $\tau\in[-1,1]$, $C_{p,\tau}(K)<\infty$ if $p\in[1,n)$. According to the proofs of (4.20) and Theorem 4.1, the desired boundedness argument follows if $C_{p,\tau}(B_n)<\infty$ is verified. To this end, let $K=B_n$ and $\varepsilon>0$. Consider

$$f_\varepsilon(x)=\begin{cases}0, & \text{if } |x|\ge 1+\varepsilon,\\ 1-\frac{|x|-1}{\varepsilon}, & \text{if } 1<|x|<1+\varepsilon,\\ 1, & \text{if } |x|\le 1.\end{cases}$$

It can be checked that $f_\varepsilon \in W_0^{1,p}$ and f_ε has its weak derivative to be

$$\nabla f_\varepsilon(x) = \begin{cases} 0, & \text{if } |x| \notin (1, 1+\varepsilon), \\ -\frac{x}{\varepsilon|x|}, & \text{if } |x| \in (1, 1+\varepsilon). \end{cases}$$

This further implies that, together with Fubini's theorem, (2.15) and (2.16),

$$\begin{aligned} \|\varphi_\tau(\nabla_u f_\varepsilon)\|_p^p &= \int_{\mathbb{R}^n} \big[\varphi_\tau(\nabla_u f_\varepsilon(x))\big]^p \, dx \\ &= \int_{\{x\in\mathbb{R}^n: 1<|x|<1+\varepsilon\}} \bigg[\varphi_\tau\Big(-\frac{u\cdot x}{\varepsilon|x|}\Big)\bigg]^p \, dx \\ &= \varepsilon^{-p} \int_1^{1+\varepsilon} r^{n-1}\, dr \cdot \int_{S^{n-1}} \big[\varphi_\tau(-u\cdot v)\big]^p \, d\sigma(v) \\ &= \frac{(1+\varepsilon)^n - 1}{\varepsilon^p} \cdot \omega_n \cdot A(n,p). \end{aligned}$$

It follows from (3.17) that

$$\mathscr{H}_{p,\tau}(f_\varepsilon) = \bigg(\int_{S^{n-1}} \|\varphi_\tau(\nabla_u f_\varepsilon)\|_p^{-n} \, du\bigg)^{-\frac{p}{n}} = \frac{(1+\varepsilon)^n - 1}{\varepsilon^p} \cdot \omega_n \cdot A(n,p).$$

By Definition 3.1, for $p \in [1, n)$,

$$C_{p,\tau}(B_n) \le \mathscr{H}_{p,\tau}(f_\varepsilon)\Big|_{\varepsilon=1} < 2^n \cdot \omega_n \cdot A(n,p) < \infty.$$

We would like to mention that the general p-affine capacity can be also defined for $p \in (0,1) \cup [n, \infty)$ along the same manner in Definition 3.1, however in these cases the general p-affine capacities are trivial. For instance, if $p \in (0,1)$,

$$C_{p,\tau}(B_n) \le \lim_{\varepsilon\to 0^+} \mathscr{H}_{p,\tau}(f_\varepsilon) = \lim_{\varepsilon\to 0^+} \frac{(1+\varepsilon)^n - 1}{\varepsilon^p} \cdot \omega_n \cdot A(n,p) = 0,$$

and hence, again due to the proofs of (4.20) and Theorem 4.1, $C_{p,\tau}(K) = 0$ for any compact set $K \subset \mathbb{R}^n$ and for any $\tau \in [-1, 1]$. The case for $p > n$ can be seen intuitively from the above estimate with $\varepsilon \to \infty$ instead, but more details for $p \ge n$ will be discussed in Theorem 5.1. The precise value of $C_{p,\tau}(B_n)$ will be provided in formulas (5.26) and (5.27). □

As $\varphi_0(t) = 2^{-1/p}|t|$, one gets the p-affine capacity defined by Xiao in [42, 43]:

$$C_{p,0}(K) = \frac{1}{2} \inf_{f\in\mathscr{E}(K)} \bigg(\int_{S^{n-1}} \|\nabla_u f\|_p^{-n} \, du\bigg)^{-\frac{p}{n}}.$$

As $\varphi_1(\nabla_u f) = \nabla_u^+ f$, one has

$$C_{p,1}(K) = \inf_{f\in\mathscr{E}(K)} \bigg(\int_{S^{n-1}} \|\nabla_u^+ f\|_p^{-n} \, du\bigg)^{-\frac{p}{n}},$$

which will be called the asymmetric p-affine capacity and denoted by $C_{p,+}$ instead of $C_{p,1}$ for better intuition. Similarly, as $\varphi_{-1}(\nabla_u f) = \nabla_u^- f$, one can have the following p-affine capacity:

$$C_{p,-}(K) = \inf_{f\in\mathscr{E}(K)} \bigg(\int_{S^{n-1}} \|\nabla_u^- f\|_p^{-n} \, du\bigg)^{-\frac{p}{n}}.$$

The following theorem plays important roles in later context. For a compact set $K \subset \mathbb{R}^n$, let

$$\mathscr{F}(K) = \Big\{ f : f \in W_0^{1,p},\ 0 \le f \le 1 \text{ in } \mathbb{R}^n,\ \text{and } f = 1 \text{ in a neighborhood of K} \Big\}.$$

Theorem 3.1. *Let $1 \leq p < n$ and K be a compact set in $\mathbb{R}^n$. Then*

$$C_{p,\tau}(K) = \inf_{f \in \mathscr{F}(K)} \mathscr{H}_{p,\tau}(f).$$

Moreover, the general p-affine capacity is upper-semicontinuous: for any $\varepsilon > 0$, there exists an open set O_ε such that for any compact set F with $K \subset F \subset O_\varepsilon$,

$$C_{p,\tau}(F) \leq C_{p,\tau}(K) + \varepsilon.$$

Proof. Our proof is based on the standard technique in [30] and is similar to that in [43, 45]. A short proof is included for completeness. Recall that $C_{p,\tau}(K) < \infty$. Due to $\mathscr{F}(K) \subset \mathscr{E}(K)$, one has

$$\inf_{f \in \mathscr{F}(K)} \mathscr{H}_{p,\tau}(f) \geq C_{p,\tau}(K).$$

On the other hand, for any $\varepsilon > 0$, let $f_\varepsilon \in \mathscr{E}(K)$ satisfy that

$$C_{p,\tau}(K) + \varepsilon \geq \mathscr{H}_{p,\tau}(f_\varepsilon).$$

For $i = 1, 2, \cdots$, there are functions $\phi_i \in C_c^\infty(\mathbb{R})$, such that, for all $t \in \mathbb{R}$,

$$0 \leq \phi_i'(t) \leq i^{-1} + 1,$$

$\phi_i = 0$ in a neighborhood of $(-\infty, 0]$, and $\phi_i = 1$ in a neighborhood of $[1, \infty)$. It follows from the chain rule in [6, Theorem 4 on p.129] and the homogeneity of φ_τ (see (2.15)) that, for all i, $\phi_i(f_\varepsilon) \in \mathscr{F}(K)$ and

$$\begin{aligned} \inf_{f \in \mathscr{F}(K)} \mathscr{H}_{p,\tau}(f) &\leq \mathscr{H}_{p,\tau}(\phi_i(f_\varepsilon)) \\ &\leq (1 + i^{-1})^p \cdot \mathscr{H}_{p,\tau}(f_\varepsilon) \\ &\leq (1 + i^{-1})^p \cdot (C_{p,\tau}(K) + \varepsilon). \end{aligned}$$

Taking $i \to \infty$ first and then letting $\varepsilon \to 0$, one gets

$$\inf_{f \in \mathscr{F}(K)} \mathscr{H}_{p,\tau}(f) \leq C_{p,\tau}(K)$$

and hence the following desired formula holds:

$$\inf_{f \in \mathscr{F}(K)} \mathscr{H}_{p,\tau}(f) = C_{p,\tau}(K).$$

Now let us prove the upper-semicontinuity. For any given $\varepsilon > 0$, let $g_\varepsilon \in \mathscr{F}(K)$ and O_ε be a neighborhood of K such that $g_\varepsilon = 1$ on O_ε and

$$C_{p,\tau}(K) + \varepsilon \geq \mathscr{H}_{p,\tau}(g_\varepsilon).$$

On the other hand, for any compact set F such that $K \subset F \subset O_\varepsilon$, one has $g_\varepsilon \in \mathscr{F}(F)$ and hence

$$\mathscr{H}_{p,\tau}(g_\varepsilon) \geq C_{p,\tau}(F),$$

by Definition 3.1. The desired inequality follows from the above two inequalities. □

Our next result regarding the definition of the general p-affine capacity for compact sets is to replace $\mathscr{E}(K)$ by the bigger set $\mathscr{D}(K)$:

$$\mathscr{D}(K) = \Big\{ f \in W_0^{1,p} \ \text{ such that } \ f \geq 1 \text{ on } K \Big\}.$$

Theorem 3.2. *Let $1 \le p < n$ and K be a compact set in $\mathbb{R}^n$. Then*

$$C_{p,\tau}(K) \;\; = \;\; \inf_{f\in\mathscr{D}(K)} \mathscr{H}_{p,\tau}(f).$$

Proof. It follows from (2.14) and [16, Lemma 1.19] that, for any $f \in W_0^{1,p}$ and for any $u \in S^{n-1}$,

$$\varphi_\tau(\nabla_u f_+(x)) = \begin{cases} \varphi_\tau(\nabla_u f(x)), & \text{if } \; f(x) > 0, \\ 0, & \text{if } \; f(x) \le 0. \end{cases}$$

