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AN ANGLE BETWEEN INTERMEDIATE SUBFACTORS AND ITS RIGIDITY

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Abstract. We introduce a new notion of an angle between intermediate subfactors and prove various interesting properties of the angle and relate it to the Jones index. We prove a uniform 60 to 90 degree bound for the angle between minimal intermediate subfactors of a finite index irreducible subfactor. From this rigidity we can bound the number of minimal (or maximal) intermediate subfactors by the kissing number in geometry. As a consequence, the number of intermediate subfactors of an irreducible subfactor has at most exponential growth with respect to the Jones index. This answers a question of Longo's published in 2003.

1. Introduction

Jones pioneered the study of modern subfactor theory in his seminal paper[Jon83]. He showed that the indices of subfactors of type II_1 lie in the set $\{4\cos^2(\frac{\pi}{n}) : n \geq 3\} \cup [4, +\infty]$. The study of intermediate subfactors $N \subset P \subset M$ for a finite index inclusion of II_1 factors plays an important role in understanding the theory of subfactors. See [Bis94, BJ97] for some early motivating results in this direction. We denote by $\mathcal{L}(N \subset M)$ the set of all intermediate von Neumann subalgebras for the subfactor $N \subset M$. The set $\mathcal{L}(N \subset M)$ forms a lattice under the two operations $P \wedge Q = P \cap Q$ and $P \vee Q = \{P \cup Q\}''$. The lattice structure of von Neumann subalgebras was first studied by Murray and von Neumann in [MVN36]. If $N \subset M$ is irreducible, that is, $N' \cap M = \mathbb{C}$, then $\mathcal{L}(N \subset M)$ is exactly the lattice of intermediate subfactors. In this case, all intermediate subalgebras are automatically factors. There is a pretty 1-1 correspondence between intermediate subfactors (of an irreducible subfactor) and biprojections introduced in [Bis94] (reformulated in planar algebraic terms in [Lan02, BJ00]).

The lattice of intermediate subfactors generalize the lattice of subgroups because of the following reason: Let G be a finite group with an outer action on the II_1 factor M. Then the intermediate subfactors of $M \subset M \rtimes G$ are given by $M \rtimes H$, where H is a subgroup of G. This leads us to the study of the lattice $\mathcal{L}(N \subset M)$ inspired by various interesting questions in group theory. See [GX11, Xu13, Xu15, Xu16] for some recent progress.

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Watatani in [Wat96], following previous work by Popa [Pop86], obtained the following remarkable result.

Theorem 1.1 ([Wat96]). *Let $N \subset M$ be an irreducible subfactor of type* II_1 *such that $[M:N] < \infty$. Then the set $\mathcal{L}(N \subset M)$ is finite.*

In the same paper Watatani remarked that in this case we can regard an intermediate subfactor lattice as a "quantization" of continuous geometry, invented by von Neumann in [vN60] as a continuous analogue of projective geometry. For a general finite index subfactor $N \subset M$, the set of all intermediate subfactors may not be finite. Even in the case in which $N' \cap M$ is abelian, the set of all intermediate subfactors may be infinite, as shown in [TW97, Theorem 5.4]. Thus, finite index irreducible subfactors may behave very differently from reducible inclusions.

Inspired by earlier works of Christensen and Watatani (see [Chr79, Wat96]), Longo gave an explicit bound for a number of intermediate subfactors for irreducible subfactors in [Lon03]. He showed that the number of intermediate subfactors is bounded by $[M:N]^{2[M:N]^2}$. Longo then asked whether the number of intermediate subfactors can be bounded by $[M:N]^{[M:N]}$ (see [Lon03, discussions at the end of section 2.2]). In this paper we answer this question positively by showing the following (see Theorem 4.7).

Theorem 1.2. *Let $N \subset M$ be a finite index, irreducible subfactor. Then the number of intermediate subfactors $|\mathcal{L}(N \subset M)|$ is bounded by* $\min\{9^{[M:N]}, [M:N]^{[M:N]}\}$.

Our bound improves the existing upper bound of the cardinality of the lattice $\mathcal{L}(N \subset M)$ and as a consequence provides another proof of Theorem 1.1. To solve this problem of finding an upper bound for the cardinality of $\mathcal{L}(N \subset M)$, our idea is to first focus on minimal intermediate subfactors. Minimal (or by duality, maximal) subfactors were extensively studied by Guralnick and Xu [GX11] inspired by Wall's conjecture [Wal62]. Our result shows that the number of minimal intermediate subfactors has at most exponential growth with respect to the index. We conjecture that this number has polynomial growth.

Conjecture 1.3. *There are constants c_1, c_2 such that for any irreducible subfactor $N \subset M$ with finite index, the number of minimal intermediate subfactors is less than $c_2[M:N]^{c_1}$.*

Furthermore, we prove that the number of minimal intermediate subfactors is bounded by the kissing number τ_n of the n-dimensional sphere, where $n = \dim(N' \cap M_1)$. A straightforward estimate of the kissing number shows that $\tau_n < 3^n$. Therefore, we get the following.

Theorem 1.4. *Suppose that $N \subset M$ is a finite index, irreducible subfactor. Then the number of minimal intermediate subfactors is less than* $3^{\dim(N' \cap M_1)}$.

We prove the above theorem by introducing a new angle (see Definition 2.2), denoted by $\alpha_M^N(P, Q)$, between intermediate subfactors P and Q of any finite index subfactor $N \subset M$. This angle is also the Fourier dual of the correlation function. We prove the following rigidity result for the angle between minimal intermediate subfactors.

Theorem 1.5. *If P, Q are two distinct minimal intermediate subfactors of a finite index, irreducible subfactor $N \subset M$, then $\frac{\pi}{3} < \alpha_M^N(P, Q) \leq \frac{\pi}{2}$.*

We can identify intermediate subfactors as unit vectors in the real vector space $(N' \cap M_1)_{s.a.}$ such that the angle between them is given by $\alpha_M^N(\cdot,\cdot)$. Then Theorem 1.4 follows from Theorem 1.5. Iterating Theorem 1.4, we obtain Theorem 1.2.

We also study the angle between two intermediate subfactors of finite index subfactors which are not necessarily irreducible and show that $\alpha_M^N(P,Q) = \frac{\pi}{2}$ if and only if the quadruple

$$\begin{matrix} Q & \subset & M \\ \cup & & \cup \\ N & \subset & P \end{matrix},$$

denoted by (N,P,Q,M), is a commuting square, namely $E_P^M E_Q^M = E_Q^M E_P^M = E_N^M$. The commuting square is a central tool in subfactor theory (see, for example, [JS97, GdlHJ89, Pop94, Pop95, Pop83, Pop89] and references therein). By Fourier duality, we define a dual angle $\beta_M^N(P,Q)$ in Definition 2.6 and show that $\beta_M^N(P,Q) = \frac{\pi}{2}$ if and only if the quadruple is a co-commuting square.

In general, the angles $\alpha_M^N(P,Q)$ and $\beta_M^N(P,Q)$ are different. Surprisingly, the following result holds.

Theorem 1.6. *Suppose that P,Q are two distinct intermediate subfactors of a finite index subfactor $N \subset M$. If $[M:Q] = [P:N]$ and hence $[M:P] = [Q:N]$, then $\alpha_M^N(P,Q) = \beta_M^N(P,Q)$.*

When the equality holds, we call the quadruple (N,P,Q,M) a *parallelogram* and consider the angles $\alpha_M^N(P,Q)$ and $\beta_M^N(P,Q)$ opposite angles of the parallelogram.

We further study the relation between angles and Pimsner–Popa bases and derive various equivalent conditions for a quadruple to be a commuting and/or co-commuting square (see Theorems 2.20, 2.28, and 2.29). As a consequence, we recover the various equivalent conditions of a "nondegenerate commuting square" by Popa in [Pop94] (see Corollary 2.30).

This paper is organized as follows. In §2 we define our notion of angle and obtain various properties, mainly related to commuting squares. In §3 we prove the main rigidity result Theorem 1.5. In §4 we estimate the number of intermediate subfactors and prove Theorems 1.4 and 1.2. In §5 we compare our angle with the Sano–Watatani angle [SW94].

2. Angle and commuting square

Suppose that $N \subset M$ is a finite index subfactor (not necessarily irreducible), suppose that P is an intermediate subfactor, and suppose that e_P is the corresponding biprojection. Let $\tau_P = \mathrm{tr}(e_P)$, let $\delta = \sqrt{[M:N]}$, and let $\tau = [M:N]^{-1}$. Note that $\mathrm{tr}(e_1) = \tau$, where e_1 denotes the Jones projection of $N \subset M$.

For two intermediate subfactors P and Q of a finite index subfactor $N \subset M$, we denote the quadruple of type II_1 factors

$$\begin{matrix} Q & \subset & M \\ \cup & & \cup \\ N & \subset & P \end{matrix}$$

by (N,P,Q,M). We call it extremal if the subfactor $N \subset M$ is extremal.

