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Reflection positivity and Levin–Wen models

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Dedicated to the memory of Richard V. Kadison

Abstract

We give a transparent algebraic formulation of our pictorial approach to the reflection positivity (RP), that we introduced in a previous paper. We apply this quantization to the $2+1$ Levin–Wen model to obtain $1+1$ anyonic/quantum spin chain theory on the boundary, possibly entangled in the bulk. The reflection positivity property has played a central role in both mathematics and physics, as well as providing a crucial link between the two subjects. In a previous paper we gave a new geometric approach to understanding reflection positivity in terms of pictures. Here we give a transparent algebraic formulation of our pictorial approach. We use insights from this translation to establish the reflection positivity property for the fashionable Levin–Wen models with respect both to vacuum and to bulk excitations. We believe these methods will be useful for understanding a variety of other problems.

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1. Introduction

In an earlier paper [3], we formulated the reflection-positivity (RP) property for expectations, see Definition 2.1, and we gave a geometric proof based on a mathematical picture language. Our proof involves a transformation $\mathfrak{F}_s$, that we call the string Fourier transform (SFT), and which acts on pictures by rotation. The SFT generalizes the usual

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Fourier transform that acts on functions, see [3]. This investigation gave rise to our mathematical picture language program [4,5].

In this paper we elaborate our previous work in two ways. Firstly, we translate our picture proof in [3] into an algebraic proof. We begin with an algebraic formulation of $\mathfrak{F}_s$ in Definition 2.2, and we prove a general theorem about RP. Secondly, we take advantage of the generality of our pictorial method to analyze other pictures that occur in the theoretical physics literature.

Levin and Wen introduced a set of lattice models to study topological order [9]. These models generalize the $\mathbb{Z}_2$ toric code of Kitaev [7]; for background see Kitaev and Kong [8]. Levin and Wen showed that ground states of their models correspond to topological quantum field theories in the sense of Turaev and Viro [12]. In their paper, Kitaev and Kong give an interesting dictionary to translate between these two sets of concepts.

In Section 3 we study Levin–Wen models for graphs on surfaces, using the data of unitary fusion categories. We then use our new methods to establish Theorem 3.2, the main new result in this paper: Levin–Wen Hamiltonians have the RP property. We establish the RP property of partition functions on the $2d$ lattice. By Wick rotation, this partition function can be interpreted as a path integral with the action induced from the Hamiltonian of the $2+1$ Levin–Wen models. The quantization by RP of this $2d$ statistical theory gives a $1 + 1$ anyonic/quantum spin chain theory on the boundary, possibly entangled in the bulk. Although we do not analyze it in detail, our method also proves the RP property for higher-dimensional pictorial models, such as the Walker–Wang models [13].

1.1. The framework of our reflection-positivity proof

We gave our pictorial proof in [3] within the framework of subfactor planar para algebras. For background see [1,2,6,10] and the extensive citations contained in these papers to work on RP by Osterwalder, Schrader, Biskup, Brydges, Dyson, Frank, Fröhlich, Israel, Jäkel, Jorgensen, Klein, Landau, Lieb, Macris, Nachtergaele, Neeb, Olafsson, Seiler, Simon, Spencer, and others.

A novel aspect of the proof of RP in [3] was our observation that the positivity of the string Fourier transform $\mathfrak{F}_s(-H)$ of H ensures the RP property. In fact when H is reflection-invariant, the positivity of $\mathfrak{F}_s(-H_0)$ is sufficient to ensure RP for H, where H_0 denotes the part of H that maps across the reflection mirror. Self-adjointness of H is unnecessary for H to have the RP property. However, RP for H and translation invariance entails the existence of a self-adjoint Hamiltonian $\mathfrak{h}$ on the Hilbert space of the boundary theory in one lower dimension defined by RP, with H as the action for $\mathfrak{h}$.

In Section 2, we present algebraic definitions of $\mathfrak{F}_s$, of the convolution product $*$, and of the RP property. While this may appear somewhat different from the standard definitions, one can recover the results in [3] by a proper choice of the Hilbert space and the Hamiltonian. We do not pursue this comparison in this paper. We attempt to make minimal assumptions in our statements, so that the methods here could be applied in a wide variety of circumstances.

1.2. Our example

In Section 3 we consider the Levin–Wen model on a surface which has a reflection mirror. The Hamiltonian is an action on the Hilbert space: it is the sum of contributions from Wilson loops on plaquettes and actions on sites. The terms in H arising from the actions on sites do not contribute to H_0. In the Levin–Wen model, H_0 is the sum of the actions on plaquettes that cross the reflection mirror.

When the plaquette p crosses the mirror P, we decompose the Wilson loop as a half circle and its mirror image. The action of $\mathfrak{F}_s$ on a picture is to rotate the picture by 90°. Pictorially we can consider the actions of the two half circles after rotation as the product of a half circle and its adjoint, namely its vertical reflection. So the $\mathfrak{F}_s(H_p)$ should be positive. The sticking point is that the actions of the two half circles are not independent, as they share boundary conditions on the mirror. So H_p is not simply a tensor product of operators on two sides of the mirror. Technically we need to take care of the boundary condition in the decomposition of H_0. Adding the boundary condition to the decomposition, we prove that $\mathfrak{F}_s(-H_0)$ is positive.

We remark that RP of the Hamiltonian H in the Levin–Wen model on a torus not only works for the expectation in the vacuum state, but also for the expectation in bulk excitations (objects in the Drinfeld center). Each bulk excitation defines its own one-dimensional lower quantized theory that are topologically entangled on the two boundary circles. We expect this realization to be useful in the study of the anomaly theory on the boundary.

2. Algebraic reflection positivity

In this section we look again at results that we proved in [3,10], using pictorial methods. Suppose $\mathcal{H}_+$ is a finite dimensional Hilbert space and $\mathcal{H}_-$ is its dual space. Let $\langle\cdot,\cdot\rangle_{\mathcal{H}_\pm}$ be the inner product of the Hilbert spaces $\mathcal{H}_\pm$. Let θ be the Riesz representation map from $\mathcal{H}_\pm$ to $\mathcal{H}_\mp$. Then for any $x, x' \in \mathcal{H}_+$, their inner product is given by

$$\langle x, x'\rangle_{\mathcal{H}_+} = \langle \theta(x'), \theta(x)\rangle_{\mathcal{H}_-} \ .$$

Let $\mathcal{H}_{-+} = \mathcal{H}_- \otimes \mathcal{H}_+$ denote the tensor product Hilbert space with the induced inner product $\langle\ ,\ \rangle_{\mathcal{H}_{-+}}$, and likewise denote $\mathcal{H}_{+-} = \mathcal{H}_+ \otimes \mathcal{H}_-$.

