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Composed inclusions of A_3 and A_4 subfactors

Zhengwei Liu

Department of Mathematics, Vanderbilt University, Nashville, TN 37240, USA

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ABSTRACT

In this article, we classify all standard invariants that can arise from a composed inclusion of an A_3 with an A_4 subfactor. More precisely, if $\mathcal{N} \subset \mathcal{P}$ is an A_3 subfactor and $\mathcal{P} \subset \mathcal{M}$ is an A_4 subfactor, then only four standard invariants can arise from the composed inclusion $\mathcal{N} \subset \mathcal{M}$. We answer a question posed by Bisch and Haagerup in 1994. The techniques of this paper also show that there are exactly four standard invariants for the composed inclusion of two A_4 subfactors.

1. Introduction

Jones classified the indices of subfactors of type II_1 in [13]. It is given by

$$\{4\cos^2(\frac{\pi}{n}), n = 3, 4, \cdots\} \cup [4, \infty].$$

For a subfactor $\mathcal{N} \subset \mathcal{M}$ of type II_1 with finite index, the Jones tower is a sequence of factors obtained by repeating *the basic construction*. The system of higher relative commutants is called the standard invariant of the subfactor [8,35]. A subfactor is said to be of finite depth, if its *principal graph* is finite. The standard invariant is a complete invariant of a finite depth subfactor [35].

E-mail address: zhengwei.liu@vanderbilt.edu.

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0001-8708/

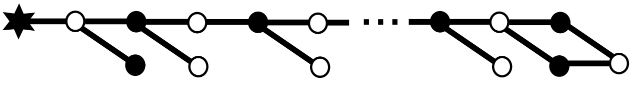
Subfactor planar algebras were introduced by Jones as a diagrammatic axiomatization of the standard invariant [17]. Other axiomatizations are known as Ocneanu's paragroups [30] and Popa's λ-lattices [37]. Each subfactor planar algebra contains a Temperley–Lieb planar subalgebra which is generated by the sequence of Jones projections. When the index of the Temperley–Lieb subfactor planar algebra is $4\cos^2(\frac{\pi}{n+1})$, its principal graph is the Coxeter–Dynkin diagram A_n.

Given two subfactors $\mathcal{N} \subset \mathcal{P}$ and $\mathcal{P} \subset \mathcal{M}$, the composed inclusion $\mathcal{N} \subset \mathcal{P} \subset \mathcal{M}$ tells the *relative position* of these factors. The *group type inclusion* $\mathcal{R}^H \subset \mathcal{R} \subset \mathcal{R} \rtimes K$ for outer actions of finite groups H and K on the hyperfinite factor $\mathcal{R}$ of type II_1 was discussed by Bisch and Haagerup [5].

We are interested in studying the composed inclusion of two subfactors of type A, i.e., a subfactor $\mathcal{N} \subset \mathcal{M}$ with an intermediate subfactor $\mathcal{P}$, such that the principal graphs of $\mathcal{N} \subset \mathcal{P}$ and $\mathcal{P} \subset \mathcal{M}$ are type A Coxeter–Dynkin diagrams. From the planar algebra point of view, the planar algebra of $\mathcal{N} \subset \mathcal{M}$ is a composition of two Temperley–Lieb subfactor planar algebras. Their tensor product is well known [17,23]. Their *free product* as a minimal composition was discovered by Bisch and Jones [6], called the Fuss–Catalan subfactor planar algebra. In general, the composition of two Temperley–Lieb subfactor planar algebras is still not understood.

The easiest case is the composed inclusion of two A_3 subfactors. In this case, the index is 4, and such subfactors are extended type D [8,36]. They also arise as a group type inclusion $\mathcal{R}^H \subset \mathcal{R} \subset \mathcal{R} \rtimes K$, where $H \cong \mathbb{Z}_2$ and $K \cong \mathbb{Z}_2$.

The first non-group-like case is the composed inclusion of an A_3 with an A_4 subfactor. Its principal graph is computed by Bisch and Haagerup in their unpublished manuscript in 1994. Either it is a free composed inclusion, then its planar algebra is Fuss–Catalan; or its principal graph is a Bisch–Haagerup fish graph as



Then they asked whether this sequence of graphs are the principal graphs of subfactors. The first Bisch–Haagerup fish graph is the principal graph of the tensor product of an A_3 and an A_4 subfactor. By considering the flip on $\mathcal{R} \otimes \mathcal{R}$, Bisch and Haagerup constructed a subfactor whose principal graph is the second Bisch–Haagerup fish graph. Later Izumi generalised the Haagerup factor [1] while considering endomorphisms of Cuntz algebras [10], and he constructed a Haagerup–Izumi subfactor for the group $\mathbb{Z}_4$ in his unpublished notes, also called the $3^{\mathbb{Z}_4}$ subfactor [31]. The third Bisch–Haagerup fish graph is the principal graph of an intermediate subfactor of a reduced subfactor of the dual of $3^{\mathbb{Z}_4}$ [12]. It turns out the even half is Morita equivalent to the even half of $3^{\mathbb{Z}_4}$.

In this paper, we prove the following classification result.

Theorem 1.1. *There are exactly four subfactor planar algebras as a composition of an A_3 with an A_4 planar algebra.*

This answers the question posed by Bisch and Haagerup. When $n \geq 4$, the nth Bisch–Haagerup fish graph is not the principal graph of a subfactor. In the meanwhile, Izumi, Morrison and Penneys have ruled out the 4th–10th Bisch–Haagerup fish graphs using a different method, see [12].

Three of the four subfactor planar algebras have finite depth which are complete invariants of subfactors of the hyperfinite factor of type II_1 [35]. The Fuss–Catalan one has infinite depth. It can also be realised from a hyperfinite subfactor [38].

By similar techniques, we also prove the following classification result.

Theorem 1.2. *There are exactly four subfactors planar algebras as a composition of two A_4 planar algebras.*

Our classification result is also important to the classification of small index subfactors. The index $3 + \sqrt{5}$ is the next frontier after 5 where the subfactor planar algebras were completed classified recently [18,28,26,11,32]. Some interesting examples and classification results are known up to this index [27,25].

Now we sketch the ideas of the proof. Following the spirit of [33,2], if the principal graph of a subfactor planar algebra is the nth Bisch–Haagerup fish graph, then by the embedding theorem [19], the planar algebra is embedded in the *graph planar algebra* of its principal graph [14]. By the existence of a "normalizer" in the Bisch–Haagerup fish graph, there will be a *biprojection* [3] in the subfactor planar algebra, and the planar subalgebra generated by the biprojection is Fuss–Catalan. The image of the biprojection is determined by the unique possible *refined principal graph*, see Definition 3.14 and Theorem 3.26. Furthermore the planar algebra is decomposed as an *annular Fuss–Catalan* module, similar to the Temperley–Lieb case, [15,20]. Comparing the principal graph of this Fuss–Catalan subfactor planar algebra and the Bisch–Haagerup fish graph, there is a lowest weight vector in the orthogonal complement of Fuss–Catalan. It will satisfy some specific relations, and there is a "unique" potential solution of these relations in the graph planar algebra.

The similarity of all the Bisch–Haagerup fish graphs admits us to compute the coefficients of loops of the potential solutions simultaneously. The coefficients of two sequences of loops have periodicity 5 and 20 with respect to n. Comparing with the coefficients of the other two sequences of loops, we will rule out the all the Bisch–Haagerup fish graphs, except the first three.

The existence of the first three follows from the construction mentioned above. The uniqueness follows from the "uniqueness" of the potential solution.

Furthermore we consider the composition of two A_4 planar algebras in the same process. In this list, there are exactly four subfactor planar algebras. They all arise from reduced subfactors of the four compositions of A_3 with A_4.

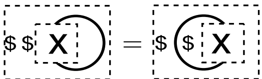
The skein theoretic construction of these subfactor planar algebras could be realised by the *Fuss–Catalan Jellyfish relations* of a generating vector space.

2. Background

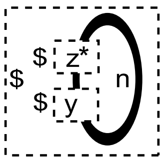
We refer the reader to [16] for the definition of planar algebras.

Notation 2.1. In a planar tangle, we use a thick string with a number k to indicate k parallel strings.

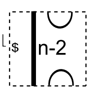
A subfactor planar algebra $\mathscr{S} = \{\mathscr{S}_{n,\pm}\}_{n\in\mathbb{N}_0}$ will be a spherical planar *-algebra over $\mathbb{C}$, such that $\dim(\mathscr{S}_{n,\pm}) < \infty$, for all n, $\dim(\mathscr{S}_{0,\pm}) = 1$, and the unnormalized Markov trace induces a positive definite inner product of $\mathscr{S}_{n,\pm}$ [16,17]. Note that $\dim(\mathscr{S}_{0,\pm}) = 1$, then $\mathscr{S}_{0,\pm}$ is isomorphic to $\mathbb{C}$ as a field. It is spherical means



as a number in $\mathbb{C}$, for any $x \in \mathscr{S}_{1,\pm}$. The inner product of $\mathscr{S}_{n,\pm}$ defined as

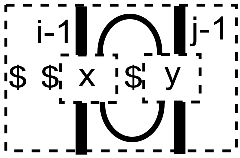
$$< y, z >= tr(z^* y) =$$ ,

the Markov trace of $z^* y$, for any $y, z \in \mathscr{S}_{n,\pm}$, is positive definite. (The diagram of the multiplication $z^* y$ is not the usual convention. The usual multiplication is from the bottom to the top.)

A subfactor planar algebra is always unital, where unital means any tangle without inner discs can be identified as a vector of $\mathscr{S}$. Note that $\mathscr{S}_{0,\pm}$ is isomorphic to $\mathbb{C}$, the (shaded or unshaded) empty diagram can be identified as the number 1 in $\mathbb{C}$. The value of a (shaded or unshaded) closed string is δ. And δ^{-1}  in $\mathscr{S}_{n,+}$, denoted by e_{n-1}, for $n \geq 2$, is the sequence of Jones projections. The filtered algebra generated by Jones projections is the smallest subfactor planar algebra, well known as the Temperley–Lieb algebra, denoted by $TL(\delta)$. Its vectors can be written as linear sums of tangles without inner discs.

Notation 2.2. We may identify $\mathscr{S}_{m,-}$ as a subspace of $\mathscr{S}_{m+1,+}$ by adding one string to the left.

Definition 2.3. Let us define the (1-string) coproduct of $x \in \mathscr{S}_{i,\pm}$ and $y \in \mathscr{S}_{j,\pm}$, for $i, j \geq 1$, to be

$x * y =$ 

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whenever the shading matched.

Let us recall some facts about the embedding theorem. Then we generalize these results to prove the embedding theorem for an intermediate subfactor in the next section.

2.1. Principal graphs

Suppose $\mathcal{N} \subset \mathcal{M}$ is an irreducible subfactor of type II_1 with finite index. Then $L^2(\mathcal{M})$ forms an irreducible $(\mathcal{N}, \mathcal{M})$ bimodule, denoted by X. Its conjugate $\overline{X}$ is an $(\mathcal{M}, \mathcal{N})$ bimodule. The tensor products $X \otimes \overline{X} \otimes \cdots \otimes \overline{X}$, $X \otimes \overline{X} \otimes \cdots \otimes X$, $\overline{X} \otimes X \otimes \cdots \otimes X$ and $\overline{X} \otimes X \otimes \cdots \otimes \overline{X}$ are decomposed into irreducible bimodules over $(\mathcal{N}, \mathcal{N})$, $(\mathcal{N}, \mathcal{M})$, $(\mathcal{M}, \mathcal{N})$ and $(\mathcal{M}, \mathcal{M})$ respectively, where $\otimes$ is Connes fusion of bimodules.

Definition 2.4. The principal graph of the subfactor $\mathcal{N} \subset \mathcal{M}$ is a bipartite graph. Its vertices are equivalence classes of irreducible bimodules over $(\mathcal{N}, \mathcal{N})$ and $(\mathcal{N}, \mathcal{M})$ in the above decomposed inclusion. The number of edges connecting two vertices, an $(\mathcal{N}, \mathcal{N})$ bimodule Y and an $(\mathcal{N}, \mathcal{M})$ bimodule Z, is the multiplicity of the equivalence class of Z as a sub bimodule of $Y \otimes X$. The vertex corresponding to the $(\mathcal{N}, \mathcal{N})$ bimodule $L^2(\mathcal{N})$ is marked by a star sign. The dimension vector of the bipartite graph is a function λ from the vertices of the graph to $\mathbb{R}^+$. Its value at a vertex is defined to be the dimension of the corresponding bimodule.

The dual principal graph is defined similarly for $(\mathcal{M}, \mathcal{M})$ and $(\mathcal{M}, \mathcal{N})$ bimodules.

Remark 2.5. By Frobenius reciprocity, the multiplicity of Z in $Y \otimes X$ equals to the multiplicity of Y in $Z \otimes \overline{X}$.

2.2. Standard invariants

For an irreducible subfactor $\mathcal{N} \subset \mathcal{M}$ of type II_1 with finite index, the Jones tower is a sequence of factors $\mathcal{N} \subset \mathcal{M} \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \cdots$ obtained by repeating the basic construction. The system of higher relative commutants

$$\begin{array}{ccccccccc} \mathbb{C} = \mathcal{N}' \cap \mathcal{N} & \subset & \mathcal{N}' \cap \mathcal{M} & \subset & \mathcal{N}' \cap \mathcal{M}_1 & \subset & \mathcal{N}' \cap \mathcal{M}_2 & \subset & \cdots \\ & & \cup & & \cup & & \cup & & \\ & & \mathrm{C} = \mathcal{M}' \cap \mathcal{M} & \subset & \mathcal{M}' \cap \mathcal{M}_1 & \subset & \mathcal{M}' \cap \mathcal{M}_2 & \subset & \cdots \end{array}$$

is called the standard invariant of the subfactor [8,35].

There is a natural isomorphism between homomorphisms of bimodules $X \otimes \overline{X} \otimes \cdots \otimes \overline{X}$, $X \otimes \overline{X} \otimes \cdots \otimes X$, $\overline{X} \otimes X \otimes \cdots \otimes X$ and $\overline{X} \otimes X \otimes \cdots \otimes \overline{X}$ and the standard invariant of the subfactor [4]. The equivalence class of a minimal projection corresponds to an irreducible bimodule. So the principal graph tells how minimal projections are decomposed after the inclusion. Then we can define the principal graph for a subfactor planar algebra without the presumed subfactor. The following two propositions are well known to experts.

Proposition 2.6. *Suppose $\mathscr{S}$ is a subfactor planar algebra. If P_1, P_2 are minimal projections of $\mathscr{S}_{m,+}$, then $P_1 e_{m+1,+}$, $P_2 e_{m+1,+}$ are minimal projections of $\mathscr{S}_{m+2,+}$. Moreover P_1 and P_2 are equivalent in $\mathscr{S}_{m,+}$ if and only if $P_1 e_{m+1}$ and $P_2 e_{m+1}$ are equivalent in $\mathscr{S}_{m+2,+}$.*

Proposition 2.7 *(Frobenius reciprocity). Suppose $\mathscr{S}$ is a subfactor planar algebra. If P is a minimal projection of $\mathscr{S}_{m,+}$ and Q is a minimal projection of $\mathscr{S}_{m+1,+}$, then* $\dim(P\mathscr{S}_{m+1,+}Q) = \dim(Pe_{m+1}\mathscr{S}_{m+2,+}Q)$.

By the above two propositions, the Bratteli diagram of $\mathscr{S}_{m,+} \subset \mathscr{S}_{m+1,+}$ is identified as a subgraph of the Bratteli diagram of $\mathscr{S}_{m+1,+} \subset \mathscr{S}_{m+2,+}$. So it makes sense to take the limit of the Bratteli diagram of $\mathscr{S}_{m,+} \subset \mathscr{S}_{m+1,+}$ as m approaches infinity.

Definition 2.8. The principal graph of a subfactor planar algebra $\mathscr{S}$ is the limit of the Bratteli diagram of $\mathscr{S}_{m,+} \subset \mathscr{S}_{m+1,+}$. The vertex corresponding to the identity in $\mathscr{S}_{0,+}$ is marked by a star sign. The dimension vector λ at a vertex is defined to be the Markov trace of the minimal projection corresponding to that vertex.

Similarly the dual principal graph of a subfactor planar algebra $\mathscr{S}$ is the limit of the Bratteli diagram of $\mathscr{S}_{m,-} \subset \mathscr{S}_{m+1,-}$. The vertex corresponding to the identity in $\mathscr{S}_{0,-}$ is marked by a star sign. The dimension vector λ' at a vertex is defined to be the Markov trace of the minimal projection corresponding to that vertex.

The Bratteli diagram of $\mathscr{S}_{m,+} \subset \mathscr{S}_{m+1,+}$, as a subgraph of the Bratteli diagram of $\mathscr{S}_{m+1,+} \subset \mathscr{S}_{m+2,+}$, corresponds to the two-sided ideal $\mathscr{I}_{m+1,+}$ of $\mathscr{S}_{m+1,+}$ generated by the Jones projection e_m. So the two graphs coincide if and only if $\mathscr{S}_{m+1,+} = \mathscr{I}_{m+1,+}$.

Definition 2.9. For a subfactor planar algebra $\mathscr{S}$, if its principal graph is finite, then the subfactor planar algebra is said to be finite depth. Furthermore it is of depth m, if m is the smallest number such that $\mathscr{S}_{m+1,+} = \mathscr{S}_{m+1,+} e_m \mathscr{S}_{m+1,+}$.

Definition 2.10. A vertex v in the principal graph has depth m if the distance between v and the star vertex is m. The vertex has multiplicity n if there are n length-m paths from the star vertex to v.

A depth-m vertex in the principal graph corresponds to a central component in $\mathscr{S}_{m,+}/\mathscr{I}_{m,+}$. The vertex has multiplicity n tells that the central component is an n by n matrix algebra. Therefore the dual of the vertex has the same multiplicity.

2.3. Finite dimensional inclusions

We refer the reader to Chapter 3 of [21] for a discussion of inclusions of finite dimensional von Neumann algebras.

Definition 2.11. Suppose $\mathcal{A}$ is a finite dimensional von Neumann algebra and τ is a trace on it. The dimension vector $\lambda_{\mathcal{A}}^{\tau}$ is a function from the set of minimal central projections (or equivalence classes of minimal projections or irreducible representations up to unitary equivalence) of $\mathcal{A}$ to $\mathbb{C}$ with following property, for any minimal central projection z, $\lambda_A^{\tau}(z) = \tau(x)$, where $x \in A$ is a minimal projection with central support z.

The trace of a minimal projection only depends on its equivalence class, so the dimension vector is well defined. On the other hand, given a function from the set of minimal central projections of $\mathcal{A}$ to $\mathbb{C}$, we can construct a trace of $\mathcal{A}$, such that the corresponding dimension vector is the given function. So the map $\lambda \to \lambda_{\mathcal{A}}^{\tau}$ is a bijection.

Let us recall some facts about the inclusion of finite dimensional von Neumann algebras $\mathcal{B}_0 \subset \mathcal{B}_1$.

The Bratteli diagram Br for the inclusion $\mathcal{B}_0 \subset \mathcal{B}_1$ is a bipartite graph. Its even or odd vertices are indexed by the equivalence classes of irreducible representations of $\mathcal{B}_0$ or $\mathcal{B}_1$ respectively. The number of edges connects a vertex corresponding to an irreducible representation U of $\mathcal{B}_0$ to a vertex corresponding to an irreducible representation V of $\mathcal{B}_1$ is given by the multiplicity of U in the restriction of V on $\mathcal{B}_0$.

Let $Br_{\pm}$ be the even/odd vertices of Br. The Bratteli diagram can be interpreted as the adjacency matrix $\Lambda = \Lambda_{\mathcal{B}_0}^{\mathcal{B}_1} : L^2(Br_-) \to L^2(Br_+)$, where $\Lambda_{u,v}$ is defined as the number of edges connects u to v for any $u \in Br_+$, $v \in Br_-$.

Proposition 2.12. *(See [13].) For the inclusion $\mathcal{B}_0 \subset \mathcal{B}_1$ and a trace τ on it, we have $\lambda_{\mathcal{B}_0}^{\tau} = \Lambda \lambda_{\mathcal{B}_1}^{\tau}$.*

If the trace τ is a faithful state, then by GNS construction we will obtain a right $\mathcal{B}_1$ module $L^2(\mathcal{B}_1)$. And $L^2(\mathcal{B}_0)$ is identified as a subspace of $L^2(\mathcal{B}_1)$. Let e be the Jones projection on to the subspace $L^2(\mathcal{B}_0)$. Let $\mathcal{B}_2$ be the von Neumann algebra $(\mathcal{B}_1 \cup \{e\})''$. Then we obtain a tower $\mathcal{B}_0 \subset \mathcal{B}_1 \subset \mathcal{B}_2$ which is called the basic construction. Furthermore if the tracial state τ satisfies the condition $\Lambda^* \Lambda \lambda_{\mathcal{B}_1}^{\tau} = \mu \lambda_{\mathcal{B}_1}^{\tau}$ for some scalar μ, then it is said to be a Markov trace. In this case the scalar μ is $||\Lambda||^2$. Then $\lambda^{\tau} = \begin{bmatrix} \lambda_{\mathcal{B}_0}^{\tau} \\ \delta \lambda_{\mathcal{B}_1}^{\tau} \end{bmatrix}$ is a Perron–Frobenius eigenvector for $\begin{bmatrix} 0 & \Lambda \\ \Lambda^* & 0 \end{bmatrix}$.

Definition 2.13. We call λ^τ the Perron–Frobenius eigenvector with respect to the Markov trace τ.

Remark 2.14. The existence of a Markov trace for the inclusion $\mathcal{B}_0 \subset \mathcal{B}_1$ follows from the Perron–Frobenius theorem. The Markov trace is unique if and only if the Bratteli diagram for the inclusion $\mathcal{B}_0 \subset \mathcal{B}_1$ is connected.

We will see the importance of the Markov trace from the following proposition.

Proposition 2.15. *If τ is a Markov trace for the inclusion $\mathcal{B}_0 \subset \mathcal{B}_1$, then τ extends uniquely to a trace on $\mathcal{B}_2$, still denoted by τ. Moreover τ is a Markov trace for the inclusion $\mathcal{B}_1 \subset \mathcal{B}_2$.*

In this case, we may repeat the basic construction to obtain a sequence of finite dimensional von Neumann algebras $\mathcal{B}_0 \subset \mathcal{B}_1 \subset \mathcal{B}_2 \subset \mathcal{B}_3 \subset \cdots$ and a sequence of Jones projections $e_1, e_2, e_3 \cdots$.

2.4. Graph planar algebras

Given a finite connected bipartite graph Γ, it can be realised as the Bratteli diagram of the inclusion of finite dimensional von Neumann algebras $\mathcal{B}_0 \subset \mathcal{B}_1$ with a (unique) Markov trace. Applying the basic construction, we will obtain the sequence of finite dimensional von Neumann algebras $\mathcal{B}_0 \subset \mathcal{B}_1 \subset \mathcal{B}_2 \subset \mathcal{B}_3 \subset \cdots$. Take $\mathscr{S}_{m,+}$ to be $\mathcal{B}_0' \cap \mathcal{B}_m$ and $\mathscr{S}_{m,-}$ to be $\mathcal{B}_1' \cap \mathcal{B}_{m+1}$. Then $\{\mathscr{S}_{m,\pm}\}$ forms a planar algebra, called the *graph planar algebra* of the bipartite graph Γ. Moreover $\mathscr{S}_{m,\pm}$ has a natural basis given by length $2m$ loops of Γ. We refer the reader to [14,19] for more details. We cite the conventions used in Section 3.4 of [19].

Definition 2.16. Let us define $\mathscr{G} = \{\mathscr{G}_{m,\pm}\}$ to be the graph planar algebra of a finite connected bipartite graph Γ. Let λ be the Perron–Frobenius eigenvector with respect to the Markov trace.

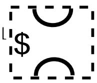
A vertex of the Γ corresponds to an equivalence class of minimal projections, so λ is also defined as a function from $\mathcal{V}_\pm$ to $\mathbb{R}^+$. If Γ is the principal graph of a subfactor, then its dimension vector is a multiple of the Perron–Frobenius eigenvector. In this paper, we only need the proportion of values of λ at vertices. We do not have to distinguish these two vectors.

Let $\mathcal{V}_\pm$ be the sets of even/odd vertices of Γ, and let $\mathcal{E}$ be the sets of all edges of Γ directed from even to odd vertices. Then we have the source and target functions $s : \mathcal{E} \to \mathcal{V}_+$ and $t : \mathcal{E} \to \mathcal{V}_-$. For a directed edge $\varepsilon \in \mathcal{E}$, we define ε^* to be the same edge with an opposite direction. The source function $s : \mathcal{E}^* = \{\varepsilon^* | \varepsilon \in \mathcal{E}\} \to \mathcal{V}_-$ and the target function $t : \mathcal{E}^* \to \mathcal{V}_+$ are defined as $s(\varepsilon^*) = t(\varepsilon)$ and $t(\varepsilon^*) = s(\varepsilon)$.

A length 2m loop in $\mathscr{G}_{m,+}$ is denoted by $[\varepsilon_1\varepsilon_2^* \cdots \varepsilon_{2m-1}\varepsilon_{2m}^*]$ satisfying

(i) $t(\varepsilon_k) = s(\varepsilon_{k+1}^*) = t(\varepsilon_{k+1})$, for all odd $k < 2m$;
(ii) $t(\varepsilon_k^*) = s(\varepsilon_k) = t(\varepsilon_{k+1})$, for all even $k < 2m$;
(iii) $t(\varepsilon_{2m}^*) = s(\varepsilon_{2m}) = t(\varepsilon_1)$.

The graph planar algebra is always unital. The unshaded empty diagram is given by $\sum_{v\in\mathcal{V}_+} v$; and the shaded empty diagram is given by $\sum_{v\in\mathcal{V}_-} v$. It is worth mentioning that the Jones projection is given by

$$e_1 = \delta^{-1}$$ 

$$= \delta^{-1} \sum_{s(\varepsilon_1)=s(\varepsilon_3)} \sqrt{\frac{\lambda(t(\varepsilon_1))\lambda(t(\varepsilon_3))}{\lambda(s(\varepsilon_1))\lambda(s(\varepsilon_3))}}[\varepsilon_1\varepsilon_1^*\varepsilon_3\varepsilon_3^*].$$

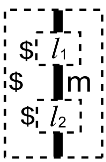
Now let us describe some elementary actions on $\mathscr{G}$.

The adjoint operation is defined as the anti-linear extension of

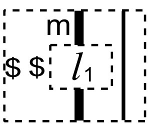
$$[\varepsilon_1\varepsilon_2^* \cdots \varepsilon_{2m-1}\varepsilon_{2m}^*]^* = [\varepsilon_{2m}\varepsilon_{2m-1}^* \cdots \varepsilon_2\varepsilon_1^*].$$

For $\mathscr{G}_{m,-}$, we have similar conventions.

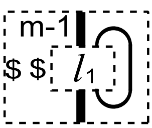
For $l_1, l_2 \in \mathscr{G}_{m,+}$, $l_1 = [\varepsilon_1\varepsilon_2^* \cdots \varepsilon_{2m-1}\varepsilon_{2m}^*]$, $l_2 = [\xi_1\xi_2^* \cdots \xi_{2m-1}\xi_{2m}]$, we have



$$= \begin{cases} \prod_{1\le k\le m} \delta_{\varepsilon_{m+k},\xi_{m+1-k}}[\varepsilon_1\varepsilon_2^* \cdots \varepsilon_m^*\xi_{m+1} \cdots \xi_{2m-1}\xi_{2m}^*] & \text{when } m \text{ is even;} \\ \prod_{1\le k\le m} \delta_{\varepsilon_{m+k},\xi_{m+1-k}}[\varepsilon_1\varepsilon_2^* \cdots \varepsilon_m\xi_{m+1}^* \cdots \xi_{2m-1}\xi_{2m}^*] & \text{when } m \text{ is odd.} \end{cases}$$



$$= \begin{cases} \sum_{s(\varepsilon)=s(\varepsilon_m)}[\varepsilon_1\varepsilon_2^* \cdots \varepsilon_m^*\varepsilon\varepsilon^*\varepsilon_{m+1} \cdots \varepsilon_{2m-1}\varepsilon_{2m}^*] & \text{when } m \text{ is even;} \\ \sum_{t(\varepsilon)=t(\varepsilon_m)}[\varepsilon_1\varepsilon_2^* \cdots \varepsilon_m\varepsilon^*\varepsilon\varepsilon_{m+1}^* \cdots \varepsilon_{2m-1}\varepsilon_{2m}^*] & \text{when } m \text{ is odd.} \end{cases}$$



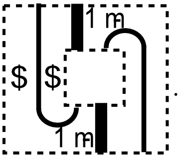
$$= \begin{cases} \delta_{\varepsilon_m,\varepsilon_{m+1}} \frac{\lambda(s(\varepsilon_m))}{\lambda(t(\varepsilon_m))}[\varepsilon_1\varepsilon_2^* \cdots \varepsilon_m^*\varepsilon\varepsilon^*\varepsilon_{m+1} \cdots \varepsilon_{2m-1}\varepsilon_{2m}^*] & \text{when } m \text{ is even;} \\ \delta_{\varepsilon_m,\varepsilon_{m+1}} \frac{\lambda(t(\varepsilon_m))}{\lambda(s(\varepsilon_m))}[\varepsilon_1\varepsilon_2^* \cdots \varepsilon_m\varepsilon^*\varepsilon\varepsilon_{m+1}^* \cdots \varepsilon_{2m-1}\varepsilon_{2m}^*] & \text{when } m \text{ is odd.} \end{cases}$$

Definition 2.17. The Fourier transform $\mathcal{F} : \mathscr{G}_{m,+} \to \mathscr{G}_{m,-}$, $m > 0$ is defined as the linear extension of

$$\mathcal{F}([\varepsilon_1\varepsilon_2^* \cdots \varepsilon_{2m-1}\varepsilon_{2m}^*]) = \begin{cases} \sqrt{\frac{\lambda(s(\varepsilon_{2m}))}{\lambda(t(\varepsilon_{2m}))}}\sqrt{\frac{\lambda(s(\varepsilon_m))}{\lambda(t(\varepsilon_m))}}[\varepsilon_{2m}^*\varepsilon_1\varepsilon_2^* \cdots \varepsilon_{2m-1}] & \text{for } m \text{ even;} \\ \sqrt{\frac{\lambda(s(\varepsilon_{2m}))}{\lambda(t(\varepsilon_{2m}))}}\sqrt{\frac{\lambda(t(\varepsilon_m))}{\lambda(s(\varepsilon_m))}}[\varepsilon_{2m}^*\varepsilon_1\varepsilon_2^* \cdots \varepsilon_{2m-1}] & \text{for } m \text{ odd.} \end{cases}$$

Similarly it is also defined from $\mathscr{G}_{m,-}$ to $\mathscr{G}_{m,+}$.