Hence, for any $u \in S^{n-1}$ and all $x \in \mathbb{R}^n$, one has

$$\varphi_\tau(\nabla_u f_+(x)) \le \varphi_\tau(\nabla_u f(x)).$$

This further implies that $\mathscr{H}_{p,\tau}(f_+) \le \mathscr{H}_{p,\tau}(f)$ for any $f \in W_0^{1,p}$. Let $\{f_k\}_{k\ge1} \subset \mathscr{D}(K)$ be such that

$$\lim_{k\to\infty} \mathscr{H}_{p,\tau}(f_k) = \inf_{f\in\mathscr{D}(K)} \mathscr{H}_{p,\tau}(f).$$

Then $\{f_{k,+}\}_{k\ge1}$ is a sequence in $\mathscr{E}(K)$. Definition 3.1 yields

$$\lim_{k\to\infty} \mathscr{H}_{p,\tau}(f_k) \;\; \ge \;\; \limsup_{k\to\infty} \mathscr{H}_{p,\tau}(f_{k,+}) \ge \inf_{f\in\mathscr{E}(K)} \mathscr{H}_{p,\tau}(f) = C_{p,\tau}(K).$$

This concludes that

$$\inf_{f\in\mathscr{D}(K)} \mathscr{H}_{p,\tau}(f) \ge C_{p,\tau}(K).$$

On the other hand, as $\mathscr{E}(K) \subset \mathscr{D}(K)$, the following inequality holds trivially:

$$\inf_{f\in\mathscr{D}(K)} \mathscr{H}_{p,\tau}(f) \le C_{p,\tau}(K).$$

Combining the above two inequalities, one has $C_{p,\tau}(K) = \inf_{f\in\mathscr{D}(K)} \mathscr{H}_{p,\tau}(f)$. □

The following result asserts that $f \in W_0^{1,p}$ in Definition 3.1, Theorems 3.1 and 3.2 could be replaced by $f \in C_c^\infty$. The smoothness of functions is convenient in establishing many properties for the general p-affine capacity.

Theorem 3.3. *Let $p \in [1, n)$ and K be a compact set in $\mathbb{R}^n$. For any $\tau \in [-1, 1]$, one has*

$$C_{p,\tau}(K) = \inf_{f\in C_c^\infty\cap\mathscr{D}(K)} \mathscr{H}_{p,\tau}(f) = \inf_{f\in C_c^\infty\cap\mathscr{E}(K)} \mathscr{H}_{p,\tau}(f) = \inf_{f\in C_c^\infty\cap\mathscr{F}(K)} \mathscr{H}_{p,\tau}(f). \tag{3.18}$$

Proof. Let $p \in [1, n)$. Let $f \in \mathscr{F}(K)$, i.e., $f \in W_0^{1,p}$ such that $0 \le f \le 1$ in $\mathbb{R}^n$ and $f = 1$ in U, a neighborhood of K. As $W_0^{1,p}$ is the closure of C_c^∞ under $\|\cdot\|_{1,p}$, there is a sequence $\{f_k\}_{k=1}^\infty \subset C_c^\infty$ such that $f_k \to f$ in $W_0^{1,p}$, i.e.,

$$\|f_k - f\|_p + \|\nabla f_k - \nabla f\|_p \to 0.$$

Without loss of generality, we can assume that $f_k \in C_c^\infty \cap \mathscr{D}(K)$ for all k. To see this, from the regularization technique (see, e.g., [16]), one can choose a cut off function $\kappa \in C^\infty$, such that, $0 \le \kappa \le 1$ on $\mathbb{R}^n$, $\kappa = 1$ on $\mathbb{R}^n \setminus U$, and $\kappa = 0$ in a neighborhood (contained in U) of K. Let

$$g_k = 1 - (1 - f_k)\kappa.$$

Clearly, $g_k \in C_c^\infty$, such that, $g_k = 1$ in a neighborhood (contained in U) of K and $g_k = f_k$ on $\mathbb{R}^n \setminus U$. This implies $g_k \in C_c^\infty \cap \mathscr{D}(K)$ for all k. Moreover, $\|g_k - f\|_{1,p} \to 0$ and hence

$$\|g_k - f\|_p \to 0 \quad \text{and} \quad \|\nabla g_k - \nabla f\|_p \to 0.$$

Let $f_k \in C_c^\infty \cap \mathscr{D}(K)$ be such that $f_k \to f$ in $W_0^{1,p}$. It can be checked that, for any $u \in S^{n-1}$,

$$|\nabla_u^+ f_k - \nabla_u^+ f| \le |\nabla f_k - \nabla f| \quad \text{and} \quad |\nabla_u^- f_k - \nabla_u^- f| \le |\nabla f_k - \nabla f|.$$

This together with (2.14) yield, for any $\tau \in [-1,1]$ and for all $k \ge 1$,

$$\begin{aligned}
\Big|\varphi_\tau(\nabla_u f_k) - \varphi_\tau(\nabla_u f)\Big| &= \Big|\Big(\frac{1+\tau}{2}\Big)^{1/p}\big[\nabla_u^+ f_k - \nabla_u^+ f\big] + \Big(\frac{1-\tau}{2}\Big)^{1/p}\big[\nabla_u^- f_k - \nabla_u^- f\big]\Big| \\
&\le \Big(\frac{1+\tau}{2}\Big)^{1/p}\Big|\nabla_u^+ f_k - \nabla_u^+ f\Big| + \Big(\frac{1-\tau}{2}\Big)^{1/p}\Big|\nabla_u^- f_k - \nabla_u^- f\Big| \\
&\le C(p,\tau) \cdot \big|\nabla f_k - \nabla f\big|,
\end{aligned}$$

where we have let $C(p,\tau)$ be the constant

$$C(p,\tau) = \Big(\frac{1+\tau}{2}\Big)^{1/p} + \Big(\frac{1-\tau}{2}\Big)^{1/p}.$$

It follows from the triangle inequality that, for any $u \in S^{n-1}$, for any $\tau \in [-1,1]$ and for any $p \in [1,n)$,

$$\begin{aligned}
\Big|\|\varphi_\tau(\nabla_u f_k)\|_p - \|\varphi_\tau(\nabla_u f)\|_p\Big| &\le \|\varphi_\tau(\nabla_u f_k) - \varphi_\tau(\nabla_u f)\|_p \\
&\le C(p,\tau) \cdot \|\nabla f_k - \nabla f\|_p.
\end{aligned}$$

Consequently, for any $u \in S^{n-1}$, for any $\tau \in [-1,1]$ and for any $p \in [1,n)$, one has

$$\lim_{k\to\infty} \|\varphi_\tau(\nabla_u f_k)\|_p = \|\varphi_\tau(\nabla_u f)\|_p.$$

By Fatou's lemma, one has

$$\begin{aligned}
\mathscr{H}_{p,\tau}(f) &= \Big(\int_{S^{n-1}} \|\varphi_\tau(\nabla_u f)\|_p^{-n}\, du\Big)^{-\frac{p}{n}} \\
&= \Big(\int_{S^{n-1}} \lim_{k\to\infty} \|\varphi_\tau(\nabla_u f_k)\|_p^{-n}\, du\Big)^{-\frac{p}{n}} \\
&\ge \Big(\liminf_{k\to\infty} \int_{S^{n-1}} \|\varphi_\tau(\nabla_u f_k)\|_p^{-n}\, du\Big)^{-\frac{p}{n}} \\
&= \limsup_{k\to\infty} \Big(\int_{S^{n-1}} \|\varphi_\tau(\nabla_u f_k)\|_p^{-n}\, du\Big)^{-\frac{p}{n}} \\
&= \limsup_{k\to\infty} \mathscr{H}_{p,\tau}(f) \\
&\ge \inf_{g \in C_c^\infty \cap \mathscr{D}(K)} \mathscr{H}_{p,\tau}(g). \qquad (3.19)
\end{aligned}$$

It follows from Theorem 3.1 that, by taking the infimum over $f \in \mathscr{F}(K)$,

$$C_{p,\tau}(K) \ge \inf_{C_c^\infty \cap \mathscr{D}(K)} \mathscr{H}_{p,\tau}(f).$$

It is easily checked that, due to $C_c^\infty \subset W_0^{1,p}$,

$$C_{p,\tau}(K) \leq \inf_{C_c^\infty \cap \mathscr{D}(K)} \mathscr{H}_{p,\tau}(f),$$

and hence equality holds, as desired.