Definition 2.1 (*Reflection-Positivity Property*)**.** The map $H \in \hom(\mathcal{H}_{-+})$ has the RP property, if for any $x', x \in \mathcal{H}_+$, and any $\beta \geqslant 0$,

$$\langle \theta(x') \otimes x', e^{-\beta H}\theta(x) \otimes x\rangle_{\mathcal{H}_{-+}} \geqslant 0 \ .$$

Definition 2.2 (*SFT*)**.** The string Fourier transform $\mathfrak{F}_s : \hom(\mathcal{H}_{-+}) \to \hom(\mathcal{H}_{+-})$ is a map such that for $T \in \hom(\mathcal{H}_{-+})$, and for arbitrary $x, x' \in \mathcal{H}_+$ and $y, y' \in \mathcal{H}_-$,

$$\langle x \otimes y, \mathfrak{F}_s(T)(x' \otimes y')\rangle_{\mathcal{H}_{+-}} = \langle \theta(x') \otimes x, T(y' \otimes \theta(y))\rangle_{\mathcal{H}_{-+}} \ .$$

Remark (*A Key Identity*)**.** Definition 2.2, with $T = e^{-\beta H}$, $x = x'$, and $y = y' = \theta(\tilde{x})$, and substituting x for $\tilde{x}$, yields

$$\langle \theta(x') \otimes x', e^{-\beta H}(\theta(x) \otimes x)\rangle_{\mathcal{H}_{-+}}$$

$$= \langle x' \otimes \theta(x), \mathfrak{F}_s(e^{-\beta H})(x' \otimes \theta(x))\rangle_{\mathcal{H}_{+-}} \,. \tag{1}$$

Thus the RP property for H is equivalent to the positivity of the expectation of $\mathfrak{F}_s(e^{-\beta H})$ in vectors that are tensor products.

Theorem 2.3 (*First RP Statement*). *A transformation $H \in \hom(\mathcal{H}_{-+})$ satisfying $\mathfrak{F}_s(-H) \geqslant 0$ has the RP property.*

The map θ defines a map from $\hom(\mathcal{H}_\pm)$ to $\hom(\mathcal{H}_\mp)$. For $H'_\pm \in \hom(\mathcal{H}_\pm)$ let

$$\theta(H'_\pm) := \theta H'_\pm \theta \,.$$

Extend the definition of θ as an anti-linear map on $\mathcal{H}_{-+}$: For any $y \otimes x \in \mathcal{H}_{-+}$, let

$$\theta(y \otimes x) := \theta(x) \otimes \theta(y) \in \mathcal{H}_{-+} \,.$$

Thus

$$\left\langle \theta(y \otimes x), \theta(y' \otimes x')\right\rangle_{\mathcal{H}_{-+}} = \left\langle y' \otimes x', y \otimes x\right\rangle_{\mathcal{H}_{-+}} \,. \tag{2}$$

A more detailed condition on H that yields the RP property depends (as in past studies) on properties of the part of H mapping between $\mathcal{H}_+$ and $\mathcal{H}_-$. For $H \in \hom(\mathcal{H}_{-+})$, let $\theta(H) := \theta H \theta \in \hom(\mathcal{H}_{-+})$.

Theorem 2.4 (*Second RP Statement*). *Suppose $H = H_- + H_0 + H_+ + \lambda I$, where $\lambda \in \mathbb{R}$, $H_+ = I_- \otimes H'_+$, for some $H'_+ \in \hom(\mathcal{H}_+)$, and where $\theta(H_+) = H_-$. If $\mathfrak{F}_s(-H_0) \geqslant 0$, then H has the RP property.*

2.1. Algebraic properties of the SFT

In this section we establish algebraic properties of $\mathfrak{F}_s$, that we use in the next section.

Proposition 2.5. *The identity I satisfies $\mathfrak{F}_s(I) \geqslant 0$.*

Proof. Let $\{x_i\}$ denote an orthonormal basis for $\mathcal{H}_+$ and $\{y_i\} = \{\theta(x_i)\}$ an orthonormal basis for $\mathcal{H}_-$. A vector $w \in \mathcal{H}_{+-}$ has an expansion $w = \sum_{ij} w_{ij}\, x_i \otimes \theta(x_j)$. According to Definition 2.2,

$$\langle w, \mathfrak{F}_s(I) w\rangle_{\mathcal{H}_{+-}} = \sum_{i,j,i',j'} \overline{w_{ij}} w_{i'j'} \left\langle \theta(x_{i'}) \otimes x_i, \theta(x_{j'}) \otimes x_j \right\rangle_{\mathcal{H}_{-+}}$$
$$= \left| \sum_{i,j} \overline{w_{ij}} \left\langle x_i, x_j \right\rangle_{\mathcal{H}_+} \right|^2 = \left| \sum_i w_{ii} \right|^2 \geqslant 0 \,,$$

showing an arbitrary expectation of $\mathfrak{F}_s(I) \geqslant 0$. □

Remark. The RP property Definition 2.1, for the case $H = 0$, is a special example of an expectation of $\mathfrak{F}_s(I)$, namely

$$\left\langle \theta(x') \otimes x', \theta(x) \otimes x\right\rangle_{\mathcal{H}_{-+}} = \left\langle x' \otimes \theta(x), \mathfrak{F}_s(I)(x' \otimes \theta(x))\right\rangle_{\mathcal{H}_{+-}}$$

$$= \left|\langle x', x\rangle_{\mathcal{H}_+}\right|^2 \geqslant 0 \,.$$

Remark. The transformation $\mathfrak{F}_s^{-1} : \hom \mathcal{H}_{+-} \to \hom \mathcal{H}_{-+}$ is,

$$\langle y \otimes x, \mathfrak{F}_s^{-1}(S)(y' \otimes x')\rangle_{\mathcal{H}_{-+}} = \langle x \otimes \theta(x'), S(\theta(y) \otimes y')\rangle_{\mathcal{H}_{+-}} \,.$$

