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A semidefinite program approach for computing the maximum eigenvalue of a class of structured tensors and its applications in hypergraphs and copositivity test

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Summary

Finding the maximum eigenvalue of a symmetric tensor is an important topic in tensor computation and numerical multilinear algebra. In this paper, we introduce a new class of structured tensors called W-tensors, which not only extends the well-studied nonnegative tensors by allowing negative entries but also covers several important tensors arising naturally from spectral hypergraph theory. We then show that finding the maximum H-eigenvalue of an even-order symmetric W-tensor is equivalent to solving a structured semidefinite program and hence can be validated in polynomial time. This yields a highly efficient semidefinite program algorithm for computing the maximum H-eigenvalue of W-tensors and is based on a new structured sums-of-squares decomposition result for a nonnegative polynomial induced by W-tensors. Numerical experiments illustrate that the proposed algorithm can successfully find the maximum H-eigenvalue of W-tensors with dimension up to 10,000, subject to machine precision. As applications, we provide a polynomial time algorithm for computing the maximum H-eigenvalues of large-size Laplacian tensors of hyperstars and hypertrees, where the algorithm can be up to 13 times faster than the state-of-the-art numerical method introduced by Ng, Qi, and Zhou in 2009. Finally, we also show that the proposed algorithm can be used to test the copositivity of a multivariate form associated with symmetric extended Z-tensors, whose order may be even or odd.

KEYWORDS

copositivity, eigenvalues, Laplacian tensor, semidefinite program, spectral hypergraph, structured tensors, sum-of-squares polynomials

1 | INTRODUCTION

Finding the extremum (maximum or minimum) eigenvalue of a tensor is an important topic in tensor computation and numerical multilinear algebra. Various applications of extremum eigenvalues have been found in the current literature.[1–4] For example, the sign of the minimum (maximum) eigenvalue plays a crucial role in checking the positive

*Abbreviations:** SDP = semidefinite program; SOS = sums of squares.

semidefiniteness (negative semidefiniteness) of a symmetric tensor, which has applications in stability analysis of nonlinear autonomous systems involved in automatic control.[5] In magnetic resonance imaging,[6,7] the principal eigenvalues of an even-order symmetric tensor associated with the fiber orientation distribution of a voxel in cerebral white matter denote volume factions of multiple nerve fibers in this voxel. For a connected even-uniform hypergraph, it has been shown that the maximum H-eigenvalues of the Laplacian tensor and the signless Laplacian tensor are equivalent if and only if the hypergraph is odd-bipartite.[8] To establish correspondences between two sets of objects, Duchenne et al.[9] constructed a hypergraph and applied the eigenvector corresponding to the largest eigenvalue of its adjacency tensor for hypergraph clustering.

In view of the importance of eigenvalues of tensors, many researchers have devoted themselves to the study of numerical methods for eigenvalues of high-order tensors.[10–17] Cui et al.[18] proposed a sophisticated Jacobian semidefinite relaxation method, which computes all of the real eigenvalues of a small symmetric tensor. Generally speaking, it is an NP-hard problem to compute the eigenvalues of a tensor even though the involved tensor is symmetric.[19] However, for tensors with specific structures, the extreme eigenvalues of large-scale tensors can be computed in reasonable time by exploiting the underlying structure. For instance, an inexact curvilinear search optimization method was established to compute the extreme eigenvalues of Hankel tensors, whose dimension may reach up to one million.[20]

Nonnegative tensors are an important class of structured tensors, which arise from the study of image science, statistics, and hypergraph theory.[21,22] Ng et al.[3] proposed an iterative method for finding the maximum H-eigenvalue of an irreducible nonnegative tensor. However, the NQZ method is not always convergent for irreducible nonnegative tensors. Liu et al.[23] and Zhou et al.[24] improved the NQZ method so that the refined algorithm is always convergent. Recently, as a more general class than nonnegative tensors, the essentially nonnegative tensors were studied.[15,25,26] Hu et al.[15] showed that the maximum H-eigenvalue of an even-order essentially nonnegative tensor can be found by solving a polynomial optimization problem, which is equivalently reformulated as a semidefinite programming problem. On the other hand, there are also many important classes of structured tensors, which need not to be nonnegative tensors or essentially nonnegative tensors, such as the Laplacian tensor of a hypergraph. This then raises the following important and natural question: can we compute the maximum H-eigenvalue of a given tensor with possibly negative off-diagonal elements? This is the main motivation of this paper.

Next, we give a sketch of the copositivity of symmetric tensors.[27] Recently, it has been found that copositive tensors have important applications in vacuum stability of a general scalar potential,[28] polynomial optimization,[29,30] and the tensor complementarity problem.[31–33] With the help of copositive tensors, Kannike[28] studied the vacuum stability of a general scalar potential of a few fields, and that study showed how to find positivity conditions for more complicated potentials. Peña et al.[29] proved that recent related results for quadratic problems can be further strengthened and generalized to higher degree polynomial optimization problems over the cone of completely positive tensors or copositive tensors. Che et al.[31] showed that the tensor complementarity problem defined by a strictly copositive tensor has a nonempty and compact solution set. Song et al.[32] proved that a real tensor is strictly semipositive if and only if the corresponding tensor complementarity problem has a unique solution for any nonnegative vector and that a real tensor is semipositive if and only if the corresponding tensor complementarity problem has a unique solution for any positive vector. Their study showed that a real symmetric tensor is a (strictly) semipositive tensor if and only if it is (strictly) copositive. Song et al.[33] further presented global error bound analysis for the tensor complementarity problem defined by a strictly semipositive tensor. Thus, copositive and strictly copositive tensors play an important role in the tensor complementarity problem. In addition, Chen et al.[34] designed a numerical algorithm for checking the copositivity of general tensors, which has applications in hypergraphs and physics. In fact, copositivity is one of the crucial properties for odd-order symmetric tensors.[34]

In this article, we propose an efficient semidefinite program (SDP) algorithm to compute the maximum H-eigenvalues of even-order symmetric W-tensors, which includes nonnegative tensors and essentially nonnegative tensors as a subclass. This algorithm heavily relies on an important sums-of-squares (SOS) representation result for a nonnegative polynomial induced from a W-tensor. To proceed, we first give the SOS representation result for nonnegativity of a homogeneous polynomial induced by an even-order symmetric W-tensor, which implies that the proposed algorithm may be much more computationally efficient when the dimension is large and the explicit expression of the subtensors is available. Another interesting feature of the W-tensor is that it also covers the Laplacian tensors of hyperstars and hypertrees,[8,35] and hence, the maximum H-eigenvalues of even-order Laplacian tensors can be efficiently computed by the proposed algorithm. Numerical examples show that the proposed algorithm can be used to compute the maximum H-eigenvalue of some large-size tensors with dimension up to 10,000. Finally, the proposed algorithm is applied to test the copositivity of a multivariate form associated with symmetric extended Z-tensors, where the order of the tensor can be either odd or even.

This paper is organized as follows. In Section 2, we recall the definitions for tensors and some basic results about homogeneous polynomials. In Section 3, we propose an algorithm and show that the maximum H-eigenvalue of an even-order symmetric W-tensor can be computed by the proposed algorithm. In Section 4, we apply the proposed algorithm to compute the maximum H-eigenvalues of large-size Laplacian tensors arising from hyperstars and hypertrees. Numerical examples are also presented to show the efficiency of our method. In Section 5, we apply the approach to test the copositivity of a multivariate form associated with a symmetric extended Z-tensor, where the order of the tensor can be either odd or even. In Section 6, we present some final remarks and possible future work.

Before moving on, we make some comments on notations that will be used in the sequel. Let $\mathbb{R}^n$ be the n-dimensional real Euclidean space and $\mathbb{R}^n_+$ be the set of all nonnegative vectors in $\mathbb{R}^n$. The set consisting of all positive integers is denoted by $\mathbb{N}$. Let $m, n \in \mathbb{N}$. Denote $[n] = \{1, 2, \cdots, n\}$. Vectors are denoted by bold lowercase letters $\mathbf{x}, \mathbf{y}, \ldots$, matrices are denoted by capital letters $A, B, \ldots$, and tensors are written as calligraphic capitals such as $\mathcal{A}, \mathcal{T}, \ldots$. The identity tensor $\mathcal{I}$ with order m and dimension n is given by $\mathcal{I}_{i_1 \cdots i_m} = 1$ if $i_1 = \cdots = i_m$ and $\mathcal{I}_{i_1 \cdots i_m} = 0$ otherwise. The ith unit coordinate vector in $\mathbb{R}^n$ is denoted by $\mathbf{e}_i$, $i \in [n]$.

2 | PRELIMINARIES

In this section, we collect some basic definitions and facts that will be used later on. Then, we introduce the definition of W-tensors.

An mth-order n-dimensional tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ is a multi-array of entries $a_{i_1 i_2 \cdots i_m}$, where $i_j \in [n]$ for $j \in [m]$. If the entries $a_{i_1 i_2 \cdots i_m}$ are invariant under any permutation of their indices, then tensor $\mathcal{A}$ is called a symmetric tensor. The entries $a_{ii \cdots i}$, $i \in [n]$, are diagonal entries of $\mathcal{A}$, and the rest are off-diagonal entries.

