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Noncommutative uncertainty principles



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ABSTRACT

The classical uncertainty principles deal with functions on abelian groups. In this paper, we discuss the uncertainty principles for finite index subfactors which include the cases for finite groups and finite dimensional Kac algebras. We prove the Hausdorff–Young inequality, Young’s inequality, the Hirschman–Beckner uncertainty principle, the Donoho–Stark uncertainty principle. We characterize the minimizers of the uncertainty principles and then we prove Hardy’s uncertainty principle by using minimizers. We also prove that the minimizer is uniquely determined by the supports of itself and its Fourier transform. The proofs take the advantage of the analytic and the categorial perspectives of subfactor planar algebras. Our method to prove the uncertainty principles also works for more general cases, such as Popa’s λ -lattices, modular tensor categories, etc.

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1. Introduction

In quantum mechanics, Heisenberg’s uncertainty principle says that the more precisely the position of some particle is determined, the less precisely its momentum can be known. This uncertainty principle is a fundamental result in quantum mechanics. By using the Shannon entropy of the measurement, entropic uncertainty principles were established, which strengthen and generalize Heisenberg’s uncertainty principle. The phenomenon of “uncertainty” also appears in mathematics. In Harmonic Analysis, various uncertainty principles are found by mathematicians such as Hardy’s uncertainty principle [14], Hirschman’s uncertainty principle [15] etc. In information theory, Donoho and Stark [11] introduced an uncertainty principle. They proved that a signal may be reconstructed with a few samples. This idea was further developed by Candes, Romberg and Tao [9] in the theory of compressed sensing.

The classical uncertainty principles deal with functions on an abelian group and its dual. A natural question is to ask whether there exist uncertainty principles for a non-abelian group. Recently the uncertainty principle of finite groups [13], compact groups [1] and Kac algebras (C^* Hopf algebras or quantum groups in von Neumann algebraic setting) [8] were discussed from different perspectives. It is natural to consider the uncertainty principles of Kac algebras due to the existence of a proper pair of observables. There is a one to one correspondence between Kac algebras and a special family of subfactors, namely irreducible *depth-2* subfactors [29]. Subfactors naturally provide vector spaces with two or more observables. In this paper, we are going to talk about the uncertainty principles of subfactors.

Modern subfactor theory was initiated by Jones. Subfactors generalize the symmetries of groups and their duals. The index of a subfactor generalizes the order of a group. All possible indices of subfactors,

$$\{4 \cos^2 \frac{\pi}{n}, n = 3, 4, \dots\} \cup [4, \infty],$$

were found by Jones in his remarkable rigidity result [18]. A deep theorem of Popa [26] showed that the *standard invariant*, a \mathbb{Z}_2 graded filtered algebra, is a complete invariant of *finite depth* subfactors of the hyperfinite factor of type II_1 . There are three axiomatizations of the standard invariant: Ocneanu’s paragroups [24]; Popa’s standard λ -lattices [27]; Jones’ subfactor planar algebras [17].

In terms of paragroups, the n -box space of the standard invariant of a (finite index, finite depth, irreducible) subfactor consists of linear sums of certain length- $2n$ loops of a bipartite graph, the *principal graph* of the subfactor. The Fourier transform of paragroups was introduced by Ocneanu [24] as the 1-click rotation of loops. It generalizes the Fourier transform of finite groups. Note that the Fourier transform of the n -box space has periodicity $2n$. Up to an anti-isomorphism, the n -box space admits n^* -algebraic structures and a proper n -tuple of observables. We will prove the uncertainty principles of n -box spaces. In particular, if we start with a subfactor arising from an outer action

of a finite abelian group, then we will recover the group and its dual from the 2-box spaces of the standard invariant of the subfactor. Moreover, the classical uncertainty principles of a finite abelian group will be recovered from those of 2-box spaces. Hence the uncertainty principles of 2-box spaces are our main interest.

For 2-box spaces, we will prove the Hausdorff–Young inequality, Young’s inequality, the Hirschman–Beckner uncertainty principle and the Donoho–Stark uncertainty principle and characterize the minimizers of the uncertainty principles. By using the minimizer, we prove Hardy’s uncertainty principle. Our proof of the Hausdorff–Young inequality also works for n -box spaces and more general cases, such as Popa’s λ -lattices, modular tensor categories, etc. The uncertainty principles follow directly from the Hausdorff–Young inequality. So far we could not find Young’s inequality for Kac algebras in the literature. Our characterization of the minimizers of the uncertainty principles is new for finite groups. The proofs of these results benefit a lot from the analytic and the categorial perspectives of subfactor planar algebras. We could not find an alternative proof without the knowledge of subfactor planar algebras.

The minimizers of the uncertainty principles of finite abelian groups are given by translations and modulations of indicator functions of its subgroups [11,25]. Unlike abelian groups, the 2-box space forms a noncommutative algebra in general. This makes extra difficulties to characterize the minimizers of the uncertainty principles. Fortunately Bisch and Jones introduced *biprojections* [3,5] in subfactors which generalize the indicator functions of subgroups. We will introduce a notion of *bi-shifts of biprojections* in terms of subfactor planar algebras as an equivalent characterization of the minimizers. Furthermore, we will prove that a bi-shift of a biprojection is uniquely determined by the *supports* of itself and its Fourier transform. We would like to see the application of this result in signal recovery.

It will be interesting to understand the uncertainty principle of infinite index subfactors. But the Fourier transform is not clear in general. We would like to refer to [6,12,20] for some known results of infinite index subfactors and [22] for locally compact quantum groups. There is no canonical choice of the measurement in general. Connes’ spatial theory [7] should be involved. These will lead us to the world of Abstract Harmonic Analysis and Noncommutative Geometry.

This paper is organized as follows. In Section 2, we briefly introduce classical uncertainty principles, subfactors and our main results. In Section 3, we review subfactor planar algebras and some notations, such as the Fourier transform, Wenzl’s formula, the local relation, etc. In Section 4, we find the Hausdorff–Young inequality and Young’s inequality for irreducible subfactor planar algebras. In Section 5, we show the Donoho–Stark and the Hirschman–Beckner uncertainty principles for irreducible subfactor planar algebras. In Section 6, we show that the two uncertainty principles found in the last section have the same minimizers and determine the minimizers in terms of biprojections. We also show Hardy’s uncertainty principle by using the minimizers. In Section 7, we generalize the first two uncertainty principles to general cases. In Sections 8 and 9, we show some applications of our uncertainty principles for groups and group actions.

2. Overview

In 1927, Heisenberg discovered the uncertainty principle

$$\Delta x \Delta p \geq \frac{\hbar}{2},$$

where x is the position of a particle, p is its momentum, \hbar is the reduced Planck constant. It has a mathematical formulation

$$\left(\int_{-\infty}^{\infty} x^2 |f(x)|^2 dx \right) \left(\int_{-\infty}^{\infty} \xi^2 |\hat{f}(\xi)|^2 d\xi \right) \geq \frac{\|f\|_2^4}{16\pi^2},$$

where $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx$.

In 1957, Hirschman [15] proved a stronger uncertainty principle for the Schwartz space on \mathbb{R} ,

$$H_s(|f|^2) + H_s(|\hat{f}|^2) \geq 0, \quad \|f\|_2 = 1,$$

where H_s is the Shannon Entropy $H_s(|f|^2) = - \int_{-\infty}^{\infty} |f|^2 \log |f|^2 dx$. He conjectured that

$$H_s(|f|^2) + H_s(|\hat{f}|^2) \geq \log \frac{e}{2}, \quad \|f\|_2 = 1.$$

This was proved by Beckner [2] in 1975.

In 1989, Donoho and Stark [11] established an uncertainty principle for finite cyclic groups \mathbb{Z}_n . Suppose f is a nonzero function on \mathbb{Z}_n and \hat{f} is its Fourier transform,

$$\hat{f}(k) = \sum_{j=0}^{n-1} f(j) e^{2\pi i j k / n}.$$

The support of f is denoted by $\text{supp}(f) = \{x \in \mathbb{Z}_n : f(x) \neq 0\}$. Then

$$|\text{supp}(f)| |\text{supp}(\hat{f})| \geq n. \tag{1}$$

They proved that the equality of (1) holds if and only if $f = c \sum_{h \in H} \chi(h) \delta_{h+k}$, where c is a nonzero constant, H is a subgroup of \mathbb{Z}_n , χ is a character of \mathbb{Z}_n , k is an element in \mathbb{Z}_n , and δ_{h+k} is the Dirac function at $h+k$.

In 1990, K. Smith [28] generalized the Donoho–Stark uncertainty principle to finite abelian groups.

In 2004, Özaydm and Przebinda [25] generalized the Hirschman–Beckner uncertainty principle and the Donoho–Stark uncertainty principle to abelian groups. They characterized all minimizers of the uncertainty principles and noticed that the minimizers of two uncertainty principles coincide.

In 2008, Alagic and Russell [1] generalized the Donoho–Stark uncertainty principles to compact groups.

In 2005, Tao [30] proved a stronger uncertainty principle for the cyclic group \mathbb{Z}_p , p prime.

In 1933, Hardy [14] proves an uncertainty principle for real line \mathbb{R} . If

$$|f(x)| \leq C e^{-\pi a x^2}, \quad |\hat{f}(\xi)| \leq C' e^{-\pi \xi^2/a},$$

for a measurable function f on \mathbb{R} and some constants $C, C', a > 0$, then

$$f \text{ is a scalar multiple of } e^{-\pi a x^2}.$$

In this paper, we discuss the uncertainty principles for finite index subfactors which cover the case for finite groups and finite dimensional Kac algebras. We prove the Hausdorff–Young inequality (see Theorem 4.8), Young’s inequality (see Theorem 4.13), the Hirschman–Beckner uncertainty principle (see Theorem 5.5), the Donoho–Stark uncertainty principle (see Theorem 5.2), and characterize the minimizers of the uncertainty principles (see Theorem 6.4,6.13). By using the minimizers, we prove the Hardy’s uncertainty principle (see Corollary 6.17)

In order to state the results, let us review some basic concepts and notations in subfactor theory. Suppose $\mathcal{N} \subset \mathcal{M}$ is an irreducible subfactor of type II₁ with finite index, denoted as $[\mathcal{M} : \mathcal{N}]$, and $\delta = \sqrt{[\mathcal{M} : \mathcal{N}]}$. Let $L^2(\mathcal{M})$ be the Gelfand–Naimark–Segal representation of \mathcal{M} derived from the unique trace of \mathcal{M} . The Jones projection $e_1 \in B(L^2(\mathcal{M}))$ is the projection onto the subspace $L^2(\mathcal{N})$ of $L^2(\mathcal{M})$. Then we have the Jones basic construction $\mathcal{N} \subset \mathcal{M} \subset \mathcal{M}_1$, where $\mathcal{M}_1 = (\mathcal{M} \cup \{e_1\})''$.

Repeating the Jones basic construction, we have the Jones tower

$$\mathcal{N} \subset \overset{e_1}{\mathcal{M}} \subset \overset{e_2}{\mathcal{M}_1} \subset \overset{e_3}{\mathcal{M}_2} \subset \dots$$

Taking the higher relative commutants, we have the standard invariant of the subfactor

$$\begin{array}{ccccccc} \mathcal{N}' \cap \mathcal{N} & \subset & \mathcal{N}' \cap \mathcal{M} & \subset & \mathcal{N}' \cap \mathcal{M}_1 & \subset & \mathcal{N}' \cap \mathcal{M}_2 \subset \dots \\ & & \cup & & \cup & & \cup \\ & & \mathcal{M}' \cap \mathcal{M} & \subset & \mathcal{M}' \cap \mathcal{M}_1 & \subset & \mathcal{M}' \cap \mathcal{M}_2 \subset \dots \end{array}$$

Jones introduced subfactor planar algebras as an axiomatization of the standard invariant. The subfactor planar algebra associated with the subfactor $\mathcal{N} \subset \mathcal{M}$ is a

collection of \mathbb{Z}_2 graded vector spaces $\{\mathcal{P}_{n,\pm}\}_{n \in \mathbb{N} \cup \{0\}}$ with actions labeled by *planar tangles*. Precisely $\mathcal{P}_{n,+} = \mathcal{N}' \cap \mathcal{M}_{n-1}$, $\mathcal{P}_{n,-} = \mathcal{M}' \cap \mathcal{M}_n$, called the n -box space of the subfactor planar algebras. An element in the n -box space is called an n -box. The action of a planar tangle is a composition of elementary operations of the standard invariant, such as multiplications, inclusions and conditional expectations.

If $\mathcal{M} = \mathcal{N} \rtimes G$, for an outer action a finite abelian group G , then the index $[\mathcal{M} : \mathcal{N}]$ is the order of the group $|G|$. We have the Jones tower

$$\mathcal{N} \subset \mathcal{N} \rtimes G \subset \mathcal{N} \rtimes G \rtimes \hat{G} \subset \mathcal{N} \rtimes G \rtimes \hat{G} \rtimes G \subset \dots$$

and the standard invariant

$$\begin{array}{ccccccc} \mathbb{C} & \subset & \mathbb{C} & \subset & L(\hat{G}) & \subset & L(\hat{G} \rtimes G) & \subset & \dots \\ & & \cup & & \cup & & \cup & & \\ & & \mathbb{C} & \subset & \mathbb{C} & \subset & L(G) & \subset & \dots \end{array}$$

The 2-box spaces of the standard invariant ($\mathcal{M}' \cap \mathcal{M}_2$ and $\mathcal{N}' \cap \mathcal{M}_1$) recover the group G and its dual \hat{G} . Hence the uncertainty principles of 2-box spaces are our main interest. Moreover, the 3-box space $\mathcal{N}' \cap \mathcal{M}_2$ naturally provides an algebra to consider G and \hat{G} simultaneously.