Recall the following definition (see, e.g., [SW94]).

Definition 2.1. A quadruple (N,P,Q,M) is called a commuting square if $E_P^M E_Q^M = E_Q^M E_P^M = E_N^M$. A quadruple (N,P,Q,M) is called a co-commuting square if the quadruple (M, Q_1, P_1, M_1) is a commuting square, where P_1, Q_1, M_1 denote the basic constructions of $P \subset M, Q \subset M$ and $N \subset M$, respectively.

Definition 2.2. For an intermediate subfactor $P \neq N$, we define the unit vector v_P in $(N' \cap M_1)_{s.a.}$ as $v_P = \frac{e_P - e_1}{\|e_P - e_1\|_2}$. Suppose that P and Q are intermediate subfactors of $N \subset M$. Define the *angle*, denoted by $\alpha_M^N(P, Q)$, between P and Q as follows:

$$\alpha_M^N(P, Q) = \cos^{-1} \langle v_P, v_Q \rangle,$$

where $\langle x, y \rangle = \mathrm{tr}(y^* x)$ and hence $||x||_2 = (\mathrm{tr}(x^* x))^{1/2}$.

If N and M are clear from the context, we may omit them from $\alpha_M^N(P, Q)$. As usual, the angle takes only the principal value: $0 \leq \alpha_M^N(P, Q) \leq \pi$. Note that $v_P, v_Q \geq 0$, so $\langle v_P, v_Q \rangle \geq 0$. Therefore, $0 \leq \alpha_M^N(P, Q) \leq \dfrac{\pi}{2}$.

Proposition 2.3. *For a quadruple (N, P, Q, M), $\alpha(P, Q) = 0$ if and only if $e_P = e_Q$.*

Proof. Note that $\alpha(P, Q) = 0$ if and only if v_P is a multiple of v_Q. Since both v_P and v_Q are positive and $\|v_P\|_2 = \|v_Q\|_2 = 1$, it follows that $v_P = v_Q$. As $(e_P - e_1)$ and $(e_Q - e_1)$ are both projections, they are equal. So $e_P = e_Q$. □

Proposition 2.4. *The quadruple (N, P, Q, M) forms a commuting square if and only if $\alpha(P, Q) = \frac{\pi}{2}$.*

Proof. Note that (N, P, Q, M) forms a commuting square iff $e_P e_Q = e_1$ iff $(e_P - e_1)(e_Q - e_1) = 0$ iff $\alpha(P, Q) = \pi/2$. □

Proposition 2.5. *For an* extremal *quadruple (N, P, Q, M),*

$$\cos \alpha_M^N(P, Q) = \mathrm{corr}(e_{P_1}, e_{Q_1}),$$

where $\mathrm{corr}(x, y) := \langle \frac{x - \mathrm{tr}(x)}{\|x - \mathrm{tr}(x)\|_2}, \frac{y - \mathrm{tr}(y)}{\|y - \mathrm{tr}(y)\|_2} \rangle$ *is the correlation function. Here P_1 and Q_1 denote the basic construction of $P \subset M$ and $Q \subset M$, respectively.*

Proof. Let $\mathfrak{F} : N' \cap M_1 \to M' \cap M_2$ be the Fourier transform (see, e.g., [Bis97]) defined as follows:

$$\mathfrak{F}(x) = E_{M'}^{N'}(e_1 e_2 x),$$

where e_1 (resp., e_2) is the Jones projection for the basic construction of $N \subset M$ (resp., $M \subset M_1$, where M_1 is the basic construction of $N \subset M$) and $E_{M'}^{N'}$ is the trace-preserving conditional expectation from N' to M'. Since the subfactor is extremal, we have

$$\cos \alpha_M^N(P, Q) = \langle v_P, v_Q \rangle = \langle \mathfrak{F}(v_P), \mathfrak{F}(v_Q) \rangle. \tag{1}$$

Note that $\mathfrak{F}(e_P)$ is a multiple of e_{P_1} and that $\mathfrak{F}(e_1)$ is a multiple of the identity. So $\mathfrak{F}(e_P - e_1) = a e_{P_1} - b$ for some constants a and b. Moreover, $\langle \mathfrak{F}(e_P - e_1), \mathfrak{F}(e_1) \rangle = \langle e_P - e_1, e_1 \rangle = 0$, so $\mathrm{tr}(\mathfrak{F}(e_P - e_1)) = 0$. Therefore, $\mathfrak{F}(e_P - e_1) = a(e_{P_1} - \mathrm{tr}(e_{P_1}))$. Recall that $\|v_P\|_2 = 1$, so $\mathfrak{F}(v_P) = \frac{e_{P_1} - \mathrm{tr}(e_{P_1})}{\|e_{P_1} - \mathrm{tr}(e_{P_1})\|_2}$. Similarly, $\mathfrak{F}(v_Q) = \frac{e_{Q_1} - \mathrm{tr}(e_{Q_1})}{\|e_{Q_1} - \mathrm{tr}(e_{Q_1})\|_2}$. By equation (1), $\cos \alpha_M^N(P, Q) = \mathrm{corr}(e_{P_1}, e_{Q_1})$. □

Definition 2.6. We define the dual angle, denoted by $\beta_M^N(P, Q)$, between P and Q as $\beta_M^N(P, Q) := \alpha_{M_1}^M(P_1, Q_1)$, where P_1 and Q_1 denote the basic construction of $P \subset M$ and $Q \subset M$, respectively.

This is similar to [SW94]. As before, if from the context it is clear what N and M are, we may omit them from $\beta_M^N(P,Q)$. By duality we have that

$$\alpha_M^N(P,Q) = \beta_{M_1}^M(P_1,Q_1). \tag{2}$$

Proposition 2.7. *The quadruple (N,P,Q,M) forms a co-commuting square if and only if $\beta(P,Q) = \frac{\pi}{2}$.*

Proof. It follows from Definitions 2.1 and 2.6 and Proposition 2.4. □

The next theorem also follows easily from the above definitions and from Proposition 2.5. We leave the details to the reader.

Theorem 2.8. *For a quadruple (N,P,Q,M), let $\tau_P = \mathrm{tr}(e_P)$, and let $\tau_Q = \mathrm{tr}(e_Q)$. Then*

$$\cos\alpha_M^N(P,Q) = \frac{\mathrm{tr}(e_Pe_Q) - \tau}{\sqrt{\tau_P - \tau}\sqrt{\tau_Q - \tau}}. \tag{3}$$

If the quadruple is extremal, then

$$\cos\beta_M^N(P,Q) = \frac{\mathrm{tr}(e_Pe_Q) - \tau_P\tau_Q}{\sqrt{\tau_P - \tau_P^2}\sqrt{\tau_Q - \tau_Q^2}}. \tag{4}$$

Remark 2.9. Theorem 2.8 implies easily the following two facts:

(1) $\mathrm{tr}(e_Pe_Q) \geq \tau$, $\mathrm{tr}(e_Pe_Q) \geq \tau_P\tau_Q$. The equalities hold if and only if $\alpha_M^N(P,Q) = \pi/2$ and $\beta_M^N(P,Q) = \pi/2$, respectively.

(2) If the extremal quadruple (N,P,Q,M) is a commuting square, then $[M:Q] \geq [P:N]$ and $[M:P] \geq [Q:N]$. This result was proved in [Pop89, Proposition 1.7], and it also follows easily from the above fact.

Definition 2.10. For a quadruple (N,P,Q,M), the following are equivalent:

(1) $\tau_P\tau_Q = \tau$,
(2) $[M:P] = [Q:N]$,
(3) $[M:Q] = [P:N]$.

We call the quadruple a parallelogram if one of the above equivalent conditions holds.

In general, it is not true that $\alpha_M^N(P,Q) = \beta_M^N(P,Q)$ (see, for instance, Fact 2.15). One can have a quadruple which is commuting, but not co-commuting. Surprisingly, the following result holds.

Theorem 2.11. *If an extremal quadruple (N,P,Q,M) is a parallelogram, then $\alpha_M^N(P,Q) = \beta_M^N(P,Q)$.*

Proof. If a quadruple (N,P,Q,M) is a parallelogram, namely $\tau = \tau_P\tau_Q$, then by Theorem 2.8, $\cos\alpha_M^N(P,Q) = \cos\beta_M^N(P,Q)$. So $\alpha_M^N(P,Q) = \beta_M^N(P,Q)$. □

Remark 2.12. Hence, we may consider $\alpha_M^N(P,Q)$ and $\beta_M^N(P,Q)$ opposite angles of the parallelogram (N,P,Q,M).

