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Jones-Wassermann subfactors for modular tensor categories

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The representation category of a conformal net is a unitary modular tensor category. We investigate the reconstruction program: whether all unitary modular tensor categories are representation categories of conformal nets. We give positive evidence: the fruitful theory of multi-interval Jones-Wassermann subfactors on conformal nets is also true for modular tensor categories. We construct multi-interval Jones-Wassermann subfactors for unitary modular tensor categories. We prove that these subfactors are symmetrically self-dual. It generalizes and categorifies the self-duality of finite abelian groups. We call this duality the modular self-duality, because the modularity of the modular tensor category appears in a crucial way. For each unitary modular tensor category, we obtain a sequence of unitary fusion categories. The cyclic group case gives examples of Tambara-Yamagami categories.

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1. Introduction

Subfactor theory provides an entry point into a world of mathematics and physics containing large parts of conformal field theory, quantum algebras and low dimensional topology, see [9]) and references therein. In [10] V. Jones has devised a renormalization program based on planar algebras as an attempt to show that all finite depth subfactors are related to CFT, i.e., the double of a finite depth subfactor should be related to CFT.

More generally, the program is the following: given a unitary modular tensor category (MTC) $\mathscr{C}$ [22,30], can we construct a (complete rational) conformal net whose representation category is isomorphic to $\mathscr{C}$? We shall call such a program "reconstruction program", analogue to a similar program in higher dimensions by Doplicher-Roberts [1].

Given a rational conformal net $\mathfrak{A}$, and let I be a union of $m \geq 1$ disconnected intervals. The Jones-Wassermann subfactor is the subfactor $\mathfrak{A}(I) \subset \mathfrak{A}(I')'$ [13,21,31,32]. This subfactor is related to permutation orbifold and a simple application of orbifold theory shows that the Jones-Wassermann subfactor is self-dual, see Remark 6.19.

If the reconstruction program works, then for any MTC $\mathscr{C}$ we can find a rational conformal net $\mathfrak{A}$ such that the category of representations of $\mathfrak{A}$ is isomorphic to $\mathscr{C}$, it will follow that there are self-dual Jones-Wassermann subfactors for each integer $m > 1$. Hence a positive solution to reconstruction program would imply that we can construct self-dual Jones-Wassermann subfactors for each integer $m > 1$ associated with any unitary MTC $\mathscr{C}$. This is the motivation for our paper.

Our main result gives a construction of self-dual Jones-Wassermann subfactors for each integer $m > 1$ associated with any unitary MTC $\mathscr{C}$. The main difficulty is the proof of the self-duality for Jones-Wassermann subfactors for MTC. Furthermore, we prove that the Jones-Wassermann subfactors are symmetrically self-dual. The proof of the self-duality and symmetrical self-duality essentially requires the modularity of $\mathscr{C}$, so we call it the *modular self-duality*. We believe that our construction will shed new light on the reconstruction program.

We construct the "m-interval" Jones-Wassermann subfactor associated with a unitary MTC $\mathscr{C}$ by a Frobenius algebra γ_m in $\mathscr{C}^m$, the m^{th} tensor power of $\mathscr{C}$. The notion of intervals is natural in conformal nets, and it is important to understand the locality. However, there is no notion of intervals in modular tensor categories. This is a major problem in the reconstruction program.

We give an explicit formula for the object and the morphism of the Frobenius algebra γ_m. When $m = 2$, the Jones-Wassermann subfactor defines the quantum double of $\mathscr{C}$ [2,21,25–27]. For people who are interested in the algebraic aspects, the unitary condition is not necessary for our construction. The unitary condition is important for analysis and the reconstruction program.

The bimodule category of a subfactor is described by a subfactor planar algebra [11]. The n-box space of the planar algebra of the m-interval Jones-Wassermann subfactor for $\mathscr{C}$ is given by the vector space $\hom_{\mathscr{C}^m}(1, \gamma_m^n)$. It turns out to be natural to represent these vectors by a 3D picture. This representation identifies $\hom_{\mathscr{C}^m}(1, \gamma_m^n)$ as a configuration space $Conf_{n,m}$ on a 2D $n \times m$ lattice. Therefore the configuration space $\{Conf_{n,m}\}_{m,n\in\mathbb{N}}$ unifies the Jones-Wassermann subfactors for all $m \geq 1$. It is a natural candidate for the configuration space of a 2D lattice model that can be used in the reconstruction program.

Moreover, we show that planar tangles can act on $\{Conf_{n,m}\}_{m,n\in\mathbb{N}}$ in two different directions independently. In one direction m is fixed. These actions are the usual ones in the planar algebra of the m-interval Jones-Wassermann subfactor. In the other direction, n is fixed. These actions relate the Jones-Wassermann subfactor with different intervals which have not been studied before.

The bi-directional actions of planar tangles are compatible with the geometric actions on the 2D lattices. We call such family of vector spaces a *bi-planar algebra*. It is a new subject in subfactor theory and it adds one additional dimension to the theory of planar algebras.

This 3D representation also leads to the discovery of a new *m-n duality* by identifying the configuration spaces $\{Conf_{n,m}\}_{m,n\in\mathbb{N}}$ and $\{Conf_{m,n}\}_{m,n\in\mathbb{N}}$, while the meaning of the actions of planar tangles in the two directions are completely different. It will be interesting to understand these additional symmetries in conformal field theory.

When $\mathscr{C}$ is the representation category of a finite abelian group G, the configuration space $Conf(\mathscr{C})_{2,2}$ becomes $L^2(G)$. Moreover, the modular self-duality coincides with the self-duality of G. The proof of the self-duality of G requires the discrete Fourier transform on G. For general MTCs, we construct the string Fourier transform (SFT) on the configuration space $\hom_{\mathscr{C}^m}(1, \gamma_m^n)$ to prove the modular self-duality. From this point of view, the modular self-duality and the SFT generalize and categorify the self-duality and the Fourier transform of finite abelian groups.

Furthermore, we prove that the m-interval Jones-Wassermann subfactor of a unitary MTC $\mathscr{C}$ is symmetrically self-dual, namely their planar algebras are unshaded in Theorems 6.18 and 6.20. The projection category of the even part of a subfactor planar algebra is a unitary fusion category [4,24]. The projection category of an unshaded planar algebra is a $\mathbb{Z}_2$-graded unitary fusion category, whose generating odd object is symmetrically self-dual [17,23]. From the unshaded planar algebra of the m-interval Jones-Wassermann subfactors of $\mathscr{C}$, we obtain a Z_2-graded extension of the full subcategory of $\mathscr{C}^m$ generated by the Frobenius algebra γ_m, such that $\gamma_m = \tau \otimes \tau$ and τ is a symmetrically self-dual odd object. The object τ appeared as a twisted sector in orbifold theory of conformal nets, see Remark 6.19 and [12]. When the Grothendieck ring of $\mathscr{C}$ is a finite cyclic group, we obtain the Tambara-Yamagami category [29] from the unshaded planar algebra of the two-interval Jones-Wasserman subfactor of $\mathscr{C}$, and τ is the only odd simple object. In general, for any unitary MTC $\mathscr{C}$ and any interval m, $m \geq 1$, we obtain a $\mathbb{Z}_2$-graded unitary fusion category [4], which generalizes Tambara-Yamagami categories.

The modular transformation S was studied in the early work of 't Hooft in rational conformal field theory as a Hopf link, a union of a Wilson loop and a Dirac string [6]. This has been formalized in the framework of conformal nets by Rehren in [28]. In the general theory of MTCs, we prove that the modular transformation S for the MTC $\mathscr{C}$ is identical to the SFT on the vector space $\hom_{\mathscr{C}^2}(1, \gamma_2^2)$ in our construction. This provides a different point of view to understand the S matrix of MTCs as a special case of the SFT. The unshaded condition of the planar algebra is necessary to define the SFT as a matrix on $\hom_{\mathscr{C}^2}(1, \gamma_2^2)$. The SFT is always a unitary, so the modularity of the MTC, namely the S matrix is a unitary [30], is crucial in this identification.

The recent progress about the Fourier analysis of the SFT on subfactors in [8,15,19] leads to many interesting inequalities for the S matrix. These unshaded planar algebras have been used in the quon 3D language for quantum information [18], where the vector space $\hom_{\mathscr{C}^2}(1, \gamma_2^2)$ is considered as the 1-quon space. This new interpretation leads to many interesting algebraic identities for the S matrix in a MTC as the Fourier duality of quons [16]. A combination of these works leads to further applications in the study of MTC.

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2. Configuration spaces

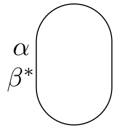
2.1. Modular tensor categories

We refer the readers to [30] for basic definitions about modular tensor categories (MTC). All categories in this paper are strict. Suppose $\mathscr{C}$ is a unitary MTC. Let Irr be the set of (representatives of) simple objects of $\mathscr{C}$ and the unit is denoted by 1. Take $\tilde{X} = \bigoplus_{X \in Irr(\mathscr{C})} X$. For an object X, its dual object is denoted by $\overline{X}$. Its quantum dimension is $d(X)$. Let $\mu = \sum_{X \in Irr} d(X)^2$ be the global dimension of $\mathscr{C}$.

The modular conjugation $J_{\mathscr{C}}$ on $\mathscr{C}$ is a horizontal reflection. We have that $J_{\mathscr{C}}(X) = \overline{X}$. Moreover, for objects X, Y, Z in $\mathscr{C}$, $J_{\mathscr{C}} : \hom_{\mathscr{C}}(X \otimes Y, Z) \to \hom_{\mathscr{C}}(\overline{Y} \otimes \overline{X}, \overline{Z})$ is an anti-linear algebroid isomorphism. The adjoint operator $*$ on $\mathscr{C}$ is a vertical reflection. We have that $X^* = X$. Moreover, $* : \hom_{\mathscr{C}}(X \otimes Y, Z) \to \hom_{\mathscr{C}}(Z, X \otimes Y)$ is an anti-linear algebroid anti-isomorphism. The contragredient map ρ_π on $\mathscr{C}$ is a rotation by π. We have that $\rho_\pi(X) = \overline{X}$. Moreover, $\rho_\pi : \hom_{\mathscr{C}}(X \otimes Y, Z) \to \hom_{\mathscr{C}}(\overline{Z}, \overline{Y} \otimes \overline{X})$ is a linear algebroid anti-isomorphism. Furthermore

$$J_{\mathscr{C}} = \rho_\pi \circ *. \tag{1}$$

For objects X, Y in $\mathscr{C}$, the inner product of $\alpha, \beta \in \hom_{\mathscr{C}}(X, Y)$ is defined as

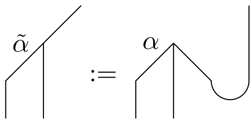
$\langle \beta, \alpha \rangle =$ 

,

which is an element in $\hom_{\mathscr{C}}(1,1) \cong \mathbb{C}$.

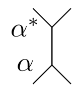
We can identify the morphism spaces $\hom_{\mathscr{C}}(\overline{Z}, X \otimes Y)$ and $\hom_{\mathscr{C}}(1, X \otimes Y \otimes Z)$ as follows: For a morphism $\alpha \in \hom_{\mathscr{C}}(1, X \otimes Y \otimes Z)$, we obtain a morphism $\tilde{\alpha} = (1_X \otimes 1_Y \otimes \phi_{Z \otimes \overline{Z}})(\alpha \otimes 1_{\overline{Z}})$ in $\hom_{\mathscr{C}}(\overline{Z}, X \otimes Y)$, where $\phi_{Z \otimes \overline{Z}} \in \hom_{\mathscr{C}}(1, Z \otimes \overline{Z})$ is the duality map.

Notation 2.1 *(Frobenius reciprocity).* Diagrammatically we represent $\tilde{\alpha}$ as



Notation 2.2. For an object X in $\mathscr{C}$, we denote an ortho-normal-basis of $\hom_{\mathscr{C}}(1, X)$ by $ONB_{\mathscr{C}}(X)$, or $ONB(X)$ for short. We denote an ortho-normal-basis of $\hom_{\mathscr{C}}(X, 1)$ by $ONB^*_{\mathscr{C}}(X)$, or $ONB^*(X)$ for short.

For two objects X and Y, we have the resolution of the identity:

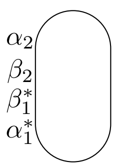
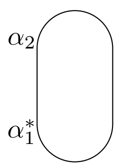
$1_X \otimes 1_Y =$  $= \sum_{Z \in Irr, \alpha \in ONB(X \otimes Y \otimes Z)} d(Z)$  (2)

Lemma 2.3. *Suppose Y and Z are objects in $\mathscr{C}$ and $X \in Irr$. Let $ONB(Y, X)$, $ONB(X, Z)$ be ONB of* $\hom_{\mathscr{C}}(Y, X)$ *and* $\hom_{\mathscr{C}}(X, Z)$*. Then*

$$\{\sqrt{d(X)}\beta\alpha \ : \ X \in Irr,\, \alpha \in ONB(Y, X),\, \beta \in ONB(X, Z)\}$$

is an ONB of $\hom_{\mathscr{C}}(Y, Z)$*.*

Proof. For $\alpha_i \in ONB(Y, X)$, $\beta_i \in ONB(X, Z)$, $i = 1, 2$,

$d(X)\langle \beta_1\alpha_1, \beta_2\alpha_2 \rangle = d(X)$  $= \langle \beta_1, \beta_2 \rangle$  $= \langle \beta_1, \beta_2 \rangle \langle \alpha_1, \alpha_2 \rangle.$

On the other hand,

$$\sum_{X \in Irr} \dim \hom_{\mathscr{C}}(Y, X) \dim \hom_{\mathscr{C}}(X, Z) = \dim \hom_{\mathscr{C}}(Y, Z).$$

Therefore, the statement holds. $\square$

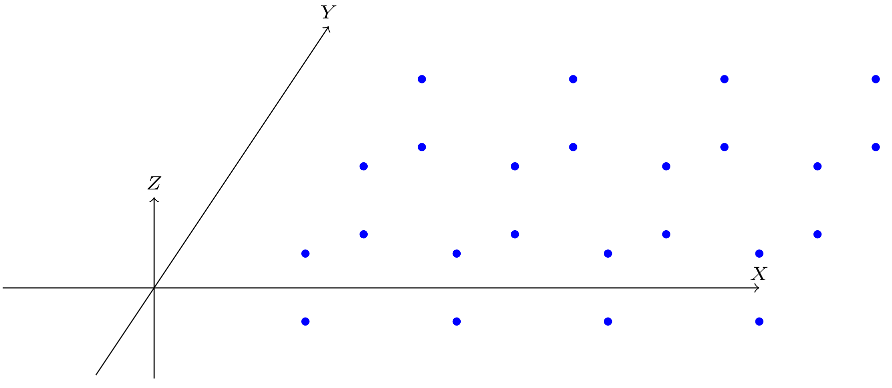


Fig. 1. Grid(n,m) for $n = 4$, $m = 3$.