The Fourier transform has a diagrammatic interpretation as a one-click rotation



.

Definition 2.18. Let us define ρ to be $\mathcal{F}^2$. Then ρ is defined from $\mathscr{G}_{m,+}$ to $\mathscr{G}_{m,+}$ as a two-click rotation for $m > 0$,

$$\rho([\varepsilon_1\varepsilon_2^* \cdots \varepsilon_{2m-1}\varepsilon_{2m}^*]) = \sqrt{\frac{\lambda(s(\varepsilon_{2m}))}{\lambda(s(\varepsilon_{2m-1}))}}\sqrt{\frac{\lambda(s(\varepsilon_m))}{\lambda(s(\varepsilon_{m-1}))}}[\varepsilon_{2m-1}\varepsilon_{2m}^*\varepsilon_1\varepsilon_2^* \cdots \varepsilon_{2m-3}\varepsilon_{2m-2}^*].$$

It is similar for $\mathscr{G}_{m,-}$.

In general, the action of a planar tangle could be realised as a composition of actions mentioned above. It has a nice formula, see page 11 in [14].

2.5. The embedding theorem

For a depth $2r$ (or $2r+1$) subfactor planar algebra $\mathscr{S}$, we have

$$\mathscr{S}_{m+1,+} = \mathscr{S}_{m+1,+}e_m\mathscr{S}_{m+1,+} = \mathscr{S}_{m,+}e_{m+1}\mathscr{S}_{m,+}, \text{ whenever } m \geq 2r+1.$$

So $\mathscr{S}_{m-1,+} \subset \mathscr{S}_{m,+} \subset \mathscr{S}_{m+1,+}$ forms a basic construction. Note that the Bratteli diagram of $\mathscr{S}_{2r,+} \subset \mathscr{S}_{2r+1,+}$ is the principal graph. So the graph planar algebra $\mathscr{G}$ of the principal graph is given by

$$\mathscr{G}_{k,+} = \mathscr{S}'_{2r,+} \cap \mathscr{S}_{2r+k,+}; \quad \mathscr{G}_{k,-} = \mathscr{S}'_{2r+1,+} \cap \mathscr{S}_{2r+k+1,+}.$$

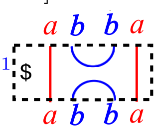
Moreover the map $\Phi : \mathscr{S} \to \mathscr{G}$ by adding $2r$ strings to the left preserves the planar algebra structure. It is not obvious that the left conditional expectation is preserved. We have the following embedding theorem, see Theorem 4.1 in [19].

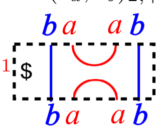
Theorem 2.19. *A finite depth subfactor planar algebra is naturally embedded into the graph planar algebra of its principal graph.*

Remark 2.20. The embedding theorem for general cases is proved in [29].

2.6. Fuss–Catalan

The Fuss–Catalan subfactor planar algebras were discovered by Bisch and Jones as *free products* of Temperley–Lieb subfactor planar algebras while studying the intermediate subfactors of a subfactor [6]. We refer the reader to [7,22] for the definition of the free product of subfactor planar algebras. It has a nice diagrammatic interpretation. For two Temperley–Lieb subfactor planar algebras $TL(\delta_a)$ and $TL(\delta_b)$, their free product $FC(\delta_a, \delta_b)$ is a subfactor planar algebra. A vector in $FC(\delta_a, \delta_b)_{m,+}$ can be expressed as a linear sum of Fuss–Catalan diagrams, a diagram consisting of disjoint a, b-colour strings whose boundary points are ordered as $\underbrace{abba\ abba \cdots abba}_{m}$, m copies of $abba$, after the dollar sign. It is similar for a vector in $FC(\delta_a, \delta_b)_{m,-}$, while the boundary points are ordered as $\underbrace{baab\ baab \cdots baab}_{m}$. For the action of a planar tangle on a simple tensor of Fuss–Catalan diagrams, first we replace each string of the planar tangle by a pair of parallel a-colour and b-colour strings which matches the a,b-colour boundary points, then the output is *gluing* the new tangle with the input diagrams. If there is an a or b-colour closed circle, then it contributes to a scalar δ_a or δ_b respectively.

The Fuss–Catalan subfactor planar algebra $FC(\delta_a, \delta_b)$ is naturally derived from an intermediate subfactor of a subfactor. Suppose $\mathcal{N} \subset \mathcal{M}$ is an irreducible subfactor with finite index, and $\mathcal{P}$ is an intermediate subfactor. Then there are two Jones projections $e_{\mathcal{N}}$ and $e_{\mathcal{P}}$ acting on $L^2(\mathcal{M})$, and we have the basic construction $\mathcal{N} \subset \mathcal{P} \subset \mathcal{M} \subset \mathcal{P}_1 \subset \mathcal{M}_1$. Repeating this process, we will obtain a sequence of factors $\mathcal{N} \subset \mathcal{P} \subset \mathcal{M} \subset \mathcal{P}_1 \subset \mathcal{M}_1 \subset \mathcal{P}_2 \subset \mathcal{M}_2 \cdots$ and a sequence of Jones projections $e_{\mathcal{N}}$, $e_{\mathcal{P}}$, $e_{\mathcal{M}}$, $e_{\mathcal{P}_1} \cdots$. The algebra generated by these Jones projections forms a planar algebra, denoted by $FC(\delta_a, \delta_b)$, where $\delta_a = \sqrt{[\mathcal{P} : \mathcal{N}]}$ and $\delta_b = \sqrt{[\mathcal{M} : \mathcal{P}]}$. Moreover $e_{\mathcal{P}} \in FC(\delta_a, \delta_b)_{2,+}$ and $e_{\mathcal{P}_1} \in FC(\delta_a, \delta_b)_{2,-}$ could be expressed as δ_b^{-1} 

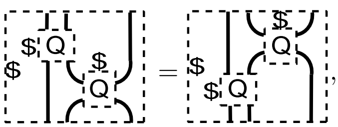
and δ_a^{-1} 

respectively. In particular, $\mathcal{F}(e_{\mathcal{P}})$ is a multiple of $e_{\mathcal{P}_1}$.

Definition 2.21. For an irreducible subfactor planar algebra $\mathscr{S}$, a projection $Q \in \mathscr{S}_{2,+}$ is called a biprojection, if $\mathcal{F}(Q)$ is a multiple of a projection.

If $\mathscr{S}$ is the planar algebra for $\mathcal{N} \subset \mathcal{M}$, then $e_{\mathcal{P}} \in \mathscr{S}_{2,+}$ is a biprojection. Conversely all the biprojections in $\mathscr{S}_{2,+}$ are realised in this way, see [6].

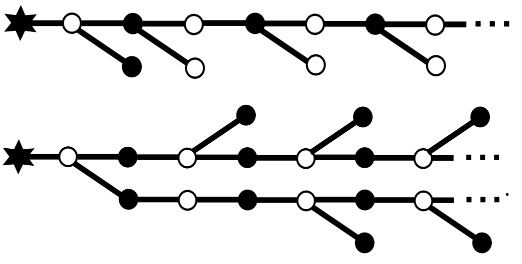
Proposition 2.22. *If we identify $\mathscr{S}_{2,-}$ as a subspace of $\mathscr{S}_{3,+}$ by adding a string to the left, then a biprojection $Q \in \mathscr{S}_{2,+}$ will satisfy $Q\mathcal{F}(Q) = \mathcal{F}(Q)Q$, i.e.*



called the exchange relation of a biprojection.

Conversely if a self-adjoint operator in $\mathscr{S}_{2,+}$ satisfies the exchange relation, then it is a multiple of a biprojection. This can be proved by adding caps at proper positions. We refer the reader to [23] for some other approaches to biprojections. The Fuss–Catalan subfactor planar algebra could also be viewed as the planar algebra generated by a biprojection with its exchange relation.

The planar algebra of a composed inclusion of an A_3 with an A_4 subfactor always contains $FC(\delta_a, \delta_b)$, where $\delta_a = \sqrt{2}$, $\delta_b = \frac{\sqrt{5}+1}{2}$, as a planar subalgebra. The principal graph and dual principal graph of $FC(\delta_a, \delta_b)$ are given as



3. The embedding theorem for an intermediate subfactor

A subfactor planar algebra is embedded in the graph planar algebra of its principal graph by the embedding theorem. If a subfactor planar algebra contains a biprojection, then we hope to know the image of the biprojection in the graph planar algebra. Recall that the image of the Jones projection e_1 is determined by the principal graph,

$$\delta e_1 = \sum_{s(\varepsilon_1)=s(\varepsilon_3)} \sqrt{\frac{\lambda(t(\varepsilon_1))}{\lambda(s(\varepsilon_1))}\frac{\lambda(t(\varepsilon_3))}{\lambda(s(\varepsilon_3))}}[\varepsilon_1\varepsilon_1^*\varepsilon_3\varepsilon_3^*].$$

We will see a similar formula for the image of the biprojection. It is determined by the *refined principal graph*. The refined principal graph is already considered by Bisch and Haagerup for bimodules, by Bisch and Jones for planar algebras. For the embedding theorem, we will use the one for planar algebras.

The lopsided version of embedding theorem for an intermediate subfactor is involved in a general embedding theorem proved by Morrison in [29]. To consider the algebraic structures, it is convenient to work with the spherical version of the embedding theorem. Their relations are described in [27]. For convenience, we prove the spherical version of embedding theorem, similar to the one proved by Jones and Penneys in [19].

In this section, we always assume $\mathcal{N} \subset \mathcal{M}$ is an irreducible subfactor of type II_1 with finite index, and $\mathcal{P}$ is an intermediate subfactor. If the subfactor has an intermediate subfactor, then its planar algebra becomes an $\mathcal{N}-\mathcal{P}-\mathcal{M}$ planar algebra. For $\mathcal{N}-\mathcal{P}-\mathcal{M}$ planar algebras, we refer the reader to Chapter 4 in [9]. In this case, the subfactor planar algebra contains a biprojection P, and a planar tangle labelled by P can be replaced by a *Fuss–Catalan planar tangle*. In this paper, we will use planar tangles labelled by P, instead of Fuss–Catalan planar tangles.

3.1. Principal graphs

For the embedding theorem, we will consider the principal graph of $\mathcal{N} \subset \mathcal{P} \subset \mathcal{M}$. It refines the principal graph of $\mathcal{N} \subset \mathcal{M}$. Instead of a bipartite graph, it will be an $(\mathcal{N}, \mathcal{P}, \mathcal{M})$ coloured graph. The following definitions and propositions are well known to experts [6,7,22,9].

Definition 3.1. An $(\mathcal{N}, \mathcal{P}, \mathcal{M})$ coloured graph Γ is a locally finite graph, such that the set $\mathcal{V}$ of its vertices is divided into three disjoint subsets $\mathcal{V}_\mathcal{N}$, $\mathcal{V}_\mathcal{P}$ and $\mathcal{V}_\mathcal{M}$, and the set $\mathcal{E}$ of its edges is divided into two disjoint subsets $\mathcal{E}_+$, $\mathcal{E}_-$. Moreover every edge in $\mathcal{E}_+$ connects a vertex in $\mathcal{V}_\mathcal{N}$ to one in $\mathcal{V}_\mathcal{P}$ and every edge in $\mathcal{E}_-$ connects a vertex in $\mathcal{V}_\mathcal{P}$ to one in $\mathcal{V}_\mathcal{M}$. Then we define the source function $s : \mathcal{E} \to \mathcal{V}_\mathcal{N} \cup \mathcal{V}_\mathcal{M}$ and the target function $t : \mathcal{E} \to \mathcal{V}_\mathcal{P}$ in the obvious way. The operation $*$ reverses the direction of an edge.

Definition 3.2. From an $(\mathcal{N}, \mathcal{P}, \mathcal{M})$ coloured graph Γ, we will obtain an $(\mathcal{N}, \mathcal{M})$ coloured bipartite graph Γ' as follows, the $\mathcal{N}/\mathcal{M}$ coloured vertices of Γ' are identical to the $\mathcal{N}/\mathcal{M}$ coloured vertices of Γ; for two vertices v_n in $\mathcal{V}_\mathcal{N}$ and $v_m \in \mathcal{V}_\mathcal{M}$, the number of edges between v_n and v_m in Γ is given by the number of length two paths from v_n to v_m in Γ'. The graph Γ' is said to be the bipartite graph induced from the graph Γ. The graph Γ is said to be a refinement of the graph Γ'.

Remark 3.3. Here we abuse the notations of Γ and Γ' which are usually reserved for the principal graph and the dual principal graph respectively.

For a factor $\mathcal{M}$ of type II_1, if $\mathcal{N} \subset \mathcal{P} \subset \mathcal{M}$ is a sequence of irreducible subfactors with finite index, then $L^2(\mathcal{P})$ forms an irreducible $(\mathcal{N}, \mathcal{P})$ bimodule, denoted by X, and $L^2(\mathcal{M})$ forms an irreducible $(\mathcal{P}, \mathcal{M})$ bimodule, denoted by Y. Their conjugates $\overline{X}$, $\overline{Y}$ are $(\mathcal{P}, \mathcal{N})$, $(\mathcal{M}, \mathcal{P})$ bimodules respectively. The tensor products $X \otimes Y \otimes \overline{Y} \otimes \overline{X} \otimes \cdots \otimes \overline{X}$, $X \otimes Y \otimes \overline{Y} \otimes \overline{X} \otimes \cdots \otimes X$, $X \otimes Y \otimes \overline{Y} \otimes \overline{X} \otimes \cdots \otimes Y$, $X \otimes Y \otimes \overline{Y} \otimes \overline{X} \otimes \cdots \otimes \overline{Y}$, are decomposed into irreducible bimodules over $(\mathcal{N}, \mathcal{N})$, $(\mathcal{N}, \mathcal{P})$, $(\mathcal{N}, \mathcal{M})$ and $(\mathcal{N}, \mathcal{P})$ respectively.

Definition 3.4. The principal graph for the inclusion of factors $\mathcal{N} \subset \mathcal{P} \subset \mathcal{M}$ is an $(\mathcal{N}, \mathcal{P}, \mathcal{M})$ coloured graph. Its vertices are equivalence classes of irreducible bimodules over $(\mathcal{N}, \mathcal{N})$, $(\mathcal{N}, \mathcal{P})$ and $(\mathcal{N}, \mathcal{M})$ in the above decomposed inclusion. The number of edges connecting two vertices, an $(\mathcal{N}, \mathcal{N})$ (or $(\mathcal{N}, \mathcal{M})$) bimodule U (or V) and an $(\mathcal{N}, \mathcal{P})$ bimodule W, is the multiplicity of the equivalence class of U (or V) as a sub bimodule of $W \otimes \overline{X}$ (or $W \otimes Y$). The vertex corresponding to the irreducible $(\mathcal{N}, \mathcal{N})$ bimodule $L^2(\mathcal{N})$ is marked by a star sign $*$. The dimension vector of the principal graph is a function λ from the vertices of the graph to $\mathbb{R}^+$. Its value at a point is defined to be the dimension of the corresponding bimodule.

Similarly the dual principal graph for the inclusion of factors is defined by considering the decomposed inclusion of $(\mathcal{M},\mathcal{M})$, $(\mathcal{M},\mathcal{P})$, $(\mathcal{M},\mathcal{N})$ bimodules.

There is another principal graph given by decomposing $(\mathcal{P},\mathcal{N})$, $(\mathcal{P},\mathcal{P})$ and $(\mathcal{P},\mathcal{M})$ bimodules under inclusions, but it is not needed in this paper.

Proposition 3.5. *The (dual) principal graph for the inclusion of factors $\mathcal{N}\subset\mathcal{P}\subset\mathcal{M}$ is a refinement of the (dual) principal graph of the subfactor $\mathcal{N}\subset\mathcal{M}$.*

Proof. If follows from the definition and the fact that $X\otimes Y$ is the $(\mathcal{N},\mathcal{M})$ bimodule $L^2(\mathcal{M})$. □

Let δ_a be $\sqrt{[\mathcal{P}:\mathcal{N}]}$, the dimension of X, and δ_b be $\sqrt{[\mathcal{M}:\mathcal{P}]}$, the dimension of Y. Then by Frobenius reciprocity theorem, we have the following proposition.

Proposition 3.6. *For the principal graph of factors $\mathcal{N}\subset\mathcal{P}\subset\mathcal{M}$ and the dimension vector λ, we have*

$$\delta_a\lambda(u)=\sum_{\varepsilon\in\mathcal{E}_+,s(\varepsilon)=u}\lambda(t(\varepsilon)),\ \forall u\in\mathcal{V}_{\mathcal{N}};\qquad \delta_b\lambda(w)=\sum_{\varepsilon\in\mathcal{E}_-,s(\varepsilon)=w}\lambda(t(\varepsilon)),\ \forall w\in\mathcal{V}_{\mathcal{M}};$$

$$\delta_a\lambda(v)=\sum_{\varepsilon\in\mathcal{E}_+,t(\varepsilon)=v}\lambda(s(\varepsilon)),\ \forall v\in\mathcal{V}_{\mathcal{P}};\qquad \delta_b\lambda(v)=\sum_{\varepsilon\in\mathcal{E}_-,t(\varepsilon)=v}\lambda(s(\varepsilon)),\ \forall v\in\mathcal{V}_{\mathcal{P}}.$$

Definition 3.7. For an $(\mathcal{N},\mathcal{P},\mathcal{M})$ coloured graph Γ, if there exits a function $\lambda:\mathcal{V}\to\mathbb{R}^+$ satisfying the proposition mentioned above, then we call it a graph with parameter (δ_a,δ_b).

Proposition 3.8. *The principal graph of factors $\mathcal{N}\subset\mathcal{P}\subset\mathcal{M}$ is a graph with parameter $(\sqrt{[\mathcal{P}:\mathcal{N}]},\sqrt{[\mathcal{M}:\mathcal{P}]})$. Consequently if $\mathcal{N}\subset\mathcal{M}$ has finite depth, then the principal graph of $\mathcal{N}\subset\mathcal{P}\subset\mathcal{M}$ is finite.*

Proof. The first statement follows from the definition. Note that the dimension of a bimodule is at least 1. By this restriction, $\mathcal{N}\subset\mathcal{M}$ has finite depth implies the principal graph of $\mathcal{N}\subset\mathcal{P}\subset\mathcal{M}$ is finite. □

3.2. Standard invariants

We will define the refined (dual) principal graph for a subfactor planar algebra with a biprojection. This definition coincides with the definition given by bimodules, but we do not need this fact in this paper. Given $\mathcal{N}\subset\mathcal{P}\subset\mathcal{M}$, there are two Jones projections $e_{\mathcal{N}}$ and $e_{\mathcal{P}}$ acting on $L^2(\mathcal{M})$. Then we have the basic construction $\mathcal{N}\subset\mathcal{P}\subset\mathcal{M}\subset\mathcal{P}_1\subset\mathcal{M}_1$. Repeating this process, we will obtain a sequence of factors $\mathcal{N}\subset\mathcal{P}\subset\mathcal{M}\subset\mathcal{P}_1\subset\mathcal{M}_1\subset$

$\mathcal{P}_2 \subset \mathcal{M}_2 \cdots$ and a sequence of Jones projections $e_{\mathcal{N}}, e_{\mathcal{P}}, e_{\mathcal{M}}, e_{\mathcal{P}_1} \cdots$. Then the standard invariant is refined as

$$\begin{array}{ccccccccc}
\mathbb{C} = \mathcal{N}' \cap \mathcal{N} & \subset & \mathcal{N}' \cap \mathcal{P} & \subset & \mathcal{N}' \cap \mathcal{M} & \subset & \mathcal{N}' \cap \mathcal{P}_1 & \subset & \mathcal{N}' \cap \mathcal{M}_1 & \subset & \cdots \\
& & \cup & & \cup & & \cup & & & & \\
& & \mathbb{C} = \mathcal{P}' \cap \mathcal{P} & \subset & \mathcal{P}' \cap \mathcal{M} & \subset & \mathcal{P}' \cap \mathcal{P}_1 & \subset & \mathcal{P}' \cap \mathcal{M}_1 & \subset & \cdots \\
& & & & \cup & & \cup & & \cup & & \\
& & & & \mathbb{C} = \mathcal{M}' \cap \mathcal{M} & \subset & \mathcal{M}' \cap \mathcal{P}_1 & \subset & \mathcal{M}' \cap \mathcal{M}_1 & \subset & \cdots
\end{array}$$

For Fuss–Catalan, the corresponding Bratteli diagram is described by the *middle patterns*, see pages 114–115 in [6].

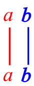
We hope to define the refined principal graph as the limit of the Bratteli diagram Br_k of $\mathcal{N}' \cap \mathcal{M}_{k-2} \subset \mathcal{N}' \cap \mathcal{P}_{k-1} \subset \mathcal{N}' \cap \mathcal{M}_{k-1}$. To show the limit is well defined, we need to prove that Br_k is identified as a subgraph of Br_{k+1}. To define it for a subfactor planar algebra with a biprojection without the presumed factors, we need to do some translations motivated by the fact

$$\mathcal{N}' \cap \mathcal{P}_k = \mathcal{N}' \cap (\mathcal{M}_k \cap \{e_{\mathcal{P}_k}\}') = (\mathcal{N}' \cap \mathcal{M}_k) \cap \{e_{\mathcal{P}_k}\}'.$$

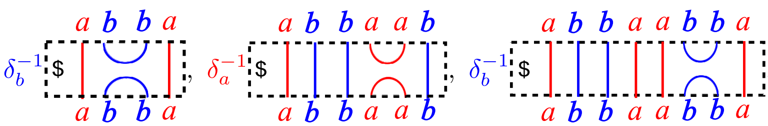
Definition 3.9. Let $\mathscr{S} = \mathscr{S}_{m,\pm}$ be a subfactor planar algebra. Let $e_1, e_2, \cdots$ be the sequence of Jones projections.

Suppose p_1 is a biprojection in $\mathscr{S}_{2,+}$. Then we obtain another sequence of Jones projections $p_1, p_2, p_3, \cdots$, corresponding to the intermediate subfactors, precisely p_2 in $\mathscr{S}_{2,-} \subset \mathscr{S}_{3,+}$ is a multiple of $\mathcal{F}(p_1)$, and p_k is obtained by adding two strings on the left side of p_{k-2}.

For $m \geq 1$, let us define $\mathscr{S}'_{m,+}$ to be $\mathscr{S}_{m,+} \cap \{p_m\}'$ and $\mathscr{S}'_{m,-}$ to be $\mathscr{S}_{m,-} \cap \{p_{m+1}\}'$.

Remark 3.10. If we interpret one string of a planar diagram as a pair of a/b-colour strings 

, then sequence of Jones projections $p_1, p_2, p_3, \cdots$, can be interpreted as following a/b-colour diagrams [6],



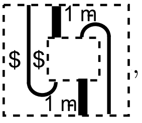
$\cdots$.

The following proposition shows that the subspace $\mathscr{S}'_{m,\pm}$ of $\mathscr{S}_{m,\pm}$ consists of diagrams with an a/b-colour through string on the rightmost.

Proposition 3.11. *For $X \in \mathscr{S}_{m,+}$, $m \geq 1$, we have*

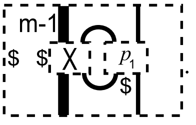
$$Xp_m = p_m X \iff \mathcal{F}(X) = \mathcal{F}(X)p_m.$$

That means $\mathscr{S}'_{m,+}$ is the invariant subspace of $\mathscr{S}_{m,+}$ under the "right action" of the biprojection. Diagrammatically its consists of vectors with one a/b-colour through string on the rightmost.

Proof. If $p_mX = Xp_m$, then take the action given by the planar tangle 

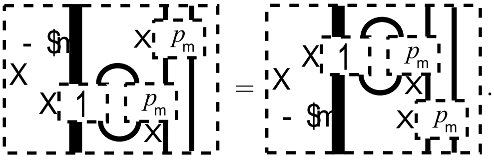
, we have $\mathcal{F}(X) = \mathcal{F}(X)p_m$.

For m odd, if $\mathcal{F}(X) = \mathcal{F}(X)p_m$, then $X = X * \mathcal{F}(p_1)$, i.e.

$X =$ 

.

By the exchange relation of the biprojection, we have



.

So $p_mX = Xp_m$.

For m even, the proof is similar. □

Note that $\mathscr{S}_{m-1,+}$ is in the commutant of $p_m{}'$. So we have the inclusion of finite dimensional von Neumann algebras

$$\mathscr{S}_{0,+} \subset \mathscr{S}'_{1,+} \subset \mathscr{S}_{1,+} \subset \mathscr{S}'_{2,+} \subset \mathscr{S}_{2,+} \subset \cdots.$$

Then we obtain the Bratteli diagram Br_m for the inclusion $\mathscr{S}_{m-1,+} \subset \mathscr{S}'_{m,+} \subset \mathscr{S}_{m,+}$. To take the limit of Br_m, we need to prove that Br_m is identified as a subgraph of Br_{m+1}.

Proposition 3.12. *If P_1, P_2 are minimal projections of $\mathscr{S}'_{m,+}$. Then P_1p_m, P_2p_m are minimal projections of $\mathscr{S}'_{m+1,+}$. Moreover P_1 and P_2 are equivalent in $\mathscr{S}'_{m,+}$ if and only if P_1p_m and P_2p_m are equivalent in $\mathscr{S}'_{m+1,+}$.*

This proposition is proved similar to Proposition 2.6.

Proposition 3.13 *(Frobenius reciprocity).*

(1) *For a minimal projection $P \in \mathscr{S}_{m-1,+}$ and a minimal projection $Q \in \mathscr{S}'_{m,+}$, we have that Qp_m is a minimal projection of $\mathscr{S}'_{m+1,+}$, Pe_m is a minimal projection*

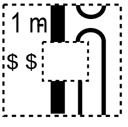
of $\mathscr{S}_{m+1,+}$, *and*

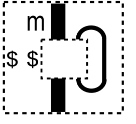
$$\dim(P(\mathscr{S}'_{m,+})Q) = \dim(Pe_m(\mathscr{S}_{m+1,+})Qp_m).$$

(2) *For a minimal projection* $P' \in \mathscr{S}'_{m,+}$ *and a minimal projection* $Q' \in \mathscr{S}_{m,+}$, *we have* $P'p_m$ *is a minimal projection of* $\mathscr{S}'_{m+1,+}$, *and*

$$\dim(P'(\mathscr{S}_{m,+})Q') = \dim(P'p_m(\mathscr{S}'_{m+1,+})Q').$$

Proof. (1) Consider the maps

$\phi_1 =$ 

$: \mathscr{S}_{m,+} \to \mathscr{S}_{m+1,+}$, $\phi_2 =$ 

$: \mathscr{S}_{m+1,+} \to \mathscr{S}_{m,+}$.