Motivated by [SW94], we try to investigate the angle $\alpha_M^N(P,Q)$ in terms of Pimsner–Popa basis [PP86]. In this paper, by a Pimsner–Popa basis we mean a *right basis*. Thus, the condition for a set $\{\lambda_i : i \in I\} \subset M$ (for some finite indexing set I) to be a right basis for M/N would be $\sum_{i=1}^n \lambda_i e_1 \lambda_i{}^* = 1$ or equivalently, $x = \sum_{i=1}^n E_N(x\lambda_i)\lambda_i{}^* = \sum_{i=1}^n \lambda_i E_N(\lambda_i{}^* x)$ for all $x \in M$. The set $\{\lambda_i : i \in I\}$ will be called a *left basis* for M/N if $\{\lambda_i{}^* : i \in I\}$ is a right basis.

Remark 2.13. A set $\{\lambda_i : i \in I\} \subset M$ is called a two-sided basis for M/N if it is both a left basis and a right basis. It is an open question as to whether any finite index (irreducible) subfactor has a two-sided basis.

Proposition 2.14. *Consider intermediate subfactors P and Q of $N \subset M$. Let $\{\lambda_i\}$ (resp., $\{\mu_j\}$) be (right) basis for P/N (resp., Q/N). Then*

$$\cos(\alpha(P,Q)) = \frac{\sum_{i,j} \operatorname{tr}(E^M_N({\lambda_i}^*\mu_j){\mu_j}^*\lambda_i) - 1}{\sqrt{[P:N]-1}\sqrt{[Q:N]-1}}. \tag{5}$$

Proof. First observe that for any intermediate subfactor, say, P, of $N \subset M$ and basis $\{\lambda_i\}$ we have $e^M_P = \sum_i \lambda_i e_1 {\lambda_i}^*$. This follows trivially from the following array of equations and is well known. For any $x \in M$ we have $(\sum_i \lambda_i e_1 {\lambda_i}^*)(x\Omega) = (\sum_i \lambda_i (E^M_N({\lambda_i}^* x)))\Omega = (\sum_i \lambda_i E^P_N({\lambda_i}^* E^M_P(x)))\Omega = E^M_P(x)\Omega = e^M_P(x\Omega)$, where Ω denotes the cyclic vector for the standard Hilbert space $L^2(M)$.

In our notation we have $e^M_Q = \sum_j \mu_j e_1 {\mu_j}^*$. Then it follows from Definition 2.2 that

$$\begin{aligned}
\cos(\alpha(P,Q)) &= \frac{\operatorname{tr}(e_P e_Q) - \tau}{\sqrt{\operatorname{tr}(e_P) - \tau}\sqrt{\operatorname{tr}(e_Q) - \tau}} \\
&= \frac{\operatorname{tr}(\sum_{i,j} \lambda_i e_1 {\lambda_i}^* \mu_j e_1 {\mu_j}^*) - \tau}{\sqrt{\operatorname{tr}(\sum_i \lambda_i e_1 {\lambda_i}^*) - \tau}\sqrt{\operatorname{tr}(\sum_j \mu_j e_1 {\mu_j}^*) - \tau}} \\
&= \frac{\sum_{i,j} \operatorname{tr}(e_1 E^M_N({\lambda_i}^*\mu_j){\mu_j}^*\lambda_i) - \tau}{\sqrt{\sum_i \operatorname{tr}(e_1 {\lambda_i}^*\lambda_i) - \tau}\sqrt{\sum_j \operatorname{tr}(e_1 {\mu_j}^*\mu_j) - \tau}} \\
&= \frac{\sum_{i,j} \operatorname{tr}(E^M_N({\lambda_i}^*\mu_j){\mu_j}^*\lambda_i) - 1}{\sqrt{[P:N]-1}\sqrt{[Q:N]-1}}.
\end{aligned}$$

This completes the proof. ☐

Fact 2.15. *Consider intermediate subfactors P and Q such that $N \subset P \subset Q \subset M$. Then the following two equations hold (as is seen from the definitions):* $\cos(\alpha(P,Q)) = \sqrt{\frac{[P:N]-1}{[Q:N]-1}}$ *and* $\cos(\beta(P,Q)) = \sqrt{\frac{[M:Q]-1}{[M:P]-1}}$. *This shows that $\alpha(P,Q)$ and $\beta(P,Q)$ may not be equal in general.*

Proposition 2.16. *Consider factors of type* II_1 *such that $R, N \subset P, Q \subset M, S$. Then $\alpha^N_M(P,Q) = \alpha^N_S(P,Q)$ and $\beta^N_M(P,Q) = \beta^R_M(P,Q)$.*

Proof. This follows from Proposition 2.14. ☐

Definition 2.17. Consider the quadruple of type II_1 factors (N,P,Q,M). Let $\{\lambda_i\}$ (resp., $\{\mu_j\}$) be a basis for P/N (resp., Q/N). Define two self-adjoint operators p and q as follows:

$$p := \sum_{i,j} \lambda_i \mu_j e_1 {\mu_j}^* {\lambda_i}^*, \; q := \sum_{i,j} \mu_j \lambda_i e_1 {\lambda_i}^* {\mu_j}^*.$$

In general, p and q are not projections. Later we will see that they always have the same spectrum and the same trace.

Lemma 2.18. *The definition above (of p and q) does not depend on the choice of the Pimsner–Popa bases.*

Proof. Suppose that $\{\psi_j : j \in I\}$ is another basis for P/N. Then it is easy to see that

$$\sum_i \lambda_i e_Q {\lambda_i}^* = \sum_i \{\sum_j \psi_j E^P_N({\psi_j}^* \lambda_i)\} e_Q {\lambda_i}^* = \sum_{i,j} \psi_j e_Q E^P_N({\psi_j}^*\lambda_i){\lambda_i}^*$$
$$= \sum_j \psi_j e_Q \{\sum_i E^P_N({\psi_j}^*\lambda_i){\lambda_i}^*\} = \sum_j \psi_j e_Q {\psi_j}^*.$$

As already observed in the proof of Proposition 2.14, $e_Q = \sum_j \mu_j e_1 {\mu_j}^*$. Thus, $p = \sum_i \lambda_i e_Q {\lambda_i}^*$. This shows that p is independent of the choice of basis. Similar proof works for q. □

Lemma 2.19. *Following the notations in Definition* 2.17, $\{\lambda_i \mu_j\}$ *is a basis for* M/N *if and only if* $p = 1$, *and* $\{\mu_j \lambda_i\}$ *is a basis for* M/N *if and only if* $q = 1$.

Proof. It follows from the definition of the Pimsner–Popa basis. □

Proposition 2.20. *Consider again* $N \subset P, Q \subset M$, *and let* $\{\lambda_i\}$ (*resp.*, $\{\mu_j\}$) *be a basis for* P/N (*resp.*, Q/N). *Then the following are equivalent:*

(1) $\alpha(P, Q) = \pi/2$.
(2) $q = \sum_{i,j} \mu_j \lambda_i e_1 {\lambda_i}^* {\mu_j}^*$ *is a projection such that* $q \geq e_P$.
(3) $p = \sum_{i,j} \lambda_i \mu_j e_1 {\mu_j}^* {\lambda_i}^*$ *is a projection such that* $p \geq e_Q$.

Proof.

(1) $\Rightarrow$ (2) That q is a projection is easy and was observed in [SW94]. We prove it for the sake of completeness. From the proof of Proposition 2.14, we have $q = \sum_i \mu_i e_P {\mu_i}^*$ and hence $q = q^*$. Then

$$q^2 = \sum_{i,j} \mu_i e_P {\mu_i}^* \mu_j e_P {\mu_j}^* = \sum_{i,j} \mu_i E^M_P({\mu_i}^* \mu_j) e_P {\mu_j}^* = \sum_{i,j} \mu_i E^M_P E^M_Q ({\mu_i}^*\mu_j) e_P {\mu_j}^*$$
$$= \sum_{i,j} \mu_i E^M_N({\mu_i}^*\mu_j) e_P {\mu_j}^* \text{ (applying Proposition 2.4)}$$
$$= \sum_j \mu_j e_P {\mu_j}^* \text{ (since}\{\mu_j\}\text{is a basis for } Q/N) = q.$$

Now we show that $(e_P)q = e_P$.

$$(e_P)q = \sum_j e_P \mu_j e_P {\mu_j}^* = \sum_j e_P E^M_P(\mu_j){\mu_j}^* = \sum_j e_P E^M_P E^M_Q(\mu_j){\mu_j}^*$$
$$= \sum_j e_P E^M_N(\mu_j){\mu_j}^* \text{ (applying Proposition 2.4)}$$
$$= e_P \text{ (since } \{\mu_j\} \text{ is a basis for } Q/N).$$

Thus, q is a projection such that $q \geq e_P$. This completes the proof of (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1) $(e_P)q = e_P$ implies that $\sum_j e_P E^M_P(\mu_j){\mu_j}^* = e_P$. Taking the trace of both sides, we get

$$\sum_j \operatorname{tr}(E^M_P(\mu_j){\mu_j}^*) = 1. \tag{6}$$

Then from the definition of angle it follows easily that

$$\cos(\alpha(P,Q)) = \frac{\operatorname{tr}(e_P e_Q) - \tau}{\sqrt{\operatorname{tr}(e_P) - \tau}\sqrt{\operatorname{tr}(e_Q) - \tau}}. \tag{7}$$