The desired formula (3.18) follows, due to $\mathscr{F}(K) \subset \mathscr{E}(K) \subset \mathscr{D}(K)$, once the following inequality is proved:

$$\inf_{f \in C_c^\infty \cap \mathscr{F}(K)} \mathscr{H}_{p,\tau}(f) \leq \inf_{f \in C_c^\infty \cap \mathscr{D}(K)} \mathscr{H}_{p,\tau}(f) = C_{p,\tau}(K).$$

This inequality follows along the same lines as the proof of Theorem 3.1. In fact, for any $\varepsilon > 0$, let $f_\varepsilon \in \mathscr{D}(K) \cap C_c^\infty$ satisfy that

$$C_{p,\tau}(K) + \varepsilon \geq \mathscr{H}_{p,\tau}(f_\varepsilon).$$

Let $\phi_i \in C_c^\infty(\mathbb{R})$ be as in Theorem 3.1. Then, $\phi_i(f_\varepsilon) \in \mathscr{F}(K) \cap C_c^\infty$ and

$$\inf_{f \in \mathscr{F}(K) \cap C_c^\infty} \mathscr{H}_{p,\tau}(f) \leq (1 + i^{-1})^p \cdot (C_{p,\tau}(K) + \varepsilon).$$

Taking $i \to \infty$ first and then letting $\varepsilon \to 0$, one gets

$$\inf_{f \in \mathscr{F}(K) \cap C_c^\infty} \mathscr{H}_{p,\tau}(f) \leq C_{p,\tau}(K)$$

as desired. □

It follows from (2.15) and $\nabla_y f = y \cdot \nabla f$ that, for all $\lambda > 0$ and $y \in \mathbb{R}^n \setminus \{o\}$,

$$\|\varphi_\tau(\nabla_{\lambda y} f)\|_p = \lambda \|\varphi_\tau(\nabla_y f)\|_p.$$

Moreover, for $p \in [1, n)$ and for any $y_1, y_2 \in \mathbb{R}^n \setminus \{o\}$, by the Minkowski's inequality, one has

$$\begin{aligned} \|\varphi_\tau(\nabla_{y_1+y_2} f)\|_p &\leq \|\varphi_\tau(\nabla_{y_1} f) + \varphi_\tau(\nabla_{y_2} f)\|_p \\ &\leq \|\varphi_\tau(\nabla_{y_1} f)\|_p + \|\varphi_\tau(\nabla_{y_2} f)\|_p. \end{aligned}$$

Hence, $\|\varphi_\tau(\nabla_y f)\|_p : \mathbb{R}^n \setminus \{o\} \to [0, \infty)$, as a function of $y \in \mathbb{R}^n \setminus \{o\}$, is sublinear. According to the proof of [31, Lemma 3.1] (or [13, Lemma 2]), if $f \in \mathscr{F}(K)$, then $\|\varphi_\tau(\nabla_u f)\|_p > 0$ and $\|\varphi_\tau(\nabla_y f)\|_p$ is the support function of a convex body in $\mathscr{K}_0$. Let $L_{f,\tau}$ be the convex body. An application of (2.9) and (2.10) yields (see also [31, (3.2)])

$$\begin{aligned} \mathscr{H}_{p,\tau}(f) &= \left(\int_{S^{n-1}} \|\varphi_\tau(\nabla_u f)\|_p^{-n} \, du \right)^{-\frac{p}{n}} \\ &= \left(\int_{S^{n-1}} \left[h_{L_{f,\tau}}(u) \right]^{-n} du \right)^{-\frac{p}{n}} \\ &= \left(\frac{1}{nV(B_n)} \int_{S^{n-1}} \left[\rho_{L_{f,\tau}^\circ}(u) \right]^n d\sigma(u) \right)^{-\frac{p}{n}} \\ &= \left(\frac{V(L_{f,\tau}^\circ)}{V(B_n)} \right)^{-\frac{p}{n}}. \end{aligned}$$

Taking the infimum over $f \in \mathscr{F}(K)$, Theorem 3.1 implies that for any compact set $K \subset \mathbb{R}^n$, for any $\tau \in [-1,1]$ and for any $p \in [1,n)$,

$$C_{p,\tau}(K) = \inf_{f \in \mathscr{F}(K)} \mathscr{H}_{p,\tau}(f) = \inf_{f \in \mathscr{F}(K)} \left(\frac{V(L^{\circ}_{f,\tau})}{V(B_n)} \right)^{-\frac{p}{n}}.$$

This provides a connection of the general p-affine capacity with the volume of convex bodies.

The general p-affine capacity of a general bounded measurable set $E \subset \mathbb{R}^n$ can be defined as well. In fact, for $O \subset \mathbb{R}^n$ a bounded open set,

$$C_{p,\tau}(O) = \sup \Big\{ C_{p,\tau}(K) : \quad K \subset O \text{ and } K \text{ is compact} \Big\}.$$

Then the general p-affine capacity of a bounded measurable set $E \subset \mathbb{R}^n$ is formulated by

$$C_{p,\tau}(E) = \inf \Big\{ C_{p,\tau}(O) : \quad E \subset O \text{ and } O \text{ is open} \Big\}.$$

In later context of this article, we only concentrate on the general p-affine capacity for compact sets. We would like to mention that many properties proved in Section 4, such as, monotonicity, affine invariance and homogeneity etc, for compact sets could work for general sets too.

4 Properties of the general p-affine capacity

This section aims to establish basic properties for the general p-affine capacity, such as, monotonicity, affine invariance, translation invariance, homogeneity and the continuity from above.

The following result provides the properties of $C_{p,\tau}(\cdot)$ as a function of $\tau \in [-1,1]$.

Corollary 4.1. *Let $p \in [1,n)$ and K be a compact set in $\mathbb{R}^n$. The following properties hold.*

i) For any $\tau \in [-1,1]$, one has

$$C_{p,\tau}(K) = C_{p,-\tau}(K).$$

ii) For any $\lambda \in [0,1]$ and for any $\tau, \gamma \in [-1,1]$, one has

$$C_{p,\lambda\tau+(1-\lambda)\gamma}(K) \geq \lambda \cdot C_{p,\tau}(K) + (1-\lambda) \cdot C_{p,\gamma}(K).$$

Proof. i) Let $v = -u$. Then for any $x \in \mathbb{R}^n$, one has

$$\nabla_u^+ f(x) = \nabla_v^- f(x) \quad \text{ and } \quad \nabla_u^- f(x) = \nabla_v^+ f(x).$$

This leads to, as $du = dv$, for any $f \in \mathscr{E}(K)$,

$$\begin{aligned}
\mathscr{H}_{p,\tau}(f) &= \left(\int_{S^{n-1}} \|\varphi_\tau(\nabla_u f)\|_p^{-n} \, du \right)^{-\frac{p}{n}} \\
&= \left(\int_{S^{n-1}} \left(\int_{\mathbb{R}^n} \left[\left(\frac{1+\tau}{2} \right) (\nabla_u^+ f(x))^p + \left(\frac{1-\tau}{2} \right) (\nabla_u^- f(x))^p \right] dx \right)^{-\frac{n}{p}} du \right)^{-\frac{p}{n}} \\
&= \left(\int_{S^{n-1}} \left(\int_{\mathbb{R}^n} \left[\left(\frac{1+\tau}{2} \right) (\nabla_v^- f(x))^p + \left(\frac{1-\tau}{2} \right) (\nabla_v^+ f(x))^p \right] dx \right)^{-\frac{n}{p}} dv \right)^{-\frac{p}{n}} \\
&= \mathscr{H}_{p,-\tau}(f).
\end{aligned}$$

It follows from Definition 3.1 that, for any $\tau \in [-1,1]$, for any $p \in [1,n)$ and for any compact set $K \subset \mathbb{R}^n$,

$$C_{p,\tau}(K) = C_{p,-\tau}(K).$$

ii) For any $\lambda \in [0,1]$ and for any $\tau, \gamma \in [-1,1]$, it follows from (1.6) that, for any $t \in \mathbb{R}$,

$$\big[\varphi_{\lambda\tau+(1-\lambda)\gamma}(t)\big]^p = \lambda\big[\varphi_\tau(t)\big]^p + (1-\lambda)\big[\varphi_\gamma(t)\big]^p,$$

which implies

$$\int_{\mathbb{R}^n} \big[\varphi_{\lambda\tau+(1-\lambda)\gamma}(\nabla_u f(x))\big]^p\,dx = \lambda\int_{\mathbb{R}^n} \big[\varphi_\tau(\nabla_u f(x))\big]^p\,dx + (1-\lambda)\int_{\mathbb{R}^n} \big[\varphi_\gamma(\nabla_u f(x))\big]^p\,dx.$$

According to the proof of [31, Lemma 3.1] (or [13, Lemma 2]), $\|\varphi_\tau(\nabla_u f)\|_p > 0$ if $f \in \mathscr{F}(K)$. The reverse Minkowski inequality yields that for any $\lambda \in [0,1]$ and for any $\tau, \gamma \in [-1,1]$,

$$\left(\int_{S^{n-1}} \|\varphi_{\lambda\tau+(1-\lambda)\gamma}(\nabla_u f)\|_p^{-n}\,du\right)^{-\frac{p}{n}} \geq \lambda\left(\int_{S^{n-1}} \|\varphi_\tau(\nabla_u f)\|_p^{-n}\,du\right)^{-\frac{p}{n}} + (1-\lambda)\left(\int_{S^{n-1}} \|\varphi_\gamma(\nabla_u f)\|_p^{-n}\,du\right)^{-\frac{p}{n}}.$$

Taking the infimum over $f \in \mathscr{F}(K)$, by Theorem 3.1,

$$C_{p,\lambda\tau+(1-\lambda)\gamma}(K) \geq \lambda \cdot C_{p,\tau}(K) + (1-\lambda)\cdot C_{p,\gamma}(K)$$

holds for any $\lambda \in [0,1]$ and for any $\tau, \gamma \in [-1,1]$. $\square$