We note that an mth-order n-dimensional symmetric tensor $\mathcal{A}$ uniquely determines an mth-degree homogeneous polynomial $f_{\mathcal{A}}(\mathbf{x})$ on $\mathbb{R}^n$: for all $\mathbf{x} = (x_1, \cdots, x_n)^T \in \mathbb{R}^n$,

$$f_{\mathcal{A}}(\mathbf{x}) = \mathcal{A}\mathbf{x}^m = \sum_{i_1, i_2, \cdots, i_m \in [n]} a_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m}. \tag{1}$$

Conversely, an mth-degree homogeneous polynomial function $f_{\mathcal{A}}(\mathbf{x})$ on $\mathbb{R}^n$ also uniquely corresponds to a symmetric tensor. Furthermore, an even-order tensor $\mathcal{A}$ is called positive semidefinite (positive definite) if

$$f_{\mathcal{A}}(\mathbf{x}) \geq 0 \ (f_{\mathcal{A}}(\mathbf{x}) > 0) \quad \text{for all} \quad \mathbf{x} \in \mathbb{R}^n \backslash \{\mathbf{0}\}.$$

Recall that for a polynomial f on $\mathbb{R}^n$, we say f is an SOS polynomial if there exist $r \in \mathbb{N}$ and polynomials f_i, $i = 1, \ldots, r$ such that $f = \sum_{i=1}^r f_i^2$. Suppose that m is even. We denote the set consisting of all SOS polynomials of degree m by $\Sigma_m^2[\mathbf{x}]$. In Equation 1, if $f_{\mathcal{A}}(\mathbf{x})$ is an SOS polynomial, then we say tensor $\mathcal{A}$ has an SOS tensor decomposition.[36,37] It is clear that a tensor with SOS tensor decomposition must be a positive semidefinite tensor, but not vice versa. For all $\mathbf{x} \in \mathbb{R}^n$, consider a homogeneous polynomial $f(\mathbf{x}) = \sum_\alpha f_\alpha \mathbf{x}^\alpha$ with degree m, where $\alpha = (\alpha_1, \cdots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n$, $\mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, and $|\alpha| := \sum_{i=1}^n \alpha_i = m$. Let $f_{m,i}$ be the coefficient associated with x_i^m, and let

$$\Omega_f = \left\{ \alpha = (\alpha_1, \cdots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n \, : f_\alpha \neq 0 \ \text{and} \ \alpha \neq m\mathbf{e}_i, \ i = 1, \ldots, n \right\}.$$

Then, f can always be written as

$$f(\mathbf{x}) = \sum_{i=1}^n f_{m,i} x_i^m + \sum_{\alpha \in \Omega_f} f_\alpha \mathbf{x}^\alpha.$$

We now recall the definitions of eigenvalues and eigenvectors for a tensor.[5,38]

Definition 1. Let $\mathbb{C}$ be the complex field. Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ be a symmetric tensor with order m and dimension n. A pair $(\lambda, \mathbf{x}) \in \mathbb{C} \times (\mathbb{C}^n \backslash \{\mathbf{0}\})$ is called an eigenvalue–eigenvector pair of tensor $\mathcal{A}$, if they satisfy

$$\mathcal{A}\mathbf{x}^{m-1} = \lambda \mathbf{x}^{[m-1]},$$

where $\mathcal{A}\mathbf{x}^{m-1}$ and $\mathbf{x}^{[m-1]}$ are all n-dimensional column vectors given by

$$\mathcal{A}\mathbf{x}^{m-1} = \left(\sum_{i_2, \cdots, i_m = 1}^n a_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m} \right)_{1 \leq i \leq n}$$

and

$$\mathbf{x}^{[m-1]} = \left(x_1^{m-1}, \dots, x_n^{m-1}\right)^T \in \mathbb{C}^n.$$

If the eigenvalue λ and the eigenvector $\mathbf{x}$ are real, then λ is called an H-eigenvalue of $\mathcal{A}$ and $\mathbf{x}$ is its corresponding H-eigenvector.[5] An important fact is that an even-order symmetric tensor is positive semidefinite (positive definite) if and only if all H-eigenvalues of the tensor are nonnegative (positive). It should be noted that even-order symmetric tensors always have H-eigenvalues.

The following lemma will play an important role in our later analysis.[5]

Lemma 1. *Let $\mathcal{A}$ be a symmetric tensor with order m and dimension n, where m is even. Denote the minimum H-eigenvalue and the maximum H-eigenvalue of $\mathcal{A}$ by $\lambda_{\min}(\mathcal{A})$ and $\lambda_{\max}(\mathcal{A})$, respectively. Then, we have*

$$\lambda_{\min}(\mathcal{A}) = \min_{\mathbf{x}\neq\mathbf{0}} \frac{\mathcal{A}\mathbf{x}^m}{\|\mathbf{x}\|_m^m} = \min_{\|\mathbf{x}\|_m=1} \mathcal{A}\mathbf{x}^m, \quad \lambda_{\max}(\mathcal{A}) = \max_{\mathbf{x}\neq\mathbf{0}} \frac{\mathcal{A}\mathbf{x}^m}{\|\mathbf{x}\|_m^m} = \max_{\|\mathbf{x}\|_m=1} \mathcal{A}\mathbf{x}^m,$$

where $\|\mathbf{x}\|_m = \left(\sum_{i=1}^n |x_i|^m\right)^{\frac{1}{m}}$.

Now, we are ready to define W-tensors formally. For $I \subseteq [n]$, we denote by $\mathbf{x}_I$ the set of variables $\{x_i : i \in I\}$ and by $\mathbb{R}[\mathbf{x}_I]$ the polynomial ring in these variables. For a set S with finitely many members, we use $|S|$ to denote its cardinality.

Definition 2. Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ be a tensor with order m and dimension n. We say $\mathcal{A}$ is a W-tensor if there exist $s \in \mathbb{N}$ with $s \le n$ and index sets $\Gamma_l \subseteq [n]$, $l \in [s]$ with $\bigcup_{l=1}^s \Gamma_l = [n]$ and $\Gamma_{l_1} \neq \Gamma_{l_2}$ for all $l_1 \neq l_2$ such that

(i) either $s = 1$ or $|(\bigcup_{l=1}^{p-1} \Gamma_l) \bigcap \Gamma_p| \le 1$ for all $2 \le p \le s$;
(ii) $\mathcal{A}\mathbf{x}^m = \sum_{l=1}^s \mathcal{A}_{\Gamma_l} \mathbf{x}_{\Gamma_l}^m$ for all $\mathbf{x} \in \mathbb{R}^n$, where for each $l \in [s]$, $\mathcal{A}_{\Gamma_l}$ is a tensor with order m and dimension $|\Gamma_l|$;
(iii) for each $l \in [s]$, the tensor $\mathcal{A}_{\Gamma_l}$ satisfies either one of the following two conditions:

(1) there exists $(\overline{i_1}, \overline{i_2}, \cdots, \overline{i_m}) \in \Gamma_l \times \Gamma_l \times \cdots \times \Gamma_l$ with $(\overline{i_1}, \overline{i_2}, \cdots, \overline{i_m}) \neq (i, i, \cdots, i)$, $i \in [n]$ such that the off-diagonal entries of $\mathcal{A}_{\Gamma_l}$ are equal to zero for all $(i_1, i_2, \cdots, i_m) \notin \{\pi(\overline{i_1}\ \overline{i_2} \cdots \overline{i_m})\}$, where $\{\pi(\overline{i_1}\ \overline{i_2} \cdots \overline{i_m})\}$ denotes a set of all the permutations of $(\overline{i_1}, \overline{i_2}, \cdots, \overline{i_m})$;
(2) all the off-diagonal entries of $\mathcal{A}_{\Gamma_l}$ are nonnegative.

Recall that a tensor $\mathcal{A}$ is called an essentially nonnegative tensor if its off-diagonal entries, $\mathcal{A}_{i_1 i_2 \cdots i_m}$ with $(i_1, i_2, \dots, i_m) \notin \{(i, i, \dots, i) : 1 \le i \le n\}$, are all nonnegative. It is obvious that essentially nonnegative tensors are W-tensors, and the converse is not true, in general. The notion of W-tensor is an extension for essentially nonnegative tensors that allows the tensor to have a block decomposition, and the off-diagonal entries of each block tensor are either all nonnegative or have only one nonzero entry (up to permutation). This special structure of W-tensors will play a key role in establishing a structured SOS decomposition for a polynomial induced by W-tensors later in Theorem 1. The following simple example illustrates the notion of W-tensors.

Example 1. Let $\mathcal{A} = (a_{i_1 i_2 i_3})$ be a third-order six-dimensional tensor such that

$$a_{111} = a_{222} = a_{333} = 1,\ a_{444} = a_{555} = a_{666} = -1,\ a_{\pi(112)} = 2,\ a_{\pi(212)} = 4,\ a_{\pi(345)} = -3,\ a_{\pi(566)} = -7$$

and $\mathcal{A}_{i_1 i_2 i_3} = 0$ for the others. We see that

$$\mathcal{A}\mathbf{x}^3 = x_1^3 + x_2^3 + x_3^3 - x_4^3 - x_5^3 - x_6^3 + 6x_1^2x_2 + 12x_1x_2^2 - 18x_3x_4x_5 - 21x_5x_6^2$$

and $\mathcal{A}\mathbf{x}^3 = f_1(x_1, x_2) + f_2(x_3, x_4, x_5) + f_3(x_5, x_6)$, where $f_1(x_1, x_2) = x_1^3 + x_2^3 + 6x_1^2x_2 + 12x_1x_2^2$, $f_2(x_3, x_4, x_5) = x_3^3 - x_4^3 - x_5^3 - 18x_3x_4x_5$, and $f_3(x_5, x_6) = -x_6^3 - 21x_5x_6^2$. Let $\Gamma_1 = \{1, 2\}, \Gamma_2 = \{3, 4, 5\}$, and $\Gamma_3 = \{5, 6\}$, and let $\mathcal{A}_{\Gamma_i}$, $i = 1, 2, 3$, be symmetric tensors such that $\mathcal{A}_{\Gamma_1}\mathbf{x}_{\Gamma_1}^3 = f_1(x_1, x_2)$, $\mathcal{A}_{\Gamma_2}\mathbf{x}_{\Gamma_2}^3 = f_2(x_3, x_4, x_5)$, and $\mathcal{A}_{\Gamma_3}\mathbf{x}_{\Gamma_3}^3 = f_3(x_5, x_6)$. Direct verification shows that $\mathcal{A}_{\Gamma_1}$ is a nonnegative tensor, $\mathcal{A}_{\Gamma_2}$ and $\mathcal{A}_{\Gamma_3}$ satisfy part (1) of (iii) in Definition 2. So, $\mathcal{A}$ is a $\mathcal{W}$-tensor.