Put $r = \sum_j {\mu_j}^* e_P \mu_j$. Thus, $\operatorname{tr}(re_1) = \operatorname{tr}(\sum_j {\mu_j}^* e_P \mu_j e_1) = \operatorname{tr}(e_P \sum_j \mu_j e_1 {\mu_j}^*) = \operatorname{tr}(e_P e_Q)$ (since $\sum_j \mu_j e_1 {\mu_j}^* = e_Q$). Thus, it follows from equation (7) that

$$\cos(\alpha(P,Q)) = \frac{\operatorname{tr}(re_1) - \tau}{\sqrt{\operatorname{tr}(e_P) - \tau}\sqrt{\operatorname{tr}(e_Q) - \tau}}. \tag{8}$$

On the other hand, $re_1 = \sum_j {\mu_j}^* e_P \mu_j e_1 = \sum_j {\mu_j}^* e_P \mu_j e_P e_1$ (since $e_P e_1 = e_1$) $= \sum_j {\mu_j}^* E^M_P(\mu_j) e_1$. Thus, $\operatorname{tr}(re_1) = \tau \operatorname{tr}({\mu_j}^* E^M_P(\mu_j)) = \tau \operatorname{tr}(E^M_P(\mu_j){\mu_j}^*)$. Then equation (6) implies that $\operatorname{tr}(re_1) = \tau$. Thus, by equation (8) we have $\alpha(P,Q) = \pi/2$, thereby completing the proof of (2) $\Rightarrow$ (1).

(1) $\Leftrightarrow$ (3) Simply observe that $\alpha(P,Q) = \alpha(Q,P)$. The rest follows from the above two implications. This completes the proof. □

Fact 2.21. *$q = e_P$ if and only if $Q = N$. Similarly, $p = e_Q$ if and only if $P = N$.*

Proof. By the Markov property of the trace, $\operatorname{tr}(q) = \operatorname{tr}(\sum_j \mu_j e_P {\mu_j}^*) = \frac{\sum_j \operatorname{tr}(\mu_j {\mu_j}^*)}{[M:P]}$. But as $\{\mu_j\}$ is a basis for Q/N, $\sum_j \mu_j {\mu_j}^* = [Q:N]$. Thus,

$$\operatorname{tr}(q) = \frac{[M:N]}{[M:P][M:Q]}. \tag{9}$$

Suppose that $q = e_P$. After taking the trace on both sides, we get $[M:N] = [M:Q]$, which implies that $Q = N$.

Conversely, $Q = N$ implies that $\operatorname{tr}(q) = \operatorname{tr}(e_P)$ (see equation (9)). Since by Proposition 2.20 $q \geq e_P$, it follows that $q = e_P$, as tr is faithful. □

Proposition 2.22. *Consider again $N \subset P, Q \subset M$, and let $\{\lambda_i\}$ (resp., $\{\mu_j\}$) be a basis for P/N (resp., Q/N). Define p and q as in Proposition 2.20. Then $JpJ = q$, where J is the usual modular conjugation operator on $L^2(M)$.*

Proof. We know that $p = \sum_i \lambda_i e_Q {\lambda_i}^*$ and that $q = \sum_j \mu_j e_P {\mu_j}^*$. Let us denote by Ω the cyclic vector for the standard Hilbert space $L^2(M)$. Then for any $x \in M$

$$\begin{aligned}
JpJ(x\Omega) &= Jp(x^*\Omega) = J(\sum_i \lambda_i e_Q({\lambda_i}^* x^* \Omega)) \\
&= \sum_i J(\lambda_i E^M_Q({\lambda_i}^* x^*)\Omega) = \sum_i (E^M_Q(x\lambda_i){\lambda_i}^*)\Omega \\
&= \sum_i (\sum_j \mu_j E^Q_N\{{\mu_j}^* E^M_Q(x\lambda_i)\}{\lambda_i}^*)\Omega \text{ (since } \{\mu_j\} \text{ is a basis for } Q/N) \\
&= \sum_{i,j} (\mu_j E^Q_N\{E^M_Q({\mu_j}^* x\lambda_i)\}{\lambda_i}^*)\Omega = \sum_{i,j} (\mu_j E^M_N({\mu_j}^* x\lambda_i){\lambda_i}^*)\Omega.
\end{aligned}$$

On the other hand, the following array of equations hold true:

$$\begin{aligned} q(x\Omega) &= (\sum_j \mu_j e_P {\mu_j}^*)(x\Omega) = \sum_j (\mu_j E^M_P({\mu_j}^* x))\Omega \\ &= \sum_j (\mu_j (\sum_i E^P_N \{E^M_P({\mu_j}^* x)\lambda_i\}{\lambda_i}^*))\Omega \text{ (since } \{\lambda_i\} \text{ is a basis for } P/N) \\ &= \sum_{i,j} (\mu_j E^P_N(E^M_P({\mu_j}^* x\lambda_i)){\lambda_i}^*)\Omega = \sum_{i,j} (\mu_j E^M_N({\mu_j}^* x\lambda_i){\lambda_i}^*)\Omega. \end{aligned}$$

Thus, we see that $JpJ = q$. This completes the proof. ☐

The following result is well known. For example, see Proposition 2.7 in [Bis97].

Lemma 2.23. *Let $N \subset M$ be an inclusion of II_1 factors with finite index, and let $\{m_i : i \in I\} \subset M$ be a Pimsner–Popa basis (not necessarily orthonormal) for M/N. Let us also denote by $\mathrm{tr}_{N'}$ the unique normalized trace on $N' = N' \cap \mathcal{B}(L^2(M))$. Then the unique $\mathrm{tr}_{N'}$-preserving conditional expectation is given by $\phi(x) = [M : N]^{-1} \sum_i m_i x {m_i}^*$, where $x \in N'$.*

Remark 2.24. Note that Proposition 2.22 may also be derived from Lemma 2.23. We thank an anonymous referee for this paper for pointing this out.

Proposition 2.25. *Let $N \subset P, Q \subset M$ be intermediate subfactors such that $[M : N]$ is finite ($N \subseteq M$ is not assumed to be irreducible for this proposition). Then the self-adjoint operator p belongs to $P' \cap Q_1$ and is given by $p = [P : N]E^{N'}_{P'}(e_Q) = [Q : N]E^{M_1}_{Q_1}(e_P)$. Similarly, $q = [Q : N]E^{N'}_{Q'}(e_P) = [P : N]E^{M_1}_{P_1}(e_Q)$. Thus, $q \in Q' \cap P_1$. Here, as usual, Q_1 (resp., P_1) denotes the basic construction of $Q \subset M$ (resp., $P \subset M$).*

Proof. Consider again $N \subset P$, $Q \subset M$, and let $\{\lambda_i\}$(resp., $\{\mu_j\}$) be a basis for P/N (resp., Q/N). By Lemma 2.23 we immediately get, for any $x \in N'$, $E^{N'}_{P'}(x) = [P : N]^{-1} \sum_i \lambda_i x {\lambda_i}^*$. Clearly $e_Q \in N'$ and hence $E^{N'}_{P'}(e_Q) = [P : N]^{-1} \sum_i \lambda_i e_Q {\lambda_i}^*$. Thus, $p = \sum_i \lambda_i e_Q {\lambda_i}^* = [P : N]E^{N'}_{P'}(e_Q)$. Similarly, we can prove $q = [Q : N]E^{N'}_{Q'}(e_P)$. Now take the modular conjugation operator J on $L^2(M)$ to get $JqJ = [Q : N]E^{M_1}_{Q_1}(e_P)$. Now, by Proposition 2.22 we immediately get $p = [Q : N]E^{M_1}_{Q_1}(e_P)$. The proof for q is similar. This completes the proof of the proposition. ☐

Proposition 2.26. *Let $\alpha = \pi/2$, and let p, q be as in Theorem 2.20. Then $\bigvee\{ve_Qv^* : v \in \mathcal{U}(P)\} = p$ and $\bigvee\{ue_Pu^* : u \in \mathcal{U}(Q)\} = q$.*

Proof. First note that (as observed in Proposition 2.20) for any basis $\{\mu_j\}$ of Q/N, $q = \sum_j \mu_j e_P {\mu_j}^*$ is a projection such that $q \geq e_P$. Consider an arbitrary unitary element $u \in \mathcal{U}(Q)$. Then it is trivial to see that $\{u^*\mu_j\}$ is a basis for Q/N. Thus, $u^*qu \geq e_P$ and hence $ue_Pu^* \leq q$. Therefore, $\bigvee\{ue_Pu^* : u \in \mathcal{U}(Q)\} \leq q$. Since $q = \sum_j \mu_j e_P {\mu_j}^*$, range$(q) \subset [\mu L^2(P) : \mu \in Q] = [uL^2(P) : u \in \mathcal{U}(Q)] =$ [range$(\{ue_Pu^* : u \in \mathcal{U}(Q)\})$]. Thus, $\bigvee\{ue_Pu^* : u \in \mathcal{U}(Q)\} \geq q$. So $\bigvee\{ue_Pu^* : u \in \mathcal{U}(Q)\} = q$. The proof for p is exactly the same. ☐

Remark 2.27. Let $\alpha = \frac{\pi}{2}$ and p, q be as in Theorem 2.20. Then it is not hard to show that $p, q \geq e_P \vee e_Q$. In general, it is not true that $e_P \vee e_Q = e_{P \vee Q}$, although $e_P \vee e_Q \leq p, q \leq e_{P \vee Q}$.

