

SYSTEMS OF INTERACTING DIFFUSIONS WITH PARTIAL ANNIHILATION THROUGH MEMBRANES¹

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We introduce an interacting particle system in which two families of reflected diffusions interact in a singular manner near a deterministic interface I . This system can be used to model the transport of positive and negative charges in a solar cell or the population dynamics of two segregated species under competition. A related interacting random walk model with discrete state spaces has recently been introduced and studied in Chen and Fan (2014). In this paper, we establish the functional law of large numbers for this new system, thereby extending the hydrodynamic limit in Chen and Fan (2014) to reflected diffusions in domains with mixed-type boundary conditions, which include absorption (harvest of electric charges). We employ a new and direct approach that avoids going through the delicate BBGKY hierarchy.

1. Introduction. With motivation to model and analyze the transport of positive and negative charges in solar cells, an interacting random walk model in domains has recently been introduced in [9]. In that model, a bounded domain in \mathbb{R}^d is divided into two adjacent subdomains D_+ and D_- by an interface I . The subdomains D_+ and D_- represent the hybrid medias which confine the positive and the negative charges, respectively. At microscopic level, positive and negative charges are modeled by independent continuous time random walks on lattices inside D_+ and D_- . These two types of particles annihilate each other at a certain rate when they come close to each other near the interface I . This interaction models the annihilation, trapping, recombination and separation phenomena of the charges. Such a stochastic system can also model population dynamics of two segregated species under competition near their boarder. Under an appropriate scaling of the lattice size, the speed of the random walks and the annihilation rate, we proved in [9] that the hydrodynamic limit is described by a system of nonlinear heat equations that are coupled on the interface.

While the random walk model in [9] is more amenable to computer simulation, it is subject to technical restrictions associated with the discrete approximations of both the diffusions performed by the particles and the underlying domains D_{\pm} .

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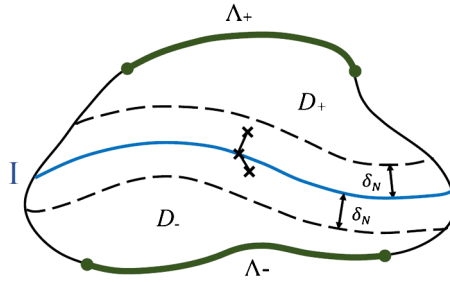


FIG. 1. $I = \text{Interface}$, $\Lambda_{\pm} = \text{Harvest sites}$.

Furthermore, that model does not consider harvest of charges, which is of practical interest.

In this paper, a new continuous state stochastic model is introduced and investigated. This model is different from that of [9] in three ways: the particles perform reflected diffusions on continuous state spaces rather than random walks over discrete state spaces, particles are absorbed (harvested) at some regions (harvest sites) away from the interface I , and the annihilation mechanism near I is different. The model in this paper allows more flexibility in modeling the underlying spatial motions performed by the particles and in the study of their various properties. In particular, it is more convenient to work with when we study the fluctuation limit (or, functional central limit theorem) of the interacting diffusion system, which is the subject of an on-going project [10].

Here is a heuristic description of our new model (see Figure 1): Let D_{\pm} and I be as above. There is a harvest region $\Lambda_{\pm} \subset \partial D_{\pm} \setminus I$ that absorbs (harvests) \pm charges, respectively, whenever it is being visited. Let N be the common initial number of particles in each of D_+ and D_- . For simplicity, we assume here that each particle in D_{\pm} performs a Brownian motion with drift in the interior of D_{\pm} . These random motions model the transport of positive (resp., negative) charges under an electric potential. When a particle hits the boundary, it is absorbed (harvested) on Λ_{\pm} , and is instantaneously reflected on $\partial D_{\pm} \setminus \Lambda_{\pm}$ along the inward normal direction of D_{\pm} . In other words, we assume that each particle in D_{\pm} performs a reflected Brownian motions (RBM) with drift in D_{\pm} that is killed upon hitting Λ_{\pm} . In addition, a pair of particles of opposite signs has a chance of being annihilated with each other when they are near I . Actually, when two particles of different types come within a small distance δ_N (which must occur near the interface I), they disappear with intensity $\frac{\lambda}{N\delta_N^{d+1}}$. Here, $\lambda > 0$ is a given parameter modeling the rate of annihilation.

The choice of the scaling $\frac{\lambda}{N\delta_N^{d+1}}$ for the per-pair annihilation intensity is to guarantee that, in the limit $N \rightarrow \infty$, a nontrivial proportion of particles is killed during the time interval $[0, t]$. Here is the heuristic reasoning. Since diffusive particles typically spread out in space, the number of pairs near the interface is of order

$N^2 \delta_N^{d+1}$ (because there are $N \delta_N$ number of particles in D_+ near I , and each of them is near to $N \delta_N^d$ number of particles in D_-). With the above choice of per-pair annihilation intensity, the expected number of pairs killed within t units of time is about $(N^2 \delta_N^{d+1}) (\frac{\lambda}{N \delta_N^{d+1}} t) = \lambda N t$ when $t > 0$ is small. This implies that a nontrivial proportion of particle is annihilated during $[0, t]$ and accounts for the boundary term in the hydrodynamic limit.

In our model, even though the boundary is static and there is no creation of particles, the interactions do affect the correlations among the particles: whether or not a positive particle disappears at a given time affects the empirical distribution of the negative particles, which in turn affects that of the positive particles. This challenge is reflected by the nonlinearity of the macroscopic limit and also by the nonproduct structure of the system of equations satisfied by the correlation functions in the pre-limit. The latter equations are computed in [10]; see also the BBGKY hierarchy in [9].

1.1. *Main result and applications.* We consider the normalized empirical measures

$$\mathfrak{X}_t^{N,+}(dx) := \frac{1}{N} \sum_{\alpha: \alpha \sim t} \mathbf{1}_{X_\alpha^+(t)}(dx) \quad \text{and} \quad \mathfrak{X}_t^{N,-}(dy) := \frac{1}{N} \sum_{\beta: \beta \sim t} \mathbf{1}_{X_\beta^-(t)}(dy).$$

Here, $\mathbf{1}_y(dx)$ stands for the Dirac measure concentrated at the point y , while $\alpha \sim t$ (resp., $\beta \sim t$) denotes the condition that particle X_α^+ (resp., X_β^-) is alive at time t .

Our main result (Theorem 5.1) implies the following: Suppose each particle in D_\pm is a RBM with gradient drift $\frac{1}{2} \nabla(\log \rho_\pm)$, where ρ_\pm is a strictly positive function on \bar{D}_\pm . Suppose δ_N tends to zero and $\liminf_{N \rightarrow \infty} N \delta_N^d \in (0, \infty]$. If $(\mathfrak{X}_0^{N,+}, \mathfrak{X}_0^{N,-})$ converges in distribution to $(f(x) dx, g(y) dy)$ where f and g are bounded continuous functions, then the random measures $(\mathfrak{X}_t^{N,+}, \mathfrak{X}_t^{N,-})$ converge in distribution to a deterministic limit $(u_+(t, x) \rho_+(x) dx, u_-(t, y) \rho_-(y) dy)$ for any $t > 0$, where (u_+, u_-) is the unique solution of the coupled heat equations [in the sense of integral equation (4.1)]

$$(1.1) \quad \begin{cases} \frac{\partial u_+}{\partial t} = \frac{1}{2} \Delta u_+ + \frac{1}{2} \nabla(\log \rho_+) \cdot \nabla u_+, & \text{on } (0, \infty) \times D_+, \\ u_+ = 0, & \text{on } (0, \infty) \times \Lambda_+, \\ \frac{\partial u_+}{\partial \vec{n}_+} = \frac{\lambda}{\rho_+} u_+ u_- \mathbf{1}_{\{I\}}, & \text{on } (0, \infty) \times \partial D_+ \setminus \Lambda_+ \end{cases}$$

and

$$(1.2) \quad \begin{cases} \frac{\partial u_-}{\partial t} = \frac{1}{2} \Delta u_- + \frac{1}{2} \nabla(\log \rho_-) \cdot \nabla u_-, & \text{on } (0, \infty) \times D_-, \\ u_- = 0, & \text{on } (0, \infty) \times \Lambda_-, \\ \frac{\partial u_-}{\partial \vec{n}_-} = \frac{\lambda}{\rho_-} u_+ u_- \mathbf{1}_{\{I\}}, & \text{on } (0, \infty) \times \partial D_- \setminus \Lambda_- \end{cases}$$

with initial condition $u_+(0, x) = f(x)$ and $u_-(0, y) = g(y)$, where \vec{n}_\pm is the inward unit normal vector field on ∂D_\pm of D_\pm and $\mathbf{1}_{\{I\}}$ is the indicator function on I . Note that $\rho_\pm = 1$ corresponds to the particular case when there is no drift.

REMARK 1.1 (Generalizations and applications). Theorem 5.1 is general enough to deal with any symmetric reflected diffusions and covers the case when λ is any continuous function $\lambda(x)$ on I . It is routine to generalize to any continuous time-dependent function $\lambda(t, x)$ and the details are left to the readers. Moreover, it is likely that a further generalization to tackle multiple deletion of particles near the interface (similar to that in [18]) can be done in an analogous way. As an immediate application of Theorem 5.1, we obtain an analytic formula for the asymptotic mass of positive charges harvested during the time interval $[0, T]$, which is

$$1 - \int_{D_+} u_+(T, x) \rho_+(x) dx - \lambda \int_0^T \int_I u_+(s, z) u_-(s, z) d\sigma(z) ds,$$

where σ denotes the surface measure on ∂D_\pm throughout this paper.

REMARK 1.2. The condition $\liminf_{N \rightarrow \infty} N \delta_N^d \in (0, \infty]$ is a lower bound for the rate at which the annihilations distance δ_N tends to 0. Such kind of condition is necessary by the following reason: The dimension of I is $d + 1$ lower than that of $D_+ \times D_-$. So we can choose δ_N small enough so that particles of different types cannot “see” each other in the limit $N \rightarrow \infty$, resulting a decoupled linear system of PDEs with Dirichlet boundary condition on Λ_\pm and Neumann boundary condition on $\partial D_\pm \setminus \Lambda_\pm$. See Example 5.3 for a rigorous statement and proof.

1.2. *Key ideas.* Theorem 3.2.39 of [22] from geometric theory asserts that

$$(1.3) \quad \lim_{\delta \rightarrow 0} \frac{\mathcal{H}^{2d}(I^\delta)}{c_{d+1} \delta^{d+1}} = \mathcal{H}^{d-1}(I),$$

where $I^\delta := \{(x, y) \in D_+ \times D_- : |x - z|^2 + |y - z|^2 < \delta^2 \text{ for some } z \in I\}$, c_{d+1} is the volume of the unit ball in \mathbb{R}^{d+1} , and \mathcal{H}^m is the m -dimensional Hausdorff measure. In Lemma 7.3, we strengthen it to

$$(1.4) \quad \lim_{\delta \rightarrow 0} \frac{1}{c_{d+1} \delta^{d+1}} \int_{I^\delta} f(x, y) dx dy = \int_I f(z, z) d\mathcal{H}^{d-1}(z)$$

uniformly in f from any equi-continuous family in $C(\overline{D}_+ \times \overline{D}_-)$. Property (1.4) leads us to the following key observation that

$$\begin{aligned} & \lim_{\delta \rightarrow 0} \lim_{N \rightarrow \infty} \frac{1}{c_{d+1} \delta^{d+1}} \mathbb{E} \int_0^T \mathfrak{X}_s^{N,+} \otimes \mathfrak{X}_s^{N,-}(I^\delta) ds \\ &= \lim_{N \rightarrow \infty} \lim_{\delta \rightarrow 0} \frac{1}{c_{d+1} \delta^{d+1}} \mathbb{E} \int_0^T \mathfrak{X}_s^{N,+} \otimes \mathfrak{X}_s^{N,-}(I) ds. \end{aligned}$$

This interchange of limit in turn allows us to characterize the mean of any subsequential limit of $(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})$ by comparing the integral equations (4.1) satisfied by the hydrodynamic limit with its stochastic counterpart (7.6). Using a similar argument, we can identify the second moment of any subsequential limit, and hence characterize any subsequential limit of $(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})$. We point out here that $\frac{1}{c_{d+1}\delta^{d+1}} \int_0^t \mathfrak{X}_s^{N,+} \otimes \mathfrak{X}_s^{N,-} (I^\delta) ds$ quantifies the amount of interaction among the two types of particles, and is related (but different from) the *collision local time* introduced in [20]. The direct approach developed in this paper to establish the hydrodynamic limit avoids going through the delicate BBGKY hierarchy as was done in [9].

1.3. *Literature.* Interacting diffusion systems have been studied by many authors and they continue to be the subject of active research. See [30] and [33] for such a system on a circle whose hydrodynamic limit is established using the entropy method. We also mention [17] for a recent large deviation result for a system of diffusions in \mathbb{R} interacting through their ranks. This large deviation principle implies convergence of the system to the hydrodynamic limit. However, the methods in these papers do not seem to work (at least not in a direct way) for our annihilating diffusion model due to the singular interaction on the interface.

An extensively studied class of stochastic particle systems is *reaction-diffusion systems* (R–D systems in short), whose hydrodynamic limits are described by R–D equations $\frac{\partial u}{\partial t} = \frac{1}{2} \Delta u + R(u)$, where $R(u)$ is a function in u which represents the reaction. R–D systems are an important class of interacting particle systems arising from various contexts. They were investigated by many authors in both the discrete setting (particles perform random walks) and the continuous setting (particles perform continuous diffusions). For instance, for the case $R(u)$ is a polynomial in u , these systems were studied in [18, 19, 28, 29] on a cube with Neumann boundary conditions, and in [3, 4] on a periodic lattice. See also [7] for a survey of a class of discrete (lattice) models called the *Polynomial Model* which contains the *Schlögl's model*. Recently, perturbations of the voter models which contain the *Lotka–Volterra systems* are considered in [14]. The authors showed that the hydrodynamic limits are R–D equations and established general conditions for the existence of nontrivial stationary measures and for extinction of the particles. Another class of stochastic particle systems which are related to our annihilation-diffusion model are the *Fleming–Viot type systems* [5, 6, 24]. In [6], Burdzy and Quastel studied an annihilating-branching system of two families of random walks on a domain. In their model, when a pair of particles of different types meet, they annihilate each other and they are immediately reborn at a site chosen randomly from the remaining particles of the same type. So the total number of particles of each type remains constant over the time, and thus this Fleming–Viot type system is different from the annihilating random walk model of [9]. The hydrodynamic limit of the model in [6] is described by a linear heat

equation with zero average temperature. An elegant result obtained by Dittrich [18] is on a system of reflected Brownian motions on the unit interval $[0, 1]$ with multiple deletion of particles. More precisely, any k -tuples ($2 \leq k \leq n$) of particles with distances between them of order ε , say $(x^{i_1}, \dots, x^{i_k})$, disappear with intensity $c_k(k-1)!\varepsilon^{k-1} \int_{[0,1]} p(\varepsilon^2, x^{i_1}, y) \cdots p(\varepsilon^2, x^{i_k}, y) dy$, where $c_k > 0$ are constants and $p(t, x, y)$ is the transition density of the reflected Brownian motion on $[0, 1]$. The hydrodynamic limit is a R–D equation with reaction term $R(u) = -\sum_{k=2}^n c_k u^k$ and Neumann boundary condition. In contrast to [18], our model has two types of particles instead of one. Moreover, the interaction of our model is singular near the boundary and gives rise to a boundary integral term in the hydrodynamic limit.

The rest of the paper is organized as follows. Preliminary materials on setup, reflecting diffusions, and notation are given in Section 2. A rigorous description of the interacting stochastic particle system introduced in this paper is presented in Section 3. In Section 4, we give an existence and uniqueness result of solution of a coupled heat equation with nonlinear boundary condition, analogous to Proposition 2.19 of [9]. The full statement and the proof of our main result (Theorem 5.1) is given in Section 5 and Section 6, respectively. The proof of a key proposition that identifies the first and second moments of subsequential limits of empirical distributions is postponed to Section 7.

2. Preliminaries.

2.1. *Reflected diffusions killed upon hitting a closed set $\Lambda \subset \overline{D}$.* Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain, and

$$W^{1,2}(D) = \{f \in L^2(D; dx) : \nabla f \in L^2(D; dx)\}.$$

Consider the bilinear form on $W^{1,2}(D)$ defined by

$$\mathcal{E}(f, g) := \frac{1}{2} \int_D \nabla f(x) \cdot \mathbf{a} \nabla g(x) \rho(x) dx,$$

where $\rho \in W^{1,2}(D)$ is a positive function on D which is bounded away from zero and infinity, $\mathbf{a} = (a^{ij})$ is a symmetric bounded uniformly elliptic $d \times d$ matrix-valued function such that $a^{ij} \in W^{1,2}(D)$ for each i, j . Since D is Lipschitz boundary, $(\mathcal{E}, W^{1,2}(D))$ is a regular symmetric Dirichlet form on $L^2(D; \rho(x) dx)$, and hence has a unique (in law) associated ρ -symmetric strong Markov process X (cf. [8]).

DEFINITION 2.1. Let (\mathbf{a}, ρ) and X be as in the preceding paragraph. We call X an (\mathbf{a}, ρ) -reflected diffusion. A special but important case is when \mathbf{a} is the identity matrix, in which X is called a reflected Brownian motion with drift $\frac{1}{2} \nabla(\log \rho)$. If in addition $\rho = 1$, then X is called a reflected Brownian motion (RBM).

Denote by \bar{n} the unit inward normal vector of D on ∂D . The Skorokhod representation of X tells us (see [8]) that X behaves like a diffusion process associated to the elliptic operator

$$(2.1) \quad \mathcal{A} := \frac{1}{2\rho} \nabla \cdot (\rho \mathbf{a} \nabla)$$

in the interior of D , and is instantaneously reflected at the boundary in the inward conormal direction $\bar{\nu} := \mathbf{a}\bar{n}$. It is well known (cf. [2, 25] and the references therein) that X has a transition density $p(t, x, y)$ with respect to the symmetrizing measure $\rho(x) dx$ [i.e., $\mathbb{P}_x(X_t \in dy) = p(t, x, y)\rho(y) dy$ and $p(t, x, y) = p(t, y, x)$], that p is locally Hölder continuous and hence $p \in C((0, \infty) \times \bar{D} \times \bar{D})$, and that we have the following: for any $T > 0$, there are constants $c_1 \geq 1$ and $c_2 \geq 1$ such that

$$(2.2) \quad \frac{1}{c_1 t^{d/2}} \exp\left(\frac{-c_2 |y - x|^2}{t}\right) \leq p(t, x, y) \leq \frac{c_1}{t^{d/2}} \exp\left(\frac{-|y - x|^2}{c_2 t}\right)$$

for every $(t, x, y) \in (0, T] \times \bar{D} \times \bar{D}$. Using (2.2) and the Lipschitz assumption for the boundary, we can check that

$$(2.3) \quad \sup_{x \in \bar{D}} \sup_{0 < \delta \leq \delta_0} \frac{1}{\delta} \int_{D^\delta} p(t, x, y) dy \leq \frac{C}{\sqrt{t}} \quad \text{for } t \in (0, T] \quad \text{and}$$

$$(2.4) \quad \sup_{x \in \bar{D}} \int_{\partial D} p(t, x, y) \sigma(dy) \leq \frac{C}{\sqrt{t}}, \quad \text{for } t \in (0, T],$$

where $C, \delta_0 > 0$ are constants which depends only on d, T , the Lipschitz characteristics of D , the ellipticity of \mathbf{a} and the lower and upper bound of ρ . Here, $D^\delta := \{x \in D : \text{dist}(x, \partial D) < \delta\}$. In fact, (2.4) follows from (2.3) via Lemma 7.1.

Now we consider an (\mathbf{a}, ρ) -reflected diffusion *killed upon hitting a closed subset* Λ of \bar{D} . In particular, Λ can be a subset of ∂D such as Λ_\pm in Figure 1. Define

$$(2.5) \quad X_t^{(\Lambda)} := \begin{cases} X_t, & t < T_\Lambda, \\ \partial, & t \geq T_\Lambda, \end{cases}$$

where ∂ is a cemetery point and $T_\Lambda := \inf\{t > 0 : X_t \in \Lambda\}$ is the first hitting time of X on Λ . Since $\bar{D} \setminus \Lambda$ is open in \bar{D} , Theorem A.2.10 of [23] asserts that $X^{(\Lambda)}$ is a Hunt process on $(\bar{D} \setminus \Lambda) \cup \partial$ with transition function $P_t^\Lambda(x, A) = \mathbb{P}^x(X_t \in A, t < T_\Lambda)$. The characterization of the Dirichlet form of $X^{(\Lambda)}$ can be found in Theorem 3.3.8 of [11] or Theorem 4.4.2 of [23]; in particular, it implies that the semigroup $\{P_t^\Lambda\}_{t \geq 0}$ of $X^{(\Lambda)}$ is symmetric and strongly continuous on $L^2(\bar{D} \setminus \Lambda, \rho(x) dx)$. Clearly, $X^{(\Lambda)}$ has a transition density $p^{(\Lambda)}$ with respect to $\rho(x) dx$ [i.e., $P_t^\Lambda(x, dy) = p^{(\Lambda)}(t, x, y)\rho(y) dy$]. Note that $p^{(\Lambda)}(t, x, y) \leq p(t, x, y)$ for all $x, y \in D$ and $t > 0$.

So far Λ is only assumed to be closed in \bar{D} . We will also need the following regularity assumption.

DEFINITION 2.2. $\Lambda \subset \overline{D}$ is said to be *regular* with respect to X if $\mathbb{P}^x(T_\Lambda = 0) = 1$ for all $x \in \Lambda$, where $T_\Lambda := \inf\{t > 0 : X_t \in \Lambda\}$.

This regularity assumption implies that $p^{(\Lambda)}(t, x, y)$ is jointly continuous in x and y up to the boundary. In particular, $p^{(\Lambda)}(t, x, y)$ is continuous for x and y in a neighborhood of I . We now gather some basic properties of $p^{(\Lambda)}(t, x, y)$ for later use.