For m odd, if $X \in P(\mathscr{S}'_{m,+})Q$, then by Proposition 3.11, we have $X = P(X' * \mathcal{F}(p_1))Q$ for some $X' \in \mathscr{S}_{m,+}$. So $\phi_1(X) \in Pe_m(\mathscr{S}_{m+1,+})Qp_m$. On the other hand, if $Y \in Pe_m(\mathscr{S}_{m+1,+})Qp_m$, then $\phi_2(Y) \in P(\mathscr{S}'_{m,+})Q$. While $\phi_1 \circ \phi_2$ is the identity map on $Pe_m(\mathscr{S}_{m+1,+})Qp_m$ and $\phi_2 \circ \phi_1$ is the identity map on $P(\mathscr{S}'_{m,+})Q$. So $\dim(P'(\mathscr{S}_{m,+})Q') = \dim(P'p_m(\mathscr{S}'_{m+1,+})Q')$.

For m even, the proof is similar.

(2) This is the same as Proposition 2.7. □

By Proposition (2.6)(3.13), the Bratteli diagram Br_m is identified as a subgraph of Br_{m+1}.

Definition 3.14. Let us define the refined principal graph of $\mathscr{S}$ with respect to the biprojection p_1 to be the limit of the Bratteli diagram of $\mathscr{S}_{m,+} \subset \mathscr{S}'_{m+1,+} \subset \mathscr{S}_{m+1,+}$. The vertex corresponding to the identity in $\mathscr{S}_{0,+}$ is marked by a star sign.

Similarly let us define the refined dual principal graph of $\mathscr{S}$ with respect to the biprojection p_1 to be the limit of the Bratteli diagram of $\mathscr{S}_{m,-} \subset \mathscr{S}'_{m+1,-} \subset \mathscr{S}_{m+1,-}$. The vertex corresponding to the identity in $\mathscr{S}_{0,-}$ is marked by a star sign.

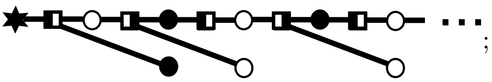
The refined principal graph is an $(\mathcal{N}, \mathcal{P}, \mathcal{M})$ coloured graph. The $\mathcal{N}$, $\mathcal{P}$, $\mathcal{M}$ coloured vertices are given by equivalence classes of minimal projections of $\mathscr{S}_{2m,-}$, $\mathscr{S}'_{2m+1,-}$, $\mathscr{S}_{2m+1,-}$ respectively as m approaches infinity. Similarly the refined dual principal graph is an $(\mathcal{M}, \mathcal{P}, \mathcal{N})$ coloured graph.

Definition 3.15. The dimension vector λ of the principal graph is defined as follows, for an $\mathcal{N}$ or $\mathcal{M}$ coloured vertex, its value is the Markov trace of the minimal projection corresponding to that vertex; for a $\mathcal{P}$ coloured vertex v, suppose $Q \in \mathscr{S}'_{m,+}$ is a minimal projection corresponding to v. Then $\lambda(v) = \delta_a^{-1} tr(Q)$, when m is even, where $\delta_a = \sqrt{tr(p_1)}$; $\lambda(v) = \delta_b^{-1} tr(Q)$, when m is odd, where $\delta_b = \delta\delta_a^{-1}$.

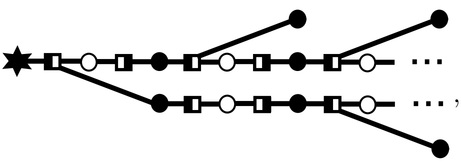
Remark 3.16. An element in $\mathscr{S}'_{m,+}$ has an a/b-colour through string on the rightmost. When we compute the dimension vector for a minimal projection in $\mathscr{S}'_{m,+}$, that string should be omitted. So there is a factor δ_a^{-1} or δ_b^{-1}.

Note that the dimension vector satisfies Proposition 3.6. So the refined principal graph is a graph with parameter (δ_a, δ_b). If the Bratteli diagram of $\mathscr{S}_{m,+} \subset \mathscr{S}_{m+1,+}$ is the same as that of $\mathscr{S}_{m+1,+} \subset \mathscr{S}_{m+2,+}$, i.e. $\mathscr{S}$ has finite depth, then $Br_{m+1} = Br_{m+2}$ by the restriction of the dimension vector. In particular, the Bratteli diagram of $\mathscr{S}'_{m+1,+} \subset \mathscr{S}_{m+1,+}$ is the same as that of $\mathscr{S}_{m+1,+} \subset \mathscr{S}'_{m+2,+}$. So $\mathscr{S}'_{m+1} \subset \mathscr{S}_{m+1,+} \subset \mathscr{S}'_{m+2,+}$ forms a basic construction, and p_{m+1} is the Jones projection. While applying the embedding theorem, the image of the Jones projection can be expressed as a linear sum of loops. We will see the formula later.

The subfactor planar algebra $FC(\sqrt{2}, \frac{1+\sqrt{5}}{2})$ contains a trace-2 biprojection. Considering the middle pattern of its minimal projections, see pages 114–115 in [6], we have its refined principal graph, as



and its refined dual principal graph as



where the black, mixed, white points are $\mathcal{N}, \mathcal{P}, \mathcal{M}$ coloured vertices. We will discuss more about these graphs in Section 4.1.

3.3. Finite dimensional inclusions

Now given an inclusion of finite dimensional von Neumann algebras $\mathcal{B}_0 \subset \mathcal{B}_1 \subset \mathcal{B}_2$, similarly we may consider its Bratteli diagram, adjacency matrixes, Markov trace if there exists one, and the basic construction.

Definition 3.17. The Bratteli diagram Br for the inclusion $\mathcal{B}_0 \subset \mathcal{B}_1 \subset \mathcal{B}_2$ is a $(\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2)$ coloured graph. Its $\mathcal{B}_i$ coloured vertices are indexed by the minimal central projections (or equivalently the irreducible representations) of $\mathcal{B}_i$, for $i = 0, 1, 2$. The subgraph of Br consisting of $\mathcal{B}_0$, $\mathcal{B}_1$ coloured vertices and the edges connecting them is the same as the Bratteli diagram for the inclusion $\mathcal{B}_0 \subset \mathcal{B}_1$. The subgraph of Br consisting of $\mathcal{B}_1$, $\mathcal{B}_2$ coloured vertices and the edges connecting them is the same as the Bratteli diagram for the inclusion $\mathcal{B}_1 \subset \mathcal{B}_2$.

Let Λ, Λ_1 and Λ_2 be the adjacency matrixes of $\mathcal{B}_0 \subset \mathcal{B}_2$, $\mathcal{B}_0 \subset \mathcal{B}_1$ and $\mathcal{B}_1 \subset \mathcal{B}_2$ respectively. Then $\Lambda = \Lambda_1 \Lambda_2$. Take a faithful tracial state τ on $\mathcal{B}_2$. Let $L^2(\mathcal{B}_2)$ be the Hilbert space given by the GNS construction with respect to τ. Then $L^2(\mathcal{B}_0)$ and $L^2(\mathcal{B}_1)$ are naturally identified as subspaces of $L^2(\mathcal{B}_2)$. Let e_1, p_1 be the Jones projections onto the subspaces $L^2(\mathcal{B}_0)$, $L^2(\mathcal{B}_1)$ respectively. Then $\mathcal{B}_3 = (\mathcal{B}_2 \cup p_1)''$, $\mathcal{B}_4 = (\mathcal{B}_2 \cup e_1)''$ are obtained by the basic construction. So $Z(\mathcal{B}_0) = Z(\mathcal{B}_4)$, $Z(\mathcal{B}_1) = Z(\mathcal{B}_3)$. And the adjacency matrixes of $\mathcal{B}_2 \subset \mathcal{B}_3$, $\mathcal{B}_2 \subset \mathcal{B}_4$ are Λ_2^T, Λ^T.

Proposition 3.18. *The adjacency matrix of $\mathcal{B}_3 \subset \mathcal{B}_4$ is Λ_1^T.*

Proof. We assume that the adjacency matrix of $\mathcal{B}_3 \subset \mathcal{B}_4$ is $\tilde{\Lambda}$. Let J denote the modular conjugation operator on $L^2(\mathcal{B}_2)$. Then $z \to Jz^*J$ is a $*$-isomorphism of $Z(\mathcal{B}_0)$ onto $Z(\mathcal{B}_4)$, of $Z(\mathcal{B}_1)$ onto $Z(\mathcal{B}_3)$. Take a minimal central projection x of $\mathcal{B}_0$ and a minimal central projection y of $\mathcal{B}_1$, we have $\tilde{x} = JxJ$ is a minimal central projection of $\mathcal{B}_4$, and $\tilde{y} = JyJ$ is a minimal central projection of $\mathcal{B}_3$. The definition of the adjacency matrix implies that

$$\Lambda_{y,x} = [\dim(xy\mathcal{B}_0'xy \cap xy\mathcal{B}_1 xy)]^{\frac{1}{2}};$$
$$\tilde{\Lambda}_{\tilde{x},\tilde{y}} = [\dim(\tilde{x}\tilde{y}\mathcal{B}_3'\tilde{x}\tilde{y} \cap \tilde{x}\tilde{y}\mathcal{B}_4\tilde{x}\tilde{y})]^{\frac{1}{2}}.$$

Note that

$$\tilde{x}\tilde{y}\mathcal{B}_3'\tilde{x}\tilde{y} \cap \tilde{x}\tilde{y}\mathcal{B}_4\tilde{x}\tilde{y} = JxyJ\mathcal{B}_3'JxyJ \cap JxyJ\mathcal{B}_4JxyJ = J(xy\mathcal{B}_0'xy \cap xy\mathcal{B}_1xy)J.$$

So $\tilde{\Lambda}_{\tilde{x},\tilde{y}} = \Lambda_{y,x} = \Lambda^T_{x,y}$. □

Definition 3.19. We say τ is a Markov trace for the inclusion $\mathcal{B}_0 \subset \mathcal{B}_1 \subset \mathcal{B}_2$, if τ is a Markov trace for the inclusions $\mathcal{B}_0 \subset \mathcal{B}_1$ and $\mathcal{B}_1 \subset \mathcal{B}_2$.

Proposition 3.20. *If τ is a Markov trace for the inclusion $\mathcal{B}_0 \subset \mathcal{B}_1 \subset \mathcal{B}_2$, then τ is a Markov trace for the inclusion $\mathcal{B}_0 \subset \mathcal{B}_2$. Moreover τ extends uniquely to a Markov trace for the inclusion $\mathcal{B}_2 \subset \mathcal{B}_3 \subset \mathcal{B}_4$.*

Proof. Let $\lambda_i = \lambda^{\tau}_{\mathcal{B}_i}$ be the dimension vectors for $i = 0, 1, 2$. If τ is a Markov trace for the inclusion $\mathcal{B}_0 \subset \mathcal{B}_1 \subset \mathcal{B}_2$, then by the definition τ is a Markov trace for the inclusions $\mathcal{B}_0 \subset \mathcal{B}_1$ and $\mathcal{B}_1 \subset \mathcal{B}_2$. So $\Lambda_2\lambda_2 = \lambda_1$; $\Lambda_1\lambda_1 = \lambda_0$; $\Lambda_1^T\lambda_0 = ||\Lambda_1||^2\lambda_1$; and $\Lambda_2^T\lambda_1 = ||\Lambda_2||^2\lambda_2$. Then $\Lambda^T\Lambda\lambda_2 = \Lambda_2^T\Lambda_1^T\Lambda_1\Lambda_2\lambda_2 = ||\Lambda_1||^2||\Lambda_2||^2\lambda^2$. So τ is a Markov trace for the inclusion $\mathcal{B}_0 \subset \mathcal{B}_2$ and $||\Lambda|| = ||\Lambda_1|| \cdot ||\Lambda_2||$. Then τ extends uniquely to a Markov trace for the inclusion $\mathcal{B}_2 \subset \mathcal{B}_4$. Let $\lambda_i = \lambda^{\tau}_{\mathcal{B}_i}$ be the dimension vectors for $i = 3, 4$. We have $\lambda_4 = ||\Lambda||^{-2}\lambda_0$ by the uniqueness of the extension of τ. And $\lambda_3 = \Lambda_1^T\lambda_4 = ||\Lambda||^{-2}\Lambda_1^T\lambda_0 = ||\Lambda_2||^{-2}\lambda_1$. Then $\Lambda_1\Lambda_1^T\lambda_4 = ||\Lambda_1||^2\lambda_4$ and $\Lambda_2\Lambda_2^T\lambda_3 = ||\Lambda_2||^2\lambda_3$ by a direct computation. That means τ extends to a Markov trace for the inclusion $\mathcal{B}_2 \subset \mathcal{B}_3 \subset \mathcal{B}_4$.

On the other hand, if τ extends to a Markov trace for the inclusion $\mathcal{B}_2 \subset \mathcal{B}_3 \subset \mathcal{B}_4$, then it also extends to a Markov trace for the inclusion $\mathcal{B}_0 \subset \mathcal{B}_2$. That implies the uniqueness of such an extension. □

Definition 3.21. Given the Bratteli diagram *Br* for the inclusion $\mathcal{B}_0 \subset \mathcal{B}_1 \subset \mathcal{B}_2$, let us define the dimension vector with respect to the Markov trace τ to be λ^τ, a function from the vertices of the Bratteli diagram the into $\mathbb{R}^+$, as follows for a $\mathcal{B}_0$ coloured vertex, its value is the trace of the minimal projection corresponding to that vertex; for a $\mathcal{B}_1$ coloured vertex, its value is $||\Lambda_1||$ times the trace of the minimal projection corresponding to that vertex; for a $\mathcal{B}_1$ coloured vertex, its value is $||\Lambda||$ times the trace of the minimal projection corresponding to that vertex.

Proposition 3.22. *The inclusion $\mathcal{B}_0 \subset \mathcal{B}_1 \subset \mathcal{B}_2$ admits a Markov trace if and only if the Bratteli diagram for the inclusion is a graph with parameter (δ_a, δ_b). In this case $\delta_a = ||\Lambda_1||$ and $\delta_b = ||\Lambda_2||$. Under this condition, the Markov trace is unique if and only if the Bratteli diagram is connected.*

Proof. The first statement follows from the definitions.

In this case, $\delta_a = ||\Lambda_1||$ and $\delta_b = ||\Lambda_2||$ follow from the fact that the eigenvalue of $\Lambda_i^T \Lambda_i$ with a positive eigenvector has to be $||\Lambda_i||^2$.

Suppose the inclusion $\mathcal{B}_0 \subset \mathcal{B}_1 \subset \mathcal{B}_2$ admits a Markov trace. If the Bratteli diagram *Br* is not connected, then we may adjust the proportion to obtain different Markov traces. If the Bratteli diagram *Br* for the inclusion $\mathcal{B}_0 \subset \mathcal{B}_1 \subset \mathcal{B}_2$ is connected, we want to show that the Bratteli diagram Br' for the inclusion $\mathcal{B}_0 \subset \mathcal{B}_2$ is connected. Actually if two $\mathcal{B}_0$ (or $\mathcal{B}_2$) coloured vertices are adjacent to the same $\mathcal{B}_1$ coloured vertex in *Br*, then they are adjacent to the same $\mathcal{B}_2$ (or $\mathcal{B}_0$) coloured vertex in Br', because any $\mathcal{B}_1$ coloured point is adjacent to a $\mathcal{B}_2$ (or $\mathcal{B}_0$) coloured vertex in *Br*. While the Bratteli diagram Br' is connected implies the uniqueness of the Markov trace for the inclusion $\mathcal{B}_0 \subset \mathcal{B}_2$. Then the dimension vectors λ_0 and λ_2 are unique. So λ_1 is also unique. That means the Markov trace for the inclusion $\mathcal{B}_0 \subset \mathcal{B}_1 \subset \mathcal{B}_2$ is unique. □

Corollary 3.23. *Given the principal graph for the inclusion $\mathcal{N} \subset \mathcal{P} \subset \mathcal{M}$, its dimension vector is uniquely determined by the graph.*

Proof. The dimension vector is a multiple of the dimension vector λ^τ with respect to the unique Markov trace τ. While the value of the marked point is 1, so the dimension vector is unique. □

Now we can repeat the basic construction to obtain the Jones tower $\mathcal{B}_0 \subset \mathcal{B}_1 \subset \mathcal{B}_2 \subset \mathcal{B}_3 \subset \mathcal{B}_4 \subset \cdots$ and a sequence of Jones projections $e_1, p_1, e_2, p_2 \cdots$.

Proposition 3.24. *The algebra generated by the sequences of projections $\{e_i\}$ and $\{p_j\}$ forms a Fuss–Catalan subfactor planar algebra.*

This proposition is essentially the same as Proposition 5.1 in [6]. In that case the Jones projections are derived from the inclusion of factors. The proof is similar. We only need the fact that the trace preserving conditional expectation induced by a Markov trace maps the Jones projections to a multiple of the identity.

3.4. Graph planar algebras and the embedding theorem

Given a connected $(\mathcal{N}, \mathcal{P}, \mathcal{M})$ coloured graph Γ with parameter (δ_a, δ_b), we have $\mathcal{V}_N$, $\mathcal{V}_P$, $\mathcal{V}_M$, $\mathcal{E}_\pm$, s, t, $*$ as in Definition 3.1. Let λ be the (unique) dimension vector. Let Γ' be the bipartite graph induced from Γ. Suppose the Bratteli diagram for the inclusion of finite dimensional von Neumann algebras $\mathcal{B}_0 \subset \mathcal{B}_1 \subset \mathcal{B}_2$ is Γ. Then the Bratteli diagram for the inclusion of $\mathcal{B}_0 \subset \mathcal{B}_2$ is Γ'. Let Λ_2 be the adjacency matrix for $\mathcal{B}_1 \subset \mathcal{B}_2$. Applying the basic construction, we will obtain the tower $\mathcal{B}_0 \subset \mathcal{B}_1 \subset \mathcal{B}_2 \subset \mathcal{B}_3 \subset \mathcal{B}_4 \subset \cdots$. Let $\{e_i\}, \{p_i\}$ be the sequences of Jones projections arising from the basic construction. Note that the relative commutant of $\mathcal{B}_0$ in the tower can be expressed as linear sums of loops of Γ. The even parts of the relative commutants form a planar algebra isomorphic to the graph planar algebra $\mathscr{G}$ of Γ' for a fixed choice of basis. So an element in $\mathscr{G}$ could be expressed as a linear sums of loops of Γ, instead of loops of Γ'. Actually an edge of Γ' is replaced by a length 2 path $\varepsilon_1\varepsilon_2^*$. It is convenient to express p_1 by loops of Γ.

Proposition 3.25. *Note that $p_1 \in \mathcal{B}_1' \cap \mathcal{B}_3$, we have*

$$p_1 = \delta_b^{-1} \sum_{\varepsilon_3, \varepsilon_7 \in \mathcal{E}_-, t(\varepsilon_3)=t(\varepsilon_7)} \sqrt{\frac{\lambda(s(\varepsilon_3))\lambda(s(\varepsilon_7))}{\lambda(t(\varepsilon_3))\lambda(t(\varepsilon_7))}} [\varepsilon_3^*\varepsilon_3\varepsilon_7^*\varepsilon_7].$$

To express p_1 as an element in $\mathscr{G}_{2,+} = \mathcal{B}_0' \cap \mathcal{B}_4$, we have

$$p_1 = \delta_b^{-1} \sum_{\substack{\varepsilon_3, \varepsilon_7 \in \mathcal{E}_- \\ \varepsilon_1, \varepsilon_5 \in \mathcal{E}_+ \\ t(\varepsilon_1)=t(\varepsilon_3)=t(\varepsilon_5)=t(\varepsilon_7)}} \sqrt{\frac{\lambda(s(\varepsilon_3))\lambda(s(\varepsilon_7))}{\lambda(t(\varepsilon_3))\lambda(t(\varepsilon_7))}} [\varepsilon_1\varepsilon_3^*\varepsilon_3\varepsilon_5^*\varepsilon_5\varepsilon_7^*\varepsilon_7\varepsilon_1^*].$$

Proof. Note that p_1 is the Jones projection for the basic construction $\mathcal{B}_1 \subset \mathcal{B}_2 \subset \mathcal{B}_3$. So we have the first formula. Take the inclusion from $\mathcal{B}_1' \cap \mathcal{B}_3$ to $\mathcal{B}_0' \cap \mathcal{B}_4$ for p_1, we obtained the second formula. □

Diagrammatically the inclusion from $\mathcal{B}_1' \cap \mathcal{B}_3$ to $\mathcal{B}_0' \cap \mathcal{B}_4$ is adding one a-colour string to the left and one to the right.

Theorem 3.26. *Suppose $\mathscr{S}$ is a finite depth subfactor planar algebra, p is a biprojection in $\mathscr{S}_{2,+}$, Γ' is the principal graph of $\mathscr{S}$, and Γ is the refined principal graph with respect to the biprojection p. Let ϕ be the embedding map from $\mathscr{S}$ to the graph planar algebra $\mathscr{G}$. Then $\phi(p) = p_1$ is the linear sum of loops as in Proposition 3.25.*

Proof. Note that p_m is the Jones projection for the basic construction $\mathscr{S}'_{m,+} \subset \mathscr{S}_{m,+} \subset \mathscr{S}'_{m+1,+}$, when m is odd and greater than the depth of $\mathscr{S}$. So $\phi(p)$ is the Jones projection for the basic construction $\mathcal{B}_1 \subset \mathcal{B}_2 \subset \mathcal{B}_3$, which implies $\phi(p) = p_1$. □

Remark 3.27. In the rest of the paper, we will identify the subfactor planar algebras $\mathscr{S}$ with its image $\phi(\mathscr{S})$ in the graph planar algebra $\mathscr{G}$. We keep the same notation for elements in $\mathscr{S}$ and $\phi(\mathscr{S})$ by ignoring the map ϕ.

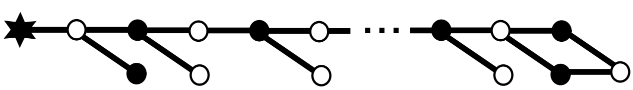
4. Bisch–Haagerup fish graphs

Suppose $\mathcal{N} \subset \mathcal{P} \subset \mathcal{M}$ is an inclusion of factors of type II_1, such that $[\mathcal{M} : \mathcal{P}] = \frac{3+\sqrt{5}}{2}$ and $[\mathcal{P} : \mathcal{N}] = 2$. Then the $(\mathcal{P},\mathcal{P})$ bimodules arisen from $\mathcal{N} \subset \mathcal{P}$ is generated by an irreducible bimodule α with the relation $\alpha^2 = 1$. Moreover, the $(\mathcal{P},\mathcal{P})$ bimodules arisen from $\mathcal{P} \subset \mathcal{M}$ is generated by an irreducible bimodule β with the relation $\beta^2 = \beta + 1$.

If there is no extra relation for α and β, then $\mathcal{N} \subset \mathcal{P}$ is called a free composed inclusion. In this case, the planar algebra of $\mathcal{N} \subset \mathcal{M}$ is Fuss–Catalan.

If there is an extra relation for α and β, then Bisch and Haagerup showed that the relation is $(\alpha\beta)^n = (\beta\alpha)^n$, for some positive integer n, in their unpublished work, see also Proposition 3.2 in [12]. It is easy to derive the principal graph of $\mathcal{N} \subset \mathcal{M}$ from the fusion rule.

Definition 4.1. In the case $(\alpha\beta)^n = (\beta\alpha)^n$, the subfactor $\mathcal{N} \subset \mathcal{M}$ has depth $2n+1$. Its principal graph was computed by Bisch and Haagerup as

,

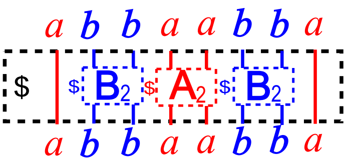
called the nth Bisch–Haagerup fish graph, when it is of depth $2n+1$.

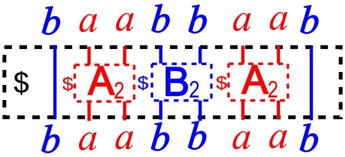
Conversely if the principal graph of a subfactor $\mathcal{N} \subset \mathcal{M}$ is a Bisch–Haagerup fish graph, then it has an intermediate subfactor $\mathcal{P}$, such that $[\mathcal{P} : \mathcal{N}] = 2$, due to the existence of a dimension one vertex at depth 2 of the Bisch–Haagerup fish graph [34].

If the principal graph of a subfactor planar algebra is the nth Bisch–Haagerup fish graph, then it contains a trace 2 biprojection. We are going to embed the subfactor planar algebra in its graph planar algebra. First we will see there is only one possible refined principal graph with respect to the biprojection. Then in the orthogonal complement of the Fuss–Catalan planar subalgebra, there is a new generator at depth $2n$. We will show that this generator satisfies some relations. We hope to solve for the generator with such relations in the graph planar algebra. In the case $n \geq 4$, there is no solution. So there is no subfactor planar algebra whose principal graph is the nth fish. In the case $n = 1, 2, 3$, there is a unique solution up to (planar algebra) isomorphism. So there is at most one subfactor planar algebra for each n. Their existence follows from three known subfactors. We can reconstruct them in the graph planar algebra by generators and Fuss–Catalan Jellyfish relations.

Notation 4.2. Take $\delta_a = \sqrt{2}$, $\delta_b = \frac{1+\sqrt{5}}{2}$ and $\delta = \delta_a\delta_b$. Then $\delta_b^2 = \delta_b + 1$. Let $FC = FC(\delta_a, \delta_b)$ be the Fuss–Catalan planar algebra with parameters (δ_a, δ_b). We assume that f_{2n} is the minimal projection in $FC_{2n,+}$ with middle pattern $\underbrace{abba\ abba\ \cdots\ abba}_{n}$, n copies of $abba$; and g_{2n} is the minimal projection in $FC_{2n,-}$ with middle pattern $\underbrace{baab\ baab\ \cdots\ baab}_{n}$.

As a/b-colour diagrams, we have, for example,

$f_4 =$ ,

$g_4 =$ ,

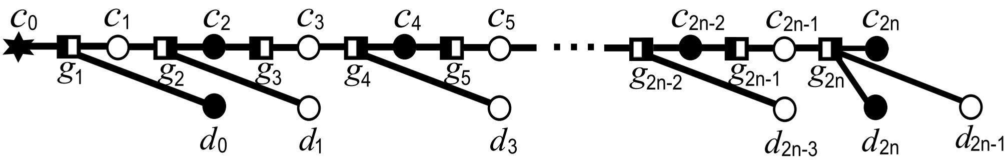
where A_2 and B_2 are the a-colour and b-colour second Jones–Wenzl projections respectively.

4.1. Principal graphs

If the nth Bisch–Haagerup fish graph is the principal graph of a subfactor $\mathcal{N} \subset \mathcal{M}$, then its index is $\delta^2 = 3 + \sqrt{5}$. Because of the existence of a normalizer, there is an intermediate subfactor $\mathcal{P}$, such that $[\mathcal{P} : \mathcal{N}] = 2$.

Definition 4.3. Let us define the subfactor planar algebra of $\mathcal{N} \subset \mathcal{M}$, if it exists, to be $\mathscr{B} = \{\mathscr{B}_{m,\pm}\}$, and $e_{\mathcal{P}}$ to be the biprojection corresponding to the intermediate subfactor $\mathcal{P}$.

Lemma 4.4. *The refined principal graph with respect to the biprojection $e_{\mathcal{P}}$ is*

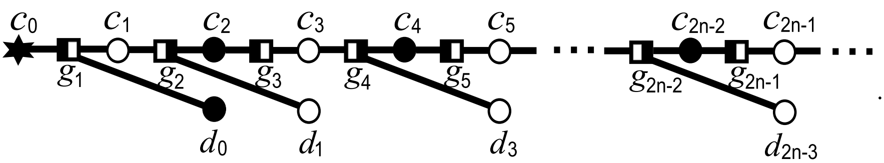


.

Its dimension vector λ is given by

$$\begin{aligned}
\lambda(c_0) &= \lambda(d_0) = 1; \\
\lambda(c_{2k-1}) &= \delta_a\delta_b^k, && \textit{for } 1 \le k \le n; \\
\lambda(d_{2k-1}) &= \delta_a\delta_b^{k-1}, && \textit{for } 1 \le k \le n; \\
\lambda(c_{2k}) &= 2\delta_b^k, && \textit{for } 1 \le k \le n-1; \\
\lambda(c_{2n}) &= \lambda(d_{2n}) = \delta_b^n; \\
\lambda(g_{2k-1}) &= \delta_a\delta_b^{k-1}, && \textit{for } 1 \le k \le n; \\
\lambda(g_{2k}) &= \delta_a\delta_b^k, && \textit{for } 1 \le k \le n.
\end{aligned}$$

Proof. Note that $\delta^2 = 3+\sqrt{5} = \delta_a^2\delta_b^2$, so the planar subalgebra generated by the trace-2 biprojection $e_{\mathcal{P}}$ is $FC = FC(\delta_a, \delta_b)$. Observe that the principal graph of FC is the same as the nth fish up to depth $2n-1$, so $\mathscr{B}_{2(n-1),+} = FC_{2(n-1),+}$. Then the refined principal graph of $\mathscr{B}$ starts as



The vertex c_{2k-1} corresponds to the minimal projection of $FC_{2k-1,+}$ with middle pattern $\underbrace{abba \ \cdots \ abba}_{k-1} ab$, $k-1$ copies of $abba$, for $1 \le k \le n$. So $\lambda(c_{2k-1}) = \delta_a\delta_b^k$.