From Corollary 4.1, one sees that, for any $p \in [1,n)$ and for any compact set $K \subset \mathbb{R}^n$, $C_{p,\tau}(K) \leq C_{p,\gamma}(K)$ holds if $-1 \leq \tau < \gamma \leq 0$, and $C_{p,\gamma}(K) \leq C_{p,\tau}(K)$ holds if $0 \leq \tau < \gamma \leq 1$. In particular, for any $\tau \in [-1,1]$, one has

$$C_{p,+}(K) = C_{p,-}(K) \leq C_{p,\tau}(K) = C_{p,-\tau}(K) \leq C_{p,0}(K).$$

Given two compact sets $K \subset L$, one sees $\mathscr{E}(L) \subset \mathscr{E}(K)$ and hence the general p-affine capacity is monotone by Definition 3.1, namely,

$$C_{p,\tau}(K) \leq C_{p,\tau}(L). \tag{4.20}$$

The general p-affine capacity is also translation invariant. To see this, let $a \in \mathbb{R}^n$ and consider the function $g(x) = f(x+a)$ for any $x \in \mathbb{R}^n$. It is easily checked that $f \in \mathscr{E}(K+a)$ if and only if $g \in \mathscr{E}(K)$. Moreover, $\nabla g(x) = \nabla f(x+a)$, and thus $\mathscr{H}_{p,\tau}(g) = \mathscr{H}_{p,\tau}(f)$. Taking the infimum over $g \in \mathscr{E}(K)$ from both sides, by Definition 3.1, for any $a \in \mathbb{R}^n$ and for any compact set $K \subset \mathbb{R}^n$,

$$C_{p,\tau}(K+a) = C_{p,\tau}(K).$$

An interesting (and common for many capacities) fact for the general p-affine capacity is that

$$C_{p,\tau}(K) = C_{p,\tau}(\partial K)$$

for any compact set $K \subset \mathbb{R}^n$. To see this, let $\varepsilon > 0$ be given. There exists $f_\varepsilon \in \mathscr{E}(\partial K)$ such that

$$C_{p,\tau}(\partial K) + \varepsilon \geq \mathscr{H}_{p,\tau}(f_\varepsilon).$$

Let $g = \max\{f_\varepsilon, 1\}$ on K and $g = f_\varepsilon$ on $\mathbb{R}^n \setminus K$. It can be checked, along the manner same as the proof of Theorem 3.2, that $g \in \mathscr{E}(K)$ and

$$\int_{\mathbb{R}^n} \big[\varphi_\tau(\nabla_u g)\big]^p\,dx \leq \int_{\mathbb{R}^n} \big[\varphi_\tau(\nabla_u f_\varepsilon)\big]^p\,dx.$$

Consequently, due to Definition 3.1,

$$C_{p,\tau}(K) \le \mathscr{H}_{p,\tau}(g) \le \mathscr{H}_{p,\tau}(f_\varepsilon) < C_{p,\tau}(\partial K) + \varepsilon.$$

Letting $\varepsilon \to 0$, one gets

$$C_{p,\tau}(K) \le C_{p,\tau}(\partial K).$$

The monotonicity of the general p-affine capacity yields that

$$C_{p,\tau}(\partial K) \le C_{p,\tau}(K)$$

and hence $C_{p,\tau}(\partial K) = C_{p,\tau}(K)$ holds for all compact set $K \subset \mathbb{R}^n$.

Let $GL(n)$ be the group of all invertible linear transforms defined on $\mathbb{R}^n$. For $T \in GL(n)$, denote by T^t and $\det(T)$ the transpose of T and the determinant of T, respectively. The affine invariance of the general p-affine capacity is stated in the following theorem.

Theorem 4.1. *The general p-affine capacity has the affine invariance and homogeneity: for any $T \in GL(n)$ and for any compact set $K \subset \mathbb{R}^n$,*

$$C_{p,\tau}(TK) = |\det(T)|^{\frac{n-p}{n}} C_{p,\tau}(K).$$

In particular, the general p-affine capacity is affine invariant: for any $T \in GL(n)$ with $|\det(T)| = 1$,

$$C_{p,\tau}(TK) = C_{p,\tau}(K).$$

Moreover, the general p-affine capacity has positive homogeneity of degree $n-p$, i.e.,

$$C_{p,\tau}(\lambda K) = \lambda^{n-p} C_{p,\tau}(K)$$

for all $\lambda > 0$, where $\lambda K = \{\lambda x : x \in K\}$.

Proof. For $T \in GL(n)$ and $f \in \mathscr{E}(TK)$, one has $g = f \circ T \in \mathscr{E}(K)$. For simplicity, assume that $|\det(T)| = 1$. Thus, by $x = Ty$,

$$\int_{\mathbb{R}^n} \left[\varphi_\tau(\nabla_u g(y))\right]^p dy = \int_{\mathbb{R}^n} \left[\varphi_\tau(\nabla_u (f\circ T)(y))\right]^p dy = \int_{\mathbb{R}^n} \left[\varphi_\tau(\nabla_{Tu}(f(x))\right]^p dx,$$

where the second equality follows from the chain rule

$$\nabla g(y) = \nabla(f \circ T)(y) = T^t \nabla f(Ty).$$

By letting $v = Tu/|Tu|$, it follows from (2.15) that

$$\begin{aligned}
\int_{S^{n-1}} \left(\int_{\mathbb{R}^n} \left[\varphi_\tau(\nabla_u g(y))\right]^p dy \right)^{-\frac{n}{p}} du &= \int_{S^{n-1}} \left(\int_{\mathbb{R}^n} \left[\varphi_\tau(\nabla_{Tu}(f(x))\right]^p dx \right)^{-\frac{n}{p}} du \\
&= \int_{S^{n-1}} \left(\int_{\mathbb{R}^n} \left[\varphi_\tau(\nabla_v(f(x))\right]^p dx \right)^{-\frac{n}{p}} |Tu|^{-n} du \\
&= \int_{S^{n-1}} \left(\int_{\mathbb{R}^n} \left[\varphi_\tau(\nabla_v(f(x))\right]^p dx \right)^{-\frac{n}{p}} dv.
\end{aligned}$$

Consequently, $\mathscr{H}_{p,\tau}(g) = \mathscr{H}_{p,\tau}(f)$. Taking the infimum over $f \in \mathscr{E}(TK)$ from both sides, which is equivalent to taking the infimum over $g \in \mathscr{E}(K)$ from the left hand side, one gets the affine invariance: for all $T \in GL(n)$ with $|\det(T)| = 1$, then

$$C_{p,\tau}(TK) = C_{p,\tau}(K).$$

For the homogeneity, let $\lambda > 0$ be given. For any $f \in \mathscr{E}(\lambda K)$, one sees $g_\lambda \geq \mathbf{1}_K$ where $g_\lambda(x) = f(\lambda x)$ for all $x \in \mathbb{R}^n$. It is easily checked, by letting $y = \lambda x$, that

$$\int_{\mathbb{R}^n} \big[\varphi_\tau(\nabla_u g_\lambda(x))\big]^p \, dx = \lambda^{p-n} \int_{\mathbb{R}^n} \big[\varphi_\tau(\nabla_u f(y))\big]^p \, dy,$$

which further implies that $\mathscr{H}_{p,\tau}(f) = \lambda^{n-p}\mathscr{H}_{p,\tau}(g_\lambda)$. The desired formula $C_{p,\tau}(\lambda K) = \lambda^{n-p} C_{p,\tau}(K)$ follows immediately by Definition 3.1 and by taking the infimum over $f \in \mathscr{E}(\lambda K)$.

Finally, we consider $T \in GL(n)$ be an invertible linear transform. Then

$$\widetilde{T} = |\det(T)|^{-1/n} T$$

has $|\det(\widetilde{T})| = 1$. Hence, the affine invariance and the homogeneity yield that, for all $T \in GL(n)$,

$$C_{p,\tau}(TK) = C_{p,\tau}(|\det(T)|^{1/n}\widetilde{T}K) = |\det(T)|^{\frac{n-p}{n}} C_{p,\tau}(\widetilde{T}K) = |\det(T)|^{\frac{n-p}{n}} C_{p,\tau}(K).$$

This concludes the proof. □

The continuity from above for the general p-affine capacity is stated in the following theorem.

Theorem 4.2. *The general p-affine capacity is continuous from above: if $\{K_i\}_{i=1}^\infty$ is a decreasing sequence of compact sets, then*

$$C_{p,\tau}(\cap_{i=1}^\infty K_i) = \lim_{i\to\infty} C_{p,\tau}(K_i). \tag{4.21}$$

Proof. Recall that the general p-affine capacity of the compact set K_1 is finite. It follows from the monotonicity that, for all i,

$$C_{p,\tau}(K_{i+1}) \leq C_{p,\tau}(K_i) \leq C_{p,\tau}(K_1) < \infty,$$

and hence $\lim_{i\to\infty} C_{p,\tau}(K_i)$ exists and is finite. Moreover, the monotonicity of the general p-affine capacity also yields

$$C_{p,\tau}(\cap_{i=1}^\infty K_i) \leq \lim_{i\to\infty} C_{p,\tau}(K_i).$$

The desired formula (4.21) follows if we prove the following inequality:

$$C_{p,\tau}(\cap_{i=1}^\infty K_i) \geq \lim_{i\to\infty} C_{p,\tau}(K_i).$$

First of all, the set $\cap_{i=1}^\infty K_i$ is clearly compact. By Definition 3.1 and Theorem 3.3, for any $\varepsilon > 0$, one can find a smooth function $f_\varepsilon \in \mathscr{E}(\cap_{i=1}^\infty K_i)$, such that, $f_\varepsilon \geq \mathbf{1}_{\cap_{i=1}^\infty K_i}$ and

$$C_{p,\tau}\big(\cap_{i=1}^\infty K_i\big) + \varepsilon \geq \mathscr{H}_{p,\tau}(f_\varepsilon).$$

Let $K_\varepsilon = \{x \in \mathbb{R}^n : f_\varepsilon(x) \geq 1 - \varepsilon\}$. Then, $\frac{f_\varepsilon}{1-\varepsilon} \in \mathscr{E}(K_\varepsilon)$ and $K_i \subset K_\varepsilon$ for i big enough. Together with (2.15), Definition 3.1 and the monotonicity of the general p-affine capacity, one has

$$\lim_{i\to\infty} C_{p,\tau}(K_i) \leq C_{p,\tau}(K_\varepsilon) \leq (1-\varepsilon)^{-p}\mathscr{H}_{p,\tau}(f_\varepsilon) \leq \frac{C_{p,\tau}(\cap_{i=1}^\infty K_i) + \varepsilon}{(1-\varepsilon)^p}.$$

Taking $\varepsilon \to 0$, one gets the desired inequality

$$\lim_{i\to\infty} C_{p,\tau}(K_i) \leq C_{p,\tau}(\cap_{i=1}^\infty K_i)$$

and this concludes the proof. □

Note that one cannot expect to have the subadditivity for the general p-affine capacity, even for $\tau = 0$; see [45] for the details. It is not clear whether the general p-affine capacity has the continuity from below.