2.2. Configuration spaces

Now let us define the configuration space on a finite 2D-lattice with the target space $\mathscr{C}$. Each configuration has three parts: Z-, X-, Y- configurations.

We use $Grid(n,m)$ to represent the grid $\mathbb{Z}_n \times \mathbb{Z}_m \times \{\pm 1\}$. We allocate the vertices of the grid at $(i, j, \pm 1)$, $1 \leq i \leq n$ and $0 \leq j \leq m-1$ in the 3D space which are indicated by the bullets in Fig. 1. To simplify the notations, we draw pictures for $n = 4$, $m = 3$. The reader can figure out the general case.

For the lattice $Lat = \mathbb{Z}_n \times \mathbb{Z}_m$, a Z**-configuration** is a map from the sites of the lattice to simple objects in $\mathscr{C}$. We denote the simple object at the site (i, j) as $X_{i,j}$. We denote this Z-configuration by $X_{\vec{i},\vec{j}}$ and represent it in the 3D space by assigning the object $X_{i,j}$ to the line from $(i, j, 1)$ to $(i, j, -1)$ as in Fig. 2:

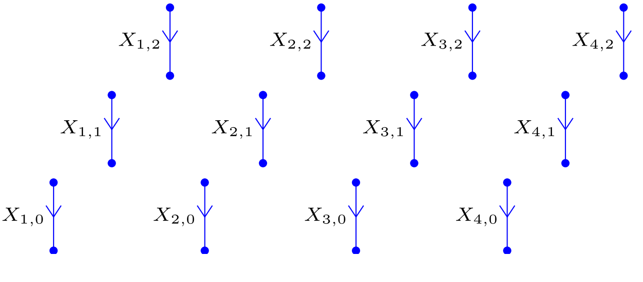


Fig. 2. Z-Configuration.

We denote $X_{\vec{i},j} = X_{1,j} \otimes \cdots \otimes X_{n,j}$ and $X_{i,\vec{j}} = X_{i,0} \otimes \cdots \otimes X_{i,m-1}$. Moreover,

$$d(X_{\vec{i},\vec{j}}) := \prod_{1 \leq i \leq n, 0 \leq j \leq m-1} d(X_{i,j}),$$

$$d(X_{\vec{i},j}) := \prod_{1 \leq i \leq n} d(X_{i,j}),$$

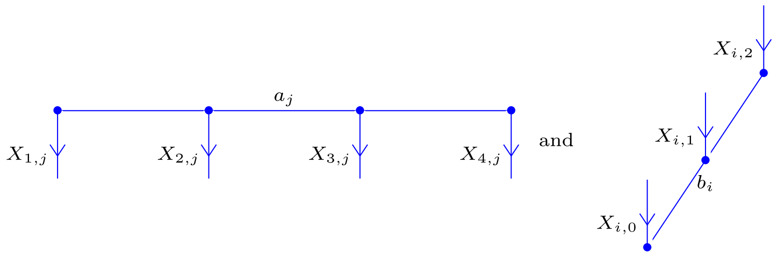


Fig. 3. X- and Y-Configurations.

$$d(X_{i,\vec{j}}) := \prod_{0\le j\le m-1} d(X_{i,j}).$$

An X**-configuration** with boundary $X_{\vec{i},j}$ is a morphism a_j in $\hom_{\mathscr{C}}(1, X_{\vec{i},j})$. We denote the boundary by $X(a_j) := X_{\vec{i},j}$. A Y**-configuration** with boundary $X_{i,\vec{j}}$ is a morphism b_i in $\hom_{\mathscr{C}}(X_{i,\vec{j}}, 1)$. We denote the boundary by $X(b_i) := X_{i,\vec{j}}$. We represent them in the 3D space in Fig. 3. Moreover, we call $a_{\vec{j}} = a_0 \otimes \cdots \otimes a_m$ an X-configuration with boundary $X_{\vec{i},\vec{j}}$ and $b_{\vec{i}} = b_1 \otimes \cdots \otimes b_n$ a Y-configuration with boundary $X_{\vec{i},\vec{j}}$. We define the X**-configuration space** with boundary $X_{\vec{i},\vec{j}}$ as

$$Conf_X(X_{\vec{i},\vec{j}}) := \bigotimes_{j=0}^{m-1} \hom_{\mathscr{C}}(1, X_{\vec{i},j})\,.$$

We define the Y**-configuration space** with boundary $X_{\vec{i},\vec{j}}$ as

$$Conf_Y(X_{\vec{i},\vec{j}}) := \bigotimes_{i=1}^{n} \hom_{\mathscr{C}}(X_{i,\vec{j}}, 1)\,.$$

We call $a_{\vec{j}} \otimes b_{\vec{i}}$ a **configuration** with boundary $X_{\vec{i},\vec{j}}$, denoted by $X(a_{\vec{j}} \otimes b_{\vec{i}}) := X_{\vec{i},\vec{j}}$. We represent it in the 3D space as in Fig. 4: We define the **configuration space** on the $n \times m$ 2D-lattice Lat to be the Hilbert space

$$\begin{aligned} Conf(Lat) = Conf(\mathscr{C})_{m,n} :&= \bigoplus_{X_{\vec{i},\vec{j}} \in Irr^{nm}} Conf_X(X_{\vec{i},\vec{j}}) \otimes Conf_Y(X_{\vec{i},\vec{j}}) \\ &= \bigoplus_{X_{\vec{i},\vec{j}} \in Irr^{nm}} \left(\bigotimes_{j=0}^{m-1} \hom_{\mathscr{C}}(1, X_{\vec{i},j}) \otimes \bigotimes_{i=1}^{n} \hom_{\mathscr{C}}(X_{i,\vec{j}}, 1) \right), \end{aligned}$$

where each hom space is considered as a Hilbert space. We simply use the notation $\sum a_{\vec{j}} \otimes b_{\vec{i}}$ to represent an element in $Conf(Lat)$.

$$\mu^{\frac{(1-n)(m-1)}{4}}\sqrt{d(X_{\vec{i},\vec{j}})}$$

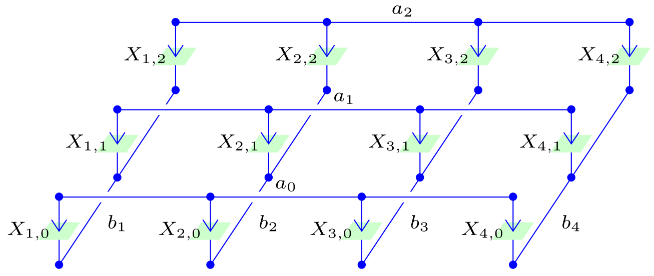


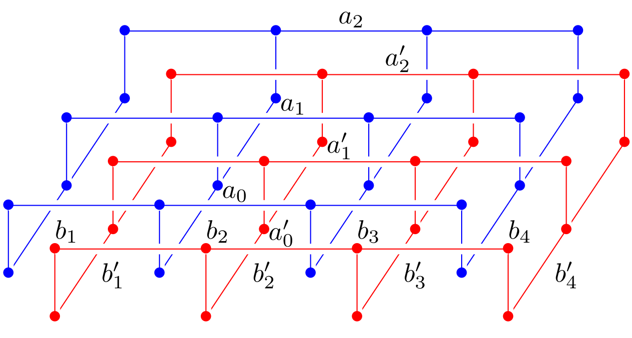
Fig. 4. Configurations for $n = 4$, $m = 3$: We use the small square (orthogonal to the Z-axis) at the coordinate $(i, j, 0)$ to indicate the vertex (i, j) in the lattice $\mathbb{Z}_m \times \mathbb{Z}_n$. Moreover, the boundaries of $a_{\vec{j}}$ and $b_{\vec{i}}$ are separated on opposite sides of the small squares. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

2.3. Duality

When we consider $Lat = \mathbb{Z}_n \times \mathbb{Z}_m$ as a lattice on a torus, its dual lattice Lat' is also $\mathbb{Z}_n \times \mathbb{Z}_m$ and the configuration space on the dual lattice is $Conf(\mathscr{C})_{m,n}$. We allocate the vertices of the corresponding $Grid(n, m)$ at $(i + 1/2, j - 1/2, \pm 1)$, $1 \leq i \leq n$ and $0 \leq j \leq m - 1$ in the 3D space.

We define a bilinear form LL on the configuration spaces of the lattice and the dual lattice $Conf(Lat) \otimes Conf(Lat') = Conf(\mathscr{C})_{m,n} \otimes Conf(\mathscr{C})_{m,n}$. For $a_{\vec{j}} \otimes b_{\vec{i}}$ with boundary $X_{\vec{i},\vec{j}}$ in $Conf(Lat)$, and $a'_{\vec{j}} \otimes b'_{\vec{i}}$ with boundary $X'_{\vec{i},\vec{j}}$ in $Conf(Lat')$, the bilinear form LL is defined as

$$LL(a_{\vec{j}} \otimes b_{\vec{i}}, a'_{\vec{j}} \otimes b'_{\vec{i}}) = \mu^{\frac{(1-n)(m-1)}{2}} \sqrt{d(X_{\vec{i},\vec{j}}) d(X'_{\vec{i},\vec{j}})}$$



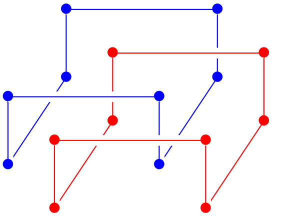
(3)

When $m = 0$ or $n = 0$, we define the configuration space as the ground field. We define LL as the multiplication of the two scalars.

Theorem 2.4. *For any unitary MTC $\mathscr{C}$, the configurations spaces of the lattice and the dual lattice are dual to each other. Precisely the map from $Conf(Lat)$ to the dual space of $Conf(Lat')$ induced by $LL(-,-)$ is an isometry.*

We first prove the case for $m = n = 2$. We prove the general case by a bi-induction in §6.

Proof for the case $m = n = 2$**:** When $m = n = 2$, the diagram in Equation (3) becomes the Hopf link and LL defines the S matrix of $\mathscr{C}$.



By the modularity of $\mathscr{C}$, namely the S matrix is a unitary [30], the map induced by LL is an isometry. □

Proposition 2.5. *Suppose V is a Hilbert space and $\{\alpha_i\}$ is an ONB. Let V' be the dual space of V. For $f \in V'$, a linear functional on V,*

$$r(f) = \sum_i \overline{f(\alpha_i)} \alpha_i \tag{4}$$

is independent of the choice of the basis.

Proof. It follows directly from the definition. □

The map $r : V^* \to V$ is an anti-isometry which is well-known as the Riesz representation. Therefore we obtain an anti-isometry $D : Conf(Lat') \to Conf(Lat)$ that we call the duality map:

Definition 2.6 *(duality maps).* We define

$$\mathfrak{D}_+(x) = \sum_{x' \in B'} \overline{LL(x, x')} x',$$

$$\mathfrak{D}_-(x') = \sum_{x \in B} \overline{LL(x, x')} x,$$

where B is an ONB of $Conf(Lat)$ and B' is an ONB of $Conf(Lat')$.

Therefore Theorem 2.4 is equivalent to the following Proposition.

Proposition 2.7. *The map $\mathfrak{D}_+$ is an anti-linear isometry from $Conf(Lat)$ to $Conf(Lat')$, and $\mathfrak{D}_-$ is its inverse.*

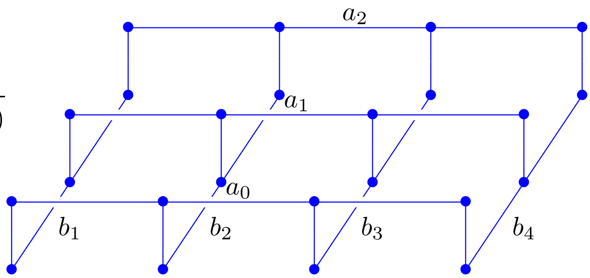
Definition 2.8. We use $1_{n,m}$ to denote the trivial configuration whose Z-, X-, Y-configurations are all 1. We define

$$\mu_{n,m} := \mathfrak{D}_{-}(1_{n,m}). \tag{5}$$

Definition 2.9. We define L as a linear functional on $Conf(Lat)$ as $L(x) = LL(x, 1_{n,m})$.

Then

$$L(a_{\vec{j}} \otimes b_{\vec{i}}) = \mu^{\frac{(1-n)(m-1)}{2}} \sqrt{d(X_{\vec{i},\vec{j}})}$$



, (6)

and

$$\mu_{n,m} = \sum_{\alpha \in B} \overline{L(\alpha)} \alpha, \tag{7}$$

where B be is an ONB of $Conf(Lat)$.

In §3, we study the actions of rotations and reflections in X- and Y-directions on the lattices and the induced actions the configuration spaces.

In §4, we fix m and study the structure of the configuration space for different n. We prove that $\mu_{3,m}$ defines a Frobenius algebra and these configuration spaces admit the action of planar tangles in the X-direction. Thus For each $m \geq 1$, $\{\mathscr{S}_{n,+} = Conf(\mathscr{C})_{m,n}\}_{n \in \mathbb{N}}$ defines a subfactor planar algebra. This defines the m-interval Jones-Wassermann subfactor for the unitary MTC $\mathscr{C}$. We prove this result in Theorem 4.13.

In §5, we describe the action of planar tangles on the dual space $\{\mathscr{S}_{n,-}\}$.

In §6, we construct a planar algebra *-isomorphism from $\mathscr{S}_{\cdot,+}$ to $\mathscr{S}_{\cdot,-}$. That means $\{\mathscr{S}_{n,\pm}\}$ defines a self-dual subfactor planar algebra, i.e., the m-interval Jones-Wassermann subfactor for $\mathscr{C}$ is self-dual. We prove this result in Theorem 6.18. Furthermore, we prove that the *-isomorphism commutes with string Fourier transform (SFT). That means $\{\mathscr{S}_n = Conf(\mathscr{C})_{m,n}\}_{n \in \mathbb{N}}$ defines an unshaded subfactor planar algebra, i.e., the m-interval Jones-Wassermann subfactor for $\mathscr{C}$ is symmetrically self-dual. (It is called symmetrically self-dual, since the SFT is a symmetric matrix [17].) We prove this result in Theorem 6.20. Moreover, the duality map is related to the SFT of the unshaded planar algebra.

Remark 2.10. If we fix n, instead of m, then all the results also work. So we also have the action of planar tangles on the configuration spaces in the Y-direction. Therefore the configuration spaces $\{Conf(\mathscr{C})_{m,n}\}_{m,n\in\mathbb{N}}\}$ admit the action of planar tangles in two different directions.