We also note that direct verification shows that condition (i) is automatically satisfied if $\Gamma_1, \cdots, \Gamma_s$ are mutually disjoint, that is, $\Gamma_{l_1} \bigcap \Gamma_{l_2} = \emptyset$ for all $l_1 \neq l_2$. On the other hand, as Example 1 illustrated, $\Gamma_1, \cdots, \Gamma_s$ need not to be mutually disjoint, in general.

Remark 1. From Definition 2, it can be verified that if $\mathcal{A}$ is a W-tensor and $\mathcal{D}$ is a diagonal tensor, then $\mathcal{D} + \mathcal{A}$ is also a W-tensor.

As we will see later in Section 4, the significance of the W-tensor is that it not only extends the essentially nonnegative tensors but also covers important structured tensors, which naturally arise in the hypergraph theory.

3 | MAXIMUM *H*-EIGENVALUE OF A SYMMETRIC *W*-TENSOR

In this section, we show that the maximum H-eigenvalue of an even-order symmetric W-tensor can be computed by solving a semidefinite programming problem and hence can be accomplished in polynomial time. We first recall a useful lemma, which provides us a simple criterion for determining whether a homogeneous polynomial with only one mixed term is an SOS polynomial or not.[39]

Lemma 2. *Assume $b_1, b_2, \cdots, b_n \geq 0$ and $d \in \mathbb{N}$. Let $a_1, a_2, \cdots, a_n \in \mathbb{N}$ and $\sum_{i=1}^n a_i = 2d$. Consider the homogeneous polynomial $f(\boldsymbol{x})$ defined by*

$$f(\mathbf{x}) = b_1 x_1^{2d} + \cdots + b_n x_n^{2d} - \mu x_1^{a_1} \cdots x_n^{a_n}.$$

Then, the following statements are equivalent:

(i) f is a nonnegative polynomial, that is, $f(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$;
(ii) f is an SOS polynomial.

To proceed, we need the following SOS representation result for nonnegativity of a polynomial induced by a symmetric W-tensor.

Theorem 1. *Let $\mathcal{A}$ be a symmetric W-tensor with even order m and dimension n, and let $f(\mathbf{x}) = -\mathcal{A}\mathbf{x}^m$. Let $\mathcal{A}_{\Gamma_l}$ and $\Gamma_l, l \in [s]$ be defined as in Definition 2. Suppose that $f(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$. Then, there exist $h_l \in \mathbb{R}[\mathbf{x}_{\Gamma_l}], l \in [s]$ and $\rho_i^l \in \mathbb{R}$, $i \in [n]$, $l \in [s]$, such that each h_l is an SOS polynomial with $h_l(\mathbf{x}_{\Gamma_l}) = -\mathcal{A}_{\Gamma_l}\mathbf{x}_{\Gamma_l}^m + \sum_{i \in \Gamma_l} \rho_i^l x_i^m$, for each $i = 1, \ldots, n$,*

$$\sum_{l \in \Lambda(i)} \rho_i^l = 0, \quad \text{with } \Lambda(i) = \{1 \leq l \leq s : i \in \Gamma_l\},$$

and

$$f(\mathbf{x}) = h_1(\mathbf{x}_{\Gamma_1}) + \cdots + h_s(\mathbf{x}_{\Gamma_s}).$$

Proof. As $\mathcal{A}$ is a symmetric W-tensor, there exist $s \in \mathbb{N}$ with $s \leq n$ and index sets $\Gamma_l \subseteq [n]$, $l \in [s]$ with $\bigcup_{l=1}^s \Gamma_l = [n]$ and $\Gamma_{l_1} \neq \Gamma_{l_2}$ for all $l_1 \neq l_2$, such that conditions (i)–(iii) hold in Definition 2. In particular, condition (ii) shows that $f = f_1 + \cdots + f_s$ with $f_l \in \mathbb{R}[\mathbf{x}_{\Gamma_l}]$, $l \in [s]$, are homogeneous polynomials with degree m given by

$$f_l(\mathbf{x}_{\Gamma_l}) = -\mathcal{A}_{\Gamma_l}\mathbf{x}_{\Gamma_l}^m.$$

Let us prove the conclusion of this theorem by induction on s.

(1) We first prove the trivial case, that is, $s = 1$. On one hand, if part (1) of condition (iii) in Definition 2 holds, we obtain that f is SOS from Lemma 2 because $f(\mathbf{x})$ is nonnegative and has only one mixed term. On the other hand, we have

$$f(\mathbf{x}) = -\mathcal{A}\mathbf{x}^m = -\sum_{i=1}^n a_{ii\cdots i} x_i^m - \sum_{\delta_{i_1 i_2 \cdots i_m} = 0} a_{i_1 i_2 \cdots i_m} x_{i_1} \cdots x_{i_m} \geq 0, \tag{2}$$

where $\delta_{i_1 i_2 \cdots i_m}$ is equal to one if $i_1 = i_2 = \cdots = i_m$ and zero otherwise. Because part (2) of condition (iii) in Definition 2 is valid, $a_{i_1 i_2 \cdots i_m} \geq 0$ for all $i_1, i_2, \cdots, i_m \in [n]$ with $\delta_{i_1 i_2 \cdots i_m} = 0$. Then, $\mathcal{A}$ is an even-order essentially nonnegative tensor. It follows from the study by Hu et al.[15] (Proposition 3.1) that f is SOS (see also a more recent study[37]), and hence, the desired result holds.

(2) Initial step. Let $1 < s \leq n$. We start with $s = 2$. Then, it holds that $f = f_1 + f_2$, where

$$f_1(\mathbf{x}_{\Gamma_1}) = -\mathcal{A}_{\Gamma_1}\mathbf{x}_{\Gamma_1}^m \text{ and } f_2(\mathbf{x}_{\Gamma_2}) = -\mathcal{A}_{\Gamma_2}\mathbf{x}_{\Gamma_2}^m.$$

From condition (i) of Definition 2, we see that there exists $i_0 \in [n]$ such that $\Gamma_1 \bigcap \Gamma_2 \subseteq \{i_0\}$.

If $\Gamma_1 \bigcap \Gamma_2 = \emptyset$, it can be easily verified that $f_l \geq 0, l \in \{1, 2\}$ because $f(\mathbf{x}) \geq 0$. So, condition (iii) of Definition 2 implies that each f_l is either a homogeneous polynomial with only one mixed term or a homogeneous polynomial such that $f_l(\mathbf{x}_{\Gamma_l}) = -\mathcal{A}_l(\mathbf{x}_{\Gamma_l})^m$ and $\mathcal{A}_l$ is an even-order essentially nonnegative tensor. This means that f_1,f_2 are SOS polynomials. Thus, $f = f_1 + f_2$ is SOS, and the desired result follows with $h_1 = f_1, h_2 = f_2$.

If $\Gamma_1 \bigcap \Gamma_2 = \{i_0\}$, denote $\widehat{\Gamma}_l = \Gamma_l \backslash \{i_0\}, l \in \{1,2\}$. Without loss of generality, we assume $\widehat{\Gamma}_l \neq \emptyset, l \in \{1,2\}$, and we order Γ_l, $l = 1, 2$, in such a way that $\mathbf{x}_{\Gamma_1} = (\mathbf{x}_{\widehat{\Gamma}_1}, x_{i_0})$ and $\mathbf{x}_{\Gamma_2} = (x_{i_0}, \mathbf{x}_{\widehat{\Gamma}_2})$. We first see that

$$\alpha = \inf\left\{f_1(\mathbf{a}, 1) : \mathbf{a} \in \mathbb{R}^{|\widehat{\Gamma}_1|}\right\} > -\infty. \tag{3}$$

Otherwise, there exists $\mathbf{a_k} \in \widehat{\Gamma}_1$ such that $f_1(\mathbf{a_k}, 1) \to -\infty$. Let $\mathbf{1}_{\widehat{\Gamma}_2} \in \mathbb{R}^{|\widehat{\Gamma}_2|}$ be the vector such that all its entries are equal to 1. Note that $(\mathbf{a_k}, 1, \mathbf{1}_{\widehat{\Gamma}_2}) \in \mathbb{R}^n$, and so

$$0 \leq f\left(\mathbf{a_k}, 1, \mathbf{1}_{\widehat{\Gamma}_2}\right) = f_1(\mathbf{a_k}, 1) + f_2\left(1, \mathbf{1}_{\widehat{\Gamma}_2}\right) \to -\infty,$$

which is impossible, and hence, Equation 3 is true. Let $q(\mathbf{x}) = \alpha x_{i_0}^m$. Define $h_1 = f_1 - q$ and $h_2 = f_2 + q$. We now verify that

$$h_1 = f_1 - q \geq 0 \quad \text{over} \quad \mathbb{R}^{|\Gamma_1|} \quad \text{and} \quad h_2 = f_2 + q \geq 0 \quad \text{over} \quad \mathbb{R}^{|\Gamma_2|}.$$