PROPOSITION 2.3. *Let X be an (\mathbf{a}, ρ) -reflected diffusion defined in Definition 2.1, and $p^{(\Lambda)}(t, x, y)$ be the transition density, with respect to $\rho(x) dx$, of X^Λ defined in (2.5). Suppose Λ is closed and regular with respect to X . Then $p^{(\Lambda)}(t, x, y) \geq 0$ and $p^{(\Lambda)}(t, x, y) = p^{(\Lambda)}(t, y, x)$ for all $x, y \in \overline{D}$ and $t > 0$. Moreover, $p^{(\Lambda)}(t, x, y)$ can be extended to be jointly continuous on $(0, \infty) \times \overline{D} \times \overline{D}$. The last property implies that the semigroup $\{P_t^\Lambda\}_{t \geq 0}$ of X^Λ is strongly continuous on the Banach space $C_\infty(\overline{D} \setminus \Lambda) := \{f \in C(\overline{D}) : f \text{ vanishes on } \Lambda\}$ equipped with the uniform norm on \overline{D} . The domain of the Feller generator of $\{P_t^\Lambda\}_{t \geq 0}$, denoted by $\text{Dom}(\mathcal{A}^{(\Lambda)})$, is dense in $C_\infty(\overline{D} \setminus \Lambda)$.*

PROOF. Define, for all $(t, x, y) \in (0, \infty) \times \overline{D} \times \overline{D}$,

$$q^{(\Lambda)}(t, x, y) := p(t, x, y) - r(t, x, y)$$

where $r(t, x, y) := \mathbb{E}^x[p(t - T_\Lambda, X_{T_\Lambda}, y); t \geq T_\Lambda]$.

Using the fact that $x \mapsto \mathbb{P}^x(T_\Lambda < t)$ is lower semicontinuous (cf. Proposition 1.10 in Chapter II of [1]), it is easy to check that if Λ is closed and regular, then

$$(2.6) \quad \lim_{n \rightarrow \infty} \mathbb{P}^{x_n}(T_\Lambda < t) = 1$$

whenever $t > 0$ and $x_n \in D$ converges to a point in Λ . Recall that $p(t, x, y)$ is symmetric in (x, y) , has two-sided Gaussian estimates (2.2), and is jointly continuous on $(0, \infty) \times \overline{D} \times \overline{D}$. Using these properties of p together with (2.6), then applying the same argument of Section 4 of Chapter II in [1], we have:

- (a) $q^{(\Lambda)}(t, x, y)$ is a density for the transition function of X^Λ ,
 - (b) $q^{(\Lambda)}(t, x, y) \geq 0$ and $q^{(\Lambda)}(t, x, y) = q^{(\Lambda)}(t, y, x)$ for all $x, y \in \overline{D}$ and $t > 0$,
- and
- (c) $q^{(\Lambda)}(t, x, y)$ is jointly continuous on $(0, \infty) \times \overline{D} \times \overline{D}$.

From (c), the semigroup $\{P_t^{(\Lambda)}\}$ of $X^{(\Lambda)}$ is strongly continuous by a standard argument. $C_\infty(\overline{D} \setminus \Lambda)$ is a Banach space since it is a closed subspace of $C(\overline{D})$. The domain of the Feller generator $\text{Dom}(\mathcal{A}^{(\Lambda)})$ of $\{P_t^{(\Lambda)}\}$ is dense in $C_\infty(\overline{D} \setminus \Lambda)$ because any $f \in C_\infty(\overline{D} \setminus \Lambda)$ is the strong limit $\lim_{t \downarrow 0} \frac{1}{t} \int_0^t P_s^{(\Lambda)} f ds$ in $C_\infty(\overline{D} \setminus \Lambda)$, and $\int_0^t P_s^{(\Lambda)} f ds \in \text{Dom}(\mathcal{A}^{(\Lambda)})$. \square

2.2. Assumptions and notation. We now return to our annihilating diffusion system. Recall that before being annihilated by a particle of the opposite kind near I , a particle in D_{\pm} performs a reflected diffusion with absorption on $\Lambda_{\pm} \subset \partial D_{\pm} \setminus I$. If a particle is absorbed (in Λ_{\pm}) rather than annihilated (near I), it is considered to be harvested.

The following assumptions are in force throughout this paper.

ASSUMPTION 2.4 (Geometric setting). Suppose D_+ and D_- are given adjacent bounded Lipschitz domains in \mathbb{R}^d such that $I := \overline{D_+} \cap \overline{D_-} = \partial D_+ \cap \partial D_-$ is \mathcal{H}^{d-1} -rectifiable. Λ_{\pm} is a closed subset of $\overline{D_{\pm}} \setminus I$ which is regular with respect to the $(\mathbf{a}_{\pm}, \rho_{\pm})$ -reflected diffusion X^{\pm} , where $\rho_{\pm} \in W^{(1,2)}(D_{\pm}) \cap C(\overline{D_{\pm}})$ is a given strictly positive function, $\mathbf{a}_{\pm} = (a_{\pm}^{ij})$ is a symmetric, uniformly elliptic $d \times d$ matrix-valued function such that $a_{\pm}^{ij} \in W^{1,2}(D_{\pm})$ for each i, j .

ASSUMPTION 2.5 (Parameter of annihilation). Suppose $\lambda \in C_+(I)$ is a given nonnegative continuous function on I . Let $\widehat{\lambda} \in C(\overline{D_+} \times \overline{D_-})$ be an arbitrary but fixed extension of λ in the sense that $\widehat{\lambda}(z, z) = \lambda(z)$ for all $z \in I$. (Such $\widehat{\lambda}$ always exists.)

ASSUMPTION 2.6 (The annihilation potential). We choose *annihilation potential functions* $\{\ell_{\delta} : \delta > 0\} \subset C_+(\overline{D_+} \times \overline{D_-})$ in such a way that $\ell_{\delta}(x, y) \leq \frac{\widehat{\lambda}(x, y)}{c_{d+1}\delta^{d+1}} \mathbf{1}_{I^{\delta}}(x, y)$ on $D_+ \times D_-$ and

$$(2.7) \quad \lim_{\delta \rightarrow 0} \left\| \ell_{\delta} - \frac{\widehat{\lambda}}{c_{d+1}\delta^{d+1}} \mathbf{1}_{I^{\delta}} \right\|_{L^2(D_+ \times D_-)} = 0,$$

where $I^{\delta} := \{(x, y) \in D_+ \times D_- : |x - z|^2 + |y - z|^2 < \delta^2 \text{ for some } z \in I\}$ and c_{d+1} is the volume of the unit ball in \mathbb{R}^{d+1} .

Assumption 2.6 is natural in view of (1.3). Intuitively, if N is the initial number of particles, then δ_N is the annihilation distance and I^{δ_N} controls the frequency of interactions. As remarked in the **Introduction**, we need to assume that the annihilation distance δ_N does not shrink too fast. This is formulated in Assumption 2.7 below.

ASSUMPTION 2.7 (The annihilation distance). $\liminf_{N \rightarrow \infty} N\delta_N^d \in (0, \infty]$, where $\{\delta_N\} \subset (0, \infty)$ converges to 0 as $N \rightarrow \infty$.

Convention: To simplify notation, we suppress Λ_{\pm} and write X^{\pm} in place of $X^{\Lambda_{\pm}}$ for a $(\mathbf{a}_{\pm}, \rho_{\pm})$ -reflected diffusions on D_{\pm} killed upon hitting Λ_{\pm} . We also use $p^{\pm}(t, x, y)$, P_t^{\pm} and \mathcal{A}^{\pm} to denote, respectively, the transition density w.r.t. ρ_{\pm} , the semigroup associated to $p^{\pm}(t, x, y)$ and the $C_{\infty}(\overline{D_{\pm}} \setminus \Lambda_{\pm})$ -generator (called

the Feller generator) of $X^\pm = X^{\Lambda^\pm}$. Under Assumption 2.4, X^\pm is a Hunt (hence strong Markov) process on

$$D_\pm^\partial := (\overline{D}_\pm \setminus \Lambda^\pm) \cup \{\partial^\pm\},$$

where ∂^\pm is the cemetery point for X^\pm (see Proposition 2.3).

For the reader's convenience, we list other notation that will be adopted in this paper:

$\mathcal{B}(E)$	Borel measurable functions on E
$\mathcal{B}_b(E)$	bounded Borel measurable functions on E
$\mathcal{B}^+(E)$	nonnegative Borel measurable functions on E
$C(E)$	continuous functions on E
$C_b(E)$	bounded continuous functions on E
$C^+(E)$	nonnegative continuous functions on E
$C_c(E)$	continuous functions on E with compact support
$D([0, \infty), E)$	space of càdlàg paths from $[0, \infty)$ to E equipped with the Skorokhod metric
$C_\infty(\overline{D} \setminus \Lambda)$	$\{f \in C(\overline{D}) : f \text{ vanishes on } \Lambda\}$
$C_\infty^{(n,m)}$	$\{\Phi \in C(\overline{D}_+^n \times \overline{D}_-^m) : \Phi \text{ vanishes outside } (\overline{D}_+ \setminus \Lambda_+)^n \times (\overline{D}_- \setminus \Lambda_-)^m\}$, see Section 7.2
\mathcal{H}^m	m -dimensional Hausdorff measure
I^δ	$\{(x, y) \in D_+ \times D_- : x - z ^2 + y - z ^2 < \delta^2 \text{ for some } z \in I\}$,
c_d	the volume of the unit ball in \mathbb{R}^d
ℓ_δ	the annihilating potential functions in Assumption 2.6
$\underline{\mathbf{X}}_t^{(N)}$	the configuration process defined in Section 3.1
S_N	$\bigcup_{m=1}^N (D_+^\partial(m) \times D_-^\partial(m)) \cup \{\partial\}$, the state space of $(\underline{\mathbf{X}}_t^{(N)})_{t \geq 0}$
$(\mathfrak{X}_t^{N,+}, \mathfrak{X}_t^{N,-})$	the normalized empirical measure defined in Section 3.2
E_N	$\bigcup_{M=1}^N E_N^{(M)} \cup \{\mathbf{0}_*\}$, the state space of $(\mathfrak{X}_t^{N,+}, \mathfrak{X}_t^{N,-})_{t \geq 0}$
$M_+(E)$	space of finite nonnegative Borel measures on E , with weak topology
$M_{\leq 1}(E)$	$\{\mu \in M_+(E) : \mu(E) \leq 1\}$
\mathfrak{M}	$M_{\leq 1}(\overline{D}_+ \setminus \Lambda_+) \times M_{\leq 1}(\overline{D}_- \setminus \Lambda_-)$, see Section 5
$\{\mathcal{F}_t^X : t \geq 0\}$	filtration induced by the process (X_t) , that is, $\mathcal{F}_t^X = \sigma(X_s, s \leq t)$
$\mathbf{1}_x$	indicator function at x or the Dirac measure at x (depending on the context)
$\xrightarrow{\mathcal{L}}$	convergence in law of random variables (or processes)
$\langle f, \mu \rangle$	$\int f(x) \mu(dx)$
$x \vee y$	$\max\{x, y\}$
$x \wedge y$	$\min\{x, y\}$

3. Annihilating diffusion system. In this section, we fix $N \in \mathbb{N}$ and construct the configuration process $\underline{X}^{(N)}$ and the normalized empirical measure process $(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})$ for our annihilating diffusion system. In the construction, we will label (rather than annihilate) pairs of particles to keep track of the annihilated particles. This provides a *coupling* of our annihilating particle system and the corresponding system without annihilation.

Let $m \in \{1, 2, \dots, N\}$ (in fact, m can be any positive integer). Starting with m points in each of D_+^∂ and D_-^∂ , we perform the following construction.

Let $\{X_i^\pm = X_i^{\Lambda_\pm}\}_{i=1}^m$ be $(\mathbf{a}_\pm, \rho_\pm)$ -reflected diffusions on D_\pm killed upon hitting Λ_\pm , starting from the given points in D_\pm^∂ and are mutually independent. In case X_i^\pm starts at the cemetery point ∂^\pm , we set $X_i^\pm(t) = \partial^\pm$ for all $t \geq 0$. Let $\{R_k\}_{k=1}^m$ be i.i.d. exponential random variables with mean one which are independent of $\{X_i^+\}_{i=1}^m$ and $\{X_i^-\}_{i=1}^m$.

Define the first time of labeling (or annihilation) to be

$$(3.1) \quad \tau_1 := \inf \left\{ t \geq 0 : \frac{1}{2N} \int_0^t \sum_{i=1}^m \sum_{j=1}^m \ell_{\delta_N}(X_i^+(s), X_j^-(s)) ds \geq R_1 \right\},$$

with the convention $\inf \emptyset = \infty$. In the above, $\ell_{\delta_N}(x, y) = 0$ if either $x = \partial^+$ or $y = \partial^-$. Hence particles absorbed at Λ_\pm do not contribute to rate of labeling (or annihilation). It is possible that $\tau_1 = \infty$, which means that there is no annihilation between positive and negatives charges. However, $\lim_{m \rightarrow \infty} \mathbb{P}(\tau_1 = \infty) = 0$. On $\{\tau_1 < \infty\}$, we label at time τ_1 exactly one pair in $\{(i, j)\}$ according to the probability distribution given by

$$\frac{\ell_{\delta_N}(X_i^+(\tau_1-), X_j^-(\tau_1-))}{\sum_{p=1}^m \sum_{q=1}^m \ell_{\delta_N}(X_p^+(\tau_1-), X_q^-(\tau_1-))} \quad \text{assigned to } (i, j).$$

Denote (i_1, j_1) to be the labeled pair at τ_1 (think of the labeled pair as begin removed due to annihilation of the corresponding particles).

On $\{\tau_1 < \infty\}$, we repeat this labeling procedure using the remaining unlabeled $2(m - 1)$ particles. Precisely, for $k = 2, 3, \dots, m$, we define

$$\tau_k := \inf \left\{ t \geq 0 : \frac{1}{2N} \int_{\tau_1 + \dots + \tau_{k-1}}^{\tau_1 + \dots + \tau_{k-1} + t} \sum_{i \notin \{i_1, \dots, i_{k-1}\}} \sum_{j \notin \{j_1, \dots, j_{k-1}\}} \ell_{\delta_N}(X_i^+(s), X_j^-(s)) ds \geq R_k \right\}$$

on $\bigcap_{j=1}^{k-1} \{\tau_j < \infty\}$ and $\tau_k = \infty$ otherwise. Define $\sigma_k := \tau_1 + \tau_2 + \dots + \tau_k$. When $\sigma_k < \infty$, we label at σ_k the k th time of labeling (annihilation), exactly one pair (i_k, j_k) in

$$\{(i, j) : i \notin \{i_1, \dots, i_{k-1}\}, j \notin \{j_1, \dots, j_{k-1}\}\}$$

according to the probability distribution given by

$$\frac{\ell_{\delta_N}(X_i^+(\sigma_k-), X_i^-(\sigma_k-))}{\sum_{i \notin \{i_1, \dots, i_{k-1}\}} \sum_{j \notin \{j_1, \dots, j_{k-1}\}} \ell_{\delta_N}(X_i^+(\sigma_k-), X_i^-(\sigma_k-))} \quad \text{assigned to } (i, j).$$

We will study the evolution of the *unlabeled (or surviving)* particles in detail below.

3.1. *Configuration process.* We denote $D_{\pm}^{\partial}(m)$ the space of unordered m -tuples of elements in $D_{\pm}^{\partial} := (\overline{D}_{\pm} \setminus \Lambda^{\pm}) \cup \{\partial^{\pm}\}$. The *configuration space* for the particles is defined as

$$(3.2) \quad S_N := \bigcup_{m=1}^N (D_+^{\partial}(m) \times D_-^{\partial}(m)) \cup \{\partial\},$$

where ∂ is a cemetery point (different from ∂^{\pm}).

We define $\underline{\mathbf{X}}_t^{(N)} \in S_N$ to be the following unordered list of (the position of) *unlabeled* (surviving) particles at time t . That is,

$$\underline{\mathbf{X}}_t^{(N)} := \begin{cases} (\{X_1^+(t), \dots, X_m^+(t)\}, \{X_1^-(t), \dots, X_m^-(t)\}), & \text{if } t \in [0, \sigma_1 = \tau_1); \\ (\{X_i^+(t)\}_{i \notin \{i_1, \dots, i_{k-1}\}}, \{X_j^-(t)\}_{j \notin \{j_1, \dots, j_{k-1}\}}), & \text{if } t \in [\sigma_{k-1}, \sigma_k), \text{ for } k = 2, 3, \dots, m; \\ \partial, & \text{if } t \in [\sigma_m, \infty). \end{cases}$$

By definition, $\underline{\mathbf{X}}_t^{(N)} \in D_+^{\partial}(m-k+1) \times D_-^{\partial}(m-k+1)$ when $t \in [\sigma_{k-1}, \sigma_k)$, and $\underline{\mathbf{X}}_t^{(N)} = \partial$ if and only if all particles are labeled (annihilated) at time t (in particular, none of them is absorbed at Λ^{\pm}). We call $\underline{\mathbf{X}}^{(N)} = (\underline{\mathbf{X}}_t^{(N)})_{t \geq 0}$ the *configuration process*.

Denote $(\Omega, \mathcal{F}, \wp)$ the ambient probability space on which the above random objects $\{X_i^+\}_{i=1}^m$, $\{X_i^-\}_{j=1}^m$, $\{R_i\}_{i=1}^m$ and $\{(i_1, j_1), \dots, (i_m, j_m)\}$ are defined. For any $z \in S_N$, we define \mathbb{P}^z to be the conditional measure $\wp(\cdot | \underline{\mathbf{X}}_0^{(N)} = z)$. From the construction, we have

PROPOSITION 3.1. $\{\underline{\mathbf{X}}^{(N)}\}$ is a strong Markov processes under $\{\mathbb{P}^z : z \in S_N\}$.

The key is to note that the choice of (i_k, j_k) depends only on the value of $\underline{\mathbf{X}}_{\sigma_k-}^{(N)}$, and that

$$\tau_{k+1} = \inf\{t \geq 0 : A_t^{(k)} > R_{k+1}\}$$

$$\text{where } A_t^{(k)} = \frac{1}{2N} \int_{\sigma_k}^{\sigma_k+t} \sum_{i=1}^m \sum_{j=1}^m \ell_{\delta_N}(X_i^+(s), X_j^-(s)) ds.$$

Hence, $\underline{\mathbf{X}}^{(N)}$ is obtained through a patching procedure reminiscent to that of Ikeda, Nagasawa and Watanabe [26]. The proof is standard and is left to the reader.

3.2. *Normalized empirical process* $(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})$. Next, we consider $E_N := \bigcup_{M=1}^N E_N^{(M)} \cup \{\mathbf{0}_*\}$, where

$$E_N^{(M)} := \left\{ \left(\frac{1}{N} \sum_{i=1}^M \mathbf{1}_{x_i}, \frac{1}{N} \sum_{j=1}^M \mathbf{1}_{y_j} \right) : x_i \in D_+^{\partial}, y_j \in D_-^{\partial} \right\}$$

and $\mathbf{0}_*$ is an abstract point isolated from $\bigcup_{M=1}^N E_N^{(M)}$. We define the *normalized empirical measure* $(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})$ by

$$(3.3) \quad (\mathfrak{X}_t^{N,+}, \mathfrak{X}_t^{N,-}) := U_N(\underline{\mathbf{X}}_t^{(N)}),$$

where $U_N : S_N \rightarrow E_N$ is the canonical map given by $U_N(\partial) := \mathbf{0}_*$ and

$$U_N : (\underline{x}, \underline{y}) = (x_1, \dots, x_m, y_1, \dots, y_m) \mapsto \left(\frac{1}{N} \sum_{i=1}^m \mathbf{1}_{x_i}, \frac{1}{N} \sum_{j=1}^m \mathbf{1}_{y_j} \right).$$

For comparison, we also consider the empirical measure for the independent reflected diffusions *without* annihilation:

$$(3.4) \quad (\bar{\mathfrak{X}}^{N,+}, \bar{\mathfrak{X}}^{N,-}) := \left(\frac{1}{N} \sum_{i=1}^m \mathbf{1}_{X_i^+(t)}, \frac{1}{N} \sum_{j=1}^m \mathbf{1}_{X_j^-(t)} \right).$$

For any $\mu \in E_N$, we define \mathbb{P}^μ to be the conditional measure $\wp(\cdot | (\mathfrak{X}_0^{N,+}, \mathfrak{X}_0^{N,-}) = \mu)$. From Proposition 3.1, we have the following.

PROPOSITION 3.2. $\{(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})\}$ is a strong Markov processes under $\{\mathbb{P}^\mu : \mu \in E_N\}$.

4. Coupled heat equation with nonlinear boundary condition. Denote by $C_\infty([0, T]; \bar{D} \setminus \Lambda)$ the space of continuous functions on $[0, T]$ taking values in $C_\infty(\bar{D} \setminus \Lambda) := \{f \in C(\bar{D}) : f \text{ vanishes on } \Lambda\}$. We equip the Banach space $C_\infty([0, T]; \bar{D}_+ \setminus \Lambda_+) \times C_\infty([0, T]; \bar{D}_- \setminus \Lambda_-)$ with norm $\|(u, v)\| := \|u\|_\infty + \|v\|_\infty$, where $\|\cdot\|_\infty$ is the uniform norm. Using a probabilistic representation and the Banach fixed point theorem in the same way as we did in the proof of the existence and uniqueness result for the PDE in [9], Propostion 2.19, we have the following.

PROPOSITION 4.1. Let $T > 0$ and $u_0^\pm \in C_\infty(\bar{D}_\pm \setminus \Lambda_\pm)$. Then there is a unique element $(u_+, u_-) \in C_\infty([0, T]; \bar{D}_+ \setminus \Lambda_+) \times C_\infty([0, T]; \bar{D}_- \setminus \Lambda_-)$ that

satisfies the coupled integral equation

$$(4.1) \quad \begin{cases} u_+(t, x) = P_t^{\Lambda^+} u_0^+(x) \\ \quad - \frac{1}{2} \int_0^t \int_I p^{\Lambda^+}(t-r, x, z) [\lambda(z) u_+(r, z) u_-(r, z)] d\sigma(z) dr, \\ u_-(t, y) = P_t^{\Lambda^-} u_0^-(y) \\ \quad - \frac{1}{2} \int_0^t \int_I p^{\Lambda^-}(t-r, y, z) [\lambda(z) u_+(r, z) u_-(r, z)] d\sigma(z) dr. \end{cases}$$

Moreover, (u_+, u_-) satisfies

$$(4.2) \quad \begin{cases} u_+(t, x) = \mathbb{E}^x \left[u_0^+(X_t^{\Lambda^+}) \exp \left(- \int_0^t (\lambda \cdot u_-)(t-s, X_s^{\Lambda^+}) dL_s^{I,+} \right) \right], \\ u_-(t, y) = \mathbb{E}^y \left[u_0^-(X_t^{\Lambda^-}) \exp \left(- \int_0^t (\lambda \cdot u_+)(t-s, X_s^{\Lambda^-}) dL_s^{I,-} \right) \right], \end{cases}$$

where $L^{I,\pm}$ is the boundary local time of X^{Λ^\pm} on the interface I , that is, the positive continuous additive functional having Revuz measure $\sigma|_I$, the surface measure σ restricted to I .