3. Actions on configuration spaces

3.1. Automorphisms on the lattice

Note that the lattice $\mathbb{Z}_m \times \mathbb{Z}_n$ is invariant under the following actions:

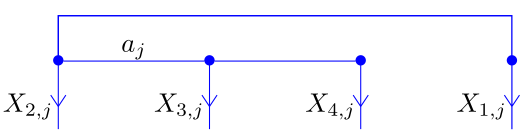
- The clockwise $2\pi/n$ rotation around the Y-direction ρ_1: $(i,j) \to (i-1,j)$.
- The reflection in the X-direction θ_1: $(i,j) \to (n+1-i,j)$.
- The clockwise $2\pi/m$ rotation around the X-direction ρ_2: $(i,j) \to (i,j+1)$.
- The reflection in the Y-direction θ_2 $(i,j) \to (i,m-1-j)$.

Now let us define the induced action on the configuration space $Conf(\mathscr{C})_{n,m}$.

For $k=1,2$, the induced actions on the Z-configurations are

$$\rho_k(X)_{i,j} = X_{\rho_k^{-1}(i,j)},$$
$$\theta_k(X)_{i,j} = J_{\mathscr{C}}(X_{\theta_k^{-1}(i,j)}).$$

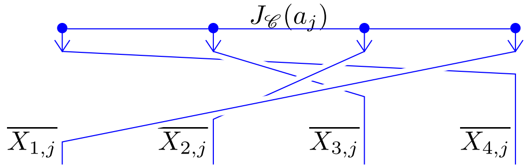
For an X-configuration a_k, we define

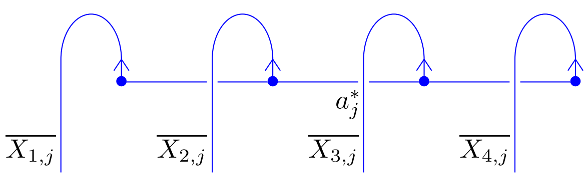
$\rho_1(a_j) =$ 

,

$\theta_1(a_j) = J_{\mathscr{C}}(a_j),$

$\rho_2(a_j) = a_j,$

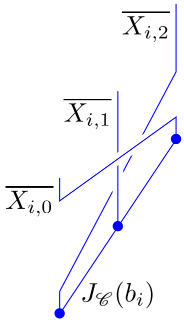
$\theta_2(a_j) =$ 

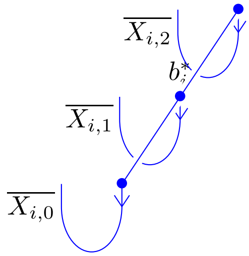
$=$ 

.

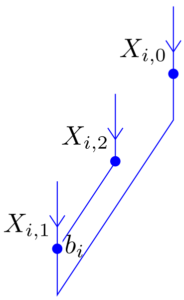
For a Y-configuration b_i, we define

$$\rho_1(b_i) = b_i,$$

$\theta_1(b_i) =$ 

$=$ 

,

$\rho_2(b_i) =$ 

,

$$\theta_2(b_i) = J_{\mathscr{C}}(b_i).$$

Definition 3.1. For a configuration $a_{\vec{j}} \otimes b_{\vec{i}}$, we define

$$\rho_1(a_{\vec{j}} \otimes b_{\vec{i}}) = (\rho_1(a_0) \otimes \cdots \otimes \rho_1(a_{m-1})) \otimes (b_2 \otimes \cdots \otimes b_n \otimes b_1),$$
$$\theta_1(a_{\vec{j}} \otimes b_{\vec{i}}) = (\theta_1(a_0) \otimes \cdots \otimes \theta_1(a_{m-1})) \otimes (\theta_1(b_n) \otimes \cdots \otimes \theta_1(b_1)),$$
$$\rho_2(a_{\vec{j}} \otimes b_{\vec{i}}) = (a_{m-1} \otimes a_1 \otimes \cdots \otimes a_{m-2}) \otimes (\rho_2(b_1) \otimes \cdots \otimes \rho_2(b_n)),$$
$$\theta_2(a_{\vec{j}} \otimes b_{\vec{i}}) = (\theta_2(a_{m-1}) \otimes \cdots \otimes \theta_2(a_0)) \otimes (\theta_2(b_1) \otimes \cdots \otimes \theta_2(b_n)).$$

The actions on the configuration space are defined by their linear or anti-linear extensions.

Note that this definition coincide with the geometric actions on the configurations in Fig. 4. Therefore, their relations also hold on the configuration space.

Proposition 3.2. *On the configuration space, ρ_1 and θ_1 commute with ρ_2 and θ_2, and for $k = 1, 2$, $\rho_k\theta_k = \theta_k\rho_k^{-1}$, $\rho_k^m = 1$, $\theta_k^2 = 1$.*

3.2. Automorphisms on the dual pair of lattices

Similarly, we also define the four actions on the dual lattice Lat' and the configuration space $Conf(Lat')$.

Proposition 3.3. *For $x \in Conf(Lat)$ and $x' \in Conf(Lat')$, we have that*

$$LL(x,x') = LL(\rho_k(x), \rho_k(x')), \quad k = 1,2 \tag{8}$$

$$\overline{LL(x,x')} = LL(\theta_1(x), \rho_1\theta_1(x')), \tag{9}$$

$$\overline{LL(x,x')} = LL(\rho_1\theta_2(x), \theta_2(x')). \tag{10}$$

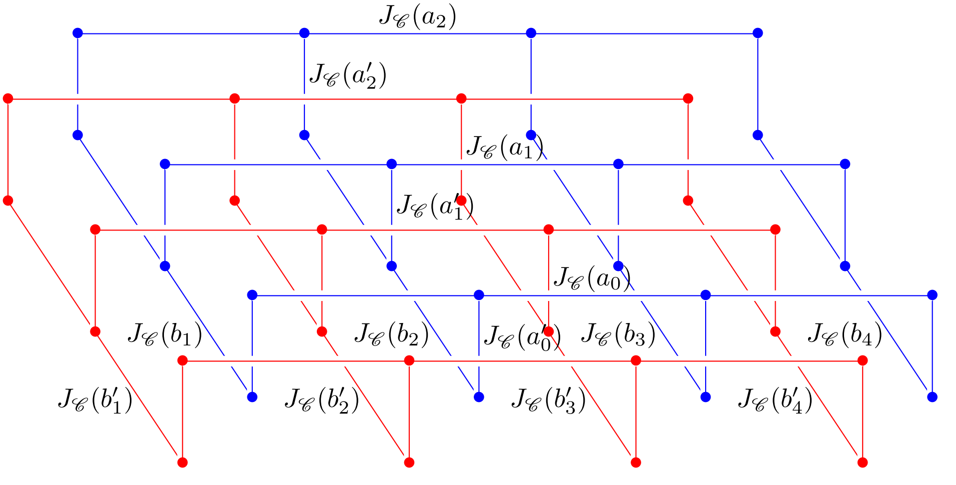
Proof. It is enough to prove the case $x = a_{\vec{j}} \otimes b_{\vec{i}}$, $x' = a'_{\vec{j}} \otimes b'_{\vec{i}}$.

Recall that LL is defined by a closed diagram in the 3D space as shown in Equation (3). Applying the rotation on the 3D diagram, we obtain Equation (8).

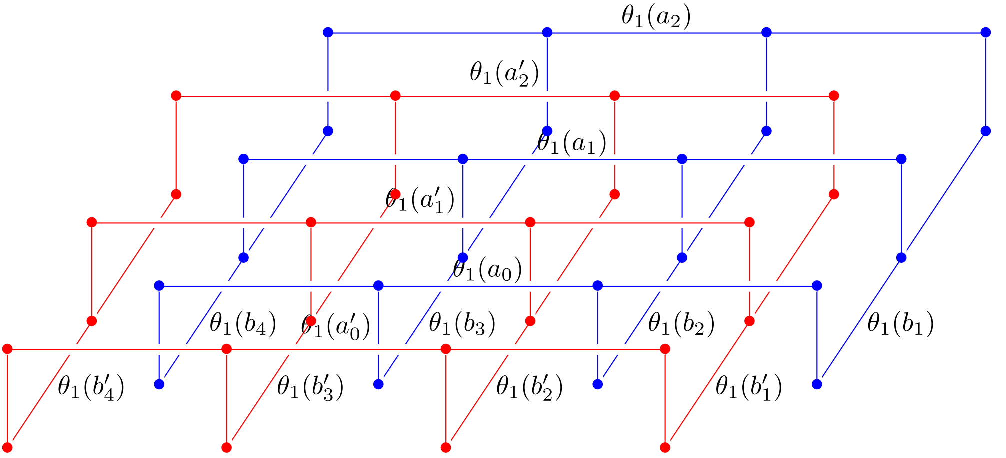
If we consider the 3D diagram as an element in $\mathscr{C}$, then we have that

$$\overline{LL(a_{\vec{j}} \otimes b_{\vec{i}}, a'_{\vec{j}} \otimes b'_{\vec{i}})}$$

$$= \mu^{\frac{(1-n)(m-1)}{2}} \sqrt{d(X_{\vec{i},\vec{j}})d(X'_{\vec{i},\vec{j}})}$$



$$= \mu^{\frac{(1-n)(m-1)}{2}} \sqrt{d(X_{\vec{i},\vec{j}})d(X'_{\vec{i},\vec{j}})}$$



$= LL(\theta_1(x), \rho_1\theta_1(x'))$

The proof of Equation (10) is similar. □

By definitions, ρ_k, θ_k, $k = 1, 2$, preserve $1_{n,m}$. Take $x' = 1_{n,m}$ in Proposition 3.3, we obtain that

Proposition 3.4. *For any x in $Conf(Lat)$, $k = 1, 2$,*

$$L(\rho_k(x)) = L(x),$$
$$L(\theta_k(x)) = \overline{L(x)}.$$

Proposition 3.5. *The four actions ρ_k, θ_k, $k = 1, 2$, preserve $\mu_{n,m}$.*

Proof. The statement follows from Proposition 3.4 and the definition of $\mu_{n,m}$ in Equation (7). □

4. Jones-Wassermann subfactors for MTC

4.1. Identification

Suppose $\mathscr{C}$ is a unitary MTC. Take $Irr^m = \{\vec{X} = X_0 \otimes \cdots \otimes X_{m-1} : X_j \in Irr,\ 1 \leq j \leq m-1\}$. Let $\mathscr{C}^m$ be the m^{th} Deligne tensor power of $\mathscr{C}$, denoted by $\mathscr{C}^m := \mathscr{C} \boxtimes \mathscr{C} \boxtimes \cdots \boxtimes \mathscr{C}$. We can consider $\vec{X}$ as an object $\vec{X}^{\boxtimes} := X_0 \boxtimes \cdots \boxtimes X_{m-1}$ in $\mathscr{C}^m$. Then the set Irr^m is identical to the set of (representatives of) simple objects of $\mathscr{C}^m$. The objects $\vec{X}$ in $\mathscr{C}$ and $\vec{X}^{\boxtimes}$ in $\mathscr{C}^m$ have the same quantum dimension, $d(\vec{X}) = d(\vec{X}^{\boxtimes}) = \prod_{j=0}^{m-1} d(X_j)$.

Definition 4.1. For a unitary MTC $\mathscr{C}$ and $m \in \mathbb{Z}_+$, we define

$$N_{\vec{X}} = \dim \hom_{\mathscr{C}}(X_0 \otimes \cdots \otimes X_{m-1}, 1); \tag{11}$$

$$\gamma = \gamma_m = \bigoplus_{\vec{X} \in Irr^m} N_{\vec{X}} \vec{X}^{\boxtimes}. \tag{12}$$

Proposition 4.2. *Recall that μ is the global dimension of $\mathscr{C}$. For $m \geq 1$,*

$$d(\gamma) = \mu^{m-1}.$$

Proof. It is obvious for $m = 1$. When $m \geq 2$,

$$\begin{aligned} d(\gamma) &= \sum_{\vec{X} \in Irr^m} N_{\vec{X}} d(\vec{X}) \\ &= \sum_{\vec{X} \in Irr_{m-1}, Y \in Irr_1} \dim \hom_{\mathscr{C}}(\vec{X}, Y) d(\vec{X}) d(Y) \qquad \text{by Frobenius reciprocity} \\ &= \sum_{\vec{X} \in Irr_{m-1}} d(\vec{X})^2 \\ &= \mu^{m-1}. \quad \square \end{aligned}$$

Notation 4.3. For a fixed m, we take $\delta = \mu^{\frac{m-1}{2}}$.

For each $\vec{X}$, let $ONB^*_{\mathscr{C}}(\vec{X})$ be an ONB of $\hom_{\mathscr{C}}(\vec{X}, 1)$. Then we can use the basis to represent the multiplicity of simple objects in γ.

$$\gamma = \bigoplus_{\vec{X} \in Irr^m, b \in ONB^*_{\mathscr{C}}(\vec{X})} \vec{X}(b), \tag{13}$$

where $\vec{X}(b) = \vec{X}^{\boxtimes} = X_0 \boxtimes \cdots \boxtimes X_{m-1}$.

The representation is covariant with respect to the choice of the ONB: For an object Y in $\mathscr{C}^m$ and a morphism $y \in \hom_{\mathscr{C}^m}(Y, N_{\vec{X}} \vec{X}^{\boxtimes})$, we take two ONB $B(1)$, $B(2)$ of $\hom_{\mathscr{C}}(\vec{X}, 1)$. Then we obtain two representations

$$y = \bigoplus_{b_1 \in B(1)} y(b_1), \qquad y(b_1) \in \hom_{\mathscr{C}^m}(Y, \vec{X}(b_1)),$$

$$y = \bigoplus_{b_2 \in B(2)} y(b_2), \qquad y(b_2) \in \hom_{\mathscr{C}^m}(Y, \vec{X}(b_2)).$$

The covariance of the representation means that

$$y(b) = \sum_{b'} \langle b', b \rangle y(b').$$

Note that

$$\hom_{\mathscr{C}^m}(1,\gamma^n) = \bigoplus_{X_{\vec{i},\vec{j}}\in Irr^{nm}} \bigoplus_{b_i\in ONB^*_{\mathscr{C}}(X_{i,\vec{j}}),1\leq i\leq n} \hom_{\mathscr{C}^m}(1, X_{\vec{i},\vec{j}}(b_{\vec{i}})), \tag{14}$$

where

$$\hom_{\mathscr{C}^m}(1, X_{\vec{i},\vec{j}}(b_{\vec{i}})) = \boxtimes_{j=0}^{m-1} \hom_{\mathscr{C}}(1, X_{\vec{i},j}(b_{\vec{i}})).$$

We call $\hom_{\mathscr{C}^m}(1,\gamma^n)$ the *n-box space* of γ. For any $a_j \in \hom_{\mathscr{C}}(1, X_{\vec{i},j})$, $0 \leq j \leq m-1$, we have $a_{\vec{j}}(b_{\vec{i}}) \in \hom_{\mathscr{C}^m}(1, X_{\vec{i},\vec{j}}(b_{\vec{i}}))$.