To see this, take any $(\mathbf{a}, 1) \in \mathbb{R}^{|\Gamma_1|}$ with $\mathbf{a} \in \mathbb{R}^{|\widehat{\Gamma}_1|}$, and so

$$h_1(\mathbf{a}, 1) = f_1(\mathbf{a}, 1) - \alpha = f_1(\mathbf{a}, 1) - \inf\left\{f_1(\mathbf{a}, 1) : \mathbf{a} \in \mathbb{R}^{|\widehat{\Gamma}_1|}\right\} \geq 0.$$

Noting that h_1 is a homogeneous polynomial with an even order degree m, it follows that

$$h_1(\mathbf{x}_{\Gamma_1}) \geq 0 \quad \text{for all} \quad \mathbf{x}_{\Gamma_1} = (\mathbf{a}, s) \in \mathbb{R}^{|\widehat{\Gamma}_1|} \times \mathbb{R} \quad \text{with} \quad s \neq 0.$$

Then, by the continuity of h_1, it shows that $h_1 \geq 0$ over $\mathbb{R}^{|\Gamma_1|}$. Moreover, take any $(1, \mathbf{b}) \in \mathbb{R}^{|\Gamma_2|}$ with $\mathbf{b} \in \mathbb{R}^{|\widehat{\Gamma}_2|}$. Fix an arbitrary $\epsilon > 0$. Take $\mathbf{z}_\epsilon \in \mathbb{R}^{|\widehat{\Gamma}_1|}$ such that $f_1(\mathbf{z}_\epsilon, 1) \leq \inf\{f_1(\mathbf{x}_{|\widehat{\Gamma}_1|}, 1) : \mathbf{x}_{|\widehat{\Gamma}_1|} \in \mathbb{R}^{|\widehat{\Gamma}_1|}\} + \epsilon$. Then,

$$\begin{aligned} h_2(1, \mathbf{b}) &= f_2(1, \mathbf{b}) + \inf\left\{f_1\left(\mathbf{x}_{|\widehat{\Gamma}_1|}, 1\right) : \mathbf{x}_{|\widehat{\Gamma}_1|} \in \mathbb{R}^{|\widehat{\Gamma}_1|}\right\} \\ &\geq f_2(1, \mathbf{b}) + f_1(\mathbf{z}_\epsilon, 1) - \epsilon \\ &= f(\mathbf{z}_\epsilon, 1, \mathbf{b}) - \epsilon \geq -\epsilon, \end{aligned}$$

where the last inequality follows by the fact that $f(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$. Letting $\epsilon \to 0$, we have $h_2(1, \mathbf{b}) \geq 0$ for all $\mathbf{b} \in \mathbb{R}^{|\widehat{\Gamma}_2|}$. Similarly, using the fact that h_2 is a continuous homogeneous polynomial with even degree m, we see that $h_2 \geq 0$ over $\mathbb{R}^{|\Gamma_2|}$.

To finish the proof of the initial step, that is, $s = 2$, it remains to be observed from condition (iii) of Definition 2 that each $h_l \in \mathbb{R}[\mathbf{x}_{\Gamma_l}]$ is either a polynomial with only one mixed term or a homogeneous polynomial such that $h_l(\mathbf{x}_{\Gamma_l}) = -\mathcal{H}_l {\mathbf{x}_{\Gamma_l}}^m$ and $\mathcal{H}_l$ is an essentially nonnegative tensor. Similar to the analysis of **(1)**, we know that $h_l, l \in \{1,2\}$ are SOS. Therefore, the conclusion holds with $s = 2$.

(3) Induction step. Suppose that the conclusion is true for $s = p - 1$. We now examine the case with $s = p$. In this case, we have

$$f = f_1 + f_2 + \cdots + f_p = \hat{f} + f_p,$$

where $\hat{f} = f_1 + f_2 + \cdots + f_{p-1}$. Then, $\hat{f}$ is a homogeneous polynomial with $\hat{f} \in \mathbb{R}[\mathbf{x}_{\bigcup_{l=1}^{p-1}\Gamma_l}]$. By the definition of W-tensor, there exists $i_p \in [n]$ such that

$$\left(\bigcup_{l=1}^{p-1} \Gamma_l\right) \bigcap \Gamma_p \subseteq \{i_p\}.$$

Similarly to the proof in initial step **(2)**, there exist $\hat{h} \in \mathbb{R}[\mathbf{x}_{\bigcup_{l=1}^{p-1}\Gamma_l}]$, $h_p \in \mathbb{R}[\mathbf{x}_{\Gamma_p}]$ and a finite number $\rho \in \mathbb{R}$ such that

$$\hat{h} = \hat{f} - \rho x_{i_p}^m \geq 0 \quad \text{over} \quad \mathbb{R}^{|\cup_{l=1}^{p-1}\Gamma_l|}, \quad h_p = f_p + \rho x_{i_p}^m \geq 0 \quad \text{over} \quad \mathbb{R}^{|\Gamma_p|}.$$

By the induction hypothesis that the conclusion holds when $s = p - 1$, we know that $\hat{h} = h_1 + h_2 + \cdots + h_{p-1}$, where $h_l \in \mathbb{R}[\mathbf{x}_{\Gamma_l}]$, $l \in [p-1]$, such that each h_l is an SOS polynomial with $h_l(\mathbf{x}_{\Gamma_l}) = \mathcal{A}_{\Gamma_l}\mathbf{x}_{\Gamma_l}^m + \sum_{i \in \Gamma_l} \rho_i^l x_i^m$ for some $\rho_i^l \in \mathbb{R}$ satisfying $\sum_{l=1}^{p-1} \sum_{i \in \Gamma_l} \rho_i^l = 0$ and

$$\hat{f}(\mathbf{x}) = h_1(\mathbf{x}_{\Gamma_1}) + \cdots + h_{p-1}(\mathbf{x}_{\Gamma_{p-1}}).$$

On the other hand, $h_p = f_p + \rho x_{i_p}^m$ is either a homogeneous polynomial with one mixed term or a homogeneous polynomial such that $h_p(\mathbf{x}) = -\mathcal{H}_p \mathbf{x}^m$ and $\mathcal{H}_p$ is an essentially nonnegative tensor. Thus, h_p is SOS, and the desired results hold. □

Theorem 2. *Let $\mathcal{A}$ be a symmetric W-tensor with even order m and dimension n. Then, it holds that*

$$\lambda_{\max}(\mathcal{A}) = \min_{t\in\mathbb{R},\mu\in\mathbb{R}} \left\{ t \mid t - \mathcal{A}\mathbf{x}^m + \mu\left(\|\mathbf{x}\|_m^m - 1\right) \in \Sigma_m^2[\mathbf{x}] \right\}. \tag{4}$$

Moreover, let $\mathcal{A}_{\Gamma_l}, l \in [s]$ be subtensors of $\mathcal{A}$ as defined in Definition 2. Then, we also have

$$\lambda_{\max}(\mathcal{A}) = \min_{t,\rho_i^l\in\mathbb{R}} \left\{ t : -\mathcal{A}_{\Gamma_l}\mathbf{x}_{\Gamma_l}^m + \sum_{i\in\Gamma_l} \rho_i^l x_i^m \in \Sigma_m^2[\mathbf{x}_{\Gamma_l}], l \in [s], \sum_{l\in\Lambda(i)} \rho_i^l \le t, i \in [n] \right\}, \tag{5}$$

where, for each $i = 1, \dots, n$, $\Lambda(i) = \{1 \le l \le s : i \in \Gamma_l\}$.

Proof. Let $t^* = \lambda_{\max}(\mathcal{A})$. By Lemma 1, we have

$$f(\mathbf{x}) = t^* \sum_{i=1}^{m} x_i^m - \mathcal{A}\mathbf{x}^m \ge 0, \quad \text{for all} \quad \mathbf{x} \in \mathbb{R}^n.$$

Let $\mathcal{B} = \mathcal{A} - t^*\mathcal{I}$. It can be easily verified that $f(\mathbf{x}) = -\mathcal{B}\mathbf{x}^m \ge 0$ for all $\mathbf{x} \in \mathbb{R}^n$. By Definition 2 and Remark 1, we know that $\mathcal{B}$ is a symmetric W-tensor because $\mathcal{A}$ is a W-tensor. Let $\mathcal{B}_{\Gamma_l}, l \in [s]$ denote the subtensors of $\mathcal{B}$. So, it is easy to verify that $\mathcal{B}_{\Gamma_l}$ only differs $\mathcal{A}_{\Gamma_l}$ by a diagonal tensor on $\mathbb{R}^{|\Gamma_l|}$. By Theorem 1, there exist SOS polynomials $h_l \in \mathbb{R}[\mathbf{x}_{\Gamma_l}], l \in [s]$ such that

$$f = h_1 + h_2 + \cdots + h_s \tag{6}$$

and

$$h_l(\mathbf{x}_{\Gamma_l}) = -\mathcal{A}_{\Gamma_l}\mathbf{x}_{\Gamma_l}^m + \sum_{i\in\Gamma_l} \bar{\rho}_i^l x_i^m \tag{7}$$

for some $\bar{\rho}_i^l \in \mathbb{R}$. Thus, $f(\mathbf{x}) = -\mathcal{B}\mathbf{x}^m = -\mathcal{A}\mathbf{x}^m + t^*\|x\|_m^m$ is an SOS polynomial, and so, $t = \mu = t^*$ is feasible for the problem

$$\min_{t\in\mathbb{R},\mu\in\mathbb{R}} \left\{ t \mid t - \mathcal{A}\mathbf{x}^m + \mu\left(\|\mathbf{x}\|_m^m - 1\right) \in \Sigma_m^2[\mathbf{x}] \right\}.$$