DEFINITION 4.2. Motivated by the probabilistic representation (4.2), we call the unique solution $(u_+, u_-) \in C_\infty([0, T]; \overline{D}_+ \setminus \Lambda_+) \times C_\infty([0, T]; \overline{D}_- \setminus \Lambda_-)$ of (4.1) the *probabilistic solution* to the following coupled PDEs starting from (u_0^+, u_0^-) :

$$(4.3) \quad \begin{cases} \frac{\partial u_+}{\partial t} = \mathcal{A}^+ u_+, & \text{on } (0, \infty) \times D_+, \\ u_+ = 0, & \text{on } (0, \infty) \times \Lambda_+, \\ \frac{\partial u_+}{\partial \vec{v}_+} = \frac{\lambda}{\rho_+} u_+ u_- \mathbf{1}_{\{I\}}, & \text{on } (0, \infty) \times \partial D_+ \setminus \Lambda_+ \end{cases}$$

and

$$(4.4) \quad \begin{cases} \frac{\partial u_-}{\partial t} = \mathcal{A}^- u_-, & \text{on } (0, \infty) \times D_-, \\ u_- = 0, & \text{on } (0, \infty) \times \Lambda_-, \\ \frac{\partial u_-}{\partial \vec{v}_-} = \frac{\lambda}{\rho_-} u_+ u_- \mathbf{1}_{\{I\}}, & \text{on } (0, \infty) \times \partial D_- \setminus \Lambda_-, \end{cases}$$

where $\vec{v}_\pm := \mathbf{a}_\pm \vec{n}_\pm$ is the inward conormal vector field on ∂D_\pm . Here, $\mathbf{1}_{\{I\}}$ is the indicator function of I .

It can be shown that the pair of continuous functions (u_+, u_-) satisfying (4.1) is weakly differentiable and satisfies the PDEs (4.3)–(4.4) in the distributional sense (see Section 3 of [12]); however, we do not need this property in this paper. Our method only requires continuity of u_+ and u_- .

5. Main result: Rigorous statement. Denote by $M_{\leq 1}(\overline{D}_{\pm} \setminus \Lambda_{\pm})$ the space of nonnegative Borel measures on $\overline{D}_{\pm} \setminus \Lambda_{\pm}$ with mass at most 1 and set

$$\mathfrak{M} := M_{\leq 1}(\overline{D}_+ \setminus \Lambda_+) \times M_{\leq 1}(\overline{D}_- \setminus \Lambda_-),$$

equipped with the topology of weak convergence. Regard $\mathbf{1}_{\partial_{\pm}}$ as $\mathbf{0}^{\pm}$ and $\mathbf{0}_*$ as $(\mathbf{0}^+, \mathbf{0}^-)$, where $\mathbf{0}^{\pm}$ is the zero measure on \overline{D}_{\pm} , respectively. Then $E_N \subset \mathfrak{M}$ for all N , and the processes $(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})$ have sample paths in $D([0, \infty), \mathfrak{M})$, the Skorokhod space of càdlàg paths in \mathfrak{M} .

We can now rigorously state our main result. In what follows, $\xrightarrow{\mathcal{L}}$ denotes convergence in law.

THEOREM 5.1 (Hydrodynamic limit). *Suppose that Assumptions 2.4 to 2.6 hold. If as $N \rightarrow \infty$, $(\mathfrak{X}_0^{N,+}, \mathfrak{X}_0^{N,-}) \xrightarrow{\mathcal{L}} (u_+^0(x)\rho_+(x) dx, u_-^0(y)\rho_-(y) dy)$ in \mathfrak{M} , where $u_{\pm}^0 \in C_{\infty}(\overline{D}_{\pm} \setminus \Lambda_{\pm})$, then*

$$(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-}) \xrightarrow{\mathcal{L}} (u_+(t, x)\rho_+(x) dx, u_-(t, y)\rho_-(y) dy) \quad \text{in } D([0, T], \mathfrak{M})$$

for any $T > 0$, where (u_+, u_-) is the probabilistic solution of (4.3)–(4.4) with initial value (u_+^0, u_-^0) .

REMARK 5.2. \mathfrak{M} is in fact a Polish space. Let $\{f_n; n \geq 1\}$ and $\{g_n; n \geq 1\}$ be sequences of continuous functions with $|f_n| \leq 1$ and $|g_n| \leq 1$ whose linear span are dense in $C_{\infty}(\overline{D}_+ \setminus \Lambda_+)$ and $C_{\infty}(\overline{D}_- \setminus \Lambda_-)$, respectively. For $\mu = (\mu_+, \mu_-)$ and $\nu = (\nu_+, \nu_-)$ in \mathfrak{M} , define

$$\varrho(\mu, \nu) := \sum_{n=1}^{\infty} 2^{-n} \left(\left| \int_{\overline{D}_+} f_n(x)(\mu_+ - \nu_+)(dx) \right| + \left| \int_{\overline{D}_-} g_n(y)(\mu_- - \nu_-)(dy) \right| \right).$$

It is well known that \mathfrak{M} is a complete separable metric space under the metric ϱ .

As mentioned in Remark 1.2 in the [Introduction](#), an assumption on the rate at which δ_N tends to zero, such as Assumption 2.7, is *necessary* for Theorem 5.1 to hold. Below is a counter-example.

EXAMPLE 5.3. Suppose that $\{X_i^+(t)\}_{i=1}^{\infty}$ and $\{X_j^-(t)\}_{j=1}^{\infty}$ are RBMs on \overline{D}_+ and \overline{D}_- , respectively, and they are all mutually independent. Note that X_i^+ and X_j^- never meet in the sense that

$$(5.1) \quad \mathbb{P}(X_i^+(t) = X_j^-(t) \text{ for some } t \in [0, \infty) \text{ and } i, j \in \{1, 2, 3, \dots\}) = 0.$$

This implies that there exists $\{\delta_N\}$ so that $\sum_{N=1}^{\infty} \alpha_N < \infty$, where

$$(5.2) \quad \alpha_N := \mathbb{P}((X_i^+(t), X_j^-(t)) \in I^{\delta_N} \text{ for some } t \in [0, \infty) \text{ and } i, j \in \{1, 2, \dots, N\}).$$

Hence, by the Borel–Cantelli lemma, we know that with probability 1, there will be no annihilation for the particle system (which occurs only when a pair of particles are in I^{δ_N}) when N is sufficiently large. In this case, $(\mathfrak{X}_t^{N,+}, \mathfrak{X}_t^{N,-})$ converges to $(P_t^+ u_0^+(x) dx, P_t^- u_0^-(y) dy)$ in distribution in $D([0, T], \mathfrak{M})$ instead, provided that $(\mathfrak{X}_0^{N,+}, \mathfrak{X}_0^{N,-})$ converges to $(u_0^+(x) dx, u_0^-(y) dy)$ in distribution in \mathfrak{M} .

Question. We will see from Theorem 6.6 below that the tightness of $(\mathfrak{X}_t^{N,+}, \mathfrak{X}_t^{N,-})$ holds without Assumption 2.7. Can we characterize all limit points of $(\mathfrak{X}_t^{N,+}, \mathfrak{X}_t^{N,-})$ without Assumption 2.7? Is $\liminf_{N \rightarrow \infty} N \delta_N^d \in (0, \infty]$ the sharpest condition for Theorem 5.1 to hold?

6. Hydrodynamic limit. Recall that Assumptions 2.4 to 2.6 are in force throughout this paper.

6.1. *Martingales and tightness.* In this subsection, we present some key martingales that are used to establish tightness of $(\mathfrak{X}_t^{N,+}, \mathfrak{X}_t^{N,-})$. More martingales related to the time dependent process $(t, (\mathfrak{X}_t^{N,+}, \mathfrak{X}_t^{N,-}))$ will be given in Section 7.2.

6.1.1. *Martingales for reflected diffusions.* We will need the following collection of fundamental martingales, together with their quadratic variations, for reflected diffusions.

LEMMA 6.1. *Suppose X^Λ is an (\mathbf{a}, ρ) -reflected diffusion in a bounded Lipschitz domain D killed upon hitting Λ . Suppose all assumptions in Proposition 2.3 hold, and f is in the domain of the Feller generator $\text{Dom}(\mathcal{A}^{(\Lambda)})$. Then*

$$(6.1) \quad M(t) := f(X^\Lambda(t)) - f(X^\Lambda(0)) - \int_0^t \mathcal{A}^{(\Lambda)} f(X^\Lambda(s)) ds$$

is a $\mathcal{F}_t^{X^\Lambda}$ -martingale that is bounded on each compact time interval and has predictable quadratic variation $\langle M \rangle_t := \int_0^t (\mathbf{a} \nabla f \cdot \nabla f)(X^\Lambda(s)) ds$ under \mathbb{P}^x for any $x \in \bar{D}$. Moreover, if X_1 and X_2 are independent copies of X^Λ , and if M_i is the above M with X^Λ replaced by X_i , then the cross variation $\langle M_1, M_2 \rangle_t = 0$.

PROOF. For $f \in \text{Dom}(\mathcal{A}^{(\Lambda)})$, $M(t)$ defined in (6.1) is an $\mathcal{F}_t^{X^\Lambda}$ -martingale that is bounded on each compact time interval. Since D is bounded, f is clearly in the domain of the L^2 -generator of X^Λ . Hence, it follows from the Fukushima decomposition of $f(X_t^\Lambda)$ (see Theorems 4.2.6 and 4.3.11 of [11]), that $M(t)$ is a martingale additive functional of X^Λ of finite energy having quadratic variation $\langle M(t) \rangle_t = \int_0^t (\mathbf{a} \nabla f \cdot \nabla f)(X^\Lambda(s)) ds$. If X_1 and X_2 are independent copies of X^Λ , then M_1 and M_2 are independent and so $\langle M_1, M_2 \rangle = 0$. \square

An immediate consequence of Lemma 6.1 is

$$(6.2) \quad \int_0^t P_s^\Lambda (\mathbf{a} \nabla f \cdot \nabla f)(x) ds = \mathbb{E}^x [M(t)^2] \leq 8(\|f\|^2 + \|\mathcal{A}^{(\Lambda)} f\|^2 t^2)$$

for $x \in \overline{D}$,

where $\|g\|$ is the uniform norm of g on \overline{D} .

6.1.2. Martingales for annihilating diffusion system.

THEOREM 6.2. *Fix any positive integer N . Suppose $F \in C_b(E_N)$ is a bounded continuous function and $G \in \mathcal{B}(E_N)$ is a Borel measurable function on E_N such that*

$$\overline{M}_t := F(\overline{\mathfrak{X}}_t^{N,+}, \overline{\mathfrak{X}}_t^{N,-}) - \int_0^t G(\overline{\mathfrak{X}}_s^{N,+}, \overline{\mathfrak{X}}_s^{N,-}) ds$$

is an $\mathcal{F}_t^{(\overline{\mathfrak{X}}^{N,+}, \overline{\mathfrak{X}}^{N,-})}$ -martingale under \mathbb{P}^μ for any $\mu \in E_N$. Then

$$M_t := F(\mathfrak{X}_t^{N,+}, \mathfrak{X}_t^{N,-}) - \int_0^t (G + KF)(\mathfrak{X}_s^{N,+}, \mathfrak{X}_s^{N,-}) ds$$

is a $\mathcal{F}_t^{(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})}$ -martingale under \mathbb{P}^μ for any $\mu \in E_N$, where

$$(6.3) \quad \begin{aligned} KF(v) := & -\frac{1}{2N} \sum_{i=1}^M \sum_{j=1}^M \ell_{\delta_N}(x_i, y_j) \\ & \times (F(v) - F(v^+ - N^{-1} \mathbf{1}_{\{x_i\}}, v^- - N^{-1} \mathbf{1}_{\{y_j\}})) \end{aligned}$$

whenever $v = (\frac{1}{N} \sum_{i=1}^M \mathbf{1}_{\{x_i\}}, \frac{1}{N} \sum_{j=1}^M \mathbf{1}_{\{y_j\}}) \in E_N^{(M)}$, and $KF(\mathbf{0}_*) := 0$.

REMARK 6.3. (i) Theorem 6.2 indicates the infinitesimal generator of $(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})$ on $C_b(E_N)$ is given by $\overline{L} + K$, where \overline{L} is the infinitesimal generator of $(\overline{\mathfrak{X}}^{N,+}, \overline{\mathfrak{X}}^{N,-})$ on $C_b(E_N)$. Note that G is merely assumed to be Borel measurable, the above provides us with a broader class of martingales (such as $N_t^{(\phi_+, \phi_-)}$ in Corollary 6.4) than from using the $C_b(E_N)$ -generator.

(ii) Theorem 6.2 can be generalized to deal with time-dependent functions $F_s \in C_b(E_N)$ ($s \geq 0$). See Theorem 7.7 in Section 7.2.

PROOF OF THEOREM 6.2. We adopt the abbreviation $\mathfrak{X} := (\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})$ when there is no confusion. In particular, we write $\mathcal{F}_t^{\mathfrak{X}}$ in place of $\mathcal{F}_t^{(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})}$. By the Markov property for \mathfrak{X} , it suffices to show that for all $t \geq 0$ and $v \in E_N$,

$$(6.4) \quad \mathbb{E}^v \left[F(\mathfrak{X}_t) - F(\mathfrak{X}_0) - \int_0^t (G + KF)(\mathfrak{X}_s) ds \right] = 0.$$

The idea is to split the time interval $[0, t]$ into pieces according to the jumping times of $F(\mathfrak{X}_s)$ ($s \in [0, t]$) caused by annihilation (excluding the jumps caused by absorption at the harvest sites Λ^\pm), then apply \overline{M} in each piece and take into account the jump distributions.

Suppose $\nu = (\nu^+, \nu^-) = (\frac{1}{N} \sum_{i=1}^m \mathbf{1}_{x_i}, \frac{1}{N} \sum_{j=1}^m \mathbf{1}_{y_j}) \in E_N^{(m)}$. Recall that $\sigma_i := \tau_1 + \dots + \tau_i$ ($i = 1, 2, \dots, m$) is the time of the i th labeling (annihilation) of particles. Then

$$(6.5) \quad \begin{aligned} & F(\mathfrak{X}_t) - F(\mathfrak{X}_0) \\ &= \sum_{i=0}^m (F(\mathfrak{X}_{(t \wedge \sigma_{i+1})^-}) - F(\mathfrak{X}_{t \wedge \sigma_i})) + \sum_{j=1}^m (F(\mathfrak{X}_{t \wedge \sigma_j}) - F(\mathfrak{X}_{(t \wedge \sigma_j)^-})), \end{aligned}$$

where $\sigma_0 := 0$, $\sigma_{m+1} := \infty$ and $\mathfrak{X}_{s-} := \lim_{r \nearrow s} \mathfrak{X}_r$. Hence, it suffices to show that

$$(6.6) \quad \mathbb{E}^\nu \left[F(\mathfrak{X}_{(t \wedge \sigma_{i+1})^-}) - F(\mathfrak{X}_{t \wedge \sigma_i}) - \int_{t \wedge \sigma_i}^{t \wedge \sigma_{i+1}} G(\mathfrak{X}_s) ds \right] = 0 \quad \text{and}$$

$$(6.7) \quad \mathbb{E}^\nu \left[F(\mathfrak{X}_{t \wedge \sigma_j}) - F(\mathfrak{X}_{(t \wedge \sigma_j)^-}) - \int_{t \wedge \sigma_{j-1}}^{t \wedge \sigma_j} K F(\mathfrak{X}_s) ds \right] = 0$$

for $i \in \{0, 1, 2, \dots, m\}$ and $j \in \{1, 2, \dots, m\}$.

The left-hand side of (6.6) equals

$$\begin{aligned} & \mathbb{E}^\nu \left[\mathbb{E}^\nu \left[F(\mathfrak{X}_{(t \wedge \sigma_{i+1})^-}) - F(\mathfrak{X}_{t \wedge \sigma_i}) - \int_{t \wedge \sigma_i}^{t \wedge \sigma_{i+1}} G(\mathfrak{X}_s) ds \middle| \mathcal{F}_{t \wedge \sigma_i}^{\mathfrak{X}} \right] \right] \\ &= \mathbb{E}^\nu \left[\mathbb{E}^{\mathfrak{X}_{\sigma_i}} \left[F(\mathfrak{X}_{(t \wedge \sigma_{i+1} - \sigma_i)^-}) - F(\mathfrak{X}_0) - \int_0^{t \wedge \sigma_{i+1} - \sigma_i} G(\mathfrak{X}_s) ds \right] \mathbf{1}_{t > \sigma_i} \right] \\ &= \mathbb{E}^\nu \left[\mathbb{E}^{\mathfrak{X}_{\sigma_i}} \left[F(\mathfrak{X}_{((t - \sigma_i) \wedge \tau_{i+1})^-}) - F(\mathfrak{X}_0) - \int_0^{(t - \sigma_i) \wedge \tau_{i+1}} G(\mathfrak{X}_s) ds \right] \mathbf{1}_{t > \sigma_i} \right]. \end{aligned}$$

The first equality follows from the strong Markov property of \mathfrak{X} (applied to the stopping time σ_i) and the fact that the expression inside the expectation vanishes when $t \leq \sigma_i$. Note that σ_i is regarded as a constant w.r.t. the expectation $\mathbb{E}^{\mathfrak{X}_{\sigma_i}}$, because $\mathcal{F}_{\sigma_i}^{\mathfrak{X}}$ contains the sigma-algebra generated by σ_i . The second equality follows from the easy fact that $(t \wedge \sigma_{i+1}) - \sigma_i = (t - \sigma_i) \wedge (\sigma_{i+1} - \sigma_i) = (t - \sigma_i) \wedge \tau_{i+1}$ on $t > \sigma_i$. Therefore, to establish (6.6), it is enough to show that for any $\eta \in E_N$ and $w \geq 0$, we have

$$(6.8) \quad \mathbb{E}^\eta \left[F(\mathfrak{X}_{(w \wedge \tau)^-}) - F(\mathfrak{X}_0) - \int_0^{w \wedge \tau} G(\mathfrak{X}_s) ds \right] = 0,$$

where τ is the time of the first annihilation for \mathfrak{X} starting from η [i.e., $\tau = \tau_1$ under \mathbb{P}^η where τ_1 is defined by (3.1)].

Equation (6.8) obviously holds if η is the zero measure since both sides vanish. Suppose $\eta \in E_N^{(n)}$. Observe that τ is a stopping time for $\tilde{\mathcal{F}}_t^{\mathfrak{X}} := \sigma(\mathcal{F}_t^{\mathfrak{X}}, \{R_i; 1 \leq$

$i \leq n$) and that \overline{M}_t is a $\tilde{\mathcal{F}}_t^{\overline{\mathfrak{X}}}$ -martingale under \mathbb{P}^η since $\{R_i\}$ is independent of $\overline{\mathfrak{X}}$ under \mathbb{P}^η . Hence, by the optional sampling theorem, (6.8) is true, and so is (6.6).

Following the same arguments as above, the left-hand side of (6.7) equals

$$\begin{aligned} & \mathbb{E}^\nu \left[\mathbb{E}^{\mathfrak{X}_{\sigma_{j-1}}} \left[F(\mathfrak{X}_{(t-\sigma_{j-1}) \wedge \tau_j}) - F(\mathfrak{X}_{((t-\sigma_{j-1}) \wedge \tau_j)-}) \right. \right. \\ & \quad \left. \left. + \int_0^{(t-\sigma_{j-1}) \wedge \tau_j} KF(\mathfrak{X}_s) ds \right] \mathbf{1}_{t > \sigma_{j-1}} \right], \end{aligned}$$

where σ_{j-1} is regarded as a constant w.r.t. the expectation $\mathbb{E}^{\mathfrak{X}_{\sigma_{j-1}}}$. Therefore, (6.7) holds if for any $\eta \in E_N$ and $\theta \geq 0$, we have

$$(6.9) \quad \mathbb{E}^\eta \left[F(\mathfrak{X}_{\theta \wedge \tau}) - F(\mathfrak{X}_{(\theta \wedge \tau)-}) - \int_0^{\theta \wedge \tau} KF(\mathfrak{X}_s) ds \right] = 0,$$

where τ is the time of the first killing for \mathfrak{X} starting from η .

Suppose $\eta = (\frac{1}{N} \sum_{i=1}^n \mathbf{1}_{x_i}, \frac{1}{N} \sum_{j=1}^n \mathbf{1}_{y_j}) \in E_N^{(n)}$ and $\mathfrak{X}_{\tau-} = (\frac{1}{N} \sum_{i=1}^n \mathbf{1}_{X_i^+(\tau-)}, \frac{1}{N} \sum_{j=1}^n \mathbf{1}_{X_j^-(\tau-)})$, where $\{X_k^\pm : k = 1, \dots, n\}$ are reflected diffusions killed upon hitting Λ^\pm in the construction of \mathfrak{X} . At time τ , one pair of particles among $\{(X_i^+, X_j^-) : 1 \leq i, j \leq n\}$ is labeled (annihilated), where the pair (X_i^+, X_j^-) is chosen to be labeled (annihilated) with probability $\frac{\ell_{\delta_N}(X_i^+(\tau-), X_j^-(\tau-))}{\sum_{p=1}^n \sum_{q=1}^n \ell_{\delta_N}(X_p^+(\tau-), X_q^-(\tau-))}$. Hence,

$$\begin{aligned} & \mathbb{E}^\eta [F(\mathfrak{X}_{(\theta \wedge \tau)-}) - F(\mathfrak{X}_{\theta \wedge \tau})] \\ &= \mathbb{E}^\eta [\mathbb{E}^\eta [F(\mathfrak{X}_{\tau-}) - F(\mathfrak{X}_\tau) | \mathcal{F}_{\tau-}^\mathfrak{X}]; \tau < \theta] \\ &= \mathbb{E}^\eta \left[\sum_{i=1}^n \sum_{j=1}^n \frac{\ell_{\delta_N}(X_i^+(\tau-), X_j^-(\tau-))}{\sum_{p=1}^n \sum_{q=1}^n \ell_{\delta_N}(X_p^+(\tau-), X_q^-(\tau-))} \right. \\ & \quad \left. \times \left(F(\mathfrak{X}_{\tau-}) - F\left(\mathfrak{X}_{\tau-} - \left(\frac{1}{N} \mathbf{1}_{X_i^+(\tau-)}, \frac{1}{N} \mathbf{1}_{X_j^-(\tau-)}\right)\right) \right); \tau < \theta \right] \\ &= \mathbb{E}^\eta \left[\frac{-(2N)KF(\mathfrak{X}_{\tau-})}{\sum_{p=1}^n \sum_{q=1}^n \ell_{\delta_N}(X_p^+(\tau-), X_q^-(\tau-))}; \tau < \theta \right] \\ &= \mathbb{E}^\eta \left[\int_0^{\theta \wedge \tau} -KF(\mathfrak{X}_s) ds \right]. \end{aligned}$$

The last equality follows from the fact that

$$\tau = \inf \left\{ t \geq 0 : \frac{1}{2N} \int_0^t \sum_{p=1}^n \sum_{q=1}^n \ell_{\delta_N}(X_p^+(s), X_q^-(s)) ds \geq R \right\},$$

where R is an independent exponential random variable of mean 1 under \mathbb{P}^η (see Proposition 2.2 of [13] for a rigorous proof). Hence, (6.9) is established and the proof is complete. \square

The following corollary is the key to the tightness of $(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})$. Recall that \mathcal{A}^\pm is the Feller generator of the diffusion $X^\pm = X^{\Lambda^\pm}$ on $\overline{D}_\pm \setminus \Lambda_\pm$, respectively.