Proposition 2.6. *For any $T \in \hom(\mathcal{H}_{-+})$, $\mathfrak{F}_s(\theta(T)) = \mathfrak{F}_s(T)^*$.*

Proof. Let $x, x' \in \mathcal{H}_+$ and $y, y' \in \mathcal{H}_-$. Then $\mathfrak{F}_s(\theta(T))$ satisfies

$$\begin{aligned}
\langle x \otimes y, \mathfrak{F}_s(\theta(T))(x' \otimes y')\rangle_{\mathcal{H}_{+-}} &= \langle \theta(x') \otimes x, \theta(T)(y' \otimes \theta(y))\rangle_{-+} \\
&= \langle \theta(x') \otimes x, \theta T \theta(y' \otimes \theta(y))\rangle_{-+} = \langle T\theta(y' \otimes \theta(y)), \theta(\theta(x') \otimes x)\rangle_{-+} \\
&= \langle T(y \otimes \theta(y')), \theta(x) \otimes x'\rangle_{-+} = \overline{\langle \theta(x) \otimes x', T(y \otimes \theta(y'))\rangle}_{-+} \\
&= \overline{\langle x' \otimes y', \mathfrak{F}_s(T)(x \otimes y)\rangle}_{+-} = \langle x \otimes y, \mathfrak{F}_s(T)^*(x' \otimes y')\rangle_{+-} \,.
\end{aligned}$$

Thus the matrix elements agree as claimed. □

Corollary 2.7 (*Reflections and Hermiticity*)**.** *Let $H \in \hom(\mathcal{H}_{-+})$. Then $\Theta(H) = H$ iff $\mathfrak{F}_s(H) = \mathfrak{F}_s(H)^*$.*

Remark. Pictorially we represent θ in [3] as a horizontal reflection, $*$ as a vertical reflection, and $\mathfrak{F}_s$ as a clockwise 90° rotation.

Definition 2.8. Let $Y : \mathcal{H}_{+-} \otimes \mathcal{H}_{+-} \to \mathcal{H}_{+-}$ be given by

$$Y(x_1 \otimes y_1 \otimes x_2 \otimes y_2) := \langle \theta(y_1), x_2\rangle_{\mathcal{H}_+} \, x_1 \otimes y_2 \,,$$

for any $x_1 \otimes y_1 \otimes x_2 \otimes y_2 \in \mathcal{H}_{+-} \otimes \mathcal{H}_{+-}$.

Lemma 2.9. *Let $\mathcal{B}$ be an orthonormal basis of $\mathcal{H}_+$. Then for any $x \in \mathcal{H}_+$ and $y \in \mathcal{H}_-$,*

$$Y^*(x \otimes y) = \sum_{\beta \in \mathcal{B}} x \otimes \theta(\beta) \otimes \beta \otimes y \,.$$

Also $YY^* = \dim(\mathcal{H}_+)\, I$, on $\mathcal{H}_+ \otimes \mathcal{H}_-$.

Proof. For any $x, x_1, x_2 \in \mathcal{H}_+$ and $y, y_1, y_2 \in \mathcal{H}_-$, and with $\mathcal{H}_{+-}^2 = \mathcal{H}_{+-} \otimes \mathcal{H}_{+-}$,

$$\begin{aligned}
&\sum_{\beta \in \mathcal{B}} \langle x \otimes \theta(\beta) \otimes \beta \otimes y, x_1 \otimes y_1 \otimes x_2 \otimes y_2\rangle_{\mathcal{H}_{+-}^2} \\
&\quad = \sum_{\beta \in \mathcal{B}} \langle \theta(\beta), y_1\rangle_{\mathcal{H}_-} \langle \beta, x_2\rangle_{\mathcal{H}_+} \langle x \otimes y, x_1 \otimes y_2\rangle_{\mathcal{H}_{+-}} \\
&\quad = \sum_{\beta \in \mathcal{B}} \langle \theta(y_1), \beta\rangle_{\mathcal{H}_+} \langle \beta, x_2\rangle_{\mathcal{H}_+} \langle x \otimes y, x_1 \otimes y_2\rangle_{\mathcal{H}_{+-}} \\
&\quad = \sum_{\beta \in \mathcal{B}} \langle \theta(y_1), x_2\rangle_{\mathcal{H}_+} \langle x \otimes y, x_1 \otimes y_2\rangle_{\mathcal{H}_{+-}} \\
&\quad = \langle x \otimes y, Y(x_1 \otimes y_1 \otimes x_2 \otimes y_2)\rangle_{\mathcal{H}_{+-}} = \langle Y^*(x \otimes y), x_1 \otimes y_1 \otimes x_2 \otimes y_2\rangle_{\mathcal{H}_{+-}^2} \,.
\end{aligned}$$

This completes the computation of Y^*. Also

$$YY^* x \otimes y = Y \sum_{\beta \in \mathcal{B}} x \otimes \theta(\beta) \otimes \beta \otimes y = \left(\sum_{\beta \in \mathcal{B}} \langle \beta, \beta \rangle \right) x \otimes y \ .$$

Note that the β are an orthonormal basis for $\mathcal{H}_+$, so the sum in parentheses equals $\dim(\mathcal{H}_+)$. □

Definition 2.10 (*Convolution*)**.** For $A, B \in \hom(\mathcal{H}_{+-})$, their convolution product is $A * B := Y(A \otimes B)Y^*$.

The convolution is associative, as a consequence of Lemma 2.9.