So, it follows that

$$\min_{t\in\mathbb{R},\mu\in\mathbb{R}} \left\{ t \mid t - \mathcal{A}\mathbf{x}^m + \mu\left(\|\mathbf{x}\|_m^m - 1\right) \in \Sigma_m^2[\mathbf{x}] \right\} \le t^* = \lambda_{\max}(\mathcal{A}).$$

On the other hand, for all $\mathbf{x} \in \mathbb{R}^n$, take any (t, μ) with $t - \mathcal{A}\mathbf{x}^m + \mu(\|\mathbf{x}\|_m^m - 1) \in \Sigma_m^2[\mathbf{x}]$. Then, it holds that

$$t - \mathcal{A}\mathbf{x}^m + \mu\left(\|\mathbf{x}\|_m^m - 1\right) \ge 0, \quad \forall\, \mathbf{x} \in \mathbb{R}^n.$$

Thus, we know that $t \ge \mathcal{A}\mathbf{x}^m$ for all $\mathbf{x} \in \mathbb{R}^n$ with $\|\mathbf{x}\|_m^m = 1$, which implies that $t \ge t^*$. Therefore, Equation 4 follows.

To see the second part, by Equations 6 and 7, we see that

$$\sum_{l\in\Lambda(i)} \bar{\rho}_i^l = t^*.$$

Combining this with the fact that each h_l is an SOS polynomial, gives us that $t = t^*$ and $\rho_i^l = \bar{\rho}_i^l$ are feasible for

$$\min_{t,\rho_i^l\in\mathbb{R}} \left\{ t : -\mathcal{A}_{\Gamma_l}\mathbf{x}_{\Gamma_l}^m + \sum_{i\in\Gamma_l} \rho_i^l x_i^m \in \Sigma_m^2[\mathbf{x}_{\Gamma_l}], l \in [s], \sum_{l\in\Lambda(i)} \rho_i^l \le t, i \in [n] \right\},$$

which implies that

$$\min_{t,\rho_i^l\in\mathbb{R}} \left\{ t : -\mathcal{A}_{\Gamma_l}\mathbf{x}_{\Gamma_l}^m + \sum_{i\in\Gamma_l} \rho_i^l x_i^m \in \Sigma_m^2[\mathbf{x}_{\Gamma_l}], l \in [s], \sum_{l\in\Lambda(i)} \rho_i^l \le t, i \in [n] \right\} \le t^*.$$

Moreover, by a direct computation, the reverse inequality always holds such that

$$\min_{t,\rho_i^l\in\mathbb{R}} \left\{ t : -\mathcal{A}_{\Gamma_l}\mathbf{x}_{\Gamma_l}^m + \sum_{i\in\Gamma_l} \rho_i^l x_i^m \in \Sigma_m^2[\mathbf{x}_{\Gamma_l}], l \in [s], \sum_{l\in\Lambda(i)} \rho_i^l \le t, i \in [n] \right\} \ge t^*.$$

Therefore, the desired results hold. □

Remark 2. (Polynomial time solvability of maximum H-eigenvalue of the symmetric W-tensor)
As explained by Hu et al.,[37] checking an SOS polynomial can be equivalently rewritten as a semidefinite programming problem. Then, Theorem 2 shows that if $\mathcal{A}$ is a symmetric W-tensor with even order, its maximum H-eigenvalue can be found by solving a semidefinite problem and hence can be validated in polynomial time.

According to the equality (4), we obtain a basic algorithm for finding the maximum H-eigenvalue of a W-tensor.

Algorithm 1

Given $m, n \in \mathbb{N}$ and let m be even.

1 Input an m-th order n-dimensional W-tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$.

2 Let $f_{\mathcal{A}}(\mathbf{x}) = \mathcal{A}\mathbf{x}^m$ and $g(\mathbf{x}) = \|\mathbf{x}\|_m^m - 1 = \sum_{i\in[n]} x_i^m - 1$. Solve the following optimization problem

$$\min_{t\in\mathbb{R},\mu\in\mathbb{R}} \{t \mid t - f_{\mathcal{A}}(\mathbf{x}) + \mu g(\mathbf{x}) \in \Sigma_m^2[\mathbf{x}]\}.$$

3 Output t.

When the subtensors $\mathcal{A}_l$ of a W-tensor $\mathcal{A}$ can be explicitly exploited, we have the following refined algorithm for computing the maximum eigenvalue of a W-tensor.

Algorithm 2

Given $m, n \in \mathbb{N}$ and let m be even.

1 Input an m-th order n-dimensional W-tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ with its subtensors $\mathcal{A}_{\Gamma_l}$, $l \in [s]$ as defined in Definition 2.

2 Solve the following optimization problem

$$\min_{t,\rho_i^l\in\mathbb{R}} \{t : -\mathcal{A}_{\Gamma_l}\mathbf{x}_{\Gamma_l}^m + \sum_{i\in\Gamma_l} \rho_i^l x_i^m \in \Sigma_m^2[\mathbf{x}_{\Gamma_l}], l \in [s], \sum_{l\in\Lambda(i)} \rho_i^l \le t,\ i \in [n]\},$$

where $\Lambda(i) = \{1 \le l \le s : i \in \Gamma_l\}$.

3 Output t.

Once the maximum H-eigenvalue t of $\mathcal{A}$ is computed, the corresponding H-eigenvector $\mathbf{x}$ could be extracted from the optimization problem in step 2, using the approach proposed by Henrion et al.[40]

Remark 3. (Comparison with Algorithms 1 and 2)
It is known that checking whether a polynomial with even degree m and dimension n is SOS or not can be equivalently reformulated as a semidefinite programming problem, which can be done via the MATLAB Toolbox YALMIP.[41,42] Thus, Algorithms 1 and 2 can both be used to compute the maximum H-eigenvalue of a W-tensor via YALMIP and the commonly used SDP solver such as SeDuMi.[43] Algorithm 2 requires the explicit expression of the subtensors of a W-tensor, whereas Algorithm 1 does not need this information. On the other hand, in the case where n is large and the explicit expression of the subtensors is available, Algorithm 2 can be much more computationally efficient. Indeed, note that checking whether a polynomial with even degree m and dimension n is SOS or not leads to a semidefinite programming problem whose size (the maximum of the number of the variables and the number of involved linear constraints) is $\binom{n+m}{m}$. So, for an mth-order n-dimensional tensor, Algorithm 1 amounts to solving a semidefinite programming problem with size $\binom{n+m}{m}$, whereas Algorithm 2 leads to a semidefinite programming problem with size $s\binom{k+m}{m}$, where $k = \max\{|\Gamma_l| : 1 \le l \le s\}$, which is much smaller than $\binom{n+m}{\mathrm{m}}$ if n is large and k is small. For example, if $s = n/4$, $k = 4$, and $m = 4$, then $\binom{n+m}{m}$ is of the order n^4, whereas $s\binom{k+m}{m}$ is of the order n.

Next, we present an example to illustrate that Algorithm 2 can be used to compute the maximum H-eigenvalue for large-size W-tensors (dimension up to 10,000).

Example 2. Let $n = 4k$ with $k \in \mathbb{N}$. Consider the symmetric tensor $\mathcal{A}$ with order 4 and dimension n, where

$$\mathcal{A}_{1111} = \mathcal{A}_{2222} = \cdots = \mathcal{A}_{nnnn} = n,$$

$$\mathcal{A}_{i_1 i_2 i_3 i_4} = -\frac{1}{6}, \quad \text{for all } (i_1, i_2, i_3, i_4) = \pi(4l-3, 4l-2, 4l-1, 4l), l = 1, \cdots, \frac{n}{4},$$

and $\mathcal{A}_{i_1 i_2 i_3 i_4} = 0$ otherwise. Here, $\pi(i_1, \cdots, i_4)$ denotes all the possible permutations of $(i_1, \cdots, i_4)$. Clearly, $\mathcal{A}$ is not a nonnegative tensor (or an essentially nonnegative tensor). The tensor $\mathcal{A}$ corresponds to a unique homogeneous polynomial

$$f_{\mathcal{A}}(\mathbf{x}) = \mathcal{A}\mathbf{x}^m = n\left(x_1^4 + \cdots + x_n^4\right) - 4\sum_{l=1}^{n/4} x_{4l-3}\, x_{4l-2}\, x_{4l-1}\, x_{4l}.$$

TABLE 1 Test results for Example 2

m	n	True $\lambda^H_{\max}(\mathcal{A})$	Est. $\lambda^H_{\max}(\mathcal{A})$	YALMIP	SeDuMi
4	500	501	501.0000	10.2	4.3
4	1,000	1,001	1,001.0000	29.9	12.6
4	2,000	2,001	2,001.0000	112.5	26.3
4	5,000	5,001	5,001.0000	650.8	66.4
4	10,000	10,001	10,001.0000	2,164.8	136.7

Let $\Gamma_l = \{4l-3, 4l-2, 4l-1, 4l\},\ l = 1, \cdots, \frac{n}{4}$. Then, by Definition 2, $\mathcal{A}$ is a W-tensor with subtensors $\mathcal{A}_{\Gamma_l}, l = 1, \ldots, \frac{n}{4}$, defined by

$$\mathcal{A}_{\Gamma_l}\mathbf{x}^4_{\Gamma_l} = n\left(x^4_{4l-3} + x^4_{4l-2} + x^4_{4l-1} + x^4_{4l}\right) - 4x_{4l-3}\,x_{4l-2}\,x_{4l-1}\,x_{4l}.$$

Moreover, using geometric mean inequality, we can directly verify that the true maximum H-eigenvalue of $\mathcal{A}$ is $\lambda^H_{\max}(\mathcal{A}) = n + 1$.