Below we give a characterization of commuting squares in terms of basis.

Theorem 2.28. *For a quadruple (N, P, Q, M) the following are equivalent:*

(1) *(N, P, Q, M) is a commuting square, that is, $\alpha(P, Q) = \pi/2$.*
(2) $p = \bigvee\{ve_Qv^* : v \in \mathcal{U}(P)\}$.
(3) $q = \bigvee\{ue_Pu^* : u \in \mathcal{U}(Q)\}$.

Proof.
$(1) \Rightarrow (2)$ This is Proposition 2.26.
$(2) \Rightarrow (1)$ Clearly $\bigvee\{ve_Qv^* : v \in \mathcal{U}(P)\} \geq e_Q$. Hence, $p \geq e_Q$. Again applying Proposition 2.20, we get $\alpha(P, Q) = \frac{\pi}{2}$.
Thus (1) and (2) are equivalent. By symmetry, (1) and (3) are equivalent. This completes the proof. □

Below we investigate a case in which $\alpha(P, Q) = \pi/2 = \beta(P, Q)$. Explicitly we characterize simultaneously commuting and co-commuting squares in terms of various equivalent conditions.

Theorem 2.29. *For a quadruple (N, P, Q, M) the following are equivalent:*

(1) *(N, P, Q, M) is a commuting and co-commuting square.*
(2) $\alpha(P, Q) = \beta(P, Q) = \pi/2$.
(3) $p = 1$.
(4) *If $\{\lambda_i\}$ (resp., $\{\mu_j\}$) is a basis for P/N (resp., Q/N), then $\{\lambda_i\mu_j\}$ is a basis for M/N.*
(5) $q = 1$.
(6) *If $\{\lambda_i\}$ (resp., $\{\mu_j\}$) is a basis for P/N (resp., Q/N),then $\{\mu_j\lambda_i\}$ is a basis for M/N.*
(7) *Any basis (not necessarily orthonormal) for P/N is a basis for M/Q.*
(8) *Any basis (not necessarily orthonormal) for Q/N is a basis for M/P.*

Proof. By Proposition 2.4 and by the definition of a co-commuting square, $(1) \iff (2)$.

Suppose that (1) holds true. Then, applying fact (2) in Remark 2.9 twice, we get $[M : Q] = [P : N]$. Thus, by equation (9) $\mathrm{tr}(q) = 1$. Since $\alpha(P, Q) = \pi/2$, from Proposition 2.20 it follows that q is a projection implying that $q = 1$. Similarly, $p = 1$. Thus, $(1) \Rightarrow (3), (5)$.

By Lemma 2.19 $(3) \iff (4)$ and $(5) \iff (6)$.

Suppose that (3) holds true, that is, $p = 1$. Thus, applying Proposition 2.20, we immediately get $\alpha(P, Q) = \pi/2$. Using again equation (9), we obtain $[M : Q] = [P : N]$. Then Theorem 2.11 implies that $\beta(P, Q) = \pi/2$. In other words, $(3) \Rightarrow (1)$.

Suppose that (4) holds true. Let $\{\lambda_i\}$ be any basis for P/N. Fix a basis $\{\mu_j\}$ for Q/N. Thus, (4) implies that $\{\lambda_i\mu_j\}$ is a basis for M/N. Hence, $\sum_{i,j} \lambda_i\mu_je_1{\mu_j}^*{\lambda_i}^* = 1$. Thus, $\sum_i \lambda_ie_Q{\lambda_i}^* = 1$ (since we know that $\sum_j \mu_je_1{\mu_j}^* = e_Q$). We thus obtain that $\{\lambda_i\}$ is a basis for M/Q. Therefore, $(4) \Rightarrow (7)$.

Simply use [JS97, Lemma 4.3.4(i)] to conclude that $(7) \Rightarrow (4)$.

Thus, we obtain that $(1) \iff (2) \iff (3) \iff (4) \iff (7)$.

By symmetry (since $\beta(P, Q) = \beta(Q, P)$) $(1) \iff (2) \iff (5) \iff (6) \iff (8)$.

This completes the proof. □

Now, the following corollary follows easily. This is the characterization of nondegenerate commuting squares due to Popa (see [Pop94]) (with slight modification).

Corollary 2.30 ([Pop94]). *For a commuting square (N, P, Q, M) of II_1 factors with $[M : N]$ finite, the following statements are equivalent:*

(1) *(N, P, Q, M) is a co-commuting square, that is, $\beta^N_M(P, Q) = \pi/2$.*
(2) $\bigvee\{ve_Qv^* : v \in \mathcal{U}(P)\} = 1$.
(3) $\bigvee\{ue_Pu^* : u \in \mathcal{U}(Q)\} = 1$.
(4) *Any basis (not necessarily orthonormal) for P/N is a basis for M/Q.*
(5) *Any basis (not necessarily orthonormal) for Q/N is a basis for M/P.*
(6) *$PQ := \mathrm{span}\{\sum_{i=1}^n x_iy_i : x_i \in P, y_i \in Q\} = M$; in particular, (N, P, Q, M) is nondegenerate.*
(7) *$QP = M$; in particular, (N, Q, P, M) is nondegenerate.*

Proof. Let $\{\lambda_i\}, \{\mu_j\}, p$, and q be as before.

By Theorem 2.29 and Proposition 2.26 it is trivial to see that conditions (1), (2), and (4) all are equivalent to the equation $p = 1$. Similarly, (1), (3), and (5) are equivalent to the equation $q = 1$.

Suppose that (3) holds true. Thus, by Theorem 2.28 $\{\mu_j\lambda_i\}$ is a basis for M/N and hence $M = QP$, implying (7). Conversely, suppose that (7) holds true. Thus, any $x \in M$ can be written as $x = \sum_k b_ka_k$, where $b_k \in Q$ and $a_k \in P$. Then it is easy to check that for any $x \in M$

$$\begin{aligned} q(x\Omega) = q((\sum_k b_ka_k)\Omega) &= \sum_{j,k} \mu_j e_P({\mu_j}^* b_k a_k \Omega) = \sum_{j,k} \mu_j E^M_P({\mu_j}^* b_k) a_k \Omega \\ &= \sum_{j,k} \mu_j E^M_P E^M_Q({\mu_j}^* b_k) a_k \Omega \\ &= \sum_{j,k} \mu_j E^Q_N({\mu_j}^* b_k) a_k \Omega \text{ (by the commuting square condition)} \\ &= \sum_k b_k a_k \Omega \text{ (since } \{\mu_j\} \text{ is a basis for } Q/N) = x\Omega. \end{aligned}$$

Thus, $q = 1$.

That (6) is equivalent to $p = 1$ is exactly the same. This completes the proof. □

Remark 2.31. It is worth mentioning that Popa has shown that if (4) of Theorem 2.29 holds for a quadruple (N, P, Q, M), then $\overline{\mathrm{sp}PQ} = M$, with the additional assumption that the quadruple is a commuting square, whereas we have shown in Theorem 2.29 that if (4) holds, then automatically the quadruple will be a nondegenerate commuting square.

Corollary 2.32. *If P/N and Q/N both have a two-sided basis, then $\alpha(P, Q) = \beta(P, Q) = \pi/2$ implies that M/N has a two-sided basis.*

Proof. Just use the fact (2) ⇔ (3) of Theorem 2.29. □

3. Boundedness of angle

In this section and the next section we assume that $N \subset M$ is a finite index, irreducible subfactor. In the irreducible case intermediate von Neumann algebras are intermediate subfactors, so the set of intermediate subfactors form a lattice under the operations $P \wedge Q = P \cap Q$ and $P \vee Q = \{P \cup Q\}''$.

Definition 3.1. Let $N \subset M$ be a subfactor. Then Q is called a maximal (resp., minimal) intermediate subfactor of $N \subset M$ if whenever there exists an intermediate subfactor P such that $N \subset Q \subset P \subset M$ (resp., $N \subset P \subset Q \subset M$), then P equals either Q or M (resp., P equals either N or Q). We exclude N and M from the definition of maximal (or minimal) intermediate subfactor for obvious reasons.