COROLLARY 6.4. *Fix any positive integer N . For any $\phi_\pm \in \text{Dom}(\mathcal{A}^\pm)$, we have*

$$\begin{aligned} M_t^{(\phi_+, \phi_-)} &:= \langle \phi_+, \mathfrak{X}_t^{N,+} \rangle + \langle \phi_-, \mathfrak{X}_t^{N,-} \rangle \\ &\quad - \int_0^t \langle \mathcal{A}^+ \phi_+, \mathfrak{X}_s^{N,+} \rangle + \langle \mathcal{A}^- \phi_-, \mathfrak{X}_s^{N,-} \rangle \\ &\quad - \frac{1}{2} \langle \ell_{\delta_N}(\phi_+ + \phi_-), \mathfrak{X}_s^{N,+} \otimes \mathfrak{X}_s^{N,-} \rangle ds \end{aligned}$$

is an $\mathcal{F}_t^{(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})}$ -martingale under \mathbb{P}^μ for any $\mu \in E_N$, where

$$\langle f(x, y), \mu^+(dx) \otimes \mu^-(dy) \rangle := \frac{1}{N^2} \sum_i \sum_j f(x_i, y_j)$$

$$\text{whenever } \mu = \left(N^{-1} \sum_i \mathbf{1}_{x_i}, N^{-1} \sum_j \mathbf{1}_{y_j} \right).$$

Moreover, $M_t^{(\phi_+, \phi_-)}$ has predictable quadratic variation

$$\begin{aligned} \langle M^{(\phi_+, \phi_-)} \rangle_t &= \frac{1}{N} \int_0^t \left(\langle \mathbf{a}_+ \nabla \phi_+ \cdot \nabla \phi_+, \mathfrak{X}_s^{N,+} \rangle + \langle \mathbf{a}_- \nabla \phi_- \cdot \nabla \phi_-, \mathfrak{X}_s^{N,-} \rangle \right. \\ (6.10) \quad &\quad \left. + \frac{1}{2} \langle \ell_{\delta_N}(\phi_+ + \phi_-)^2, \mathfrak{X}_s^{N,+} \otimes \mathfrak{X}_s^{N,-} \rangle \right) ds \end{aligned}$$

and $\sup_{t \in [0, T]} \mathbb{E}^\mu [(M_t^{(\phi_+, \phi_-)})^2] \leq \frac{C}{N}$ for some constant C that is independent of N and μ .

PROOF. From Lemma 6.1, we have the following two $\mathcal{F}_t^{(\overline{\mathfrak{X}}^{N,+}, \overline{\mathfrak{X}}^{N,-})}$ -martingales for $\phi_\pm \in \text{Dom}(\mathcal{A}^\pm)$:

$$\begin{aligned} \overline{M}_t^{(\phi_+, \phi_-)} &:= \langle \phi_+, \overline{\mathfrak{X}}_t^{N,+} \rangle + \langle \phi_-, \overline{\mathfrak{X}}_t^{N,-} \rangle - \int_0^t \langle \mathcal{A}^+ \phi_+, \overline{\mathfrak{X}}_s^{N,+} \rangle \\ &\quad + \langle \mathcal{A}^- \phi_-, \overline{\mathfrak{X}}_s^{N,-} \rangle ds \quad \text{and} \\ \overline{N}_t^{(\phi_+, \phi_-)} &:= (\langle \phi_+, \overline{\mathfrak{X}}_t^{N,+} \rangle + \langle \phi_-, \overline{\mathfrak{X}}_t^{N,-} \rangle)^2 \end{aligned}$$

$$\begin{aligned}
& - \int_0^t 2(\langle \phi_+, \bar{\mathfrak{X}}_s^{N,+} \rangle + \langle \phi_-, \bar{\mathfrak{X}}_s^{N,-} \rangle)(\langle \mathcal{A}^+ \phi_+, \bar{\mathfrak{X}}_s^{N,+} \rangle + \langle \mathcal{A}^- \phi_-, \bar{\mathfrak{X}}_s^{N,-} \rangle) \\
& + \frac{1}{N} (\langle \mathbf{a}_+ \nabla \phi_+ \cdot \nabla \phi_+, \bar{\mathfrak{X}}_s^{N,+} \rangle + \langle \mathbf{a}_- \nabla \phi_- \cdot \nabla \phi_-, \bar{\mathfrak{X}}_s^{N,-} \rangle) ds.
\end{aligned}$$

Note that $F_1(\mu) = F_1(\mu^+, \mu^-) := \langle \phi_+, \mu^+ \rangle + \langle \phi_-, \mu^- \rangle$ is a function in $C(E_N)$, with the convention that $\phi_{\pm}(\partial^{\pm}) := 0$ and $F_1(\mathbf{0}_*) := 0$. A direct calculation shows that

$$KF_1(\mu) = \frac{-1}{2} \langle \ell_{\delta_N}(\phi_+ + \phi_-), \mu^+ \otimes \mu^- \rangle.$$

Therefore, by Theorem 6.2, $M_t^{(\phi_+, \phi_-)}$ is a martingale. Similarly, $F_2(\mu) := (\langle \phi_+, \mu^+ \rangle + \langle \phi_-, \mu^- \rangle)^2 \in C(E_N)$ and

$$\begin{aligned}
KF_2(\mu) &= -(\langle \phi_+, \mu^+ \rangle + \langle \phi_-, \mu^- \rangle) \langle \ell_{\delta_N}(\phi_+ + \phi_-), \mu^+ \otimes \mu^- \rangle \\
&+ \frac{1}{2N} \langle \ell_{\delta_N}(\phi_+ + \phi_-)^2, \mu^+ \otimes \mu^- \rangle.
\end{aligned}$$

Hence, Theorem 6.2 asserts that

$$\begin{aligned}
N_t^{(\phi_+, \phi_-)} &:= (\langle \phi_+, \mathfrak{X}_t^{N,+} \rangle + \langle \phi_-, \mathfrak{X}_t^{N,-} \rangle)^2 \\
&- \int_0^t 2(\langle \phi_+, \mathfrak{X}_s^{N,+} \rangle + \langle \phi_-, \mathfrak{X}_s^{N,-} \rangle) \\
&\times (\langle \mathcal{A}^+ \phi_+, \mathfrak{X}_s^{N,+} \rangle + \langle \mathcal{A}^- \phi_-, \mathfrak{X}_s^{N,-} \rangle) \\
&+ \frac{1}{N} (\langle \mathbf{a}_+ \nabla \phi_+ \cdot \nabla \phi_+, \mathfrak{X}_s^{N,+} \rangle + \langle \mathbf{a}_- \nabla \phi_- \cdot \nabla \phi_-, \mathfrak{X}_s^{N,-} \rangle) \\
&- (\langle \phi_+, \mathfrak{X}_s^{N,+} \rangle + \langle \phi_-, \mathfrak{X}_s^{N,-} \rangle) \langle \ell_{\delta_N}(\phi_+ + \phi_-), \mathfrak{X}_s^{N,+} \otimes \mathfrak{X}_s^{N,-} \rangle \\
&+ \frac{1}{2N} \langle \ell_{\delta_N}(\phi_+ + \phi_-)^2, \mathfrak{X}_s^{N,+} \otimes \mathfrak{X}_s^{N,-} \rangle ds
\end{aligned}$$

is a martingale. Denote Θ_t to be the expression on the right-hand side of (6.10). We claim that $(M_t^{(\phi_+, \phi_-)})^2 - N_t^{(\phi_+, \phi_-)} - \Theta_t$ is a martingale. By definition, $M_t^{(\phi_+, \phi_-)} = f_t - \int_0^t g_s ds$, where

$$\begin{aligned}
f_t &= \langle \phi_+, \mathfrak{X}_t^{N,+} \rangle + \langle \phi_-, \mathfrak{X}_t^{N,-} \rangle, \\
g_s &= \langle \mathcal{A}^+ \phi_+, \mathfrak{X}_s^{N,+} \rangle + \langle \mathcal{A}^- \phi_-, \mathfrak{X}_s^{N,-} \rangle - \frac{1}{2} \langle \ell_{\delta_N}(\phi_+ + \phi_-), \mathfrak{X}_s^{N,+} \otimes \mathfrak{X}_s^{N,-} \rangle.
\end{aligned}$$

Then $N_t^{(\phi_+, \phi_-)} = f_t^2 - 2 \int_0^t f_s g_s ds - \Theta_t$ and

$$\begin{aligned}
& (M_t^{(\phi_+, \phi_-)})^2 - N_t^{(\phi_+, \phi_-)} - \Theta_t \\
&= \left(\int_0^t g_s ds \right)^2 - 2 f_t \int_0^t g_s ds + 2 \int_0^t f_s g_s ds
\end{aligned}$$

$$\begin{aligned}
 &= \left(\int_0^t g_s ds \right)^2 - 2 \int_0^t G_s (dM_s^{(\phi_+, \phi_-)} - 2g_s ds) + [f, G]_t \\
 &= -2 \int_0^t G_s dM_s^{(\phi_+, \phi_-)},
 \end{aligned}$$

which is a martingale, where $G_t = \int_0^t g_s ds$ and $[f, G]_t$ is the quadratic covariation (also called square bracket process) of f and G . The second equality follows from integration by parts applied to $f_t G_t$. See, for example, Corollary 2 in Chapter 2, Section 6 of [32]. In the last equality, we used the fact that $[f, G]_t = 0$ since G has bounded variation. Therefore, $(M_t^{(\phi_+, \phi_-)})^2 - \Theta_t$ is a martingale. Since Θ_t is a continuous process of finite variation, we have $\langle M^{(\phi_+, \phi_-)} \rangle_t = \Theta_t$, proving (6.10).

Clearly, (6.10) implies that

$$\begin{aligned}
 &\mathbb{E}^\mu [(M_t^{(\phi_+, \phi_-)})^2] \\
 &= \mathbb{E}^\mu [\langle M^{(\phi_+, \phi_-)} \rangle_t] \\
 &\leq \frac{1}{N} \left(\int_0^t \langle P_s^+ (\mathbf{a}_+ \nabla \phi_+ \cdot \nabla \phi_+), \mu^+ \rangle ds + \int_0^t \langle P_s^- (\mathbf{a}_- \nabla \phi_- \cdot \nabla \phi_-), \mu^- \rangle ds \right. \\
 &\quad \left. + \frac{1}{2} \|(\phi_+ + \phi_-)^2\| \int_0^t \mathbb{E}^\mu [\ell_{\delta_N}, \mathfrak{X}_s^{N,+} \otimes \mathfrak{X}_s^{N,-}] ds \right) \\
 &\leq \frac{1}{N} \left(8(\|\phi_+\|^2 + \|\mathcal{A}^+ \phi_+\|^2 t^2) + 8(\|\phi_-\|^2 + \|\mathcal{A}^- \phi_-\|^2 t^2) \right. \\
 &\quad \left. + \frac{1}{2} \|(\phi_+ + \phi_-)^2\| \int_0^t \mathbb{E}^\mu [\ell_{\delta_N}, \mathfrak{X}_s^{N,+} \otimes \mathfrak{X}_s^{N,-}] ds \right),
 \end{aligned}$$

where we have used (6.2) in the last inequality. Finally, we show that

$$(6.11) \quad \sup_{\mu \in \tilde{E}_N} \int_0^t \mathbb{E}^\mu [\ell_{\delta_N}, \mathfrak{X}_s^{N,+} \otimes \mathfrak{X}_s^{N,-}] ds \leq 1.$$

Let $(\tilde{\mathfrak{X}}^{N,+}, \tilde{\mathfrak{X}}^{N,-})$ be the normalized empirical measure corresponding to the case Λ_\pm being empty sets. By applying the martingale $M_t^{(\phi_+, \phi_-)}$ to the case Λ_\pm being empty sets and $\phi_\pm = 1$ (now 1 is in the domain of the Feller generator), we have

$$\int_0^t \mathbb{E} [\ell_{\delta_N}, \tilde{\mathfrak{X}}_s^{N,+} \otimes \tilde{\mathfrak{X}}_s^{N,-}] ds = \langle 1, \tilde{\mathfrak{X}}_0^{N,+} \rangle - \mathbb{E} [\langle 1, \tilde{\mathfrak{X}}_t^{N,+} \rangle] \leq 1.$$

We then obtain (6.11) by a coupling of $(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})$ and $(\tilde{\mathfrak{X}}^{N,+}, \tilde{\mathfrak{X}}^{N,-})$. The idea is that $(\tilde{\mathfrak{X}}^{N,+}, \tilde{\mathfrak{X}}^{N,-})$ dominates $(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})$. This coupling can be constructed by labeling (rather than killing) particles which hit Λ_\pm , using the same method of Section 3.1. Hence, we obtain the desired bound for $\mathbb{E}^\mu [(M_t^{(\phi_+, \phi_-)})^2]$. \square

6.1.3. *Tightness.* The proof of tightness for $(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})$ is nontrivial because the natural bound $\langle \ell_{\delta_N}, \bar{\mathfrak{X}}_s^{N,+} \otimes \bar{\mathfrak{X}}_s^{N,-} \rangle^2$ for $\langle \ell_{\delta_N}, \mathfrak{X}_s^{N,+} \otimes \mathfrak{X}_s^{N,-} \rangle^2$ blows up near $s = 0$ in such a way that

$$\lim_{N \rightarrow \infty} \int_0^t \mathbb{E}[\langle \ell_{\delta_N}, \bar{\mathfrak{X}}_s^{N,+} \otimes \bar{\mathfrak{X}}_s^{N,-} \rangle^2] ds = \infty,$$

which follows directly from the Gaussian bound (2.2). Here, $(\bar{\mathfrak{X}}^{N,+}, \bar{\mathfrak{X}}^{N,-})$ is the empirical measure for the independent reflected diffusions *without* annihilation, defined in (3.4). To deal with this singularity at $s = 0$, we will use the following lemma whose proof is based on the Prohorov's theorem. We omit the proof here since it is simple. A proof can be found in [21].

LEMMA 6.5. *Let $\{Y_N\}$ be a sequence of real-valued processes such that $t \mapsto \int_0^t Y_N(r) dr$ is continuous on $[0, T]$ a.s., where $T \in [0, \infty)$. Suppose the following two conditions hold:*

(i) *There exists $q > 1$ such that $\overline{\lim}_{N \rightarrow \infty} \mathbb{E}[\int_h^T |Y_N(r)|^q dr] < \infty$ for any $h > 0$,*

(ii) *$\lim_{\alpha \searrow 0} \overline{\lim}_{N \rightarrow \infty} \mathbb{P}(\int_0^\alpha |Y_N(r)| dr > \varepsilon_0) = 0$ for any $\varepsilon_0 > 0$.*

Then $\{\int_0^t Y_N(r) dr; t \in [0, T]\}_{N \in \mathbb{N}}$ is tight in $C([0, T], \mathbb{R})$.

Here is our tightness result for $(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})$. Note that it does not require Assumption 2.7.

THEOREM 6.6 (Tightness). *Suppose $\{\delta_N\}$ tends to 0. Then $\{(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})\}$ is tight in $D([0, T], \mathfrak{M})$ and any of subsequential limits is carried on $C_{\mathfrak{M}}[0, T]$. Moreover, $\{J_N\}$ is tight in $C([0, T])$, where $J_N(t) := \int_0^t \langle \ell_{\delta_N}, \mathfrak{X}_s^{N,+} \otimes \mathfrak{X}_s^{N,-} \rangle ds$.*

PROOF. Recall from Remark 5.2 that \mathfrak{M} is a complete separable metric space. Since $\text{Dom}(\mathcal{A}^\pm)$ is dense in $C_\infty(\bar{D}_\pm \setminus \Lambda_\pm)$, we only need to check a “weak tightness criteria” (cf. Proposition 1.7 of [27]), that is, it suffices to check that $\{(\langle \phi_+, \mathfrak{X}^{N,+} \rangle, \langle \phi_-, \mathfrak{X}^{N,-} \rangle)\}_N$ is tight in $D([0, T], \mathbb{R}^2)$ for any $\phi_\pm \in \text{Dom}(\mathcal{A}^\pm)$. By Prohorov's theorem (see Theorem 1.3 and Remark 1.4 of [27]), $\{(\langle \phi_+, \mathfrak{X}^{N,+} \rangle, \langle \phi_-, \mathfrak{X}^{N,-} \rangle)\}_N$ is tight in $D([0, T], \mathbb{R}^2)$ if the following two properties (a) and (b) hold:

(a) For all $t \in [0, T]$ and $\varepsilon_0 > 0$, there exists a compact set $K(t, \varepsilon_0) \subset \mathbb{R}^2$ such that

$$\sup_N \mathbb{P}(\langle \phi_+, \mathfrak{X}_t^{N,+} \rangle, \langle \phi_-, \mathfrak{X}_t^{N,-} \rangle \notin K(t, \varepsilon_0)) < \varepsilon_0.$$

(b) For all $\varepsilon_0 > 0$,

$$\begin{aligned} & \lim_{\gamma \rightarrow 0} \overline{\lim}_{N \rightarrow \infty} \mathbb{P} \left(\sup_{\substack{|t-s| < \gamma \\ 0 \leq s, t \leq T}} |(\phi_+, \mathfrak{X}_t^{N,+}), (\phi_-, \mathfrak{X}_t^{N,-})| \right. \\ & \quad \left. - ((\phi_+, \mathfrak{X}_s^{N,+}), (\phi_-, X_s^{N,-})) \Big|_{\mathbb{R}^2} > \varepsilon_0 \right) \\ & = 0. \end{aligned}$$

Property (a) is true since we can always take $K = [-\|\phi_+\|_\infty, \|\phi_+\|_\infty] \times [-\|\phi_-\|_\infty, \|\phi_-\|_\infty]$. To verify property (b), we only need to focus on $\mathfrak{X}^{N,+}$. Note that (writing $\phi = \phi_+$ for simplicity) by Corollary 6.4, we have

$$\begin{aligned} & \langle \phi, \mathfrak{X}_t^{N,+} \rangle - \langle \phi, \mathfrak{X}_s^{N,+} \rangle \\ (6.12) \quad & = \int_s^t \langle \mathcal{A}^+ \phi, \mathfrak{X}_r^{N,+} \rangle dr - \frac{1}{2} \int_s^t \langle \ell_{\delta_N} \phi, \mathfrak{X}_r^{N,+} \otimes \mathfrak{X}_r^{N,-} \rangle dr \\ & \quad + (M_N(t) - M_N(s)), \end{aligned}$$

where $M_N(t)$ is a martingale. So we only need to verify (b) with $\langle \phi, \mathfrak{X}_t^{N,+} \rangle - \langle \phi, \mathfrak{X}_s^{N,+} \rangle$ replaced by each of the three terms on the right-hand side of (6.12).

The first term of (6.12) is obvious since $\langle \mathcal{A}^+ \phi, \mathfrak{X}_r^{N,+} \rangle \leq \|\mathcal{A}^+ \phi\|$. For the third term of (6.12), recall that $\lim_{N \rightarrow \infty} \mathbb{E}[M_N(t)^2] = 0$ by Corollary 6.4. Hence, by applying Chebyshev’s inequality and then Doob’s maximal inequality, we see that (b) is satisfied by the third term of (6.12).

For the second term of (6.12), we show that

$$(6.13) \quad \lim_{\gamma \rightarrow 0} \overline{\lim}_{N \rightarrow \infty} \mathbb{P} \left(\sup_{\substack{|t-s| < \gamma \\ 0 \leq s, t \leq T}} \int_s^t \langle \ell_{\delta_N}, \mathfrak{X}_r^{N,+} \otimes \mathfrak{X}_r^{N,-} \rangle dr > \varepsilon_0 \right) = 0.$$

Observe that, since $\langle \ell_{\delta_N}, \mathfrak{X}_r^{N,+} \otimes \mathfrak{X}_r^{N,-} \rangle$ is nonnegative, it suffices to prove (6.13) for the dominating case where Λ_\pm are empty. We now prove this together with the tightness of $\{J_N\}$ at one stroke by applying Lemma 6.5 to the special case $q = 2$ and $Y_N(r) = \langle \ell_{\delta_N}, \mathfrak{X}_r^{N,+} \otimes \mathfrak{X}_r^{N,-} \rangle$.

Using the Gaussian upper bound (2.2) for the heat kernel of the reflected diffusions, we have

$$\begin{aligned} & \overline{\lim}_{N \rightarrow \infty} \int_h^T \mathbb{E}[\langle \ell_{\delta_N}, \overline{\mathfrak{X}}_s^{N,+} \otimes \overline{\mathfrak{X}}_s^{N,-} \rangle^2] ds \leq C(d, D_+, D_-) \|\rho_+\| \|\rho_-\| \int_h^T s^{-2d} ds \\ & < \infty. \end{aligned}$$

The hypothesis (i) of Lemma 6.5 is therefore satisfied, since $(\overline{\mathfrak{X}}^{N,+}, \overline{\mathfrak{X}}^{N,-})$ dominates $(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})$.