We compute the maximum H-eigenvalue of $\mathcal{A}$ using Algorithm 2 for the cases of $n = 500$ to 10,000, where True $\lambda^H_{\max}(\mathcal{A})$ and Est. $\lambda^H_{\max}(\mathcal{A})$ denote the true maximum H-eigenvalue and an estimated maximum H-eigenvalue, respectively. The results are summarized in Table 1. Obviously, Algorithm 2 finds the maximum H-eigenvalue of $\mathcal{A}$ exactly. The CPU time (measured in seconds) for converting the SOS problem to SDP via the MATLAB toolbox YALMIP and solving SDP via the commonly used SDP software SeDuMi[43] is reported in columns YALMIP and SeDuMi, respectively. When the dimension of the tensor increases, the CPU time for YALMIP and SeDuMi grows steadily.

4 | APPLICATIONS IN SPECTRA OF HYPERGRAPHS

Throughout this part, unless stated otherwise, a hypergraph means an undirected simple k-uniform hypergraph $G = (V, E)$, where $E \subseteq 2^V$. The elements of $V = V(G)$, which is labeled as $[n] = \{1, 2, \cdots, n\}$, are referred to as vertices, and the elements of $E = E(G)$ are called edges. Recall that a simple hypergraph is a hypergraph where none of its edges are contained within another. We say a hypergraph is m-uniform if, for every edge $e \in E$, it holds that $|e| = m$. For a subset $S \subseteq [n]$, we denote by E_S the set of edges $\{e \in E | S \cap e \neq \emptyset\}$. For a vertex $i \in V$, we simplify $E_{\{i\}}$ as E_i. The cardinality of the set E_i is defined as the degree of the vertex i, which is denoted by d_i.

Now, we first introduce several basic definitions that will be studied. The following definition for Laplacian and signless Laplacian tensors was proposed by Qi.[4] For other related definitions, see studies by Hu et al.[44] and Li et al.[45]

Definition 3. (Laplacian and signless Laplacian of hypergraphs)
Let $G = (V, E)$ be an m-uniform hypergraph, where $V = \{1, 2, \cdots, n\}$. The adjacency tensor of G is defined as the mth-order n-dimensional tensor $\mathcal{A}$ with

$$a_{i_1 i_2 \cdots i_m} = \begin{cases} \frac{1}{(m-1)!}, & \{i_1, i_2, \cdots, i_m\} \in E, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathcal{D}$ be an mth-order n-dimensional diagonal tensor with its ith diagonal elements equal to the degree of vertex i, for all $i \in [n]$. Then $\mathcal{L} = \mathcal{D} - \mathcal{A}$ is the Laplacian tensor of hypergraph G, and $\mathcal{Q} = \mathcal{D} + \mathcal{A}$ is the signless Laplacian tensor of hypergraph G.

It can be easily verified that the signless Laplacian tensor of a hypergraph is a nonnegative tensor and hence, in particular, a W-tensor. On the other hand, off-diagonal elements of the Laplacian tensor of a hypergraph can be negative. In the following, we show that the Laplacian tensors of two important types of hypergraphs are indeed W-tensors, and hence, their maximum H-eigenvalues can be found in polynomial time via Algorithm 2.

4.1 | Laplacian tensors of hyperstars

Moving on, we first recall the concept of a hyperstar.[8]

Definition 4. Let $G=(V,E)$ be an m-uniform hypergraph. If there is a disjoint partition of the vertex set V as $V = V_0 \cup V_1 \cup \cdots \cup V_s$ such that $|V_0| = 1$ and $|V_1| = |V_2| = \cdots = |V_s| = m-1$, and $E = \{V_0 \cup V_{\rm i} \mid i \in [s]\}$, then G is called a *hyperstar*.

It is an immediate fact that, with a possible renumbering of vertices, all the hyperstars with the same size are identical.

Theorem 3. *Let $G=(V,E)$ be a hyperstar. Then, its Laplacian tensor is a symmetric W-tensor.*

Proof. Let $\mathcal{A}$ and $\mathcal{L}$ be the adjacent tensor and the Laplacian tensor of G, respectively. Then, $\mathcal{L} = \mathcal{D} - \mathcal{A}$, where $\mathcal{D}$ is the diagonal tensor with its diagonal entries d_i, that is, the degree of the vertex $i \in [n]$. Assume $V = [n]$ and $|E| = s$. Let $V_0 = \{i^0\}$ and $V_l = \{i_1^l, i_2^l, \cdots, i_{m-1}^l\}, l \in [s]$. Define $\Gamma_l = \{i^0, i_1^l, i_2^l, \cdots, i_{m-1}^l\}$, $l \in [s]$. It holds that $\left(\bigcup_{l=1}^{p-1} \Gamma_l\right) \bigcap \Gamma_p = \{i^0\}$, for all $2 \le p \le s$. By Definitions 3 and 4, we know that

$$\mathcal{L}_{i_1 i_2 \cdots i_m} = \begin{cases} d_i, & \text{if } \ i_1 = i_2 = \cdots = i_m = i, \\ -\frac{1}{(m-1)!}, & \text{if } \ \{i_1, i_2, \cdots, i_m\} = V_0 \cup V_l \ \text{ for some } \ l \in [s], \\ 0, & \text{otherwise}, \end{cases}$$

and

$$\mathcal{L}\mathbf{x}^m = \sum_{i=1}^m d_i x_i^m - \sum_{\{i_1,\cdots,i_m\}\in E} \frac{1}{(m-1)!} x_{i_1} x_{i_2} \cdots x_{i_m}.$$

For any $l \in [s]$, define the mth-order $|\Gamma_l|$-dimensional tensors $\mathcal{L}_{\Gamma_l}$ such that

$$(\mathcal{L}_{\Gamma_l})_{i_1 i_2 \cdots i_m} = \begin{cases} \frac{d_{i^0}}{s}, & \text{if } \ i_1 = i_2 = \cdots = i_m = i^0, \\ d_{i_j^l}, & \text{if } \ i_1 = i_2 = \cdots = i_m = i_j^l, \ j = 1, 2, \cdots, m-1, \\ -\frac{1}{(m-1)!}, & \text{if } \ \{i_1, i_2, \cdots, i_m\} = V_0 \cup V_l, \\ 0, & \text{otherwise}. \end{cases}$$

Direct verification gives us that $\mathcal{L}\mathbf{x}^m = \sum_{l=1}^s \mathcal{L}_{\Gamma_l} \mathbf{x}_{\Gamma_l}^m$ for all $\mathbf{x} \in \mathbb{R}^n$ and $\mathcal{L}$ is a symmetric W-tensor. □

4.2 | Laplacian tensors of hypertrees

The notion of hypertree is defined on the basis of a usual graph. Let $G = (V, E)$ be a usual graph, which is a two-uniform hypergraph. If any two vertices of G are connected by exactly one path, then G is called a tree. In particular, a tree is called a rooted oriented tree if one vertex has been designated the root, in which case the edges have a natural orientation that orders the edge from the top of the rooted tree (root) to the bottom and from the left to the right.

According to Hu et al.,[35] an m-uniform hypertree $G = (V, E)$ is the mth power of a tree such that there exist a tree $T = (V_0, E_0)$ and additional vertices $\bar{V} = \{i_{e,1}, i_{e,2}, \cdots, i_{e,m-2} \mid e \in E_0\}$ satisfying $V = V_0 \cup \bar{V}$ and $E = \{e \cup \{i_{e,1}, \cdots, i_{e,m-2}\} \mid e \in E_0\}$. If the vertices of $\bar{V}$ are all distinct and the tree is an oriented rooted tree, we call it an *m-uniform oriented rooted hypertree generated by independent vertices*. As an illustration, the following hypergraph $V = \{1, \ldots, 19\}$ with $E = \{(1,2,3,4), (4,5,6,7), (4,8,9,10), (1,11,12,13), (13,14,15,16), (16,17,18,19)\}$ is a four-uniform hypertree generated by independent vertices, because it can be formed by the tree $T = (V_0, E_0)$, where

$$V_0 = \{1,4,7,10,13,16,19\}, \quad E_0 = \{(1,4),(4,7),(4,10),(1,13),(13,16),(16,19)\},$$

and the additional vertices $\bar{V} = \{2,3,5,6,8,9,11,12,14,15,17,18\}$.