There is an (unnormalized) trace of the 2-box space, denoted by tr_2 . For a 2-box x , its p -norm is given by

$$\|x\|_p = tr_2(|x|^p)^{\frac{1}{p}}, \quad p \geq 1.$$

The Fourier transform \mathcal{F} was introduced by Ocneanu. In terms of planar algebras, it is a 1-click rotation diagrammatically.

We have the Hausdorff–Young inequality for a 2-box x (see [Theorem 4.8](#)),

$$\|\mathcal{F}(x)\|_p \leq \left(\frac{1}{\delta}\right)^{1-\frac{2}{p}} \|x\|_q, \quad 2 \leq p \leq \infty, \frac{1}{p} + \frac{1}{q} = 1.$$

Recall that δ is the square root of the index.

The Hausdorff–Young inequality becomes non-trivial for subfactor planar algebras, since the Fourier transform is no longer an integral. When $p = \infty$, we will apply the local relation, see relation (3), and the C^* structure of the 3-box space to give a categorical proof. When $p = 2$, the spherical isotopy of subfactor planar algebras implies the Plancherel’s formula of the Fourier transform,

$$\|\mathcal{F}(x)\|_2 = \|x\|_2.$$

With the help of the interpolation theorem [21], we obtain the general case.

Applying a similar method in the 4-box space, we obtain Young’s inequality for 2-boxes x and y (see [Theorem 4.13](#)),

$$\|x * y\|_r \leq \frac{\|x\|_p \|y\|_q}{\delta}, \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1.$$

For the group case, $\delta x * y$ is the convolution of functions of the group.

By the Hausdorff–Young inequality in subfactor planar algebras, we are able to give the following uncertainty principles.

Main Theorem 1. (See *Theorems 5.2, 5.5.*) *Suppose x is a nonzero 2-box in an irreducible subfactor planar algebra. Then we have the Hirschman–Beckner uncertainty principle,*

$$H(|x|^2) + H(|\mathcal{F}(x)|^2) \geq \|x\|_2^2(2 \log \delta - 4 \log \|x\|_2),$$

where $H(\cdot)$ is the von Neumann entropy, $H(|x|^2) = -\text{tr}_2(|x|^2 \log |x|^2)$; and the Donoho–Stark uncertainty principle,

$$\mathcal{S}(x)\mathcal{S}(\mathcal{F}(x)) \geq \delta^2,$$

where $\mathcal{S}(x)$ is the trace of the range projection of x .

Taking the derivative of the Hausdorff–Young inequality with respect to p at $p = 2$, we derive the Hirschman–Beckner uncertainty principle. By the concavity of $-t \log t$, we have $\log \mathcal{S}(x) \geq \log H(|x|^2)$, when $\|x\|_2 = 1$. Then we obtain the Donoho–Stark uncertainty principle.

We apply the noncommutative version of Özaydm and Przebinda [25] to show the minimizers of the two uncertainty principles coincide. To characterize the minimizers of Donoho–Stark uncertainty principle, we apply the noncommutative version of Tao’s proof in [30]. We show that the minimizers are *extremal bi-partial isometries* (see Definition 6.1 and Theorem 6.4). Extremal bi-partial isometries are convenient to work with theoretically, but not convenient to construct.

Recall that the minimizer of uncertainty principles of finite abelian groups is given by a nonzero constant multiple of a translation and a modulation of the indicator function of its subgroup. The original technique for finite abelian groups is not enough to give this kind of characterization for the 2-box space, because the 2-box space may not be a commutative algebra and there is no translation or modulation to shift the minimizer to a good position. Fortunately Bisch and Jones introduced biprojections [3,5] which generalized the indicator functions of subgroups. We expect to construct minimizers with the knowledge of biprojections. We prove that an extremal bi-partial isometry x is uniquely determined by the range projections of x and its Fourier transform $\mathcal{F}(x)$ (see Theorem 6.16). Moreover, two range projections are *shifts* of a pair of biprojections (see Definition 6.5), a generalization of the translation or the modulation of the indicator function of a subgroup. To construct such an extremal bi-partial isometry, we introduce the notion of a *bi-shift* of a biprojection (see Definition 6.6), a generalization of a translation and a modulation of the indicator function of a subgroup. Bi-shifts of biprojections are easy to construct.

Main Theorem 2. (See Theorems 6.4, 6.13 and Proposition 6.10.) For a nonzero 2-box x in an irreducible subfactor planar algebra, the following statements are equivalent:

- (1) $\mathcal{S}(x)\mathcal{S}(\mathcal{F}(x)) = \delta^2$;
- (2) $H(|x|^2) + H(|\mathcal{F}(x)|^2) = \|x\|_2^2(2 \log \delta - 4 \log \|x\|_2)$;
- (3) x is an extremal bi-partial isometry;
- (3') x is a partial isometry and $\mathcal{F}^{-1}(x)$ is extremal;
- (4) x is a bi-shift of a biprojection,

As a corollary, we obtain a new characterization of biprojections.

Corollary 2.1. (See Corollary 6.14.) For a nonzero 2-box x in an irreducible subfactor planar algebra. If x and $\mathcal{F}(x)$ are positive operators and $\mathcal{S}(x)\mathcal{S}(\mathcal{F}(x)) = \delta^2$, then x is a biprojection.

Then by using the minimizers, we have Hardy’s uncertainty principle

Main Theorem 3. (See Corollary 6.17.) Suppose a 2-box w is a minimizer for the (Hirschman–Beckner/Donoho–Stark) uncertainty principle. For a 2-box x and constants C, C' , if

$$|x| \leq C|w|, \quad |\mathcal{F}(x)| \leq C'|\mathcal{F}(w)|,$$

then x is a scalar multiple of w .

As mentioned above, we have the uncertainty principle for 2-boxes of an irreducible subfactor planar algebra. The irreducibility is not necessary, in Section 7, we have the uncertainty principle for 2-boxes of arbitrary subfactor planar algebra. Moreover, we obtain the uncertainty principles for n -boxes.

Main Theorem 4. (See Theorem 7.6.) For a nonzero n -box x in an irreducible subfactor planar algebra, we have

$$\sum_{k=0}^{n-1} H(|\mathcal{F}^k(x)|^2) \geq \|x\|_2^2(n \log \delta - 2n \log \|x\|_2);$$

$$\prod_{k=0}^{n-1} \mathcal{S}(\mathcal{F}^k(x)) \geq \delta^n.$$

The equalities can be achieved on a special n -box.

When the subfactor planar algebra is the group subfactor planar algebra associated with a finite group G , we will have the uncertainty principle for G . It will recover the classical uncertainty principle when G is abelian. Moreover, its minimizers are given as follows.

Proposition 2.2. (See Proposition 8.1.) Suppose G is a finite group. Take a subgroup H , a one dimensional representation χ of H , an element $g \in G$, a nonzero constant $c \in \mathbb{C}$. Then

$$x = c \sum_{h \in H} \chi(h)hg$$

is a bi-shift of a biprojection. Conversely any bi-shift of a biprojection is of this form.

Let a group G act on a finite set S . The fixed point algebra of the spin model associated with S under the group action of G is a subfactor planar algebra. The 2-box space of this subfactor planar algebra consists of $S \times S$ matrices which commute with the action of G . We may apply all the results above to these $S \times S$ matrices. For example, we have an inequality for the Hadamand product \circ as follows.

Corollary 2.3. (See Corollary 9.1.) If A, B are two $S \times S$ matrices commuting with a group action of G on S , then

$$\|A \circ B\|_r \leq \frac{1}{n_0^2} \|A\|_p \|B\|_q,$$

where $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$ and n_0 is the minimal cardinality of G -orbits of S .

In particular, when $S = G$ and the action is the group multiplication, we will recover the group subfactor planar algebra and related results.

We would like to emphasize the importance of irreducibility of subfactors while studying its Fourier analysis, although our method also works without this condition. (It even works for planar algebras with multiple kinds of regions and strings.) Non-irreducibility changes the constants in the inequalities and uncertainty principles. (See Section 7 for more details.)

We also consider the maximum of sum of entropies in the Hirschman–Beckner uncertainty principle. If there exists a biunitary in a subfactor planar algebra, then the maximizer is a biunitary and biunitaries are maximizers. But the subfactor planar algebra may not have any biunitary.

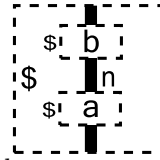
3. Preliminaries

3.1. Planar algebras

We refer the reader to [19] for the definition of subfactor planar algebras. Now let us recall some notations in planar algebras.

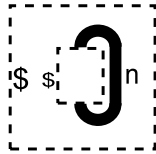
Notation 3.1. In planar tangles, we use a thick string with a number n to indicate n parallel strings.

Suppose \mathcal{P} is a subfactor planar algebra, then the n -box space $\mathcal{P}_{n,\pm}$ forms a C^* algebra. For $a, b \in \mathcal{P}_{n,\pm}$, the product of a and b is $ab =$



The adjoint operation $*$ is a vertical reflection. The unnormalized Markov trace tr_n of the n -box

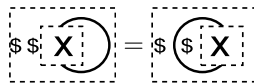
space is given by the tangle $\$ \$ \int_n$. It induces a positive definite inner product of



n -boxes

$$\langle a, b \rangle = tr_n(b^*a).$$

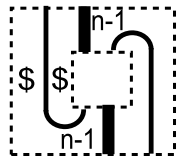
Note that $\dim(\mathcal{P}_{0,\pm}) = 1$, then $\mathcal{P}_{0,\pm}$ is isomorphic to \mathbb{C} as a field. The sphericity of the planar algebra means



as a scalar in \mathbb{C} , for any $x \in \mathcal{P}_{1,\pm}$.

Note that $\mathcal{P}_{0,\pm}$ is isomorphic to \mathbb{C} , the (shaded or unshaded) empty diagram can be identified as the number 1 in \mathbb{C} . The value of a (shaded or unshaded) closed string is δ , the square root of the index. Moreover, $\delta^{-1} \int_{n-2}$ in $\mathcal{P}_{n,+}$, denoted by e_{n-1} , for $n \geq 2$, is the sequences of Jones projections. The filtered algebra generated by Jones projections is the smallest subfactor planar algebra, well known as the Temperley–Lieb algebra, denoted by $TL(\delta)$.

The Fourier transform from $\mathcal{P}_{n,\pm}$ to $\mathcal{P}_{n,\mp}$ is the 1-click rotation

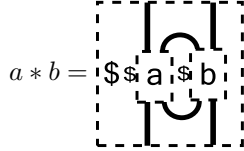


The Fourier transform and inverse Fourier transform has the following relation.

Proposition 3.2. For any x in $\mathcal{P}_{n,\pm}$, we have $\mathcal{F}^{-1}(x^*) = (\mathcal{F}(x))^*$.

For an n -box x , its contragredient, denoted by \bar{x} , is defined as its 180° rotation, i.e. $\mathcal{F}^n(x)$.

If $a, b \in \mathcal{P}_{2,\pm}$, then the pull back of the product of $\mathcal{P}_{2,\mp}$ by the Fourier transform gives another product of 2-boxes, called the coproduct, denoted by $a * b$, and $\mathcal{F}(a * b) = \mathcal{F}(b)\mathcal{F}(a)$; $\mathcal{F}^{-1}(a * b) = \mathcal{F}^{-1}(a)\mathcal{F}^{-1}(b)$. The corresponding tangle is as follows



We collect some properties of coproduct in the following proposition and lemmas.

Proposition 3.3 (Schur Product Theorem). *Let \mathcal{P} be a subfactor planar algebra. If a, b are positive operators in $\mathcal{P}_{2,\pm}$, then $a * b$ is positive.*

Proof. This is Theorem 4.1 in [23]. \square

Lemma 3.4. *Let \mathcal{P} be a subfactor planar algebra and $a, b, c \in \mathcal{P}_{2,\pm}$. Then*

$$\begin{aligned} \text{tr}_2((a * b)\bar{c}) &= \text{tr}_2((b * c)\bar{a}) = \text{tr}_2((c * a)\bar{b}) \\ &= \text{tr}_2((\bar{c} * \bar{b})a) = \text{tr}_2((\bar{a} * \bar{c})b) = \text{tr}_2((\bar{b} * \bar{a})c). \end{aligned}$$

Proof. This is Lemma 4.6 in [23]. \square

Lemma 3.5. *For $x, y \in \mathcal{P}_{2,\pm}$, we have $\mathcal{R}(x * y) \leq \mathcal{R}(\mathcal{R}(x) * \mathcal{R}(y))$, where $\mathcal{R}(\cdot)$ is the range projection of the variable.*

Proof. This is equivalent to Lemma 4.7 in [23]. \square

Notation 3.6. When the tangle is written as a box and the dollar sign is on the left side, we may omit the boundary and the dollar sign.

Definition 3.7. (See [3,5].) For an irreducible subfactor planar algebra \mathcal{P} , a projection $Q \in \mathcal{P}_{2,\pm}$ is called a biprojection if $\mathcal{F}(Q)$ is a multiple of a projection.

It is shown that there is a one to one correspondence between biprojections and intermediate subfactors (see [3,5]). If the subfactor is the group subfactor $\mathcal{N} \subset \mathcal{N} \rtimes G$, for an outer action of the group G on the factor \mathcal{N} of type II_1 , then each intermediate subfactor is $\mathcal{N} \rtimes H$, for some subgroup H of G . Let \mathcal{P} be the subfactor planar algebra of the subfactor $\mathcal{N} \subset \mathcal{N} \rtimes G$, then the corresponding biprojection in $\mathcal{P}_{2,+}$ is the indicator function of H .