Definition 4.4. We define a map $\Phi : \hom_{\mathscr{C}^m}(1,\gamma^n) \to Conf(\mathscr{C})_{n,m}$ as a linear extension of

$$\Phi(a_{\vec{j}}(b_{\vec{i}})) = a_{\vec{j}} \otimes b_{\vec{i}}.$$

The definition is independent of the choice of the ONB $b_{\vec{i}}$, since the representation is covariant. Moreover,

$$\langle a_{\vec{j}}(b_{\vec{i}}), c_{\vec{j}}(d_{\vec{i}})\rangle = \langle a_{\vec{j}}, c_{\vec{j}}\rangle\langle b_{\vec{i}}, d_{\vec{i}}\rangle = \langle a_{\vec{j}} \otimes b_{\vec{i}}, c_{\vec{j}} \otimes d_{\vec{i}}\rangle.$$

So Φ is an isometry. Therefore we can identify the vectors in the two Hilbert spaces $\hom_{\mathscr{C}^m}(1,\gamma^n)$ and $Conf(\mathscr{C})_{n,m}$. We use the notation $\sum a_{\vec{j}}(b_{\vec{i}})$ to represent an element in $\hom_{\mathscr{C}^m}(1,\gamma^n)$.

Definition 4.5. Induced by the isometry Φ, the four actions ρ_k, θ_k, $k = 1, 2$, and the contractions $\wedge_k$, $k \geq 0$, are also defined on $\hom_{\mathscr{C}^m}(1,\gamma^n)$, still denoted by ρ_k, θ_k.

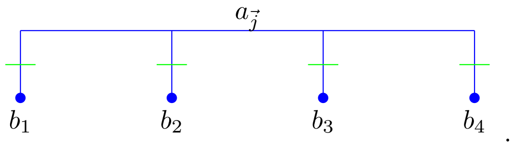
Recall that the multiplicity of $X_{\_,\vec{j}}$ in γ is represented by b in $ONB^*(X_{\_,\vec{j}})$. We need an anti-isometric involution on $ONB^*(X_{\_,\vec{j}})$ to specify the dual of $X_{\_,\vec{j}}(b)$. To be compatible with the geometric interpretation of the configuration in the 3D space, we define the dual by θ_1 as follows:

Definition 4.6. For an object $X_{\_,\vec{j}}(b)$, we define its dual object as $\overline{X_{\_,\vec{j}}}(\theta_1(b))$.

Note that $\theta_1^2(b) = b$, thus $\overline{\overline{X_{\_,\vec{j}}(b)}} = X_{\_,\vec{j}}(b)$. By Frobenius reciprocity, the modular conjugation on $\mathscr{C}^m$ is given by θ_1.

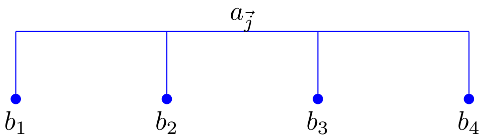
The element in $\hom_{\mathscr{C}^m}(1,\gamma^n)$ is usually represented by a diagram on the 2D plane. To be compatible with the isometry Φ, we simplify the 3D pictures for configurations by their projections on the plane $Y = 0$ as follows:

(1) The configuration in Fig. 4 is simplified as its projection on the plane $Y = 0$,



(2) Induced by the isometry Φ, L becomes a linear functional on $\hom_{\mathscr{C}^m}(1, \gamma^n)$,

$$L(a_{\vec{j}}(b_{\vec{i}})) := L(a_{\vec{j}} \otimes b_{\vec{i}}) = \delta^{1-n}\sqrt{d(X_{\vec{i},\vec{j}})}$$



where we simplify the diagram in Equation (6) by its projection on the plane $Y = 0$.

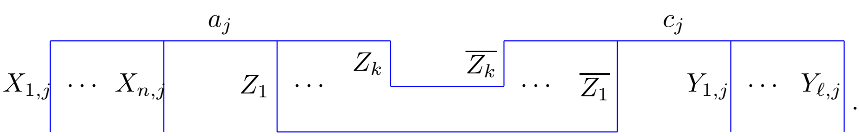
4.2. Contractions

The multiplication on $\mathscr{C}^m$ defines a map from $\hom_{\mathscr{C}^m}(\gamma^n, \gamma^k) \otimes \hom_{\mathscr{C}^m}(\gamma^k, \gamma^l)$ to $\hom_{\mathscr{C}^m}(\gamma^n, \gamma^l)$. Applying Frobenius reciprocity, we obtain a contraction $\wedge_k$: $\hom_{\mathscr{C}^m}(1, \gamma^{n+k}) \otimes \hom_{\mathscr{C}^m}(1, \gamma^{k+l}) \to \hom_{\mathscr{C}^m}(1, \gamma^{n+l})$. Then $\wedge_k$ is also defined on the configuration spaces induced by Φ. We give the definition in detail here.

Remark 4.7. The notation $\wedge_k$ comes from the graded multiplication in [5].

Suppose $X_{\vec{i},\vec{j}}$, $Y_{\vec{i},\vec{j}}$, $Z_{\vec{i},\vec{j}}$ are Z-configurations of size $n \times m$, $\ell \times m$, $k \times m$. For X-configurations $a_j \in \hom_{\mathscr{C}}(1, X_{\vec{i},j} \otimes Z_{\vec{i},j})$ and $c_j \in \hom_{\mathscr{C}}(1, \theta_1(Z_{\vec{i},j}) \otimes Y_{\vec{i},j})$, $0 \le j \le m-1$, we define the k-string contraction, $k \ge 0$, as

$$a_j \wedge_k c_j :=$$



Moreover, $a_{\vec{j}} \wedge_k c_{\vec{j}} := (a_1 \wedge_k c_1) \otimes \cdots \otimes (a_m \wedge_k c_m)$. Suppose

$$\begin{aligned}
&b_i \in \hom_{\mathscr{C}}(1, X_{i,\vec{j}}), && \text{for } 1 \le i \le n; \\
&b_i \in \hom_{\mathscr{C}}(1, Z_{i,\vec{j}}), && \text{for } n+1 \le i \le n+k; \\
&d_i \in \hom_{\mathscr{C}}(1, \theta_1(Z_{i,\vec{j}})), && \text{for } 1 \le i \le k; \\
&d_i \in \hom_{\mathscr{C}}(1, Y_{i,\vec{j}}), && \text{for } k+1 \le i \le k+\ell,
\end{aligned}$$

namely $b_{\vec{i}} \in Conf_Y(X_{\vec{i},\vec{j}} \otimes Z_{\vec{i},\vec{j}})$ and $d_{\vec{i}} \in Conf_Y(\theta_1(Z_{\vec{i},\vec{j}}) \otimes Y_{i,\vec{j}})$. We define the k-string contraction $\wedge_k$ on the configurations $a_{\vec{j}} \otimes b_{\vec{i}}$ and $c_{\vec{j}} \otimes d_{\vec{i}}$ as

$$(a_{\vec{j}} \otimes b_{\vec{i}}) \wedge_k (c_{\vec{j}} \otimes d_{\vec{i}}) = \prod_{s=1}^{k} \langle \theta_1(d_{k+1-s}), b_{n+s} \rangle \left(a_{\vec{j}} \wedge_k c_{\vec{j}} \right) \otimes \left(\bigotimes_{i=1}^{n} b_i \otimes \bigotimes_{i=k+1}^{k+\ell} d_i \right). \quad (15)$$

Definition 4.8. We define the k-string contraction on the configuration spaces $\wedge_k$: $Conf(\mathscr{C})_{n+k,m} \otimes Conf(\mathscr{C})_{k+\ell,m} \to Conf(\mathscr{C})_{k+\ell,m}$ as a linear extension of Equation (15).

When $\ell = 0$, we can identity $Conf(\mathscr{C})_{n+k,m}$ as operators from $Conf(\mathscr{C})_{k,m}$ to $Conf(\mathscr{C})_{n,m}$ corresponding to Frobenius reciprocity. Moreover, the composition of these operators is associative. So we obtain a C^*-algebroid for a fixed m.

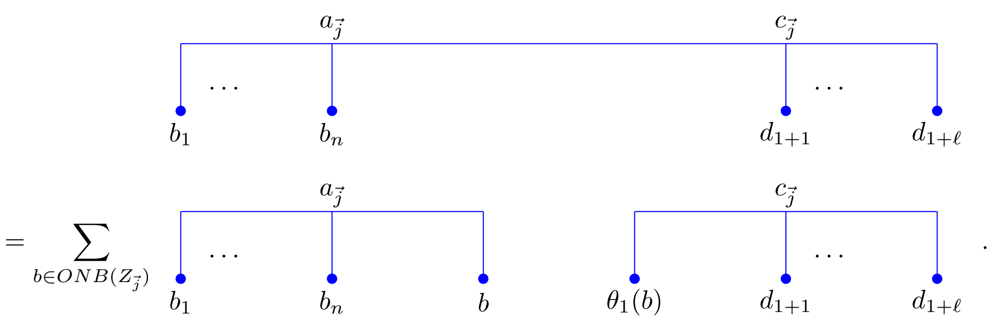
Proposition 4.9. *Recall that ρ_2 and θ_2 are actions in the Y-directions. They commute with the contraction $\wedge_k$ in the X-direction.*

Proof. Recall that ρ_2 is an isometry and it commutes with θ_1 by Proposition 3.2, so

$$\begin{aligned}
&\rho_2(a_{\vec{j}} \otimes b_{\vec{i}}) \wedge_k \rho_2(c_{\vec{j}} \otimes d_{\vec{i}}) \\
&= \prod_{s=1}^{k} \langle \theta_1 \rho_2(d_{k+1-s}), \rho_2(b_{n+s}) \rangle \left(\rho_2(a_{\vec{j}}) \wedge_k \rho_2(c_{\vec{j}}) \right) \otimes \left(\bigotimes_{i=1}^{n} \rho_2(b_i) \otimes \bigotimes_{i=k+1}^{k+\ell} \rho_2(d_i) \right) \\
&= \prod_{s=1}^{k} \langle \theta_1(d_{k+1-s}), (b_{n+s}) \rangle \rho_2(a_{\vec{j}} \wedge_k c_{\vec{j}}) \otimes \left(\bigotimes_{i=1}^{n} \rho_2(b_i) \otimes \bigotimes_{i=k+1}^{k+\ell} \rho_2(d_i) \right) \\
&= \rho_2((a_{\vec{j}} \otimes b_{\vec{i}}) \wedge_k (c_{\vec{j}} \otimes d_{\vec{i}})).
\end{aligned}$$

The proof for θ_2 is similar. □

Lemma 4.10. *When $k = 1$, we have*



Proof. We apply Equation (2), the resolution of identity in $\mathscr{C}$, to $1_{Z_{\vec{j}}}$ in the first diagram. Only the component equivalent to 1 remains non-zero, since the first diagram has no boundary on the left. This component gives the second diagram. □

4.3. Frobenius algebras

Definition 4.11. We define $\mu_n = \Phi^{-1}(\mu_{n,m})$, for $n \geq 1$, and $\mu_0 = 1$. Then

$$\mu_n = \sum_{\alpha \in B} \overline{L(\alpha)}\alpha,$$

where B be is an ONB of $\hom_{\mathscr{C}^m}(1, \gamma^n)$.

Moreover, μ_1 is the canonical inclusion from 1 to γ, and μ_2 is the canonical inclusion from 1 to $\gamma \otimes \overline{\gamma}$ which defines the dual of objects. By Proposition 3.5, for $k = 1, 2$,

$$\rho_k(\mu_n) = \mu_n, \tag{16}$$

$$\theta_k(\mu_n) = \mu_n. \tag{17}$$

Let us prove that μ_3 defines a Frobenius algebra.

Lemma 4.12. *For $n \geq 2$, $\ell \geq 1$,*

$$\mu_n \wedge_1 \mu_\ell = \delta^{-1}\mu_{n+\ell-2}. \tag{18}$$

Proof. Suppose $X_{\vec{i},\vec{j}}$, $Y_{\vec{i},\vec{j}}$, $Z_{\vec{j}}$ are Z-configurations of size $(n-1) \times m$, $(\ell - 1) \times m$, $1 \times m$ respectively. By Lemma 2.3,

$$\left\{ \sqrt{d(Z_j)} a_j \wedge_1 c_j \ : \ Z_j \in Irr^m, \ a_j \in ONB(X_{\vec{i},j} \otimes Z_j), \ c_j \in ONB(\overline{Z_j} \otimes Y_{\vec{i},j}) \right\}$$

forms an $ONB(X_{\vec{i},j} \otimes Y_{\vec{i},j})$. Take $b_{\vec{i}} = b_1 \otimes \cdots \otimes b_n$ and $d_{\vec{i}} = d_1 \otimes \cdots \otimes d_\ell$, where $b_i \in ONB(X_{i,\vec{j}})$ and $d_i \in ONB(Y_{i,\vec{j}})$. By Lemma 4.10, we have

$$\begin{aligned}
&\overline{L(\sqrt{d(Z_j)} a_{\vec{j}} \wedge_1 c_{\vec{j}}(b_{\vec{i}} \otimes d_{\vec{i}}))} \sqrt{d(Z_j)} a_{\vec{j}} \wedge_1 c_{\vec{j}}(b_{\vec{i}} \otimes d_{\vec{i}}) \\
=&\delta \sum_{b \in ONB^*(Z_{\vec{j}})} \overline{L(a_{\vec{j}}(b_{\vec{i}} \otimes b)) L(c_{\vec{j}}(\theta_1(b) \otimes d_{\vec{i}}))} a_{\vec{j}} \wedge_1 c_{\vec{j}}(b_{\vec{i}} \otimes d_{\vec{i}}) \\
=&\delta \left(\sum_{b \in ONB^*(Z_{\vec{j}})} \overline{L(a_{\vec{j}}(b_{\vec{i}} \otimes b))} a_{\vec{j}}(b_{\vec{i}} \otimes b) \right) \\
&\quad \wedge_1 \left(\sum_{b \in ONB^*(Z_{\vec{j}})} \overline{L(c_{\vec{j}}(\theta_1(b) \otimes d_{\vec{i}}))} c_{\vec{j}}(\theta_1(b) \otimes d_{\vec{i}}) \right).
\end{aligned}$$

Summing over $a_{\vec{j}}(b_{\vec{i}})$, $c_{\vec{j}}(d_{\vec{i}})$, we have

$$\mu_{n+\ell-2} = \delta\mu_n \wedge_1 \mu_\ell. \quad \square$$

For morphisms $v \in \hom_{\mathscr{C}^m}(1,\gamma)$, $w \in \hom_{\mathscr{C}^m}(\gamma,\gamma^2)$, we call (γ,v,w) a Q-system [20], or (γ,v,w,v^*,w^*) a Frobenius algebra [24] in the *-category $\mathscr{C}^m$, if

$$\begin{aligned}(w\otimes 1_\gamma)w &= (1_\gamma \otimes w)w;\\ (v^*\otimes 1_\gamma)w &= 1_\gamma = (1_\gamma\otimes v^*)w;\\ (w^*\otimes 1_\gamma)1_\gamma\otimes w &= ww^*.\end{aligned}$$

Theorem 4.13. *By Frobenius reciprocity, we can identify μ_{n+k} as a morphism $\mu_n^k \in \hom_{\mathscr{C}^m}(\gamma^k,\gamma^n)$. Then $(\gamma,\mu_1,\delta^2\mu_2^1)$ is a Q-system in $\mathscr{C}^m$.*

Proof. By Equations (16), (17) and (18),

$$\begin{aligned}(\mu_2^1\otimes 1_\gamma)\mu_2^1 &= \mu_3^1 = (1_\gamma\otimes\mu_2^1)\mu_2^1;\\ (\mu_0^1\otimes 1_\gamma)\mu_2^1 &= \delta^{-2}1_\gamma = (1_\gamma\otimes\mu_0^1)\mu_2^1;\\ (\mu_1^2\otimes 1_\gamma)1_\gamma\otimes\mu_2^1 &= \mu_{2,2} = \mu_2^1\mu_1^2.\end{aligned}$$

So $(\gamma,\mu_1,\delta^2\mu_2^1)$ is a Q-system in $\mathscr{C}^m$. $\square$

Notation 4.14. Since μ_1 is the canonical inclusion, we simply denote this Q-system or Frobenius algebra by (γ,μ_3).