Note that maximal intermediate subfactors in $N \subset M$ correspond to minimal intermediate subfactors in $M \subset M_1$, where M_1 denotes the basic construction of $N \subset M$.

Lemma 3.2. *For a quadruple (N, P, Q, M) with $N' \cap M = \mathbb{C}$, the self-adjoint operator p (resp., q) is a multiple of a projection.*

Proof. We have $p = \sum_i \lambda_i e_Q \lambda_i^*$. Now by the pushdown lemma [PP86, Lemma 1.2] $pe_Q = [M : Q]E_M(pe_Q)e_Q$. But clearly, $E_M(pe_Q) \in N' \cap M = \mathbb{C}$. Thus, $pe_Q = \lambda e_Q$ (say). Using Proposition 2.25, we obtain that $p = [P : N]E^{N'}_{P'}(e_Q)$. Thus, $p^2 = [P : N]pE^{N'}_{P'}(e_Q)$. But since $p \in P' \cap Q_1$, we get that $p^2 = [P : N]E^{N'}_{P'}(pe_Q) = \lambda[P : N]E^{N'}_{P'}(e_Q) = \lambda p$. This completes the proof. The proof for q is similar. □

Remark 3.3. Lemma 3.2 implies that $\frac{1}{\lambda}p$ is a projection, where $\lambda = \|p\| = [M : Q]\mathrm{tr}(pe_Q)$. Thus, $\frac{p}{\|p\|}$ and $\frac{q}{\|q\|}$ both are projections. Observe, by Proposition 2.22, that $\|q\| = \|p\|$.

Lemma 3.4. *Let $N \subset M$ be a finite index irreducible inclusion of II_1 factors, and let P, Q be intermediate subfactors. Suppose that e_P and e_Q are two biprojections; then $e_P \vee e_Q$ is a subprojection of the projection $\frac{p}{\|p\|}$ (or $\frac{q}{\|q\|}$).*

Proof. From the proof of Lemma 3.2 we obtain $pe_Q = \|p\|e_Q$. Thus, $\frac{p}{\|p\|} \geq e_Q$. Similarly, $qe_P = \|q\|e_P = \|p\|e_P$. Now, by Proposition 2.22 $pe_P = JqJe_P = Jqe_PJ$. Hence, $pe_P = \|p\|Je_PJ = \|p\|e_P$. In other words, $\frac{p}{\|p\|} \geq e_P$. In conclusion, $\frac{p}{\|p\|} \geq e_P \vee e_Q$. The proof for q is similar. This completes the proof of the lemma. □

Proposition 3.5. *Suppose that P, Q are distinct minimal intermediate subfactors of a finite index, irreducible subfactor $N \subset M$. Then $\frac{\tau_P \tau_Q}{\mathrm{tr}(e_P e_Q)} \geq \tau_P + \tau_Q - \tau$.*

Proof. If P and Q are minimal intermediate subfactors, then $P \cap Q = N$. Thus, $e_P \wedge e_Q = e_1$. Now, by Remark 3.3 and Lemma 3.4 we have $\frac{p}{[M:Q]\mathrm{tr}(pe_Q)} \geq e_P \vee e_Q$.

Now, by Proposition 2.25 we get $pe_Q = [Q : N]E^{M_1}_{Q_1}(e_P e_Q)$. Thus, $\mathrm{tr}(pe_Q) = [Q : N]\mathrm{tr}(e_P e_Q)$. Hence, we get that $\frac{p}{[M:N]\mathrm{tr}(e_P e_Q)} \geq e_P \vee e_Q$.

Computing the trace of both sides and observing by equation (9) $\mathrm{tr}(p) = \frac{[M:N]}{[M:P][M:Q]}$, we get

$$\frac{\tau_P \tau_Q}{\mathrm{tr}(e_P e_Q)} \geq \mathrm{tr}(e_P \vee e_Q) = \tau_P + \tau_Q - \mathrm{tr}(e_P \wedge e_Q) = \tau_P + \tau_Q - \mathrm{tr}(e_1) = \tau_P + \tau_Q - \tau.$$

□

Theorem 3.6. *Let P, Q be distinct minimal intermediate subfactors of a finite index, irreducible subfactor $N \subset M$. Then $\alpha(P, Q) > \frac{\pi}{3}$.*

Proof. First observe that $(\tau_P + \tau_Q - \tau) > 0$. By the inequality in Proposition 3.5 we have

$$\mathrm{tr}(e_P e_Q) - \tau \leq \frac{\tau_P \tau_Q}{\tau_P + \tau_Q - \tau} - \tau = \frac{\tau_P \tau_Q - \tau(\tau_P + \tau_Q) + \tau^2}{\tau_P + \tau_Q - \tau} = \frac{(\tau_P - \tau)(\tau_Q - \tau)}{\tau_P + \tau_Q - \tau}.$$

By Theorem 2.8 we have

$$\cos(\alpha(P,Q)) = \frac{\mathrm{tr}(e_P e_Q) - \tau}{\sqrt{\tau_P - \tau}\sqrt{\tau_Q - \tau}} \leq \frac{(\tau_P - \tau)^{1/2}(\tau_Q - \tau)^{1/2}}{\tau_P + \tau_Q - \tau}$$
$$< \frac{(\tau_P - \tau)^{1/2}(\tau_Q - \tau)^{1/2}}{\tau_P - \tau + \tau_Q - \tau} \leq 1/2.$$

Therefore, $\alpha(P,Q) > \frac{\pi}{3}$. □

4. Number of intermediate subfactors

In geometry the kissing number problem asks for the maximum number τ_n of unit spheres that can simultaneously touch the unit sphere in n-dimensional Euclidean space without pairwise overlapping. The value of τ_n is known only for $n = 1, 2, 3, 4, 8, 24$ (even though upper bounds for τ_n are known for $n \leq 24$.) While its determination for $n = 1, 2$ is trivial, it is not the case for other values of n. The case $n = 3$ was the object of a famous discussion between Isaac Newton and David Gregory in 1694. See [Cas04], for instance.

Theorem 4.1. *Suppose that $N \subset M$ is a finite index, irreducible inclusion of* II_1 *factors. Let M_1 denote the basic construction for $N \subset M$. Let $\mathcal{L}_m(N,M)$ be the set of all minimal intermediate subfactors of $N \subset M$. Then the number of minimal intermediate subfactors $|\mathcal{L}_m(N,M)|$ is bounded by the kissing number τ_n, where $n = \dim(N' \cap M_1)$. In particular, $|\mathcal{L}_m(N,M)| < 3^n$.*

Proof. Note that $\{v_P : P \in \mathcal{L}_m(N,M)\}$ is a set of unit vectors in $(N' \cap M_1)_{s.a}$, a real inner product space of dimension n. Consider the n-dimensional unit ball B_P with center at $2v_P$. Each B_P is adjacent to the unit ball $B(1)$ with a center at the origin.

By Theorem 3.6 $\|v_P - v_Q\|_2 > 1$ for distinct P and Q in $\mathcal{L}_m(N,M)$. So B_P and B_Q are disjoint. Therefore, $|\mathcal{L}_m(N,M)| \leq \tau_n$.

Furthermore, for any $P \in \mathcal{L}_m(N,M)$, $B_P \subset \overline{B(3) \setminus B(1)}$, where $B(3)$ stands for the n-dimensional ball with a center at the origin and radius 3. Thus, we have

$$|\mathcal{L}_m(N,M)| \leq \frac{Vol(B(3)) - Vol(B(1))}{Vol(B(1))} = 3^n - 1. \qquad \square$$

Remark 4.2. For an irreducible subfactor $N \subset M$ one has $\dim(N' \cap M_1) \leq [M : N]$. Hence, the number of minimal intermediate subfactors is also bounded by $3^{[M:N]}$.

Definition 4.3. Suppose that δ^2 is a real number greater than or equal to 2. We define

$$I(\delta^2) = \sup_{N \subset M} \{|\mathcal{L}(N,M)| : N \subset M \text{ is a subfactor with } [M:N] \leq \delta^2\},$$
$$m(\delta^2) = \sup_{N \subset M} \{|\mathcal{L}_m(N,M)| : N \subset M \text{ is a subfactor with } [M:N] \leq \delta^2\}.$$

Corollary 4.4. *Let δ^2 be a real number greater than or equal to 2. Then we have $m(\delta^2) \leq 3^{\delta^2}$.*

Lemma 4.5. *If $\delta^2 \geq 4$, then we have $I(\delta^2) \leq m(\delta^2) I(\delta^2/2)$.*

Proof. Note that the inclusion $R \subset R \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$ is of index 4 and that $\mathbb{Z}_2 \times \mathbb{Z}_2$ has two nontrivial proper subgroups. Thus, $m(\delta^2) \geq 2$ when $\delta^2 \geq 4$.