It remains to verify hypothesis (ii) of Lemma 6.5, that is, to prove that for any $\varepsilon_0 > 0$, $\lim_{\alpha \rightarrow 0} \overline{\lim}_{N \rightarrow \infty} \mathbb{P}(J_n(\alpha) > \varepsilon_0) = 0$. By Corollary 6.4 again, for any $\phi \in$

$\text{Dom}(\mathcal{A}^+)$, we have

$$(6.14) \quad \begin{aligned} & \frac{1}{2} \int_0^t \langle \ell_{\delta_N} \phi, \mathfrak{X}_s^{N,+} \otimes \mathfrak{X}_s^{N,-} \rangle ds \\ &= \langle \phi, \mathfrak{X}_0^{N,+} \rangle - \langle \phi, \mathfrak{X}_t^{N,+} \rangle + \int_0^t \langle \mathcal{A}^+ \phi, \mathfrak{X}_s^{N,+} \rangle ds + M_N(t), \end{aligned}$$

where $M_N(t)$ is a martingale and $\lim_{N \rightarrow \infty} \mathbb{E}[(M_N(t))^2] = 0$ for all $t > 0$. Note that the left-hand side of (6.14) is comparable to $J_N(t)$ whenever we pick $\phi \in \text{Dom}(\mathcal{A}^+)$ in such a way that $\ell_{\delta_N} \phi \approx \ell_{\delta_N}$. The idea is to pick $\phi \approx \mathbf{1}_{(D_+)_r}$, then let $r \rightarrow 0$ to bound $J_N(t)$ from above. Here, $(D_+)_r$ is the set of points in D_+ whose distance from the boundary is less than r . More specifically, for any $r > 0$, let $\psi_r \in C(\overline{D_+})$ be such that $\psi_r = 1$ on $(D_+)_r$, $\psi_r = 0$ on $D_+ \setminus (D_+)_{2r}$ and $0 \leq \psi \leq 1$. Let $\phi_r \in \text{Dom}(\mathcal{A}^+) \cap C^+(\overline{D_+})$ be such that $\|\phi_r - \psi_r\|_\infty = o(r)$. Such ϕ_r exists since $\text{Dom}(\mathcal{A}^+)$ is dense in $C(\overline{D_+})$. Then (6.14) implies

$$\begin{aligned} 0 &\leq J_N(\alpha) \\ &\leq \left| \int_0^\alpha \langle \ell_{\delta_N} - \ell_{\delta_N} \phi_r, \mathfrak{X}_s^{N,+} \otimes \mathfrak{X}_s^{N,-} \rangle ds \right| + \langle \phi_r, \mathfrak{X}_0^{N,+} \rangle - \langle \phi_r, \mathfrak{X}_\alpha^{N,+} \rangle \\ &\quad + \|\mathcal{A}^+ \phi_r\| \alpha + |M_N(\alpha)| \\ &\leq o(r) J_N(\alpha) + \langle \phi_r, \mathfrak{X}_0^{N,+} \rangle + \|\mathcal{A}^+ \phi_r\| \alpha + |M_N(\alpha)| \quad \text{whenever } r > 2\delta_N. \end{aligned}$$

This is because when $r > 2\delta_N$, $\phi_r(x)$ is close to 1 on $(D_+)_{\delta_N}$. Hence, we have, for $r > 2\delta_N$,

$$(1 - o(r)) J_N(\alpha) \leq \langle \phi_r, \mathfrak{X}_0^{N,+} \rangle + \|\mathcal{A}^+ \phi_r\| \alpha + |M_N(\alpha)|.$$

From this, we have

$$\lim_{\alpha \rightarrow 0} \overline{\lim}_{N \rightarrow \infty} \mathbb{P}(J_N(\alpha) > 3\varepsilon_0) \leq \overline{\lim}_{N \rightarrow \infty} \mathbb{P}(\langle \phi_r, \mathfrak{X}_0^{N,+} \rangle > \varepsilon_0(1 - o(r))).$$

Note that $0 \leq \phi_r \leq \mathbf{1}_{(D_+)_{2r}} + o(r)$. So for $r > 0$ small enough,

$$\mathbb{P}(\langle \phi_r, \mathfrak{X}_0^{N,+} \rangle > \varepsilon_0(1 - o(r))) \leq \mathbb{P}(\langle \mathbf{1}_{(D_+)_{2r}}, \mathfrak{X}_0^{N,+} \rangle > \varepsilon_0/2).$$

Moreover, since $\mathfrak{X}_0^{N,+} \xrightarrow{\mathcal{L}} u_0^+(x) dx$ with $u_0^+ \in C(\overline{D})$, we have

$$\lim_{r \rightarrow 0} \overline{\lim}_{N \rightarrow \infty} \mathbb{P}(\langle \mathbf{1}_{(D_+)_{2r}}, \mathfrak{X}_0^{N,+} \rangle > \varepsilon_0/2) = 0.$$

Hence, the second hypothesis of Lemma 6.5 is verified. We have shown that (ii) is true. Thus, $(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})$ is relatively compact. Property (ii) above also tells us that any subsequential limit has law concentrated on $C([0, \infty), \mathfrak{M})$ (details can be found in [21]). \square

6.2. *Identifying subsequential limits.* Recall that we have already established tightness of $\{(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-}); N \geq 1\}$ in Theorem 6.6. Hence, any subsequence has a further subsequence which converges in distribution in $D([0, T], \mathfrak{M})$. Let \mathbb{P}^∞ be the law of an arbitrary subsequential limit $(\mathfrak{X}^{\infty,+}, \mathfrak{X}^{\infty,-})$. Then $\mathbb{P}^\infty((\mathfrak{X}^{\infty,+}, \mathfrak{X}^{\infty,-}) \in C([0, \infty), \mathfrak{M})) = 1$ by Theorem 6.6. Our goal is to show that

$$(\mathfrak{X}^{\infty,+}, \mathfrak{X}^{\infty,-}) = (u_+(t, x)\rho_+(x) dx, u_-(t, y)\rho_-(y) dy), \quad \mathbb{P}^\infty\text{-a.s.}$$

An immediate question is whether $\mathfrak{X}^{\infty,+}$ and $\mathfrak{X}^{\infty,-}$ have densities with respect to the Lebesgue measure. For this, we can compare $(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})$ with $(\bar{\mathfrak{X}}^{N,+}, \bar{\mathfrak{X}}^{N,-})$ to get an affirmative answer. The construction in Section 3.1 provides a natural coupling between $\{(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})\}$ and $\{(\bar{\mathfrak{X}}^{N,+}, \bar{\mathfrak{X}}^{N,-})\}$. We summarize some preliminary information about $(\mathfrak{X}^{\infty,+}, \mathfrak{X}^{\infty,-})$ in the following lemma. Its proof can be found in [21]. Denote $\langle f, g \rangle_\rho := \int f(x)g(x)\rho(x) dx$ and $\langle f, \mu \rangle_\rho := \int f(x)\rho(x)\mu(dx)$ if f, g, ρ are functions on a domain D and μ is a measure on D .

LEMMA 6.7.

$$\begin{aligned} \mathbb{P}^\infty(\langle \phi_+, \mathfrak{X}_t^{\infty,+} \rangle &\leq \langle \phi_+, P_t^+ u_0^+ \rangle_{\rho_+} \text{ and } \langle \phi_-, \mathfrak{X}_t^{\infty,-} \rangle \leq \langle \phi_-, P_t^- u_0^- \rangle_{\rho_-} \\ \text{for } t \geq 0 \text{ and } \phi_\pm &\in C_\infty(\bar{D}_\pm \setminus \Lambda_\pm) = 1. \end{aligned}$$

In particular, both $\mathfrak{X}_t^{\infty,+}$ and $\mathfrak{X}_t^{\infty,-}$ are absolutely continuous with respect to the Lebesgue measure for $t \geq 0$. Moreover, $(\mathfrak{X}_t^{\infty,+}, \mathfrak{X}_t^{\infty,-}) = (v_+(t, x)\rho_+(x) dx, v_-(t, y)\rho_-(y) dy)$ for some $v_\pm(t) \in \mathcal{B}_b(D_\pm)$ with $v_+(t, x) \leq P_t^+ u_0^+(x)$ and $v_-(t, y) \leq P_t^- u_0^-(y)$ for a.e. $(x, y) \in D_+ \times D_-$.

The characterization $(\mathfrak{X}^{\infty,+}, \mathfrak{X}^{\infty,-})$ will be accomplished by the following result of ‘‘mean–variance analysis’’:

PROPOSITION 6.8. For all $\phi_\pm \in C_\infty(\bar{D}_\pm \setminus \Lambda_\pm)$ and $t \geq 0$, we have

$$(6.15) \quad \mathbb{E}^\infty[\langle v_\pm(t), \phi_\pm \rangle_{\rho_\pm}] = \langle u_\pm(t), \phi_\pm \rangle_{\rho_\pm},$$

$$(6.16) \quad \mathbb{E}^\infty[\langle v_\pm(t), \phi_\pm \rangle_{\rho_\pm}^2] = \langle u_\pm(t), \phi_\pm \rangle_{\rho_\pm}^2,$$

where v_\pm is the density of $\mathfrak{X}^{\infty,\pm}$ w.r.t. $\rho_\pm(x) dx$ stated in Lemma 6.7, and u_\pm is the function defined in Section 4.

We postpone the proof of Proposition 6.8 to Section 7, and proceed to present the proof of Theorem 5.1.

6.3. *Proof of Theorem 5.1.*

PROOF. Tightness of $\{(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})\}$ was proved in Theorem 6.6. It remains to identify any subsequential limit. We conclude from (6.15) and (6.16) that

$$\langle \mathfrak{X}_t^{\infty,+}, \phi_+ \rangle = \langle u_+(t), \phi_+ \rangle_{\rho_+} \quad \text{and} \quad \langle \mathfrak{X}_t^{\infty,-}, \phi_- \rangle = \langle u_-(t), \phi_- \rangle_{\rho_-}, \quad \mathbb{P}^\infty\text{-a.s.}$$

for any fixed $t > 0$ and $\phi_\pm \in C_\infty(\overline{D}_\pm \setminus \Lambda_\pm)$. Recall that $(\mathfrak{X}^{\infty,+}, \mathfrak{X}^{\infty,-}) \in C([0, \infty), \mathfrak{M})$ by Theorem 6.6 and that $C_\infty(\overline{D}_\pm \setminus \Lambda_\pm)$ is separable. Hence, through rational numbers and a countable dense subset of $C_\infty(\overline{D}_\pm \setminus \Lambda_\pm)$ to strengthen the previous statement to

$$\begin{aligned} & \mathbb{P}^\infty((\mathfrak{X}_t^{\infty,+}, \mathfrak{X}_t^{\infty,-}) \\ & = (u_+(t, x)\rho_+(x) dx, u_-(t, y)\rho_-(y) dy) \in \mathfrak{M} \text{ for every } t \geq 0) = 1. \end{aligned}$$

This completes the proof of Theorem 5.1. \square

7. Characterization of the mean and the variance. The goal of this last section is to prove Proposition 6.8.

7.1. *Results about Minkowski content.* We first look at a single domain and strengthen some results from geometric measure theory.

7.1.1. *Minkowski content for ∂D .*

LEMMA 7.1. *Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain and D_ϵ is the set of points in D whose distance from the boundary is less than ϵ . If $\mathcal{F} \subset C(\overline{D})$ is an equi-continuous and uniformly bounded family of functions on \overline{D} , then*

$$\lim_{\epsilon \rightarrow 0} \sup_{f \in \mathcal{F}} \left| \frac{1}{\epsilon} \int_{D_\epsilon} f(x) dx - \int_{\partial D} f(x) \sigma(dx) \right| = 0.$$

PROOF. The result holds trivially when $d = 1$, by the uniform continuity of f . We will only consider $d \geq 2$. The idea is to cut ∂D into small pieces so that f is almost constant in each piece, and then apply (1.3) in each piece.

Fix $\eta > 0$. There exists $\delta > 0$ such that $|f(x) - f(y)| < \eta$ whenever $|x - y| \leq \delta$. Since D is bounded and Lipschitz (or by a more general result by David in [15] or Section 2 of [16]), we can reduce to local coordinates to obtain a partition $\{Q_i\}_{i=1}^N$ of ∂D in such a way that for any i , Q_i is the Lipschitz image of a bounded subset of \mathbb{R}^{d-1} [hence it is (\mathcal{H}^{d-1}) -rectifiable], $\text{diam}(Q_i) \leq \delta$ and ∂Q_i is (\mathcal{H}^{d-2}) -rectifiable. Here, ∂Q_i is the boundary of Q_i with respect to the topology induced by ∂D .

Let $(Q_i)_\varepsilon := \{x \in D : \text{dist}(x, Q_i) < \varepsilon\}$ and $(\partial Q_i)_\varepsilon := \{x \in D : \text{dist}(x, \partial Q_i) < \varepsilon\}$. Since $\{(Q_i)_\varepsilon \setminus (\partial Q_i)_\varepsilon\}_{i=1}^N$ are disjoint and $\bigcup_{i=1}^N (Q_i)_\varepsilon \setminus (\partial Q_i)_\varepsilon \subset D_\varepsilon \subset \bigcup_{i=1}^N (Q_i)_\varepsilon$, we have

$$(7.1) \quad \left| \sum_{i=1}^N \int_{(Q_i)_\varepsilon} f \, dx - \int_{D_\varepsilon} f \, dx \right| \leq \sum_{i=1}^N \int_{(\partial Q_i)_\varepsilon} |f| \, dx.$$

Therefore, we have

$$\begin{aligned} & \left| \frac{1}{\varepsilon} \int_{D_\varepsilon} f \, dx - \int_{\partial D} f \, d\sigma \right| \\ & \leq \left| \frac{1}{\varepsilon} \int_{D_\varepsilon} f \, dx - \frac{1}{\varepsilon} \sum_{i=1}^N \int_{(Q_i)_\varepsilon} f \, dx \right| + \left| \frac{1}{\varepsilon} \sum_{i=1}^N \int_{(Q_i)_\varepsilon} f \, dx - \int_{\partial D} f \, d\sigma \right| \\ & \leq \frac{1}{\varepsilon} \sum_{i=1}^N \int_{(\partial Q_i)_\varepsilon} |f| \, dx + \sum_{i=1}^N \left| \frac{1}{\varepsilon} \int_{(Q_i)_\varepsilon} f \, dx - \int_{Q_i} f \, d\sigma \right| \quad \text{by (7.1)} \\ & \leq \sum_{i=1}^N \left(\|f\|_\infty \frac{|(\partial Q_i)_\varepsilon|}{\varepsilon} + \left| \frac{1}{\varepsilon} \int_{(Q_i)_\varepsilon} f - f(\xi_i) \, dx \right| \right. \\ & \quad \left. + |f(\xi_i)| \left| \frac{|(Q_i)_\varepsilon|}{\varepsilon} - \sigma(Q_i) \right| + \left| \int_{Q_i} f - f(\xi_i) \, d\sigma \right| \right) \\ & \leq \eta \sum_{i=1}^N \left(\frac{|(Q_i)_\varepsilon|}{\varepsilon} + \sigma(Q_i) \right) + \|f\|_\infty \sum_{i=1}^N \left(\frac{|(\partial Q_i)_\varepsilon|}{\varepsilon} + \left| \frac{|(Q_i)_\varepsilon|}{\varepsilon} - \sigma(Q_i) \right| \right). \end{aligned}$$

In the third inequality, ξ_i is an arbitrary point in Q_i . Since ∂Q_i and $(Q_i)_\varepsilon$ are (\mathcal{H}^{d-2}) -rectifiable and (\mathcal{H}^{d-1}) -rectifiable, respectively, Theorem 3.2.39 of [22] tells us that

$$\lim_{\varepsilon \rightarrow 0} \frac{|(\partial Q_i)_\varepsilon|}{c_2 \varepsilon^2} = \mathcal{H}^{d-2}(\partial Q_i) \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \frac{|(Q_i)_\varepsilon|}{\varepsilon} = \mathcal{H}^{d-1}(Q_i),$$

where $c_m := |\{x \in \mathbb{R}^m : |x| < 1\}|$. Thus,

$$\overline{\lim}_{\varepsilon \rightarrow 0} \left| \frac{1}{\varepsilon} \int_{D_\varepsilon} f \, dx - \int_{\partial D} f \, d\sigma \right| \leq 2\eta \sum_i \sigma(Q_i) = 2\sigma(\partial D)\eta.$$

Since $\eta > 0$ is arbitrary and the above estimate is uniform over $f \in \mathcal{F}$, we get the desired result. \square

By the same proof of Lemma 7.1, we obtain the following stronger result.

LEMMA 7.2. *Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $k \in \mathbb{N}$. If $\mathcal{F} \subset C(\overline{D}^k)$ is an equi-continuous and uniformly bounded family of functions, then*

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^k} \int_{(D_\varepsilon)^k} f(z_1, \dots, z_k) \, dz_1 \cdots dz_k = \int_{(\partial D)^k} f(z_1, \dots, z_k) \sigma(dz_1) \cdots \sigma(dz_k)$$

uniformly for $f \in \mathcal{F}$, where σ is the surface measure on ∂D .

7.1.2. *Minkowski content for $\{(z, z) : z \in I\}$.* Now we prove analogous results for the interface I for our annihilation model.

LEMMA 7.3. *Under our geometric setting in Assumption 2.4, if $\mathcal{F} \subset C(\overline{D}_+ \times \overline{D}_-)$ is an equi-continuous and uniformly bounded family of functions on $\overline{D}_+ \times \overline{D}_-$, then*

$$\limsup_{\delta \rightarrow 0} \sup_{f \in \mathcal{F}} \left| (c_{d+1} \delta^{d+1})^{-1} \int_{I_\delta} f(x, y) dx dy - \int_I f(z, z) d\sigma(z) \right| = 0.$$

PROOF. By the same argument as in the proof of Lemma 7.1, we can construct a nice partition $\{Q_i\}_{i=1}^N$ of I and apply Theorem 3.2.39 (page 275) of [22]. The only essential difference is that now we require $\partial Q_i \setminus \partial I$ to be (\mathcal{H}^{d-2}) -rectifiable, where ∂I is the boundary of I with respect to the topology induced by ∂D_+ , or equivalently by ∂D_- . Moreover, instead of (7.1), we now have

$$(7.2) \quad \left| \sum_{i=1}^N \int_{(Q_i)_\delta} f dx dy - \int_{I_\delta} f dx dy \right| \leq \sum_{i=1}^N \int_{(\partial Q_i \setminus \partial I)_\delta} |f| dx dy.$$

Note that we do not need any assumption on ∂I . \square

By the same proof of Lemma 7.1, we obtain the following stronger result.

LEMMA 7.4. *Suppose Assumptions 2.4, 2.5 and 2.6 hold. Suppose $k \in \mathbb{N}$ and $\mathcal{F} \subset C((\overline{D}_+ \times \overline{D}_-)^k)$ is an equi-continuous and uniformly bounded family of functions on $(\overline{D}_+ \times \overline{D}_-)^k$. Then as $\delta \rightarrow 0$, we have*

$$\begin{aligned} & \int_{(x_1, y_1) \in D_+ \times D_-} \cdots \int_{(x_k, y_k) \in D_+ \times D_-} f(x_1, y_1, \dots, x_k, y_k) \\ & \times \prod_{i=1}^k \ell_\delta(x_i, y_i) d(x_1, y_1, \dots, x_k, y_k) \\ & \rightarrow \int_{z_1 \in I} \cdots \int_{z_k \in I} f(z_1, z_1, \dots, z_k, z_k) \\ & \times \prod_{i=1}^k \lambda(z_i) d\sigma(z_1) \cdots d\sigma(z_k) \end{aligned}$$

uniformly for $f \in \mathcal{F}$.

REMARK 7.5. Following the same proof as above, clearly we can strengthen Lemmas 7.3 and 7.4 by only requiring \mathcal{F} to be equi-continuous and uniformly

bounded on a neighborhood of the interface I . We can also generalize Lemmas 7.1 and 7.2 to deal with $\int_J f(x) d\sigma(x)$ for any closed \mathcal{H}^{d-1} -rectifiable subset of J of ∂D (rather than the whole boundary ∂D), and by requiring \mathcal{F} to be equicontinuous and uniformly bounded on a neighborhood of J .

7.2. *Martingales for space–time processes.* In this subsection, we collect some integral equations satisfied by $(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})$ that will be used later to identify the limit. These integral equations can be viewed as the Dynkins’ formulae for our annihilating diffusion system, and will be proved rigorously by considering suitable martingales associated with the process $(t, (\mathfrak{X}_t^{N,+}, \mathfrak{X}_t^{N,-}))$.

LEMMA 7.6. *Suppose X^Λ is an (\mathbf{a}, ρ) -reflected diffusion in a bounded Lipschitz domain D killed upon hitting a closed subset Λ of ∂D that is regular with respect to X . Then for any $T > 0$ and bounded measurable function ϕ on $\overline{D} \setminus \Lambda$, we have*

$$(7.3) \quad P_{T-s}^\Lambda \phi(X_s^\Lambda) \quad \text{is a } \mathcal{F}_s^{X^\Lambda} \text{-martingale for } s \in [0, T],$$

under \mathbb{P}^x for any $x \in \overline{D} \setminus \Lambda$. Moreover, its quadratic variation is $\int_0^s \mathbf{a} \nabla P_{T-r}^\Lambda \phi \cdot \nabla P_{T-r}^\Lambda \phi(X^\Lambda(r)) dr$.

PROOF. Equation (7.3) follows from the Markov property of X^Λ . Denote by $\mathcal{L}^{(\Lambda)}$ the L^2 -generator of $X^{(\Lambda)}$. Then for every $t \in [0, T)$, $P_{T-s}^\Lambda \phi \in \text{Dom}(\mathcal{L}^{(\Lambda)})$. It follows from the spectral representation of $\mathcal{L}^{(\Lambda)}$ that

$$\left\| \frac{\partial P_{T-s}^\Lambda \phi}{\partial s} \right\|_{L^2} = \left\| -\mathcal{L}^{(\Lambda)} P_{T-s}^\Lambda \phi \right\|_{L^2} \leq \frac{\|\phi\|_{L^2}}{T-s}.$$

Thus, $(s, x) \mapsto P_{T-s}^\Lambda \phi(x)$ for $s \in [0, T)$ and $x \in \overline{D} \setminus \Lambda$ is in the domain of the Dirichlet form for the space–time process $(s, X_s^{(\Lambda)})$. By an application of the Fukushima decomposition in the context of time-dependent Dirichlet forms, one concludes that the quadratic variation of the martingale $s \mapsto P_{T-s}^\Lambda \phi(X_s^\Lambda)$ is $\int_0^s \mathbf{a} \nabla P_{T-r}^\Lambda \phi \cdot \nabla P_{T-r}^\Lambda \phi(X^\Lambda(r)) dr$; see Example 6.5.6 of [31]. \square

As mentioned in Remark 6.3, a time-dependent version of Theorem 6.2 is valid. We now state it precisely. A proof can be obtained by following the same argument in the proof of Theorem 6.2, but now to the time dependent process $(t, (\mathfrak{X}_t^{N,+}, \mathfrak{X}_t^{N,-}))$. The detail is left to the reader.