5 Sharp geometric inequalities for the general p-affine capacity

This section aims to establish several sharp geometric inequalities for the general p-affine capacity. In particular, the general p-affine capacity is compared with the p-variational capacity, the general p-integral affine surface areas and the volume.

5.1 Comparison with the p-variational capacity

This subsection aims to compare the general p-affine capacity and the p-variational capacity. For $p \in [1, n)$ and a compact set $K \subset \mathbb{R}^n$, the p-variational capacity of K, denoted by $C_p(K)$, is formulated by

$$C_p(K) = \inf_{f \in \mathscr{D}(K)} \int_{\mathbb{R}^n} |\nabla f|^p \, dx = \inf_{f \in \mathscr{D}(K) \cap C_c^\infty} \int_{\mathbb{R}^n} |\nabla f|^p \, dx.$$

Of course, the set $\mathscr{D}(K)$ in the above definition for the p-variational capacity could be replaced by $\mathscr{E}(K)$ and $\mathscr{F}(K)$ (see e.g., [6, 30]). The p-variational capacity is fundamental in many areas, such as, analysis, geometry and physics. It has many properties similar to those for the general p-affine capacity, such as, homogeneity, monotonicity; however the p-variational capacity does not have the affine invariance.

The comparison between the general p-affine capacity and the p-variational capacity is stated in the following theorem. The case $\tau = 0$ was discussed in [43, Remark 2.7] and [42, Theorem 1.5']. Let $A(n, p)$ be the constant given in (2.16).

Theorem 5.1. *Let $p \in [1, n)$ and $K \subset \mathbb{R}^n$ be a compact set. For any $\tau \in [-1, 1]$, one has*

$$C_{p,\tau}(K) \leq A(n, p) \cdot C_p(K).$$

Proof. According to the proof of [31, Lemma 3.1] (or [13, Lemma 2]), $\|\varphi_\tau(\nabla_u f)\|_p > 0$ for any $f \in \mathscr{F}(K) \cap C_c^\infty$, for any $\tau \in [-1, 1]$ and for any $u \in S^{n-1}$. By Jensen's inequality, Fubini's theorem, (2.15) and (2.16), one has, for any $f \in \mathscr{F}(K) \cap C_c^\infty$,

$$\begin{aligned}
\mathscr{H}_{p,\tau}(f) &= \left(\int_{S^{n-1}} \left(\int_{\mathbb{R}^n} \left[\varphi_\tau(\nabla_u f)\right]^p dx \right)^{-\frac{n}{p}} du \right)^{-\frac{p}{n}} \\
&\leq \int_{S^{n-1}} \left(\int_{\mathbb{R}^n} \left[\varphi_\tau(\nabla_u f)\right]^p dx \right) du \\
&= \int_{\mathbb{R}^n} \left(\int_{S^{n-1}} \left[\varphi_\tau(\nabla_u f)\right]^p du \right) dx \\
&= \left(\int_{S^{n-1}} \left[\varphi_\tau(u \cdot v)\right]^p du \right) \cdot \left(\int_{\mathbb{R}^n} |\nabla f|^p \, dx \right) \\
&= A(n, p) \cdot \int_{\mathbb{R}^n} |\nabla f|^p \, dx,
\end{aligned}$$

where $v \in S^{n-1}$ (depending on $x \in \mathbb{R}^n$) is given by

$$v = \frac{\nabla f(x)}{|\nabla f(x)|} \quad \text{on} \quad \{x \in \mathbb{R}^n : \nabla f \neq 0\}.$$

Taking the infimum over $f \in \mathscr{F}(K) \cap C_c^\infty$, one has, by Theorem 3.3 and the definition of the p-variational capacity,

$$\begin{aligned} C_{p,\tau}(K) &= \inf_{f \in \mathscr{F}(K) \cap C_c^\infty} \mathscr{H}_{p,\tau}(f) \\ &\leq A(n,p) \cdot \inf_{f \in \mathscr{F}(K) \cap C_c^\infty} \int_{\mathbb{R}^n} |\nabla f|^p \, dx \\ &= A(n,p) \cdot C_p(K) \end{aligned}$$

holds for any $\tau \in [-1,1]$, for any $p \in [1,n)$ and for any compact set $K \subset \mathbb{R}^n$. □

It is well known (see e.g., [30, (2.2.13) and (2.2.14)]) that

$$C_p(B_n) = n\omega_n \cdot \left(\frac{n-p}{p-1}\right)^{p-1} \tag{5.22}$$

for $p \in (1,n)$, $C_p(B_n) = 0$ for $p \geq n$, and $C_1(B_n) = \lim_{p \to 1^+} C_p(B_n) = n\omega_n$. Hence, for any $\tau \in [-1,1]$,

$$C_{p,\tau}(B_n) \leq A(n,p) C_p(B_n) = A(n,p) \cdot n\omega_n \cdot \left(\frac{n-p}{p-1}\right)^{p-1} \tag{5.23}$$

holds for any $p \in (1,n)$, and

$$C_{1,\tau}(B_n) \leq A(n,1) C_1(B_n) = A(n,1) \cdot n\omega_n. \tag{5.24}$$

Following along the same lines as the proof of Theorem 5.1, one has, for any $\tau \in [-1,1]$ and for any $p \geq n$,

$$0 \leq C_{p,\tau}(B_n) \leq A(n,p) C_p(B_n) = 0.$$

Again due to the proofs of (4.20) and Theorem 4.1, $C_{p,\tau}(K) = 0$ for any $\tau \in [-1,1]$, for any $p \geq n$ and for any compact set $K \subset \mathbb{R}^n$.

5.2 Affine isocapacitary inequalities

This subsection dedicates to establish the affine isocapacitary inequality which compares the general p-affine capacity with the volume. An ellipsoid is a convex body of form $TB_n + x_0$ for some $T \in GL(n)$ and $x_0 \in \mathbb{R}^n$.

Theorem 5.2. *Let $p \in [1,n)$. For any $\tau \in [-1,1]$ and for any compact set $K \subset \mathbb{R}^n$, one has*

$$\left(\frac{C_{p,\tau}(K)}{C_{p,\tau}(B_n)}\right)^{\frac{1}{n-p}} \geq \left(\frac{V(K)}{V(B_n)}\right)^{\frac{1}{n}}$$

with equality if K is an ellipsoid.

Proof. Let $p \in (1,n)$, $\tau \in [-1,1]$ and $K \subset \mathbb{R}^n$ be a compact set. It follows from [13, inequality (5.8)] that for $f \in C_c^\infty \cap \mathscr{F}(K)$, $\|f\|_\infty = 1$ and

$$\left(\int_{S^{n-1}} \|\nabla_u^+ f\|_p^{-n} \, du\right)^{-\frac{p}{n}} \geq n^p \omega_n^{\frac{p}{n}} A(n,p) \int_0^1 \frac{V([f]_t)^{\frac{np-p}{n}}}{[-V([f]_t)']^{p-1}} \, dt,$$

where $V([f]_t)'$ is the derivative of $V([f]_t)$ with respect to t. Recall that for any real number $t > 0$ and for any $f \in C_c^\infty$,

$$[f]_t = \{x \in \mathbb{R}^n : |f(x)| \geq t\}.$$

Note that $V(K) \leq V([f]_1) \leq V([f]_0)$. Together with Jensen's inequality, one has, for $p \in (1, n)$,

$$\begin{aligned} \int_0^1 \frac{V([f]_t)^{\frac{np-p}{n}}}{\big[-V([f]_t)'\big]^{p-1}}\,dt &\geq \left(\int_0^1 V([f]_t)^{\frac{np-p}{n-np}}(-dV([f]_t)\right)^{1-p} \\ &= \left(\frac{np-n}{n-p}\cdot V([f]_t)^{\frac{n-p}{n-np}}\Big|_0^1\right)^{1-p} \\ &\geq \left(\frac{np-n}{n-p}\right)^{1-p} V([f]_1)^{\frac{n-p}{n}} \\ &\geq \left(\frac{np-n}{n-p}\right)^{1-p} V(K)^{\frac{n-p}{n}}. \end{aligned}$$

Together with (5.22), Theorem 3.3 and Corollary 4.1, for any $p \in (1, n)$ and for any $\tau \in [-1, 1]$,

$$\begin{aligned} C_{p,\tau}(K) &\geq C_{p,+}(K) \\ &= \inf_{f \in \mathscr{F}(K)\cap C_c^\infty} \left(\int_{S^{n-1}} \|\nabla_u^+ f\|_p^{-n}\,du\right)^{-\frac{p}{n}} \\ &\geq n\omega_n^{\frac{p}{n}} \cdot A(n,p)\cdot \left(\frac{n-p}{p-1}\right)^{p-1} V(K)^{\frac{n-p}{n}} \\ &= A(n,p)\cdot C_p(B_n)\cdot \left(\frac{V(K)}{V(B_n)}\right)^{\frac{n-p}{n}}. \end{aligned} \tag{5.25}$$