Theorem 4. *Let $G = (V, E)$ be an m-uniform oriented rooted hypertree generated by independent vertices. Then, its Laplacian tensor is a symmetric W-tensor.*

Proof. Let the adjacent tensor and the Laplacian tensor of G be denoted by $\mathcal{A}$ and $\mathcal{L}$, respectively. Let $\mathcal{D}$ be the diagonal tensor with its diagonal entries d_i, that is, the degree of the vertex $i \in [n]$. So, it holds that $\mathcal{L} = \mathcal{D} - \mathcal{A}$. From Remark 1, it suffices to show that $-\mathcal{A}$ is a W-tensor. As $G = (V, E)$ is an m-uniform oriented rooted hypertree by independent vertices, there exist an oriented tree $T = (V_0, E_0)$ and additional distinct vertices $\bar{V} = \{i_{e,1}, i_{e,2}, \cdots, i_{e,m-2} \mid e \in E_0\}$ satisfying $V = V_0 \cup \bar{V}$. Without loss of generality, suppose that

$$V_0 = \{1, 2, \cdots, n_0\} \ \text{ and } \ E_0 = \left\{e_0^1, e_0^2, \cdots, e_0^s\right\}, \ \text{ where } \ s = |E_0|.$$

Let $\bar{\Gamma}_l = \{i \in V_0 \mid i \in e_0^l\}, l \in [s]$. Then, it satisfies that $V_0 = \bigcup_{l=1}^{s} \bar{\Gamma}_l$ and $|\left(\bigcup_{l=1}^{p-1} \bar{\Gamma}_l\right) \bigcap \bar{\Gamma}_p| \leq 1$ for all $2 \leq p \leq s$. Now, define $\Gamma_l = \bar{\Gamma}_l \bigcup \{i_{e_0^l,1}, i_{e_0^l,2}, \cdots, i_{e_0^l,m-2}\}, l \in [s]$. Then, we see that each Γ_l corresponds to an edge of the hypertree G, and it can be easily verified that all conditions (i)–(iii) of Definition 2 are satisfied. Hence, $-\mathcal{A}$ is a W-tensor, and it follows that the Laplacian tensor $\mathcal{L}$ is a W-tensor. □

4.3 | Numerical examples

Throughout this section, all numerical experiments are performed on a desktop, with 3.47-GHz quad-core Intel E5620 Xeon 64-bit CPUs and 4 GB of RAM, equipped with MATLAB 2015.

Example 3. Let $G = (V, E)$ be a four-uniform hyperstar. Suppose that $E = \{e_1, e_2, \cdots, e_k\}$, where $k \in \mathbb{N}$. Then, it holds that $|V| = 3k + 1$. Without loss of generality, assume

$$e_j = \{1, 3j-1, 3j, 3j+1\},\ j \in [k].$$

So, the Laplacian tensor $\mathcal{L}$ of the hyperstar is a fourth-order $(3k+1)$-dimensional tensor such that

$$\mathcal{L}_{i_1 i_2 i_3 i_4} = \begin{cases} k, & \text{if} \quad i_1 = i_2 = i_3 = i_4 = 1, \\ 1, & \text{if} \quad i_1 = i_2 = i_3 = i_4 = i, \ \ i \in \{2, 3, \cdots, 3k+1\}, \\ -\frac{1}{3!}, & \text{if} \quad (i_1, i_2, i_3, i_4) = \pi(1, 3j-1, 3j, 3j+1), \ \text{for some } j \in [k], \\ 0, & \text{otherwise.} \end{cases}$$

So, the homogeneous polynomial corresponding to $\mathcal{L}$ is

$$\mathcal{L}\mathbf{x}^4 = kx_1^4 + x_2^4 + \cdots + x_n^4 - 4\sum_{j=1}^{k} x_1 x_{3j-1} x_{3j} x_{3j+1}.$$

We now compute the maximum H-eigenvalue of $\mathcal{L}$ using Algorithm 2.

Using Algorithm 2, we compute the maximum H-eigenvalue of the Laplacian tensor $\lambda_{\max}^{H}(\mathcal{L})$ of four-uniform hyperstars with edges ranging from 10 to 2,000. For each case, we obtain $\lambda_{\max}^{H}(\mathcal{L})$ within 36 min. It is known that the true maximum H-eigenvalue $\lambda_{\max}^{H}(\mathcal{L})$ is the unique root of the polynomial equation $(1-x)^{m-1}(x-k) + k = 0$ in an open interval $(k, k+1)$, where k is the number of edges and m is the degree of the hypergraph. The preceding polynomial equation is solved by the MATLAB command "vpnsolve." The results are summarized in Table 2, where the meanings of the data are the same as in Table 1. It can be easily seen that the maximum H-eigenvalues computed by Algorithm 2 are consistent with the true maximum H-eigenvalues.

Example 4. Let $k, m \in \mathbb{N}$. Suppose that m is an even number. Assume that $G = (V, E)$ is an m-uniform hypergraph with vertices and edges such that

$$V = \{1, 2, \cdots, k(m-1)+1\}, \quad E = \{e_1, e_2, \cdots, e_k\},$$

where each edge $e_l = \{(l-1)(m-1)+1, (l-1)(m-1)+2, \cdots, l(m-1)+1\}, l \in [k]$. So, the hypergraph $G = (V, E)$ is the case of a hypertree with its Laplacian tensor $\mathcal{L}$ being an mth-order $n = k(m-1)+1$-dimensional tensor. By a

TABLE 2 Test results for the Laplacian tensor of Example 3

m	k	n	True $\lambda_{\max}^{H}(\mathcal{L})$	Est. $\lambda_{\max}^{H}(\mathcal{L})$	YALMIP	SeDuMi
4	10	31	10.0137	10.0137	10.1	3.0
4	100	301	100.0001	100.0001	97.4	5.0
4	500	1501	500.0000	500.0000	557.9	28.6
4	1,000	3,001	1,000.0000	1,000.0000	1,112.1	64.0
4	2,000	6,001	2,000.0000	2,000.0000	1,907.2	205.1

TABLE 3 Test results for the Laplacian tensor of Example 4

m	k	n	Est. $\lambda^H_{\max}(\mathcal{L})$	YALMIP	SeDuMi	Est. $\lambda^H_{\max}(\mathcal{Q})$	NQZ
4	100	301	2.9997	4.4	3.3	2.9997	9.4
4	400	1201	3.0000	28.4	12.7	3.0000	337.0
4	1,000	3001	3.0000	160.6	36.4	3.0000	2,678.3
6	100	501	2.6954	19.9	14.2	2.6954	24.3
6	400	2,001	2.6956	282.1	61.7	2.6956	923.9
6	1,000	5,001	2.6956	2,828.3	144.2	2.6956	6,959.4

direct computation, we obtain

$$\mathcal{L}_{i_1 i_2 \cdots i_m} = \begin{cases} 2, & \text{if} \quad i_1 = i_2 = \cdots = i_m = l(m-1)+1, l \in \{1,2,\cdots,k-1\}, \\ 1, & \text{if} \quad i_1 = i_2 = \cdots = i_m = i, \;\; i \in [n] \setminus \{(m-1)+1, 2(m-1)+1, \cdots, (k-1)(m-1)+1\}, \\ -\frac{1}{(m-1)!}, & \text{if} \quad \{i_1, i_2, \cdots, i_m\} = e_l, \;\text{for some } l \in [k], \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, we have that

$$\mathcal{L}\mathbf{x}^m = \sum_{i \in [n] \setminus \{l(m-1)+1 | l \in [k-1]\}} x_i^m + 2\sum_{l=1}^{k-1} x_{l(m-1)+1}^m - m \sum_{l=1}^{k} x_{(l-1)(m-1)+1} x_{(l-1)(m-1)+2} \cdots x_{l(m-1)+1}.$$

Combining this with Algorithm 2, we compute the maximum H-eigenvalues of $\mathcal{L}$ with different m and n, and the results are listed in the Table 3. It should be noted that Hu et al.[8] showed that the maximum H-eigenvalues of the Laplacian tensor $\mathcal{L}$ and the signless Laplacian tensor $\mathcal{Q}$ are equivalent when the even-uniform hypergraph is connected and odd-bipartite. On the other hand, because the signless Laplacian tensor is nonnegative, its maximum H-eigenvalue could be computed by the classical NQZ algorithm.[3] Direct verification shows that the hypergraph discussed in this example is connected and odd-bipartite, and hence, the maximum H-eigenvalue of its Laplacian tensor $\mathcal{L}$ can also be computed by the NQZ method. To compare the performance of our method and the NQZ method, we list the estimated maximum H-eigenvalue of the signless Laplacian tensor $\lambda^H_{\max}(\mathcal{Q})$ as well as the CPU time of the NQZ algorithm in Table 3. It can be seen that the estimated maximum H-eigenvalues of $\lambda^H_{\max}(\mathcal{L})$ and $\lambda^H_{\max}(\mathcal{Q})$ coincide, which numerically verifies the assertion. Interestingly, one also observes that Algorithm 2 is indeed *faster than* the first-order NQZ algorithm,[3] as it exploits the structure of the underlying problem. Particularly, for the Laplacian tensor of a four-uniform hypertree with 3,001 vertices, Algorithm 2 is up to 13 times faster compared to the NQZ method.

5 | APPLICATIONS IN COPOSITIVITY TEST OF TENSORS

In this section, we present further applications on testing the copositivity of a multivariate form associated with symmetric extended Z-tensors, where the order of the tensor can be either odd or even. The copositivity of Z-tensors was applied to design algorithms to test the copositivity of general tensors, which has been applied to hypergraphs and physics.[34] Now, the challenging problem is how to check the copositivity of Z-tensors efficiently.