Definition 3.8. Let \mathcal{P} be a subfactor planar algebra. A biunitary U in $\mathcal{P}_{2,\pm}$ is a unitary element in $\mathcal{P}_{2,\pm}$ such that $\mathcal{F}(U)$ is unitary.

3.3. The rank-one decomposition

We would like to introduce a decomposition of an element in a finite dimensional C^* -algebra. Suppose \mathcal{A} is a finite dimensional C^* -algebra and x is an element in \mathcal{A} . Let $x = v|x|$ be the polar decomposition of x , where $|x| = (x^*x)^{1/2}$, v is the unique partial isometry from the range of x^* onto the range of x . Let $|x| = \sum_j \lambda_j p_j$ be a decomposition of $|x|$ such that $\lambda_j \geq 0$ and p_j are minimal projections in \mathcal{A} . Then x can be uniquely written as $\sum_j \lambda_j v p_j = \sum_j \lambda_j v_j$ subject to the decomposition of $|x|$ as above, where $v_j = v p_j$ is a rank-one partial isometry. We will say this unique decomposition of x subject to a decomposition of $|x|$ as the rank-one decomposition of x .

4. Inequalities

In this section, \mathcal{P} is an irreducible subfactor planar algebra. We will prove the Hausdorff–Young inequality and Young’s inequality for \mathcal{P} .

Definition 4.1. For x in $\mathcal{P}_{2,\pm}$, we define its p -norm, $1 \leq p < \infty$, as

$$\|x\|_p = \text{tr}_2((x^*x)^{p/2})^{1/p} = \text{tr}_2(|x|^p)^{1/p};$$

$$\|x\|_\infty = \|x\|,$$

the operator norm of x .

Note that $\mathcal{P}_{2,\pm}$ are finite dimensional C^* algebras, so all the topologies induced by $\|\cdot\|_p$, $1 \leq p \leq \infty$, are identical to its norm topology.

Remark. The Fourier transforms \mathcal{F} and \mathcal{F}^{-1} are continuous.

Proposition 4.2. For any x in $\mathcal{P}_{2,\pm}$ and $1 \leq p \leq \infty$, we have

$$\|x\|_p = \||x|\|_p = \|x^*\|_p = \|\bar{x}\|_p.$$

Proof. We omit its proof here. \square

Proposition 4.3 (Hölder’s inequality). For any x, y, z in $\mathcal{P}_{2,\pm}$, we have

- (1) $|\text{tr}_2(xy)| \leq \|x\|_p \|y\|_q$, where $1 \leq p \leq \infty, \frac{1}{p} + \frac{1}{q} = 1$.
- (2) $|\text{tr}_2(xyz)| \leq \|x\|_p \|y\|_q \|z\|_r$, where $1 \leq p, q \leq \infty, \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$.
- (3) $\|xy\|_r \leq \|x\|_p \|y\|_q$, where $1 \leq p, q, r \leq \infty, \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.

We will use the first inequality frequently.

Proof. The proof can be found in [32]. \square

Proposition 4.4. For any x in $\mathcal{P}_{2,\pm}$, we have

$$\|x\|_p = \sup\{|\tau(xy)| : y \in \mathcal{P}_{2,\pm}, \|y\|_q \leq 1\},$$

where $1 \leq p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. The proof can be found in [32]. \square

Proposition 4.5. For any x, y in $\mathcal{P}_{2,\pm}$, $\frac{1}{p} + \frac{1}{q} = 1$, $p, q \geq 1$, we have

$$|\text{tr}_2(x^*y)| = \|x\|_p \|y\|_q$$

if and only if

$$x = u|x|, y = \lambda u|y|, \quad \frac{|x|^p}{\|x\|_p^p} = \frac{|y|^q}{\|y\|_q^q}$$

for some unitary element u and some complex number λ with $|\lambda| = 1$. When $p = \infty$, $\frac{|x|^p}{\|x\|_p^p}$ is defined to be the spectral projection of $|x|$ corresponding to its maximal spectrum (which is $\|x\|_\infty$) over its trace.

Proof. This follows from the proof of Proposition 1.9 in [32] and the conditions for the equalities to hold in Hölder’s inequality and the Cauchy–Schwartz inequality. \square

Proposition 4.6 (Interpolation theorem). Let \mathcal{M} be a finite von Neumann algebra with a faithful normal tracial state τ . Suppose $T : \mathcal{M} \rightarrow \mathcal{M}$ is a linear map. If

$$\|Tx\|_{p_1} \leq K_1 \|x\|_{q_1} \text{ and } \|Tx\|_{q_1} \leq K_2 \|x\|_{q_2},$$

then

$$\|Tx\|_{p_\theta} \leq K_1^{1-\theta} K_2^\theta \|x\|_{q_\theta},$$

where $\frac{1}{p_\theta} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$, $\frac{1}{q_\theta} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}$, $0 \leq \theta \leq 1$.

Proof. This is a special case of the interpolation theorem in [21]. \square

Proposition 4.7. For any x in $\mathcal{P}_{2,\pm}$, we have

$$\|\mathcal{F}(x)\|_\infty \leq \frac{\text{tr}_2(|x|)}{\delta} = \frac{\|x\|_1}{\delta}.$$

Proof. Note that $\mathcal{P}_{2,\pm} = \mathcal{I}_{2,\pm} \oplus \mathcal{P}_{2,\pm}/\mathcal{I}_{2,\pm} = \mathbb{C}e_1 \oplus \mathcal{P}_{2,\pm}/\mathcal{I}_{2,\pm}$. If $x = e_1$, $\mathcal{F}(x) = 1$ and $\text{tr}_2(e_1) = \delta$, then $\|\mathcal{F}(x)\|_\infty = 1 = \frac{\text{tr}_2(|x|)}{\delta}$.

Theorem 4.8 (Hausdorff–Young inequality). For any x in $\mathcal{P}_{2,\pm}$,

$$\|\mathcal{F}(x)\|_p \leq \left(\frac{1}{\delta}\right)^{1-\frac{2}{p}} \|x\|_q,$$

where $2 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. By Proposition 4.7 and the sphericity of subfactor planar algebras, we have

$$\|\mathcal{F}(x)\|_\infty \leq \frac{\|x\|_1}{\delta}, \text{ and } \|\mathcal{F}(x)\|_2 = \|x\|_2.$$

By Proposition 4.6, we have

$$\|\mathcal{F}(x)\|_p \leq \left(\frac{1}{\delta}\right)^{1-\frac{2}{p}} \|x\|_q,$$

where $2 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. \square

Now we are going to prove Young’s inequality.

Lemma 4.9. For any x, y in $\mathcal{P}_{2,\pm}$, we have

$$\|x * y\|_\infty \leq \frac{\|x\|_\infty \|y\|_1}{\delta}, \quad \|y * x\|_\infty \leq \frac{\|x\|_\infty \|y\|_1}{\delta}.$$

Proof. If y is a rank-one partial isometry v in $\mathcal{P}_{2,\pm}/\mathcal{I}_{2,\pm}$, we have

$$\frac{\delta}{\|v\|_1} \left| \begin{array}{c} \boxed{v} \\ \boxed{x} \\ \boxed{v^*} \end{array} \right| \leq \left| \begin{array}{c} \boxed{v} \\ \boxed{x} \end{array} \right|.$$

Then

$$\frac{\delta}{\|v\|_1} \left| \begin{array}{c} \boxed{v} \\ \boxed{x} \\ \boxed{v^*} \end{array} \right| \leq \left| \begin{array}{c} \boxed{v} \\ \boxed{x} \end{array} \right| \leq \left| \begin{array}{c} \boxed{v} \\ \boxed{x} \\ \boxed{v^*} \end{array} \right| \leq \|x\|_\infty^2 \left| \begin{array}{c} \boxed{v} \\ \boxed{v^*} \end{array} \right| = \|x\|_\infty^2 \frac{\|v\|_1}{\delta},$$

the last inequality follows from Schur Product Theorem (Proposition 3.3). Hence

$$\|(v * x)^*(v * x)\|_\infty \leq (\|x\|_\infty \frac{\|v\|_1}{\delta})^2,$$

and

$$\|v * x\|_\infty \leq \|x\|_\infty \frac{\|v\|_1}{\delta}.$$

For an arbitrary y in $\mathcal{P}_{2,\pm}$, let $y = \sum_k \lambda_k v_k$ be the rank-one decomposition. Then $\|y\|_1 = \sum_k \lambda_k \|v_k\|_1$. We have proved that $\|v_k * x\|_\infty \leq \|x\|_\infty \frac{\|v_k\|_1}{\delta}$, so

$$\|y * x\|_\infty \leq \sum_k \lambda_k \|v_k * x\|_\infty \leq \sum_k \lambda_k \|x\|_\infty \frac{\|v_k\|_1}{\delta} = \|x\|_\infty \frac{\|y\|_1}{\delta}.$$

Note that $\|x * y\|_\infty = \|\overline{y * x}\|_\infty$, $\|x\|_\infty = \|\overline{x}\|_\infty$, and $\|y\|_1 = \|\overline{y}\|_1$, we obtain that

$$\|x * y\|_\infty \leq \frac{\|x\|_\infty \|y\|_1}{\delta}. \quad \square$$

Lemma 4.10. For any x, y in $\mathcal{P}_{2,\pm}$, we have

$$\|x * y\|_1 \leq \frac{\|x\|_1 \|y\|_1}{\delta}.$$

Proof. For any x, y in $\mathcal{P}_{2,\pm}$, by Proposition 4.4 and Lemma 3.4, we have

$$\begin{aligned} \|x * y\|_1 &= \sup_{\|z\|_\infty=1} |tr_2((x * y)z)| \\ &= \sup_{\|z\|_\infty=1} |tr_2(x(z * \overline{y}))| \\ &\leq \|x\|_1 \|z * \overline{y}\|_\infty && \text{H\"older's inequality} \\ &\leq \|x\|_1 \frac{\|y\|_1}{\delta} && \text{Lemma 4.9} \quad \square \end{aligned}$$

Lemma 4.11. For any x, y in $\mathcal{P}_{2,\pm}$, we have

$$\|x * y\|_p \leq \frac{\|x\|_p \|y\|_1}{\delta}, \quad \|y * x\|_p \leq \frac{\|x\|_p \|y\|_1}{\delta},$$

where $1 \leq p \leq \infty$.

Proof. It follows from Lemmas 4.9, 4.10 and Proposition 4.6. \square

Lemma 4.12. For any x, y in $\mathcal{P}_{2,\pm}$, we have

$$\|x * y\|_\infty \leq \frac{\|x\|_p \|y\|_q}{\delta},$$

where $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Suppose that $x * y = \sum_k \lambda_k v_k$ is the rank-one decomposition. Then

$$\|x * y\|_\infty = \sup_k \lambda_k = \sup_k \frac{\text{tr}_2((x * y)v_k^*)}{\text{tr}_2(|v_k|)}.$$

By Lemma 3.4 and Proposition 4.3, we have that

$$|\text{tr}_2((x * y)v_k^*)| = |\text{tr}_2((\overline{v_k^*} * a)\overline{y})| \leq \|\overline{v_k^*} * x\|_p \|\overline{y}\|_q$$

By Lemma 4.11, we have

$$\|\overline{v_k^*} * x\|_p \leq \|x\|_p \frac{\text{tr}_2(|\overline{v_k^*}|)}{\delta}.$$

Moreover, since $\|b\|_q = \|\overline{b}\|_q$, we obtain

$$\|x * y\|_\infty \leq \frac{\|x\|_p \text{tr}_2(|\overline{v_k^*}|) \|y\|_q}{\delta \text{tr}_2(|v_k^*|)} = \frac{\|x\|_p \|y\|_q}{\delta}.$$

This completes the proof of the lemma. \square

Theorem 4.13 (Young’s inequality). For any x, y in $\mathcal{P}_{2,\pm}$, we have

$$\|x * y\|_r \leq \frac{\|x\|_p \|y\|_q}{\delta},$$

where $1 \leq p, q, r \leq \infty, \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$.

Proof. It follows from Lemmas 4.11, 4.12 and Proposition 4.6. \square

Remark. The inequalities listed as above are sharp. This can be easily checked if one lets $x = y = 1$.

5. Noncommutative uncertainty principles

In this section, we will prove the Donoho–Stark uncertainty principle and the Hirschman–Beckner uncertainty principle for irreducible subfactor planar algebras. It was shown [11,28] that for a finite abelian group G and a nonzero function f on G ,

$$|\text{supp}(f)| |\text{supp}(\hat{f})| \geq |G|,$$

where \hat{f} is the Fourier transform of f , $\text{supp}(f) = \{x \in G : f(x) \neq 0\}$. We will generalize this uncertainty principle in subfactor planar algebras.

Notation 5.1. Suppose \mathcal{P} be a subfactor planar algebra. For any x in $\mathcal{P}_{n,\pm}$, recall that $\mathcal{R}(x)$ is the range projection of x . Let us define $\mathcal{S}(x) = \text{tr}_n(\mathcal{R}(x))$.

Theorem 5.2. *Suppose \mathcal{P} is an irreducible subfactor planar algebra. Then for any nonzero x in $\mathcal{P}_{2,\pm}$, we have*

$$\mathcal{S}(x)\mathcal{S}(\mathcal{F}(x)) \geq \delta^2.$$

This partially proves Main [Theorem 1](#).