Let $K = B_n$ in inequality (5.25). Then, for any $p \in (1, n)$ and for any $\tau \in [-1, 1]$,

$$C_{p,\tau}(B_n) \geq A(n,p)\cdot C_p(B_n).$$

Together with (5.23), one gets, for any $p \in (1, n)$ and for any $\tau \in [-1, 1]$,

$$C_{p,\tau}(B_n) = A(n,p)\cdot C_p(B_n) = A(n,p)\cdot n\omega_n \cdot \left(\frac{n-p}{p-1}\right)^{p-1}. \tag{5.26}$$

Hence, inequality (5.25) can be rewritten as, for any $p \in (1, n)$, for any $\tau \in [-1, 1]$ and for any compact set $K \subset \mathbb{R}^n$,

$$\left(\frac{C_{p,\tau}(K)}{C_{p,\tau}(B_n)}\right)^{\frac{1}{n-p}} \geq \left(\frac{V(K)}{V(B_n)}\right)^{\frac{1}{n}}.$$

Now let us consider the case $p = 1$. For $f \in C_c^\infty \cap \mathscr{F}(K)$, it can be checked, due to the dominated convergence theorem, that for any $u \in S^{n-1}$ and for any $\tau \in [-1, 1]$,

$$\lim_{p\to 1^+} \|\varphi_\tau(\nabla_u f)\|_p = \|\varphi_\tau(\nabla_u f)\|_1.$$

By Fatou's lemma, one has

$$\begin{aligned}
\left(\int_{S^{n-1}} \|\varphi_\tau(\nabla_u f)\|_1^{-n}\, du\right)^{-\frac{1}{n}} &= \left(\int_{S^{n-1}} \lim_{p\to 1^+} \|\varphi_\tau(\nabla_u f)\|_p^{-n}\, du\right)^{-\frac{1}{n}} \\
&\geq \left(\liminf_{p\to 1^+} \int_{S^{n-1}} \|\varphi_\tau(\nabla_u f)\|_p^{-n}\, du\right)^{-\frac{1}{n}} \\
&= \limsup_{p\to 1^+} \left(\int_{S^{n-1}} \|\varphi_\tau(\nabla_u f)\|_p^{-n}\, du\right)^{-\frac{p}{n}} \\
&\geq \limsup_{p\to 1^+} C_{p,\tau}(K).
\end{aligned}$$

It follows from Theorem 3.3, after taking the infimum over $f \in C_c^\infty \cap \mathscr{F}(K)$, that for any $\tau \in [-1, 1]$ and for any compact set $K \subset \mathbb{R}^n$,

$$C_{1,\tau}(K) \geq \limsup_{p\to 1^+} C_{p,\tau}(K).$$

In particular, by (5.24) and (5.26), one has

$$A(n,1)\cdot n\omega_n \geq C_{1,\tau}(B_n) \geq \limsup_{p\to 1^+} C_{p,\tau}(B_n) = A(n,1)\cdot n\omega_n.$$

This gives the precise value of $C_{1,\tau}(B_n)$:

$$C_{1,\tau}(B_n) = A(n,1)\cdot n\omega_n = \lim_{p\to 1^+} C_{p,\tau}(B_n), \tag{5.27}$$

and hence inequality (5.25) yields

$$\left(\frac{C_{1,\tau}(K)}{C_{1,\tau}(B_n)}\right)^{\frac{1}{n-1}} \geq \limsup_{p\to 1^+} \left(\frac{C_{p,\tau}(K)}{C_{p,\tau}(B_n)}\right)^{\frac{1}{n-p}} \geq \left(\frac{V(K)}{V(B_n)}\right)^{\frac{1}{n}}$$

for any $\tau \in [-1, 1]$ and for any compact set $K \subset \mathbb{R}^n$, as desired.

Due to the affine invariance and the translation invariance, it is trivial to see that equality holds if K is an ellipsoid. □

Theorem 5.2 asserts that the general p-affine capacity attains the minimum, among all compact sets with fixed volume, at ellipsoids. It also asserts that ellipsoids have the maximal volumes among all compact sets with fixed general p-affine capacity. When $\tau = 0$, one recovers the affine isocapacitary inequality for the p-affine capacity proved in [43, Theorem 3.2] and [42, Theorem 1.3']. Recall that the isocapacitary inequality for the p-variational capacity reads: for any $p \in [1, n)$ and any compact set $K \subset \mathbb{R}^n$,

$$\left(\frac{C_p(K)}{C_p(B_n)}\right)^{\frac{1}{n-p}} \geq \left(\frac{V(K)}{V(B_n)}\right)^{\frac{1}{n}}.$$

It follows from Theorem 5.1 that the affine isocapacitary inequality in Theorem 5.2 is stronger than the isocapacitary inequality for the p-variational capacity. That is, for any $p \in [1, n)$, for any $\tau \in [-1, 1]$ and for any compact set $K \subset \mathbb{R}^n$,

$$\left(\frac{C_p(K)}{C_p(B_n)}\right)^{\frac{1}{n-p}} \geq \left(\frac{C_{p,\tau}(K)}{C_{p,\tau}(B_n)}\right)^{\frac{1}{n-p}} \geq \left(\frac{V(K)}{V(B_n)}\right)^{\frac{1}{n}}.$$

Moreover, combining the above inequality with [22, (12)], when $K \subset \mathbb{R}^n$ is a Lipschitz star body with the origin in its interior, the following inequality holds: for any $p \in [1, n)$ and for any $\tau \in [-1, 1]$,

$$\left(\frac{S_p(K)}{S_p(B_n)}\right)^{\frac{1}{n-p}} \geq \left(\frac{C_p(K)}{C_p(B_n)}\right)^{\frac{1}{n-p}} \geq \left(\frac{C_{p,\tau}(K)}{C_{p,\tau}(B_n)}\right)^{\frac{1}{n-p}} \geq \left(\frac{V(K)}{V(B_n)}\right)^{\frac{1}{n}},$$

where $S_p(K)$ denotes the p-surface area of K given by formula (1.3).

5.3 Connection with the general p-integral affine surface area

In this subsection, we explore the relation between the general p-affine capacity and the general p-integral affine surface area. Throughout, denote by $\mathscr{L}_0$ the set of all Lipschitz star bodies (with respect to the origin o) containing o in their interiors. For a Lipschitz star body $K \in \mathscr{L}_0$, let $\nu_K(x)$ denote the unit outer normal vector of ∂K at x (sometimes may be abbreviated as $\nu(x)$). Let D_K, the core of K, be given by

$$D_K = \big\{tx : \ t > 0, \ x \in \partial K, \ |x \cdot \nu(x)| > 0\big\}.$$

According to [22, Lemma 5], for each Lipschitz star body $K \subset \mathbb{R}^n$, one has

$$\nu_K(x) = -\frac{\nabla \rho_K(x)}{|\nabla \rho_K(x)|} \quad \text{and} \quad \nabla \rho_K(x) = -\frac{\nu_K(x)}{x \cdot \nu_K(x)}$$

for almost all $x \in \partial K \cap D_K$.

For $p \geq 1$ and $\tau \in [-1, 1]$, define $\Pi_{p,\tau}(K)$, the general L_p projection body of $K \in \mathscr{L}_0$, to be the convex body with support function $h_{\Pi_{p,\tau}(K)}$; namely, for any $\theta \in S^{n-1}$,

$$h_{\Pi_{p,\tau}(K)}(\theta) = \left(\int_{\partial K} \big[\varphi_\tau(\theta \cdot \nu_K(x))\big]^p \cdot |x \cdot \nu_K(x)|^{1-p}\, d\mathscr{H}^{n-1}(x)\right)^{\frac{1}{p}}.$$

Note that $|x \cdot \nu_K(x)|^{-1} = |\nabla \rho_K(x)|$ is bounded on ∂K because $\rho_K(x)$ is Lipschitz continuous on ∂K, and hence $h_{\Pi_{p,\tau}(K)}$ is finite. The general L_1 projection body can be defined for more general sets in $\mathbb{R}^n$, such as compact domains (i.e., the closure of bounded open sets) with piecewise C^1 boundaries (or compact domains with finite perimeters). When $K \in \mathscr{K}_0$, formula (2.11) yields that, for any $\theta \in S^{n-1}$,

$$h_{\Pi_{p,\tau}(K)}(\theta) = \left(\int_{S^{n-1}} \big[\varphi_\tau(u \cdot \theta)\big]^p h_K(u)^{1-p}\, dS(K, u)\right)^{\frac{1}{p}}.$$

Denote by $v_{p,\tau}(K, \cdot) = h^p_{\Pi_{p,\tau}(K)}(\cdot)$ the general p-projection function of K. The general p-integral affine surface area of $K \in \mathscr{L}_0$ is defined by

$$\Phi_{p,\tau}(K) = \left(\int_{S^{n-1}} \big[v_{p,\tau}(K, u)\big]^{-\frac{n}{p}}\, du\right)^{-\frac{p}{n}} = \omega_n^{\frac{p}{n}} V(\Pi^\circ_{p,\tau}(K))^{-\frac{p}{n}}, \tag{5.28}$$

where du is the normalized spherical measure and $\Pi^\circ_{p,\tau}(K)$ is the polar body of $\Pi_{p,\tau}(K)$. When $\tau = 0$, one gets the p-integral affine surface area of $K \in \mathscr{L}_0$ in, e.g., [22, 47]. The case $\tau = 1$ defines the asymmetric p-integral affine surface area, denoted by $\Phi_{p,+}(K)$, of $K \in \mathscr{L}_0$. Similarly, one can also define $\Phi_{p,-}(K)$ if $\tau = -1$. When $K = B_n$, by (2.16), (5.26) and (5.27), for any $p \geq 1$ and for any $\tau \in [-1, 1]$,

$$\Phi_{p,\tau}(B_n) = \left(\frac{n-p}{p-1}\right)^{1-p} C_{p,\tau}(B_n). \tag{5.29}$$

It can be checked that for any $T \in GL(n)$,

$$\Phi_{p,\tau}(TK) = |\det T|^{\frac{n-p}{n}} \Phi_{p,\tau}(K).$$

Similar to the proof of Corollary 4.1, the following properties for the general p-integral affine surface area can be proved. One cannot expect that the general p-integral affine surface area has the translation invariance (unless $p = 1$, see following Proposition 5.1) and monotonicity.