THEOREM 7.7. *Let $T > 0$, and $f_s \in C_b(E_N)$ and $g_s \in \mathcal{B}(E_N)$ for $s \in [0, T]$. Suppose*

$$\overline{M}_s := f_s(\overline{\mathfrak{X}}_s^{N,+}, \overline{\mathfrak{X}}_s^{N,-}) - \int_0^s g_r(\overline{\mathfrak{X}}_r^{N,+}, \overline{\mathfrak{X}}_r^{N,-}) dr$$

is a $\mathcal{F}_s^{\bar{\mathfrak{X}}^{N,+}, \bar{\mathfrak{X}}^{N,-}}$ -martingale for $s \in [0, T]$, under \mathbb{P}^μ for any $\mu \in E_N$. Then

$$M_s := f_s(\mathfrak{X}_s^{N,+}, \mathfrak{X}_s^{N,-}) - \int_0^s (g_r + Kf_r)(\mathfrak{X}_r^{N,+}, \mathfrak{X}_r^{N,-}) dr$$

is a $\mathcal{F}_r^{\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-}}$ -martingale for $s \in [0, T]$, under \mathbb{P}^μ for any $\mu \in E_N$, where the operator K is given by (6.3).

Consider $X_{(n,m)} := (X_1^+, \dots, X_n^+, \mathfrak{X}_1^-, \dots, \mathfrak{X}_m^-) \in (D_+^\partial)^n \times (D_-^\partial)^m$, which consists of independent copies of X^\pm 's. The transition density of $X_{(n,m)}$ w.r.t. $\rho_{(n,m)}$ is $p^{(n,m)}$, where

$$p^{(n,m)}(t, (\vec{x}, \vec{y}), (\vec{x}', \vec{y}')) := \prod_{i=1}^n p^+(t, x_i, x'_i) \prod_{j=1}^m p^-(t, y_j, y'_j),$$

$$\rho_{(n,m)}(\vec{x}, \vec{y}) := \prod_{i=1}^n \rho_+(x_i) \prod_{j=1}^m \rho_-(y_j).$$

The semigroup of $X_{(n,m)}$, denoted by $P_t^{(n,m)}$, is strongly continuous on

$$(7.4) \quad C_\infty^{(n,m)} := \{ \Phi \in C(\bar{D}_+^n \times \bar{D}_-^m) : \\ \Phi \text{ vanishes outside } (\bar{D}_+ \setminus \Lambda_+)^n \times (\bar{D}_- \setminus \Lambda_-)^m \}.$$

Clearly, $C_\infty^{(1,0)} = C_\infty(\bar{D}_+ \setminus \Lambda_+)$ and $C_\infty^{(0,1)} = C_\infty(\bar{D}_- \setminus \Lambda_-)$.

COROLLARY 7.8. *Let n and m be any nonnegative integers, $T > 0$ be any positive number and $\Phi \in C_\infty^{(n,m)}$. Consider the function $f : [0, T] \times E_N \rightarrow \mathbb{R}$ defined as follows: $f(s, \mathbf{0}_*) := 0$ and for an arbitrary element $\mu \in E_N \setminus \{\mathbf{0}_*\}$, we can write $\mu = (\frac{1}{N} \sum_{i \in A_+} \mathbf{1}_{x_i}, \frac{1}{N} \sum_{j \in A_-} \mathbf{1}_{y_j})$ for some index sets A_+ and A_- , then*

$$f(s, \mu) := \sum_{\substack{i_1, \dots, i_n \\ \text{distinct}}} \sum_{\substack{j_1, \dots, j_m \\ \text{distinct}}} P_{T-s}^{(n,m)} \Phi(x^{i_1}, \dots, x^{i_n}, y^{j_1}, \dots, y^{j_m}),$$

where the first summation is on the collection of all n -tuples (i_1, \dots, i_n) chosen from distinct elements of A_+ , the second summation is on the collection of all m -tuples (j_1, \dots, j_m) chosen from distinct elements of A_- . Then we have

$$f(s, (\mathfrak{X}_s^{N,+}, \mathfrak{X}_s^{N,-})) - \int_0^s Kf(r, \cdot)(\mathfrak{X}_r^{N,+}, \mathfrak{X}_r^{N,-}) dr$$

is a $\mathcal{F}_s^{\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-}}$ -martingale for $s \in [0, T]$, under \mathbb{P}^ν , for any $\nu \in E_N$.

PROOF. Clearly, $f(s, \cdot) \in C_b(E_N)$ for $s \in [0, T]$. By Lemma 7.6, we have $f(s, \bar{\mathfrak{X}}_s)$ is a $\mathcal{F}_s^{\bar{\mathfrak{X}}}$ -martingale for $s \in [0, T]$ for all $T \geq 0$. Hence, we can take g_r to be constant zero and f_r to be $f(r, \cdot)$ in Theorem 7.7 to complete the proof. \square

As an immediate consequence, we obtain the Dynkin's formula for our system: For $0 \leq t \leq T$, we have

$$(7.5) \quad \mathbb{E} \left[f(T, (\mathfrak{X}_T^{N,+}, \mathfrak{X}_T^{N,-})) - f(t, (\mathfrak{X}_t^{N,+}, \mathfrak{X}_t^{N,-})) - \int_t^T Kf(r, \cdot)(\mathfrak{X}_r^{N,+}, \mathfrak{X}_r^{N,-}) dr \right] = 0.$$

Corollary 7.8 is the key to obtain the system of equations satisfied by the correlation functions of the particles in the annihilating diffusion system. This system of equations, usually called BBGKY hierarchy, will be formulated in the forthcoming paper [10]. The specific integral equations that we need to identify subsequential limits of $\{(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})\}$ are stated in the following lemmas. These equations are a part of the BBGKY hierarchy.

LEMMA 7.9. *For any $\phi_{\pm} \in C_{\infty}(\overline{D}_{\pm} \setminus \Lambda_{\pm})$ and $0 \leq t \leq T < \infty$, we have*

$$(7.6) \quad \mathbb{E}[\langle \phi_+, \mathfrak{X}_T^{N,+} \rangle + \langle \phi_-, \mathfrak{X}_T^{N,-} \rangle] - \mathbb{E}[\langle P_{T-t}^+ \phi_+, \mathfrak{X}_t^{N,+} \rangle + \langle P_{T-t}^- \phi_-, \mathfrak{X}_t^{N,-} \rangle] \\ = -\frac{1}{2} \int_t^T \mathbb{E}[\langle \ell_{\delta_N}(P_{T-r}^+ \phi_+ + P_{T-r}^- \phi_-), \mathfrak{X}_r^{N,+} \otimes \mathfrak{X}_r^{N,-} \rangle] dr$$

and

$$(7.7) \quad \mathbb{E}[\langle \phi_+, \mathfrak{X}_T^{N,+} \rangle^2] - \mathbb{E}[\langle P_{T-t}^+ \phi_+, \mathfrak{X}_t^{N,+} \rangle^2] \\ = - \int_t^T \mathbb{E}[\langle P_{T-r}^+ \phi_+, \mathfrak{X}_r^{N,+} \rangle \langle \ell_{\delta_N}(P_{T-r}^+ \phi_+), \mathfrak{X}_r^{N,+} \otimes \mathfrak{X}_r^{N,-} \rangle] dr \\ + o(N),$$

where $o(N)$ is a term which tends to zero as $N \rightarrow \infty$. A similar formula for (7.7) holds for $\mathfrak{X}^{N,-}$.

PROOF. Since $\text{Dom}(\mathcal{A}^{\pm})$ is dense in $C_{\infty}(\overline{D}_{\pm} \setminus \Lambda_{\pm})$, it suffices to check the lemma for $\phi_{\pm} \in \text{Dom}(\mathcal{A}^{\pm})$.

Identity (7.6) follows directly from Corollary 7.8 by taking $f(s, \mu) = \langle P_{T-s}^+ \phi_+, \mu^+ \rangle + \langle P_{T-s}^- \phi_-, \mu^- \rangle$.

For (7.7), we can apply Lemma 7.6 and Theorem 7.7, with $f_s(\mu) = \langle P_{T-s}^+ \phi_+, \mu^+ \rangle^2$ and $g_s(\mu) = \frac{1}{N} \langle \mathbf{a}_+ \nabla P_{T-s}^+ \phi_+ \cdot \nabla P_{T-s}^+ \phi_+, \mu^+ \rangle$, to obtain

$$\mathbb{E}[\langle \phi_+, \mathfrak{X}_T^{N,+} \rangle^2] - \mathbb{E}[\langle P_{T-t}^+ \phi_+, \mathfrak{X}_t^{N,+} \rangle^2] \\ = - \int_t^T \mathbb{E}[\langle P_{T-r}^+ \phi_+, \mathfrak{X}_r^{N,+} \rangle \langle \ell_{\delta_N}(P_{T-r}^+ \phi_+), \mathfrak{X}_r^{N,+} \otimes \mathfrak{X}_r^{N,-} \rangle] dr \\ + \frac{1}{2N} \int_t^T \mathbb{E}[\langle 2\mathbf{a}_+ \nabla P_{T-s}^+ \phi_+ \cdot \nabla P_{T-s}^+ \phi_+, \mathfrak{X}_r^{N,+} \rangle \\ + \langle \ell_{\delta_N}(P_{T-r}^+ \phi_+)^2, \mathfrak{X}_r^{N,+} \otimes \mathfrak{X}_r^{N,-} \rangle] dr.$$

Note that the term with a factor $\frac{1}{N}$ converges to zero as $N \rightarrow \infty$. This can be proved by the same argument for the bound of the quadratic variation $\mathbb{E}^\mu[(M_t^{(\phi_+, \phi_-)})^2]$ in Corollary 6.4. Hence, we have (7.7). \square

We now derive the integral equations satisfied by the integrands (with respect to dr) on the right-hand side of (7.6) and (7.7). The integrand (with respect to dr) on the right-hand side of (7.7) is of the form

$$\begin{aligned} & \langle \phi, \mu^+ \rangle \langle \phi, \mu^+ \otimes \mu^- \rangle \\ &= \frac{1}{N^3} \left(\sum_i \sum_j \phi(x_i) \varphi(x_i, y_j) + \sum_\ell \sum_{i \neq \ell} \sum_j \phi(x_\ell) \varphi(x_i, y_j) \right), \end{aligned}$$

where $\varphi \in \mathcal{B}(\overline{D}_+ \times \overline{D}_-)$, $\phi = \phi_+ \in \mathcal{B}(\overline{D}_+)$ and $\mu = (\frac{1}{N} \sum_i \mathbf{1}_{x_i}, \frac{1}{N} \sum_j \mathbf{1}_{y_j}) \in E_N$. We define

$$\begin{aligned} & P_t^{(*)}(\langle \phi, \mu^+ \rangle \langle \phi, \mu^+ \otimes \mu^- \rangle) \\ &:= \frac{1}{N^3} \left(\sum_i \sum_j P_t^{(1,1)}(\phi\varphi)(x_i, y_j) + \sum_\ell \sum_{i \neq \ell} \sum_j P_t^{(2,1)}(\phi\varphi)(x_\ell, x_i, y_j) \right) \\ (7.8) \quad &= \langle P_t^{(2,1)}(\phi\varphi)(x_1, x_2, y), \mu^+(dx_1) \otimes \mu^+(dx_2) \otimes \mu^-(dy) \rangle \\ &+ \frac{1}{N} \langle P_t^{(1,1)}(\phi\varphi)(x, y) - P_t^{(2,1)}(\phi\varphi)(x, x, y), \mu^+(dx) \otimes \mu^-(dy) \rangle. \end{aligned}$$

In $P_t^{(1,1)}(\phi\varphi)$, we view $\phi\varphi$ as the function of two variables $(a, b) \mapsto \phi(a)\varphi(a, b)$; in $P_t^{(2,1)}(\phi\varphi)$, we view $\phi\varphi$ as the function of three variables $(a_1, a_2, b) \mapsto \phi(a_1)\varphi(a_2, b)$. The definition of $P_t^{(*)}$ is motivated by the fact that $f(s, \mu) := P_{T-s}^{(*)}(\langle \phi_+, \mu^+ \otimes \mu^+ \otimes \mu^- \rangle)$ is of the same form as the function in Corollary 6.4.

LEMMA 7.10. *Suppose $\varphi \in C_\infty^{(1,1)}$, $\phi_\pm \in C_\infty(\overline{D}_\pm \setminus \Lambda_\pm)$ and $0 \leq t \leq T < \infty$. Let $F_r = P_{T-r}^{(1,1)}\varphi$, $G_r = P_{T-r}^{(1,1)}(\phi_+\varphi)$ and $H_r = P_{T-r}^{(2,1)}(\phi_+\varphi)$. Then we have*

$$\begin{aligned} & \mathbb{E}[\langle \varphi, \mathfrak{X}_T^{N,+} \otimes \mathfrak{X}_T^{N,-} \rangle] - \mathbb{E}[\langle P_{T-t}^{(1,1)}\varphi, \mathfrak{X}_t^{N,+} \otimes \mathfrak{X}_t^{N,-} \rangle] \\ (7.9) \quad &= -\frac{1}{2} \int_t^T \mathbb{E} \left[\left\langle \ell_{\delta_N}(x, y) \right. \right. \\ &\quad \times \left(\langle F_r(x, \cdot), \mathfrak{X}_r^{N,-} \rangle + \langle F_r(\cdot, y), \mathfrak{X}_r^{N,+} \rangle - \frac{1}{N} F_r(x, y) \right), \\ &\quad \left. \left. \mathfrak{X}_r^{N,+}(dx) \otimes \mathfrak{X}_r^{N,-}(dy) \right\rangle \right] dr \end{aligned}$$

and

$$\begin{aligned}
 & \mathbb{E}[\langle \phi_+, \mathfrak{X}_T^{N,+} \rangle \langle \varphi, \mathfrak{X}_T^{N,+} \otimes \mathfrak{X}_T^{N,-} \rangle] - \mathbb{E}[P_{T-t}^{(*)} \langle \phi_+ \varphi, \mathfrak{X}_t^{N,+} \otimes \mathfrak{X}_t^{N,+} \otimes \mathfrak{X}_t^{N,-} \rangle] \\
 &= -\frac{1}{2} \int_t^T \mathbb{E} \left[\left\langle \ell_{\delta_N}(x, y) \left(\langle H_r(x, \cdot, \cdot), \mathfrak{X}_r^{N,+} \otimes \mathfrak{X}_r^{N,-} \rangle \right. \right. \right. \\
 (7.10) \quad & \left. \left. \left. + \langle H_r(\cdot, x, \cdot), \mathfrak{X}_r^{N,+} \otimes \mathfrak{X}_r^{N,-} \rangle + \langle H_r(\cdot, \cdot, y), \mathfrak{X}_r^{N,+} \otimes \mathfrak{X}_r^{N,+} \rangle \right. \right. \right. \\
 & \left. \left. \left. - \frac{1}{N} [\langle 2H_r(x, x, \cdot), \mathfrak{X}_r^{N,-} \rangle + \langle H_r(\cdot, x, y), \mathfrak{X}_r^{N,+} \rangle + \langle H_r(x, \cdot, y), \mathfrak{X}_r^{N,+} \rangle] \right. \right. \right. \\
 & \left. \left. \left. + \frac{1}{N} [\langle G_r(x, \cdot), \mathfrak{X}_r^{N,-} \rangle + \langle G_r(\cdot, y), \mathfrak{X}_r^{N,+} \rangle - \langle H_r(\cdot, \cdot, y), \mathfrak{X}_r^{N,+} \rangle] \right. \right. \right. \\
 & \left. \left. \left. + \frac{1}{N^2} [2H_r(x, x, y) - G_r(x, y)] \right) \right\rangle, \mathfrak{X}_r^{N,+}(dx) \otimes \mathfrak{X}_r^{N,-}(dy) \right] dr.
 \end{aligned}$$

In (7.10), $\langle H_r(\cdot, \cdot, y), \mathfrak{X}_r^{N,+} \rangle$ is the integral of the function $x \mapsto H_r(x, x, y)$ with respect to $\mathfrak{X}_r^{N,+}$. A similar formula for (7.10) holds for $\mathbb{E}[\langle \phi_-, \mathfrak{X}_T^{N,-} \rangle \langle \varphi, \mathfrak{X}_T^{N,+} \otimes \mathfrak{X}_T^{N,-} \rangle]$.

PROOF. We first prove (7.9). Consider, for $s \in [0, T]$, $f_s(\mu) = f(s, \mu) := \langle P_{T-s}^{(1,1)} \varphi, \mu^+ \otimes \mu^- \rangle$. Then (7.9) follows from Corollary 6.4 by directly calculating $\mathbb{E}[K(f_r)(\mathfrak{X}_r^{N,+}, \mathfrak{X}_r^{N,-})]$ as follows: If $U_N(\vec{x}, \vec{y}) = \mu$ where $(\vec{x}, \vec{y}) \in E_N^{(m)}$, then

$$\begin{aligned}
 -Kf_r(\mu) &= \frac{1}{2N} \sum_{i=1}^m \sum_{j=1}^m \ell_{\delta_N}(x_i, y_j) \left(f_r(\mu) - f_r \left(\mu^+ - \frac{1}{N} \mathbf{1}_{\{x_i\}}, \mu^- - \frac{1}{N} \mathbf{1}_{\{y_j\}} \right) \right) \\
 &= \frac{1}{2N} \sum_{i=1}^m \sum_{j=1}^m \ell_{\delta_N}(x_i, y_j) \\
 & \quad \times \left(\frac{1}{N^2} \left(\sum_l F_r(x_i, y_l) + \sum_k F_r(x_k, y_j) - F_r(x_i, y_j) \right) \right) \\
 &= \frac{1}{2N} \sum_{i=1}^m \sum_{j=1}^m \ell_{\delta_N}(x_i, y_j) \\
 & \quad \times \left(\frac{1}{N} \langle F_r(x_i), \mu^- \rangle + \frac{1}{N} \langle F_r(y_j), \mu^+ \rangle - \frac{1}{N^2} F_r(x_i, y_j) \right) \\
 &= \frac{1}{2} \langle \ell_{\delta_N}(\langle F_r, \mu^- \rangle + \langle F_r, \mu^+ \rangle - N^{-1} F_r), \mu^+ \otimes \mu^- \rangle.
 \end{aligned}$$

For (7.10), we choose $g_s(\mu) := P_{T-s}^{(*)} \langle \phi_+ \varphi, \mu^+ \otimes \mu^+ \otimes \mu^- \rangle$ instead and follow the same argument as above. The expression on the right-hand side of (7.10)

follows from the observation that, for fixed (i, j) , we have

$$\begin{aligned}
& N^3 \left(g_r(\mu) - g_r \left(\mu^+ - \frac{1}{N} \mathbf{1}_{\{x_i\}}, \mu^- - \frac{1}{N} \mathbf{1}_{\{y_j\}} \right) \right) \\
&= \sum_q \sum_\ell H_r(x_i, x_q, y_\ell) + \sum_p \sum_\ell H_r(x_p, x_i, y_\ell) + \sum_p \sum_q H_r(x_p, x_q, y_j) \\
&\quad - \sum_\ell H_r(x_i, x_i, y_\ell) - \sum_p H_r(x_p, x_i, y_j) \\
&\quad - \sum_q H_r(x_i, x_q, y_j) + H_r(x_i, x_i, y_j) \\
&\quad + \sum_\ell G_r(x_i, y_\ell) + \sum_p G_r(x_p, y_j) - G_r(x_i, y_j) \\
&\quad - \sum_\ell H_r(x_i, x_i, y_\ell) - \sum_p H_r(x_p, x_p, y_j) + H_r(x_i, x_i, y_j).
\end{aligned}$$

The above expression can be obtained by using the inclusion-exclusion principle. \square

The next two sections will be devoted to the proof of (6.15) and (6.16), respectively.

7.3. First moment. The goal of this subsection is to prove (6.15) in Proposition 6.8. The following key lemma allows us to interchange limits. This is a crucial step in our characterization of $(\mathfrak{X}^{\infty,+}, \mathfrak{X}^{\infty,-})$, and is the step where Assumption 2.7 that $\liminf_{N \rightarrow \infty} N \delta_N^d \in (0, \infty]$ is used.

LEMMA 7.11. *Suppose Assumption 2.7 holds. Then for any $t > 0$ and any $\phi \in C_\infty^{(1,1)}$, as $\varepsilon \rightarrow 0$, each of $\mathbb{E}^\infty[\langle \ell_\varepsilon \phi, v_+(t) \rho_+ \otimes v_-(t) \rho_- \rangle]$ and $\mathbb{E}[\langle \ell_\varepsilon \phi, \mathfrak{X}_t^{N,+} \otimes \mathfrak{X}_t^{N,-} \rangle]$ converges uniformly in $N \in \mathbb{N}$ and in any initial distributions $\{(\mathfrak{X}_0^{N,+}, \mathfrak{X}_0^{N,-})\}$. Moreover,*

$$\begin{aligned}
A^\phi(t) &:= \lim_{\varepsilon \rightarrow 0} \mathbb{E}[\langle \ell_\varepsilon \phi, v_+(t) \rho_+ \otimes v_-(t) \rho_- \rangle] \\
&= \lim_{N' \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \mathbb{E}[\langle \ell_\varepsilon \phi, \mathfrak{X}_t^{N',+} \otimes \mathfrak{X}_t^{N',-} \rangle]
\end{aligned}$$

for any subsequence $\{N'\}$ along which $\{(\mathfrak{X}^{N',+}, \mathfrak{X}^{N',-})\}_N$ converges to $(\mathfrak{X}^{\infty,+}, \mathfrak{X}^{\infty,-})$ in distribution in $D([0, T], \mathfrak{M})$. Furthermore, $|A^\phi(t)| \leq \|\phi\| \|P_t^+ u_0^+\| \times \|P_t^- u_0^-\| \|\rho_+\| \|\rho_-\| \sigma(I)$.