Remark. Let $\mathcal{B}$ be an orthonormal basis for $\mathcal{H}_+$ and $\theta(\mathcal{B})$ a corresponding basis for $\mathcal{H}_-$. Then for $i, j \in \mathcal{B}$, the vectors $i \otimes \theta(j)$ are an orthonormal basis for $\mathcal{H}_{+-}$. A matrix unit $E_{ii'jj'} \in \hom(\mathcal{H}_{+-})$ is zero except on $i' \otimes \theta(j')$ and maps that vector to the vector $i \otimes \theta(j)$. The transformations $A, B \in \hom(\mathcal{H}_{+-})$ can be written

$$A = \sum_{i,i',j,j' \in \mathcal{B}} a_{ii'jj'} E_{ii'jj'} \ , \quad B = \sum_{k,k',\ell,\ell' \in \mathcal{B}} b_{kk'\ell\ell'} E_{kk'\ell\ell'} \ .$$

One can compare the matrix elements of AB with those of $A * B$, namely

$$\left\langle \alpha \otimes \theta(\beta), (AB)\alpha' \otimes \theta(\beta') \right\rangle_{\mathcal{H}_{+-}} = \sum_{k,k' \in \mathcal{B}} a_{\alpha k \beta k'} \, b_{k\alpha' k' \beta'} \ ,$$

$$\left\langle \alpha \otimes \theta(\beta), (A * B)\alpha' \otimes \theta(\beta') \right\rangle_{\mathcal{H}_{+-}} = \sum_{k,k' \in \mathcal{B}} a_{\alpha\alpha' kk'} \, b_{kk'\beta\beta'} \ .$$

In particular on $\mathcal{H}_{+-}$, one has $I = \sum_{ij} E_{iijj}$ and

$$I * I = \dim(\mathcal{H}_+) I \ . \tag{3}$$

In [3] we represent A and B pictorially as "two-box" pictures. The multiplication AB is given by vertical composition of the two-box pictures, while the multiplication $A * B$ is given by the corresponding horizontal composition of the same pictures.

Theorem 2.11 (*SFT on Products*)**.** *The SFT maps products in* $\hom(\mathcal{H}_{-+})$ *to convolutions in* $\hom(\mathcal{H}_{+-})$. *For* $S, T \in \hom(\mathcal{H}_{-+})$*,*

$$\mathfrak{F}_s(ST) = \mathfrak{F}_s(S) * \mathfrak{F}_s(T) \ .$$

Proof. Let $x_1, x_2 \in \mathcal{H}_+$ and $y_1, y_2 \in \mathcal{H}_-$. By Definition 2.2,

$$\begin{aligned} &\langle x_1 \otimes y_1, \mathfrak{F}_s(ST)(x_2 \otimes y_2) \rangle_{\mathcal{H}_{+-}} = \langle \theta(x_2) \otimes x_1, ST(y_2 \otimes \theta(y_1)) \rangle_{\mathcal{H}_{-+}} \\ &\quad = \sum_{\beta_1, \beta_2 \in \mathcal{B}} \langle \theta(x_2) \otimes x_1, S(\theta(\beta_1) \otimes \beta_2) \rangle_{\mathcal{H}_{-+}} \langle \theta(\beta_1) \otimes \beta_2, T(y_2 \otimes \theta(y_1)) \rangle_{\mathcal{H}_{-+}} \\ &\quad = \sum_{\beta_1, \beta_2 \in \mathcal{B}} \langle x_1 \otimes \theta(\beta_2), \mathfrak{F}_s(S)(x_2 \otimes \theta(\beta_1)) \rangle_{\mathcal{H}_{+-}} \langle \beta_2 \otimes y_1, \mathfrak{F}_s(T)(\beta_1 \otimes y_2) \rangle_{\mathcal{H}_{+-}} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\beta_1,\beta_2\in\mathcal{B}} \langle x_1\otimes\theta(\beta_2)\otimes\beta_2\otimes y_1, (\mathfrak{F}_s(S)\otimes\mathfrak{F}_s(T))(x_2\otimes\theta(\beta_1)\otimes\beta_1\otimes y_2)\rangle_{\mathcal{H}_{+-}^2}\\
&= \langle Y^*(x_1\otimes y_1), (\mathfrak{F}_s(S)\otimes\mathfrak{F}_s(T))Y^*(x_2\otimes y_2)\rangle_{\mathcal{H}_{+-}^2}\\
&= \langle x_1\otimes y_1, Y(\mathfrak{F}_s(S)\otimes\mathfrak{F}_s(T))Y^*(x_2\otimes y_2)\rangle_{\mathcal{H}_{+-}}\\
&= \langle x_1\otimes y_1, \mathfrak{F}_s(S)*\mathfrak{F}_s(T)(x_2\otimes y_2)\rangle_{\mathcal{H}_{+-}} .
\end{aligned}
$$

We infer the last three equalities from Lemma 2.9 and Definition 2.2. Therefore, the operators agree as claimed. □

Theorem 2.12 (*Schur Product Theorem*). *Let $S, T \in \hom(\mathcal{H}_{+-})$. If $S \geqslant 0$ and $T \geqslant 0$, then $S * T \geqslant 0$.*

Proof. Let $\sqrt{S}$ and $\sqrt{T}$ denote the positive square roots of S and T. By Definition 2.10, one has $S*T = (Y(\sqrt{S}\otimes\sqrt{T}))(Y(\sqrt{S}\otimes\sqrt{T}))^* \geqslant 0$. □

Corollary 2.13 (*Exponentials, Products*). *If $\mathfrak{F}_s(S) \geqslant 0$, then $\mathfrak{F}_s(e^S) \geqslant 0$. If $\mathfrak{F}_s(S) \geqslant 0$, $\mathfrak{F}_s(T) \geqslant 0$, then $\mathfrak{F}_s(ST) \geqslant 0$.*

Proof. From Theorem 2.11, $\mathfrak{F}_s(ST) = \mathfrak{F}_s(S) * \mathfrak{F}_s(T)$. We then infer $\mathfrak{F}_s(ST) \geqslant 0$ from Theorem 2.12. Likewise $\mathfrak{F}_s(S) \geqslant 0$ ensures $\mathfrak{F}_s(S^n) \geqslant 0$ for any natural number n. Since $\mathfrak{F}_s$ is a linear transformation, and the exponential power series has positive coefficients, so $\mathfrak{F}_s(e^S - I) \geqslant 0$. But from Proposition 2.5 we know $\mathfrak{F}_s(I) \geqslant 0$, hence $\mathfrak{F}_s(e^S) \geqslant 0$. □