Proof. Let $\mathcal{F}(x) = \sum_j \lambda_j v_j$ be the rank-one decomposition of $\mathcal{F}(x)$. Then $\|\mathcal{F}(x)\|_\infty = \sup_j \lambda_j$. Now we have

$$\begin{aligned} \sup_j \lambda_j &= \|\mathcal{F}(x)\|_\infty \\ &\leq \frac{\|x\|_1}{\delta} \leq \frac{\|\mathcal{R}(x)\|_2 \|x\|_2}{\delta} \\ &= \frac{\mathcal{S}(x)^{1/2}}{\delta} \|\mathcal{F}(x)\|_2 \\ &= \frac{\mathcal{S}(x)^{1/2}}{\delta} \left(\sum_j \lambda_j^2 \|v_j\|_1\right)^{1/2} \\ &\leq \frac{\mathcal{S}(x)^{1/2}}{\delta} (\sup_j \lambda_j^2)^{1/2} \|\mathcal{R}(\mathcal{F}(x))\|_1^{1/2} \\ &= \frac{\mathcal{S}(x)^{1/2} \mathcal{S}(\mathcal{F}(x))^{1/2}}{\delta} \sup_j \lambda_j. \end{aligned}$$

Thus

$$\mathcal{S}(x)\mathcal{S}(\mathcal{F}(x)) \geq \delta^2. \quad \square$$

If we apply [Theorem 5.2](#) to group subfactor planar algebras, we have

Corollary 5.3. *For any finite abelian group G and a nonzero function f on G , we have*

$$|\text{supp}(f)| |\text{supp}(\hat{f})| \geq |G|,$$

where \hat{f} is the Fourier transform of f .

In [\[30\]](#), T. Tao shows that for the group \mathbb{Z}_p , p prime, and a nonzero function f on it,

$$|\text{supp}(f)| + |\text{supp}(\hat{f})| \geq p + 1.$$

Similarly we can ask if a parallel inequality

$$\mathcal{S}(x) + \mathcal{S}(\mathcal{F}(x)) \geq \delta^2 + 1 \tag{4}$$

holds in the 2-box space of subfactor planar algebras. The statement is false if a subfactor planar algebra has a non-trivial biprojection.

Suppose P is a nontrivial biprojection in $\mathcal{P}_{2,\pm}$. Then $\mathcal{S}(P)\mathcal{S}(\mathcal{F}(P)) = \delta^2$ and $1 < \mathcal{S}(P) < \delta^2$. Hence $\mathcal{S}(P) + \mathcal{S}(\mathcal{F}(P)) = \mathcal{S}(P) + \frac{\delta^2}{\mathcal{S}(P)} < \delta^2 + 1$. In the case of group subgroup subfactor planar algebra, the inequality (4) is true only when the index is a prime number p which corresponds the finite group case for Tao’s uncertainty principle. In general, we do not have inequality (4) even if the index is a prime number. For example, let $\mathcal{P}^1, \mathcal{P}^2$ be Temperley Lieb algebra with index a and p/a , where p is a prime number, $a > 4, p/a > 4$. Then the free product $\mathcal{P}^1 * \mathcal{P}^2$ is a Fuss–Catalan algebra with index p (see [5]). Since $\mathcal{P}^1 * \mathcal{P}^2$ has nontrivial biprojection, the inequality (4) is false for $\mathcal{P}^1 * \mathcal{P}^2$.

We need a “prime” condition. It is proved by Bisch in [4] that a finite depth subfactor planar algebra with index p, p prime, has no non-trivial biprojection. We may ask the following question.

Question. Suppose \mathcal{P} is an irreducible finite depth subfactor planar algebra with index p, p prime. Is inequality (4) true?

Remark. If \mathcal{P} is a group subgroup subfactor planar algebra with index p , then the argument reduces to the group case, where it is true.

Remark. If $\mathcal{S}(\mathcal{F}(x))$ is replaced by the trace of the central support of $\mathcal{F}(x)$, then the corresponding weaker inequality for finite groups was proved in Corollary 5.1 in [13].

For Hirschman’s uncertainty principle [15], the Shannon entropy is used to describe his uncertainty principle in abelian groups. In subfactors, we will use the von Neumann entropy in place of the Shannon entropy.

Notation 5.4. For any x in $\mathcal{P}_{n,\pm}$, the von Neumann entropy of $|x|^2$ is

$$H(|x|^2) = -tr_n(|x|^2 \log |x|^2) = -tr_n(x^*x \log x^*x).$$

Von Neumann entropy is widely used as an important measurement in quantum information theory and it is also important for our next uncertainty principle.

Theorem 5.5. *Suppose that \mathcal{P} is an irreducible subfactor planar algebra and $x \in \mathcal{P}_{2,\pm}$. Then*

$$H(|x|^2) + H(|\mathcal{F}(x)|^2) \geq \|x\|_2^2(2 \log \delta - 4 \log \|x\|_2).$$

Specifically

$$H(|x|^2) + H(|\mathcal{F}(x)|^2) \geq 2 \log \delta$$

whenever $x \in \mathcal{P}_{2,\pm}$ and $\|x\|_2 = 1$.

This together with [Theorem 5.2](#) proves Main [Theorem 1](#).

Proof. Without loss of generality, we assume that $x \neq 0$. By [Theorem 4.8](#),

$$\|\mathcal{F}(x)\|_p \leq \left(\frac{1}{\delta}\right)^{1-\frac{2}{p}} \|x\|_q,$$

where $2 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Take

$$f(p) = \log \|\mathcal{F}(x)\|_p - \log \|x\|_q - \log\left(\frac{1}{\delta}\right)^{1-\frac{2}{p}}.$$

Then $f(p) \leq 0$. Note that $\|\mathcal{F}(x)\|_2 = \|x\|_2$, so $f(2) = 0$. Then $f'(2) \leq 0$.

Note that

$$\frac{d}{dp} \sum_k x_k^p \Big|_{p=2} = \sum_k x_k^2 \log x_k,$$

so

$$\frac{d}{dp} \|\mathcal{F}(x)\|_p^p \Big|_{p=2} = -\frac{1}{2} H(|\mathcal{F}(x)|^2).$$

Then

$$\frac{d}{dp} \log \|\mathcal{F}(x)\|_p \Big|_{p=2} = \frac{d}{dp} \frac{1}{p} \log \|\mathcal{F}(x)\|_p^p \Big|_{p=2} = -\frac{1}{4} \log \|\mathcal{F}(x)\|_2^2 - \frac{1}{4} \frac{H(|\mathcal{F}(x)|^2)}{\|\mathcal{F}(x)\|_2^2}.$$

Similarly

$$\frac{d}{dp} \log \|x\|_q \Big|_{p=2} = \left(\frac{d}{dq} \log \|x\|_q \Big|_{q=2}\right) \left(\frac{dq}{dp} \Big|_{p=2}\right) = \frac{1}{4} \log \|x\|_2^2 + \frac{1}{4} \frac{H(|x|^2)}{\|x\|_2^2}.$$

Moreover,

$$\frac{d}{dp} \log\left(\frac{1}{\delta}\right)^{1-\frac{2}{p}} \Big|_{p=2} = -\frac{1}{2} \log \delta.$$

Therefore

$$f'(2) = \left(-\frac{1}{4} \log \|\mathcal{F}(x)\|_2^2 - \frac{1}{4} \frac{H(|\mathcal{F}(x)|^2)}{\|\mathcal{F}(x)\|_2^2}\right) - \left(\frac{1}{4} \log \|x\|_2^2 + \frac{1}{4} \frac{H(|x|^2)}{\|x\|_2^2}\right) + \frac{1}{2} \log \delta.$$

Recall that $f'(2) \leq 0$ and $\|\mathcal{F}(x)\|_2 = \|x\|_2$, so

$$H(|x|^2) + H(|\mathcal{F}(x)|^2) \geq \|x\|_2^2 (2 \log \delta - 4 \log \|x\|_2).$$

Specifically

$$H(|x|^2) + H(|\mathcal{F}(x)|^2) \geq 2 \log \delta,$$

when $\|x\|_2 = 1$. \square

Remark. This is a noncommutative version of the proof of Theorem 23 in [10].

Corollary 5.6. *Suppose \mathcal{P} is an irreducible subfactor planar algebra. Then for any nonzero x in $\mathcal{P}_{2,\pm}$, we have*

$$\mathcal{S}(x)\mathcal{S}(\mathcal{F}(x)) \geq \delta^2.$$

Proof. We will prove the inequality $\log \mathcal{S}(x) \geq H(|x|^2)$ when $\|x\|_2 = 1$. With this inequality, it is easy to see that the corollary follows from Theorem 5.5.

Let $x = \sum_j \lambda_j v_j$ be the rank-one decomposition. Since $\|x\|_2 = 1$, we have $\|x\|_2^2 = \sum_j \lambda_j^2 \text{tr}_2(|v_j|) = 1$. Then

$$\begin{aligned} H(|x|^2) &= -\text{tr}_2(|x|^2 \log |x|^2) = -\sum_j \lambda_j^2 \log \lambda_j^2 \text{tr}_2(|v_j|) \\ &= -\text{tr}_2(\mathcal{R}(|x|)) \sum_j \frac{\text{tr}_2(|v_j|)}{\text{tr}_2(\mathcal{R}(|x|))} \lambda_j^2 \log \lambda_j^2 \\ &\leq -\text{tr}_2(\mathcal{R}(|x|)) \left(\sum_j \frac{\text{tr}_2(|v_j|)}{\text{tr}_2(\mathcal{R}(|x|))} \lambda_j^2 \right) \log \left(\sum_j \frac{\text{tr}_2(|v_j|)}{\text{tr}_2(\mathcal{R}(|x|))} \lambda_j^2 \right) \\ &\quad \text{Jensen's inequality} \\ &= \log \text{tr}_2(\mathcal{R}(|x|)) = \log \text{tr}_2(\mathcal{R}(x)) = \log \mathcal{S}(x). \quad \square \end{aligned}$$

Remark. The Hirschman–Beckner uncertainty principle is stronger than the Donoho–Stark uncertainty principle.

Remark. If $\|x\|_2 = \delta$, then $H(|x|^2) \leq 0$. Moreover, $H(|x|^2) = 0$ if and only if x is a unitary. Therefore the maximizer of $H(|x|^2) + H(|\mathcal{F}(x)|^2)$ is a biunitary if there is one. In the case $\|x\|_2 = \delta$, we have

$$-2\delta \log(2\delta) \leq H(|x|^2) + H(|\mathcal{F}(x)|^2) \leq 0$$

6. Minimizers for noncommutative uncertainty principles

Throughout this section, \mathcal{P} is an irreducible subfactor planar algebra. We will discuss the minimizers of its two uncertainty principles shown in the last section. First, we would like to introduce some notions.

Definition 6.1. An element x in $\mathcal{P}_{2,\pm}$ is said to be extremal if $\|\mathcal{F}(x)\|_\infty = \frac{\|x\|_1}{\delta}$. We say a nonzero element x is an (extremal) bi-partial isometry if x and $\mathcal{F}(x)$ are multiplies of (extremal) partial isometries.

For example, a positive operator is always extremal, since the norm of its Fourier transform is achieved on the Jones projection; a biprojection is an extremal bi-partial isometry.

Proposition 6.2. *If x in $\mathcal{P}_{2,\pm}$ is extremal, then x^* and \bar{x} are extremal.*

Proof. It follows from the facts that

$$\|\mathcal{F}(x^*)\|_\infty = \|\mathcal{F}^{-1}(x)^*\|_\infty = \|\mathcal{F}^{-1}(x)\|_\infty = \|\mathcal{F}(x)\|_\infty$$

and

$$\|x^*\|_1 = \|x\|_1 = \|\bar{x}\|_1. \quad \square$$

We will first show that the two uncertainty principles have the same minimizers which are extremal bi-partial isometries. The proof of the following theorem benefits a lot from [25], and we also need Hopf’s maximum principle. For readers’ convenience, we state it here:

Proposition 6.3 (Hopf’s maximum principle). (See [16].) *Let $D \subset \mathbb{C}$ be an open unit disc, and let $u : D \rightarrow \mathbb{R}$ be a harmonic function which extends to a continuous function on the closure \bar{D} of D , $u : \bar{D} \rightarrow \mathbb{R}$. Suppose z is a point on the boundary of D such that $u(z) \geq u(z')$ for all $z' \in \bar{D}$, and the directional derivative of u at z along the radius which ends at z , is zero. Then $u(z) = u(z')$ for all $z' \in \bar{D}$.*

Theorem 6.4. *For a non-zero element x in $\mathcal{P}_{2,\pm}$, the following statements are equivalent:*

- (1) $H(|x|^2) + H(|\mathcal{F}(x)|^2) = \|x\|_2^2(2 \log \delta - 4 \log \|x\|_2)$;
- (2) $\mathcal{S}(x)\mathcal{S}(\mathcal{F}(x)) = \delta^2$;
- (3) x is an extremal bi-partial isometry.

This proves part of Main Theorem 2.

Proof. “(1) \Rightarrow (3)”. Suppose x is in $\mathcal{P}_{2,\pm}$ such that $\|x\|_2 = \sqrt{\delta}$ and $H(|x|^2) + H(|\mathcal{F}(x)|^2) = 0$. Then λx also satisfies equation (1) in the theorem. Without loss of generality, we assume that $\|x\|_2 = \sqrt{\delta}$. We define a complex function

$$F(z) = tr_2 \left(\mathcal{F} \left(w_x |x|^{2z} \right) |\mathcal{F}(x)|^{2z} w_{\mathcal{F}(x)}^* \right),$$

where w_x means the partial isometry in the of polar decomposition of x .