PROOF. Since $\rho_\pm \in C(\overline{D}_\pm)$ and is strictly positive, for notational simplicity, we assume without loss of generality that $\rho_\pm = 1$. (The general case can be proved

in the same way.) Recall from (7.9) that for any $\varphi \in C_\infty^{(1,1)}$, $\phi_\pm \in C_\infty(\overline{D}_\pm \setminus \Lambda_\pm)$ and $0 \leq s \leq t < \infty$, we have

$$\begin{aligned}
 & \mathbb{E}[\langle \varphi, \mathfrak{X}_t^{N,+} \otimes \mathfrak{X}_t^{N,-} \rangle] - \mathbb{E}[\langle P_{t-s}^{(1,1)} \varphi, \mathfrak{X}_s^{N,+} \otimes \mathfrak{X}_s^{N,-} \rangle] \\
 (7.11) \quad &= -\frac{1}{2} \int_s^t \mathbb{E} \left[\left\langle \ell_{\delta_N} \left(\langle P_{t-r}^{(1,1)} \varphi, \mathfrak{X}_r^{N,-} \rangle + \langle P_{t-r}^{(1,1)} \varphi, \mathfrak{X}_r^{N,+} \rangle - \frac{1}{N} P_{t-r}^{(1,1)} \varphi \right), \right. \right. \\
 & \quad \left. \left. \mathfrak{X}_r^{N,+} \otimes \mathfrak{X}_r^{N,-} \right\rangle \right] dr.
 \end{aligned}$$

Note that $\ell_\varepsilon \phi \in C_\infty^{(1,1)}$ for ε small enough since I is disjoint from Λ_\pm . We fix $s \in (0, t)$. Putting $\ell_{\varepsilon_1} \phi$ and $\ell_{\varepsilon_2} \phi$, respectively, in the place of φ in (7.11) and then subtract, we have

$$\begin{aligned}
 \Theta &:= |\mathbb{E}[\langle \ell_{\varepsilon_1} \phi, \mathfrak{X}_t^{N,+} \otimes \mathfrak{X}_t^{N,-} \rangle] - \mathbb{E}[\langle \ell_{\varepsilon_2} \phi, \mathfrak{X}_t^{N,+} \otimes \mathfrak{X}_t^{N,-} \rangle]| \\
 &= \left| \mathbb{E}[\langle F_s, \mathfrak{X}_s^{N,+} \otimes \mathfrak{X}_s^{N,-} \rangle] \right. \\
 & \quad \left. - \frac{1}{2} \int_s^t \mathbb{E} \left[\left\langle \ell_{\delta_N} \left(\langle F_r, \mathfrak{X}_r^{N,-} \rangle + \langle F_r, \mathfrak{X}_r^{N,+} \rangle - \frac{1}{N} F_r \right), \mathfrak{X}_r^{N,+} \otimes \mathfrak{X}_r^{N,-} \right\rangle \right] dr \right| \\
 &\leq \mathbb{E}[\langle |F_s|, \overline{\mathfrak{X}}_s^{N,+} \otimes \overline{\mathfrak{X}}_s^{N,-} \rangle] \\
 & \quad + \frac{1}{2} \int_s^t \mathbb{E}[\langle \ell_{\delta_N} \langle |F_r|, \overline{\mathfrak{X}}_r^{N,-} \rangle, \overline{\mathfrak{X}}_r^{N,+} \otimes \overline{\mathfrak{X}}_r^{N,-} \rangle] \\
 & \quad + \frac{1}{2} \mathbb{E}[\langle \ell_{\delta_N} \langle |F_r|, \overline{\mathfrak{X}}_r^{N,+} \rangle, \overline{\mathfrak{X}}_r^{N,+} \otimes \overline{\mathfrak{X}}_r^{N,-} \rangle] \\
 & \quad + \frac{1}{2N} \mathbb{E}[\langle |\ell_{\delta_N} F_r|, \overline{\mathfrak{X}}_r^{N,+} \otimes \overline{\mathfrak{X}}_r^{N,-} \rangle] dr \\
 &\leq \|P_s^{(1,1)}(|F_s|)\| + \frac{1}{2} \int_s^t (A_1 + A_2 + A_3) dr,
 \end{aligned}$$

where $F_r := P_{t-r}^{(1,1)}(\ell_{\varepsilon_1} \phi - \ell_{\varepsilon_2} \phi)$, $A_1 := \|P_r^{(1,1)}(\ell_{\delta_N} P_r^{(0,1)}(|F_r|))\|$, $A_2 := \|P_r^{(1,1)}(\ell_{\delta_N} P_r^{(1,0)}(|F_r|))\|$, and $A_3 := \frac{1}{N} \|P_r^{(1,1)}(|\ell_{\delta_N} F_r|)\|$.

Clearly, $\|P_s^{(1,1)}(|F_s|)\| \leq \|F_s\|$. By applying Lemma 7.3 to the equi-continuous and uniformly bounded family,

$$\begin{aligned}
 & \{(x, y) \mapsto \phi(x) p(t-s, (a, b), (x, y)) : (a, b) \in \overline{D}_+ \times \overline{D}_-\} \\
 & \subset C_\infty^{(1,1)} \subset C(\overline{D}_+ \times \overline{D}_-),
 \end{aligned}$$

we see that $\|F_s\|$ converges to zero uniformly for $N \in \mathbb{N}$ and for any initial configuration, as ε_1 and ε_2 both tend to zero.

By definition of A_1 , (1.3), the Gaussian upper bound estimate (2.2) for the transition density p of the reflected diffusion, we have

$$\begin{aligned} A_1 &= \sup_{(a,b)} \int_{\overline{D}_+} \int_{\overline{D}_-} \ell_{\delta_N}(x, y) \left(\sup_y P_r^- (|F_r|)(x, y) \right) p(r, (a, b), (x, y)) dx dy \\ &\leq \left(\sup_{(x,y)} P_r^- (|F_r|)(x, y) \right) \frac{C(d, D_+, D_-)}{s^d} \quad \text{if } N \geq N(d, D_+, D_-). \end{aligned}$$

Using this bound, we have

$$\begin{aligned} (7.12) \quad &\int_s^t A_1 dr \\ &\leq \frac{C}{s^d} \int_s^t \sup_{(x,y)} P_r^- (|P_{t-r}^{(1,1)}(\ell_{\varepsilon_1}\phi - \ell_{\varepsilon_2}\phi)|)(x, y) dr \\ &= \frac{C}{s^d} \int_0^{t-s} \sup_{(x,y)} P_{t-w}^- (|P_w^{(1,1)}(\ell_{\varepsilon_1}\phi - \ell_{\varepsilon_2}\phi)|)(x, y) dw \\ &= \frac{C}{s^d} \int_0^{t-s} \left(\sup_{(x,y)} \int_{D_-} \left| \int_{D_+} \int_{D_-} (\ell_{\varepsilon_1}\phi - \ell_{\varepsilon_2}\phi)(\tilde{x}, \tilde{y}) \right. \right. \\ &\quad \left. \left. \times p(w, (x, b), (\tilde{x}, \tilde{y})) d\tilde{x} d\tilde{y} \right| p^-(t-w, y, b) db \right) dw \\ &\leq \frac{C}{s^d} \left(\int_0^\alpha \frac{2C}{\sqrt{w}} t^{-d/2} dw + \int_\alpha^{t-s} \|P_w^{(1,1)}(\ell_{\varepsilon_1}\phi - \ell_{\varepsilon_2}\phi)\| dw \right). \end{aligned}$$

The last inequality holds for any $\alpha \in (0, t - s)$. This is because for $\epsilon > 0$ and $w \in [0, T]$,

$$\begin{aligned} &\sup_{(x,y)} \int_{D_-} \int_{D_+} \int_{D_-} \ell_\epsilon(\tilde{x}, \tilde{y}) p(w, (x, b), (\tilde{x}, \tilde{y})) d\tilde{x} d\tilde{y} p^-(t-w, y, b) db \\ &= \sup_{(x,y)} \int_{D_-} \int_{D_+} \ell_\epsilon(\tilde{x}, \tilde{y}) p^+(w, x, \tilde{x}) p^-(t, y, \tilde{y}) d\tilde{x} d\tilde{y} \\ &\quad \text{by Chapman–Kolmogorov equation for } p^- \\ &\leq \frac{2C(d, D_+, D_-, T)}{\sqrt{w}} t^{-d/2} \quad \text{by applying the bound (2.3) on } D_+. \end{aligned}$$

Hence, from (7.12), by letting $\alpha \downarrow 0$ suitably and applying Lemma 7.3 to the equi-continuous and uniformly bounded family

$$\begin{aligned} &\{(x, y) \mapsto \phi(x) p(w, (a, b), (x, y)) : (a, b) \in \overline{D}_+ \times \overline{D}_-, w \in [\alpha, t - s]\} \\ &\subset C(\overline{D}_+ \times \overline{D}_-), \end{aligned}$$

we see that $\int_s^t A_1 dr$ converges to zero uniformly for N large enough, as ε_1 and ε_2 tend to 0. The same conclusion hold for $\int_s^t A_2 dr$ by the same argument.

So far we have not used the Assumption 2.7 of $\liminf_{N \rightarrow \infty} N\delta_N^d \in (0, \infty]$. We now use this assumption to show that $\int_s^t A_3 dr$ tends to 0 uniformly for N large enough, as ε_1 and ε_2 tend to 0. By a change of variable $r \mapsto t - w$,

$$\begin{aligned} \int_s^t A_3 dr &\leq \int_0^{t-s} \sup_{(a,b)} \int_{D_+} \int_{D_-} p(t-w, (a,b), (x,y)) \frac{1}{N} \ell_{\delta_N}(x,y) \\ &\quad \times |P_w^{(1,1)}(\ell_{\varepsilon_1}\phi - \ell_{\varepsilon_2}\phi)(x,y)| dx dy dw \\ &\leq \frac{2C_1}{s^{d/2}t^{d/2}} \int_0^\alpha \frac{1}{\sqrt{w}} dw + \frac{C_2}{Ns^d} \int_\alpha^{t-s} \|P_w^{(1,1)}(\ell_{\varepsilon_1}\phi - \ell_{\varepsilon_2}\phi)\| dw. \end{aligned}$$

The last inequality holds for any $\alpha \in (0, t-s)$, where $C_1 = C_1(d, D_+, D_-, T, \phi)$ and $C_2 = C_2(d, D_+, D_-)$. This is because for $\varepsilon > 0$ and $w \in [0, t-s]$,

$$\begin{aligned} &\sup_{(a,b)} \int \int \left(\int \int p(w, (x,y), (\tilde{x}, \tilde{y})) \ell_\varepsilon(\tilde{x}, \tilde{y}) d\tilde{x} d\tilde{y} \right) \\ &\quad \times p(t-w, (a,b), (x,y)) \frac{1}{N} \ell_{\delta_N}(x,y) dx dy \\ &\leq \frac{|I^\varepsilon|}{c_{d+1}\varepsilon^{d+1}} \sup_{(a,b)} \sup_{(\tilde{x}, \tilde{y})} \frac{1}{c_{d+1}N\delta_N^{d+1}} \int_{D_+^{\delta_N}} \int_{D_- \cap B(x, \delta_N)} p(w, (x,y), (\tilde{x}, \tilde{y})) \\ &\quad \times p(t-w, (a,b), (x,y)) dy dx \\ &\leq \frac{|I^\varepsilon|}{c_{d+1}\varepsilon^{d+1}} \frac{C(d, D_-)}{t^{d/2}} \sup_a \sup_{\tilde{x}} \frac{1}{c_{d+1}N\delta_N^{d+1}} \int_{D_+^{\delta_N}} p^+(w, x, \tilde{x}) \\ &\quad \times p^+(t-w, a, x) dx \\ &\leq \frac{|I^\varepsilon|}{c_{d+1}\varepsilon^{d+1}} \frac{C(d, D_-) C(d, D_+)}{t^{d/2} s^{d/2}} \sup_{\tilde{x}} \frac{1}{c_{d+1}N\delta_N^{d+1}} \int_{D_+^{\delta_N}} p^+(w, x, \tilde{x}) dx \\ &\quad \text{by the Gaussian upper bound (2.2) for } p^+ \\ &\leq \frac{|I^\varepsilon|}{c_{d+1}\varepsilon^{d+1}} \frac{C(d, D_+, D_-)}{s^{d/2}t^{d/2}} \frac{1}{\sqrt{w}} \quad \text{for } N \geq N(d, D_+), \end{aligned}$$

by the assumption $\liminf_{N \rightarrow \infty} N\delta_N^d \in (0, \infty]$ and the bound (2.3) on D_+ .

In conclusion, we have shown that $\{\mathbb{E}[(\ell_\varepsilon\phi, \mathfrak{X}_t^{N,+} \otimes \mathfrak{X}_t^{N,-})]\}_{\varepsilon>0}$ is a Cauchy family and converges as $\varepsilon \rightarrow 0$ to a number in $[-\infty, \infty]$. Furthermore, the convergence is uniformly for N large enough and for any initial configuration. On other hand, $\{(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})\}_N$ converges along a subsequence N' in distribution to a continuous process to $(v_+(\cdot, x) dx, v_-(\cdot, y) dy)$. Since $(\mu^+, \mu^-) \mapsto (\ell_\varepsilon\phi, \mu^+ \otimes \mu^-)$ is a bounded continuous function on \mathfrak{M} , we have

$$\mathbb{E}^\infty[(\ell_\varepsilon\phi, v_+(t) \otimes v_-(t))] = \lim_{N' \rightarrow \infty} \mathbb{E}[(\ell_\varepsilon\phi, \mathfrak{X}_t^{N',+} \otimes \mathfrak{X}_t^{N',-})]$$

for all $t \geq 0$. Hence, the proof for the convergence of $\lim_{\varepsilon \rightarrow 0} \mathbb{E}^\infty[\langle \ell_\varepsilon \phi, v_+(t) \otimes v_-(t) \rangle]$ is the same. Finally, the bound for $|A^\phi(t)|$ follows directly from Lemmas 6.7 and 7.3 as

$$\begin{aligned} & \mathbb{E}[\langle \ell_\varepsilon \phi, v_+(t) \rho_+ \otimes v_-(t) \rho_- \rangle] \\ & \leq \|\phi\| \|P_t^+ u_0^+\| \|P_t^- u_0^-\| \|\rho_+\| \|\rho_-\| \int_{D_-} \int_{D_+} \ell_\varepsilon(x, y) dx dy \end{aligned}$$

and $\int_{D_-} \int_{D_+} \ell_\varepsilon(x, y) dx dy \rightarrow \sigma(I)$ as $\varepsilon \rightarrow 0$. This bound also tells us that $A^\phi(t)$ actually lies in \mathbb{R} . \square

From the above lemma, we immediately have the following.

COROLLARY 7.12. *Suppose that Assumption 2.7 holds and $\{N'\}$ is any subsequence along which $\{(\mathfrak{X}^{N',+}, \mathfrak{X}^{N',-})\}_N$ converges to $(\mathfrak{X}^{\infty,+}, \mathfrak{X}^{\infty,-})$ in distribution in $D([0, T], \mathfrak{M})$. Then for $\phi \in C_\infty(\overline{D}_+ \setminus \Lambda_+) \cup C_\infty(\overline{D}_- \setminus \Lambda_-)$, we have*

$$\begin{aligned} & \lim_{N' \rightarrow \infty} \mathbb{E}[\langle \ell_{\delta_{N'}} \phi, \mathfrak{X}_r^{N',+} \otimes \mathfrak{X}_r^{N',-} \rangle] = A^\phi(r) \quad \text{for } r > 0 \quad \text{and} \\ (7.13) \quad & \lim_{N' \rightarrow \infty} \int_s^t \mathbb{E}[\langle \ell_{\delta_{N'}} \phi, \mathfrak{X}_r^{N',+} \otimes \mathfrak{X}_r^{N',-} \rangle] dr = \int_s^t A^\phi(r) dr \\ & \text{for } 0 < s \leq t < \infty. \end{aligned}$$

Question. It is an interesting question if one can strengthen (7.13) to include $s = 0$.

We can now present our proof for (6.15) by applying a Gronwall type argument to (7.18).

Proof of (6.15). Without loss of generality, we continue to assume $\rho_\pm = 1$. Recall from (7.6) that for $\phi_+ \in C_\infty(\overline{D}_+ \setminus \Lambda_+)$ and $0 < s \leq t < \infty$, we have

$$\begin{aligned} (7.14) \quad & \mathbb{E}[\langle \phi_+, \mathfrak{X}_t^{N,+} \rangle] - \mathbb{E}[\langle P_{t-s}^+ \phi_+, \mathfrak{X}_s^{N,+} \rangle] \\ & = -\frac{1}{2} \int_s^t \mathbb{E}[\langle \ell_{\delta_N} P_{t-r}^+ \phi_+, \mathfrak{X}_r^{N,+} \otimes \mathfrak{X}_r^{N,-} \rangle] dr. \end{aligned}$$

By (7.13), we can let $N \rightarrow \infty$ to obtain

$$(7.15) \quad \mathbb{E}^\infty[\langle \phi_+, v_+(t) \rangle] - \mathbb{E}^\infty[\langle P_{t-s}^+ \phi_+, v_+(s) \rangle] = -\frac{1}{2} \int_s^t A^{P_{t-r}^+ \phi_+}(r) dr$$

for $0 < s \leq t < \infty$. Now let $s \rightarrow 0$. By the uniform bound for (v_+, v_-) given by Lemma 6.7, the continuity of $(v_+(s), v_-(s))$ in s and Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned} (7.16) \quad & \mathbb{E}^\infty[\langle \phi_+, v_+(t) \rangle] - \langle P_t^+ \phi_+, u_0^+ \rangle \\ & = -\frac{1}{2} \int_0^t \lim_{\varepsilon \rightarrow 0} \mathbb{E}^\infty[\langle \ell_\varepsilon P_{t-r}^+ \phi_+, v_+(r) \otimes v_-(r) \rangle] dr. \end{aligned}$$

On other hand, the first equation in (4.1) reads as

$$(7.17) \quad u_+(t, x) = P_t^+ u_0^+(x) - \frac{1}{2} \int_0^t \int_I p^+(t-r, x, z) g_r(z) d\sigma(z) dr,$$

where $g_r(z) := \lambda(z)u_+(r, z)u_-(r, z)$. Multiply both sides by the ϕ_+ and then integrate over D_+ w.r.t. Lebesgue measure, we obtain

$$\begin{aligned} \langle \phi_+, u_+(t) \rangle &= \langle \phi_+, P_t^+ u_0^+ \rangle - \frac{1}{2} \int_0^t \int_I P_{t-r}^+ \phi_+(z) g_r(z) d\sigma(z) dr \\ &= \langle P_t^+ \phi_+, u_0^+ \rangle - \frac{1}{2} \int_0^t \lim_{\varepsilon \rightarrow 0} \langle \ell_\varepsilon P_{t-r}^+ \phi_+, u_+(r) \otimes u_-(r) \rangle dr, \end{aligned}$$

where we used the symmetry of P_t^+ (guaranteed by Proposition 2.3) and Lemma 7.3. This equation is (7.16) with (v_+, v_-) replaced by (u_+, u_-) .

Subtracting (7.16) from its counterpart for (u_+, u_-) , we get

$$(7.18) \quad \begin{aligned} &\langle \phi_+, u_+(t) - \mathbb{E}^\infty[v_+(t)] \rangle \\ &= -\frac{1}{2} \int_0^t \lim_{\varepsilon \rightarrow 0} \int_{D_-} \int_{D_+} \ell_\varepsilon(x, y) P_{t-r}^+ \phi_+(x) \\ &\quad \times (u_+(r, x)u_-(r, y) - \mathbb{E}^\infty[v_+(r, x)v_-(r, y)]) dx dy dr. \end{aligned}$$

The above equation holds for $\phi_+ \in C_\infty(\overline{D}_+ \setminus \Lambda_+)$ (and since ρ_+ has support in the entire domain \overline{D}_+), so we have

$$(7.19) \quad \begin{aligned} u_+(t) - \mathbb{E}^\infty[v_+(t)] &= -\frac{1}{2} \int_0^t \lim_{\varepsilon \rightarrow 0} \int_{D_-} \int_{D_+} \ell_\varepsilon(x, y) p^+(t-r, x, \cdot) \\ &\quad \times (u_+(r, x)u_-(r, y) - \mathbb{E}^\infty[v_+(r, x)v_-(r, y)]) dx dy dr \end{aligned}$$

almost everywhere in D_+ .

Let $w_\pm(t) := u_\pm(t) - \mathbb{E}^\infty[v_\pm(t)] \in \mathcal{B}_b(D_\pm)$ and $\|w_\pm(r)\|_\pm$ be the L^∞ norm in D_\pm . Then by the a.s. bound of v_\pm in Lemma 6.7 and a simple use of triangle inequality, we have $\|u_+(r, x)u_-(r, y) - \mathbb{E}^\infty[v_+(r, x)v_-(r, y)]\| \leq (\|u_0^+\| \|w_-(r)\| + \|u_0^-\| \|w_+(r)\|)$. On other hand,

$$(7.20) \quad \begin{aligned} &\int_{D_-} \int_{D_+} \ell_\varepsilon(x, y) p^+(t-r, x, a) dx dy \\ &= \frac{1}{c_{d+1}\varepsilon^{d+1}} \int_{I^\varepsilon} p^+(t-r, x, a) dx dy \\ &\leq \frac{1}{c_{d+1}\varepsilon^{d+1}} \int_{D_+^\varepsilon} \int_{B(x, \varepsilon) \cap D_-^\varepsilon} p^+(t-r, x, a) dy dx \\ &\leq \frac{|B(x, \varepsilon) \cap D_-^\varepsilon|}{c_{d+1}\varepsilon^{d+1}} \int_{D_+^\varepsilon} p^+(t-r, x, a) dx \\ &\leq \frac{C(d, D_+)}{\sqrt{t-r}} + \tilde{C}(d, D_+) \quad \text{uniformly for } a \in \overline{D}_+, \text{ for } \varepsilon < \varepsilon(d, D_+). \end{aligned}$$

Using these observations, it is easy to check that (7.19) implies

$$(7.21) \quad \|w_+(t)\|_+ \leq \int_0^t (\|u_0^+\| \|w_-(r)\| + \|u_0^-\| \|w_+(r)\|) \frac{C(d, D_+, T)}{\sqrt{t-r}} dr.$$

By the same argument, we have

$$(7.22) \quad \|w_-(t)\|_- \leq \int_0^t (\|u_0^+\| \|w_-(r)\| + \|u_0^-\| \|w_+(r)\|) \frac{C(d, D_-, T)}{\sqrt{t-r}} dr.$$

Adding (7.21) and (7.22), we have, for $C = C(\|u_0^+\|, \|u_0^-\|, d, D_+, D_-, T)$,

$$(7.23) \quad \|w_+(t)\|_+ + \|w_-(t)\|_- \leq C \int_0^t (\|w_-(r)\| + \|w_+(r)\|) \frac{1}{\sqrt{t-r}} dr.$$

By a ‘‘Gronwall type’’ argument (cf. [21]), we have $\|w_+(t)\|_+ + \|w_-(t)\|_- = 0$ for all $t \in [0, T]$. Since $T > 0$ is arbitrary, we have $\|w_+(t)\|_+ + \|w_-(t)\|_- = 0$ for all $t \geq 0$. This completes the proof for (6.15).