Corollary 5.1. *Let $p \geq 1$ and $K \in \mathscr{L}_0$. The following statements hold:*

i) for any $\tau \in [-1, 1]$,

$$\Phi_{p,\tau}(K) = \Phi_{p,-\tau}(K);$$

ii) for any $\lambda \in [0, 1]$ and for any $\tau, \gamma \in [-1, 1]$,

$$\Phi_{p,\lambda\tau+(1-\lambda)\gamma}(K) \geq \lambda \cdot \Phi_{p,\tau}(K) + (1 - \lambda) \cdot \Phi_{p,\gamma}(K);$$

iii) for any $\tau \in [-1, 1]$,

$$\Phi_{p,+}(K) = \Phi_{p,-}(K) \leq \Phi_{p,\tau}(K) \leq \Phi_{p,0}(K);$$

iv) if $-1 < \tau < \gamma \leq 0$, then

$$\Phi_{p,\tau}(K) \leq \Phi_{p,\gamma}(K)$$

and if $0 < \tau \leq \gamma < 1$, then

$$\Phi_{p,\gamma}(K) \leq \Phi_{p,\tau}(K).$$

By $\mathscr{C}_1$, we mean the set of all compact domains with piecewise C^1 boundaries. Again, for $M \in \mathscr{C}_1$, its outer unit normal vector is denoted by $\nu_M(x)$ for $x \in \partial M$. In the following proposition, we show that the general 1-affine capacity and the general 1-integral affine surface area are all equal to the 1-affine capacity (or equivalently, the 1-integral affine surface area) for any $M \in \mathscr{C}_1$.

Proposition 5.1. *Let $M \in \mathscr{C}_1$ be a compact domain with piecewise C^1 boundary. For any $\tau \in [-1, 1]$, one has*

$$C_{1,0}(M) = C_{1,\tau}(M) = \Phi_{1,\tau}(M) = \Phi_{1,0}(M).$$

Proof. We first prove $C_{1,0}(M) = C_{1,\tau}(M)$ for $M \in \mathscr{C}_1$; it follows immediately from Theorem 3.3 once $\|\varphi_\tau(\nabla_u f)\|_1 = \|\varphi_0(\nabla_u f)\|_1$ is established for any $f \in C_c^\infty \cap \mathscr{F}(M)$. To this end, for any $M_0 \in \mathscr{C}_1$ and for any $u \in S^{n-1}$,

$$\int_{\partial M_0} (u \cdot \nu_{M_0}(x))\, d\mathscr{H}^{n-1}(x) = 0 \quad \text{and} \quad \int_{\partial M_0} |u \cdot \nu_{M_0}(x)|\, d\mathscr{H}^{n-1}(x) > 0. \tag{5.30}$$

Note that (5.30) together with the Minkowski existence theorem leads to the powerful convexification technique, see e.g., [46, p.189-190]. For almost every $t \in (0, 1)$ with $f \in C_c^\infty \cap \mathscr{F}(M)$, it follows from the Sard's theorem, (2.13) and (5.30) that, for any $\tau \in [-1, 1]$,

$$\begin{aligned}
\|\varphi_\tau(\nabla_u f)\|_1 &= \int_{\mathbb{R}^n} \left(\frac{1}{2}|u \cdot \nabla f| + \frac{\tau}{2} u \cdot \nabla f\right) dx \\
&= \int_0^1 \int_{\partial[f]_t} \left(\frac{1}{2}|u \cdot \nu(x)| + \frac{\tau}{2} u \cdot \nu(x)\right) d\mathscr{H}^{n-1}(x)dt \\
&= \int_0^1 \int_{\partial[f]_t} \frac{|u \cdot \nu(x)|}{2}\, d\mathscr{H}^{n-1}(x)dt \\
&= \int_{\mathbb{R}^n} \frac{|u \cdot \nabla f|}{2}\, dx \\
&= \|\varphi_0(\nabla_u f)\|_1.
\end{aligned}$$

This concludes the proof of $C_{1,0}(M) = C_{1,\tau}(M)$ for $M \in \mathscr{C}_1$.

On the other hand, for any $\tau \in [-1,1]$,

$$\begin{aligned}
v_{1,\tau}(M,\theta) &= \int_{\partial M} \varphi_\tau(\theta \cdot \nu_M(x))\, d\mathscr{H}^{n-1}(x) \\
&= \int_{\partial M} \left(\frac{|\theta \cdot \nu_M(x)|}{2} + \frac{\tau}{2}\big(\theta \cdot \nu_M(x)\big) \right) d\mathscr{H}^{n-1}(x) \\
&= \int_{\partial M} \frac{|\theta \cdot \nu_M(x)|}{2}\, d\mathscr{H}^{n-1}(x) \\
&= v_{0,\tau}(M,\theta),
\end{aligned}$$

where the third equality follows again from (5.30). Consequently, for any $\tau \in [-1,1]$ and for any $M \in \mathscr{C}_1$,

$$\Phi_{1,\tau}(M) = \left(\int_{S^{n-1}} \big[v_{1,\tau}(K,u)\big]^{-n} du \right)^{-\frac{1}{n}} = \Phi_{1,0}(M).$$

Finally, let us prove that $C_{1,0}(M) = \Phi_{1,0}(M)$ holds for any $M \in \mathscr{C}_1$. For each function $f \in C_c^\infty \cap \mathscr{F}(M)$, it follows from (2.13), (2.15), and $M \subset [f]_t$ for any $t \in [0,1]$ that

$$\begin{aligned}
\|\varphi_0(\nabla_u f)\|_1 &= \int_{\mathbb{R}^n} \varphi_0(\nabla_u f)\, dx \\
&= \frac{1}{2} \int_0^1 \int_{\partial [f]_t} |u \cdot \nu(x)|\ d\mathscr{H}^{n-1}(x)\ dt \\
&= \frac{1}{2} \int_0^1 \int_{\Pi_u [f]_t} \#([f]_t \cap (y + u\mathbb{R}))\, d\mathscr{H}^{n-1}(y) dt \\
&\geq \frac{1}{2} \int_0^1 \int_{\Pi_u M} \#(M \cap (y + u\mathbb{R}))\, d\mathscr{H}^{n-1}(y) dt \\
&= \frac{1}{2} \int_0^1 \int_{\partial M} |u \cdot \nu_M(x)|\ d\mathscr{H}^{n-1}(x)\ dt \\
&= v_{1,0}(M,u),
\end{aligned}$$

where $\Pi_u K$ is the projection of $K \subset \mathbb{R}^n$ onto $u^\perp = \{x \in \mathbb{R}^n :\ x \cdot u = 0\}$ and $\#$ denotes the number of elements of a set (see e.g., [47]). Thus, for any $M \in \mathscr{C}_1$ and for any $f \in C_c^\infty \cap \mathscr{F}(M)$,

$$\left(\int_{S^{n-1}} \left(\int_{\mathbb{R}^n} \varphi_0(\nabla_u f)\ dx \right)^{-n} du \right)^{-\frac{1}{n}} \geq \left(\int_{S^{n-1}} v_{1,0}(M,u)^{-n}\ du \right)^{-\frac{1}{n}} = \Phi_{1,0}(M).$$

Due to Theorem 3.3, by taking the infimum over $f \in C_c^\infty \cap \mathscr{F}(M)$, one gets, for any $M \in \mathscr{C}_1$,

$$C_{1,0}(M) \geq \Phi_{1,0}(M).$$

For the opposite direction, let $\varepsilon > 0$ be small enough and consider

$$f_\varepsilon(x) = \begin{cases} 0 & \text{if } \operatorname{dist}(x,K) \geq \varepsilon, \\ 1 - \frac{\operatorname{dist}(x,K)}{\varepsilon} & \text{if } \operatorname{dist}(x,K) < \varepsilon. \end{cases}$$

It has been proved in [46] that for any $u \in S^{n-1}$,

$$\lim_{\varepsilon \to 0} \|\varphi_0(\nabla_u f_\varepsilon)\|_1 = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \varphi_0(\nabla_u f_\varepsilon)\, dx = v_{1,0}(M,u).$$