Now we will show that $F(z)$ is analytic and bounded on the strip $\frac{1}{2} < \sigma < 1, z = \sigma + it$. By Proposition 4.3 and Theorem 4.8,

$$\begin{aligned} |F(\sigma + it)| &\leq \|\mathcal{F}(w_x|x|^{2z})\|_{\frac{1}{1-\sigma}} \|\mathcal{F}(x)^{2z}\|_{\frac{1}{\sigma}} \\ &\leq \left(\frac{1}{\delta}\right)^{1-2(1-\sigma)} \| |x|^{2\sigma} \|_{1/\sigma} \| |\mathcal{F}(x)|^{2\sigma} \|_{1/\sigma} \\ &= \left(\frac{1}{\delta}\right)^{2\sigma-1} \delta^\sigma \delta^\sigma = \delta. \end{aligned}$$

This implies that $F(z)$ is bounded on $\frac{1}{2} < \sigma < 1$. Differentiating the function with respect to z , we have

$$\begin{aligned} F'(z) &= \text{tr}_2 \left(\mathcal{F}(w_x|x|^{2z}(\log|x|^2)) |\mathcal{F}(x)|^{2z} w_{\mathcal{F}(x)}^* \right) \\ &\quad + \text{tr}_2 \left(\mathcal{F}(w_x|x|^{2z}) |\mathcal{F}(x)|^{2z} (\log|\mathcal{F}(x)|^2) w_{\mathcal{F}(x)}^* \right). \end{aligned}$$

Then evaluating the function at $z = \frac{1}{2}$, we obtain

$$\begin{aligned} F'\left(\frac{1}{2}\right) &= \text{tr}_2 \left(\mathcal{F}(w_x|x|(\log|x|^2)) |\mathcal{F}(x)| w_{\mathcal{F}(x)}^* \right) \\ &\quad + \text{tr}_2 \left(\mathcal{F}(w_x|x|) |\mathcal{F}(x)| (\log|\mathcal{F}(x)|^2) w_{\mathcal{F}(x)}^* \right) \\ &= \text{tr}_2(\mathcal{F}(x(\log|x|^2))\mathcal{F}(x)^*) + \text{tr}_2(\mathcal{F}(x)(\log\mathcal{F}(x)^2)\mathcal{F}(x)^*) \\ &= \text{tr}_2(x(\log|x|^2)x^*) - H(|\mathcal{F}(x)|^2) \\ &= -H(|x|^2) - H(|\mathcal{F}(x)|^2) = 0. \end{aligned}$$

By Hopf’s maximum principle, Proposition 6.3, we have that $F(z)$ is constant on the disc center at $\frac{3}{4}$ with radius $\frac{1}{4}$. Hence

$$F(1) = F\left(\frac{1}{2}\right) = \text{tr}_2(\mathcal{F}(x)\mathcal{F}(x)^*) = \delta.$$

On the other hand, we have

$$\begin{aligned} F(1) &= \text{tr}_2 \left(\mathcal{F}(w_x|x|^2) |\mathcal{F}(x)|^2 w_{\mathcal{F}(x)}^* \right) \\ &= \text{tr}_2(\mathcal{F}(x|x|)|\mathcal{F}(x)|\mathcal{F}(x)^*). \end{aligned}$$

Let $x = \sum_j \mu_j w_j$ and $\mathcal{F}(x) = \sum_k \lambda_k v_k$ be the rank-one decompositions. Then

$$\delta = \|x\|_2^2 = \sum_j \mu_j^2 \|w_j\|_1, \quad \delta = \|\mathcal{F}(x)\|_2^2 = \sum_k \lambda_k^2 \|v_k\|_1,$$

and

$$\mathcal{F}(x|x) = \sum_j \mu_j^2 \mathcal{F}(w_j), \quad |\mathcal{F}(x)|\mathcal{F}(x)^* = \sum_k \lambda_k^2 v_k^*.$$

Therefore

$$\begin{aligned} \delta = F(1) &= \text{tr}_2\left(\sum_j \mu_j^2 \mathcal{F}(w_j) \sum_k \lambda_k^2 v_k^*\right) \\ &= \sum_{j,k} \lambda_k^2 \mu_j^2 \text{tr}_2(\mathcal{F}(w_j)v_k^*) \end{aligned}$$

On the other hand,

$$\delta^2 = \|x\|_2^2 \|\mathcal{F}(x)\|_2^2 = \sum_{j,k} \lambda_k^2 \mu_j^2 \|w_j\|_1 \|v_k\|_1.$$

Combining the two equations above, we see that

$$\sum_{j,k} \lambda_k^2 \mu_j^2 \left(\frac{1}{\delta} \|w_j\|_1 \|v_k\|_1 - \text{tr}_2(\mathcal{F}(w_j)v_k^*) \right) = 0. \tag{5}$$

Note that $\lambda_k > 0, \mu_j > 0$, and by [Propositions 4.3, 4.7](#),

$$|\text{tr}_2(\mathcal{F}(w_j)v_k^*)| \leq \|\mathcal{F}(w_j)\|_\infty \|v_k^*\|_1 \leq \frac{\|w_j\|_1}{\delta} \|v_k\|_1.$$

By Equation (5), we see that $\text{tr}_2(\mathcal{F}(w_j)v_k^*) = \frac{1}{\delta} \|w_j\|_1 \|v_k\|_1$. Therefore

$$\lambda_k \|v_k\|_1 = \text{tr}_2(\mathcal{F}(x)v_k^*) = \sum_j \mu_j \text{tr}_2(\mathcal{F}(w_j)v_k^*) = \frac{1}{\delta} \sum_j \mu_j \|w_j\|_1 \|v_k\|_1 = \frac{1}{\delta} \|x\|_1 \|v_k\|_1.$$

Then $\lambda_k = \frac{\|x\|_1}{\delta}$ for any k . So x is extremal and $\mathcal{F}(x)$ is a multiple of a partial isometry.

Similarly, we have $\mu_j = \frac{\|\mathcal{F}(x)\|_1}{\delta}$ for any j . So x is an extremal bi-partial isometry.

“(3) \Rightarrow (2)” If x is an extremal bi-partial isometry, then it is easy to check that all the equalities of the inequalities hold in the proof of [Theorem 5.2](#).

“(2) \Rightarrow (1)” From the proof of [Corollary 5.6](#), we see that the Hirschman–Beckner uncertainty principle is stronger than the Donoho–Stark uncertainty principle. So the minimizer of the latter one has to be that of the former one. \square

[Theorem 6.4](#) is a noncommutative version of Theorem 1.5 in [\[25\]](#) when A is a finite abelian group. As showed in [\[25\]](#), the minimizer of the classical uncertainty principle is

Proof. Note that $x = \mathcal{F}(\tilde{B}_h) * (yB_g)$, we then have $\mathcal{F}^{-1}(x) = \tilde{B}_h \mathcal{F}^{-1}(yB_g)$. This implies that $\mathcal{R}(\mathcal{F}^{-1}(x)) \leq \tilde{B}_h$. On the other hand, we have

$$\begin{aligned} x^* &= \mathcal{F}(\tilde{B}_h)^* * (B_g y^*) \\ &= \mathcal{F}^{-1}(\tilde{B}_h) * (B_g y^*), \\ \mathcal{R}(\mathcal{F}^{-1}(\tilde{B}_h)) &= \mathcal{R}(\mathcal{F}^{-1}(\tilde{B})) \\ &= \mathcal{R}(B) = B, \\ \mathcal{R}(x^*) &\leq \mathcal{R}(\mathcal{R}(\mathcal{F}^{-1}(\tilde{B}_h)) * \mathcal{R}(B_g y^*)) \quad \text{Proposition 3.5} \\ &\leq \mathcal{R}(B * B_g) = B_g. \end{aligned}$$

By Theorem 5.2, we have

$$\begin{aligned} \delta^2 &\leq \text{tr}_2(\mathcal{R}(x)) \text{tr}_2(\mathcal{R}(\mathcal{F}^{-1}(x))) \\ &= \text{tr}_2(\mathcal{R}(x^*)) \text{tr}_2(\mathcal{R}(\mathcal{F}^{-1}(x))) \\ &\leq \text{tr}_2(B_g) \text{tr}_2(\tilde{B}_h) = \text{tr}_2(B) \text{tr}_2(\tilde{B}) = \delta^2. \end{aligned}$$

This means that $\mathcal{R}(x^*) = B_g$ and $\mathcal{R}(\mathcal{F}^{-1}(x)) = \tilde{B}_h$. Moreover, x is a minimizer of the uncertainty principles. \square

Lemma 6.8. Suppose $x \in \mathcal{P}_{2,\pm}$ is extremal. Let $x = \sum_k \lambda_k v_k$ and $\mathcal{F}(x) = \sum_l \mu_l w_l$ be rank-one decompositions. Then

$$w_l^* \mathcal{F}(v_k) = \frac{\|v_k\|_1}{\delta} |w_l|, \text{ for all } k,$$

whenever $\mu_l = \|\mathcal{F}(x)\|_\infty$.

Proof. If $\mu_l = \|\mathcal{F}(x)\|_\infty = \frac{\|x\|_1}{\delta}$, we have

$$\text{tr}_2(w_l^* \mathcal{F}(x)) = \mu_l \|w_l\|_1 = \frac{\|x\|_1}{\delta} \|w_l\|_1.$$

Recall that $x = \sum_k \lambda_k v_k$, we obtain

$$\sum_k \lambda_k \text{tr}_2(w_l^* \mathcal{F}(v_k)) = \sum_k \lambda_k \frac{\|v_k\|_1}{\delta} \|w_l\|_1.$$

By Propositions 4.3 and 4.7,

$$|\text{tr}_2(w_l^* \mathcal{F}(v_k))| \leq \text{tr}_2(|w_l^* \mathcal{F}(v_k)|) \leq \|w_l\|_1 \|\mathcal{F}(v_k)\|_\infty \leq \|w_l\|_1 \frac{\|v_k\|_1}{\delta}.$$

Then

$$\operatorname{tr}_2(w_i^* \mathcal{F}(v_k)) = \operatorname{tr}_2(|w_i^* \mathcal{F}(v_k)|) = \|w_l\|_1 \frac{\|v_k\|_1}{\delta} = \operatorname{tr}_2(|w_l| \frac{\|v_k\|_1}{\delta}), \forall k.$$

Note that $\mathcal{R}(w_i^* \mathcal{F}(v_k)) = |w_l|$, and

$$\|w_i^* \mathcal{F}(v_k)\|_\infty = \|\mathcal{F}(v_k)\|_\infty \leq \frac{\|v_k\|_1}{\delta},$$

so $w_i^* \mathcal{F}(v_k) = \frac{\|v_k\|_1}{\delta} |w_l|$ for all k . \square

Definition 6.9. Suppose $v \in \mathcal{P}_{2,+}, w \in \mathcal{P}_{2,-}$, we say v is extremal with respect to w if

$$\operatorname{tr}_2(w^* \mathcal{F}(v)) = \overline{\operatorname{tr}_2(v^* \mathcal{F}^{-1}(w))} = \frac{\|v\|_1 \|w\|_1}{\delta}.$$

Remark. In general, we have

$$|\operatorname{tr}_2(w^* \mathcal{F}(v))| = |\overline{\operatorname{tr}_2(v^* \mathcal{F}^{-1}(w))}| \leq \frac{\|v\|_1 \|w\|_1}{\delta}.$$

Proposition 6.10. If w is a partial isometry and $\mathcal{F}^{-1}(w)$ is extremal, then w is an extremal bi-partial isometry.

This proves part of Main [Theorem 2](#).

Proof. By [Proposition 4.5](#), w is a multiple of a partial isometry if and only if $\|w\|_2^2 = \|w\|_\infty \|w\|_1$. To see $\mathcal{F}(w)$ is a partial isometry, we are going to check $\|\mathcal{F}(w)\|_2^2 = \|\mathcal{F}(w)\|_\infty \|\mathcal{F}(w)\|_1$. If $\mathcal{F}^{-1}(w)$ is extremal, then

$$\|w\|_\infty = \|\mathcal{F}(\mathcal{F}^{-1}(w))\|_\infty = \frac{\|\mathcal{F}^{-1}(w)\|_1}{\delta}.$$

If w is a partial isometry, then

$$\begin{aligned} \|\mathcal{F}(w)\|_\infty \|\mathcal{F}(w)\|_1 &\geq \|\mathcal{F}(w)\|_2^2 = \|w\|_2^2 \\ &= \|w\|_\infty \|w\|_1 \\ &\geq \frac{\|\mathcal{F}^{-1}(w)\|_1}{\delta} \delta \|\mathcal{F}(w)\|_\infty \\ &= \|\mathcal{F}(w)\|_\infty \|\mathcal{F}(w)\|_1 \end{aligned}$$

Hence $\|\mathcal{F}(w)\|_2^2 = \|\mathcal{F}(w)\|_\infty \|\mathcal{F}(w)\|_1$ and $\|\mathcal{F}(w)\|_\infty = \frac{1}{\delta} \|w\|_1$. Then $\mathcal{F}(w)$ is a multiple of a partial isometry and w is extremal. \square

We are going to prove

$$\begin{array}{c} \boxed{W_{l_1}} \\ \boxed{V_k} \\ \boxed{W_{l_1}^*} \end{array} = \frac{\delta}{\|v_k\|_1} \begin{array}{c} \boxed{W_{l_1}} \\ \boxed{V_k} \\ \boxed{V_k^*} \\ \boxed{W_{l_1}^*} \end{array}, \tag{8}$$

which implies the rest summands of the right side of Equation (7) are zeros. The left hand side of Equation (8) is

$$\frac{\delta}{\|v_k\|_1} w_{l_1}^* w_{l_1} = \frac{\delta}{\|v_k\|_1} |w_{l_1}|.$$

The right hand side of Equation (8) is

$$\frac{\delta}{\|v_k\|_1} w_{l_1}^* \mathcal{F}(v_k^*) \mathcal{F}^{-1}(v_k) w_{l_1} = \frac{\delta}{\|v_k\|_1} w_{l_1}^* \mathcal{F}(v_k^*) (\mathcal{F}(v_k^*))^* w_{l_1}.$$

From Lemma 6.8, we have

$$\frac{\|v_k\|_1}{\delta} |w_{l_1}| = w_{l_1}^* \mathcal{F}(v_k^*).$$

Thus

$$\frac{\delta}{\|v_k\|_1} w_{l_1}^* \mathcal{F}(v_k^*) (\mathcal{F}(v_k^*))^* w_{l_1} = \frac{\delta}{\|v_k\|_1} \left(\frac{\|v_k\|_1}{\delta} |w_{l_1}| \right)^2 = \frac{\|v_k\|_1}{\delta} |w_{l_1}|.$$

So Equation (8) holds.