Corollary 4.15. *For $k\geq 1$, $n\geq k+1$, $\ell\geq k$,*

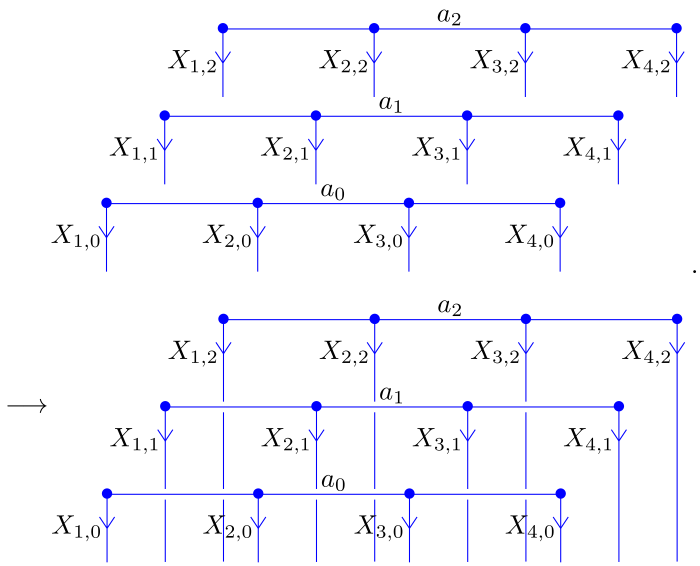
$$\begin{aligned}&\mu_n\wedge_k\mu_\ell = \delta^{-1}\mu_{n+\ell-2k};\\ &\mu_k\wedge_k\mu_k = 1.\end{aligned}$$

Definition 4.16. We call the subfactor associated with the Frobenius algebra (γ,μ_3) the m-interval Jones-Wassermann subfactor of $\mathscr{C}$.

The modularity is not used in the construction of the Jones-Wassermann subfactors, but it is crucial in the proof of the self-duality of the Jones-Wassermann subfactor. The formula of (γ,μ_3) in terms of the 3D configuration is intuitive in the proof of self-duality.

Remark 4.17. For an X-configuration $a_{\vec{j}}$ with boundary $X_{\vec{i},\vec{j}}$, we project it to the 2D space, such that the boundary points are ordered on a line as

$$X_{1,0},\ldots,X_{1,m-1},X_{2,0},\ldots,X_{2,m-1},\ldots X_{n,0},\ldots,X_{n,m-1}:$$



This defines a map $\mathscr{F} : \hom_{\mathscr{C}^m}(1, \bigotimes_{i=1}^n X_{i,\vec{j}}^{\boxtimes}) \to \hom_{\mathscr{C}}(1, \bigotimes_{i=1}^n X_{i,\vec{j}})$. Furthermore, $\mathscr{F}$ extends to a monoidal functor from $\mathscr{C}^m$ to $\mathscr{C}$. We can also derive the Frobenius algebra (γ, μ_3) from the tensor functor $\mathscr{F}$. (1) The adjoint functor of $\mathscr{F}$ sends the trivial Frobenius algebra in $\mathscr{C}$ to a Frobenius algebra in $\mathscr{C}^m$ [14], which is our (γ, μ_3). (2) Take $X_{i,j} = \tilde{X}$, then $\mathscr{F}(\hom_{\mathscr{C}^m}(1, \bigotimes_{i=1}^n X_{i,\vec{j}}^{\boxtimes})) \subseteq \hom_{\mathscr{C}}(\tilde{X}^{mn})$. The inductive limit of this inclusion for $n \to \infty$ defines a subfactor, which was studied by Erlijman and Wenzl in [3]. The corresponding Frobenius algebra is (γ, μ_3).

The Frobenius algebra (γ, μ_3) defines a $\gamma - \gamma$ bimodule category induced by $\mathscr{C}^m$. It is a unitary fusion category, called the dual of $\mathscr{C}^m$ with respect to (γ, μ_3). When $m = 2$, the dual of $\mathscr{C}^2$ is known as the quantum double of $\mathscr{C}$.

Definition 4.18. For a general m, we call the dual of $\mathscr{C}^m$ with respect to the Frobenius algebra (γ, μ_3) the quantum m-party, or quantum multiparty, of $\mathscr{C}$.

Definition 4.19. By Equation (14), the n-box space of the m-interval Jones-Wassermann subfactor is isomorphic to the m-box space of the n-interval Jones-Wassermann subfactor. The identification is switching the m, n coordinates of the 3D configurations. We call this identification the *m-n duality*.

Remark 4.20. Suppose the Grothendieck ring of the MTC $\mathscr{C}$ is the group $\mathbb{Z}_d$ and $\vec{k} = (k_1, k_2, \ldots, k_n) \in \mathbb{Z}_d^n$. Note that $N_{\vec{k}} = 1$ if $|\vec{k}| = 0$, where $|\vec{k}| := k_1 + k_2 + \cdots + k_n$ in $\mathbb{Z}_d$, and $N_{\vec{k}} = 0$ elsewhere. So the object of Frobenius algebra of the n-interval Jones-Wassermann subfactor for $\mathscr{C}$ is

$$\gamma_n = \bigoplus_{|\vec{k}|=0} k_1 \boxtimes \cdots \boxtimes k_n. \tag{19}$$

A similar formula has been considered as a resource state in quantum information in [7],

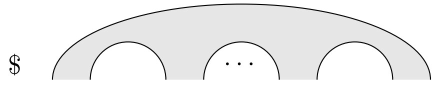
$$|\mathrm{Max}\rangle = \sum_{|\vec{k}|=0} |k_1 k_2 \cdots k_n\rangle. \tag{20}$$

The two formulas are identical, but they have completely different meanings. By Frobenius reciprocity, the formula γ_n can be considered as μ_2, a 2-box of the n-interval Jones-Wassermann subfactor. The formula $|Max\rangle$ is μ_n, an n-box of the 2-interval Jones-Wassermann subfactor in quon language [18]. The identification between γ_n and $|Max\rangle$ could be realized through the m-n duality, for $m = 2$.

5. The string Fourier transform on planar algebras

Once we obtain a Frobenius algebra (γ, μ_3), we can define a subfactor planar algebra $\mathscr{S} = \{\mathscr{S}_{n,\pm}\}_{n\in\mathbb{N}}$, such that $\mathscr{S}_{n,+} = \hom_{\mathscr{C}^m}(1, \gamma^n)$ [24]. This provides partial structures of the configuration spaces in Theorem 2.4. We show that the planar algebra is unshaded by constructing a planar algebraic *-isomorphism from $\mathscr{S}_{n,-}$ to $\mathscr{S}_{n,+}$ in §6.

The modular conjugation θ_1 defines the involution $*$ of the subfactor planar algebra $\mathscr{S}$. In the planar algebra $\mathscr{S}_{n,+}$, the element $\delta^{\frac{n}{2}}\mu_n$ is represented by



,

where the diagram has $2n$ boundary points at the bottom.

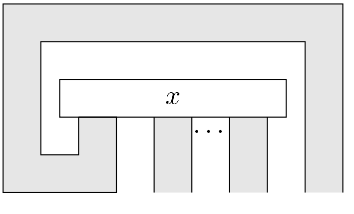
Remark 5.1. Convention: We omit the $ sign of the planar diagram if it is on the left.

Remark 5.2. Using the above diagrammatic representation of μ_n, it is easy to see that μ_1 is the canonical inclusion from 1 to γ; μ_2 is the canonical inclusion from 1 to $\gamma \otimes \overline{\gamma}$; and μ_n, $n \geq 1$, satisfy Equations (16), (17) and (18). Conversely, these conditions imply that $(\gamma, \mu_1, \delta^2\mu_2^1)$ is a Frobenius algebra by Theorem 4.13.

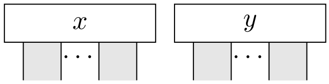
The action of any planar tangle on $\mathscr{S}_{\cdot,+}$ is a composition of the following 6 elementary ones, for $n, \ell \geq 0$:

- The rotation $\rho : \mathscr{S}_{n,+} \to \mathscr{S}_{n,+}$,

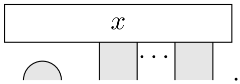
$\rho(x) =$



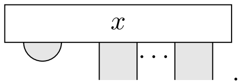
- The wedge product $\wedge : \mathscr{S}_{n,+} \otimes \mathscr{S}_{\ell,+} \to \mathscr{S}_{n\ell,+}$,

$x \wedge y =$ 

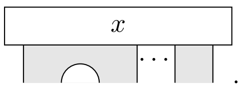
- The inclusion $\iota_0 : \mathscr{S}_{n,\pm} \to \mathscr{S}_{n+1,\pm}$,

$\iota_0(x) = \delta^{-1/2}$ 

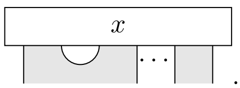
- The contraction $\phi_0 : \mathscr{S}_{n+1,\pm} \to \mathscr{S}_{n,\pm}$,

$\phi_0(x) = \delta^{-1/2}$ 

- The inclusion $\iota_1 : \mathscr{S}_{n,\pm} \to \mathscr{S}_{n+1,\pm}$,

$\iota_1(x) = \delta^{-1/2}$ 

- The contraction $\phi_1 : \mathscr{S}_{n+1,\pm} \to \mathscr{S}_{n,\pm}$,

$\phi_1(x) = \delta^{-1/2}$ 

The first two are isometries. The last four are partial isometries. These actions except the rotation can be written as contractions:

$$\iota_0(x) = \mu_1 \wedge x$$
$$\phi_0(x) = \mu_1 \wedge_1 x$$
$$\iota_1(x) = \delta\mu_3 \wedge_1 x$$
$$\phi_1(x) = \delta\mu_3 \wedge_2 x.$$

Moreover, ϕ_k is the adjoint operator of ι_k:

Proposition 5.3. *For $x \in \mathscr{S}_{n,\pm}, y \in \mathscr{S}_{n+2,\pm}$, $k = 0, 1$, we have*

$$< \iota_k(x), y >=< x, \phi_k(y) > .$$

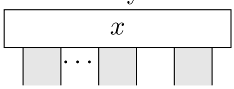
As a subfactor planar algebra, we have the involution on $\mathscr{S}_{n,+}$ defined by the reflection θ_1 which is an anti-isometry.

We have also these actions on the configuration space in the Y-direction. In particular, the rotation ρ_2 and the reflection θ_2 preserves the size m, and they are defined on $\mathscr{S}_{n,+} = \hom_{\mathscr{C}^m}(1, \gamma^n)$.

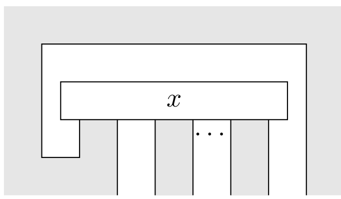
Theorem 5.4. *The action ρ_2 on $\mathscr{S}$ is a planar algebraic $*$-isomorphism. The action θ_2 on $\mathscr{S}$ is an anti-linear planar algebraic $*$-isomorphism.*

Proof. By Propositions 3.2, ρ_2 and θ_2 commute with ρ and θ_1. By Propositions 3.5 and 4.9, we have that ρ_2 and θ_2 commute with $\wedge$, ι_k and ϕ_k, for $k = 1, 2$. Therefore they are (anti-linear) planar algebraic *-isomorphisms. □

Similarly, we have the 6+1 elementary actions on $\mathscr{S}_{\cdot,-}$ by switching the shading. The string Fourier transform (SFT) $\mathfrak{F} : \mathscr{S}_{n,\pm} \to \mathscr{S}_{n,\mp}$ is an isometry given by a clockwise one-string rotation. Applying the SFT, we can represent the element in $\mathscr{S}_{n,-}$ by $\mathfrak{F}(x)$ for x in $\mathscr{S}_{n,+}$ and derive the six elementary actions on $\mathscr{S}_{\cdot,-}$ by actions on $\mathscr{S}_{\cdot,+}$.

For an element $x \in \mathscr{S}_{n,+}$ 

, its SFT $\mathfrak{F}(x) \in \mathscr{S}_{n,-}$ is given by

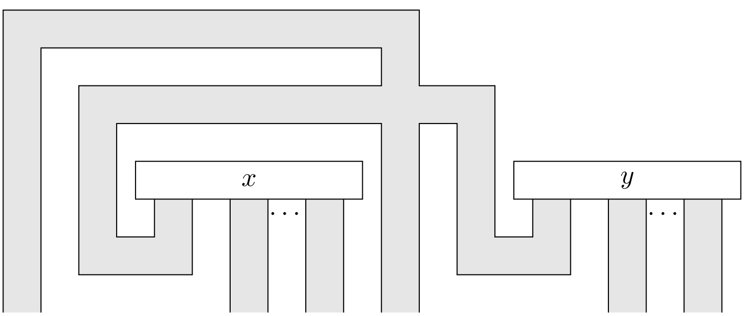
$\mathfrak{F}(x) =$ 

. (21)

Then $\rho = \mathfrak{F}^2$. Moreover, $\iota_k := \mathfrak{F}^{-k}\iota_0\mathfrak{F}^k$, $1 \leq k \leq 2n$, is adding a cap before the k^{th} boundary points, and $\phi_k := \mathfrak{F}^{-k}\phi_0\mathfrak{F}^k$, $1 \leq k \leq 2n$, is a contraction between the $k+1^{\text{th}}$ and $k+2^{\text{th}}$ boundary points.