To prove the lemma, we need to show that for an arbitrary subfactor $N \subset M$ with $[M : N] \leq \delta^2$, $|\mathcal{L}(N, M)| \leq m(\delta^2) I(\delta^2/2)$.

Case 1. If $|\mathcal{L}_m(N, M)| = 0$, then $|\mathcal{L}(N, M)| = 2$ (since in this case $\mathcal{L}(N, M) = \{N, M\}$). Note that $m(\delta^2) \geq 2$ and $I(\delta^2) \geq 2$, and the lemma follows directly.

Case 2. Suppose that $|\mathcal{L}_m(N, M)| = 1$. Let P be the minimal intermediate subfactor. Then we have $\mathcal{L}(N, M) = \mathcal{L}(P, M) \cup \{N\}$. Thus, we have $|\mathcal{L}(N, M)| = |\mathcal{L}(P, M)| + 1 \leq I([M : P]) + 1$.

Since $[M : P] = [M : N]/[P : N]$ and $[P : N] \geq 2$, we have $[M : P] \leq [M : N]/2 \leq \delta^2/2$. Therefore,

$$|\mathcal{L}(N, M)| \leq I(\delta^2/2) + 1 \leq 2I(\delta^2/2) \leq m(\delta^2) I(\delta^2/2).$$

Case 3. Suppose that $|\mathcal{L}_m(N, M)| \geq 2$. It follows that $\mathcal{L}(N, M) \backslash \{N, M\} \subset \bigcup_{P \in \mathcal{L}_m(N,M)} (\mathcal{L}(P, M) \backslash M)$. Therefore,

$$\begin{aligned} |\mathcal{L}(N, M)| &\leq \sum_{P \in \mathcal{L}_m(N,M)} (|\mathcal{L}(P, M)| - 1) + 2 \leq \sum_{P \in \mathcal{L}_m(N,M)} (I([M : P]) - 1) + 2 \\ &\leq \sum_{P \in \mathcal{L}_m(N,M)} (I(\delta^2/2) - 1) + 2 \leq |\mathcal{L}_m(N, M)| I(\delta^2/2) - |\mathcal{L}_m(N, M)| + 2 \\ &\leq |\mathcal{L}_m(N, M)| I(\delta^2/2) \leq m(\delta^2) I(\delta^2/2). \end{aligned}$$

□

Theorem 4.6. *Let $N \subset M$ be a finite index, irreducible inclusion of type* II_1 *factors. Then the number of intermediate subfactors is at most* $9^{[M:N]}$.

Proof. First note that if we have $2 \leq [M : N] < 4$, then there there are no nontrivial intermediate subfactors for $N \subset M$. Therefore, $|\mathcal{L}(N, M)| = 2 < 9^2 \leq 9^{[M:N]}$. Suppose that $\delta^2 = [M : N] \geq 4$. By Lemma 4.5 we have

$$\begin{aligned} |\mathcal{L}(N, M)| &\leq I(\delta^2) \leq m(\delta^2) I(\delta^2/2) \leq m(\delta^2) m(\delta^2/2) I(\delta^2/2^2) \\ &\leq m(\delta^2) m(\delta^2/2) m(\delta^2/4) \cdots m(\delta^2/2^k) I(\delta^2/2^{k+1}), \end{aligned}$$

where k is the smallest integer such that $2 \leq \delta^2/2^{k+1} < 4$.

By Theorem 4.1 we have

$$\begin{aligned} |\mathcal{L}(N, M)| &\leq I(\delta^2/2^{k+1}) \prod_{j=0}^{k} 3^{\delta^4/2^j} \leq \prod_{j=0}^{k+1} 3^{\delta^2/2^j} \text{ (since } I(\delta^2/2^{k+1}) = 2 < 3^{\delta^2/2^{k+1}}) \\ &\leq \prod_{j=0}^{+\infty} 3^{\delta^2/2^j} \leq 3^{2\delta^2} = 9^{\delta^2}. \end{aligned}$$

This completes the proof. □

In 2003 Longo asked whether the number of intermediate subfactors can be bounded by $[M : N]^{[M:N]}$ (see [Lon03]). Theorem 4.6 provides a better estimate for the number of intermediate subfactors for an index greater than 9. In fact, we can use our techniques to give a positive answer to Longo's question for any value of the index $[M : N]$.

Theorem 4.7. *Let $N \subset M$ be an irreducible subfactor of finite index. Then the number of intermediate subfactors is bounded by* $\min\{9^{[M:N]},\ [M:N]^{[M:N]}\}$.

Proof. By Lemma 4.5 we know that

$$|\mathcal{L}(N,M)| \leq I(\delta^2) \leq m(\delta^2) I(\delta^2/2) \leq 3^{\delta^2} I(\delta^2/2). \tag{10}$$

If $[M:N] < 4$, then from Jones's index theorem [Jon83] we know that there are no nontrivial intermediate subfactors, and hence the statement holds. If $4 \leq \delta^2 < 8$, then we have $I(\delta^2/2) = 2$. Therefore, equation (10) yields

$$|\mathcal{L}(N,M)| \leq 3^{\delta^2} I(\delta^2/2) = 2 \cdot 3^{\delta^2} \leq 4^{\delta^2} \leq [M:N]^{[M:N]}.$$

Now if $8 \leq \delta^2 \leq 9$, we have $|\mathcal{L}(N,M)| \leq I(\delta^2) \leq m(\delta^2) I(\delta^2/2) \leq m(\delta^2) m(\delta^2/2) I(\delta^2/4)$. Note that $\delta^2/4 < 4$ and hence $I(\delta^2/4) = 2$. We also have $m(\delta^2)m(\delta^2/2) \leq 3^{\delta^2} 3^{\delta^2/2} \leq 3^{14}$. So $|\mathcal{L}(N,M)| \leq 2 \cdot 3^{14} \leq 8^8 \leq [M:N]^{[M:N]}$.

Finally, if $[M:N] \geq 9$, then from Theorem 4.6 we have $|\mathcal{L}(N,M)| \leq 9^{[M:N]} \leq [M:N]^{[M:N]}$. ◻

Remark 4.8. In various cases we can get better estimates for the number of intermediate subfactors, as explained below:

(1) We can use the estimate $m(\delta^2) \leq \tau_n$, where n is the smallest integer bigger than δ^2 and τ_n denotes the kissing number (at dimension n). Since upper bounds of the values of kissing number are known for small dimensions, the above calculation yields that for $[M:N] < 16$, $|\mathcal{L}(N,M)| \leq 2 \cdot \tau_9 \cdot \tau_5 \leq 2 \times 44 \times 364 = 32032 < 2^{15}$.
(2) Suppose that $N \subset M$ is an irreducible subfactor, and suppose that $N' \cap M_1$ is abelian. (For example, $R \subset R \rtimes G$, where G is a finite abelian group acting outerly on R. Therefore, Theorem 4.9 provides a bound for the cardinality of the set of all subgroups of a finite abelian group.) Then for two distinct minimal intermediate subfactors P and Q it is trivial to check that $\alpha_M^N(P,Q) = \frac{\pi}{2}$. Thus, the set $\{v_P : P \in \mathcal{L}_m(N,M)\}$ forms an orthonormal set, and hence the number of minimal intermediate subfactors is bounded by $\dim(N' \cap M_1) \leq [M:N]$. After that, doing an iteration as above, we obtain a better bound than [TW97, Proposition 5.1] for the cardinality of the lattice $\mathcal{L}(N \subset M)$, as explained below.

Theorem 4.9. *Suppose that $N \subset M$ is an irreducible subfactor, and suppose that $N' \cap M_1$ is abelian. Then*

$$|\mathcal{L}(N,M)| \leq \left(\frac{[M:N]}{\sqrt{2}}\right)^{\frac{\log([M:N])}{2}}.$$

Proof. Let P and Q be two minimal intermediate subfactors. Then we have

$$\cos(\alpha(P,Q)) = \frac{\operatorname{tr}((e_P - e_1)(e_Q - e_1))}{\|e_P - e_1\|_2 \|e_Q - e_1\|_2} = \frac{\operatorname{tr}(e_P e_Q) - \operatorname{tr}(e_1)}{\|e_P - e_1\|_2 \|e_Q - e_1\|_2}.$$

Note that $e_P, e_Q \in N' \cap M_1$, which is abelian, and that $P \cap Q = N$. Thus, we have $\operatorname{tr}(e_P e_Q) = \operatorname{tr}(e_P \wedge e_Q) = \operatorname{tr}(e_1)$, and this further implies that $\cos(\alpha(P,Q)) = 0$. Therefore, for any two minimal intermediate subfactors P and Q, $\alpha(P,Q) = \pi/2$. In particular, this means that the set $\{v_P : P \in \mathcal{L}_m(N,M)\}$ is an orthonormal set.