Now we have $\begin{array}{c} \boxed{W_{l_1}} \\ \boxed{U_j} \\ \boxed{U_j^*} \\ \boxed{W_{l_2}^*} \end{array} = 0$, for $j \neq 1$, so $\begin{array}{c} \boxed{W_{l_1}} \\ \boxed{U_j} \end{array} = 0$. Note that $\begin{array}{c} \boxed{V_k} \\ \boxed{S_3} \end{array}$ is a projection,

so $\begin{array}{c} \boxed{W_{l_1}} \\ \boxed{V_k} \\ \boxed{S_3} \end{array} = 0$. By Equation (6), we have

$$\begin{aligned}
 \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} &= \sum_{j=1}^n \frac{\delta}{\|v_k\|_1} \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} + \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} = \frac{\delta}{\|v_k\|_1} \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} = \frac{\delta}{\|v_k\|_1} \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} \\
 &= \frac{\delta}{\|v_k\|_1} \left(\frac{\|v_k\|_1}{\delta} \right)^2 \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array},
 \end{aligned}$$

the last equality follows from Lemma 6.8. Adding a cap to the right and then taking the Fourier transform (the 1-click rotation), we have

$$\begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} = \frac{\|v_k\|_1}{\delta} \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array}.$$

Then

$$\sum_{k,l_1,l_2} \lambda_k^2 \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} = \sum_{k,l_1,l_2} \lambda_k^2 \frac{\|v_k\|_1}{\delta} \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array}.$$

Note that $\sum_l w_l = w$, $\sum_k \lambda_k^2 \|v_k\|_1 = \|x\|_2^2$, and

$$\mathcal{F}\left(\sum_k \lambda_k^2 |v_k|\right) = \mathcal{F}\left(\left(\sum_k \lambda_k v_k\right)^* \left(\sum_k \lambda_k v_k\right)\right) = \mathcal{F}(x^* x) = w * \bar{w}^*,$$

so

$$\begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} = \sum_{l_1,l_2} \frac{\|x\|_2^2}{\delta} \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array}.$$

Computing the trace on both sides, we have

$$\begin{aligned} \text{tr}_2((w^* * \bar{w})(w * \bar{w}^*)) &= \frac{\|x\|_2^2}{\delta} \sum_{l_1, l_2} \text{tr}_2(|w_{l_2}| * \overline{|w_{l_1}|}) \\ &= \frac{\|x\|_2^2}{\delta} \sum_{l_1, l_2} \frac{\|w_{l_2}\|_1 \|w_{l_1}\|_1}{\delta} \\ &= \frac{\|x\|_2^2}{\delta} \frac{\|w\|_1^2}{\delta}. \end{aligned}$$

By Hölder’s inequality (Proposition 4.3), we have

$$\text{tr}_2((w^* * \bar{w})(w * \bar{w}^*)) \leq \|w^* * \bar{w}\|_\infty \|w * \bar{w}^*\|_1. \tag{9}$$

So

$$\frac{\|x\|_2^2}{\delta} \frac{\|w\|_1^2}{\delta} \leq \|w^* * \bar{w}\|_\infty \|w * \bar{w}^*\|_1.$$

On the other hand, by Proposition 4.7 and Lemma 4.9, we have

$$\|w * \bar{w}^*\|_\infty \leq \frac{\|x^* x\|_1}{\delta} = \frac{\|x\|_2^2}{\delta} \tag{10}$$

and

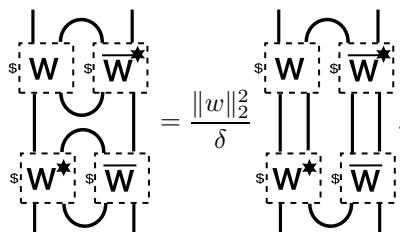
$$\|w * \bar{w}^*\|_1 \leq \frac{\|w\|_1 \|\bar{w}^*\|_1}{\delta} = \frac{\|w\|_1^2}{\delta}.$$

Hence all equalities of the inequalities hold. Note that $(w^* * \bar{w}) = (w * \bar{w}^*)^*$. Then by the equality of (9), we have that

$$\|w * \bar{w}^*\|_2^2 = \|w * \bar{w}^*\|_\infty \|w * \bar{w}^*\|_1.$$

By Proposition 4.4, $w * \bar{w}^*$ is a multiple of a partial isometry. By the equality of Equation (10), we have that $\frac{\delta}{\|w\|_2^2} w * \bar{w}^*$ is a partial isometry.

Furthermore, since $\sum_l |w_l| = w^* w$ and $\|x\|_2^2 = \|w\|_2^2$, we see that



By Theorem 6.11, we have $\frac{\delta}{\|w\|_2^2} w * \overline{w^*}$ is a partial isometry and B is a projection. Note that $\mathcal{F}(w^*)$ is an extremal bi-partial isometry, by Theorem 6.11, one obtains $\mathcal{F}(w^*) * \overline{\mathcal{F}(w^*)}^*$ is a multiple of a partial isometry. Moreover,

$$\begin{aligned} \mathcal{F}(w^*) * \overline{\mathcal{F}(w^*)}^* &= \mathcal{F}(w^*) * \overline{\mathcal{F}^{-1}(w)} \\ &= \mathcal{F}(w^*) * \mathcal{F}(w) = \mathcal{F}(w^*w). \end{aligned}$$

and

$$\begin{aligned} \frac{\|w\|_2^2}{\delta} \mathcal{F}(B) &= \frac{\delta}{\|w\|_2^2} \mathcal{F}((w * \overline{w^*}) * (w * \overline{w^*})) \\ &= \mathcal{F}((w^*w) * (\overline{w\overline{w^*}})) \\ &= \mathcal{F}(\overline{w\overline{w^*}}) \mathcal{F}(w^*w) \\ &= (\mathcal{F}(w^*w))^* \mathcal{F}(w^*w). \end{aligned}$$

We see that $\mathcal{F}(B)$ is a multiple of a projection, and B is a biprojection.

We define an element $B_g = w^*w$, then B_g is a projection. We are going to show B_g is a right shift of the biprojection B . By Theorem 6.11, we have $\frac{\delta}{\|w\|_2^2} B_g * \overline{B_g} = B$. Computing the trace on both sides, we have $\frac{\delta}{\|w\|_2^2} \frac{tr_2(B_g)^2}{\delta} = tr_2(B)$. Note that $\|w\|_2^2 = tr_2(B_g)$, so $tr_2(B_g) = tr_2(B)$.

Recall that $\mathcal{F}(w^*)$ is an extremal bi-partial isometry, so $\frac{\delta}{\|w\|_1} \mathcal{F}(w^*)$ is a partial isometry. By Theorem 6.11, we see that

$$\frac{\delta}{\|w\|_2^2} \mathcal{F}(B_g) = \frac{\delta}{\|\frac{\delta}{\|w\|_1} \mathcal{F}(w^*)\|_2^2} \left(\frac{\delta}{\|w\|_1} \mathcal{F}(w^*) \right) * \left(\frac{\delta}{\|w\|_1} \overline{\mathcal{F}(w^*)}^* \right)$$

is a partial isometry. Hence we obtain

$$\begin{aligned} \mathcal{F}(B_g) &= \left(\frac{\delta}{\|w\|_2^2} \right)^2 \mathcal{F}(B_g) \mathcal{F}(B_g)^* \mathcal{F}(B_g) \\ &= \left(\frac{\delta}{\|w\|_2^2} \right)^2 \mathcal{F}(B_g) \mathcal{F}(\overline{B_g}) \mathcal{F}(B_g) \\ &= \left(\frac{\delta}{\|w\|_2^2} \right)^2 \mathcal{F}(B_g * \overline{B_g} * B_g). \end{aligned}$$

and $\left(\frac{\delta}{\|w\|_2^2} \right)^2 B_g * \overline{B_g} * B_g = B_g$. Then

$$B * B_g = \frac{\delta}{\|w\|_2^2} B_g * \overline{B_g} * B_g = \frac{\|w\|_2^2}{\delta} B_g = \frac{\|B_g\|_1}{\delta} B_g.$$

Therefore B_g is a right shift of the biprojection B .

Note that $\mathcal{F}^{-1}(\overline{w}^*)$ is an extremal bi-partial isometry, so $\frac{\delta}{\|w\|_1} \mathcal{F}^{-1}(\overline{w}^*)$ is a partial isometry. Take $\tilde{B} = \mathcal{R}(\mathcal{F}(B))$ and $\tilde{B}_h = \left(\frac{\delta}{\|w\|_1} \mathcal{F}^{-1}(\overline{w}^*)\right)^* \left(\frac{\delta}{\|w\|_1} \mathcal{F}^{-1}(\overline{w}^*)\right)$. Similarly \tilde{B}_h is a right shift of \tilde{B} .

Recall that $\frac{\delta}{\|w\|_1} \mathcal{F}^{-1}(\overline{w}^*)$ is a partial isometry, we have

$$\mathcal{F}^{-1}(\overline{w}^*)^* = \mathcal{F}(\overline{w}) = \mathcal{F}^{-1}(w), \quad \mathcal{F}(\tilde{B}_h) = \left(\frac{\delta}{\|w\|_1}\right)^2 w * \overline{w}^*,$$

and

$$\begin{aligned} \mathcal{F}^{-1}(\overline{w}^*)^* &= \left(\frac{\delta}{\|w\|_1}\right)^2 \mathcal{F}^{-1}(\overline{w}^*)^* \mathcal{F}^{-1}(\overline{w}^*) \mathcal{F}^{-1}(\overline{w}^*)^* \\ &= \left(\frac{\delta}{\|w\|_1}\right)^2 \mathcal{F}^{-1}(w * \overline{w}^* * w). \end{aligned}$$

So $w = \left(\frac{\delta}{\|w\|_1}\right)^2 w * \overline{w}^* * w$. Then $w = \mathcal{F}(\tilde{B}_h) * w = \mathcal{F}(\tilde{B}_h) * (wB_g)$, i.e.

The diagrammatic equation shows the element w as a product of a scalar $\left(\frac{\delta}{\|w\|_1}\right)^2$ and a composition of two bishifts. The first bishift is \tilde{B}_h , represented by a box with two vertical lines on the left and two on the right, with a dashed box inside containing w^* and w . The second bishift is B_g , represented by a box with two vertical lines on the left and two on the right, with a dashed box inside containing w . The composition is indicated by an equals sign and a larger dashed box containing the two bishifts.

is a bi-shift of the biprojection B .

By Lemma 6.7, a bi-shift of a biprojection gets the minimal value of the Donoho–Stark uncertainty principle, so it is an extremal bi-partial isometry by Theorem 6.4.

Furthermore, if w is positive, then w is identical to B_g in the above argument. \square

As a corollary, we obtain a new characterization of biprojections.

Corollary 6.14. For a nonzero 2-box x in an irreducible subfactor planar algebra. If x and $\mathcal{F}(x)$ are positive operators and $\mathcal{S}(x)\mathcal{S}(\mathcal{F}(x)) = \delta^2$, then x is a biprojection.

Proof. It follows from Theorems 6.4, 6.13. \square

Note that the proof of Theorem 6.13 is independent of the choice of the form of a bi-shift of a biprojection. We refer the reader to Appendix A for the eight forms of bishifts of biprojections.

We have shown that an element is the minimizer of the uncertainty principles if and only if it is an extremal bi-partial isometry if and only if it is a bi-shift of a biprojection.

Now let us prove the uniqueness of a bi-shift of a biprojection in the following sense. Given a 2-box, we will obtain four range projections, the left, top, right, bottom ones. An extremal bi-partial isometry, or equivalently a bi-shift of a biprojection is determined by two adjacent range projections up to a scalar.