The vertex d_{2k-1} corresponds to the minimal projection of $FC_{2k+1,+}$ with middle pattern $\underbrace{abba \ \cdots \ abba}_{k-1} abbb$, for $1 \le k \le n-1$. So $\lambda(d_{2k-1}) = \delta_a\delta_b^{k-1}$.

The vertex c_{2k} corresponds to the minimal projection of $FC_{2k,+}$ with middle pattern $\underbrace{abba \ \cdots \ abba}_{k}$, for $1 \le k \le n-1$. So $\lambda(c_{2k}) = 2\delta_b^k$;

The vertex c_0 is the marked point. So $\lambda(c_0) = 1$; The vertex d_0 corresponds to the minimal projection of $FC_{2,+}$ with middle pattern aa. So $\lambda(d_0) = 1$;

The vertex g_{2k-1} corresponds to the minimal projection of $FC'_{2k-1,+}$ with middle pattern $\underbrace{abba \ \cdots \ abba}_{k-1}\ a$, for $1 \le k \le n$. So $\lambda(g_{2k-1}) = \delta_a\delta_b^{k-1}$;

The vertex g_{2k} corresponds to the minimal projection of $FC'_{2k,+}$ with middle pattern $\underbrace{abba \ \cdots \ abba}_{k-1}\ abb$, for $1 \le k \le n-1$. So $\lambda(g_{2k}) = \delta_a\delta_b^k$.

All these vertices are not adjacent to a new point in the refined principal graph except c_{2n-1}, because they are identical to the vertices of the refined principal graph of FC.

Note that

$$\delta_b\lambda(c_{2n-1}) - \lambda(g_{2n-1}) = \delta_a\delta_b^{n+1} - \delta_a\delta_b^{n-1} = \delta_a\delta_b^n,$$

so there is a new $\mathcal{P}$ coloured vertex, denoted by g_{2n}, adjacent to c_{2n-1}. Then $\lambda(g_{2n}) \leq \delta_a\delta_b^n$. On the other hand

$$\lambda(g_{2n}) \geq \delta_b^{-1}\lambda(c_{2n-1}) = \delta_b^n > \frac{1}{2}\delta_a\delta_b^n,$$

so g_{2n} is unique new $\mathcal{P}$ coloured vertex adjacent to c_{2n-1} and $\lambda(g_{2n}) = \delta_a\delta_b^n$.

Note that

$$\delta_b\lambda(g_{2n}) - \lambda(c_{2n-1}) = \delta_a\delta_b^{n+1} - \delta_a\delta_b^n = \delta_a\delta_b^{n-1},$$

so there is a new $\mathcal{N}$ coloured vertex, denoted by d_{2n-1}, adjacent to g_{2n}. Then $\lambda(d_{2n-1}) \leq \delta_a\delta_b^{n-1}$. On the other hand

$$\lambda(d_{2n-1}) \geq \delta_b^{-1}\lambda(g_{2n}) = \delta_a\delta_b^{n-1},$$

so d_{2n-1} is unique new $\mathcal{N}$ coloured vertex adjacent to g_{2n} and $\lambda(d_{2n-1}) = \delta_a\delta_b^{n-1}$.

Now $\delta_b\lambda(d_{2n-1}) = \lambda(g_{2n})$, so there is no new $\mathcal{P}$ coloured vertex adjacent to d_{2n-1}.

In the principal graph, there are two $\mathcal{M}$ coloured vertices, denoted by c_{2n}, d_{2n}, adjacent to c_{2n-1}. Thus c_{2n}, d_{2n} are adjacent to g_{2n} in the refined principal graph. Moreover

$$\lambda(c_{2n}) = \lambda(d_{2n}) = \frac{1}{\delta}(\lambda(c_{2n-1}) + \lambda(d_{2n-1})) = \delta_b^n.$$

Then $\delta_a\lambda(c_{2n}) = \delta_a\lambda(d_{2n}) = \lambda(g_{2n})$. So there is no new $\mathcal{P}$ coloured vertices adjacent to c_{2n} or d_{2n}.

Therefore we have the unique possible refined principal graph and its dimension vector as mentioned in the statement. □

Because $\mathscr{B}$ contains a biprojection, it is decomposed as an *Annular Fuss–Catalan module* [24], similar to the Temperley–Lieb case [15,20]. The Fuss–Catalan planar subalgebra FC is already a submodule of $\mathscr{B}$. There is a lowest weight vector in $\mathscr{B}_{2n,+}$ which is orthogonal to FC. So this vector is rotation invariant up to a phase. Moreover it is *totally uncappable*, see [24]. In this special case, we have a direct proof of this result.

Definition 4.5. An element $x \in \mathscr{B}_{m,+}$ is said to be totally uncappable, if

$$\rho^k(x)e_{\mathcal{P}} = 0, \ \rho^k(\mathcal{F}(x))\mathcal{F}(e_{\mathcal{P}}) = 0, \quad \forall k \geq 0;$$

An element $y \in \mathscr{B}_{m,-}$ is said to be totally uncappable, if $\mathcal{F}(y)$ is totally uncappable.

If we consider x as an a,b-colour diagram, then an element is totally uncappable means it becomes zero whenever it is capped by an a/b-colour string.

Now let us construct the totally uncappable element $S \in \mathscr{B}_{2n,+}$. If S is totally uncappable, then S is orthogonal to $FC_{2n,+}$. While the minimal projection f_{2n} of $FC_{2n,+}$ is separated into two minimal projections in $\mathscr{B}_{2n,+}$, denoted by P_c and P_d, with the same trace. So S has to be a multiple of $P_c - P_d$. Take S to be $P_c - P_d$, then S satisfies the following propositions.

Proposition 4.6. *For $S = P_c - P_d$ in $\mathscr{B}_{2n,+}$, we have*

(1) $S^* = S$;
(2) $S^2 = f_{2n}$;
(3) *S is totally uncappable;*
(4) *$\rho(S) = \omega S$, for some $\omega \in \mathbb{C}$ satisfying $|\omega| = 1$.*

Proof. (1) $S^* = (P_c - P_d)^* = S$.

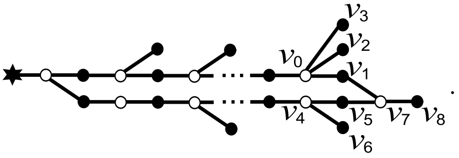
(2) $S^2 = (P_c - P_d)^2 = P_c + P_d = f_{2n}$.

(4) Note that ρ preserves the inner product of $S \in \mathscr{B}_{2n,+}$, and $FC_{2n,+}$ is rotation invariant, so both S and $\rho(S)$ are in the orthogonal complement of $FC_{2n,+}$ which is a one-dimensional subspace. Then we have $\rho(S) = \omega S$ for some $\omega \in \mathbb{C}$. Moreover $||\rho(S)||_2 = ||S||_2$, so $|\omega| = 1$.

(3) From the refined principal graph, we have $S * P$ is a multiple of f_{2n}. By computing the trace, we have $S * P = 0$. On the other hand $tr((SP)^*(SP)) = tr(f_{2n}P) = 0$, so $SP = 0$. By proposition (4), we have S is totally uncappable. □

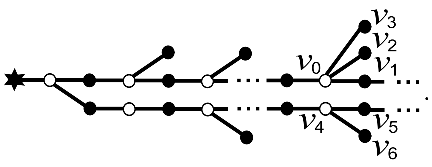
If $S \in \mathscr{B}_{2n,+}$ is totally uncappable, then $\mathcal{F}(S) \in \mathscr{B}_{2n,-}$ is also totally uncappable. To describe its relations, we need the dual principal graph of $\mathscr{B}$.

Lemma 4.7. *If the principal graph of $\mathscr{B}$ is the nth Bisch–Haagerup fish graph, then the dual principal graph of $\mathscr{B}$ is*



For its dimension vector λ', we have $\lambda'(v_1) = \delta_b^n$, $\lambda'(v_2) = \delta_b^{n-1}$.

Proof. Note that $\mathscr{B}_{2n-1,+} = FC_{2n-1,+}$, so $\mathscr{B}_{2n-1,-} = FC_{2n-1,-}$. Then the dual principal graph of $\mathscr{B}$ is the same as the dual principal graph of FC up to depth $2n - 1$. In $\mathscr{B}_{2n,-}$, there is a totally uncappable element, so the minimal projection g_{2n} of $FC_{2n,-}$ is separated into two minimal projections of $\mathscr{B}_{2n,-}$, denoted by P'_c, P'_d. Then we have the dual principal graph up to depth $2n$ as



The vertex v_0 corresponds to the minimal projection of $FC_{2n-1,-}$ with middle pattern $\underbrace{baab \ \cdots \ baab}_{n-1} ba$. So $\lambda'(v_0) = \delta_a \delta_b^n$;

The vertex v_1 corresponds to the minimal projection P'_c; The vertex v_2 corresponds to the minimal projection P'_d;

In the case $n = 1$, there is no vertex v_3; In the case $n \geq 2$, the vertex v_3 corresponds to the minimal projection of $FC_{2n,-}$ with middle pattern $\underbrace{baab \ \cdots \ baab}_{n-1} bb$. So $\lambda'(v_3) = \delta_b^{n-1}$.

In the case $n = 1$, there is no vertex v_4; In the case $n \geq 2$ the vertex v_4 corresponds to the minimal projection of $FC_{2n-1,-}$ with middle pattern $bb \underbrace{baab \ \cdots \ baab}_{n-2} \ ba$. So $\lambda'(v_4) = \delta_a \delta_b^{n-2}$.

The vertex v_5 corresponds to the minimal projection of $FC_{2n,-}$ with middle pattern $bb \ \underbrace{baab \ \cdots \ baab}_{n-1}$. So $\lambda'(v_5) = \delta_b^{n-1}$.

In the case $n \leq 2$, there is no vertex v_6; In the case $n \geq 3$, the vertex v_5 corresponds to the minimal projection of $FC_{2n,-}$ with middle pattern $bb \underbrace{baab \ \cdots \ baab}_{n-2} bb$. So $\lambda'(v_6) = \delta_b^{n-3}$.

In the principal graph, there is one vertex at depth $2n + 1$ with multiplicity 2 (Definition 2.9). So in the dual principal graph, there is one vertex at depth $2n + 1$ with multiplicity 2, denoted by v_7.

While $\delta\lambda'(v_5) - \lambda'(v_4) = \delta_a \delta_b^n - \delta_a \delta_b^{n-2} = \delta_a \delta_b^{n-1}$. So v_5 is adjacent to v_7. Then at most one of v_1 and v_2 is adjacent to v_7. Without loss of generality, we assume that v_2 is not adjacent to v_7. Then $\lambda'(v_2) = \frac{1}{\delta}\lambda'(v_0) = \delta_b^{n-1}$. So $\lambda'(v_1) = tr(g_{2n}) - \lambda'(v_2) = \delta_b^{n+1} - \delta_b^{n-1} = \delta_b^n$. Then $\delta\lambda'(v_1) - \lambda'(v_0) = \delta_a \delta_b^{n+1} - \delta_a \delta_b^n = \delta_a \delta_b^{n-1}$. So v_1 is adjacent to v_7, and $\lambda'(v_7) = \delta_a \delta_b^{n-1}$. While $\delta\lambda'(v_7) - \lambda'(v_1) - \lambda'(v_5) = 2\delta_b^n - \delta_b^n - \delta_b^{n-1} = \delta_b^{n-2}$. So there is a new $\mathcal{N}$ coloured vertex, denoted by v_8, adjacent to v_7. Then $\lambda'(v_8) \leq \delta_b^{n-2}$. On the other hand $\lambda'(v_8) \geq \delta^{-1}\lambda'(v_7) = \delta_b^{n-2}$. So $\lambda'(v_8) = \delta_b^{n-2}$. And there is no new vertices in the dual principal graph.

Therefore we obtain the unique possible dual principal graph. $\square$

Definition 4.8. Let us define Γ_n to be the (potential) dual principal graph of $\mathscr{B}$.

Note that the minimal projection g_{2n} of $FC_{2n,-}$ is separated into two minimal projections P'_c, P'_d in $\mathscr{B}_{2n,-}$. And $tr(P'_c) = \lambda(v_1) = \delta_b^n$, $tr(P'_d) = \lambda(v_2) = \delta_b^{n-1}$. Take R to be $\delta_b^{-1}P'_c - \delta_b^{-2}P'_d$, then R is orthogonal to $FC_{2n,-}$ in $\mathscr{B}_{2n,-}$. Recall that $\mathcal{F}(S) \in FC_{2n,-}$ is totally uncappable, so $\mathcal{F}(S)$ is also orthogonal to $FC_{2n,-}$ in $\mathscr{B}_{2n,-}$. While the orthogonal complement of $FC_{2n,-}$ in $\mathscr{B}_{2n,-}$ is one dimensional. So $\mathcal{F}(S)$ is a multiple of R. Then we have the following propositions.

Proposition 4.9. *For $R = \delta_b^{-1}P'_d - \delta_b^{-2}P'_c$ in $\mathscr{B}_{2n,-}$, we have*

(0) *$R = \omega_0\delta^{-1}\mathcal{F}(S)$, for a constant ω_0 satisfying $\omega_0^{-2} = \omega$, where S and ω are given in Proposition 4.6;*
(1') *$R^* = R$;*
(2') *$R + \delta_b^{-2}g_{2n}$ is a projection;*
(3') *R is totally uncappable;*
(4') *$\rho(R) = \omega R$.*

Proof. (1') $R^* = (\delta_b^{-1}P'_d - \delta_b^{-2}P'_c)^* = R$.

(0) By the argument above, we have $\mathcal{F}(S)$ is a multiple of R. Note that

$$\begin{aligned}
||\mathcal{F}(S)||_2^2 &= tr(S * S) \\
&= tr(f_{2n}) \\
&= \delta_a^2\delta_b^n, \\
||R||_2^2 &= tr(R^*R) \\
&= \delta_b^{-2}tr(P'_c) + \delta_b^{-4}tr(P'_d) \\
&= \delta_b^{-2}\delta_b^{n-1} + \delta_b^{-4}\delta_b^n \\
&= \delta_b^{n-2} \\
&= \delta^{-2}||\mathcal{F}(S)||_2^2.
\end{aligned}$$

So $R = \omega_0\delta^{-1}\mathcal{F}(S)$, for some phase ω_0, i.e. $\omega_0 \in \mathbb{C}$ and $|\omega_0| = 1$.

Note that

$$(\mathcal{F}(R))^* = \mathcal{F}^{-1}(R^*) = \mathcal{F}^{-1}(R).$$

So

$$\begin{aligned}
(\omega_0\delta^{-1}\mathcal{F}^2(S))^* &= (\mathcal{F}(R))^* \\
&= \mathcal{F}^{-1}(R) \\
&= \omega_0\delta^{-1}(S).
\end{aligned}$$

Then

$$\omega_0 \rho(S) = (\omega_0 S)^* = \overline{\omega_0} S.$$

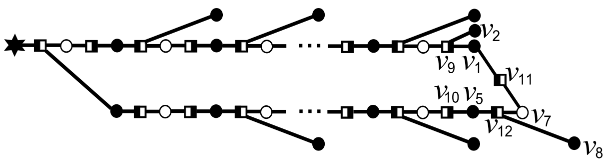
Recall that $\rho(S) = \omega S$. Thus $\omega_0^{-2} = \omega$.

(2') $R + \delta_b^{-2} g_{2n} = P'_d$ is a projection.

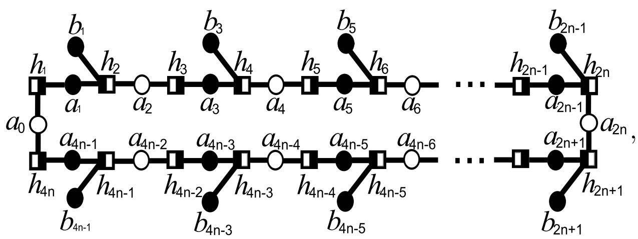
(3') and (4') follow from (0). □

By the embedding theorem, we hope to solve for (S, R, ω_0) in the graph planar algebra, such that (S, R, ω_0) satisfies the propositions (0)(1)(2)(3)(4)(1')(2')(3')(4') listed in Proposition (4.6)(4.9). In this case, there is no essential difference to solve it in the graph planar algebra of the principal graph or the dual principal graph. But for computations, we may avoid a factor $\frac{1}{2}$ in the graph planar algebra of the dual principal graph. The factor $\frac{1}{2}$ comes from the symmetry of c_0, d_0 and c_{2n}, d_{2n} in the principal graph. Now let us describe the refined dual principal graph of $\mathscr{B}$.

Lemma 4.10. *The refined principal graph of $\mathscr{B}$ with respect to the biprojection $e_{\mathcal{P}}$ is*



For computations, let us adjust the refined principal graph and relabel its the vertices as



where the marked vertex is b_1. For convenience, we assume that $a_{4n} = a_0$. Then its dimension vector λ' is given by

$$\begin{aligned}
\lambda'(a_{2k-1}) &= \lambda'(a_{4n-2k+1}) = \delta_b^k, && \textit{for } 1 \le k \le n;\\
\lambda'(b_{2k-1}) &= \lambda'(b_{4n-2k+1}) = \delta_b^{k-1}, && \textit{for } 1 \le k \le n;\\
\lambda'(a_{2k}) &= \lambda'(a_{4n-2k}) = \delta_a \delta_b^k, && \textit{for } 0 \le k \le n;\\
\lambda'(h_{2k-1}) &= \lambda'(h_{4n-2k+2}) = \delta_b^{k-1}, && \textit{for } 1 \le k \le n;\\
\lambda'(h_{2k}) &= \lambda'(h_{4n-2k+1}) = \delta_b^k, && \textit{for } 1 \le k \le n.
\end{aligned}$$

The even vertices of the principal graph have odd indices and the odd vertices have even indices.

Proof. The proof is similar to that of Lemma 4.4.

We have known that $\mathscr{B}_{2n,-} = FC_{2n,-} \oplus \mathbb{C}(R)$, where $\mathbb{C}(R)$ is the one dimensional vector space generated by the totally uncappable element R. So we obtain the refined principal graph up to depth $2n$ as mentioned in the statement.

For the vertices v_9, v_{10} as marked in the statement, we have

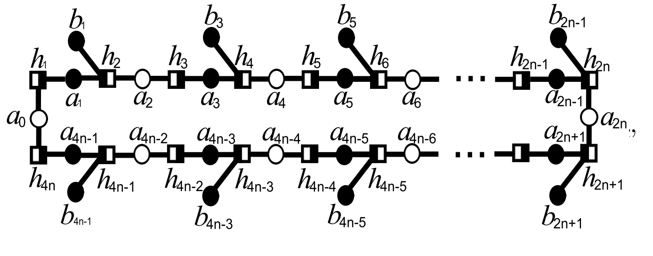
$$\lambda'(v_9) = \delta_b \lambda'(v_2) = \delta_b \delta_b^{n-1} = \delta_b^n,$$
$$\lambda'(v_{10}) = \delta_b^{-1} \lambda'(v_5) = \delta_b^{-1} \delta_b^{n-1} = \delta_b^{n-2}.$$

Then $\delta_b \lambda'(v_1) - \lambda(v_9) = \delta_b \delta_b^n - \delta_b^n = \delta_b^{n-1}$. So v_1 is adjacent to a new $\mathcal{P}$ coloured vertex, denoted by v_{11}. Then $\lambda'(v_{11}) \leq \delta_b^{n-1}$. On the other hand $\lambda'(v_{11}) \geq \delta_b^{-1}\lambda'(v_1) = \delta_b^{n-1}$. So v_{11} is the unique new $\mathcal{P}$ coloured vertex adjacent to v_1 and $\lambda'(v_{11}) = \delta_b^{n-1}$. Then $\delta_b \lambda'(v_{11}) = \lambda'(v_1)$ implies v_8 is not adjacent to v_{11}. And the $\mathcal{N}$ coloured vertex adjacent to v_{11} has to be v_7.

Moreover $\delta_b \lambda'(v_5) - \lambda(v_{10}) = \delta_b \delta_b^{n-1} - \delta_b^{n-2} = \delta_b^{n-1}$. So v_1 is adjacent to a new $\mathcal{P}$ coloured vertex, denoted by v_{12}. Then $\lambda'(v_{12}) \leq \delta_b^{n-1}$. On the other hand $\lambda'(v_{12}) \geq \delta_b^{-1}\lambda'(v_5) = \delta_b^{n-2} > \frac{1}{2}\delta_b^{n-1}$. So v_{12} is the unique new $\mathcal{P}$ coloured vertex adjacent to v_5 and $\lambda'(v_{12}) = \delta_b^{n-1}$. Then $\delta_b \lambda'(v_{12}) - \lambda'(v_5) = \lambda'(v_8)$ implies v_8 is adjacent to v_{11}. And the $\mathcal{N}$ coloured vertex adjacent to v_{11} has to be v_7.

While $\delta_a \lambda(v_7) = 2\delta_b^{n-1} = \lambda'(v_{11}) + \lambda'(v_{12})$, $\delta_b \lambda'(v_8) = \delta_b^{n-1} = \lambda'(v_{12})$. So there is no new $\mathcal{P}$ coloured vertices. Then we have the unique possible refined dual principal of $\mathscr{B}$.

Now we adjust the refined principal graph and relabel its the vertices as



where the marked vertex is b_1.

The graph is vertically symmetrical, by Corollary 3.23, the dimension vector λ' is also symmetric. So we only need to compute the value of λ' for the upper half vertices.

The vertex a_1 corresponds to the minimal projection of $FC_{2,-}$ with middle pattern bb. So $\lambda'(a_1) = \delta_b$; The vertex a_{2k-1} corresponds to the minimal projection of $FC_{2k-2,-}$ with middle pattern $baab \cdots baab$, $k-1$ copies of $baab$, for $2 \leq k \leq n$. So $\lambda'(a_{2k-1}) = \delta_b^k$, for $2 \leq k \leq n$;

The vertex b_1 is the marked vertex. So $\lambda'(b_1) = 1$; The vertex b_{2k-1} corresponds to the minimal projection of $FC_{2k-1,-}$ with middle pattern $baab \cdots baab\ bb$, $k-1$ copies of $baab$, for $2 \leq k \leq n$. So $\lambda'(b_{2k-1}) = \delta_b^{k-1}$, for $2 \leq k \leq n$;

The vertex a_0 corresponds to the minimal projection of $FC_{3,-}$ with middle pattern $bbba$. So $\lambda'(a_0) = \delta_a$; The vertex a_{2k} corresponds to the minimal projection of $FC_{2k-1,-}$ with middle pattern $baab \cdots baab\ ba$, $k-1$ copies of $baab$, for $1 \leq k \leq n$. So $\lambda'(a_{2k}) = \delta_a\delta_b^k$, for $1 \leq k \leq n$;

The vertex h_1 corresponds to the minimal projection of $FC'_{3,-}$ with middle pattern bbb. So $\lambda'(h_1) = 1$; The vertex h_{2k-1} corresponds to the minimal projection of $FC'_{2k-1,-}$ with middle pattern $baab \cdots baab\ baa$, $k-2$ copies of $baab$, for $2 \leq k \leq n$. So $\lambda'(a_{2k}) = \delta_b^{k-1}$, for $2 \leq k \leq n$;

The vertex h_1 corresponds to the minimal projection of $FC'_{3,-}$ with middle pattern bbb. So $\lambda'(h_1) = 1$; The vertex h_{2k} corresponds to the minimal projection of $FC'_{2k-1,-}$ with middle pattern $baab \cdots baab\ b$, $k-1$ copies of $baab$, for $1 \leq k \leq n$. So $\lambda'(a_{2k}) = \delta_b^{k-1}$, for $1 \leq k \leq n$; □

We hope to embed $\mathscr{B}_{m,\mp}$ in the graph planar algebra of the dual principal graph, so we will consider the biprojection $e_{\mathcal{P}_1} = \delta_a^{-1}\delta_b\mathcal{F}(e_{\mathcal{P}})$ in $\mathscr{B}_{2,-}$.

Definition 4.11. Let us define $\mathscr{G} = \mathscr{G}_{m,\pm}$ to be the graph planar algebra of the dual principal graph Γ_n. Then $\mathscr{B}_{m,\mp}$ is embedded in $\mathscr{G}_{m,\pm}$. Let $p_1 \in \mathscr{G}_{2,+}$ be the image of $e_{\mathcal{P}_1}$. Then the planar subalgebra $FC(\delta_b, \delta_a)_{m,\pm}$ of $\mathscr{G}$ generated by p_1 is identical to the image of $FC(\delta_a, \delta_b)_{m,\mp}$. The images of f_{2n} and g_{2n} are still denoted by f_{2n} and g_{2n}.

Notation 4.12. Note that the dual principal graph Γ is simply laced. An edge ε of Γ_n is determined by $s(\varepsilon)$ and $t(\varepsilon)$, so we may use

$$[s(\varepsilon_1)t(\varepsilon_1)s(\varepsilon_3)t(\varepsilon_3)\cdots s(\varepsilon_{2m-1})t(\varepsilon_{2m-1})]$$

to express a loop $[\varepsilon_1\varepsilon_2^*\varepsilon_3\varepsilon_4^*\cdots\varepsilon_{2m-1}\varepsilon_{2m}^*]$ in $\mathscr{G}_{2m,+}$, similarly for loops in $\mathscr{G}_{2m,-}$.

Proposition 4.13.

$$\begin{aligned}
p_1 = \sum_{k=1}^{n} & [a_{2k-1}a_{2k-2}a_{2k-1}a_{2k-2}] + [a_{4n-2k+1}a_{4n-2k+2}a_{4n-2k+1}a_{4n-2k+2}] \\
&+ [a_{2k-1}a_{2k}a_{2k-1}a_{2k}] + [a_{4n-2k+1}a_{4n-2k}a_{4n-2k+1}a_{4n-2k}] \\
&+ [a_{2k-1}a_{2k}b_{2k-1}a_{2k}] + [a_{4n-2k+1}a_{4n-2k}b_{4n-2k+1}a_{4n-2k}] \\
&+ [b_{2k-1}a_{2k}b_{2k-1}a_{2k}] + [b_{4n-2k+1}a_{4n-2k}b_{4n-2k+1}a_{4n-2k}] \\
&+ [b_{2k-1}a_{2k}a_{2k-1}a_{2k}] + [b_{4n-2k+1}a_{4n-2k}a_{4n-2k+1}a_{4n-2k}].
\end{aligned}$$

Proof. It follows from Theorem 3.26 and Lemma 4.10. □

Definition 4.14. Note that $\mathscr{G}_{0,+}$ is abelian. Let us define A_k, B_k to be the minimal projections corresponding to the vertices a_{2k-1}, b_{2k-1} respectively, for $1 \leq k \leq 2n$.

Note that $\mathscr{G}_{1,+}$ is abelian. Let us decompose A_k into minimal projections A_k^- and A_k^+ as follows,

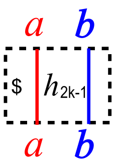
$$A_k^- = [a_{2k-1}a_{2k-2}], \qquad A_{2n-k}^- = [a_{4n-2k+1}a_{4n-2k+2}],$$
$$A_k^+ = [a_{2k-1}a_{2k}], \qquad A_{2n-k}^+ = [a_{4n-2k+1}a_{4n-2k}],$$

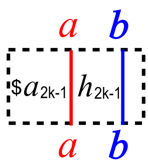
for $1 \leq k \leq n$.

Let us define H_{2k-1}, $H_{4n-2k+1}$, H_{2k} and H_{4n-2k} in $\mathscr{G}_{1,-}$, for $1 \leq k \leq n$, as follows

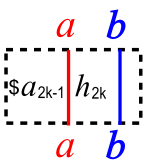
$$H_{2k-1} = [a_{2k-2}a_{2k-1}], \qquad H_{2k} = [a_{2k}a_{2k-1}] + [a_{2k}b_{2k-1}],$$
$$H_{4n-2k+2} = [a_{4n-2k+2}a_{4n-2k+1}], \quad H_{4n-2k+1} = [a_{4n-2k}a_{4n-2k+1}] + [a_{4n-2k}b_{4n-2k+1}].$$

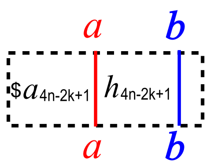
Remark 4.15. If we apply the general embedding theorem [29] for the graph planar algebra of the refined principal graph equipped with actions of Fuss–Catalan tangles (with $\mathcal{N} - \mathcal{P} - \mathcal{M}$ shadings of regions and a/b-colour strings), then

$A_k^- =$ 

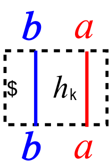
, $\qquad A_{2n-k}^- =$ 

, $\qquad 1 \leq k \leq n;$

$A_k^+ =$ 

, $\qquad A_{2n-k}^+ =$ 

, $\qquad 1 \leq k \leq n;$

$H_k =$ 

, $\qquad 1 \leq k \leq 4n,$

while identifying the dual spaces of loop algebras as themselves for a finite graph.