Proposition 2.14. *If $T_+ \in \hom(\mathcal{H}_+)$, then*

$$\mathfrak{F}_s(\theta(T_+)\otimes T_+) \geqslant 0 .$$

Proof. Let $\{x_i\}$ be an orthonormal basis for $\mathcal{H}_+$ and $\{y_j\}$ an orthonormal basis for $\mathcal{H}_-$. Let $s_{ij} = \langle x_i, T\theta(y_j)\rangle_{\mathcal{H}_+}$. A vector $a \in \mathcal{H}_{+-}$ has the form $a = \sum_{i,j} a_{ij}\, x_i\otimes y_j$. According to Definition 2.2, the matrix elements of $\mathfrak{F}_s(\theta(T_+)\otimes T_+)$ on $\mathcal{H}_{+-}$ on the basis $x_i\otimes y_j$ are

$$
\begin{aligned}
&\langle x_i\otimes y_j, \mathfrak{F}_s(\theta(T_+)\otimes T_+)(x_{i'}\otimes y_{j'})\rangle_{\mathcal{H}_{+-}}\\
&\quad= \langle \theta(x_{i'})\otimes x_i, (\theta(T_+)\otimes T_+)(y_{j'}\otimes\theta(y_j))\rangle_{\mathcal{H}_{-+}}\\
&\quad= \langle \theta(x_{i'}), \theta(T_+)y_{j'}\rangle_{\mathcal{H}_-} \langle x_i, T_+\theta(y_j)\rangle_{\mathcal{H}_+}\\
&\quad= \langle T_+\theta(y_{j'}), x_{i'}\rangle_{\mathcal{H}_+} \langle x_i, T_+\theta(y_j)\rangle_{\mathcal{H}_+} = \overline{s_{i'j'}}\, s_{ij} .
\end{aligned}
$$

Thus

$$\langle a, \mathfrak{F}_s(\theta(T_+)\otimes T_+)a\rangle_{\mathcal{H}_{+-}} = \sum_{i,j,i',j'} \overline{a_{ij}} a_{i'j'} \overline{s_{i'j'}}\, s_{ij} = \left| \sum_{i,j} \overline{a_{ij}}\, s_{ij} \right|^2 \geqslant 0 ,$$

to complete the proof. □

2.2. Proof of the RP property

We apply the above properties of $\mathfrak{F}_s$ to establish the reflection positivity property for H.

Proof of Theorem 2.3. We assume $\mathfrak{F}_s(-H) \geqslant 0$, so Corollary 2.13 and $\beta \geqslant 0$ ensures $\mathfrak{F}_s(e^{-\beta H}) \geqslant 0$. Hence (1) is the expectation of a positive operator, which establishes the RP property for H. □

Proof of Theorem 2.4. See also the proof of Theorem 4.2 in [1]. Assume $\mathfrak{F}_s(H_0) \geqslant 0$. For $s > 0$, define

$$\begin{aligned} -H(s) &= -H_0 + s(H_- - s^{-1}I)(H_+ - s^{-1}I) \\ &= -H_0 + s\,\theta(T_+) \otimes T_+ \,. \end{aligned}$$

Here $T_+ = H_+ - s^{-1}I$. As $H_+ = I_- \otimes H'_+$ acts on $\mathcal{H}_+$, we infer that T_+ satisfies the hypotheses of Proposition 2.14. Hence $\mathfrak{F}_s(\theta(T_+) \otimes T_+) \geqslant 0$, and consequently $\mathfrak{F}_s(-H(s)) \geqslant 0$. We then conclude from Theorem 2.3, that $H(s)$ has the RP property. Adding a constant to $H(s)$ does not affect RP, so $H(s) + (\lambda + s^{-1})I = H - s\,\theta(H_+)H_+$ also has the RP property. Namely for all $x', x \in \mathcal{H}_+$ and all $\beta \geqslant 0$,

$$\langle \theta(x') \otimes x', e^{-\beta H + s\,\beta\theta(H_+)H_+} \theta(x) \otimes x \rangle_{\mathcal{H}_{-+}} \geqslant 0 \,.$$

This representation is continuous in s, also at $s = 0$. So let $s \to 0+$ to ensure the RP property for H. □

3. Levin–Wen models

In this section, we define the Levin–Wen model for graphs in surfaces using the data of unitary fusion categories. Our main result is proving reflection positivity for the Hamiltonian in the Levin–Wen model. We consider this $2+1$ Hamiltonian H as a $1+1$ action in a path integral of the configurations of a $1+1$ statistical mechanical theory. We prove the RP for the partition function. Now we consider the 2d surface as a $1+1$ space time with boundary at time zero. This leads to a $1+1$ quantum theory on the boundary. The quantized Hilbert space for the $1+1$ quantum theory is given by the RP. The transfer matrix e^{-th} on the quantized space $\widetilde{\mathcal{H}}$ is defined as a quantization of translation by t in the 2d model, and t denotes the imaginary time parameter. Moreover, h is the $1+1$ Hamiltonian on the quantized space. In order to implement this concept mathematically, we need the graphs in the surface to be invariant locally, under translation in the time direction by t.

3.1. Graphs in surfaces

Let M_+ be a surface in the half space $\mathbb{R}^3_+ = \{(x_1, x_2, x_3)|x_1 \geqslant 0\}$ with boundary ∂M on the plane $P = \{(x_1, x_2, x_3)|x_1 = 0\}$. Let Γ_+ be an oriented graph embedded in the surface M_+, such that $\Gamma_+ \cap \partial M_+ = \partial \Gamma_+$, namely the boundary points of Γ_+. Let θ_P be the reflection by the hyperplane P. Take $M_- = \theta_P(M_+)$. Then $\partial M_- = \partial M_+$.

Let $M = M_+ \cup M_-$. Take $\Gamma_- = \theta_P(\Gamma_+)$, and the orientation is reversed by θ_P. Then $\partial\Gamma_- = \partial\Gamma_+$. Take $\Gamma = \Gamma_+ \cup \Gamma_-$. Then M is a closed surface and Γ is a closed oriented graph in M.

Denote $E_+ = E(\Gamma_+)$ to be the edges of Γ_+ and $V_+ = V(\Gamma_+)$ to be the vertices of Γ_+. (The boundary points in $\partial\Gamma_+$ are not vertices of Γ_+.) Similarly define $E_- = E(\Gamma_-)$, $E = E(\Gamma)$, $V_- = V(\Gamma_-)$ and $V = V(\Gamma)$. Then $V = V_+ \cup V_-$ and $V_+ \cap V_- = \emptyset$. Take $E_0 = \{e \in E | e \cap P \neq \emptyset\}$, the set of edges go across the plane P. Then for any $e \in E_0$, its positive half is an edge in E_+ and its negative half is an edge in E_-. We identify the three edges as the same edge. Then $E_+ \cap E_- = E_0$, $E_+ \cup E_- = E$.