Lemma 6.15. *Suppose B is a biprojection in $\mathcal{P}_{2,+}$, and \tilde{B} is the range projection of $\mathcal{F}(B)$. For $x \in \mathcal{P}_{2,+}$, if $\mathcal{R}(x^*) = B$ and $\mathcal{R}(\mathcal{F}^{-1}(x)) = \tilde{B}$, then x is a multiple of B .*

Proof. It is shown in [5] that the biprojection B can be expressed as a multiple of the

(a, b-colored) Fuss–Catalan diagram $\left| \begin{array}{cc} ab & ba \\ \cup & \cup \\ ab & ba \end{array} \right|$. If $\mathcal{R}(x^*) = B$ and $\mathcal{R}(\mathcal{F}^{-1}(x)) = \tilde{B}$, then

x is a multiple of the Fuss–Catalan diagram $\left| \begin{array}{cc} ab & ba \\ \cup & \cup \\ ab & ba \end{array} \right|$. Recall that \mathcal{P} is assumed to be

irreducible, so $\left| \begin{array}{cc} ab & ba \\ \cup & \cup \\ ab & ba \end{array} \right|$ is a multiple of $\left| \begin{array}{cc} ab & ba \\ \cup & \cup \\ ab & ba \end{array} \right|$. Hence x is a multiple of B . \square

Theorem 6.16. *Suppose $B \in \mathcal{P}_{2,+}$ is a biprojection and \tilde{B} is the range projection of $\mathcal{F}(B)$. Take a right shift B_g of B , and a right shift \tilde{B}_h of \tilde{B} . Then there is at most one element $x \in \mathcal{P}_{2,+}$ up to a scalar such that the range projection of $|x|$ is contained in B_g and the range projection of $\mathcal{F}^{-1}(x)$ is contained in \tilde{B}_h .*

Proof. If x and z are two nonzero elements of $\mathcal{P}_{2,+}$, such that the range projections $\mathcal{R}(|x|)$ and $\mathcal{R}(|z|)$ are contained in B_g and the range projections $\mathcal{R}(\mathcal{F}^{-1}(x))$ and $\mathcal{R}(\mathcal{F}^{-1}(z))$ are contained in \tilde{B}_h , then by Theorems 6.4, 6.13, we have $\mathcal{R}(|x|) = \mathcal{R}(|z|) = B_g$ and $\mathcal{R}(\mathcal{F}^{-1}(x)) = \mathcal{R}(\mathcal{F}^{-1}(z)) = \tilde{B}_h$. Thus zx^* and xx^* are nonzero and $\mathcal{R}(|zx^*|) = \mathcal{R}(|xx^*|) = \mathcal{R}(x)$. By Lemma 3.5, we have

$$\mathcal{R}(\mathcal{F}^{-1}(zx^*)) = \mathcal{R}(\overline{\mathcal{F}^{-1}(x)}^* * \mathcal{F}^{-1}(z)) \leq \mathcal{R}(\overline{\tilde{B}_h} * \tilde{B}_h).$$

By Theorem 6.11,

$$\mathcal{S}(\overline{\tilde{B}_h} * \tilde{B}_h) = \mathcal{S}(\tilde{B}_h) = \mathcal{S}(\mathcal{F}^{-1}(x)).$$

Then

$$\mathcal{S}(zx^*)\mathcal{S}(\mathcal{F}^{-1}(zx^*)) \leq \mathcal{S}(x)\mathcal{S}(\mathcal{F}^{-1}(x)) = \delta^2.$$

By Theorem 5.2, we have

$$\mathcal{S}(zx^*) = \mathcal{S}(x); \quad \mathcal{S}(\mathcal{F}^{-1}(zx^*)) = \mathcal{S}(\mathcal{F}^{-1}(x)) = \mathcal{S}(\overline{\widetilde{B}_h} * \widetilde{B}_h).$$

Therefore

$$\mathcal{R}(zx^*) = \mathcal{R}(x); \quad \mathcal{R}(\mathcal{F}^{-1}(zx^*)) = \mathcal{R}(\overline{\widetilde{B}_h} * \widetilde{B}_h).$$

By Theorems 5.2, 6.4, we have that zx^* is a bi-shift of a biprojection. Similarly xx^* is a bi-shift of a biprojection. Moreover,

$$\mathcal{R}(|zx^*|) = \mathcal{R}(|xx^*|); \quad \mathcal{R}(\mathcal{F}^{-1}(zx^*)) = \mathcal{R}(\mathcal{F}^{-1}(xx^*)).$$

By a similar argument, we have $\overline{(xx^*)}^* * (zx^*)$ and $\overline{(xx^*)}^* * (xx^*)$ are bi-shifts of biprojections, and

$$\begin{aligned} \mathcal{R}(|\overline{(xx^*)}^* * (zx^*)|) &= \mathcal{R}(|\overline{(xx^*)}^* * (xx^*)|); \\ \mathcal{R}(\mathcal{F}^{-1}(\overline{(xx^*)}^* * (zx^*))) &= \mathcal{R}(\mathcal{F}^{-1}(\overline{(xx^*)}^* * (xx^*))). \end{aligned}$$

By Theorem 6.11, $\overline{(xx^*)}^* * (xx^*)$ is a multiple of a biprojection, denoted by Q . By Lemma 6.15, $\overline{(xx^*)}^* * (zx^*)$ is also a multiple of Q . Observe that both z and x are multiples of $((xx^*) * Q)x$, so z is a multiple of x . \square

Finally we are able to formulate Hardy’s uncertainty principle as follows.

Corollary 6.17. *Suppose a 2-box w is a minimizer for the (Hirschman–Beckner/Donoho–Stark) uncertainty principle. For a 2-box x and constants $C, C',$ if*

$$|x| \leq C|w|, \quad |\mathcal{F}(x)| \leq C'|\mathcal{F}(w)|,$$

then x is a scalar multiple of w .

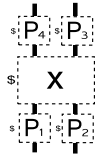
This proves Main Theorem 3.

7. Uncertainty principles for n -boxes

In this section, we will prove the uncertainty principles for general cases, in particular for reducible subfactor planar algebras and n -boxes. Suppose \mathcal{P} is a subfactor planar algebra. For a projection $P \in \mathcal{P}_{n,+}$, we define $P\mathcal{P} = \{Px|x \in \mathcal{P}_{n,+}\}$ to be the Hilbert subspace of $\mathcal{P}_{n,+}$. We make a similar definition for a projection in $\mathcal{P}_{n,-}$. For projections $P_i \in \mathcal{P}_{n_i,\pm}$, $i = 1, 2, 3, 4$, we define

$$P_i \otimes P_j = \begin{array}{c} \begin{array}{|c|} \hline \mathbf{P}_i \\ \hline \end{array} \begin{array}{|c|} \hline \mathbf{P}_j \\ \hline \end{array} \\ \hline \end{array},$$

when the shadings match. We define $\mathcal{P}_{P_1 \otimes P_2}^{P_4 \otimes P_3}$ to be the span of elements in the following form:



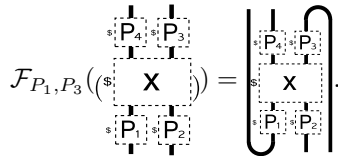
For an element in $\mathcal{P}_{P_1 \otimes P_2}^{P_3 \otimes P_4}$, we also have the rank-one decomposition similar to the rank one decomposition in Section 2.

Note that $|x| = (x^*x)^{1/2} \in \mathcal{P}_{n_3+n_4, \pm}$. Let us define $\|x\|_p = \||x|\|_p, 1 \leq p \leq \infty; \mathcal{S}(x) = \mathcal{S}(|x|)$.

Proposition 7.1. *For any x in $\mathcal{P}_{n, \pm}$, we have*

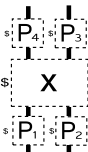

- (1) $\|x\|_p = \|x^*\|_p = \|\bar{x}\|_p, 1 \leq p \leq \infty;$
- (2) $\mathcal{S}(x) = \mathcal{S}(x^*) = \mathcal{S}(\bar{x});$
- (3) $H(|x|^2) = H(|x^*|^2) = H(|\bar{x}|^2).$


Let us define the Fourier transform $\mathcal{F}_{P_1, P_3} : \mathcal{P}_{P_1 \otimes P_2}^{P_4 \otimes P_3} \rightarrow \mathcal{P}_{P_2 \otimes P_3}^{\bar{P}_1 \otimes P_4}$ as follows



Proposition 7.2. *For any x in $\mathcal{P}_{P_1 \otimes P_2}^{P_3 \otimes P_4}$, we have $\|\mathcal{F}_{P_1, P_3}(x)\|_2 = \|x\|_2.$*

Proof. It follows from the sphericity of subfactor planar algebras. \square

In the rest of the section, we will write the diagram  as , or simply as

 if there is no confusion. By this notation, we can see that our method also works for a planar algebra with more kinds of regions and strings. The planar algebra should be C^* and spherical. We will not give a formal definition here and skip the details.

Let us define

$$\delta_i = \min\{tr_{n_i}(P)|P \text{ is a nonzero subprojection of } P_i\}, 1 \leq i \leq 4;$$

$$\delta_0 = \max\{\sqrt{\delta_1\delta_3}, \sqrt{\delta_2\delta_4}\}.$$

Therefore

$$\|\mathcal{F}_{P_1, P_3}(v)\|_\infty \leq \frac{\|v\|_1}{\delta_0},$$

where $\delta_0 = \max\{\sqrt{\delta_1\delta_3}, \sqrt{\delta_2\delta_4}\}$.

If x is arbitrary in $\mathcal{P}_{2,\pm}$, let $x = \sum_k \lambda_k v_k$ be the rank-one decomposition, then $\|x\|_1 = \sum_k \lambda_k \|v_k\|_1$. We have proved that $\|\mathcal{F}_{P_1, P_3}(v_k)\|_\infty \leq \frac{\|v_k\|_1}{\delta_0}$ for the rank-one partial isometry v_k , so

$$\|\mathcal{F}_{P_1, P_3}(x)\|_\infty \leq \sum_k \lambda_k \|\mathcal{F}_{P_1, P_3}(v_k)\|_\infty \leq \sum_k \lambda_k \frac{\|v_k\|_1}{\delta_0} = \frac{\|x\|_1}{\delta_0}. \quad \square \quad (13)$$

Notation 7.4. We call this general version of 2-box x extremal if it satisfies the equality

$$\|\mathcal{F}_{P_1, P_3}(x)\|_\infty = \frac{1}{\delta_0} \|x\|_1$$

for this general Fourier transform.

Lemma 7.5. Suppose $x \in \mathcal{P}_{P_1 \otimes P_2}^{P_4 \otimes P_3}$ is extremal. If $\delta_0 = \sqrt{\delta_1\delta_3}$, then there are subprojections Q_i of P_i with trace δ_i , for $i = 1, 3$, such that $x = (Q_1 \otimes P_2)x(P_4 \otimes Q_3)$; If $\delta_0 = \sqrt{\delta_2\delta_4}$, then there are subprojections Q_i of P_i with trace δ_i , for $i = 2, 4$, such that $x = (P_1 \otimes Q_2)x(Q_4 \otimes P_3)$.

Proof. Suppose x is extremal. When $\delta_0 = \sqrt{\delta_1\delta_3}$, if x is a rank-one partial isometry, then Equations (11), (12) hold in the proof of 7.3. So there are subprojections Q_i of P_i with trace δ_i , for $i = 1, 3$, such that $x = (Q_1 \otimes P_2)x(P_4 \otimes Q_3)$.

If x is arbitrary, let $x = \sum_k \lambda_k v_k$ be the rank-one decomposition, then each v_k is a rank-one extremal partial isometry. Without loss of generality, we take v_1 and v_2 , then there are subprojections Q_i, Q'_i of P_i with trace δ_i , for $i = 1, 3$, such that $v_1 = (Q_1 \otimes 1)v_1(1 \otimes Q_3)$; $v_2 = (Q'_1 \otimes 1)v_2(1 \otimes Q'_3)$. It is enough to show that $Q_i = Q'_i$. Since x is extremal, from Equation (13) in the proof of 7.3, we have

$$\|\mathcal{F}_{P_1, P_3}(v_1) + \mathcal{F}_{P_1, P_3}(v_2)\|_\infty = \|\mathcal{F}_{P_1, P_3}(v_1)\|_\infty + \|\mathcal{F}_{P_1, P_3}(v_2)\|_\infty.$$

Taking the square on both sides,

$$\begin{aligned} & \|(\mathcal{F}_{P_1, P_3}(v_1) + \mathcal{F}_{P_1, P_3}(v_2))^*(\mathcal{F}_{P_1, P_3}(v_1) + \mathcal{F}_{P_1, P_3}(v_2))\|_\infty \\ &= (\|\mathcal{F}_{P_1, P_3}(v_1)\|_\infty + \|\mathcal{F}_{P_1, P_3}(v_2)\|_\infty)^2. \end{aligned}$$

So

$$\| |\mathcal{F}_{P_1, P_3}(v_1)|^2 + |\mathcal{F}_{P_1, P_3}(v_2)|^2 \|_\infty = \|\mathcal{F}_{P_1, P_3}(v_1)\|_\infty^2 + \|\mathcal{F}_{P_1, P_3}(v_2)\|_\infty^2.$$

Then

$$\begin{aligned} & \| \mathcal{F}_{P_1, P_3}(v_1) \|_\infty^2 (\overline{Q_1} \otimes P_4) + \| \mathcal{F}_{P_1, P_3}(v_2) \|_\infty^2 (\overline{Q'_1} \otimes P_4) \|_\infty \\ & \geq \| \mathcal{F}_{P_1, P_3}(v_1) \|_\infty^2 + \| \mathcal{F}_{P_1, P_3}(v_1) \|_\infty^2. \end{aligned}$$

So

$$\| \mathcal{F}_{P_1, P_3}(v_1) \|_\infty^2 \overline{Q_1} + \| \mathcal{F}_{P_1, P_3}(v_2) \|_\infty^2 \overline{Q'_1} \|_\infty \geq \| \mathcal{F}_{P_1, P_3}(v_1) \|_\infty^2 + \| \mathcal{F}_{P_1, P_3}(v_1) \|_\infty^2.$$

That implies $Q_1 = Q'_1$. Similarly $Q_3 = Q'_3$.

The proof is similar if $\delta_0 = \sqrt{\delta_2 \delta_4}$. \square

For $x \in \mathcal{P}_{P_1 \otimes P_2}^{P_4 \otimes P_3}, y \in \mathcal{P}_{P_2 \otimes P'_1}^{\overline{P_3} \otimes P'_4}$, let us define the coproduct $*_{P_2}^{P_3}$, simply denoted by $*$, as follows

$$x * y = \begin{array}{c} \begin{array}{|c|} \hline \overline{P_3} \\ \hline \end{array} \\ \left. \begin{array}{|c|} \hline x \\ \hline \end{array} \right\} \begin{array}{|c|} \hline y \\ \hline \end{array} \\ \begin{array}{|c|} \hline P_2 \\ \hline \end{array} \end{array}$$

By a similar argument in Section 4 and the technique in the proof of Theorem 7.3, we have the following Young’s inequality.