Note that $f_\varepsilon \in \mathscr{F}(M)$ for any $\varepsilon > 0$ small enough. It follows from Theorem 3.1 that, for any $M \in \mathscr{C}_1$,

$$\begin{aligned} C_{1,0}(M) &\le \limsup_{\varepsilon\to 0}\left(\int_{S^{n-1}}\left(\int_{\mathbb{R}^n}\varphi_0(\nabla_u f_\varepsilon)\,dx\right)^{-n}du\right)^{-\frac{1}{n}} \\ &= \left(\liminf_{\varepsilon\to 0}\int_{S^{n-1}}\left(\int_{\mathbb{R}^n}\varphi_0(\nabla_u f_\varepsilon)\,dx\right)^{-n}du\right)^{-\frac{1}{n}} \\ &\le \left(\int_{S^{n-1}}\lim_{\varepsilon\to 0}\left(\int_{\mathbb{R}^n}\varphi_0(\nabla_u f_\varepsilon)\,dx\right)^{-n}du\right)^{-\frac{1}{n}} \\ &= \left(\int_{S^{n-1}}\left(\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n}\varphi_0(\nabla_u f_\varepsilon)\,dx\right)^{-n}du\right)^{-\frac{1}{n}} \\ &= \left(\int_{S^{n-1}}(v_{1,0}(M,u))^{-n}\,du\right)^{-\frac{1}{n}} \\ &= \Phi_{1,0}(M), \end{aligned}$$

where the second inequality is due to Fatou's lemma. This concludes the proof of

$$C_{1,0}(M) = \Phi_{1,0}(M)$$

for any $M \in \mathscr{C}_1$. □

When M is an origin-symmetric convex body, the equality $C_{1,0}(M) = \Phi_{1,0}(M)$ was proved in [44, Theorem 2]; Proposition 5.1 extends it to all Lipschitz star bodies $M \in \mathscr{L}_0$. The proof of $C_{1,0}(M) = C_{1,\tau}(M)$ basically relies on the smoothness (and the convexification) of $\partial[f]_t$ instead of the compact domain M itself; hence, the argument $C_{1,0}(M) = C_{1,\tau}(M)$ holds for any compact set $M \subset \mathbb{R}^n$ and for any $\tau \in [-1,1]$. The assumption $M \in \mathscr{C}_1$ is imposed here mainly in order to have $\Phi_{1,0}(M)$ well defined and finite. As commented in [47, p.247], the assumption $M \in \mathscr{C}_1$ could be relaxed to more general compact domains (such as compact domains with finite perimeters). Recall the affine isoperimetric inequality for the 1-integral affine surface area: for $M \in \mathscr{C}_1$,

$$\left(\frac{V(M)}{V(B_n)}\right)^{\frac{1}{n}} \le \left(\frac{\Phi_{1,0}(M)}{\Phi_{1,0}(B_n)}\right)^{\frac{1}{n-1}}$$

with equality if and only if M is an ellipsoid. Then Proposition 5.1 yields that for any $M \in \mathscr{C}_1$ and for any $\tau \in [-1,1]$,

$$\left(\frac{V(M)}{V(B_n)}\right)^{\frac{1}{n}} \le \left(\frac{\Phi_{1,\tau}(M)}{\Phi_{1,\tau}(B_n)}\right)^{\frac{1}{n-1}} = \left(\frac{C_{1,\tau}(M)}{C_{1,\tau}(B_n)}\right)^{\frac{1}{n-1}}$$

with equality if and only if M is an ellipsoid.

The following theorem compares the general p-affine capacity and the general p-integral affine surface area. We only concentrate on $p \in (1,n)$ as the case $p = 1$ has been discussed in Proposition 5.1. When $\tau = 0$ and K is an origin-symmetric convex body, it recovers [43, Theorem 3.5].

Theorem 5.3. *Let $K \in \mathscr{L}_0$ and $1 < p < n$. The following inequality*

$$\frac{C_{p,\tau}(K)}{C_{p,\tau}(B_n)} \le \frac{\Phi_{p,\tau}(K)}{\Phi_{p,\tau}(B_n)} \quad \text{for any} \quad \tau \in [-1,1]$$

holds with equality if K is an origin-symmetric ellipsoid.

Proof. Let $K \in \mathscr{L}_0$ and $p \in (1, n)$. Define the function g by: for $s > 0$,

$$g(s) = \min\Big\{1, \quad s^{\frac{n-p}{1-p}}\Big\}.$$

Let $f(x) = g\big(\frac{1}{\rho_K(x)}\big)$. Then $f(x) \geq \mathbf{1}_K$ and $\|f\|_\infty = 1$. From (2.12) and the fact that g is strictly decreasing on $s \in (1, \infty)$, it follows that, for all $t \in (0, 1)$ with $t = g(s) = s^{\frac{n-p}{1-p}}$,

$$[f]_t = \{x \in \mathbb{R}^n : 1/\rho_K(x) \leq s\}.$$

That is, $[f]_t = [f]_{g(s)} = sK$ for any $s > 1$. Together with [22, Lemma 6], for any $x \in \partial[f]_t$, there exists $z \in \partial K$ with $x = sz$ such that

$$|\nabla f(x)| = \frac{|g'(s)|}{|z \cdot \nu_K(z)|} \quad \text{and} \quad \nu_K(z) = \nu_{[f]_t}(x) = -\frac{\nabla f(x)}{|\nabla f(x)|}.$$

By (2.13), one has, for any $u \in S^{n-1}$,

$$\begin{aligned}
\|\varphi_\tau(\nabla_u f)\|_p^p &= \int_0^1 \int_{\partial[f]_t} \big[\varphi_\tau(-u \cdot \nu_{[f]_t}(x))\big]^p \cdot |\nabla f(x)|^{p-1}\, d\mathscr{H}^{n-1}(x)\, dt \\
&= \int_1^\infty |g'(s)| \int_{\partial[f]_{g(s)}} \big[\varphi_\tau(-u \cdot \nu_{[f]_{g(s)}}(x))\big]^p \cdot |\nabla f(x)|^{p-1}\, d\mathscr{H}^{n-1}(x)\, ds \\
&= \int_1^\infty |g'(s)|^p s^{n-1} \int_{\partial K} \big[\varphi_\tau(-u \cdot \nu_K(z))\big]^p \cdot |z \cdot \nu_K(z)|^{1-p}\, d\mathscr{H}^{n-1}(z)\, ds \\
&= \Big(\frac{n-p}{p-1}\Big)^p \Big(\int_1^\infty s^{\frac{n-1}{1-p}}\, ds\Big)\Big(\int_{\partial K} \big[\varphi_\tau(-u \cdot \nu_K(z))\big]^p \cdot |z \cdot \nu_K(z)|^{1-p}\, d\mathscr{H}^{n-1}(z)\Big) \\
&= \Big(\frac{n-p}{p-1}\Big)^{p-1} v_{p,\tau}(K, -u).
\end{aligned}$$

It follows from (3.17) and (5.28) that

$$\begin{aligned}
\mathscr{H}_{p,\tau}(f) &= \Big(\int_{S^{n-1}} \|\varphi_\tau(\nabla_u f)\|_p^{-n}\, du\Big)^{-\frac{p}{n}} \\
&= \Big(\frac{n-p}{p-1}\Big)^{p-1} \Big(\int_{S^{n-1}} v_{p,\tau}(K, -u)^{-\frac{n}{p}}\, du\Big)^{-\frac{p}{n}} \\
&= \Big(\frac{n-p}{p-1}\Big)^{p-1} \Phi_{p,\tau}(K).
\end{aligned}$$

A standard limiting argument together with Definition 3.1 show that, for any $p \in (1, n)$, for any $\tau \in [-1, 1]$ and for any $K \in \mathscr{L}_0$,

$$C_{p,\tau}(K) \leq \Big(\frac{n-p}{p-1}\Big)^{p-1} \Phi_{p,\tau}(K).$$

By (5.29), the above inequality can be rewritten as

$$\frac{C_{p,\tau}(K)}{C_{p,\tau}(B_n)} \leq \frac{\Phi_{p,\tau}(K)}{\Phi_{p,\tau}(B_n)}.$$

Clearly equality holds in the above inequality if $K = B_n$. Due to the affine invariance of both $C_{p,\tau}(\cdot)$ and $\Phi_{p,\tau}(\cdot)$, equality holds in the above inequality if K is an origin-symmetric ellipsoid. $\square$

Together with [22, (13)], Corollary 5.1 and Theorem 5.2, for any $K \in \mathscr{L}_0$, for any $p \in (1, n)$ and for any $\tau \in [-1, 1]$, one has,

$$\left(\frac{S_p(K)}{S_p(B_n)}\right)^{\frac{1}{n-p}} \geq \left(\frac{\Phi_{p,0}(K)}{\Phi_{p,0}(B_n)}\right)^{\frac{1}{n-p}} \geq \left(\frac{\Phi_{p,\tau}(K)}{\Phi_{p,\tau}(B_n)}\right)^{\frac{1}{n-p}} \geq \left(\frac{C_{p,\tau}(K)}{C_{p,\tau}(B_n)}\right)^{\frac{1}{n-p}} \geq \left(\frac{V(K)}{V(B_n)}\right)^{\frac{1}{n}} \tag{5.31}$$

with equality if K is an origin-symmetric ellipsoid. Inequality (5.31) extends several known results in the literature. For example, inequality (5.31) strengthens the following (affine) isoperimetric inequality (see [22, inequality (13)]): for $\tau = 0$ and for any $K \in \mathscr{L}_0$,

$$\left(\frac{S_p(K)}{S_p(B_n)}\right)^{\frac{1}{n-p}} \geq \left(\frac{\Phi_{p,0}(K)}{\Phi_{p,0}(B_n)}\right)^{\frac{1}{n-p}} \geq \left(\frac{V(K)}{V(B_n)}\right)^{\frac{1}{n}}.$$

Moreover, inequality (5.31) holds for all $K \in \mathscr{K}_0 \subset \mathscr{L}_0$, and hence it extends the following affine isoperimetric inequality (5.32) for convex bodies to Lipschitz star bodies: for any $K \in \mathscr{K}_0$, for any $\tau \in [-1, 1]$ and for any $p \in (1, n)$,

$$\left(\frac{\Phi_{p,\tau}(K)}{\Phi_{p,\tau}(B_n)}\right)^{\frac{1}{n-p}} \geq \left(\frac{V(K)}{V(B_n)}\right)^{\frac{1}{n}}, \tag{5.32}$$

which is an immediate consequence of the general L_p affine isoperimetric inequality for the general L_p projection body [12].

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