Proposition 4.16. *In the graph planar algebra, A_k, B_k are in the centre of $\mathscr{G}_{2n,+}$. Moreover, g_{2n} commutes with A_k^+ and A_k^-.*

Proof. The first statement is obvious.

By Proposition 4.13, for $1 \leq k \leq n$, we have

$$p_1 A_k^+ = [a_{2k-1}a_{2k}a_{2k}a_{2k-1}a_{2k}] = A_k^+ p_1.$$

Note that $p_1 + g_1$ is the identity, so $g_1 A_k^+ = A_k^+ g_1$. Adding one through string to the left of f_{2n-1}, denoted by $1 \otimes f_{2n-1}$, we have $(1 \otimes f_{2n-1})A_k^+ = A_k^+(1 \otimes f_{2n-1})$ and $g_{2n} = g_1(1 \otimes f_{2n-1})$. Therefore $g_{2n}A_k^+ = A_k^+ g_{2n}$.

Similar formulas hold for other cases. □

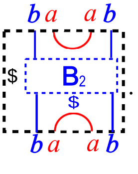
Remark 4.17. As a/b-colour diagrams, $A_k^{\pm}$ has a b-colour through string on the right and g_{2n} has an a-colour through string on the left, so they commute with each other.

4.2. The potential generator

Now we sketch the idea of solving the generator R in $\mathscr{G}$. Essentially we are considering the length $8n$ loops on the refined dual principal graph. Observe that if a loop contains a word $h_k a_k h_k$, for $1 \leq k \leq 2n$, then the vertex a_k could be replaced by an a/b-colour cap, because a_k is the unique $\mathcal{N}/\mathcal{M}$ coloured vertex adjacent to h_k. The coefficient of such a loop in the totally uncappable element R has to be 0. Therefore for a loop l with non-zero coefficient in R, if it goes to the right, then it will not return until passing the vertex a_{2n}. Among these loops, there is exactly one in $A_1^- \mathscr{G}_{2n,+} A_1^+$, that tells the initial condition of R. By proposition (2'), $A_k R A_k$ is determined by $A_k^- R A_k^+$. By proposition (3'), $B_k R$ is determined by $A_k^+ R A_k^+$. By proposition (4'), $A_{k+1}^- R A_{k+1}$ is determined by $(A_k + B_k)R(A_k + B_k)$. That means R could be computed inductively by the initial condition.

Definition 4.18. Let us define $F \in \mathscr{G}_{2,+}$ to be the image of $\mathcal{F}(id - e_{\mathcal{P}})$, i.e. $F = \delta e_1 - \delta_a \delta_b^{-1} p_1$.

Remark 4.19. In terms of a/b-colour diagrams, we have

$F =$ .

It is easy to check that $F * F = F$ and $F * g_{2n} = g_{2n} * F = 0$.

Note that e_1 and p_1 could be expressed as linear sums of loops, then we have

$$
\begin{aligned}
F = \sum_{1 \leq k \leq n} & \delta_a \delta_b^{-0.5}([a_{2k-1}a_{2k-2}a_{2k-1}a_{2k}] + [a_{4n-2k+1}a_{4n-2k+2}a_{4n-2k+1}a_{4n-2k}] \\
& + [a_{2k-1}a_{2k}a_{2k-1}a_{2k-2}] + [a_{4n-2k+1}a_{4n-2k}a_{4n-2k+1}a_{4n-2k+2}]) \\
& + \delta_a \delta_b^{-2}([a_{2k-1}a_{2k}a_{2k-1}a_{2k}] + [a_{4n-2k+1}a_{4n-2k}a_{4n-2k+1}a_{4n-2k}]) \\
& - \delta_a \delta_b^{-1}([a_{2k-1}a_{2k}b_{2k-1}a_{2k}] + [a_{4n-2k+1}a_{4n-2k}b_{4n-2k+1}a_{4n-2k}]) \\
& + \delta_a([b_{2k-1}a_{2k}b_{2k-1}a_{2k}] + [b_{4n-2k+1}a_{4n-2k}b_{4n-2k+1}a_{4n-2k}])
\end{aligned}
$$

$$- \delta_a\delta_b^{-1}([b_{2k-1}a_{2k}a_{2k-1}a_{2k}] + [b_{4n-2k+1}a_{4n-2k}a_{4n-2k+1}a_{4n-2k}]).$$

We can compute $F * l$ for a loop $l \in \mathscr{G}_{2n,+}$ by the following fact,

$$[y_0y_1y_2y_3] * [x_0x_1\cdots x_{4n-1}] = \delta_{y_1x_1}\delta_{y_2x_0}\delta_{y_3x_{4n-1}}\sqrt{\frac{\lambda'(y_2)\lambda'(y_2)}{\lambda'(y_1)\lambda'(y_3)}}[y_0x_1\cdots x_{4n-1}].$$

Proposition 4.20. *For a loop $l \in \mathscr{G}_{2n,+}$ and $1 \leq k \leq 2n$, we have*

$$\begin{aligned} &F * l = 0, && \text{when } l = A_k^- l A_k^-, \\ &F * l = l, && \text{when } l = A_k^- l A_k^+ \text{ or } l = A_k^+ l A_k^-; \\ &F * l = (A_k^+ + B_k)(F * l)(A_k^+ + B_k), && \text{when } l = (A_k^+ + B_k)l(A_k^+ + B_k). \end{aligned}$$

So $\mathscr{G}_{2n,+}$ is separated into $6n$ invariant subspaces under the action $F*$. Moreover the set of length $4n$ loops, as a basis of $\mathscr{G}_{2n,+}$, is separated into $6n$ subsets simultaneously.

Proof. It could be checked by a direct computation. □

Definition 4.21. Let $\beta : A_k^+\mathscr{G}_{2n,+}A_k^+ \to B_k\mathscr{G}_{2n,+}$, $\forall\, 1 \leq k \leq 2n$, be the linear extension of

$$\beta([a_{2k-1}a_{2k}x_3x_4\cdots x_{2n-1}a_{2k}]) = [b_{2k-1}a_{2k}x_3x_4\cdots x_{2n-1}a_{2k}],$$

for any loop $[a_{2k-1}a_{2k-2}x_3x_4\cdots x_{2n-1}a_{2k-2}] \in A_k^+\mathscr{G}_{2n,+}A_k^+$, $1 \leq k \leq n$;

$$\beta([a_{2k-1}a_{2k-2}x_3x_4\cdots x_{2n-1}a_{2k-2}]) = [b_{2k-1}a_{2k-2}x_3x_4\cdots x_{2n-1}a_{2k-2}],$$

for any loop $[a_{2k-1}a_{2k-2}x_3x_4\cdots x_{2n-1}a_{2k-2}] \in A_k^+\mathscr{G}_{2n,+}A_k^+$, $n+1 \leq k \leq 2n$.

Proposition 4.22. *The linear map $\beta : A_k^+\mathscr{G}_{2n,+}A_k^+ \to B_k\mathscr{G}_{2n,+}$ is a *-isomorphism. Moreover,*

$$\begin{aligned} F * x &= \delta_b^{-2}x - \delta_b^{-1}\beta(x), && \forall\, x \in A_k^+\mathscr{G}_{2n,+}A_k^+; \\ F * y &= \delta_b^{-1}y - \delta_b^{-2}\beta^{-1}(y), && \forall\, y \in B_k\mathscr{G}_{2n,+}; \\ \beta(A_k^+g_{2n}) &= B_kg_{2n}. \end{aligned}$$

Proof. It is obvious that β is a *-isomorphism. It is easy to check the first two formulas by a direct computation. For the third formula, by Proposition 4.20 and the fact that $F * g_{2n} = 0$, we have

$$F * ((A_k^+ + B_k)g_{2n}(A_k^+ + B_k)) = 0.$$

By Proposition 4.16, we have

$$F * (A_k^+ g_{2n}) = -F * (B_k g_{2n}).$$

Then

$$\delta_b^{-2}(A_k^+ g_{2n}) - \delta_b^{-1}\beta(A_k^+ g_{2n}) = -\delta_b^{-1}(B_k g_{2n}) + \delta_b^{-2}\beta^{-1}(B_k g_{2n}).$$

So

$$\beta(A_k^+ g_{2n}) = B_k g_{2n}. \qquad \square$$

Lemma 4.23. *In the graph planar algebra,*

$$A_k^- R A_k^- = 0, \ \forall\, 1 \leq k \leq 2n;$$
$$H_i \mathcal{F}(R) H_i = 0, \ \forall\, 1 \leq i \leq 4n.$$

Proof. By proposition (3'), R is totally uncappable, so $R = F * R$. By Proposition 4.20, we have

$$(A_k^- R A_k^-) = F * (A_k^- R A_k^-) = 0.$$

Note that

$$\sum_{1 \leq i \leq 4n} H_i \mathcal{F}(R) H_i = \mathcal{F}(R p_1) = 0,$$

so

$$H_i \mathcal{F}(R) H_i = 0, \quad \forall\, 1 \leq i \leq 4n. \qquad \square$$

Lemma 4.24. *In the graph planar algebra,*

$A_1^- R A_1^+$ is a multiple of the loop $[a_1 a_{4n} a_{4n-1} \cdots a_2]$, denote by L_1;
$A_{2n}^- R A_{2n}^+$ is a multiple of the loop $[a_{4n-1} a_0 a_1 \cdots a_{4n-2}]$, denote by L_2.

Proof. Note that the coefficient of a loop $l = [a_1 a_{4n} x_3 x_4 \cdots x_{4n-1} a_2]$ in $A_1^- R A_1^+$ is the same as the coefficient of l in R. If it is non-zero, then by proposition (4'), the coefficient of $\mathcal{F}^{-2k+1}(l)$ in $\mathcal{F}(R)$ is non-zero and the coefficient of $\mathcal{F}^{-2k}(l)$ in R is non-zero. Applying Lemma 4.23, we have

$$H_1 \mathcal{F}(R) H_1 = 0 \Rightarrow x_3 = a_{4n-1};$$
$$A_{4n-1}^- R A_{4n-1}^- = 0 \Rightarrow x_4 = a_{4n-2};$$

and for $k = 1, 2, \cdots, n$,

$$H_{4n+3-2k}\mathcal{F}(R)H_{4n+3-2k} = 0 \Rightarrow x_{2k+1} = a_{4n+1-2k};$$
$$A^-_{4n+1-2k}RA^-_{4n+1-2k} = 0 \Rightarrow x_{2k+2} = a_{4n-2k}.$$

For the rest, there is only one length $2n-2$ path from a_{2n} to a_2. So

$$l = [a_1a_{4n}a_{4n-1}\cdots a_2] = L_1.$$

That means $A^-_1RA^+_1$ is a multiple of L_1. Similarly $A^-_{2n}RA^+_{2n}$ is a multiple of L_2. □

Definition 4.25. For a loop $l = [x_0x_1\cdots x_{4n-1}]$ and $0 \le k \le 4n-1$, the point x_k is said to be a cusp point of the loop l, if $x_{k-1} = x_{k+1}$, where $x_{-1} = x_{2n-1}$, $x_{2n} = x_0$. Otherwise it is said to be a flat point.

Similar to the proof of Lemma 4.24, Lemma 4.23 tells that if the coefficient of a loop $l = [x_0x_1\cdots x_{4n-1}]$ in R is non-zero, then a cusp point x_k of l has to be b_{2i-1} or a_{2i-1}. In this case, we have $x_{k-1} = x_{k+1} = a_{2i}$, when $1 \le i \le n$; Or $x_{k-1} = x_{k+1} = a_{2i-2}$, when $n+1 \le i \le 2n$. Furthermore if l passes the point a_0, then it is unique up to rotation and the adjoint operation $*$; If l does not pass the point a_0, then it is determined by its first point and cusp points. So we can simplify the expression of a loop by its first point and cusp points. To compute the product of two loops, we also need the middle point x_{2n}. Then the loop is separated into two length $2n$ paths from the first point to the middle point. We can label the two paths by the first point, cusp points and the middle point.

Definition 4.26. For a loop $l = [x_0x_1\cdots x_{4n-1}]$, $x_k \neq a_0$, $\forall\, 0 \le k \le 4n-1$, we assume that $y_1, y_2, \cdots, y_i$ are the cusp points from x_1 to x_{2n-1} and $z_1, z_2, \cdots, z_j$ are the cusp points from x_{2n+1} to x_{4n-1}. Then we use $[x_0y_1y_2\cdots y_ix_{2n}\rangle_c$ to express the first length $2n$ path of l, $\langle x_{2n}z_1z_2\cdots z_jx_0]_c$ to express the second length $2n$ path of l and $[x_0y_1y_2\cdots y_ix_{2n}\rangle_c\langle x_{2n}z_1z_2\cdots z_jx_0]_c$ to express the loop l. Furthermore if x_{2n} is a cusp point, then it could be simplified as $[x_0y_1y_2\cdots y_ix_{2n}z_1z_2\cdots z_jx_0]_c$; if x_{2n} is a flat point, then it could be simplified as $[x_0y_1y_2\cdots y_iz_1z_2\cdots z_jx_0]_c$.

All the loops in the rest of the paper have length $4n$.

Definition 4.27. Suppose $R \in \mathscr{G}_{2n,+}$ is a solution of Proposition 4.9, i.e. R satisfies the following propositions,

(1') $R^* = R$;
(2') $R + \delta_b^{-2}g_{2n}$ is a projection;
(3') R is totally uncappable;
(4') $\rho(R) = \omega R$, for some $\omega \in \mathbb{C}$ satisfying $|\omega| = 1$.

Let us define U_k, P_k, Q_k, $\overline{P}_k$, $\overline{Q}_k$, R_k for $1 \leq k \leq 2n$ as follows

$$\begin{aligned} U_k &= A_k^- R A_k^+; \\ \overline{P}_k &= \delta_b^{-2}(R - \delta_b^{-1} g_{2n}) B_k; \\ \overline{Q}_k &= \delta_b^{-1}(R + \delta_b^{-2} g_{2n}) B_k; \\ P_k &= -\delta_b^{-1}\beta^{-1}(\overline{P}_k); \\ Q_k &= -\delta_b^{-1}\beta^{-1}(\overline{Q}_k); \\ R_k &= (A_k^+ + B_k) R (A_k^+ + B_k). \end{aligned}$$

The following lemma is the key to solve the generator R in the graph planar algebra $\mathscr{G}_{2n,+}$. It allows us to construct the unique possible solution in $\mathscr{G}_{2n,+}$ inductively for all n simultaneously.

Lemma 4.28. *The element R is uniquely determined by μ_1, μ_2 and ω. Precisely*

$$\begin{aligned} U_1 &= \mu_1 \delta_b^{-1.5} L_1, && \text{for some } \mu_1 \in \mathbb{C},\ |\mu_1| = 1; \\ U_{2n} &= \mu_2 \delta_b^{-1.5} L_2, && \text{for some } \mu_2 \in \mathbb{C},\ |\mu_2| = 1; \\ P_k &= U_k^* U_k, && \text{for } 1 \leq k \leq 2n; \\ R_k &= \delta_b^4 F * P_k * F, && \text{for } 1 \leq k \leq 2n; \\ U_{k+1} &= \omega^{-1}\rho(R_k + U_k) && \text{and} \\ U_{2n-k} &= \omega^{-1}\rho(R_{2n-k+1} + U_{2n-k+1}), && \text{for } 1 \leq k \leq n-1; \\ R &= \sum_{1 \leq k \leq 2n} U_k + U_k^* + R_k. \end{aligned}$$

Proof. For $1 \leq k \leq 2n$, by definition, we have

$$RB_k = -\delta_b^{-2}(\delta_b^{-1} g_{2n} - R) B_k + \delta_b^{-1}(R + \delta_b^{-2}) B_k = \overline{P}_k + \overline{Q}_k.$$

By proposition (2')(3'), we have $R + \delta_b^{-2} g_{2n}$ is a subprojection of g_{2n}. Then

$$g_{2n} - (R + \delta_b^{-2} g_{2n}) = \delta_b^{-1} g_{2n} - R$$

is a projection. So

$$\delta_b \overline{Q}_k = (R + \delta_b^{-2}) B_k,\ -\delta_b^2 \overline{P}_k = (\delta_b^{-1} g_{2n} - R) B_k$$

are projections, by Proposition 4.16. Note that

$$R_k = (A_k^+ + B_k) R (A_k^+ + B_k) = A_k^+ R A_k^+ + B_k R B_k,$$

so $F * R_k = R_k$, by Proposition 4.20. Furthermore by Proposition 4.22, we have

$$F * R_k = \delta_b^{-2} A_k^+ R A_k^+ - \delta_b^{-1}\beta(A_k^+ R A_k^+) + \delta_b^{-1} B_k R B_k - \delta_B^{-2}\beta^{-1}(B_k R B_k).$$

Thus

$$A_k^+ R A_k^+ = \delta_b^{-2} A_k^+ R A_k^+ - \delta_b^{-2}\beta^{-1}(B_k R B_k).$$

Then

$$A_k^+ R A_k^+ = -\delta_b^{-1}\beta^{-1}(B_k R B_k) = -\delta_b^{-1}\beta^{-1}(\overline{P}_k + \overline{Q}_k) = P_k + Q_k.$$

By Proposition 4.22, we have

$$A_k^+ g_{2n} = \beta^{-1}(B_k g_{2n}) = \beta^{-1}(-\delta_b^2 \overline{P}_k + \delta_b \overline{Q}_k) = \delta_b^3 P_k - \delta_b^2 Q_k,$$

and $\delta_b^3 P_k$, $-\delta_b^2 Q_k$ are projections. Then

$$A_k^+(R + \delta_b^{-2} g_{2n})A_k^+ = (P_k + Q_k) + (\delta_b P_k - Q_k) = \delta_b^2 P_k.$$

By Proposition (4.16)(4.23) and proposition (1'), we have

$$\begin{bmatrix} A_k^-(R+\delta_b^{-2}g_{2n})A_k^- & A_k^-(R+\delta_b^{-2}g_{2n})A_k^+ \\ A_k^+(R+\delta_b^{-2}g_{2n})A_k^- & A_k^+(R+\delta_b^{-2}g_{2n})A_k^+ \end{bmatrix} = \begin{bmatrix} \delta_b^{-2}A_k^- g_{2n} & U_k \\ U_k^* & \delta_b^2 P_k \end{bmatrix}$$

Recall that $R + \delta_b^{-2} g_{2n}$ is a projection, so $A_k(R + \delta_b^{-2} g_{2n})$ is a projection. Then the matrix

$$\begin{bmatrix} \delta_b^{-2}A_k^- g_{2n} & U_k \\ U_k^* & \delta_b^2 P_k \end{bmatrix}$$

is a projection. While $A_k^- g_{2n}$ and $\delta_b^3 P_1$ are projections, so $\delta_b^{1.5} U_k$ is a partial isometry from $\delta_b^3 P_1$ to $A_k^- g_{2n}$. Then

$$(\delta_b^{1.5} U_k)^*(\delta_b^{1.5} U_k) = \delta_b^3 P_k; \quad (\delta_b^{1.5} U_k)(\delta_b^{1.5} U_k)^* = A_k^- g_{2n}.$$

Therefore

$$U_k^* U_k = P_k \text{ and } U_1 U_1^* = \delta_b^{-3} A_1^- g_{2n}.$$

Observe that $[a_1 a_{4n} a_{4n-1} \cdots a_{2n+2} a_{2n+1} a_{2n+2} \cdots a_{4n}]_c$ is a subprojection of $A_1^- g_{2n}$. So $A_1^- g_{2n} \neq 0$. Then $U_1 \neq 0$. By Lemma 4.24, we have

$$U_1 = \mu_1 \delta_b^{-1.5} L_1, \text{ for some } \mu_1 \in \mathbb{C},\ |\mu_1| = 1;$$

Symmetrically

$$U_{2n} = \mu_2 \delta_b^{-1.5} L_2, \text{ for some } \mu_2 \in \mathbb{C}, \ |\mu_2| = 1;$$

Note that

$$B_k(x * F) = (B_k x) * F, \ \forall x \in \mathscr{G}_{2n,+},$$

so

$$\delta_b^2 \overline{P}_k * F = (B_k R) * F - \delta_b(B_k g_{2n}) * F = B_k(R * F) - \delta_b B_k(g_{2n} * F) = B_k R = \overline{P}_k + \overline{Q}_k.$$

Observe that

$$\beta^{-1}(y * F) = \beta^{-1}(y) * F, \ \forall y \in B_k \mathscr{G}_{2n,+},$$

so

$$\delta_b^2 P_k * F = P_k + Q_k.$$

By Proposition 4.22, we have

$$\delta_b^2 F * P_k = P_k - \delta_b \beta(P_k) = P_k + \overline{P}_k.$$

So

$$\begin{aligned}
\delta_b^4 F * P_k * F &= \delta_b^2 (P_k + \overline{P}_k) * F \\
&= P_k + Q_k + \overline{P}_k + \overline{Q}_k \\
&= A_k^+ R A_k^+ + R B_k \\
&= (A_k^+ + B_k) R (A_k^+ + B_k) \\
&= R_k.
\end{aligned}$$

Note that ρ induces a one onto one map from the loops of $\mathscr{G}_{2n,+}(A_k^+ + B_k)$ to loops of $A_{k+1}^- \mathscr{G}_{2n,+} A_{k+1}^+$, for $1 \le k \le n-1$. So

$$\rho(R(A_k^+ + B_k)) = A_{k+1}^- \rho(R) A_{k+1}^+.$$

Then by proposition (4'), we have

$$\rho(R(A_k^+ + B_k)) = \omega A_{k+1}^- R A_{k+1}^+.$$

While

$$R(A_k^+ + B_k) = (A_k^+ + B_k) R (A_k^+ + B_k) + A_k^- R (A_k^+) = R_k + U_k,$$

thus

$$U_{k+1} = \omega^{-1}\rho(R_k + U_k).$$

By a similar calculation, we see that

$$U_{2n-k} = \omega^{-1}\rho(R_{2n-k+1} + U_{2n-k+1}).$$

Finally

$$\begin{aligned} R &= \sum_{1\leq k\leq 2n} (A_k + B_k)R(A_k + B_k) \\ &= \sum_{1\leq k\leq 2n} (A_k^- + A_k^+ + B_k)R(A_k^- + A_k^+ + B_k) \\ &= \sum_{1\leq k\leq 2n} A_k^- R A_k^+ + A_k^+ R A_k^- + (A_k^+ + B_k)R(A_k^+ + B_k) \\ &= \sum_{1\leq k\leq 2n} U_k + U_k^* + R_k. \end{aligned}$$

Given μ_1, μ_2 and ω, U_k, P_k, R_k could be obtained inductively. So R is uniquely determined by μ_1, μ_2 and ω. □

Remark 4.29. Note that the dual principal graph has a $\mathbb{Z}_2$ symmetry. When $\mu_1 = \mu_2 = \omega = 1$, the above process for calculating R is also symmetric. Thus the coefficient of a loop in R does not change by switching a_{2k-1} to $a_{4n-2k+1}$.

4.3. Solutions

Definition 4.30. Based on Lemma 4.28, for fixed $\mu_1, \mu_2, \omega \in \mathbb{C}$, $|\mu_1| = |\mu_2| = |\omega| = 1$, let us construct the unique possible generator $R_{\mu_1\mu_2\omega} \in \mathscr{G}_{2n,+}$ inductively,

$$\begin{aligned} U_1 &= \mu_1 \delta_b^{-1.5} L_1; && \\ U_{2n} &= \mu_2 \delta_b^{-1.5} L_2; && \\ P_k &= U_k^* U_k, && \text{for } 1 \leq k \leq 2n; \\ R_k &= \delta_b^4 F * P_k * F, && \text{for } 1 \leq k \leq 2n; \\ U_{k+1} &= \omega^{-1}\rho(R_k + U_k) && \text{and} \\ U_{2n-k} &= \omega^{-1}\rho(R_{2n-k+1} + U_{2n-k+1}), && \text{for } 1 \leq k \leq n-1; \\ R_{\mu_1\mu_2\omega} &= \sum_{1\leq k\leq 2n} U_k + U_k^* + R_k. && \end{aligned}$$

We hope to check proposition (1')(2')(3')(4') for $R_{\mu_1\mu_2\omega}$. Actually propositions (1')(2')(3') are satisfied, but not obvious. Proposition (4') fails, when $n \geq 4$. We are going to compute the coefficients of loops in $R_{\mu_1\mu_2\omega}$. If proposition (4') is satisfied, then their absolute values are determined by the coefficients of loops in R_k.

Lemma 4.31. *$R_{\mu_1\mu_2\omega}$ is totally uncappable.*

Proof. Note that U_1 is totally uncappable. So

$$g_{2n}U_1g_{2n} = U_1.$$

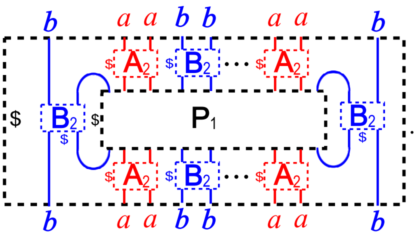
Then

$$g_{2n}P_1g_{2n} = P_1.$$

By the exchange relation of the biprojection, we have

$$g_{2n}(F * P_1 * F)g_{2n} = F * (g_{2n} * P_1 * g_{2n}) * F = F * p_1 * F.$$

As an a/b-colour diagram, it is



Therefore $R_1 = F * P_1 * F$ is totally uncappable. Then $U_2 = \omega^{-1}\rho(R_1)$ is totally uncappable. Inductively we have U_k, R_k are totally uncappable, for $k = 1, 2, \cdots, n$. Symmetrically U_i, R_i are totally uncappable, for $i = 2n, 2n-1, \cdots, n+1$. So $R_{\mu_1\mu_2\omega} = \sum_{1\leq k\leq 2n} U_k + U_k^* + R_k$ is totally uncappable. $\square$

Lemma 4.32. *For $1 \leq k \leq 2n$, R_k does not depend on the parameters μ_1, μ_2 and ω.*

Proof. Note that $P_1 = U_1^*U_1$ does not depend on the parameters, so $R_1 = \delta_b^4 F * P_1 * F$ does not depend on the parameters. We use the second principle of mathematical induction. For $k = 1, 2, \cdots, n-1$, assume that R_i, for any $i \leq k$, does not depend on the parameters. Note that

$$\begin{aligned} P_{k+1} &= U_k^*U_k \\ &= \rho(R_k + U_k)^*\rho(R_k + U_k) \end{aligned}$$

$$= \rho(R_k)^*\rho(R_k) + \rho(U_k)^*\rho(U_k)$$
$$= \cdots$$
$$= \rho(R_k)^*\rho(R_k) + \rho^2(R_{k-1})^*\rho^2(R_{k-1}) + \cdots + \rho^k(R_1)^*\rho^k(R_1) + \rho^k(U_1)^*\rho^k(U_1)^k.$$

Moreover $\rho^k(U_1)^*\rho^k(U_1)$ does not depend on the parameters. So P_{k+1} does not depend on the parameters. Then $R_{k+1} = \delta_b^4 F * P_{k+1} * F$ does not depend on the parameters. For $n+1 \leq k \leq 2n$, the proof is similar. □

To compute R_k, we can fix the parameters as $\mu_1 = \mu_2 = \omega = 1$ first. Now let us compute the coefficients of loops in $R = R_{111}$.

Definition 4.33. For a loop $l \in \mathscr{G}_{2n,+}$, let us define $C_R(l)$ to be the coefficient of l in $R = R_{111}$. Let us define $C_P(l)$ to be the coefficient of l in $P = \sum_{1\leq k\leq 2n} P_k$.

If a loop l' has a cusp point b_{2i-1}, then we can substitute b_{2i-1} by a_{2i-1} to obtain another loop l. By Proposition (4.20)(4.22) and Lemma 4.31, we have $C_R(l')$ is determined by $C_R(l)$. Essentially we only need to compute the coefficients of loops whose cusp points are just $a_j's$. Their relations are given by the following lemma.

Lemma 4.34. *For a loop $l_1' \in \mathscr{G}_{2n,+}$, $l_1' = [x_0 \cdots b_{2i-1} \cdots x_{2n}\rangle_c\langle x_{2n} \cdots x_0]_c$, we have*

$$C_R(l_1') = -\delta_b^{\frac{1}{2}} C_R(l_1),$$

where $l_1 = [x_0 \cdots a_{2i-1} \cdots x_{2n}\rangle_c\langle x_{2n} \cdots x_0]_c$ is the loop replacing the given point b_{2i-1} by a_{2i-1} in l_1'.

For a loop $l_2 \in A_k^+\mathscr{G}_{2n+}A_k^+$, $l_2 = [a_{2k-1} \cdots a_{2m-1}\rangle_c\langle a_{2m-1} \cdots a_{2k-1}]_c$, we have

$$C_R(l_2) = \begin{cases} \delta_b^2 C_P(l_2), & \text{when the middle point } a_{2m-1} \text{ is a flat point;} \\ C_P(l_2) - C_P(l_2'), & \text{when the middle point } a_{2m-1} \text{ is a cusp point,} \end{cases}$$

where $l_2' = [a_{2k-1} \cdots b_{2m-1}\rangle_c\langle b_{2m-1} \cdots a_{2k-1}]_c$ is the loop replacing the middle point a_{2m-1} by b_{2m-1} in l_2.