Theorem 7.6. For any x in $\mathcal{P}_{P_1 \otimes P_2}^{P_4 \otimes P_3}$, y in $\mathcal{P}_{P_2 \otimes P'_1}^{\overline{P_3} \otimes P'_4}$, we have

$$\|x * y\|_r \leq \left(\frac{1}{\sqrt{\delta_2 \delta_3}} \right)^{\frac{1}{r}} \left(\frac{1}{\sqrt{\delta'_1 \delta'_4}} \right)^{1 - \frac{1}{q}} \left(\frac{1}{\sqrt{\delta_1 \delta_4}} \right)^{1 - \frac{1}{p}} \|x\|_q \|y\|_p,$$

where $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$, and δ'_1, δ'_4 are defined in a similar way for P'_1, P'_4 respectively.

Repeating the arguments in Sections 5, 6, we have the following results.

Theorem 7.7. For a nonzero element x in $\mathcal{P}_{P_1 \otimes P_2}^{P_3 \otimes P_4}$,

- (1) $\mathcal{S}(x)\mathcal{S}(\mathcal{F}_{P_1, P_3}(x)) \geq \delta_0^2$;
- (2) $H(|x|^2) + H(|\mathcal{F}_{P_1, P_3}(x)|^2) \geq \|x\|_2^2(2 \log \delta_0 - 4 \log \|x\|_2)$.

Moreover, the following are equivalent

- (1) $\mathcal{S}(x)\mathcal{S}(\mathcal{F}_{P_1, P_3}(x)) = \delta_0^2$;
- (2) $H(|x|^2) + H(|\mathcal{F}_{P_1, P_3}(x)|^2) = \|x\|_2^2(2 \log \delta_0 - 4 \log \|x\|_2)$;
- (3) x is an extremal bi-partial isometry.

Furthermore, if $\delta_1\delta_3 = \delta_2\delta_4$, then by Lemma 7.5, the projection P_i on the boundary of an extremal bi-partial isometry could be replaced by a minimal projection Q_i with trace δ_i , for $1 \leq i \leq 4$. Then the element B constructed in Theorem 6.13 is a 2-box in an irreducible subfactor planar algebra, so the definition and properties of the biprojection B are inherited. By a similar argument in Section 6, we have the following results.

Theorem 7.8. *With above notations and assumptions, for a nonzero element x in $\mathcal{P}_{P_1 \otimes P_2}^{P_3 \otimes P_4}$, $\delta_1\delta_3 = \delta_2\delta_4$, the following statements are equivalent:*

- (1) $\mathcal{S}(x)\mathcal{S}(\mathcal{F}_{P_1, P_3}(x)) = \delta_0^2$;
- (2) $H(|x|^2) + H(|\mathcal{F}_{P_1, P_3}(x)|^2) = \|x\|_2^2(2 \log \delta_0 - 4 \log \|x\|_2)$;
- (3) x is an extremal bi-partial isometry;
- (3') x is a partial isometry and $\mathcal{F}_{P_1, P_3}^{-1}(x)$ is extremal;
- (4) x is a bi-shift of a biprojection.

Theorem 7.9. *Suppose \mathcal{P} is an irreducible subfactor planar algebra. For a nonzero element $x \in \mathcal{P}_{n, \pm}$, we have*

$$\prod_{k=0}^{n-1} \mathcal{S}(\mathcal{F}^k(x)) \geq \delta^n;$$

$$\sum_{k=0}^{n-1} H(|\mathcal{F}^k(x)|^2) \geq \|x\|_2^2(n \log \delta - 2n \log \|x\|_2).$$

This is Main Theorem 4.

Proof. Considering P_1, P_3 as one through string, P_2, P_4 as $n - 1$ through strings in Theorem 7.7, then $\delta_0 \geq \sqrt{\delta_1\delta_3} = \delta$ and $\mathcal{S}(x)\mathcal{S}(\mathcal{F}_{P_1, P_3}(x)) \geq \delta^2$. So

$$\prod_{k=0}^{2n-1} \mathcal{S}(\mathcal{F}^k(x)) \geq \delta^{2n}.$$

Note that $\mathcal{S}(\mathcal{F}^{n+k}(x)) = \mathcal{S}(\mathcal{F}^k(x))$, so

$$\prod_{k=0}^{n-1} \mathcal{S}(\mathcal{F}^k(x)) \geq \delta^n.$$

Similarly we have

$$\sum_{k=0}^{n-1} H(|\mathcal{F}^k(x)|^2) \geq \|x\|_2^2(n \log \delta - 2n \log \|x\|_2). \quad \square$$

The equalities of the above inequalities are obtained on the Temperley–Lieb diagram with caps around the boundary, for example the 5-box $\left| \begin{array}{c} \cup \\ \cup \\ \cup \\ \cup \\ \cup \end{array} \right.$. If there is a biprojection, then the equality is also obtained on the Fuss–Catalan diagram with caps around.

Definition 7.10. Suppose \mathcal{P} is an irreducible subfactor planar algebra. A projection P in $\mathcal{P}_{n,\pm}$ is called an n -projection if $\mathcal{F}^k(P)$ is a multiple of a projection, for $0 \leq k \leq n - 1$. An element x in $\mathcal{P}_{n,\pm}$ is called an extremal n -partial isometry if $\mathcal{F}^k(x)$ is a multiple of a partial isometry and $\|\mathcal{F}^{k\pm 1}(x)\| = \frac{\|\mathcal{F}^k(x)\|_1}{\delta}$, for $0 \leq k \leq n - 1$.

Question. Are all n -projections minimizers of the uncertainty principle? Are all minimizers extremal n -partial isometries? Are all minimizers “ n -shifts of n -projections”?

8. Application to group algebras

Suppose $\mathcal{M} = \mathcal{N} \rtimes G$, for an outer action a finite group G . Let \mathcal{P} be the subfactor planar algebra of the subfactor $\mathcal{N} \subset \mathcal{M}$. Then $\mathcal{P}_{2,+}$ is the C^* -algebra of functions over G and $\mathcal{P}_{2,-}$ is the group C^* -algebra of G . The value of a closed circle δ is $\sqrt{|G|}$. The trace tr_2 of $\mathcal{P}_{2,+}$ is the integral of functions on G over the counting measure. The trace tr_2 of $\mathcal{P}_{2,-}$ is the trace of the group algebra of G on its left regular representation. For a group element $g \in G$, the Fourier transform from $\mathcal{P}_{2,\mp}$ to $\mathcal{P}_{2,\pm}$ maps the group element g to the function $\sqrt{|G|}\delta_g$; the function $\sqrt{|G|}\delta_g$ to the group element g^{-1} , where δ_g is the Dirac function at g . The coproduct of $\mathcal{P}_{2,+}$ is a multiple of the convolution. Precisely for two functions $f_1, f_2 : G \rightarrow \mathbb{C}$,

$$f_1 * f_2(g_0) = \frac{1}{\sqrt{|G|}} \sum_{g \in G} f_1(g_0 g^{-1}) f_2(g).$$

Theorem 4.13 becomes Young’s inequality for the convolution of groups. The Plancherel formula is a spherical isotopy.

Proposition 8.1. Suppose G is a finite group. Take a subgroup H , a one dimensional representation χ of H , an element $g \in G$, a nonzero constant $c \in \mathbb{C}$. Then

$$x = c \sum_{h \in H} \chi(h)hg$$

is a bi-shift of a biprojection. Conversely any bi-shift of a biprojection is of this form.

Proof. For a subgroup H , the corresponding biprojection is $B = \sum_{h \in H} \delta_h$. The range projection \tilde{B} of $\mathcal{F}(B)$ is $\frac{1}{|H|} \sum_{h \in H} h$. By a direct computation, we have that a right shift

of $B = \sum_{h \in H} \delta_h$ is $B_g = \sum_{h \in H} \delta_{hg}$ for some $g \in G$. Let Q be a right shift of \tilde{B} . Then Q is an element in the group algebra of H , and it minimizes the uncertainty principle. Note that \tilde{B} is a central minimal projection in the group algebra of H , so its right shift Q is also a rank-one projection. Then

$$\frac{\text{tr}_2(hQ)}{\text{tr}_2(Q)}, h \in H,$$

is a one dimensional representation of H . Take χ to be its contragredient representation, then $\chi(h) = \frac{\text{tr}_2(h^{-1}Q)}{\text{tr}_2(Q)}$, and $Q = \frac{1}{|H|} \sum_{h \in H} \chi(h)h$. Take $x = \sum_{h \in H} \chi(h)hg$. Then $\mathcal{R}(x) = Q$ and $\mathcal{R}(\mathcal{F}(x)) = B_g$. So x is a bi-shift of a biprojection. By Theorem 6.16, any bi-shift of a biprojection is of this form. \square

Remark. Note that χ is the pull back of a character of $H/[H, H]$, where $[H, H]$ is the commutator subgroup.

By Theorems 6.11, 6.13, we have the following corollary.

Corollary 8.2. *Take a finite group G acting on its left regular representation, a subset S of G , an operator $x = \sum_{g \in S} \omega_g g$, $|\omega_g| = 1, \forall g$. Then*

- (1) x is a bi-shift of a biprojection $\iff x$ is extremal.
 Furthermore, if $x = \sum_{g \in S} g$, then
- (2) S is a coset $\iff x$ is extremal $\iff \|x\|_1 = |S|$.
- (3) S is a subgroup $\iff x$ is positive.

9. Applications to group actions

The spin model [17] of an n -dimensional vector space V with an orthonormal basis S is a subfactor planar algebra, denoted by $Spin$. Its 2-box space $Spin_{2,+}$ is the C^* algebra $End(V)$; $Spin_{2,-}$ is the C^* algebra of functions on the set $S \times S$. For a matrix $A = (a_{ij})_{i,j \in S}$ in $Spin_{2,+}$, $\text{tr}_2(A) = \sum_{i \in S} a_{ii}$; For a function $f(i, j) = f_{ij}$ in $Spin_{2,-}$, $\text{tr}_2(f) = \frac{1}{n} \sum_{i,j \in S} f_{ij}$. The Fourier transform $\mathcal{F} : Spin_{2,\pm} \rightarrow Spin_{2,\mp}$ is

$$\mathcal{F}(A)(i, j) = \sqrt{n}a_{ij}; \mathcal{F}(f) = \left(\frac{f_{ji}}{\sqrt{n}}\right)_{i,j \in S}.$$

Let $A = (a_{ij}), B = (b_{ij})$ be $n \times n$ matrices. The Hadamard product of A and B is the matrix $C = (c_{ij})$ (denoted by $A \circ B$) given by $c_{ij} = a_{ij}a_{ij}$. The coproduct $A * B$ is realized as $\sqrt{n}A \circ B$.

If there is a group action G on S , then the fixed point algebra of $Spin$ under the induced group action of G is also a subfactor planar algebra, denoted by \mathcal{P} . Moreover, $\mathcal{P}_{2,+}$ consists of $S \times S$ matrices commuting with the action of G . Let δ_0 be the minimal value of the trace of a nonzero minimal projection in $\mathcal{P}_{1,\pm}$. Then $\delta_0 = \frac{n_0}{\sqrt{n}}$, where n_0 is the minimal cardinality of G -orbits of S . Then we obtain the Hausdorff–Young inequality, Young’s inequality, the uncertainty principles, and the characterizations of minimizers for elements in $\mathcal{P}_{2,+}$. For example, we have the following inequality for Hadamard product.

Corollary 9.1. *If A, B are two $S \times S$ matrices commuting with a group action of G on S , then we have*

$$\|A \circ B\|_r \leq \frac{1}{n_0^2} \|A\|_p \|B\|_q,$$

where $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$ and n_0 is the minimal value of the number of elements in an orbit of S under the action of G .

If the group action is transitive, then \mathcal{P} is irreducible and it could be viewed as a group subgroup subfactor planar algebra. If $S = G$, and the action of G on G is the group multiplication, then \mathcal{P} becomes the group subfactor planar algebra of G .

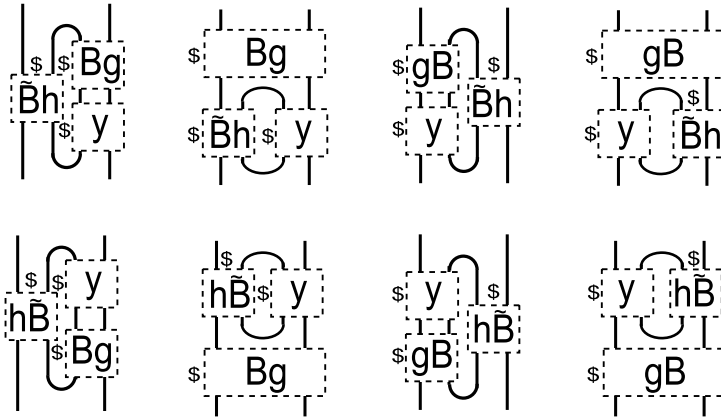
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Appendix A

Suppose B is a biprojection, and \tilde{B} is the range projection of $\mathcal{F}(B)$. Let ${}_g B, B_g$ be a left and a right shift of B respectively. Let ${}_h \tilde{B}, \tilde{B}_h$ be a left and a right shift of \tilde{B} respectively.

The point to construct a bi-shift of the biprojection B is to make sure the pair of biprojections B and \tilde{B} are adjacent to the same corner of the 2-box. There are eight forms of bi-shifts of biprojections as follows:



Although the definition of bishifts of biprojections is given by the first form. By similar arguments, the proof of [Theorem 6.13](#) also works for the other forms of bishifts of biprojections. Consequently any form of bi-shift of a biprojection could be expressed as any other form of bi-shift of a possibly different biprojection. Therefore it does not matter what we choose as the definition of bishifts of biprojections.

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