Therefore, $|\mathcal{L}_m(N,M)| \le \dim(N' \cap M_1) \le [M:N]$. This implies that $m(\delta^2) \le \delta^2$ and therefore, by Lemma 4.5, we have

$$\begin{aligned}|\mathcal{L}(N,M)| &\le I(\delta^2) \le m(\delta^2)I(\delta^2/2) \le m(\delta^2)m(\delta^2/2)I(\delta^2/2^2)\\ &\le m(\delta^2)m(\delta^2/2)m(\delta^2/4)\cdots m(\delta^2/2^k)I(\delta^2/2^{k+1}),\end{aligned}$$

where k is the smallest integer such that $2 \le \delta^2/2^{k+1} < 4$, i.e., $k+1 \le \log(\delta^2/2)$. Since $m(\delta^2) \le \delta^2$, we have

$$\begin{aligned}|\mathcal{L}(N,M)| &\le m(\delta^2)m(\delta^2/2)m(\delta^2/4)\cdots m(\delta^2/2^k)I(\delta^2/2^{k+1})\\ &\le I(\delta^2/2^{k+1})\prod_{j=0}^{k}\delta^2/2^j \le \prod_{j=0}^{k+1}\delta^2/2^j\\ &= (\delta^2)^{k+1}\frac{1}{2^{(k+1)(k+2)/2}} \le (\delta^2)^{k+1}\frac{1}{2^{(k+1)/2}} = (\frac{\delta^2}{\sqrt{2}})^{k+1} \le (\frac{\delta^2}{\sqrt{2}})^{\log(\delta^2/2)}.\end{aligned}$$

This completes the proof. □

5. Final remarks: Comparison with the Sano–Watatani angle

In this section we compare our notion of the angle between intermediate subfactors with the notion of the angle operator due to Sano and Watatani [SW94]. The authors would like to thank an anonymous referee of this paper for suggesting that we include a discussion on the comparison between these two notions of angles.

In [SW94] Sano and Watatani introduced the notion of angles between a pair of subalgebras of a given finite von Neumann algebra as the spectrum of an angle operator (see below). We define only the angle between *intermediate subfactors* of a finite index subfactor $N \subset M$. In theory, the Sano–Watatani angle is a set of values, whereas our notion always gives a single number. This is perhaps the main difference between these two notions. As we show later, the two notions are not comparable in general, even though they coincide when we consider quadrilaterals which are commuting squares. Recall that a quadruple (N,P,Q,M) of II_1 factors is called a quadrilateral if $N = P \cap Q$ and M is generated by P and Q.

Let us now briefly recall the definition of a Sano–Watatani angle. Motivated by the relative position of two different subspaces $\mathcal{K}$ and $\mathcal{L}$ of a Hilbert space $\mathcal{H}$, see, for example, [Hal69], Sano and Watatani defined the angle operator $\theta(p,q)$ (where p (resp., q) is the projection onto $\mathcal{K}$ (resp., $\mathcal{L}$)) as $\cos^{-1}\sqrt{pqp - p\wedge q}$, where $(pqp - p\wedge q)$ is regarded as the operator acting on its support. The set of angles $\mathrm{Ang}(p,q)$ between p and q is the subset of $[0,\pi/2]$ defined by [SW94, Definition 2.1]

$$\mathrm{Ang}(p,q) = \begin{cases}\mathrm{sp}\ \theta(p,q) & \text{if } pq \ne qp.\\ \{\pi/2\} & \text{otherwise.}\end{cases} \tag{11}$$

Definition 5.1 ([SW94])**.** Let M be a finite von Neumann algebra with a faithful normal tracial state tr, and let P,Q be von Neumann subalgebras. Then the Sano–Watatani angle $\mathrm{Ang}_M(P,Q)$ between two subalgebras P and Q of M is defined as follows:

$$\mathrm{Ang}_M(P,Q) = \mathrm{Ang}(e_P,e_Q).$$

(Here e_P and e_Q are corresponding Jones projections.)

Most of the paper [SW94] is devoted to the case in which (N, P, Q, M) is a quadrilateral of type II_1 factors. In this scenario, by [SW94, Proposition 3.2], $\mathrm{Ang}_M(P, Q)$ is a finite set. Thus, in this case we might hope to relate the two definitions of angles. However, as we show below, the two notions of angles are not comparable in general.

- Consider a quadruple $N \subset P \subset Q \subset M$ of type II_1 factors with $[M : N] < \infty$. Then by Fact 2.15 $\cos(\alpha_M^N(P, Q)) = \sqrt{\frac{[P:N]-1}{[Q:N]-1}}$. Whereas, since in this situation $e_P e_Q = e_Q e_P = e_P$, we conclude that $\mathrm{Ang}_M(P, Q) = \{\pi/2\}$. Therefore, $\alpha_M^N(P, Q)$ and $\mathrm{Ang}_M(P, Q)$ are different in general.
- By [SW94, Corollary 3.1] we see that for a quadrilateral (N, P, Q, M) of type II_1 factors with $[M : N] < \infty$, if the operator $s = e_P e_Q e_P - e_N$ is 0, then $\mathrm{Ang}_M(P, Q) = \{\pi/2\}$, and hence the quadrilateral is a commuting square. Therefore, $\alpha_M^N(P, Q) = \pi/2$, and two notions of angles coincide in this case.
- Consider again the quadrilateral (N, P, Q, M) of type II_1 factors with $[M : N] < \infty$, and let s be the operator as described above. In [SW94, Corollary 3.1] Sano and Watatani proved that if $s^2 = \mu s \neq 0$ for some scalar μ, then $\mathrm{Ang}_M(P, Q)$ consists of one point, say, $\theta(P, Q)$. Then $\theta(P, Q) = \cos^{-1}\sqrt{\mu}$. In [SW94, proof of Lemma 3.6] the authors have shown that $\mathrm{tr}(e_P e_Q - e_N) = \mu\, \mathrm{tr}(r)$, where r is the projection $e_P - e_P \wedge e_Q - e_P \wedge {e_Q}^{\perp}$. Then an easy calculation shows that

$$\cos(\alpha(P, Q)) = \cos^2(\theta(P, Q)) \left\{ \frac{\mathrm{tr}(r)}{\sqrt{\mathrm{tr}(e_P) - \tau}\sqrt{\mathrm{tr}(e_Q) - \tau}} \right\}.$$

  In conclusion, even if $\mathrm{Ang}_M(P, Q)$ is a singleton, it can be different from $\alpha(P, Q)$. More precisely, in this case $\cos^2(\theta(P, Q)) = \cos(\alpha(P, Q))$ if and only if the following equation holds:

$$\mathrm{tr}(e_P \wedge {e_Q}^{\perp}) = \sqrt{\mathrm{tr}(e_P - e_N)} \left\{ \sqrt{\mathrm{tr}(e_P - e_N)} - \sqrt{\mathrm{tr}(e_Q - e_N)} \right\}.$$

- Recall that a finite index subfactor $N \subset M$ is called 2-supertransitive if $N' \cap M_1 = \{1, e_N\}''$. Now let us discuss the two notions of angles for the 2-supertransitive case which was extensively studied in [GJ07, GI08]. Suppose that (N, P, Q, M) is a quadrilateral such that $N \subset M$ is an irreducible, finite index subfactor and $N \subset P$ and $N \subset Q$ are 2-supertransitive. Then by [GJ07, Lemma 4.14] we see that $e_P e_Q e_P - e_N = \lambda(e_P - e_N)$, where $\lambda = \frac{\mathrm{tr}(e_{P_1 Q_1})^{-1} - 1}{[P:N]-1}$. Therefore, we get that $s^2 = \lambda s$. If $\lambda \neq 0$, then $\mathrm{Ang}_M(P, Q)$ is a singleton $\theta(P, Q)$, and $\cos^2(\theta(P, Q)) = \lambda$. On the other hand, an easy calculation shows that $\cos(\alpha(P, Q)) = \lambda\sqrt{\frac{[P:N]-1}{[Q:N]-1}}$. If the quadrilateral is not a commuting square, then by [GJ07, Lemma 4.8] we have $[P : N] = [Q : N]$. Therefore, in this case $\cos(\alpha(P, Q)) = \cos^2(\theta(P, Q))$.

  Furthermore, if the quadrilateral is a co-commuting square, then $\mathrm{tr}(e_P e_Q) = \frac{1}{[M:P][M:Q]}$ by the first fact in Remark 2.9. Therefore, by Theorem 2.8

and the fact that $\cos(\alpha(P,Q)) = \cos^2(\theta(P,Q))$, we see that

$$\cos^2(\theta(P,Q) = \frac{[P:N]-[M:Q]}{[M:Q]\Big([P:N]-1\Big)} = \frac{[P:N]-[M:P]}{[M:P]\Big([P:N]-1\Big)}.$$

This result is [GI08, Lemma 2.1]. Here we have reproved this result by relating our angle to the Sano–Watatani angle.

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