Notation 5.5. By the spherical property, we define ϕ_1 on $\mathscr{S}_{1,\pm}$ by ϕ_0.

For $x \in \mathscr{S}_{n,+}, y \in \mathscr{S}_{n',+}$, we define $x \star y \in \mathscr{S}_{n+n',+}$ as

$x \star y =$ 

(22)

Then

$$\rho\mathfrak{F}(x) = \mathfrak{F}\rho(x), \tag{23}$$

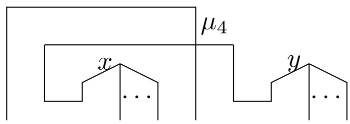
$$\mathfrak{F}(x) \wedge \mathfrak{F}(y) = \mathfrak{F}(x \star y), \tag{24}$$

$$\phi_k\mathfrak{F}(x) = \mathfrak{F}\phi_{k+1}(x), \tag{25}$$

$$\iota_{k+1}\mathfrak{F}(x) = \mathfrak{F}\iota_k(x), \tag{26}$$

$$\theta_1(\mathfrak{F}(x)) = \mathfrak{F}\rho^{-1}\theta_1(x). \tag{27}$$

Recall that $\mathscr{S}_{n,+} = \hom_{\mathscr{C}^m}(1, \gamma^n)$, thus the induced map $\star : \hom_{\mathscr{C}^m}(1, \gamma^n) \otimes \hom_{\mathscr{C}^m}(1, \gamma^{n'}) \to \hom_{\mathscr{C}^m}(1, \gamma^{n+n'})$ is also defined. From the planar algebra $\mathscr{S}$ to category $\mathscr{C}^m$, the shaded strip becomes a γ-string. Then Equation (22) becomes

$x \star y = \delta^2$ 

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6. Modular self-duality

In this Section, we prove that the Jones-Wassermann subfactor is self-dual in Theorem 6.18 and symmetrically self-dual in Theorem 6.20. We apply Theorem 2.4 to prove Proposition 6.1 and 6.2. Proposition 6.2 will be used in Lemma 6.13. Then we prove Theorem 6.18 which implies Theorem 2.4. We prove Theorems 2.4, 6.18 in the following order:

Theorem 2.4 for $m = 2$, $n = 2$;

$\to$Theorem 6.18 for $m = 2$;

$\to$Theorem 2.4 for $m = 2$, $n \geq 1$;

$\leftrightarrow$Theorem 2.4 for $m \geq 1$, $n = 2$;

$\to$Theorem 6.18 for $m \geq 1$;

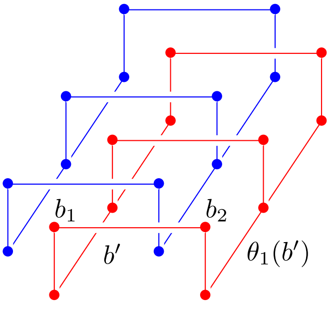
$\to$Theorem 2.4 for $m \geq 1$, $n \geq 1$.

By induction, Theorems 2.4, 6.18 hold for all m and n. Then we apply them to prove Theorem 6.20. (When $m = 1$, the configuration space $Conf(\mathscr{C})_{m,n}$ is $\mathbb{C}$. The theorems are obvious.)

6.1. Killing relation

Proposition 6.1. *For a fixed m and two Y-configurations $b_1, b_2 \in \hom_{\mathscr{C}}(\tilde{X}^m, 1)$, we have*

$$\delta^{-2} \sum_{X_{\vec{j}} \in Irr(\mathscr{C}^m)} d(X_{\vec{j}}) \sum_{b' \in ONB(X_{\vec{j}})}$$



$$= \langle b_1, 1_m^* \rangle \langle b_2, 1_m^* \rangle,$$

where 1_m is the canonical inclusion from 1^m to $\tilde{X}^m$ in $\mathscr{C}^m$.

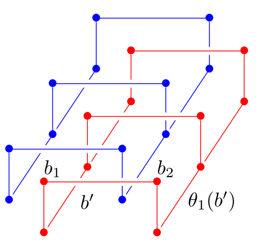
Proof. Without loss of generality, we assume that b_1 and b_2 are unit vectors. Note that if $X(b_1) \neq X(\theta_1(b_2))$, then both sides are zero. We assume that $X(b_2) = X(\theta_1(b_1))$.

If a Y-configuration $b \in \hom_{\mathscr{C}}(\tilde{X}^m, 1)$ is a unit vector, then

$$\dim \hom_{\mathscr{C}^m}(1, X(b) \otimes X(\theta(b))) = 1.$$

So there is only one X-configuration with boundary $X(b) \otimes X(\theta(b))$ up to a scalar. Let a_b be the canonical inclusion from 1 to $X(b) \otimes X(\theta(b))$ in $\mathscr{C}^m$. Let $C' = \{a'_{\vec{j}} \otimes b'_{\vec{i}}\}$ be an ONB of $Conf(Lat')$. Applying Theorem 2.4 for $n = 2$, we have that

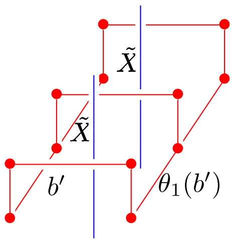
$$\begin{aligned}
&\langle b_1, 1_m^* \rangle \langle b_2, 1_m^* \rangle \\
=&\langle a_{b_1} \otimes (b_1 \otimes b_2), 1_{m,2} \rangle \\
=& \sum_{a'_{\vec{j}} \otimes b'_{\vec{i}} \in C'} LL(a_{b_1} \otimes (b_1 \otimes b_2), a'_{\vec{j}} \otimes b'_{\vec{i}}) \overline{LL(1_{m,2}, a'_{\vec{j}} \otimes b'_{\vec{i}})} \\
=& \sum_{b' \in B} LL(a_{b_1} \otimes (b_1 \otimes b_2), a'_{b'} \otimes (b' \otimes \theta_1(b'))) \overline{LL(1_{m,2}, a'_{b'} \otimes (b' \otimes \theta_1(b')))} \\
=&\delta^{-2} \sqrt{d(X(b_1))} \sum_{b' \in B} d(X(b'))
\end{aligned}$$



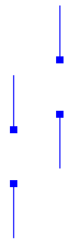
If $X(b_1) \neq 1$, then both sides are zero. If $X(b_1) = 1$, then $d(X(b_1)) = 1$ and the statement holds. $\square$

Proposition 6.2. *For a fixed m, we have the following identity in $\mathscr{C}$:*

$$\mu^{1-m} \sum_{X_{\vec{j}} \in Irr^m} d(X_{\vec{j}}) \sum_{b' \in ONB(X_{\vec{j}})} \quad (28)$$



$=$

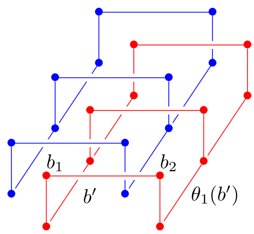


,

where the one-valent vertex labeled by a square (orthogonal to the Y-axis) on the right represents the conical inclusion from 1 to $\tilde{X}$.

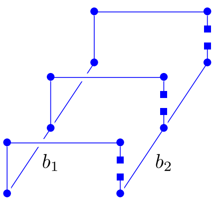
Proof. Note that $\delta^{-2} = \mu^{1-m}$. By Proposition 6.1, for any $b_1, b_2 \in \hom_{\mathscr{C}}(\tilde{X}^m, 1)$,

$$\delta^{-2} \sum_{X_{\vec{j}} \in Irr(\mathscr{C}^m)} d(X_{\vec{j}}) \sum_{b' \in ONB(X_{\vec{j}})}$$

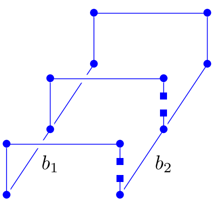


$=\langle b_1, 1_m^* \rangle \langle b_2, 1_m^* \rangle,$

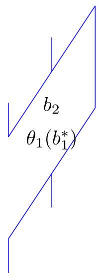
$=$



$=$



So the inner product of both sides of Equation (28) with

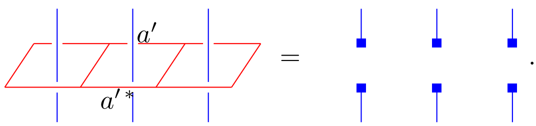


are the same. Note that such elements form a generating set of $\hom_{\mathscr{C}}(\tilde{X}^{m-1}, \tilde{X}^{m-1})$, so Equation (28) holds. □

If we switch n and m in Proposition 6.2, then we have obtain the following equivalent result:

Proposition 6.3. *For a fixed m, take $\tilde{X} = \bigoplus_{X \in Irr(\mathscr{C})} X$ and $1_{\tilde{X}}$ to be the conical inclusion from 1 to $\tilde{X}$. Then*

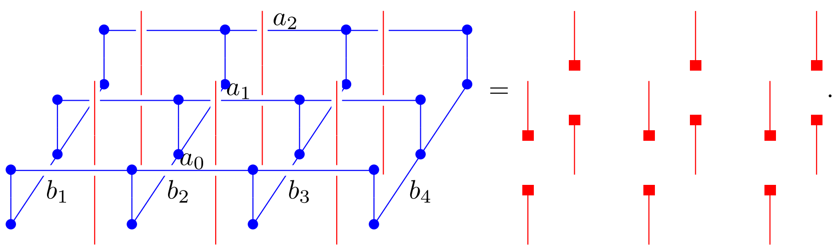
$$\mu^{1-n} \sum_{X_{\vec{i}} \in Irr^n} d(X_{\vec{i}}) \sum_{a' \in ONB(X_{\vec{i}})} \quad (29)$$



After the bi-induction argument in this section, we obtain Theorem 2.4 for any m, n. Then we prove the following general situation.

Theorem 6.4. *For $m \geq 2$ and $n \geq 2$, we have*

$$\mu^{\frac{(1-n)(m-1)}{2}} \sum_{X_{\vec{i},\vec{j}} \in Irr^{nm}} \sqrt{d(X_{\vec{i},\vec{j}})} \sum_{a_j \in ONB(X_{\vec{i},j}), b_i \in ONB(X_{i,\vec{j}})} \overline{LL(a_{\vec{j}} \otimes b_{\vec{i}})}$$



Proof. The proof is similar to that of Propositions 6.1 and 6.2 using Theorem 2.4. □

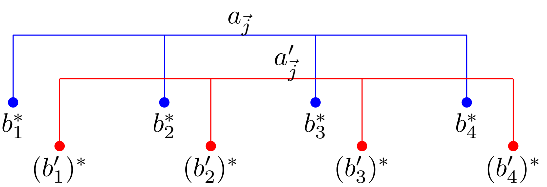
Remark 6.5. When $m = n = 2$, Equation (29) is the killing relation.

6.2. The self-duality of Jones-Wassermann subfactors

Suppose that $\mathscr{S}$ is the subfactor planar algebras of the Jones-Wassermann subfactor for a unitary MTC $\mathscr{C}$. In this section, we construct a planar algebraic *-isomorphism from $\mathscr{S}_{n,-}$ to $\mathscr{S}_{n,+}$. That means the Jones-Wassermann subfactor is self-dual. Furthermore, we show that the *-isomorphism commutes with the SFT, so the subfactor planar algebra $\mathscr{S}$ is unshaded, i.e., the Jones-Wassermann subfactor is symmetrically self-dual.

Induced by Φ, we define LL on $\hom_{\mathscr{C}^m}(1,\gamma^n)\otimes\hom_{\mathscr{C}^m}(1,\gamma^n)$. Moreover, we use the following notation to simplify the diagram in Equation (3):

$$\begin{aligned}&LL(a_{\vec{j}}(b_{\vec{i}}), a'_{\vec{j}}(b'_{\vec{i}}))\\ =&LL(a_{\vec{j}}\otimes b_{\vec{i}}, a'_{\vec{j}}\otimes b'_{\vec{i}})\\ =&\delta^{1-n}\sqrt{d(X_{\vec{i},\vec{j}})d(X'_{\vec{i},\vec{j}})}\end{aligned}$$



(30)

Notation 6.6. We use A_n to denote an ONB of $\hom_{\mathscr{C}^m}(1,\gamma^n)$. We use B to denote an ONB of $\hom_{\mathscr{C}}(\gamma,1)$.

Definition 6.7 *(string Fourier transform).* We represent elements in $\mathscr{S}_{n,-}$ as $\mathfrak{F}(x)$ for $x\in\mathscr{S}_{n,+}=\hom_{\mathscr{C}}(1,\gamma^n)$. We define $\Psi:\mathscr{S}_{n,-}\to\mathscr{S}_{n,+}$, $n\geq 0$,

$$\Psi(\mathfrak{F}(x))=\sum_{x'\in B_n}LL(x,\theta_2(x'))x'. \tag{31}$$

When $n=0$, Ψ maps 1 to 1. When $n=1$, Ψ maps the canonical inclusion from 1 to γ in $\mathscr{S}_{1,-}$ to the canonical inclusion in $\mathscr{S}_{1,+}$. Let us prove that Ψ commutes with the 6+1 elementary actions, so Ψ is a planar algebraic *-isomorphism from $\mathscr{S}_{n,-}$ to $\mathscr{S}_{n,+}$. Then $\mathscr{S}$ becomes an unshaded planar algebra. Moreover, the map $\Psi\mathfrak{F}$ in Equation (31) defines the SFT on the unshaded planar algebra $\mathscr{S}_n$.

When $m=n=2$, $\gamma=\bigoplus_{X\in Irr(\mathscr{C})}X\otimes\overline{X}$. The vectors $\{v_X\}_{X\in Irr(\mathscr{C})}$ form an ONB of $\hom_{\mathscr{C}^2}(1,\gamma^2)$, where v_X is the canonical inclusion from 1 to $(X\otimes\overline{X})\otimes\overline{(X\otimes\overline{X})}$ in $\mathscr{C}^2$. We call the ONB $\{v_X\}_{X\in Irr(\mathscr{C})}$ the *quantum coordinate* of $\hom_{\mathscr{C}^2}(1,\gamma^2)$.