7.4. Second moment. In this subsection, we give a proof for (6.16) in Proposition 6.8. We start with a key lemma that is analogous to Lemma 7.11.

LEMMA 7.13. *Suppose Assumption 2.7 holds. Then for any $t > 0$ and any $\phi \in C_\infty(\overline{D}_+ \setminus \Lambda_+)$, as $\varepsilon \rightarrow 0$, each of $\mathbb{E}^\infty[\langle \phi, v_+(t) \rangle_{\rho_+} \langle \ell_\varepsilon \phi, v_+(t) \rho_+ \otimes v_-(t) \rho_- \rangle]$ and $\mathbb{E}[\langle \phi, \mathfrak{X}_t^{N,+} \rangle \langle \ell_\varepsilon \phi, \mathfrak{X}_t^{N,+} \otimes \mathfrak{X}_t^{N,-} \rangle]$ converges uniformly for $N \in \mathbb{N}$ and for any initial distributions $\{(X_0^{N,+}, \mathfrak{X}_0^{N,-})\}$. Moreover, we have*

$$\begin{aligned} B^\phi(t) &:= \lim_{\varepsilon \rightarrow 0} \mathbb{E}^\infty[\langle \phi, v_+(t) \rangle_{\rho_+} \langle \ell_\varepsilon \phi, v_+(t) \rho_+ \otimes v_-(t) \rho_- \rangle] \\ &= \lim_{N' \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \mathbb{E}[\langle \phi, \mathfrak{X}_t^{N',+} \rangle \langle \ell_\varepsilon \phi, \mathfrak{X}_t^{N',+} \otimes \mathfrak{X}_t^{N',-} \rangle] \in \mathbb{R} \end{aligned}$$

for any subsequence $\{N'\}$ along which $\{(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})\}_N$ converges to $(\mathfrak{X}^{\infty,+}, \mathfrak{X}^{\infty,-})$ in distribution in $D([0, T], \mathfrak{M})$. Similar results hold for $\phi \in C_\infty(\overline{D}_- \setminus \Lambda_-)$, but with $\langle \phi, v_-(t) \rangle_{\rho_-}$ and $\langle \phi, \mathfrak{X}_t^{N,-} \rangle$ in place of $\langle \phi, v_+(t) \rangle_{\rho_+}$ and $\langle \phi, \mathfrak{X}_t^{N,+} \rangle$, respectively.

PROOF. The proof follows the same strategy as that of Lemma 7.11, based on (7.10) rather than (7.9). We only provide the main steps. Without loss of generality, assume $\phi = \phi_+ \in C_\infty(\overline{D}_+ \setminus \Lambda_+)$ and $\rho_\pm = 1$.

Suppose $t > 0$ and $s \in (0, t)$ are fixed. Then (7.10) implies that

$$\begin{aligned} \Theta &:= |\mathbb{E}(\langle \phi, \mathfrak{X}_t^{N,+} \rangle \langle \ell_{\varepsilon_1} \phi, \mathfrak{X}_t^{N,+} \otimes \mathfrak{X}_t^{N,-} \rangle - \langle \phi, \mathfrak{X}_t^{N,+} \rangle \langle \ell_{\varepsilon_2} \phi, \mathfrak{X}_t^{N,+} \otimes \mathfrak{X}_t^{N,-} \rangle)| \\ &\leq |\mathbb{E}(P_{t-s}^{(*)}(\phi(x_1))(\ell_{\varepsilon_1}(x_2, y) - \ell_{\varepsilon_2}(x_2, y))\phi(x_2), \\ (7.24) \quad &\mathfrak{X}_t^{N,+}(dx_1) \otimes \mathfrak{X}_t^{N,+}(dx_2) \otimes \mathfrak{X}_t^{N,-}(dy))| \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{2} \int_s^t \mathbb{E} \left[\left\langle \ell_{\delta_N}(x, y) \left((H_r(x, \cdot, \cdot), \mathfrak{X}_r^{N,+} \otimes \mathfrak{X}_r^{N,-}) \right. \right. \right. \\
 & + \langle H_r(\cdot, x, \cdot), \mathfrak{X}_r^{N,+} \otimes \mathfrak{X}_r^{N,-} \rangle + \langle H_r(\cdot, \cdot, y), \mathfrak{X}_r^{N,+} \otimes \mathfrak{X}_r^{N,+} \rangle \\
 & + \frac{1}{N} [2H_r(x, x, \cdot), \mathfrak{X}_r^{N,-}] + \langle H_r(\cdot, x, y), \mathfrak{X}_r^{N,+} \rangle + \langle H_r(x, \cdot, y), \mathfrak{X}_r^{N,+} \rangle] \\
 & + \frac{1}{N} [\langle G_r(x, \cdot), \mathfrak{X}_r^{N,-} \rangle + \langle G_r(\cdot, y), \mathfrak{X}_r^{N,+} \rangle + \langle H_r(\cdot, \cdot, y), \mathfrak{X}_r^{N,+} \rangle] \\
 & \left. \left. \left. + \frac{1}{N^2} [2H_r(x, x, y) - G_r(x, y)] \right), \mathfrak{X}_r^{N,+}(dx) \otimes \mathfrak{X}_r^{N,-}(dy) \right\rangle \right] dr,
 \end{aligned}$$

where the operator $P_{t-s}^{(*)}$ is defined in (7.8),

$$\begin{aligned}
 G_r & := |P_{t-r}^{(1,1)}(\phi^2(x)(\ell_{\varepsilon_1}(x, y) - \ell_{\varepsilon_2}(x, y)))| \in C_\infty^{(1,1)} \subset C(\overline{D}_+ \times \overline{D}_-) \quad \text{and} \\
 H_r & := |P_{t-r}^{(2,1)}(\phi(x_1)\phi(x_2)(\ell_{\varepsilon_1}(x_2, y_1) - \ell_{\varepsilon_2}(x_2, y_1)))| \\
 & \in C_\infty^{(2,1)} \subset C(\overline{D}_+^2 \times \overline{D}_-).
 \end{aligned}$$

Since $(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})$ is dominated by $(\overline{\mathfrak{X}}^{N,+}, \overline{\mathfrak{X}}^{N,-})$ (see Lemma 6.7), the absolute value of each term on the right-hand side of 7.24 can be bounded by the corresponding expression with $(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})$ replaced by $(\overline{\mathfrak{X}}^{N,+}, \overline{\mathfrak{X}}^{N,-})$. Hence,

$$(7.25) \quad \Theta \leq \left(1 + \frac{1}{N}\right) \|H_s\| + \frac{1}{N} \|G_s\| + \frac{1}{2} \int_s^t \left(\sum_{i=1}^9 A_i + B_1 + B_2 \right) dr,$$

where, with abbreviations that will be explained,

$$\begin{aligned}
 A_1 & := \|P_r^{(1,1)}(\ell_{\delta_N}(x, y) \|P_r^{(1,1)} H_r(x, \cdot, \cdot)\|)\|, \\
 A_2 & := \|P_r^{(1,1)}(\ell_{\delta_N}(x, y) \|P_r^{(1,1)} H_r(\cdot, x, \cdot)\|)\|, \\
 A_3 & := \|P_r^{(1,1)}(\ell_{\delta_N}(x, y) \|P_r^{(2,0)} H_r(\cdot, \cdot, y)\|)\|, \\
 A_4 & := \frac{2}{N} \|P_r^{(1,1)}(\ell_{\delta_N}(x, y) \|P_r^{(0,1)} H_r(x, x, \cdot)\|)\|, \\
 A_5 & := \frac{1}{N} \|P_r^{(1,1)}(\ell_{\delta_N}(x, y) \|P_r^{(1,0)} H_r(\cdot, x, y)\|)\|, \\
 A_6 & := \frac{1}{N} \|P_r^{(1,1)}(\ell_{\delta_N}(x, y) \|P_r^{(1,0)} H_r(x, \cdot, y)\|)\|, \\
 A_7 & := \frac{1}{N} \|P_r^{(1,1)}(\ell_{\delta_N}(x, y) \|P_r^{(0,1)} G_r(x, \cdot)\|)\|, \\
 A_8 & := \frac{1}{N} \|P_r^{(1,1)}(\ell_{\delta_N}(x, y) \|P_r^{(1,0)} G_r(\cdot, y)\|)\|,
 \end{aligned}$$

$$\begin{aligned}
 A_9 &:= \frac{1}{N} \left\| P_r^{(1,1)}(\ell_{\delta_N}(x, y) \| P_r^{(1,0)} H_r(\cdot, \cdot, y) \|) \right\|, \\
 B_1 &:= \frac{2}{N^2} \left\| P_r^{(1,1)}(\ell_{\delta_N}(x, y) H_r(x, x, y)) \right\|, \\
 B_2 &:= \frac{1}{N^2} \left\| P_r^{(1,1)}(\ell_{\delta_N}(x, y) G_r(x, y)) \right\|.
 \end{aligned}$$

In the above, the first $P_r^{(1,1)}$ acts on the (x, y) variable, while the second $P_r^{(i,j)}$ in each A_i acts on the “ \cdot ” variable. Beware of the difference between $P_r^{(2,0)} H_r(\cdot, \cdot, y)$ and $P_r^{(1,0)} H_r(\cdot, \cdot, y)$ in A_3 and A_9 , respectively. In fact, $P_r^{(2,0)} H_r(\cdot, \cdot, y)$ is defined as the function on \overline{D}_+^2 which maps (a_1, a_2) to $\int_{D_+^2} p^{(2,0)}(r, (a_1, a_2), (x_1, x_2)) \times H_r(x_1, x_2, y) d(x_1, x_2)$, while $P_r^{(1,0)} H_r(\cdot, \cdot, y)$ is defined as the function on \overline{D}_+ which maps a_1 to $\int_{D_+} p^{(1,0)}(r, a_1, x) H_r(x, x, y) dx$.

The rest of the proof goes in the same way as that for Lemma 7.11. For example, note that

$$\begin{aligned}
 \|H_s\| &= \sup_{(a_1, a_2, b_1)} \left| \int_{D_+^2 \times D_-} \phi(x_1) \phi(x_2) (\ell_{\varepsilon_1}(x_2, y_1) - \ell_{\varepsilon_2}(x_2, y_1)) \right. \\
 &\quad \left. \times p^{(2,1)}(t - s, (a_1, a_2, b_1), (x_1, x_2, y_1)) d(x_1, x_2, y_1) \right|.
 \end{aligned}$$

By applying Lemma 7.3 to the equi-continuous and uniformly bounded family

$$\begin{aligned}
 &\{(x_1, x_2, y) \mapsto \phi(x_1) \phi(x_2) p^{(2,1)}(t - s, (a_1, a_2, b), (x_1, x_2, y)) : \\
 &\quad (a_1, a_2, b) \in \overline{D}_+^2 \times \overline{D}_-\} \\
 &\subset C(\overline{D}_+^2 \times \overline{D}_-),
 \end{aligned}$$

we see that $\|H_s\|$ converges to zero uniformly for N large enough and for any initial configuration, as ε_1 and ε_2 both tend to zero. The integral term with respect to dr can be estimated as in the proof of Lemma 7.11, using the bound (2.3), Lemma 7.3 and Assumption 2.7 that $\liminf_{N \rightarrow \infty} N \delta_N^d \in (0, \infty]$.

We have shown that $\{\mathbb{E}[\langle \phi, \mathfrak{X}_t^{N,+} \rangle \langle \ell_\varepsilon \phi, \mathfrak{X}_t^{N,+} \otimes \mathfrak{X}_t^{N,-} \rangle]\}_{\varepsilon > 0}$ is a Cauchy family which converges, as $\varepsilon \rightarrow 0$, uniformly for N large enough and for any initial configuration. Hence $B^\phi(t)$ in the statement of the lemma exists in $[-\infty, \infty]$. Finally, we have $B^\phi(t) \in \mathbb{R}$ since $|B^\phi(t)| < \infty$ by Lemmas 6.7 and 7.3. \square

From the above lemma, we immediately obtain the following.

COROLLARY 7.14. *Suppose Assumption 2.7 holds and $\{N'\}$ is a subsequence along which $\{\mathfrak{X}^{N',+}, \mathfrak{X}^{N',-}\}$ converges to $(\mathfrak{X}^{\infty,+}, \mathfrak{X}^{\infty,-})$ in distribution*

in $D([0, T], \mathfrak{M})$. Then for $\phi \in C_\infty(\overline{D}_+ \setminus \Lambda_+)$,

$$(7.26) \quad \begin{aligned} & \lim_{N' \rightarrow \infty} \mathbb{E}[\langle \phi, \mathfrak{X}_r^{N', +} \rangle \langle \ell_{\delta_{N'}} \phi, \mathfrak{X}_r^{N', +} \otimes \mathfrak{X}_r^{N', -} \rangle] = B^\phi(r) \quad \text{for } r > 0 \quad \text{and} \\ & \lim_{N' \rightarrow \infty} \int_s^t \mathbb{E}[\langle \phi, \mathfrak{X}_r^{N', +} \rangle \langle \ell_{\delta_{N'}} \phi, \mathfrak{X}_r^{N', +} \otimes \mathfrak{X}_r^{N', -} \rangle] dr \\ & = \int_s^t B^\phi(r) dr \quad \text{for } 0 < s \leq t < \infty. \end{aligned}$$

We are now ready to give the following.

PROOF OF (6.16). As before, without loss of generality we assume $\rho_\pm = 1$. Recall from (7.7) that for $\phi = \phi_+ \in C_\infty(\overline{D}_+ \setminus \Lambda_+)$ and $0 < s \leq t < \infty$, we have

$$\begin{aligned} & \mathbb{E}[\langle \phi, \mathfrak{X}_t^{N, +} \rangle^2] - \mathbb{E}[\langle P_{t-s}^+ \phi, \mathfrak{X}_s^{N, +} \rangle^2] \\ & = - \int_s^t \mathbb{E}[\langle P_{t-r}^+ \phi, \mathfrak{X}_r^{N, +} \rangle \langle \ell_{\delta_N}(P_{t-r}^+ \phi), \mathfrak{X}_r^{N, +} \otimes \mathfrak{X}_r^{N, -} \rangle] dr + o(N). \end{aligned}$$

Letting $N' \rightarrow \infty$ in (7.26), we get

$$\begin{aligned} & \mathbb{E}^\infty[\langle \phi, v_+(t) \rangle^2] - \mathbb{E}^\infty[\langle P_{t-s}^+ \phi, v_+(s) \rangle^2] \\ & = - \frac{1}{2} \int_s^t B^{P_{t-r}^+ \phi}(r) dr \end{aligned}$$

for $0 < s \leq t < \infty$. Now let $s \rightarrow 0$. By the uniform bound for (v_+, v_-) given by Lemma 6.7, the continuity of $(v_+(s), v_-(s))$ in s (guaranteed by Theorem 6.6) and the Lebesgue dominated convergence theorem, we obtain

$$(7.27) \quad \begin{aligned} & \mathbb{E}^\infty[\langle \phi_+, v_+(t) \rangle^2] - \langle P_t^+ \phi, u_0^+ \rangle^2 \\ & = - \int_0^t \lim_{\varepsilon \rightarrow 0} \mathbb{E}^\infty[\langle P_{t-r}^+ \phi, v_+(r) \rangle \langle \ell_\varepsilon P_{t-r}^+ \phi, v_+(r) \otimes v_-(r) \rangle] dr. \end{aligned}$$

On other hand, the first equation in (4.1) reads as (7.17) Chapman–Kolmogorov’s equation implies that for $t \geq s \geq 0$,

$$P_{t-s}^+ u_+(s)(x) = P_t^+ u_0^+(x) - \frac{1}{2} \int_0^s \int_I p^+(t-r, x, z) g_r(z) d\sigma(z) dr.$$

Since $g_r(z) = \lambda(z)u_+(r, z)u_-(r, z)$ is bounded and continuous for $(r, z) \in [0, T] \times \overline{D}_+$, we have

$$\frac{d}{ds} \langle \phi, P_{t-s}^+ u_+(s) \rangle = 0 - \frac{1}{2} \int_I P_{t-s}^+ \phi(z) g_s(z) d\sigma(z)$$

for all $\phi = \phi_+ \in C_\infty(\overline{D}_+ \setminus \Lambda_+)$. Therefore,

$$\begin{aligned} \langle \phi_+, u_+(t) \rangle^2 - \langle \phi_+, P_t^+ u_0^+ \rangle^2 &= \int_0^t \frac{d}{ds} \langle \phi, P_{t-s}^+ u_+(s) \rangle^2 ds \\ &= \int_0^t 2 \langle \phi, P_{t-s}^+ u_+(s) \rangle \frac{d}{ds} \langle \phi, P_{t-s}^+ u_+(s) \rangle ds \\ &= - \int_0^t \langle \phi, P_{t-s}^+ u_+(s) \rangle \int_I P_{t-s}^+ \phi(z) g_s(z) d\sigma(z) ds. \end{aligned}$$

In view of Lemma 7.3, the above equation is (7.27) with (v_+, v_-) replaced by (u_+, u_-) .

Subtracting (7.27) from its counterpart for (u_+, u_-) , we get

$$\begin{aligned} (7.28) \quad & \mathbb{E}^\infty[\langle \phi, v_+(t) \rangle^2] - \langle \phi, u_+(t) \rangle^2 \\ &= \int_0^t \lim_{\varepsilon \rightarrow 0} \mathbb{E}^\infty[\langle P_{t-r}^+ \phi, u_+(r) \rangle \langle \ell_\varepsilon P_{t-r}^+ \phi, u_+(r) \otimes u_-(r) \rangle \\ & \quad - \langle P_{t-r}^+ \phi, v_+(r) \rangle \langle \ell_\varepsilon P_{t-r}^+ \phi, v_+(r) \otimes v_-(r) \rangle] dr. \end{aligned}$$

The left-hand side of (7.28) equals $\mathbb{E}^\infty[\langle \phi, v_+(t) - u_+(t) \rangle^2]$ because $\mathbb{E}^\infty[\langle \phi, v_+(t) \rangle] = \langle \phi, u_+(t) \rangle$. Since $\mathbb{E}^\infty[\langle \ell_\varepsilon P_{t-r}^+ \phi, v_+(r) \otimes v_-(r) \rangle] = \langle \ell_\varepsilon P_{t-r}^+ \phi, u_+(r) \otimes u_-(r) \rangle$, the integrand in the right-hand side of (7.28) with respect to dr equals

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \mathbb{E}^\infty[\langle \ell_\varepsilon P_{t-r}^+ \phi, v_+(r) \otimes v_-(r) \rangle \langle P_{t-r}^+ \phi, u_+(r) - v_+(r) \rangle] \\ & \leq C \mathbb{E}^\infty[\langle P_{t-r}^+ \phi, u_+(r) - v_+(r) \rangle]. \end{aligned}$$

The constant $C = C(\phi, f, g, D_+, D_-)$ above arises from the uniform bound for $v(r)$ in Lemma 6.7 and the bound (2.3). Hence, we have

$$\mathbb{E}^\infty[\langle \phi, v_+(t) - u_+(t) \rangle^2] \leq C \int_0^t \mathbb{E}^\infty[\langle P_{t-r}^+ \phi, u_+(r) - v_+(r) \rangle] dr.$$

Letting $w_+(t) = u_+(t) - v_+(t)$, we obtain

$$(7.29) \quad \mathbb{E}^\infty[\langle \phi, w_+(t) \rangle^2] \leq C \int_0^t \mathbb{E}^\infty[\langle P_{t-r}^+ \phi, w_+(r) \rangle^2] dr.$$

We can then deduce by a ‘‘Gronwall-type’’ argument that $\mathbb{E}^\infty[\langle \phi, w_+(t) \rangle^2] = 0$ for all $t \geq 0$. In fact, by Fubinni’s theorem, the left-hand side of (7.29) equals

$$(7.30) \quad \int_{D_+} \int_{D_+} \phi(x_1) \phi(x_2) \mathbb{E}^\infty[w_+(t, x_1) w_+(t, x_2)] dx_1 dx_2,$$

and the integrand with respect to dr of the right-hand side of (7.29) is

$$\begin{aligned} & \int_{D_+} \int_{D_+} \phi(a_1) \phi(a_2) \int_{D_+} \int_{D_+} P^+(t-r, x_1, a_1) P^+(t-r, x_2, a_2) \\ & \quad \times \mathbb{E}^\infty[w_+(t, x_1) w_+(t, x_2)] dx_1 dx_2 da_1 da_2. \end{aligned}$$

Hence, for a.e. $a_1, a_2 \in D_+$, we have

$$\begin{aligned} & \mathbb{E}^\infty[w_+(t, a_1)w_+(t, a_2)] \\ & \leq C \int_0^t \int_{D_+} \int_{D_+} p^+(t-r, x_1, a_1)p^+(t-r, x_2, a_2) \\ & \quad \times \mathbb{E}^\infty[w_+(t, x_1)w_+(t, x_2)] dx_1 dx_2 dr. \end{aligned}$$

Let $\bar{f}(t) \triangleq \sup_{(a_1, a_2) \in \bar{D}_+^2} |\mathbb{E}^\infty[w_+(t, a_1)w_+(t, a_2)]|$, then the above equation asserts that $\bar{f}(t) \leq C \int_0^t \bar{f}(r) dr$. Note that $\bar{f}(r) \in L^1[0, t]$ since it is bounded. Hence, by Gronwall's lemma, we have $\bar{f}(t) = 0$ for all $t \geq 0$. This together with (7.30) yields $\mathbb{E}^\infty[\langle \phi, w_+(t) \rangle^2] = 0$. Hence $\mathbb{E}^\infty[\langle \phi, v_+(t) \rangle^2] = \langle \phi, u_+(t) \rangle^2$. The same holds for v_- . This completes the proof for (6.16). \square

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