Let $s, t : E \to V$ be the source function and the target function. For any edge $e \in E$, the end points of e are $\partial e = \{s(e), t(e)\}$. Since the orientation is reversed by θ_P, we have

$$s(\theta_P(e)) = \theta_P\, t(e).$$

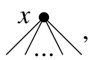
For any vertex $v \in V$, we define the set of adjacent edges $E(v) = \{e \in E | v \in \partial e\}$. The cardinality of $E(v)$ is called the degree of the vertex v, denoted by $|v|$. Let κ_v be a bijection from $\{1, 2, \ldots, |v|\}$ to $E(V)$, so that the numbers go from 1 to $|v|$ anti-clockwise around the vertices. The order κ_v is determined by the choice of the edge $\kappa_v(1)$. Define $\varepsilon_v(e) = +$ if $s(e) = v$; $\varepsilon_v(e) = -$ if $t(e) = v$.

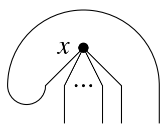
3.2. Unitary fusion categories

Suppose $\mathscr{C}$ is a unitary fusion category, (corresponding to a unitary tensor category in [8]). Let Irr be the set of irreducible objects (i.e., simple objects) of $\mathscr{C}$, and let $1 \in Irr$ be the trivial object. Take $A = \bigoplus_{X \in Irr} X$ and $A^n := \otimes_{k=1}^n A$. For any object X, let $ONB(X)$ denote an orthonormal basis of $\hom_{\mathscr{C}}(1, X)$. Let $d(X)$ be the quantum dimension of X. Let 1_X be the identity map in $\hom_{\mathscr{C}}(X, X)$. Define $X^+ = X$ and X^- to be the dual object of X. For any objects X, Y, Z in $\mathscr{C}$, let $\theta_{\mathscr{C}} : \hom_{\mathscr{C}}(X \otimes Y, Z) \to \hom_{\mathscr{C}}(Y^- \otimes X^-, Z^-)$ be the modular conjugation on $\mathscr{C}$. Pictorially $\theta_{\mathscr{C}}$ is a horizontal reflection.

Let $\cap_A$ be the co-evaluation map from 1 to A^2 and $\cup_A$ be the evaluation map from A^2 to 1. Then $\cup_A \cap_A = d(A)$ and $(1_A \otimes \cup_A)(\cap_A \otimes 1_A) = 1_A$. Define $\rho : \hom_{\mathscr{C}}(1, A^n) \to \hom_{\mathscr{C}}(1, A^n)$: for $x \in \hom_{\mathscr{C}}(1, A^n)$, let

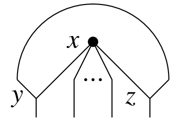
$$\rho(x) = (\cup_A \otimes 1_{A^n})(1_A \otimes x \otimes 1_A) \cap_A .$$

Pictorially, we represent x as , where the n edges are all labeled by the object A. Then

$\rho(x) :=$  .

For any $y, z \in \hom_{\mathscr{C}}(A^2, A)$, define $C_{y,z} : \hom_{\mathscr{C}}(1, A^n) \to \hom_{\mathscr{C}}(1, A^n)$: for any $x \in \hom_{\mathscr{C}}(1, A^n)$, $n \geqslant 2$, take the algebraic expression to be

$$C_{y,z}(x) := (y \otimes 1_{A^{n-2}} \otimes z)(1_A \otimes x \otimes 1_A) \cap_A .$$

The corresponding pictorial representation is, $C_{y,z}(x) =$  .

3.3. Configuration spaces

For every edge $e \in E$, we define $\mathcal{H}_e = L^2(Irr)$. Moreover, the delta functions δ_j, $j \in Irr$, form an ONB of $L^2(Irr)$. For every vertex $v \in V$, we define $\mathcal{H}_v = \hom_{\mathscr{C}}(1, A^{|v|})$.

Definition 3.1 (*LW Hilbert Spaces*)**.** Define the Hilbert spaces for the Levin–Wen model as

$$\mathcal{H}_+ := (\bigotimes_{v\in V_+} \mathcal{H}_v) \otimes (\bigotimes_{e\in E_+} \mathcal{H}_e) ,$$

$$\mathcal{H}_- := (\bigotimes_{v\in V_-} \mathcal{H}_v) \otimes (\bigotimes_{e\in E_-} \mathcal{H}_e) ,$$

$$\mathcal{H} := (\bigotimes_{v\in V} \mathcal{H}_v) \otimes (\bigotimes_{e\in E} \mathcal{H}_e).$$

The two Hilbert spaces $\mathcal{H}_-$ and $\mathcal{H}_+$ are dual to each other with respect to the Riesz representation θ. Define the embedding map

$$\iota : \mathcal{H} \to \mathcal{H}_{-+} = \mathcal{H}_- \boxtimes \mathcal{H}_+ , \tag{5}$$

as a multilinear extension of the map on an ONB:

$$\iota \left((\bigotimes_{v\in V} \beta_v) \otimes (\bigotimes_{e\in E} \delta_{j(e)}) \right)$$
$$= (\bigotimes_{v\in V_-} \beta_v) \otimes (\bigotimes_{e\in E_-} \delta_{j(e)}) \boxtimes (\bigotimes_{v\in V_+} \beta_v) \otimes (\bigotimes_{e\in E_+} \delta_{j(e)}),$$

for any $\beta_v \in ONB(\mathcal{H}_v)$ and any $j(e) \in Irr$. Extend the reflection θ_P to an anti-unitary $\theta : \mathcal{H}_+ \to \mathcal{H}_-$ as follows,

$$\theta \left((\bigotimes_{v\in V_+} \beta_v) \otimes (\bigotimes_{e\in E_+} \delta_{j(e)}) \right)$$
$$= \bigotimes_{v\in V_-} \theta_{\mathscr{C}}(\beta_{\theta_P(v)}) \otimes (\bigotimes_{e\in E_+} \delta_{j(\theta_P(e))}) .$$