The vector v_X is independent of the choice of the representative of $X\otimes\overline{X}$ in γ. For convenience, we take b^* to be the canonical inclusion from 1 to $X\otimes\overline{X}$ to indicate the multiplicity of $X\otimes\overline{X}$ in γ, then $\overline{(X\otimes\overline{X})(b)}=(\overline{X}\otimes X)(\theta_1(b))$.

The following result is a consequence of the modular self-duality and our definition of the SFT.

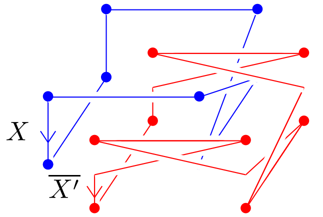
Theorem 6.8. *The SFT on the quantum coordinate of* $\hom_{\mathscr{C}^2}(1,\gamma^2)$ *is the same as the modular S matrix of the MTC $\mathscr{C}$: for any $X, X' \in IrrC$,*

$$\langle \Psi\mathfrak{F}(v_X), v_{X'}\rangle = S_{XX'}. \tag{32}$$

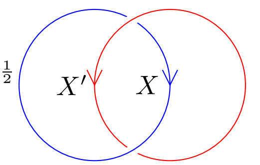
Proof. By Definition 6.7, the matrix units of $\Psi\mathfrak{F}$ on the basis $\{v_X\}_{X\in Irr(\mathscr{C})}$ is

$$\langle \Psi\mathfrak{F}(v_X), v_{X'}\rangle = LL(x, \theta_2(x'))$$

$$= \mu^{-\frac{1}{2}}$$



$$= \mu^{-\frac{1}{2}}$$



$$= S_{XX'} \quad \square$$

Proposition 6.9. *For $x \in \hom_{\mathscr{C}}(1,\gamma^n)$, $n \geq 1$,*

$$\Psi(\mathfrak{F}(\rho(x)) = \rho\Psi(\mathfrak{F}(x)),$$
$$\Psi(\mathfrak{F}\rho^{-1}\theta_1(x)) = \theta_1(\Psi(\mathfrak{F}(x))).$$

Proof. By Propositions 3.2 and 3.3, we have

$$\Psi(\mathfrak{F}(\rho(x)) \tag{33}$$
$$= \sum_{x'\in A_n} LL(\rho(x), \theta_2(x'))x' \tag{34}$$
$$= \sum_{x'\in A_n} LL(x, \rho^{-1}\theta_2(x')))x' \tag{35}$$
$$= \sum_{x'\in A_n} LL(x, \theta_2\rho^{-1}(x'))\rho\rho^{-1}(x') \tag{36}$$
$$=\rho\Psi(\mathfrak{F}(x)). \tag{37}$$

Similarly, we have $\Psi(\mathfrak{F}\rho^{-1}\theta_1(x)) = \theta_1(\Psi(\mathfrak{F}(x)))$. $\square$

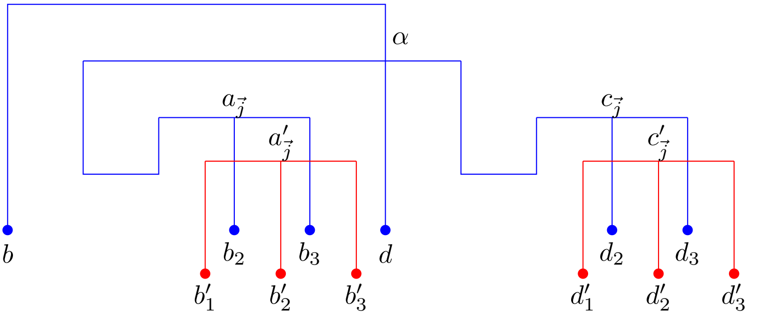
Lemma 6.10. *For $x, x' \in \hom_{\mathscr{C}^m}(1, \gamma^n)$, $y, y' \in \hom_{\mathscr{C}^m}(1, \gamma^\ell)$, $n, \ell \geq 1$,*

$$LL(x \star y, x' \wedge y') = LL(x, x')LL(y, y').$$

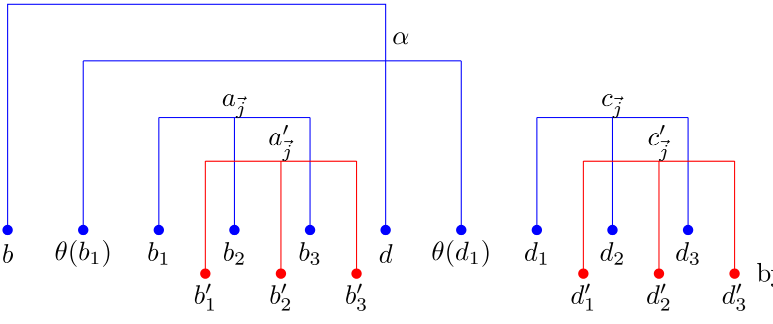
Proof. Take $x = a_{\vec{j}}(b_{\vec{i}})$, $x = a'_{\vec{j}}(b'_{\vec{i}})$, $y = c_{\vec{j}}(d_{\vec{i}})$ and $y' = c'_{\vec{j}}(d'_{\vec{i}})$. Note that the boundary of a Y-configuration b is a Z-configuration, denoted by $\vec{X}(b)$. It represents a simple sub object of γ in $\mathscr{C}^m$. For Y-configurations $b, d \in B$, we define $A_{b,d}$ to be an ONB of $\hom_{\mathscr{C}^m}(1, X(b) \otimes X(\theta_1(b_1)) \otimes X(d) \otimes X(\theta_1(d_1)))$, a sub space of $\hom_{\mathscr{C}^m}(1, \gamma^4)$. Then

$$LL(\vec{x} \star \vec{y}, \vec{x'} \wedge \vec{y'})$$

$$= \sum_{b,d \in B_1, \alpha \in A_{b,d}} \delta^{1-n-\ell} \delta^2 \sqrt{\frac{d(b)d(d)d(b_{\vec{j}})d(b'_{\vec{j}})d(d_{\vec{j}})d(d'_{\vec{j}})}{d(b_1)d(d_1)}} \overline{L(\alpha)}$$



$$= \sum_{b,d \in B_1, \alpha \in A_{b,d}} \delta^{3-n-\ell} \sqrt{\frac{d(b)d(d)d(b_{\vec{j}})d(b'_{\vec{j}})d(d_{\vec{j}})d(d'_{\vec{j}})}{d(b_1)d(d_1)}} \overline{L(\alpha)}$$



by Lemma 4.10

$$= \sum_{b,d \in B_1, \alpha \in A_{b,d}} \delta^4 \frac{1}{d(b_1)d(d_1)} |L(\alpha)|^2 LL(x, x')LL(y, y')$$

$$= \sum_{d \in B_1} \delta^{-2} d(d) LL(x, x')LL(y, y') \qquad \text{by Lemma 4.10 and Equation (2)}$$

$$= LL(x, x')LL(y, y') \qquad \text{by Proposition 4.2.}$$

By the linearity, the equation holds for any x and x'. (Here we give the pictures for $n = \ell = 3$. One can figure out the general case.) $\square$

Proposition 6.11. *For $x \in \hom_{\mathscr{C}^m}(1, \gamma^n)$, $y \in \hom_{\mathscr{C}^m}(1, \gamma^{n'})$, $n, n' \geq 1$,*

$$\Psi(\mathfrak{F}(x \star y)) = \Psi(\mathfrak{F}(x)) \wedge \Psi(\mathfrak{F}(y)).$$

Proof. By Lemma 6.10,

$$\begin{aligned} &\Psi(\mathfrak{F}(x \star y)) \\ =&\sum_{x', y' \in B} LL(\vec{x} \star \vec{y}, \vec{x'} \otimes \vec{y'}) x' \otimes y' \\ =&\sum_{x', y' \in B} LL(\vec{x}, \vec{x'}) LL(\vec{y}, \vec{y'}) x' \otimes y' \\ =&\Psi(\mathfrak{F}(x)) \wedge \Psi(\mathfrak{F}(y)). \quad \square \end{aligned}$$

Lemma 6.12. *For $x \in \hom_{\mathscr{C}^m}(1, \gamma^n)$ and $x' \in \hom_{\mathscr{C}^m}(1, \gamma^{n-1})$, $n \geq 1$,*

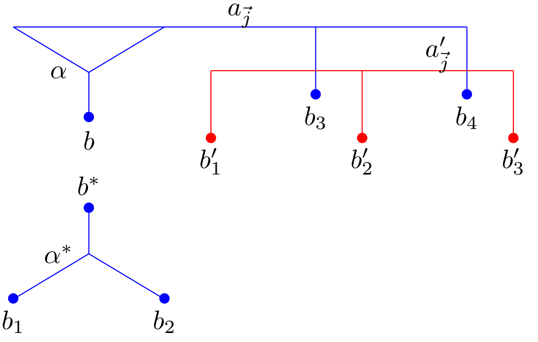
$$LL(\phi_1(x), x') = LL(x, \iota_0(x')).$$

Proof. When $n = 1$, the statement is obvious.

When $n \geq 2$, suppose $x = a_{\vec{j}}(b_{\vec{i}})$ and $x' = a'_{\vec{j}}(b'_{\vec{i}})$. For Y-configurations $b \in B$, we define A_b to be an ONB of $\hom_{\mathscr{C}^m}(1, X(b) \otimes X(\theta_1(b_2)) \otimes X(\theta_1(b_1)))$, a sub space of $\hom_{\mathscr{C}^m}(1, \gamma^3)$. Then by Lemma 4.10 and Equation (2), we have

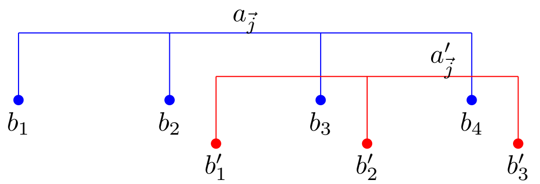
$$LL(\phi_1(x), x') \tag{38}$$

$$= \sum_{b \in B_1, \alpha \in A_b} \delta^{1-(n-1)} \delta \delta^{-2} d(b) \sqrt{d(b_{\vec{j}}) d(b'_{\vec{j}})} \tag{39}$$



(40)

$$=\delta^{1-n} \sqrt{d(b_{\vec{j}})} \sqrt{d(b'_{\vec{j}})} \tag{41}$$



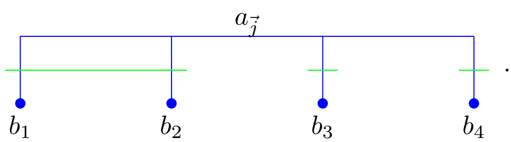
(42)

$$= LL(x, \iota_0(x')) \tag{43}$$

The general case follows from the linearity. □

From the proof of Lemma 6.12, we see that the contraction ϕ_1 on the configuration space is contracting the Z-configurations $X_{1,\vec{j}}$ and $X_{2,\vec{j}}$. The diagrammatic representation of the contracted configuration is given by

$$\delta^{1-n}\sqrt{d(X_{\vec{i},\vec{j}})}$$



. (44)

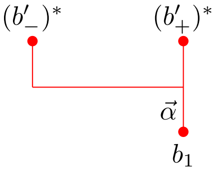
Lemma 6.13. *For $x \in \hom_{\mathscr{C}^m}(1, \gamma^n)$ and $x' \in \hom_{\mathscr{C}^m}(1, \gamma^{n-1})$, $n \geq 2$,*

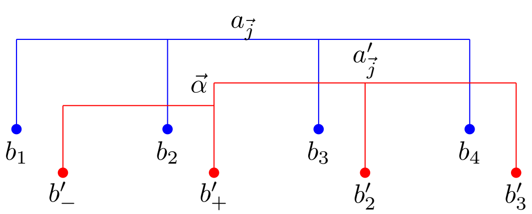
$$LL(\phi_2(x), x') = LL(x, \iota_1(x')).$$

Proof. For $b'_-, b'_+ \in B$, take $A_{b'_-,b'_+}$ to be an ONB of $\hom_{\mathscr{C}^m}(1, X(b'_-) \otimes X(b'_+) \otimes X(\theta_1(b'_1)))$. Then by Lemma 4.10, Equation (2) and Proposition 6.2, we have

$$LL(x, \iota_1(x')) \tag{45}$$

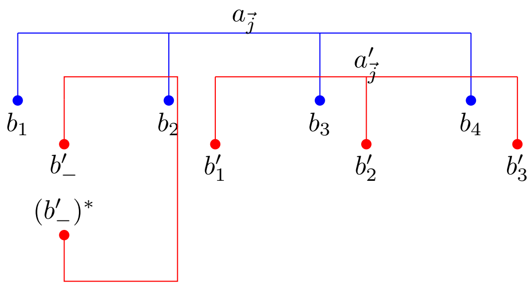
$$= \sum_{b'_-,b'_+ \in B, \alpha \in A_{b'_-,b'_+}} \delta^{1-n}\delta\delta^{-2} d(b'_-)d(b'_+)\sqrt{d(b_{\vec{j}})d(b'_{\vec{j}})} \tag{46}$$





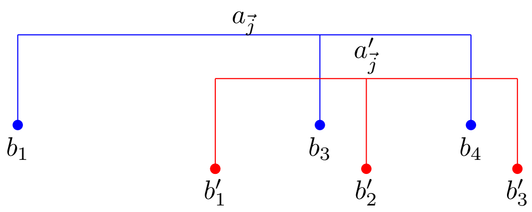
(47)

$$= \sum_{b'_- \in B} \delta^{-n} d(b'_-)\sqrt{d(b_{\vec{j}})d(b'_{\vec{j}})} \tag{48}$$



$$\tag{49}$$

$$= \delta_{b_2,1^*_m}\delta^2\delta^{-n} \tag{50}$$



$$\tag{51}$$

$$= LL(\phi_2(x), x') \quad \square \tag{52}$$

Lemma 6.14. *Suppose H_1 and H_2 are Hilbert spaces, and T is an operator from H_1 to H_2. If $x \perp T(H_1)$ in H_2, then $T^*x = 0$.*

Proof. If $x \perp T(H_1)$ in H_2, i.e., $< x, Ty >= 0$, $\forall y \in H_2$, then $< T^*x, y >= 0$. Thus $T^*x = 0$. $\square$