Proof. For a loop $l_1' \in \mathscr{G}_{2n+}$,

$$l_1' = [x_0 \cdots x_{2k-1} b_{2i-1} x_{2k+1} \cdots x_{2n}\rangle_c\langle x_{2n} x_{2n+1} \cdots x_{4n-1} x_0]_c,$$

we take l_1 to be the loop

$$l_1 = [x_0 \cdots x_{2k-1} a_{2i-1} x_{2k+1} \cdots x_{2n}\rangle_c\langle x_{2n} x_{2n+1} \cdots x_{4n-1} x_0]_c.$$

Assume that

$$l_0' = [b_{2i-1} x_{2k+1} \cdots x_{2n+2k}\rangle_c\langle x_{2n+2k} \cdots x_{4n-1} x_0 \cdots x_{2k-1} b_{2i-1}]_c$$

and

$$l_0 = [a_{2i-1}x_{2k+1}\cdots x_{2n+2k}\rangle_c\langle x_{2n+2k}\cdots x_{4n-1}x_0\cdots x_{2k-1}a_{2i-1}]_c.$$

Then the coefficient of l_0' in $\rho^{-k}(R)$ is

$$\sqrt{\frac{\lambda'(x_0)\lambda'(x_{2n})}{\lambda'(b_{2i-1})\lambda'(x_{2n+2k})}}C_R(l_1');$$

and the coefficient of l_0 in $\rho^{-k}(R)$ is

$$\sqrt{\frac{\lambda'(x_0)\lambda'(x_{2n})}{\lambda'(a_{2i-1})\lambda'(x_{2n+2k})}}C_R(l_1).$$

By Proposition 4.20, the linear space spanned by l_0, l_0' is invariant under the coproduct of F on the left side. By Lemma 4.31, we have

$$F*(\rho^{-k}(R)) = \rho^{-k}(R).$$

So

$$\sqrt{\frac{\lambda'(x_0)\lambda'(x_{2n})}{\lambda'(b_{2i-1})\lambda'(x_{2n+2k})}}C_R(l_1')l_0' + \sqrt{\frac{\lambda'(x_0)\lambda'(x_{2n})}{\lambda'(a_{2i-1})\lambda'(x_{2n+2k})}}C_R(l_1)l_0$$

is invariant under the coproduct of F on the left side. By Proposition 4.22, we have

$$\sqrt{\frac{\lambda'(x_0)\lambda'(x_{2n})}{\lambda'(b_{2i-1})\lambda'(x_{2n+2k})}}C_R(l_1') + \delta_b\sqrt{\frac{\lambda'(x_0)\lambda'(x_{2n})}{\lambda'(a_{2i-1})\lambda'(x_{2n+2k})}}C_R(l_1) = 0.$$

Thus

$$C_R(l_1') = -\delta_b^{\frac{1}{2}}C_R(l_1).$$

For a loop $l_2 \in A_k^+\mathscr{G}_{2n+}A_k^+$, $l_2 = [a_{2k-1}\cdots a_{2m-1}\rangle_c\langle a_{2m-1}\cdots a_{2k-1}]_c$, we have

$$C_R(l_2) = \frac{tr(Rl_2^*)}{tr(l_2l_2^*)} = \frac{tr(R_kl_2^*)}{tr(l_2l_2^*)} = \delta_b^4\frac{tr((F*P_k*F)l_2^*)}{tr(l_2l_2^*)}.$$

Note that

$$tr((F*P_k*F)l_2^*) = tr(P_k(F*l_2^**F))$$

by a diagram isotopy. So

$$C_R(l_2) = \delta_b^4 \frac{tr(P_k(F * l_2^* * F))}{tr(l_2 l_2^*)} = \delta_b^4 \frac{tr(P_k(F * l_2 * F)^*)}{tr(l_2 l_2^*)}.$$

If a_{2m-1} is a flat point, then $l_2 * F = l_2$, by a direct computation. By Proposition 4.22, we have

$$F * l_2 = \delta_b^{-2} l_2 - \delta_b^{-1} \beta(l_2).$$

So

$$C_R(l_2) = \delta_b^4 \delta_b^{-2} \frac{tr(P_k l_2^*)}{tr(l_2 l_2^*)} = \delta_b^2 C_P(l_2).$$

If a_{2m-1} is a cusp point, then

$$l_2 * F = \delta_b^{-2} l_2 - \delta_b^{-1} l_2',$$

by Proposition 4.22 and an 180° rotation, where $l_2' = [a_{2k-1} \cdots b_{2m-1}\rangle_c \langle b_{2m-1} \cdots a_{2k-1}]_c$ is the loop replacing the middle point a_{2m-1} by b_{2m-1} in l_2. Again by Proposition 4.22, we have

$$F * l_2 * F = \delta_b^{-4} l_2 - \delta_b^{-3} \beta(l_2) - \delta_b^{-3} l_2' + \delta_b^{-2} \beta(l_2').$$

So

$$C_R(l_2) = \delta_b^4 \delta_b^{-4} \frac{tr(P_k l_2^*)}{tr(l_2 l_2^*)} - \delta_b^4 \delta_b^{-3} \frac{tr(P_k l_2'^*)}{tr(l_2 l_2^*)}.$$

Observe that

$$tr(l_2 l_2^*) = \delta_b tr(l_2' l_2'^*).$$

Therefore

$$C_R(l_2) = C_P(l_2) - C_P(l_2'). \qquad \square$$

Note that $P_k = U_k^* U_k$, to compute the coefficient of a loop in P_k we only need the coefficients of loops in U_k. They are determined by the coefficients of loops in R_{k-1}.

Definition 4.35. For $1 \leq k \leq n$, let us define $[a_{2k-1}, y\rangle_c$ to be the set of all length $2n$ paths from a_{2k-1} to y starting with $a_{2k-1}a_{2k-2}$. For a path $\eta = [z_0 z_1 \cdots z_{k-1} z_k\rangle_c$, let us define η^* to be the path $\langle z_k, z_{k-1}, \cdots, z_1, z_0]_c$.

Lemma 4.36. *For a loop $\eta_1\eta_2^* \in A_k^+\mathscr{G}_{2n,+}A_k^+$ whose first point is a_{2k-1}, suppose its middle point is y. Then we have*

$$C_P(\eta_1\eta_2^*) = \sum_{\eta\in[a_{2k-1}y\rangle_c} C_R(\eta_1\eta^*)C_R(\eta\eta_2^*).$$

Proof. Note that a length $2n$ path $\eta \in [a_{2k-1}y\rangle_c$ starts with $a_{2k-1}a_{2k-2}$, so $C_R(\eta^*\eta_2)$ is the coefficient of $\eta^*\eta_2$ in U_k and $C_R(\eta_1\eta^*)$ is the coefficient of $\eta_1\eta^*$ in U_k^*. Then the statement follows from the fact $P_k = U_k^*U_k$. □

When the initial condition $\mu_1 = \mu_2 = \omega = 1$ is fixed, given a loop

$$l = [a_{k_1}a_{2n+k_2}a_{2n-k_3}\cdots a_{2n+k_{2t}}a_{k_1}]_c, \text{ for } 1 \leq k_1, k_2, \cdots, k_{2t} \leq 2n-1,$$

we can compute $C_R(l)$ by repeating Lemma (4.34)(4.36). A significant fact is that the computation only depends on $k_1, k_2, \cdots, k_{2t}$, in other words, $C_R(l)$ is independent of n. We list all the coefficients for $k_1 \leq 7$ in Appendix A. This is enough to rule out the 4th fish by comparing the coefficients $C_R([a_5a_9a_5a_9a_5]_c)$ and $C_R([a_7a_{11}a_7a_{11}a_7]_c)$. It is possible to rule out finitely many Bisch–Haagerup fish graphs by computing more coefficients. To rule out the nth Bisch–Haagerup fish graph, for all $n \geq 4$, we need formulas for the coefficients of two families of loops which do not match the proposition (4'). Then only the first three Bisch–Haagerup fish graphs are the principal graphs of subfactors.

Lemma 4.37.

$$C_R([a_{2k-1}a_{2n+2k-1}a_{2k-1}]_c) = \delta_b^{-3}, \forall\, 1 \leq k \leq n.$$

Proof. For $1 \leq k \leq n$, by Lemma 4.34, we have

$$\begin{aligned} &C_R([a_{2k-1}a_{2n+2k-1}a_{2k-1}]_c) \\ &= C_P([a_{2k-1}a_{2n+2k-1}a_{2k-1}]_c) - C_P([a_{2k-1}b_{2n+2k-1}a_{2k-1}]_c). \end{aligned}$$

By Lemma 4.36, we have

$$\begin{aligned} &C_P([a_{2k-1}a_{2n+2k-1}a_{2k-1}]_c) \\ &= C_R([a_{2k-1}\cdots a_{4n-1}a_0\cdots a_{2k-1}a_{2k-2}]_c)C_R([a_{2k-1}a_{2k-2}\cdots a_0a_{4n-1}\cdots a_{2k}]_c), \end{aligned}$$

because

$$[a_{2k-1}a_{2n+2k-1}a_{2k-1}]_c = [a_{2k-1}a_{2n+2k-1}\rangle_c\langle a_{2n+2k-1}a_{2k-1}]_c,$$

and $a_{2k-1}a_{2k-2}\cdots a_0a_{4n-1}\cdots a_{2n+2k-1}$ is the unique path in $[a_{2k-1}, a_{2n+2k-1}\rangle_c$. Note that

$$[a_{2k-1}\cdots a_{4n-1}a_0\cdots a_{2k-1}a_{2k-2}]_c^* = [a_{2k-1}a_{2k-2}\cdots a_0a_{4n-1}\cdots a_{2k}]_c,$$

and $R = R^*$, so

$$C_R([a_{2k-1} \cdots a_{4n-1}a_0 \cdots a_{2k-1}a_{2k-2}]_c) = \overline{C_R([a_{2k-1}a_{2k-2} \cdots a_0a_{4n-1} \cdots a_{2k}]_c)}.$$

Observe that

$$\begin{aligned}
&\rho[a_1a_0a_{4n-1} \cdots a_2]_c \\
&= \sqrt{\frac{\lambda'(a_1)\lambda'(a_{2n+1})}{\lambda'(a_{2k-1})\lambda'(a_{2n+2k-1})}}[a_{2k-1}a_{2k-2} \cdots a_1a_0a_{4n-1} \cdots a_{2k}]_c \\
&= [a_{2k-1}a_{2k-2} \cdots a_1a_0a_{4n-1} \cdots a_{2k}]_c,
\end{aligned}$$

and $\rho(R) = R$, (we assumed that $\mu_1 = \mu_2 = \omega = 1$), so

$$C_R([a_{2k-1}a_{2k-2} \cdots a_0a_{4n-1} \cdots a_{2k}]_c) = C_R([a_1a_0a_{4n-1} \cdots a_2]_c) = \delta_b^{-1.5}.$$

Then

$$C_P([a_{2k-1}a_{2n+2k-1}a_{2k-1}]_c) = \delta_b^{-3}.$$

On the other hand,

$$[a_{2k-1}b_{2n+2k-1}a_{2k-1}]_c = [a_{2k-1}b_{2n+2k-1}\rangle_c\langle b_{2n+2k-1}a_{2k-1}]_c,$$

but there is no path in $[a_{2k-1}, b_{2n+2k-1}]_c$, so

$$C_P([a_{2k-1}b_{2n+2k-1}a_{2k-1}]_c) = 0.$$

Then

$$C_R([a_{2k-1}a_{2n+2k-1}a_{2k-1}]_c) = \delta_b^{-3}. \qquad \square$$

Lemma 4.38.

$$\begin{aligned}
&C_R([a_{2k-1}a_{2n+1}a_{2n-2k+3}a_{2n+1}a_{2k-1}]_c) = \delta_b^{-5}, && \forall\, 2 \le k \le n; \\
&C_R([a_{2k-1}a_{2n+1}a_{2n-1}a_{2n+2k-3}a_{2k-1}]_c) = \delta_b^{-5.5}, && \forall\, 3 \le k \le n; \\
&C_R([a_{2k-1}a_{2n+2k-3}a_{2n-1}a_{2n+1}a_{2k-1}]_c) = \delta_b^{-5.5}, && \forall\, 3 \le k \le n.
\end{aligned}$$

Proof. For $2 \le k \le n$, by Lemma 4.34, we have

$$\begin{aligned}
&C_P([a_{2k-1}a_{2n+1}a_{2n-2k+3}a_{2n+1}a_{2k-1}]_c) \\
&= C_P([a_{2k-1}a_{2n+1}a_{2n-2k+3}\rangle_c\langle a_{2n-2k+3}a_{2n+1}a_{2k-1}]_c)
\end{aligned}$$

$$
\begin{aligned}
&= C_R([a_{2k-1}a_{2n+1}a_{2n-2k+3}\rangle_c\langle a_{2n-2k+3}a_1a_{2k-1}]_c) \\
&\times C_R([a_{2k-1}a_1a_{2n-2k+3}\rangle_c\langle a_{2n-2k+3}a_{2n+1}a_{2k-1}]_c) \\
&+ C_R([a_{2k-1}a_{2n+1}a_{2n-2k+3}\rangle_c\langle a_{2n-2k+3}b_1a_{2k-1}]_c) \\
&\times C_R([a_{2k-1}b_1a_{2n-2k+3}\rangle_c\langle a_{2n-2k+3}a_{2n+1}a_{2k-1}]_c)
\end{aligned}
$$

By Lemma 4.34, we have

$$
\begin{aligned}
&C_R([a_{2k-1}b_1a_{2n-2k+3}\rangle_c\langle a_{2n-2k+3}a_{2n+1}a_{2k-1}]_c \\
&= -\delta_b^{0.5}C_R([a_{2k-1}a_1a_{2n-2k+3}\rangle_c\langle a_{2n-2k+3}a_{2n+1}a_{2k-1}]_c.
\end{aligned}
$$

So the formula is simplified as

$$
\begin{aligned}
&C_P([a_{2k-1}a_{2n+1}a_{2n-2k+3}a_{2n+1}a_{2k-1}]_c) \\
&= \delta_b^2 C_R([a_{2k-1}a_{2n+1}a_{2n-2k+3}\rangle_c\langle a_{2n-2k+3}a_1a_{2k-1}]_c) \\
&\times C_R([a_{2k-1}a_1a_{2n-2k+3}\rangle_c\langle a_{2n-2k+3}a_{2n+1}a_{2k-1}]_c),
\end{aligned}
$$

where δ_b^2 is given by $1 + (-\delta_b^{0.5})^2 = \delta_b^2$.

We see that the cusp point of a path in $[a_{2k-1}a_{2n-2k+3}\rangle_c$ could be a_1 or b_1, but we can ignore the path with the cusp point b_1 by adding a factor δ_b^2.

Moreover,

$$
\begin{aligned}
&C_R([a_{2k-1}a_1a_{2n+1}a_{2k-1}]_c) \\
&= \sqrt{\frac{\lambda'(a_1)\lambda'(a_{2n+1})}{\lambda'(a_{2k-1})\lambda'(a_{2n-2k+3})}} C_R([a_1a_{2n+1}a_1]_c) \\
&= \delta_b^{-0.5}\delta_b^{-3} \\
&= \delta_b^{-3.5}.
\end{aligned}
$$

So

$$
C_P([a_{2k-1}a_{2n+1}a_{2n-2k+3}a_{2n+1}a_{2k-1}]_c) = \delta_b^2(\delta_b^{-3.5})^2 = \delta_b^{-5}.
$$

On the other hand, there is no path in $[a_{2k-1}b_{2n-2k+3}\rangle_c$, so

$$
C_P([a_{2k-1}a_{2n+1}b_{2n-2k+3}a_{2n+1}a_{2k-1}]_c) = 0.
$$

Then

$$
C_R([a_{2k-1}a_{2n+1}a_{2n-2k+3}a_{2n+1}a_{2k-1}]_c) = \delta_b^{-5}.
$$

For the formula $C_R([a_{2k-1}a_{2n+1}a_{2n-1}a_{2n+2k-3}a_{2k-1}]_c)$, when $k=3$, we have

$$C_R([a_3a_{2n+1}a_{2n-1}a_{2n+1}a_3]_c)=\delta_b^{-5}.$$

When $k\geq 3$, by Lemma 4.34, we have

$$\begin{aligned}
&C_P([a_{2k-1}a_{2n+1}a_{2n-1}a_{2n+2k-3}a_{2k-1}]_c)\\
&=C_P([a_{2k-1}a_{2n+1}a_{2n-1}a_{2n+2k-5}\rangle_c\langle a_{2n+2k-5}a_{2n+2k-3}a_{2k-1}]_c)\\
&=\delta_b^2C_R([a_{2k-1}a_{2n+1}a_{2n-1}a_{2n+2k-5}\rangle_c\langle a_{2n+2k-5}a_{2k-3}a_{2k-1}]_c)\\
&\times C_R([a_{2k-1}a_{2k-3}a_{2n+2k-5}\rangle_c\langle a_{2n+2k-5}a_{2n+2k-3}a_{2k-1}]_c),
\end{aligned}$$

where the factor δ_b^2 comes from the choice the cusp point a_{2k-3}. Moreover,

$$\begin{aligned}
&C_R([a_{2k-1}a_{2n+1}a_{2n-1}a_{2n+2k-3}a_{2k-1}]_c)\\
&=C_R([a_{2k-1}a_{2n+1}a_{2n-1}a_{2n+2k-5}\rangle_c\langle a_{2n+2k-5}a_{2k-3}a_{2k-1}]_c)\\
&=\sqrt{\frac{\lambda'(a_{2k-3})\lambda'(a_{2n+2k-7})}{\lambda'(a_{2k-1})\lambda'(a_{2n+2k-5})}}C_R([a_{2k-3}a_{2n+1}a_{2n-1}a_{2n+2k-5}a_{2k-3}]_c)\\
&=\begin{cases}\delta_b^{-0.5}C_R([a_3a_{2n+1}a_{2n-1}a_{2n+1}a_3]_c)=\delta_b^{-5.5} & \text{when } k=3;\\ C_R([a_{2k-3}a_{2n+1}a_{2n-1}a_{2n+2k-5}a_{2k-3}]_c) & \text{when } k\geq 4.\end{cases}\\
&C_R([a_{2k-1}a_{2k-3}a_{2n+2k-3}a_{2k-1}]_c)\\
&=\sqrt{\frac{\lambda'(a_{2k-3})\lambda'(a_{2n+2k-3})}{\lambda'(a_{2k-1})\lambda'(a_{2n+2k-5})}}C_R([a_{2k-3}a_{2n+2k-3}a_{2k-3}]_c)=\delta_b^{-1}\delta_b^{-3}=\delta_b^{-4}.
\end{aligned}$$

Note that the middle point $a_{2n+2k-5}$ is a flat point, by Lemma 4.34, we have

$$C_R([a_{2k-1}a_{2n+1}a_{2n-1}a_{2n+2k-3}a_{2k-1}]_c)=\delta_b^2C_P([a_{2k-1}a_{2n+1}a_{2n-1}a_{2n+2k-3}a_{2k-1}]_c).$$

Then

$$C_R([a_{2k-1}a_{2n+1}a_{2n-1}a_{2n+2k-3}a_{2k-1}]_c)=\delta_b^{-5.5}\qquad \text{when } k=3;$$

$$\begin{aligned}
&C_R([a_{2k-1}a_{2n+1}a_{2n-1}a_{2n+2k-3}a_{2k-1}]_c)\\
&=C_R([a_{2k-3}a_{2n+1}a_{2n-1}a_{2n+2k-5}a_{2k-3}]_c)\qquad \text{when } k\geq 4.
\end{aligned}$$

Therefore we have $C_R([a_{2k-1}a_{2n+1}a_{2n-1}a_{2n+2k-3}a_{2k-1}]_c)=\delta_b^{-5.5}$ inductively, for $3\leq k\leq n$.

Take the adjoint, we have $C_R([a_{2k-1}a_{2n+2k-3}a_{2n-1}a_{2n+1}a_{2k-1}]_c)=\delta_b^{-5.5}$. □

Lemma 4.39.

$$C_R([a_{2k-1}a_{2n+1}a_{2n-1}a_{2n+2k-5}a_{2n-1}a_{2n+1}a_{2k-1}]_c) = -\delta_b^{-8}, \forall\, 3 \leq k \leq n.$$

Proof. For $3 \leq k \leq n$, by Lemma 4.36, we have

$$\begin{aligned} & C_P([a_{2k-1}a_{2n+1}a_{2n-1}a_{2n+2k-5}a_{2n-1}a_{2n+1}a_{2k-1}]_c) \\ = & C_P([a_{2k-1}a_{2n+1}a_{2n-1}a_{2n+2k-5}\rangle_c\langle a_{2n+2k-5}a_{2n-1}a_{2n+1}a_{2k-1}]_c) \\ = & \delta_b^2 C_R([a_{2k-1}a_{2n+1}a_{2n-1}a_{2n+2k-5}\rangle_c\langle a_{2n+2k-5}a_{2k-3}a_{2k-1}]_c) \\ \times & C_R([a_{2k-1}a_{2k-3}a_{2n+2k-5}\rangle_c\langle a_{2n+2k-5}a_{2n-1}a_{2n+1}a_{2k-1}]_c), \end{aligned}$$

where δ_b^2 is given by the choice of a_{2k-3}.

On the other hand

$$\begin{aligned} & C_P([a_{2k-1}a_{2n+1}a_{2n-1}b_{2n+2k-5}a_{2n-1}a_{2n+1}a_{2k-1}]_c) \\ = & C_P([a_{2k-1}a_{2n+1}a_{2n-1}b_{2n+2k-5}\rangle_c\langle b_{2n+2k-5}a_{2n-1}a_{2n+1}a_{2k-1}]_c) \\ = & \delta_b^2 C_R([a_{2k-1}a_{2n+1}a_{2n-1}b_{2n+2k-5}\rangle_c\langle b_{2n+2k-5}a_{2k-3}a_{2k-1}]_c) \\ \times & C_R([a_{2k-1}a_{2k-3}b_{2n+2k-5}\rangle_c\langle b_{2n+2k-5}a_{2n-1}a_{2n+1}a_{2k-1}]_c), \end{aligned}$$

where δ_b^2 is given by the choice of a_{2k-3}.

Note that

$$\begin{aligned} & C_R([a_{2k-1}a_{2n+1}a_{2n-1}b_{2n+2k-5}a_{2k-3}a_{2k-1}]_c) \\ = & \delta_b^{-1} C_R([a_{2k-1}a_{2n+1}a_{2n-1}a_{2n+2k-5}a_{2k-3}a_{2k-1}]_c); \\ & C_R([a_{2k-1}a_{2k-3}b_{2n+2k-5}a_{2n-1}a_{2n+1}a_{2k-1}]_c) \\ = & \delta_b^{-1} C_R([a_{2k-1}a_{2k-3}a_{2n+2k-5}a_{2n-1}a_{2n+1}a_{2k-1}]_c). \end{aligned}$$

By Lemma 4.34, we have

$$\begin{aligned} & C_R([a_{2k-1}a_{2n+1}a_{2n-1}a_{2n+2k-5}a_{2n-1}a_{2n+1}a_{2k-1}]_c) \\ = & C_P([a_{2k-1}a_{2n+1}a_{2n-1}a_{2n+2k-5}a_{2n-1}a_{2n+1}a_{2k-1}]_c) \\ - & C_P([a_{2k-1}a_{2n+1}a_{2n-1}b_{2n+2k-5}a_{2n-1}a_{2n+1}a_{2k-1}]_c) \\ = & \delta_b^{-1}\delta_b^2 C_R([a_{2k-1}a_{2n+1}a_{2n-1}a_{2n+2k-5}\rangle_c\langle a_{2n+2k-5}a_{2k-3}a_{2k-1}]_c) \\ \times & C_R([a_{2k-1}a_{2k-3}a_{2n+2k-5}\rangle_c\langle a_{2n+2k-5}a_{2n-1}a_{2n+1}a_{2k-1}]_c), \end{aligned}$$

where $-\delta_b$ is given by $1 - (\delta_b^{-1})^2 = -\delta_b$.

We see that if the middle point is a cusp point, and both $a_{2n+2k-5}$ and $b_{2n+2k-5}$ contribute to the middle point of a loop in the multiplication, then we can ignore the loop with middle point $b_{2n+2k-5}$ by adding a factor $-\delta_b$.