3.4. Hamiltonians

Let Irr^n denote the tensor product,

$$Irr^n := \{j_1 \otimes j_2 \otimes \cdots \otimes j_n | j_k \in Irr, 1 \leqslant k \leqslant n\} .$$

Define $P_{v,\vec{j}}$ to be the projection from $\hom_{\mathscr{C}}(1, A^{|v|})$ onto $\hom_{\mathscr{C}}(1, \vec{j})$ at the vertex v. Define $P_{e,j}$ to be the projection from $L^2(Irr)$ on to $\mathbb{C}\delta_j$ at the edge e. For any $v \in V$, the action on the vertex is given by the operator H_v on $\mathcal{H}$:

$$H_v = \sum_{\vec{j}\in Irr^{|v|}} P_{v,\vec{j}} \prod_{k=1}^{|v|} P_{\kappa_v(k), j_k^{\varepsilon_v \kappa_v(k)}} \, .$$

One calls each connected component of $M\setminus\Gamma$ a plaquette. Let $\mathfrak{P}$ be the set of plaquettes. For any $p \in \mathfrak{P}$, let us denote the vertices and edges on ∂p by $v_1, e_1, v_2, e_2 \dots, v_m, e_m$ clockwise.

For any $j \in Irr$, the action on the plaquette is given by the operator $H_{p,j}$ on $\mathcal{H}$:

$$H_{p,j} = \sum_{\vec{j}\in Irr^{|v|}} \prod_{\ell=1}^{m} P_{e_\ell, j_\ell} \left(\sum_{k=1}^{m} \sum_{j'_k \in Irr} \sum_{y_k \in ONB\,\hom(j\otimes j_k^{\varepsilon v_k(e_k)},\,(j'_k)^{\varepsilon v_k(e_k)})} \times d(j'_k) \prod_{k=1}^{|v|} \rho_{v_k}^{1-\kappa_v^{-1}(e_k)} C_{v_k, y_k, \theta_{\mathscr{C}}(y_{k-1})} \rho_{v_k}^{\kappa_v^{-1}(e_k)-1} \right),$$

where $y_0 = y_n$ and ρ_{v_k}, $C_{v_k, y_k, \theta(y_{k-1})}$ are the actions of ρ and $C_{y_k, \theta(y_{k-1})}$ at the vertex v_k respectively. Here also

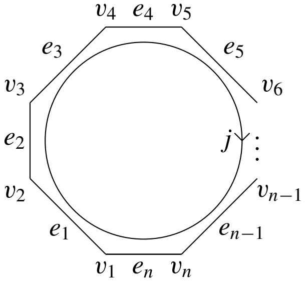
$$H_p = \sum_{j\in Irr} \frac{d(j)^2}{\mu} H_{p,j},$$

where $\mu = \sum_{j\in Irr} d(j)^2$ is the global dimension of $\mathscr{C}$. It is known that H_p, for $p \in \mathfrak{P}$, and H_v, for $v \in V$ are mutually commuting projections [8,9]. In the Levin–Wen model, the Hamiltonian H on $\mathcal{H}$ is

$$H = \lambda_{\mathfrak{P}} \sum_{p\in\mathfrak{P}} (1 - H_p) + \lambda_V \sum_{v\in V} (1 - H_v) \, , \tag{6}$$

for some $\lambda_{\mathfrak{P}} \geqslant 0$ and $\lambda_V \geqslant 0$.

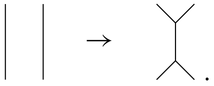
Pictorially, the action of $H_{p,j}$ is contracting a loop labeled by j in the plaquette p with morphisms in $\mathscr{C}$ on ∂p:



The contraction is induced from the relation

$$1_j \otimes 1_{j_k^{\pm}} = \sum_{j'_k\in Irr} d(j'_k) \sum_{y_k\in ONB\,\hom(j\otimes j_k^{\pm},\, j'_k)} y_k^* y_k$$

in $\mathscr{C}$. See §3 of [8] for more details. Pictorially, this relation changes the shape of a pair of lines labeled by e_i and j and as follows:



Then around each vertex v_i, the shape of the picture looks like 3.2.

The definition of $H_{p,j}$ is independent of the choice of the starting vertex v_1. It is also independent of the order κ_v. When we change the orientation of an edge e in the oriented graph Γ, we replace $P_{e,X}$ by P_{e,X^-}. Then the operators H_v and $H_{p,j}$ are not changed. So the operators are essentially independent of the orientation of the graph Γ.

3.5. *Reflection positivity*

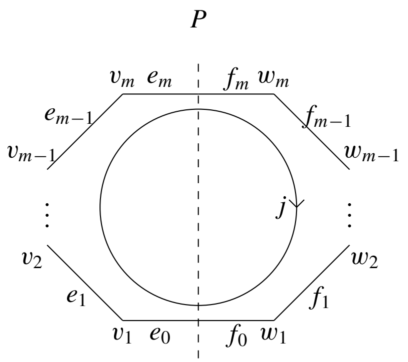
The main new result of this paper is the following:

Theorem 3.2 (*RP Property for Levin–Wen Models*)**.** *The Hamiltonian H in (6), acting on the Hilbert space $\mathcal{H}$ of Definition 3.1, has the RP property: for any $h_+, \Omega_+ \in \mathcal{H}_+$, and $\beta \geqslant 0$,*

$$\langle e^{-\beta H}\iota^*(\theta(h_+)\boxtimes h_+), \iota^*(\theta(\Omega_+)\boxtimes \Omega_+)\rangle_{\mathcal{H}} \geqslant 0. \tag{7}$$

Lemma 3.3. *For any plaquette p across the plane P, namely $p \cap P \neq \emptyset$, we have $\mathfrak{F}_s(-\iota H_{p,j}\iota^*) \geqslant 0$.*

Proof. For any plaquette p across the plane P, let us denote the vertices and edges in $\partial p \cap \Gamma_-$ clockwise as $e_0, v_1, e_1 v_2, \ldots, v_m, e_m$; the vertices and edges in $\partial p \cap \Gamma_+$ anti-clockwise as $f_0, w_1, f_1, w_2, \ldots, w_m, f_m$. Then $w_k = \theta(v_k)$, for $1 \leqslant k \leqslant m$; and $f_k = \theta(e_k)$, for $0 \leqslant k \leqslant m$. Moreover, $\varepsilon_{w_1}(f_0) = -\varepsilon_{v_1}(e_0)$, $\varepsilon_{w_n}(f_n) = -\varepsilon_{v_n}(e_n)$.