Proposition 6.15. *For $0 \leq k \leq 2n-2$, $x \in \hom_{\mathscr{C}^m}(1, \gamma^n)$,*

$$\Psi(\mathfrak{F}(\phi_{k+1}(x)) = \phi_k \Psi(\mathfrak{F}(x)).$$

Proof. For $k = 0, 1$,

$$\begin{aligned}
&\Psi(\mathfrak{F}(\phi_{k+1}(x)) \\
&= \sum_{x' \in B_{n-1}} LL(\phi_{k+1}(x), \theta_2(x'))x' \\
&= \sum_{x' \in B_{n-1}} LL(x, \iota_k\theta_2(x')))x' && \text{by Lemmas 6.12, 6.13} \\
&= \sum_{x' \in B_{n-1}} LL(x, \theta_2\iota_k(x'))x' && \text{by Proposition 3.2} \\
&= \sum_{x'' \in \iota_k(B_{n-1})} LL(x, \theta_2(x''))\phi_k(x'') \\
&= \sum_{x'' \in B_n} LL(x, \theta_2(x''))\phi_k(x'') && \text{by Proposition 5.3 and Lemma 6.14}
\end{aligned}$$

$$=\phi_k\Psi(\mathfrak{F}(x)).$$

The general case follows from Proposition 6.9. $\square$

Proposition 6.16. *The map $\Psi : \mathscr{S}_{n,-} \to \mathscr{S}_{n,+}$ is an isometry.*

Proof. It is true for $n = 0, 1$ by definition. When $n \geq 2$, for x, y in $\mathscr{S}_{n,+}$,

$$\begin{aligned}
&\langle \Psi\mathfrak{F}(x), \Psi\mathfrak{F}(y)\rangle \\
=&\delta^{n/2}\phi_0\phi_1\cdots\phi_{2n-1}(\Psi\mathfrak{F}(x) \wedge \Psi\mathfrak{F}(y)) \\
=&\delta^{n/2}\phi_0\Psi\mathfrak{F}(\phi_2\cdots\phi_{2n}(x \star y)) && \text{by Propositions 6.11, 6.15} \\
=&\delta^{n/2}\phi_0\phi_2\cdots\phi_{2n}(x \star y) \\
=&\langle x, y\rangle \\
=&\langle \mathfrak{F}(x), \mathfrak{F}(y)\rangle. \quad \square
\end{aligned}$$

Proposition 6.17. *For $0 \leq k \leq 2n-2$, $x \in \hom_{\mathscr{C}^m}(1, \gamma^n)$,*

$$\Psi(\mathfrak{F}(\iota_k(x)) = \iota_{k+1}\Psi(\mathfrak{F}(x)).$$

Proof. By Propositions 6.16, 5.3, and 6.15, we have

$$\begin{aligned}
&\langle \Psi\mathfrak{F}\iota_k(x), y)\rangle. \\
=&\langle \iota_k(x), \Psi\mathfrak{F}(y)\rangle \\
=&\langle x, \phi_k\Psi\mathfrak{F}(y)\rangle \\
=&\langle x, \Psi\mathfrak{F}(\phi_{k+1}(y))\rangle \\
=&\langle \iota_{k+1}\Psi\mathfrak{F}(x), y\rangle
\end{aligned}$$

Therefore $\Psi(\mathfrak{F}(\iota_k(x)) = \iota_{k+1}\Psi(\mathfrak{F}(x))$. $\square$

Theorem 6.18. *The map Ψ is a planar algebraic *-isomorphism from $\mathscr{S}_{n,-}$ to $\mathscr{S}_{n,+}$. Therefore, the m-interval Jones-Wassermann subfactor is self-dual for any $m \geq 1$.*

Proof. We write an elements in $\mathscr{S}_{\cdot,-}$ as $x' = \mathfrak{F}(x)$, $y' = \mathfrak{F}(y)$, for $x, y \in \mathscr{S}_{\cdot,+}$.

By Equation (23) and Proposition 6.9, $\Psi(\rho(x')) = \Psi(\mathfrak{F}(\rho(x)) = \rho\Psi(x')$.

By Equation (24) and Proposition 6.11, $\Psi(x' \wedge y') = \Psi(F(x \star y)) = \Psi(x') \wedge \Psi(y')$.

By Equation (25) and Proposition 6.15, $\Psi(\phi_k(x')) = \Psi(\mathfrak{F}(\phi_{k+1}(x)) = \phi_k\Psi(x')$.

By Equation (26) and Proposition 6.17, $\Psi(\iota_k(x')) = \Psi(\mathfrak{F}(\iota_{k+1}(x)) = \iota_k\Psi(x')$.

By Equation (27) and Proposition 6.9, $\Psi(\theta_1(x')) = \Psi(\mathfrak{F}\rho^{-1}\theta_1(x)) = \theta_1(\Psi(x'))$.

That means Ψ commutes with the 6+1 elementary actions of planar algebras. So for any planar tangles T_+ on $\mathscr{S}_{\cdot,+}$, we have a tangle T_- on $\mathscr{S}_-$ with opposite shading and the following commutative diagram

$$\begin{array}{ccc} \mathscr{S}_{\cdot,+} & \xrightarrow{\Psi} & \mathscr{S}_{\cdot,-} \\ T\Big\downarrow & & T\Big\downarrow \\ \mathscr{S}_{\cdot,+} & \xrightarrow{\Psi} & \mathscr{S}_{\cdot,-} \end{array}.$$

So Ψ is a planar algebraic *-isomorphism. □

When Theorem 6.18 holds for some m, we obtain Theorem 2.4 for m, since the SFT of a subfactor planar algebra is a unitary.

Remark 6.19. From orbifold theory it is easy to see that the Jones-Wassermann subfactors for n disjoint intervals are isomorphic to its dual as subfactors. Here is a proof using orbifold theory: we refer the reader to Section 6.2 of [12] for notations. By construction the Jones-Wassermann subfactor is represented by representation $\pi_{1,\{0,1,...,n-1\}}$. By (2) of Prop. 6.2 in [12], the dual of $\pi_{1,\{0,1,...,n-1\}}$ is $\pi_{1,\{n-1,n-2,...,0\}}$, but $\{n-1, n-2, ..., 0\}$ is conjugate to $\{0, 1, ..., n-1\}$ in S_n via $g(i) = n-i-1, i = 0, 1, ..., n-1$, hence $\pi_{1,\{n-1,n-2,...,0\}} \simeq g\pi_{1,\{0,1,...,n-1\}}g^{-1}$.

Theorem 6.20. *The following commutative diagram holds,*

$$\begin{array}{ccc} \mathscr{S}_{\cdot,+} & \xrightarrow{\Psi^{-1}} & \mathscr{S}_{\cdot,-} \\ \mathfrak{F}\Big\downarrow & & \mathfrak{F}\Big\downarrow \\ \mathscr{S}_{\cdot,-} & \xrightarrow{\Psi} & \mathscr{S}_{\cdot,+} \end{array}.$$

Therefore the planar algebra $\mathscr{S}_\bullet$ is unshaded and the Jones-Wassermann subfactor is symmetrically self-dual.

Proof. We can consider both $\mathfrak{F}$ and Ψ as a map from the lattice to the dual lattice. By the definition of $\Psi\mathfrak{F}$ in Equation (31) and Theorem 2.4, we have that $(\Psi\mathfrak{F})\rho^{-1}(\Psi\mathfrak{F})$ is the identity map. Therefore, the commutative diagram holds. So the isomorphism $\Psi^{\pm 1}$ commutes with all planar tangles on $\mathscr{S}_\bullet$. Then we can lift the shading of the planar algebra and obtain an unshaded planar algebra. □

Remark 6.21. The modularity is essential in the proof of the symmetrically self-duality of Jones-Wassermann subfactors for the unitary MTC $\mathscr{C}$, so we call this property the modular self-duality of the MTC.

Remark 6.22. Recall that ρ_2 is a planar algebraic *-isomorphism of $\mathscr{S}_{\cdot,+}$ with periodicity m, then for each $k \in \mathbb{Z}_m$, $\Psi\rho_2^k$ is a planar algebraic *-isomorphism from $\mathscr{S}_{\cdot,-}$ to $\mathscr{S}_{\cdot,+}$.

Therefore there are k different ways to lift the shading of $\mathscr{S}_{n,\pm}$. Each choice defines an unshaded subfactor planar algebra.

Example. If we take $\mathscr{C}$ to be the unitary MTC, such that its Grothendieck ring is the cyclic group $\mathbb{Z}_d$ and its S matrix is the discrete Fourier transform of $\mathbb{Z}_d$, then the Frobenius algebra object is

$$\gamma = \bigoplus_{k_i \in \mathbb{Z}_d,\ \sum_{i=1}^{n} k_i = 0} k_1 \boxtimes \cdots \boxtimes k_n. \tag{53}$$

Moreover, the m-interval Jones-Wassermann subfactor is the group subfactor for $\mathbb{Z}_d^{m-1}$. The projection category of the unshaded planar algebra is a Tambara-Yamagami category [29]. It has only one odd simple object τ and $\tau^2 = \gamma$. Its even simple objects are $\{k_1 \boxtimes \cdots \boxtimes k_n : k_i \in \mathbb{Z}_d, \ \sum_{i=1}^{n} k_i = 0\}$.

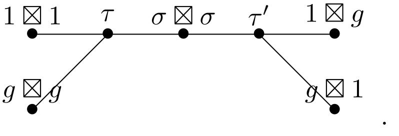
Moreover, the two-box space of the two-interval Jones-Wassermann subfactor is isomorphic to $L^2(\mathbb{Z}_d)$. It is known that the usual multiplication and coproduct on the 2-box space in subfactor theory coincide with the multiplication and convolution on $L^2(\mathbb{Z}_d)$. The SFT $\mathfrak{F}$ intertwines the multiplication and the convolution. We have shown that $\mathfrak{F} = S$, which is the discrete Fourier transform. The modular self-duality reduces to the self-duality of $\mathbb{Z}_d$ on $L^2(\mathbb{Z}_d)$. Therefore the modular self-duality generalizes and categorifies the self-duality of finite abelian groups.

Example. If $\mathscr{C}$ is the representation category of $SU(2)_2$, then it has three simple objects $Irr = \{1, \sigma, g\}$, and $d(1) = d(g) = 1$, $d(\sigma) = \sqrt{2}$. Let $\mathscr{D}$ be the $\mathbb{Z}_2$-graded projection category of the unshaded planar algebra of the m-interval Jones-Wassermann subfactor of $\mathscr{C}$.

When $m = 2$,

$$\tau^2 = \gamma = 1 \boxtimes 1 \oplus \sigma \boxtimes \sigma \oplus g \boxtimes g. \tag{54}$$

The category $\mathscr{D}$ has two odd objects and five even objects. The fusion rule of even objects follows from the fusion rule of $\mathscr{C}$. The fusion graph of tensoring τ is given by the principal graph of the subfactor:



The left half of the graph is given by Equation (54) and the right half is given by the left half tensoring $1 \boxtimes g$. Since $\tau' = \tau \otimes (1 \boxtimes g)$, the fusion rule of tensoring τ' can be derived from the fusion rule for τ. Therefore, we obtain the fusion rule of $\mathfrak{D}$.

When $m \geq 2$, we denote S_k to be the set of Deligne tensors of m simple objects with k multiples of τ. Denote S_+ (and S_-) to be the elements in S_0 that are equivalent to 1 (and g) while changing the Deligne tensor $\boxtimes$ to the tensor $\otimes$ in $\mathscr{C}$ respectively. Then

$$\tau^2 = \gamma = \bigoplus_{\vec{X} \in S_+} \vec{X} \oplus \left(\bigoplus_{k=1}^{[\frac{m}{2}]} \bigoplus_{\vec{Y} \in S_{2k}} 2^{k-1} \vec{Y} \right). \tag{55}$$

It is easy to compute that

$$d(\gamma) = 2^{m-1} + \frac{3^m + (-1)^m - 2}{4}.$$

Similar to the case $m = 2$, we obtain the left half from Equation (54) and the right half from tensoring an element $\in S_-$. There are two odd simple objects τ and τ' in $\mathscr{D}$. The even simple objects in $\mathscr{D}$ are given by $\bigcup_{k=0}^{[\frac{m}{2}]} S_{2k}$. For any $\vec{X} \in S_+$,

$$\vec{X} \otimes \tau = \tau;$$
$$\vec{X} \otimes \tau' = \tau'.$$

For any $\vec{X} \in S_-$,

$$\vec{Z} \otimes \tau = \tau';$$
$$\vec{Z} \otimes \tau' = \tau.$$

For any $\vec{Y} \in S_{2k}$, $k \geq 1$,

$$\vec{Y} \otimes \tau = 2^{k-1}(\tau \oplus \tau');$$
$$\vec{Y} \otimes \tau' = 2^{k-1}(\tau \oplus \tau').$$

And

$$\tau' \otimes \tau' = \gamma;$$
$$\tau' \otimes \tau = \gamma \otimes \vec{Z} = \bigoplus_{\vec{Z} \in S_-} \vec{Z} \oplus \left(\bigoplus_{k=1}^{[\frac{m}{2}]} \bigoplus_{\vec{Y} \in S_{2k}} 2^{k-1} \vec{Y} \right).$$

6.3. Actions of planar tangles on the configuration space

Motivated by the Jones-Wassermann subfactor, we obtain actions of planar tangles on the configuration spaces $\{Conf_{n,m}\}_{n,m \in \mathbb{N}}$ in both X- and Y-directions. Moreover, these actions coincide with the geometric action on the lattices: The contraction tangle ϕ_1 corresponds contractions of lattices to as shown in Equation (44). The correspondence for the other 6+1 elementary tangles are more straightforward. Thus the actions of planar tangles in two different directions commute. We call the (Hilbert) space $\{Conf_{n,m}\}_{n,m\in\mathbb{N}}$ equipped with such commutative actions of bidirectional planar tangles a *bi-planar algebra* which we will study in the future.

Note that

$$\mathfrak{D}_+(x) = \sum_{x'\in B} \overline{LL(x,x')}x' = \theta_2\Psi(\mathfrak{F}(x)). \tag{56}$$

Since θ_2 is anti-isometry, we obtain Theorem 2.4 from Proposition 6.16. Moreover, θ_2 commute with the action of planar tangles, we have the following result corresponding to Propositions 6.9, 6.11, 6.15, 6.17:

Proposition 6.23. *For $x \in Conf(\mathscr{C})_{n,m}$, $y \in Conf(\mathscr{C})_{\ell,m}$,*

$$\begin{aligned}
\mathfrak{D}_+\rho(x) &= \rho\mathfrak{D}_+(x),\\
\mathfrak{D}_+\rho^{-1}\theta_1(x) &= \theta_1\mathfrak{D}_+(x),\\
\mathfrak{D}_+(x\star y) &= \mathfrak{D}_+(x)\wedge\mathfrak{D}_+(y),\\
\mathfrak{D}_+\phi_{k+1}(x) &= \phi_k\mathfrak{D}_+(x),\\
\mathfrak{D}_+\iota_k(x) &= \iota_{k+1}\mathfrak{D}_+(x).
\end{aligned}$$

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