While

$$\begin{aligned}
&C_R([a_{2k-1}a_{2k-3}a_{2n+2k-5}a_{2n-1}a_{2n+1}a_{2k-1}]_c\\
&=\sqrt{\frac{\lambda'(a_{2k-3})\lambda'(a_{2n+2k-7})}{\lambda'(a_{2k-1})\lambda'(a_{2n+2k-5})}}C_R([a_{2k-3}a_{2n+2k-5}a_{2n-1}a_{2n+1}a_{2k-1}]_c)\\
&=\begin{cases}\delta_b^{-0.5}C_R([a_3a_{2n+1}a_{2n-1}a_{2n+1}a_3]_c)=\delta_b^{-5.5} & \text{when } k=3;\\ C_R([a_{2k-3}a_{2n+2k-5}a_{2n-1}a_{2n+1}a_{2k-1}]_c)=\delta_b^{-5.5} & \text{when } k\geq 4.\end{cases}
\end{aligned}$$

So

$$\begin{aligned}
C_R([a_{2k-1}a_{2n+1}a_{2n-1}a_{2n+2k-5}a_{2n-1}a_{2n+1}a_{2k-1}]_c)&=-\delta_b\delta_b^2(\delta_b^{-5.5})^2\\
&=-\delta_b^{-8},\ \forall k\geq 3. \qquad \square
\end{aligned}$$

Lemma 4.40. *For $5\leq k\leq n$, we assume that*

$$\begin{aligned}
\eta_{k1}&=[a_{2k-1}a_{2n+2k-5}a_{2n+2k-9}\rangle_c;\\
\eta_{k2}&=[a_{2k-1}a_{2n+1}a_{2n-1}a_{2n+2k-7}a_{2n+2k-9}\rangle_c;\\
\tilde{\eta}_{k1}&=[a_{2k-1}a_{2k-5}a_{2n+2k-9}\rangle_c;\\
\tilde{\eta}_{k2}&=[a_{2k-1}a_{2k-3}a_{2n+1}a_{2n-1}a_{2n+2k-9}\rangle_c.
\end{aligned}$$

Then

$$\begin{bmatrix} C_R(\eta_{k1}\tilde{\eta}_{k1}^*) & C_R(\eta_{k1}\tilde{\eta}_{k2}^*)\\ C_R(\eta_{k2}\tilde{\eta}_{k1}^*) & C_R(\eta_{k2}\tilde{\eta}_{k2}^*)\end{bmatrix}=\begin{bmatrix}\delta_b^{-5} & \delta_b^{-6.5}\\ \delta_b^{-6.5} & -\delta_b^{-9}\end{bmatrix}.$$

Proof.

$$\begin{aligned}
C_R(\eta_{k1}\tilde{\eta}_{k1}^*)&=C_R([a_{2k-1}a_{2n+2k-5}a_{2k-5}a_{2k-1}]_c)\\
&=\delta_b^{-2}C_R([a_{2k-5}a_{2n+2k-5}a_{2k-5}]_c)\\
&=\delta_b^{-5}, && \text{by Lemma 4.37}\\
C_R(\eta_{k1}\tilde{\eta}_{k2}^*)&=C_R([a_{2k-1}a_{2n+2k-5}a_{2n-1}a_{2n+1}a_{2k-3}a_{2k-1}]_c)\\
&=\delta_b^{-1}C_R([a_{2k-3}a_{2n+2k-5}a_{2n-1}a_{2n+1}a_{2k-3}]_c)\\
&=\delta_b^{-6.5}, && \text{by Lemma 4.38;}\\
C_R(\eta_{k2}\tilde{\eta}_{k1}^*)&=C_R([a_{2k-1}a_{2n+1}a_{2n-1}a_{2n+2k-7}a_{2k-5}a_{2k-1}]_c)\\
&=\delta_b^{-1}C_R([a_{2k-5}a_{2n+1}a_{2n-1}a_{2n+2k-7}a_{2k-5}]_c)\\
&=\delta_b^{-6.5}, && \text{by Lemma 4.38;}\\
C_R(\eta_{k2}\tilde{\eta}_{k2}^*)&=C_R([a_{2k-1}a_{2n+1}a_{2n-1}a_{2n+2k-7}a_{2n-1}a_{2n+1}a_{2k-3}a_{2k-1}]_c)\\
&=\delta_b^{-1}C_R([a_{2k-3}a_{2n+1}a_{2n-1}a_{2n+2k-7}a_{2n-1}a_{2n+1}a_{2k-3}]_c)\\
&=-\delta_b^{-9}, && \text{by Lemma 4.39.} \qquad \square
\end{aligned}$$

Lemma 4.41. *For $5 \leq k \leq n$, we assume that*

$$\eta_{k3} = [a_{2k-1}a_{2n+1}a_{2n-1}a_{2n+1}a_{2n-1}a_{2n+2k-9}\rangle_c;$$
$$\eta_{k4} = [a_{2k-1}a_{2n+3}a_{2n-1}a_{2n+2k-9}\rangle_c;$$
$$\eta_{k5} = [a_{2k-1}a_{2n+1}a_{2n-3}a_{2n+2k-9}\rangle_c.$$

Then

$k=$	$5l+5$	$5l+6$	$5l+7$	$5l+8$	$5l+9$
$C_R(\eta_{k1}\eta_{k3}^*)$	0	0	δ_b^{-8}	$-\delta_b^{-9}$	δ_b^{-8}
$C_R(\eta_{k2}\eta_{k3}^*)$	$-\delta_b^{-10.5}$	$\delta_b^{-9.5}$	$-\delta_b^{-10.5}$	$\delta_b^{-11.5}$	$\delta_b^{-11.5}$
$C_R(\eta_{k1}\eta_{k4}^*)$	$\delta_b^{-5.5}$	0	$\delta_b^{-6.5}$	$\delta_b^{-6.5}$	0
$C_R(\eta_{k2}\eta_{k4}^*)$	0	0	δ_b^{-8}	$-\delta_b^{-9}$	δ_b^{-8}
$C_R(\eta_{k1}\eta_{k5}^*)$	0	$\delta_b^{-5.5}$	0	$\delta_b^{-6.5}$	$\delta_b^{-6.5}$
$C_R(\eta_{k2}\eta_{k5}^*)$	δ_b^{-8}	0	0	δ_b^{-8}	$-\delta_b^{-9}$
$C_R(\eta_{k3}\eta_{k3}^*)$	δ_b^{-13}	$-\delta_b^{-12}$	$-\delta_b^{-12}$	δ_b^{-13}	$-\delta_b^{-14}$
$C_R(\eta_{k3}\eta_{k4}^*)$	0	0	0	$\delta_b^{-9.5}$	$-\delta_b^{-10.5}$
$C_R(\eta_{k3}\eta_{k5}^*)$	$\delta_b^{-9.5}$	0	0	0	$-\delta_b^{-10.5}$
$C_R(\eta_{k4}\eta_{k4}^*)$	$-\delta_b^{-8}$	δ_b^{-7}	$-\delta_b^{-8}$	0	0
$C_R(\eta_{k4}\eta_{k5}^*)$	0	0	0	0	δ_b^{-8}
$C_R(\eta_{k5}\eta_{k5}^*)$	0	$-\delta_b^{-8}$	δ_b^{-7}	$-\delta_b^{-8}$	0

Proof. For $5 \leq k \leq n$, $i = 3, 4, 5$, we assume that

$$\begin{bmatrix} \alpha_{ki} \\ \beta_{ki} \end{bmatrix} = \begin{bmatrix} C_R(\eta_{k1}\eta_{ki}^*) \\ C_R(\eta_{k2}\eta_{ki}^*) \end{bmatrix}.$$

Then

$$\begin{bmatrix} \alpha_{ki} \\ \beta_{ki} \end{bmatrix} = \begin{bmatrix} C_R(\eta_{k1}\eta_{ki}^*) \\ C_R(\eta_{k2}\eta_{ki}^*) \end{bmatrix} = \delta_b^2 \begin{bmatrix} C_R(\eta_{k1}\tilde{\eta}_{k1}^*) & C_R(\eta_{k1}\tilde{\eta}_{k2}^*) \\ C_R(\eta_{k2}\tilde{\eta}_{k1}^*) & C_R(\eta_{k2}\tilde{\eta}_{k2}^*) \end{bmatrix} \begin{bmatrix} \delta_b^2 C_R(\tilde{\eta}_{k1}\eta_{ki}^*) \\ \delta_b^6 C_R(\tilde{\eta}_{k2}\eta_{ki}^*) \end{bmatrix}$$

Furthermore we have

$$C_R(\tilde{\eta}_{k1}\eta_{ki}^*) = C_R(\rho^{-2}(\tilde{\eta}_{(k-2)1}\eta_{(k-2)i}^*)) = C_R(\tilde{\eta}_{(k-2)1}\eta_{(k-2)i}^*) = \alpha_{(k-2)i}, \text{ when } k \geq 7.$$
$$C_R(\tilde{\eta}_{k2}\eta_{k2}^*) = C_R(\rho^{-1}(\tilde{\eta}_{(k-1)2}\eta_{(k-1)i}^*)) = C_R(\tilde{\eta}_{(k-1)2}\eta_{(k-1)i}^*) = \beta_{(k-1)i}, \text{ when } k \geq 6.$$

So

$$\begin{bmatrix} \alpha_{ki} \\ \beta_{ki} \end{bmatrix} = \delta_b^2 \begin{bmatrix} \delta_b^{-5} & \delta_b^{-6.5} \\ \delta_b^{-6.5} & -\delta_b^{-9} \end{bmatrix} \begin{bmatrix} \delta_b^2 \alpha_{(k-2)i} \\ \delta_b^6 \beta_{(k-1)i} \end{bmatrix} = \begin{bmatrix} \delta_b^{-1} & \delta_b^{1.5} \\ \delta_b^{-2.5} & -\delta_b^{-1} \end{bmatrix} \begin{bmatrix} \alpha_{(k-2)i} \\ \beta_{(k-1)i} \end{bmatrix}.$$

Substituting β_{ki} by α_{ki}, we have

$$\alpha_{(k+1)i} + \delta_b^{-1}\alpha_{ki} - \delta_b^{-1}\alpha_{(k-1)i} - \alpha_{(k-2)i} = 0.$$

Recall that $\delta_b = \frac{1+\sqrt{5}}{2}$, so $x^3 + \delta_b^{-1}x^2 - \delta_b^{-1}x - 1 = 0$ has three roots 1, $e^{\frac{4\pi i}{5}}$, $e^{\frac{6\pi i}{5}}$. So

$$\alpha_{ki} = r_{1i} + r_{2i}e^{\frac{4k\pi i}{5}} + r_{3i}e^{\frac{6k\pi i}{5}},$$

for some constant r_{1i}, r_{2i}, r_{3i}. Therefore the periodicity of α_{ki} is 5 with respect to k.

Based on the results listed in Appendix A, the initial condition is

$$\begin{bmatrix}\alpha_{33}\\ \alpha_{43}\\ \beta_{43}\end{bmatrix} = \begin{bmatrix}C_R(\tilde{\eta}_{51}\eta_{53}^*)\\ C_R(\tilde{\eta}_{61}\eta_{53}^*)\\ C_R(\tilde{\eta}_{52}\eta_{53}^*)\end{bmatrix} = \begin{bmatrix}-\delta_b^{-9}\\ \delta_b^{-8}\\ \delta_b^{-11.5}\end{bmatrix};$$
$$\begin{bmatrix}\alpha_{34}\\ \alpha_{44}\\ \beta_{44}\end{bmatrix} = \begin{bmatrix}C_R(\tilde{\eta}_{51}\eta_{54}^*)\\ C_R(\tilde{\eta}_{61}\eta_{54}^*)\\ C_R(\tilde{\eta}_{52}\eta_{54}^*)\end{bmatrix} = \begin{bmatrix}\delta_b^{-6.5}\\ 0\\ \delta_b^{-8}\end{bmatrix};$$
$$\begin{bmatrix}\alpha_{35}\\ \alpha_{45}\\ \beta_{45}\end{bmatrix} = \begin{bmatrix}C_R(\tilde{\eta}_{51}\eta_{55}^*)\\ C_R(\tilde{\eta}_{61}\eta_{55}^*)\\ C_R(\tilde{\eta}_{52}\eta_{55}^*)\end{bmatrix} = \begin{bmatrix}\delta_b^{-6.5}\\ \delta_b^{-6.5}\\ -\delta_b^{-9}\end{bmatrix}.$$

For example,

$$\begin{aligned}\alpha_{33} &= C_R(\tilde{\eta}_{51}\eta_{53}^*)\\ &= C_R([a_9a_5a_{2n+1}a_{2n-1}a_{2n+1}a_{2n-1}a_{2n+1}a_9)\\ &= \delta_b^{-1}C_R([a_5a_{2n+1}a_{2n-1}a_{2n+1}a_{2n-1}a_{2n+1}a_5]_c)\\ &= -\delta_b^{-9}.\end{aligned}$$

The others are similar.

Then $\begin{bmatrix}\alpha_{ki}\\ \beta_{ki}\end{bmatrix}$ is obtained inductively. The result is listed in the following table

$k =$	$5l+5$	$5l+6$	$5l+7$	$5l+8$	$5l+9$
α_{k3}	0	0	δ_b^{-8}	$-\delta_b^{-9}$	δ_b^{-8}
β_{k3}	$-\delta_b^{-10.5}$	$\delta_b^{-9.5}$	$-\delta_b^{-10.5}$	$\delta_b^{-11.5}$	$\delta_b^{-11.5}$
α_{k4}	$\delta_b^{-5.5}$	0	$\delta_b^{-6.5}$	$\delta_b^{-6.5}$	0
β_{k4}	0	0	δ_b^{-8}	$-\delta_b^{-9}$	δ_b^{-8}
α_{k5}	0	$\delta_b^{-5.5}$	0	$\delta_b^{-6.5}$	$\delta_b^{-6.5}$
β_{k5}	δ_b^{-8}	0	0	δ_b^{-8}	$-\delta_b^{-9}$

For $5 \leq k \leq n$, $3 \leq i, j \leq 5$, by Lemma (4.34)(4.36), we have

$$\begin{aligned}
C_R(\eta_{ki}\eta_{kj}^*) &= -\delta_b(\delta_b^2 C_R(\eta_{ki}\tilde{\eta}_{k1}^*)C_R(\tilde{\eta}_{k1}\eta_{kj}^*) + \delta_b^6 C_R(\eta_{ki}\tilde{\eta}_{k2}^*)C_R(\tilde{\eta}_{k2}\eta_{kj}^*)) \\
&+ \delta_b^4 C_R(\eta_{ki}\langle a_{2n+2k-9}a_{2n+2k-7}a_{2k-3}a_{2k-1}]_c) \\
&\times C_R([a_{2k-1}a_{2k-3}a_{2n+2k-7}a_{2n+2k-9}\rangle_c \eta_{kj}^*) \\
&= -\delta_b(\delta_b^2 C_R(\eta_{ki}\tilde{\eta}_{k1}^*)C_R(\tilde{\eta}_{k1}\eta_{kj}^*) + \delta_b^6 C_R(\eta_{ki}\tilde{\eta}_{k2}^*)C_R(\tilde{\eta}_{k2}\eta_{kj}^*)) \\
&+ \delta_b^4 C_R(\eta_{(k+1)i}\tilde{\eta}_{(k+1)1}^*)C_R(\tilde{\eta}_{(k+1)1}\eta_{kj}^*). \\
&= -\delta_b^3\alpha_{(k-2)i}\alpha_{(k-2)j} - \delta_b^7\beta_{(k-1)i}\beta_{(k-1)j} + \delta_b^4\alpha_{(k-1)i}\alpha_{(k-1)j}.
\end{aligned}$$

Then

$k =$	$5l+5$	$5l+6$	$5l+7$	$5l+8$	$5l+9$
$C_R(\eta_{k3}\eta_{k3}^*)$	δ_b^{-13}	$-\delta_b^{-12}$	$-\delta_b^{-12}$	δ_b^{-13}	$-\delta_b^{-14}$
$C_R(\eta_{k3}\eta_{k4}^*)$	0	0	0	$\delta_b^{-9.5}$	$-\delta_b^{-10.5}$
$C_R(\eta_{k3}\eta_{k5}^*)$	$\delta_b^{-9.5}$	0	0	0	$-\delta_b^{-10.5}$
$C_R(\eta_{k4}\eta_{k4}^*)$	$-\delta_b^{-8}$	δ_b^{-7}	$-\delta_b^{-8}$	0	0
$C_R(\eta_{k4}\eta_{k5}^*)$	0	0	0	0	δ_b^{-8}
$C_R(\eta_{k5}\eta_{k5}^*)$	0	$-\delta_b^{-8}$	δ_b^{-7}	$-\delta_b^{-8}$	0

□

Lemma 4.42.

$$C_R(a_{2n-1}a_{4n-7}a_{2n-1}a_{2n+1}a_{2n-1}a_{2n+1}a_{2n-1}a_{2n+1}a_{2n-1})$$
$$= \begin{cases} -\delta_b^{-13.5} & \text{when } n = 20l+8; \\ -\delta_b^{-13.5} & \text{when } n = 20l+13; \\ -\delta_b^{-11.5} & \text{when } n = 20l+18; \\ -\delta_b^{-11.5} & \text{when } n = 20l+23; \end{cases} \quad \forall\, l \geq 0.$$

Proof. When $7 \leq k \leq n$, we assume that

$$\begin{aligned}
\xi_{k1} &= [a_{2k-1}a_{2n+2k-7}a_{2n+2k-13}\rangle_c; \\
\xi_{k2} &= [a_{2k-1}a_{2n+1}a_{2n-1}a_{2n+2k-9}a_{2n+2k-13}\rangle_c; \\
\xi_{k3} &= [a_{2k-1}a_{2n+1}a_{2n-1}a_{2n+1}a_{2n-1}a_{2n+2k-11}a_{2n+2k-13}\rangle_c; \\
\xi_{k4} &= [a_{2k-1}a_{2n+3}a_{2n-1}a_{2n+2k-11}a_{2n+2k-13}\rangle_c; \\
\xi_{k5} &= [a_{2k-1}a_{2n+1}a_{2n-3}a_{2n+2k-11}a_{2n+2k-13}\rangle_c; \\
\tilde{\xi}_{k1} &= [a_{2k-1}a_{2k-7}a_{2n+2k-13}\rangle_c; \\
\tilde{\xi}_{k2} &= [a_{2k-1}a_{2k-5}a_{2n+1}a_{2n-1}a_{2n+2k-13}\rangle_c;
\end{aligned}$$

$$\tilde{\xi}_{k3} = [a_{2k-1}a_{2k-3}a_{2n+1}a_{2n-1}a_{2n+1}a_{2n-1}a_{2n+2k-13}\rangle_c;$$

$$\tilde{\xi}_{k4} = [a_{2k-1}a_{2k-3}a_{2n+3}a_{2n-1}a_{2n+2k-13}\rangle_c;$$

$$\tilde{\xi}_{k5} = [a_{2k-1}a_{2k-3}a_{2n+1}a_{2n-3}a_{2n+2k-13}\rangle_c.$$

By Lemma (4.40)(4.41), we can compute T_k, for $k \geq 7$, where

$$T_k = \begin{bmatrix} C_R(\xi_{k1}\tilde{\xi}_{k1}^*) & C_R(\xi_{k1}\tilde{\xi}_{k2}^*) & C_R(\xi_{k1}\tilde{\xi}_{k3}^*) & C_R(\xi_{k1}\tilde{\xi}_{k4}^*) & C_R(\xi_{k1}\tilde{\xi}_{k5}^*) \\ C_R(\xi_{k2}\tilde{\xi}_{k1}^*) & C_R(\xi_{k2}\tilde{\xi}_{k2}^*) & C_R(\xi_{k2}\tilde{\xi}_{k3}^*) & C_R(\xi_{k2}\tilde{\xi}_{k4}^*) & C_R(\xi_{k2}\tilde{\xi}_{k5}^*) \\ C_R(\xi_{k3}\tilde{\xi}_{k1}^*) & C_R(\xi_{k3}\tilde{\xi}_{k2}^*) & C_R(\xi_{k3}\tilde{\xi}_{k3}^*) & C_R(\xi_{k3}\tilde{\xi}_{k4}^*) & C_R(\xi_{k3}\tilde{\xi}_{k5}^*) \\ C_R(\xi_{k4}\tilde{\xi}_{k1}^*) & C_R(\xi_{k4}\tilde{\xi}_{k2}^*) & C_R(\xi_{k4}\tilde{\xi}_{k3}^*) & C_R(\xi_{k4}\tilde{\xi}_{k4}^*) & C_R(\xi_{k4}\tilde{\xi}_{k5}^*) \\ C_R(\xi_{k5}\tilde{\xi}_{k1}^*) & C_R(\xi_{k5}\tilde{\xi}_{k2}^*) & C_R(\xi_{k5}\tilde{\xi}_{k3}^*) & C_R(\xi_{k5}\tilde{\xi}_{k4}^*) & C_R(\xi_{k5}\tilde{\xi}_{k5}^*) \end{bmatrix}.$$

For $1 \leq i, j \leq 2$, we have

$$C_R(\xi_{ki}\tilde{\xi}_{kj}^*) = \delta_b^{-1} C_R(\eta_{(k-1)i}\tilde{\eta}_{(k-1)j}^*) \quad \forall k \geq 7.$$

For $1 \leq i \leq 5$, $3 \leq j \leq 5$, we have

$$C_R(\xi_{ki}\tilde{\xi}_{kj}^*) = \delta_b^{-1} C_R(\eta_{(k-1)i}\tilde{\eta}_{(k-1)j}^*) \quad \forall k \geq 7.$$

For $3 \leq i \leq 5$, $j = 2$, we have

$$C_R(\xi_{ki}\tilde{\xi}_{kj}^*) = \delta_b^{-1} C_R(\eta_{(k-2)i}\tilde{\eta}_{(k-2)j}^*) \quad \forall k \geq 7.$$

For $3 \leq i \leq 5$, $j = 1$, we have

$$C_R(\xi_{ki}\tilde{\xi}_{kj}^*) = \delta_b^{-1} C_R(\eta_{(k-3)i}\tilde{\eta}_{(k-3)j}^*) \quad \forall k \geq 8.$$

Based on the results listed in Appendix A, we have

$$C_R(\xi_{73}\tilde{\xi}_{71}^*) = \delta_b^{-9}; \quad C_R(\xi_{74}\tilde{\xi}_{71}^*) = 0; \quad C_R(\xi_{75}\tilde{\xi}_{71}^*) = \delta_b^{-7.5}.$$

For example,

$$\begin{aligned} C_R(\xi_{73}\tilde{\xi}_{71}^*) &= C_R([a_{13}a_{2n+1}a_{2n-1}a_{2n+1}a_{2n-1}a_{2n+3}a_7a_{13}]_c) \\ &= \delta_b^{-1.5} C_R([a_7a_{2n+1}a_{2n-1}a_{2n+1}a_{2n-1}a_{2n+3}]_c) \\ &= \delta_b^{-9}. \end{aligned}$$

The others are similar. Then

$$T_k = \begin{bmatrix} \delta_b^{-6} & \delta_b^{-7.5} & 0 & 0 & \delta_b^{-6.5} \\ \delta_b^{-7.5} & -\delta_b^{-10} & \delta_b^{-10.5} & 0 & 0 \\ \delta_b^{-9} & -\delta_b^{-11.5} & -\delta_b^{-13} & 0 & 0 \\ 0 & 0 & 0 & \delta_b^{-8} & 0 \\ \delta_b^{-7.5} & \delta_b^{-9} & 0 & 0 & -\delta_b^{-9} \end{bmatrix}, \qquad \text{when } k = 5l+7;$$

$$T_k = \begin{bmatrix} \delta_b^{-6} & \delta_b^{-7.5} & \delta_b^{-9} & \delta_b^{-7.5} & 0 \\ \delta_b^{-7.5} & -\delta_b^{-10} & -\delta_b^{-11.5} & \delta_b^{-9} & 0 \\ 0 & \delta_b^{-10.5} & -\delta_b^{-13} & 0 & 0 \\ \delta_b^{-6.5} & 0 & 0 & -\delta_b^{-9} & 0 \\ 0 & 0 & 0 & 0 & \delta_b^{-8} \end{bmatrix}, \qquad \text{when } k = 5l+8;$$

$$T_k = \begin{bmatrix} \delta_b^{-6} & \delta_b^{-7.5} & -\delta_b^{-10} & \delta_b^{-7.5} & \delta_b^{-7.5} \\ \delta_b^{-7.5} & -\delta_b^{-10} & \delta_b^{-12.5} & -\delta_b^{-10} & \delta_b^{-9} \\ 0 & -\delta_b^{-11.5} & \delta_b^{-14} & \delta_b^{10.5} & 0 \\ 0 & \delta_b^{-9} & \delta_b^{10.5} & 0 & 0 \\ \delta_b^{-6.5} & 0 & 0 & 0 & -\delta_b^{-9} \end{bmatrix}, \qquad \text{when } k = 5l+9;$$

$$T_k = \begin{bmatrix} \delta_b^{-6} & \delta_b^{-7.5} & \delta_b^{-9} & 0 & \delta_b^{-7.5} \\ \delta_b^{-7.5} & -\delta_b^{-10} & \delta_b^{-12.5} & \delta_b^{-9} & -\delta_b^{-10} \\ \delta_b^{-9} & \delta_b^{-12.5} & -\delta_b^{-15} & -\delta_b^{-11.5} & -\delta_b^{-11.5} \\ \delta_b^{-7.5} & -\delta_b^{-10} & -\delta_b^{-11.5} & 0 & \delta_b^{-9} \\ 0 & \delta_b^{-9} & -\delta_b^{-11.5} & \delta_b^{-9} & 0 \end{bmatrix}, \qquad \text{when } k = 5l+10;$$

$$T_k = \begin{bmatrix} \delta_b^{-6} & \delta_b^{-7.5} & 0 & \delta_b^{-6.5} & 0 \\ \delta_b^{-7.5} & -\delta_b^{-10} & -\delta_b^{-11.5} & 0 & \delta_b^{-9} \\ -\delta_b^{-10} & \delta_b^{-12.5} & \delta_b^{-14} & 0 & \delta_b^{-10.5} \\ \delta_b^{-7.5} & \delta_b^{-9} & 0 & -\delta_b^{-9} & 0 \\ \delta_b^{-7.5} & -\delta_b^{-10} & \delta_b^{10.5} & 0 & 0 \end{bmatrix}, \qquad \text{when } k = 5l+11.$$

Take ξ_k to be $[a_{2k-1}a_{2n+1}a_{2n-1}a_{2n+1}a_{2n-1}a_{2n+1}a_{2n-1}a_{2n+2k-13}\rangle_c$, then

$$\begin{bmatrix} C_R(\xi_{k1}\xi_k^*) \\ C_R(\xi_{k2}\xi_k^*) \\ C_R(\xi_{k3}\xi_k^*) \\ C_R(\xi_{k4}\xi_k^*) \\ C_R(\xi_{k5}\xi_k^*) \end{bmatrix} = \delta_b^2 T_k \begin{bmatrix} \delta_b^2 C_R(\tilde{\xi}_{k1}\xi_k^*) \\ \delta_b^6 C_R(\tilde{\xi}_{k2}\xi_k^*) \\ \delta_b^{10} C_R(\tilde{\xi}_{k3}\xi_k^*) \\ \delta_b^6 C_R(\tilde{\xi}_{k4}\xi_k^*) \\ \delta_b^6 C_R(\tilde{\xi}_{k5}\xi_k^*) \end{bmatrix}, \ \forall k \geq 7.$$

Furthermore

$$\begin{aligned}
&C_R(\tilde{\xi}_{k1}\xi_k^*) = C_R(\xi_{(k-3)1}\xi_{k-3}^*), \text{ when } k \geq 10;\\
&C_R(\tilde{\xi}_{k2}\xi_k^*) = C_R(\xi_{(k-2)2}\xi_{k-2}^*), \text{ when } k \geq 9;\\
&C_R(\tilde{\xi}_{k3}\xi_k^*) = C_R(\xi_{(k-1)3}\xi_{k-1}^*), \text{ when } k \geq 8;\\
&C_R(\tilde{\xi}_{k4}\xi_k^*) = C_R(\xi_{(k-1)4}\xi_{k-1}^*), \text{ when } k \geq 8;\\
&C_R(\tilde{\xi}_{k5}\xi_k^*) = C_R(\xi_{(k-1)5}\xi_{k-1}^*), \text{ when } k \geq 8.
\end{aligned}$$

So we can compute it inductively. By Lemma (4.40)(4.41) and a direct computation, the initial condition is

$$\begin{bmatrix} C_R(\tilde{\xi}_{71}\xi_7^*) & C_R(\tilde{\xi}_{81}\xi_8^*) & C_R(\tilde{\xi}_{91}\xi_9^*) \\ & C_R(\tilde{\xi}_{72}\xi_7^*) & C_R(\tilde{\xi}_{82}\xi_8^*) \\ & & C_R(\tilde{\xi}_{73}\xi_7^*) \\ & & C_R(\tilde{\xi}_{74}\xi_7^*) \\ & & C_R(\tilde{\xi}_{75}\xi_7^*) \end{bmatrix} = \begin{bmatrix} \delta_b^{-12.5} & 0 & -\delta_b^{-12.5} \\ & \delta_b^{-14} & \delta_b^{-13} \\ & & \delta_b^{-14.5} \\ & & -\delta_b^{-14} \\ & & -\delta_b^{-14} \end{bmatrix}.$$

For example,

$$\begin{aligned}
& C_R(\tilde{\xi}_{91}\xi_9^*)\\
&= C_R([a_{17}a_{11}a_{2n+5}a_{2n-1}a_{2n+1}a_{2n-1}a_{2n+1}a_{2n-1}a_{2n+1}a_{17}]_c)\\
&= \delta_b^{-0.5}C_R([a_{11}a_{2n+5}a_{2n-1}a_{2n+1}a_{2n-1}a_{2n+1}a_{2n-1}a_{2n+1}a_{11}]_c)\\
&= \delta_b^{-0.5}(\delta_b^2 C_R([a_{11}a_{2n+5}a_5a_{11}]_c)C_R([a_{11}a_5a_{2n+1}a_{2n-1}a_{2n+1}a_{2n-1}a_{2n+1}a_{11}]_c)\\
&+ \delta_b^6 C_R([a_{11}a_{2n+5}a_{2n-3}a_{2n+1}a_9a_{11}]_c)\\
&\times C_R([a_{11}a_9a_{2n+1}a_{2n-3}a_{2n+1}a_{2n-1}a_{2n+1}a_{2n-1}a_{2n+1}a_{11}]_c)\\
&- \delta_b\delta_b^4 C_R([a_{11}a_{2n+5}a_{2n-1}a_{2n+1}a_7a_{11}]_c)\\
&\times C_R([a_{11}a_7a_{2n+1}a_{2n-1}a_{2n+1}a_{2n-1}a_{2n+1}a_{2n-1}a_{2n+1}a_{11}]_c)\\
&- \delta_b\delta_b^4 C_R([a_{11}a_{2n+5}a_{2n-1}a_{2n+3}a_9a_{11}]_c)\\
&\times C_R([a_{11}a_9a_{2n+3}a_{2n-1}a_{2n+1}a_{2n-1}a_{2n+1}a_{2n-1}a_{2n+1}a_{11}]_c)\\
&- \delta_b\delta_b^8 C_R([a_{11}a_{2n+5}a_{2n-1}a_{2n+1}a_{2n-1}a_{2n+1}a_9a_{11}]_c)\\
&\times C_R([a_{11}a_9a_{2n+1}a_{2n-1}a_{2n+1}a_{2n-1}a_{2n+1}a_{2n-1}a_{2n+1}a_{2n-1}a_{2n+1}a_{11}]_c))\\
&= \delta_b^{-0.5}(\delta_b^2\delta_b^{-2.5}\delta_b^{-3}\delta_b^{-1.5}(-\delta_b^{-8}) + \delta_b^6\delta_b^{-0.5}C_R(\eta_{51}\eta_{55}^*)\delta_b^{-0.5}C_R(\eta_{55}\eta_{53}^*)\\
&- \delta_b\delta_b^4\delta_b^{-1.5}\delta_b^{-7.5}\delta_b^{-1}\delta_b^{-11} - \delta_b\delta_b^4\delta_b^{-0.5}C_R(\eta_{51}\eta_{54}^*)\delta_b^{-0.5}C_R(\eta_{54}\eta_{53}^*)\\
&- \delta_b\delta_b^8\delta_b^{-0.5}C_R(\eta_{51}\eta_{53}^*)\delta_b^{-0.5}C_R(\eta_{53}\eta_{53}^*))
\end{aligned}$$

$$= \delta_b^{-0.5}(\delta_b^2\delta_b^{-2.5}\delta_b^{-3}\delta_b^{-1.5}(-\delta_b^{-8}) + 0 - \delta_b\delta_b^4\delta_b^{-1.5}\delta_b^{-5.5}\delta_b^{-1}\delta_b^{-11} + 0 + 0)$$
$$= -\delta_b^{-12.5}.$$

The others are similar.