By the definitions of $H_{p,j}$ and ι, we have

$$-\iota H_{p,j}\iota^* = \sum_{j_0, j_0', j_m, j_m' \in Irr} T_{j_0,j_0',j_m,j_m'} \boxtimes \theta(T_{j_0,j_0',j_m,j_m'}),$$

where

$$T_{j_0,j_0',j_m,j_m'} = P_{e_0,j_0} P_{e_m,j_m} \sum_{y_0 \in ONB\, \hom(j\otimes j_0^{-\varepsilon_{v_1}(e_0)}, (j_0')^{-\varepsilon_{v_1}(e_0)})}$$

$$\sum_{y_m \in ONB\, \hom(j\otimes j_0^{\varepsilon_{v_m}(e_m)},(j_m')^{\varepsilon_{v_m}(e_m)})} \sum_{\vec{j}\in Irr^{|v|-1}}$$
$$\prod_{\ell=1}^{m-1} P_{e_\ell, j_\ell} \left(\sum_{k=1}^{m-1} \sum_{j_k' \in Irr} \sum_{y_k \in ONB\, \hom(j\otimes j_k^{\varepsilon_{v_k}(e_k)},(j_k')^{\varepsilon_{v_k}(e_k)})} \right.$$
$$\left. d(j_k') \prod_{k=1}^{|v|} \rho_{v_k}^{1-\kappa_v^{-1}(e_k)} C_{v_k, y_k, \theta_{\mathscr{C}}(y_{k-1})} \rho_{v_k}^{\kappa_v^{-1}(e_k)-1} \right).$$

By Proposition 2.14, $\mathfrak{F}_s(-\iota H_{p,j}\iota^*) \geqslant 0$. □

Proof of Theorem 3.2. Take $\tilde{H} = \iota H \iota^*$. We have the decomposition $\tilde{H} = H_0 + H_+ + H_- + \lambda I$, such that

$$H_0 = \lambda_{\mathfrak{P}} \sum_{p\in\mathfrak{P},\, p\cap P\neq\emptyset} -H_p;$$
$$H_\pm = \lambda_{\mathfrak{P}} \sum_{p\in\mathfrak{P},\, p\subset M_\pm} (1-H_p) + \lambda_V \sum_{v\in V_\pm} (1-H_v);$$
$$\lambda = \lambda_{\mathfrak{P}} \sum_{p\in\mathfrak{P},\, p\cap P\neq\emptyset} 1.$$

Then $\theta(H_+) = H_-$ and $H_+ = I \otimes H_+'$, for some $H_+' \in \hom(\mathcal{H}_+)$. By Lemma 3.3, $\mathfrak{F}_s(-H_0) \geqslant 0$. By Theorem 2.4, $\tilde{H}$ has the RP property. For any $h_+, \Omega_+ \in \mathcal{H}_+, \beta \geqslant 0$,

$$\begin{aligned}&\langle e^{-\beta H}\iota^*(\theta(h_+)\boxtimes h_+), \iota^*(\theta(\Omega_+)\boxtimes \Omega_+)\rangle_{\mathcal{H}} \\ &\quad = \langle e^{-\beta \tilde{H}}(\theta(h_+)\boxtimes h_+), (\theta(\Omega_+)\boxtimes \Omega_+)\rangle_{\mathcal{H}_{-+}} \geqslant 0.\end{aligned}$$

Therefore H has the RP property. □

3.6. An interpretation

Let us explain an elementary example: let M_+ be isotopic to a cylinder, so M is a torus. Take the graph Γ to be a square lattice in M.

If Ω_+ is the vacuum vector in $\mathcal{H}_+$, namely all objects and morphisms are trivial, then $\iota^*(\theta(\Omega_+)\boxtimes\Omega_+)$ is the vacuum vector in $\mathcal{H}$. We can consider the expectation in $\iota^*(\theta(\Omega_+)\boxtimes\Omega_+)$, as a path integral over configurations, where the $2+1$ Hamiltonian acts diagonally (as an action) on the configurations in the $1+1$ theory. These configurations can be identified as closed string nets on the dual lattice through the modular self-duality proved in [11], when $\mathscr{C}$ is a unitary modular tensor category. The RP condition for the path integral in the bulk induces a one-dimensional lower quantum theory on the boundary of M_+, which is a union of two circles. The quantized space is spanned by quantum spin chains on the circles. In this case the Hamiltonian $\widetilde{H}$ acts diagonally on the quantum spin chains, and it has no local interactions. The basic reason is that the vacuum is not entangled.

For the Levin–Wen model on a torus M, it is known that the excitations in the bulk are objects of the Drinfeld center $Z(\mathscr{C})$. If Ω_+ is a translation-invariant open string labeled by an object X in $\mathscr{C}$ along the imaginary time direction, with end points on the two boundary circles of M_+, then $\iota^*(\theta(\Omega_+) \boxtimes \Omega_+)$ is a closed string in M, corresponding to a bulk excitation. We consider the expectation on $\iota^*(\theta(\Omega_+) \boxtimes \Omega_+)$ as a *non-local* path integral. The RP condition for the path integral in the bulk induces a quantum theory topologically entangled on the two boundary circles. The quantized spaces are given by quantum spin chains on the two circles sharing a fiber labeled by X. The Hamiltonian $\widetilde{H}$ acts diagonally as well.

In general, the quantized spaces consist of quantum spin chains on the boundary, possibly entangled in the bulk. If $\theta(\Omega_+) \boxtimes \Omega_+$ is translation invariant, then we obtain a time-independent Hamiltonian $\widetilde{H}$ acting on the quantized space. If $\theta(\Omega_+) \boxtimes \Omega_+$ is not a simple tensor product, namely it is entangled, then the Hamiltonian $\widetilde{H}$ has non-trivial interactions on the quantum spin chains.

The RP, symmetries, and locality of $\theta(\Omega_+)\boxtimes\Omega_+$ reflects the positivity, symmetries, and locality of the one-dimensional lower quantum theory on the boundary. This coincides with the philosophy of Osterwalder–Schrader quantization using RP for quantum field theory.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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