Then we have

$k =$	7	8	9	10	11	12	13
$C_R(\xi_{k1}\xi_k^*)$	0	$-\delta_b^{-13.5}$	$\delta_b^{-12.5}$	$-\delta_b^{-12.5}$	$-\delta_b^{-12.5}$	$\delta_b^{-12.5}$	$-\delta_b^{-13.5}$
$C_R(\xi_{k2}\xi_k^*)$	δ_b^{-13}	$-\delta_b^{-15}$	δ_b^{-16}	δ_b^{-12}	δ_b^{-15}	0	δ_b^{-16}
$C_R(\xi_{k3}\xi_k^*)$	$-\delta_b^{-15.5}$	$\delta_b^{-14.5}$	$\delta_b^{-17.5}$	$-\delta_b^{-16.5}$	$\delta_b^{-16.5}$	$-\delta_b^{-15.5}$	$\delta_b^{-15.5}$
$C_R(\xi_{k4}\xi_k^*)$	$-\delta_b^{-14}$	δ_b^{-15}	δ_b^{-12}	$-\delta_b^{-14}$	δ_b^{-15}	δ_b^{-15}	$-\delta_b^{-14}$
$C_R(\xi_{k5}\xi_k^*)$	δ_b^{-13}	δ_b^{-13}	$-\delta_b^{-13}$	δ_b^{-14}	$-\delta_b^{-14}$	δ_b^{-12}	δ_b^{-12}

14	15	16	17	18	19	20	21
0	$-\delta_b^{-12.5}$	0	$\delta_b^{-12.5}$	$-\delta_b^{-11.5}$	0	0	0
$\delta_b^{-13} + \delta_b^{-16}$	δ_b^{-13}	$-\delta_b^{-14}$	δ_b^{-14}	$\delta_b^{-13} + \delta_b^{-16}$	δ_b^{-14}	0	0
$-\delta_b^{-18.5}$	$\delta_b^{-15.5}$	$\delta_b^{-14.5}$	$-\delta_b^{-14.5}$	$\delta_b^{-17.5}$	$-\delta_b^{-15.5} - \delta_b^{-18.5}$	$\delta_b^{-15.5}$	$\delta_b^{-13.5}$
δ_b^{-14}	$-\delta_b^{-14}$	δ_b^{-13}	δ_b^{-13}	$-\delta_b^{-13}$	δ_b^{-14}	0	0
$-\delta_b^{-13} - \delta_b^{-15}$	δ_b^{-14}	0	δ_b^{-14}	δ_b^{-14}	$-\delta_b^{-15}$	δ_b^{-12}	0

22	23	24	25	26	27	28	29
0	$-\delta_b^{-11.5}$	$\delta_b^{-12.5}$	0	$-\delta_b^{-12.5}$	0	$-\delta_b^{-13.5}$	$\delta_b^{-12.5}$
δ_b^{-12}	δ_b^{-14}	$-\delta_b^{-15}$	δ_b^{-14}	δ_b^{-13}	δ_b^{-13}	$-\delta_b^{-15}$	δ_b^{-16}
$-\delta_b^{-14.5}$	$\delta_b^{-15.5}$	$-\delta_b^{-16.5}$	$-\delta_b^{-16.5}$	$\delta_b^{-14.5}$	$-\delta_b^{-15.5}$	$\delta_b^{-14.5}$	$\delta_b^{-17.5}$
0	0	δ_b^{-12}	0	$-\delta_b^{-14}$	$-\delta_b^{-14}$	δ_b^{-15}	δ_b^{-12}
0	0	0	δ_b^{-12}	$-\delta_b^{-14}$	δ_b^{-13}	δ_b^{-13}	$-\delta_b^{-13}$

Note that the periodicity is 20. So

$$\begin{aligned} &C_R(a_{2n-1}a_{4n-7}a_{2n-1}a_{2n+1}a_{2n-1}a_{2n+1}a_{2n-1}a_{2n+1}a_{2n-1}) \\ &= C_R(\xi_{n1}\xi_n^*) \\ &= \begin{cases} -\delta_b^{-13.5} & \text{when } n = 20l + 8; \\ -\delta_b^{-13.5} & \text{when } n = 20l + 13; \\ -\delta_b^{-11.5} & \text{when } n = 20l + 18; \\ -\delta_b^{-11.5} & \text{when } n = 20l + 23. \end{cases} \qquad \square \end{aligned}$$

Theorem 4.43. *When $n \geq 4$, the nth Bisch–Haagerup fish graph is not the principal graph of a subfactor.*

Proof. By Lemma 4.32, to compute the coefficients C_R of loops in $A_k^+\mathscr{G}_{2n,+}A_k^+$, we can fix the initial condition as $\mu_1 = \mu_2 = \omega = 1$.

When $n = 4$, from Appendix A, we have $C_R([a_5a_9a_5a_9a_5]_c) = \delta_b^{-5}$ and $C_R([a_7a_{11}a_7a_{11}a_7]_c) = 0$. Recall that the coefficient of loops in R can be computed inductively from the initial condition $\mu_1 = \mu_2 = \omega = 1$. By the symmetry of the dual principal graph and the symmetry of the initial condition $\mu_1 = \mu_2 = \omega = 1$, we can substitute $2k-1$ by $4n-2k+1$, see Remark 4.29. Then $C_R([a_9a_5a_9a_5a_9]_c)=0$. By Lemma 4.32, these coefficients are independent of the parameters μ_1, μ_2, ω. If $R_{\mu_1\mu_2\omega}$ is a solution of Proposition 4.9, then

$$\frac{\lambda'(a_5)}{\lambda'(a_9)}C_R([a_5a_9a_5a_9a_5]_c)=\omega^2C_R([a_9a_5a_9a_5a_9]_c).$$

So

$$|\delta_b^{-1}C_R([a_5a_9a_5a_9a_5]_c)|=|C_R([a_9a_5a_9a_5a_9]_c)|.$$

It is a contradiction. That means the 4th Bisch–Haagerup fish graph is not the principal graph of a subfactor.

When $n\geq 5$, by Lemma 4.39, we have

$$C_R([a_5a_{2n+1}a_{2n-1}a_{2n+1}a_{2n-1}a_{2n+1}a_5]_c)=-\delta_b^{-8}.$$

By the symmetries of the dual principal graph and the initial condition, we have

$$C_R([a_{4n-5}a_{2n-1}a_{2n+1}a_{2n-1}a_{2n+1}a_{2n-1}a_{4n-5}]_c)=-\delta_b^{-8}.$$

If $R_{\mu_1\mu_2\omega}$ is a solution of Proposition 4.9, then by Lemma 4.32, we have

$$\begin{aligned}&|C_R([a_{2n-1}a_{4n-5}a_{2n-1}a_{2n+1}a_{2n-1}a_{2n+1}a_{2n-1}]_c)|\\&=|\delta_b^{-1}C_R([a_{4n-5}a_{2n-1}a_{2n+1}a_{2n-1}a_{2n+1}a_{2n-1}a_{4n-5}]_c)|\\&=\delta_b^{-9}.\end{aligned}$$

On the other hand, by Lemma 4.41, we have

$$|C_R(\eta_{n1}\eta_{n3}^*)|=|C_R([a_{2n-1}a_{4n-5}a_{2n-1}a_{2n+1}a_{2n-1}a_{2n+1}a_{2n-1}]_c)|=\delta_b^{-9}$$

which implies $5|n-3$.

When $n\geq 8$ and $5|n-3$, from Appendix A, we have

$$C_R([a_7a_{2n+1}a_{2n-1}a_{2n+1}a_{2n-1}a_{2n+1}a_{2n-1}a_{2n+1}a_7]_c)=\delta_b^{-11}.$$

By the symmetries of the dual principal graph and the initial condition $\mu_1=\mu_2=\omega=1$, we have

$$C_R([a_{4n-7}a_{2n-1}a_{2n+1}a_{2n-1}a_{2n+1}a_{2n-1}a_{2n+1}a_{2n-1}a_{4n-7}]_c)=\delta_b^{-11}.$$

So

$$|C_R([a_{2n-1}a_{4n-7}a_{2n-1}a_{2n+1}a_{2n-1}a_{2n+1}a_{2n-1}a_{2n+1}a_{2n-1}]_c)|=\delta_b^{-12.5}.$$

On the other hand, by Lemma 4.42, we have

$$|C_R([a_{2n-1}a_{4n-7}a_{2n-1}a_{2n+1}a_{2n-1}a_{2n+1}a_{2n-1}a_{2n+1}a_{2n-1}]_c)| = \delta_b^{-11.5} \text{ or } \delta_b^{-13.5}.$$

It is a contradiction.

Therefore the nth Bisch–Haagerup fish graph is not the principal graph of a subfactor whenever $n \geq 4$. □

4.4. Uniqueness

Theorem 4.44. *There is only one subfactor planar algebra whose principal graph is the nth Bisch–Haagerup fish graph, for $n = 1, 2, 3$.*

It is easy to generalize the Jellyfish technic [2] for Fuss–Catalan tangles, or tangles labelled by the biprojection. We are going to check the Fuss–Catalan Jellyfish relations for the generators S and R. Before that let us prove two lemmas which tell the Fuss–Catalan Jellyfish relations.

Lemma 4.45. *If R is a solution of Proposition 4.9 in a subfactor planar algebra with a biprojection, then*

$$P = \delta^2 P e_{2n} P,$$

where $P = \delta_b^{-1} g_{2n} - R$.

Proof. Note that $P = \delta_b^{-1} g_{2n} - R$ is a projection. It is easy to check that $\delta^2 P e_{2n} P$ is a subprojection of P. Moreover they have the same trace. So $P = \delta^2 P e_{2n} P$. □

Remark 4.46. This is Wenzl's formula [39,23] for the minimal projection P.

Lemma 4.47. *If S is a solution of Proposition 4.6 in a subfactor planar algebra with a biprojection, then*

$$Q = \delta\delta_a Q p_{2n} Q,$$

where $Q = \frac{1}{2}(f_{2n} + S)$.

Proof. Note that $Q = \frac{1}{2}(f_{2n} + S)$ is a projection. It is easy to check that $\delta\delta_a Q p_{2n} Q$ is a subprojection of Q. Moreover they have the same trace. So $Q = \delta\delta_a Q p_{2n} Q$. □

Proof of Theorem 4.44. We have known three examples whose principal graphs are the first three Bisch–Haagerup fish graphs. We only need to prove the uniqueness.

For $n = 1, 2, 3$, suppose $R_{\mu_1\mu_2\omega}$ is a solution of Proposition 4.9. Note that the loop

$$[\underbrace{a_{2n-1}a_{2n+1}\cdots a_{2n-1}a_{2n+1}}_{n}]_c$$

is rotation invariant. Moreover its coefficient in R is non-zero. So $\omega = 1$.

If (S, R, ω_0) is a solution of Proposition (4.6)(4.9), then $(-S, R, -\omega_0)$ is also a solution. Up to this isomorphism, we can assume $\omega_0 = 1$.

Suppose $\mathscr{B}$ is a subfactor planar algebra whose principal graph is the nth Bisch–Haagerup fish graph, and its generators R, S satisfy Proposition (4.6)(4.9), such that $\omega_0 = 1$. Let us consider the linear subspaces $V_\pm$ of $\mathscr{B}_{2n+1,\pm}$ generated by annular Fuss–Catalan tangles acting on R. We claim that the space $V_\pm$ satisfies Fuss–Catalan Jellyfish relations. Therefore the subfactor planar algebra is unique.

Obviously $V_\pm$ is * closed and rotation invariant. The multiplication on $V_\pm$ is implied by Lemma 4.45. Now let us check the Fuss–Catalan Jellyfish relations.

When we add one string in an unshaded region, for example, we add one string on the right of $\tilde{R}$, where $\tilde{R} \in V_-$ is the diagram adding one string on the right of R. Then by Lemma 4.45, we have $\delta_b^{-1} g_{2n} - R \in \mathscr{I}_{2n+2,-}$, where $\mathscr{I}_{2n+2,-}$ is the two sided ideal of $\mathscr{B}_{2n,-}$ generated by the Jones projection. That implies the Jellyfish relation of $\tilde{R}$ while adding one string on the right. Other Jellyfish relations are similar.

When we add one string in a shaded region, for example, we add one string on the right of $\tilde{S}$, where $\tilde{S} \in V_+$ is the diagram adding one string on the right of S. Then by Lemma 4.47, and the fact that $p_{2n} \in \mathscr{I}_{2n+2,+}$, where $\mathscr{I}_{2n+2,+}$ is the two sided ideal of $\mathscr{B}_{2n,+}$ generated by the Jones projection, we have $\frac{1}{2}(f_{2n} + S) \in \mathscr{I}_{2n+2,+}$. That implies the Jellyfish relation of $\tilde{S}$ while adding one string on the right. Other Jellyfish relations are similar. □

It is easy to check that the possible solution (R, S), for $\mu_1 = \mu_2 = \pm 1$, $\omega_0 = 1$, in the graph planar algebra does satisfy Proposition (4.6)(4.9). The skein theoretic construction of the three subfactor planar algebras corresponding to the first three Bisch–Haagerup fish graphs could be realised by the Fuss–Catalan Jellyfish relations of the generating vector space $V_\pm$ mentioned above. We leave the details to the reader.

5. Composed inclusions of two A_4 subfactors

In this section, we will consider composed inclusions $\mathcal{N} \subset \mathcal{P} \subset \mathcal{M}$ of two A_4 subfactors. Let id be the trivial $(\mathcal{P}, \mathcal{P})$ bimodule, and ρ_1, ρ_2 be the non-trivial $(\mathcal{P}, \mathcal{P})$ bimodules arise from $\mathcal{N} \subset \mathcal{P}$, $\mathcal{P} \subset \mathcal{M}$ respectively. Then $\rho_i^2 = \rho_i \oplus id$, for $i = 1, 2$. If it is a free composed inclusion, i.e., there is no relation between ρ_1 and ρ_2, then its planar algebra is $FC(\delta_b, \delta_b)$; Otherwise take w to be a shortest word of ρ_1, ρ_2 which contains id. If $w = (\rho_1\rho_2)^n\rho_1$, and n is even, then by Frobenius reciprocity, we have

$$\dim(\hom((\rho_1\rho_2)^{\frac{n}{2}}\rho_1, (\rho_1\rho_2)^{\frac{n}{2}})) = c \geq 1.$$

So

$$\dim(\hom((\rho_1\rho_2)^{\frac{n}{2}}\rho_1^2,(\rho_1\rho_2)^{\frac{n}{2}}))=\dim(\hom((\rho_1\rho_2)^{\frac{n}{2}}\rho_1,(\rho_1\rho_2)^{\frac{n}{2}}\rho_1))\geq c+1.$$

Note that $\rho_1^2=\rho\oplus id$, we have

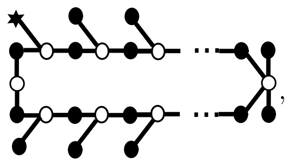
$$\dim(\hom((\rho_1\rho_2)^{\frac{n}{2}},(\rho_1\rho_2)^{\frac{n}{2}}))\geq 1.$$

So $(\rho_1\rho_2)^n$ contains id, which contradicts to the assumption that w is shortest. It is similar for the other cases. Without loss of generality, we have $w=(\rho_1\rho_2)^n$, for some $n\geq 1$.

Considering the planar algebra $\mathscr{B}$ of $\mathcal{N}\subset\mathcal{M}$ as an annular Fuss–Catalan module, then it contains a lowest weight vector $T\in\mathscr{B}_{n,+}$ which induces a morphism from $(\rho_1\rho_2)^n$ to id. So T is totally uncappable.

Remark 5.1. There is another proof without using bimodules. The lowest weight vector $T\in\mathscr{B}_{n,+}$ is totally uncappable, for $n\geq 2$, see [24]. For the case $n=1$, to show it is totally uncappable, we need the fact that the biprojection cutdown induces a planar algebra isomorphism [7].

Definition 5.2. Let us define Ω_n, for $n\geq 1$, to be the $(\mathcal{N},\mathcal{P},\mathcal{M})$ coloured graph with parameter (δ_b,δ_b) as

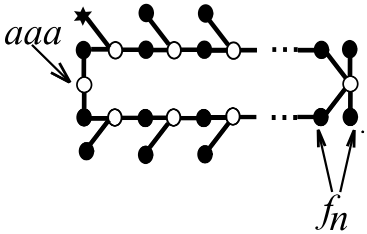


where the black vertices are $\mathcal{N}$, $\mathcal{M}$ coloured, and the white vertices are $\mathcal{P}$ coloured, and the number of white vertices is $2n$.

Lemma 5.3. *Suppose $\mathscr{B}$ is a composition of two A_4 Temperley–Lieb planar algebras. Then either $\mathscr{B}$ is Fuss–Catalan, or its refined principal graph is Ω_n, for some $n\geq 1$.*

Proof. If $\mathscr{B}$ is not Fuss–Catalan, then it contains a lowest weight vector $T\in\mathscr{B}_{n,+}$ which is totally uncappable, for some $n\geq 1$. So the refined principal graph of $\mathscr{B}$ is the same as that of $FC(\delta_b,\delta_b)$, until the vertex corresponding to f_n splits, where f_n the minimal projection of $FC(\delta_b,\delta_b)_{n,+}$ with middle pattern $abba\cdots abba(ab)$.

By the embedding theorem, T is embedded in the graph planar algebra. Similar to the proof of Lemma 4.28, the loop passing the vertex, corresponding to the middle pattern aaa, has non-zero coefficient in S. Similar to the proof of Lemma 4.24, it has to be a length $2n$ flat loop, a loop whose vertices are all flat. Via computing the trace, there is a unique way to complete the refined principal graph as



□

For $n = 1, 2, 3$, it is easy to check that Ω_n is the refined principal graph of the reduced subfactor from the vertex a_3, corresponding to the middle pattern $baab$, in the (refined) dual principal graph of the nth fish factor.

Comparing this refine principal graph with the one obtained in Lemma 4.10, they share the same black and white vertice and the same dimension vector on these vertices. Similar to Proposition 4.9, we have the following result.

Proposition 5.4. *Suppose $\mathscr{B}$ is a planar algebra as a composition of two A_4 planar algebras, and it is not Fuss–Catalan. Then there is a lowest weight vector $T \in \mathscr{B}_{n,+}$, such that*

(1) $T* = T$;
(2) $T + \delta_b^{-2} f_n$ *is a projection;*
(3) T *is totally uncappable;*
(4) $\rho(T) = \omega T$,

where f_n is the minimal projection of $FC(\delta_b, \delta_b)_{n,+}$ with middle pattern $abba \cdots abba(ab)$.

Note that the dual of $\mathscr{B}$ is still a composition of two A_4 planar algebras. So the refined dual principal graph is the same as Ω_n. Then there is a lowest weight vector $T' \in \mathscr{B}_{n,-}$ satisfying similar propositions. Solving this generators T, T' in the graph planar algebra is the same as solving R for the compositions of A_3 with A_4, while the rotation is replaced by the Fourier transform. Therefore we have the following result.

Theorem 5.5. *There are exactly four subfactor planar algebras as a composition of two A_4 planar algebras.*

Proof. Suppose $\mathscr{B}$ is a planar algebra as a composition of two A_4 planar algebras. If $\mathscr{B}$ is not Fuss–Catalan, then there is a lowest weight vector $T \in \mathscr{B}_{n,+}$ satisfying proposition (1)(2)(3)(4), and $T' \in \mathscr{B}_{n,+}$ satisfying similar propositions. Comparing with the process of solving R in the graph planar algebra for the composition of A_3 and A_4, we have the Ω_n, for $n \geq 4$, is not the refined principal graph of a subfactor.

For $n = 1, 2, 3$, three examples are known as reduced subfactors. We only need to prove the uniqueness. Similar to the proof of Theorem 4.44, by comparing the coefficient of the rotation invariant loop, we have $T = \mathcal{F}(T') = \rho(T)$. So $\omega = 1$. Furthermore the linear subspaces $V_{\pm}$ of $\mathscr{B}_{n+1,\pm}$ generated by annular Fuss–Catalan tangles acting on T satisfy Fuss–Catalan Jellyfish relations, which are derived from Wenzl's formula similar to Lemma 4.47 and Theorem 4.44. Therefore the subfactor planar algebra is unique. □

Similarly we can construct the generators (T, T') in the graph planar algebra. The skein theoretic construction of the three subfactor planar algebras could be realised by the Fuss–Catalan Jellyfish relations of the generating vector space $V_{\pm}$.

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Appendix A. The initial conditions

Up to the rotation, we only need $C_R(l)$ for a loop $l \in A_k^+ \mathscr{G}_{2n,+} A_k^+$. Now we list of results up to adjoint for $1 \leq k \leq 4$. They are obtained by a direct computation by Lemma (4.34)(4.36).

When $n \geq 1$,

$$C_R([a_1 a_{2n+1}]_c) = \delta_b^{-3}.$$

When $n \geq 2$,

$$C_R([a_3 a_{2n+3}]_c) = \delta_b^{-3};$$
$$C_R([a_3 a_{2n+1} a_{2n-1} a_{2n+1}]_c) = \delta_b^{-5}.$$

When $n \geq 3$,

$$C_R([a_5 a_{2n+5}]_c) = \delta_b^{-3};$$
$$C_R([a_5 a_{2n+1} a_{2n-3} a_{2n+1}]_c) = \delta_b^{-5};$$
$$C_R([a_5 a_{2n+1} a_{2n-1} a_{2n+1} a_{2n-1} a_{2n+1}]_c) = -\delta_b^{-8};$$
$$C_R([a_5 a_{2n+1} a_{2n-1} a_{2n+3}]_c) = \delta_b^{-5.5}.$$

When $n \geq 4$,

$$C_R([a_7 a_{2n+7}]_c) = \delta_b^{-3};$$
$$C_R([a_7 a_{2n+1} a_{2n-5} a_{2n+1}]_c) = \delta_b^{-5};$$
$$C_R([a_7 a_{2n+3} a_{2n-1} a_{2n+3}]_c) = 0;$$

$$C_R([a_7a_{2n+3}a_{2n-1}a_{2n+1}a_{2n-1}a_{2n+1}]_c) = \delta_b^{-7.5};$$
$$C_R([a_7a_{2n+1}a_{2n-1}a_{2n+1}a_{2n-1}a_{2n+1}a_{2n-1}a_{2n+1}]_c) = \delta_b^{-11};$$
$$C_R([a_7a_{2n+1}a_{2n-3}a_{2n+1}a_{2n-1}a_{2n+1}]_c) = -\delta_b^{-8.5};$$
$$C_R([a_7a_{2n+1}a_{2n-3}a_{2n+3}]_c) = \delta_b^{-6};$$
$$C_R([a_7a_{2n+5}a_{2n-1}a_{2n+1}]_c) = \delta_b^{-5.5};$$
$$C_R([a_7a_{2n+1}a_{2n-1}a_{2n+3}a_{2n-1}a_{2n+1}]_c) = -\delta_b^{-8}.$$

References

[1] M. Asaeda, U. Haagerup, Exotic subfactors of finite depth with Jones indices $(5+\sqrt{13})/2$ and $(5+\sqrt{17})/2$, Comm. Math. Phys. 202 (1999) 1–63.
[2] S. Bigelow, S. Morrison, E. Peters, N. Snyder, Constructing the extended Haagerup planar algebra, Acta Math. (2012) 29–82.
[3] D. Bisch, A note on intermediate subfactors, Pacific J. Math. 163 (1994) 201–216.
[4] D. Bisch, Bimodules, higher relative commutants and the fusion algebra associated to a subfactor, in: Operator Algebras and Their Applications, Waterloo, ON, 1994/1995, in: Fields Inst. Commun., vol. 13, Amer. Math. Soc., Providence, RI, 1997, pp. 13–63.
[5] D. Bisch, U. Haagerup, Composition of subfactors: new examples of infinite depth subfactors, Ann. Sci. Éc. Norm. Supér. 29 (1996) 329–383.
[6] D. Bisch, V.F.R. Jones, Algebras associated to intermediate subfactors, Invent. Math. 128 (1997) 89–157.
[7] D. Bisch, V.F.R. Jones, The free product of planar algebras, and subfactors, unpublished.
[8] F. Goodman, P. de la Harpe, V.F.R. Jones, Coxeter Graphs and Towers of Algebras, Math. Sci. Res. Inst. Publ., vol. 14, Springer-Verlag, 1989.
[9] M. Hartglass, The GJS construction for Fuss Catalan and N-P-M planar algebras, Dissertation Thesis.
[10] M. Izumi, The structure of sectors associated with Longo–Rehren inclusions II. Examples, Rev. Math. Phys. 13 (2001) 603–674.
[11] M. Izumi, V.F.R. Jones, S. Morrison, N. Snyder, Subfactors of index less than 5, part 3: quadruple points, Comm. Math. Phys. 316 (2) (2012) 531–554.
[12] M. Izumi, S. Morrison, D. Penneys, Fusion categories between $\mathcal{C} \boxtimes \mathcal{D}$ and $\mathcal{C} * \mathcal{D}$, arXiv:1308.5723.
[13] V.F.R. Jones, Index for subfactors, Invent. Math. 72 (1983) 1–25.
[14] V.F.R. Jones, The planar algebra of a bipartite graph, in: Knots in Hellas '98 (Delphi), in: Ser. Knots Everything, vol. 24, World Sci. Publ., River Edge, NJ, 2000, pp. 94–117.
[15] V.F.R. Jones, The annular structure of subfactors, in: Essays on Geometry and Related Topics, Vol. 1, 2, in: Monogr. Enseign. Math., vol. 38, Enseignement Math, Geneva, 2001, pp. 401–463.
[16] V.F.R. Jones, Quadratic tangles in planar algebras, Duke Math. J. 161 (2012) 2257–2295.
[17] V.F.R. Jones, Planar algebras, I, arXiv:math.QA/9909027.
[18] V.F.R. Jones, S. Morrison, N. Snyder, The classification of subfactors of index at most 5, Bull. Amer. Math. Soc. (N.S.) 51 (2) (2014) 277–327.
[19] V.F.R. Jones, D. Penneys, The embedding theorem for finite depth subfactor planar algebras, Quantum Topol. 2 (2011) 301–337.
[20] V.F.R. Jones, S. Reznikoff, Hilbert space representations of the annular Temperley–Lieb algebra, Pacific J. Math. 228 (2) (2006) 219–249.
[21] V.F.R. Jones, V.S. Sunder, Introduction to Subfactors, London Math. Soc. Lecture Note Ser., vol. 234, Cambridge University Press, 1997.
[22] Z. Landau, Exchange relation planar algebras, Geom. Dedicata 95 (2002) 183–214.
[23] Z. Liu, Exchange relation planar algebras of small rank, Trans. Amer. Math. Soc. (2015), in press, arXiv:1308.5656.
[24] Z. Liu, Hilbert space representations of the annular Fuss–Catalan algebra, in preparation.
[25] S. Morrison, D. Penneys, Constructing spoke subfactors using the jellyfish algorithm, Trans. Amer. Math. Soc. (2012), http://dx.doi.org/10.1090/S0002-9947-2014-06109-6, in press, arXiv:1208.3637.

[26] S. Morrison, D. Penneys, E. Peters, N. Snyder, Subfactors of index less than 5, part 2: triple points, Int. J. Math. 23 (3) (2012).
[27] S. Morrison, E. Peters, The little desert? some subfactors with index in the interval $(5, 3 + \sqrt{5})$, Internat. J. Math. 25 (8) (2014) 1450080, pp. 51.
[28] S. Morrison, N. Snyder, Subfactors of index less than 5, part 1: the principal graph odometer, Comm. Math. Phys. 117 (2012) 1–35.
[29] S. Morrison, K. Walker, The graph planar algebra embedding theorem, http://tqft.net/gpa, 2010.
[30] A. Ocneanu, Quantized groups, string algebras and Galois theory for algebras, in: Operator Algebras and Applications, Vol. 2, in: London Math. Soc. Lecture Note Ser., vol. 136, Cambridge Univ. Press, Cambridge, 1988, pp. 119–172.
[31] D. Penneys, E. Peters, Calculating two-strand jellyfish relations, Pacific J. Math. (2015), in press, arXiv:1308.5197.
[32] D. Penneys, J. Tener, Subfactors of index less than 5, part 4: quadruple points, Int. J. Math. 23 (3) (2012), 18 pages.
[33] E. Peters, A planar algebra construction of the Haagerup subfactor, Int. J. Math. 21 (2010) 987–1045.
[34] M. Pimsner, S. Popa, Entropy and index for subfactors, Ann. Sci. Éc. Norm. Supér. 19 (1986) 57–106.
[35] S. Popa, Classification of subfactors: reduction to commuting squares, Invent. Math. 101 (1990) 19–43.
[36] S. Popa, Classification of amenable subfactors of type II, Acta Math. 172 (1994) 352–445.
[37] S. Popa, An axiomatization of the lattice of higher relative commutants, Invent. Math. 120 (1995) 237–252.
[38] S. Popa, Free-independent sequences in type ii1 factors and related problems, Astérisque 232 (1995) 187.
[39] H. Wenzl, On sequences of projections, C. R. Math. Rep. Acad. Sci. Can. 9 (1) (1987) 5–9.