We first recall the definition of extended Z-tensors.[36]

Definition 5. A symmetric tensor $\mathcal{A}$ is called an *extended Z-tensor* if its associated polynomial $f_{\mathcal{A}}(\mathbf{x}) = \mathcal{A}\mathbf{x}^m$ satisfies that there exist $s \in \mathbb{N}$ with $s \le n$ and index sets $\Gamma_l \subseteq \{1, \cdots, n\}$, $l = 1, \cdots, s$ with $\cup_{l=1}^s \Gamma_l = \{1, \cdots, n\}$ and $\Gamma_{l_1} \cap \Gamma_{l_2} = \emptyset$ for all $l_1 \neq l_2$, such that

$$f(\mathbf{x}) = \sum_{i=1}^n f_{m,i} x_i^m + \sum_{l=1}^s \sum_{\alpha_l \in \Omega_l} f_{\alpha_l} \mathbf{x}^{\alpha_l},$$

where

$$\Omega_l = \left\{\alpha \in ([n] \cup \{0\})^n \, : \, |\alpha| = m, \mathbf{x}^\alpha = x_{i_1} x_{i_2} \cdots x_{i_m}, \{i_1, \cdots, i_m\} \subseteq \Gamma_l, \;\; \text{and} \;\; \alpha \neq m\mathbf{e_i}, \; i = 1, \cdots, n\right\}$$

for each $l = 1, \cdots, s$ and either one of the following two conditions holds:

(1) $f_{\alpha_l} = 0$ for all but one $\alpha_l \in \Omega_l$;
(2) $f_{\alpha_l} \le 0$ for all $\alpha_l \in \Omega_l$.

Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ be a symmetric tensor with order m and dimension n. Then, $\mathcal{A}$ is copositive if and only if

$$\mathcal{A}\mathbf{x}^m = \sum_{i_1,\dots,i_m=1}^{n} a_{i_1\dots i_m} x_{i_1}\cdots x_{i_m} \geq 0, \;\; \forall\, \mathbf{x} \in \mathbb{R}^n_+,$$

which is equivalent to

$$\mathcal{A}_h\mathbf{x}^{2m} = \sum_{i_1,\dots,i_m=1}^{n} a_{i_1\dots i_m} x_{i_1}^2\cdots x_{i_m}^2 \geq 0, \quad \forall\, \mathbf{x} \in \mathbb{R}^n, \tag{8}$$

where $\mathcal{A}_h$ is a symmetric tensor with order $2m$ and dimension n.

In particular, if $\mathcal{A}$ is a symmetric extended Z-tensor (odd or even order), then $\mathcal{A}_h$ is also an even-order extended Z-tensor.[36] Thus, $-\mathcal{A}_h$ is an even-order W-tensor. Let $h(\mathbf{x}) = -\mathcal{A}_h\mathbf{x}^{2m}$. Then, by Theorem 2 and Equation 8, we have the following corollary.

Corollary 1. *Let $\mathcal{A}$ be a symmetric extended Z-tensor with order m and dimension n. For $\mathbf{x} \in \mathbb{R}^n$, suppose that $\mathcal{A}_h$ and $h(\boldsymbol{x})$ are defined as aforementioned. Then, $\mathcal{A}$ is copositive if and only if*

$$\min_{t\in\mathbb{R}} \left\{ t \mid t\|\mathbf{x}\|_{2m}^{2m} - h(\mathbf{x}) \in \Sigma_{2m}^2[\mathbf{x}] \right\} \leq 0. \tag{9}$$

We now use the aforementioned corollary to test the copositivity of symmetric extended Z-tensors with order m and dimension n. We first start with a simple numerical example verifying the aforementioned copositivity check.

Example 5. Consider the symmetric tensor $\mathcal{A} = (a_{i_1 i_2 i_3 i_4 i_5})_{1\leq i_1,\cdots,i_5\leq 4}$ with order 5 and dimension 4, such that

$$a_{1\cdots 1} = a_{2\cdots 2} = a_{3\cdots 3} = a_{4\cdots 4} = 1, \;\; a_{\pi(12222)} = -\frac{1}{5}, \;\; a_{\pi(34444)} = \frac{2}{5},$$

and $a_{i_1\cdots i_5} = 0$ otherwise, where $\pi(i_1,\cdots,i_5)$ denotes all permutations of $i_1,\cdots,i_5$. Then, $\mathcal{A}$ is a symmetric W-tensor, and the associated polynomial is

$$\mathcal{A}\mathbf{x}^5 = x_1^5 + x_2^5 + x_3^5 + x_4^5 - x_1x_2^4 + 2x_3x_4^4.$$

It can be easily verified that $\mathcal{A}$ is a symmetric extended Z-tensor but not an essentially nonnegative tensor. Moreover, geometric inequality shows that $\mathcal{A}$ is copositive.

Let

$$h(x) = -\mathcal{A}_h\mathbf{x}^{10} = -x_1^{10} - x_2^{10} - x_3^{10} - x_4^{10} + x_1^2x_2^8 - 2x_3^2x_4^8,$$

and compute $\min_{t\in\mathbb{R}}\{t \mid t\|\mathbf{x}\|_{10}^{10} - h(\mathbf{x}) \in \Sigma_{10}^2[\mathbf{x}]\}$ via YALMIP. We obtain an optimal value of -2.2126, which confirms that $\mathcal{A}$ is copositive by the corollary.

Next, we generate extended Z-tensors, using the following procedure involving randomizing techniques, and then test the copositivity accordingly. The concrete process of generating extended Z-tensors is listed as follows.

Procedure

1. Given (m, n, s, k, M) with $n = sk$, where n and m are the dimension and the order of the randomly generated tensor, respectively, and M is a large positive constant.
2. Randomly generate a partition of the index set $\{1,\cdots,n\}$, $\{\Gamma_1,\cdots,\Gamma_s\}$, such that $|\Gamma_i| = k$, $i = 1,\cdots,s$ and $\Gamma_i \cap \Gamma_{i'} = \emptyset$ for all $i \neq i'$. For each $i = 1,\cdots,s-1$, generate a random multi-index $(l_1^i,\cdots,l_m^i)$ with $l_j^i \in \Gamma_i$, $j = 1,\cdots,m$ and a random number $\bar{a}_{l_1^i\cdots l_m^i} \in [0,1]$. Generate one randomly mth-order k-dimensional symmetric tensor $\mathcal{B}$, such that all elements of $\mathcal{B}$ are in the interval $[0,1]$.
3. We define extended Z-tensor $\mathcal{A} = (a_{i_1 i_2\cdots i_m})$ such that
$$a_{i_1\cdots i_m} = \begin{cases} M, & \text{if} \quad i_1 = \cdots = i_m = i \text{ for all } i = 1,\cdots,n, \\ \bar{a}_{l_1^i\cdots l_m^i}, & \text{if} \quad (i_1,\cdots,i_m) = \pi(l_1^i,\cdots,l_m^i) \text{ with } l_1^i,\cdots,l_m^i \in \Gamma_i, i \in [s-1], \\ -\mathcal{B}_{i_1\cdots i_m}, & \text{if} \quad i_1,\cdots,i_m \in \Gamma_s, \\ 0, & \text{otherwise.} \end{cases}$$
Here, $\pi(i_1,\cdots,i_m)$ denotes all the possible permutations of $(i_1,\cdots,i_m)$.
4. Let $\mathcal{A}_h = (a^h_{i_1 i_2\cdots i_{2m}})$ be an extended Z-tensor with order $2m$ and dimension n such that
$$a^h_{\pi(i_1 i_1 i_2 i_2\cdots i_m i_m)} = a_{i_1 i_2\cdots i_m}, \quad \forall\; i_1, i_2,\cdots,i_m \in [n],$$
and $a^h_{i_1 i_2\cdots i_{2m}} = 0$ otherwise.

TABLE 4 The percentage of copositive instances of randomly generated extended Z-tensors

			$m = 3$ and $n = 2{,}500$			
M	11	12	13	14	15	16
Copositivity	12%	35%	68%	91%	96%	99%
			$m = 4$ and $n = 2{,}000$			
M	25	28	31	34	37	40
Copositivity	2%	11%	36%	63%	89%	99%
			$m = 5$ and $n = 1{,}500$			
M	30	35	40	45	50	55
Copositivity	6%	19%	50%	74%	93%	99%
			$m = 6$ and $n = 1{,}000$			
M	9	12	15	18	21	24
Copositivity	5%	20%	44%	67%	87%	99%

5. Let $f(\mathbf{x}) = -\mathcal{A}_h\mathbf{x}^{2m}$, $\mathbf{x} \in \mathbb{R}^n$. Then, solve the SOS programming problem (2) by MATLAB Toolbox YALMIP[41,42] and SeDuMi.[43]

We perform the aforementioned procedure to test extended Z-tensors with order $m = 3, 4, 5, 6$ and dimension $n = 2500, 2000, 1500, 1000$, respectively. For each case, we fix $s = 500$ and run 100 tests. Table 4 summarizes the percentage of copositive instances of these extended Z-tensors. Clearly, for fixed order m and dimension n, the percentage of copositive extended Z-tensors increases as diagonal elements M increase. Moreover, if M is large enough, the generated extended Z-tensor must be positive definite and hence, in particular, copositive.

6 | CONCLUSIONS AND REMARKS

In this article, we propose an efficient SDP algorithm to compute the maximum H-eigenvalues of even-order symmetric W-tensors based on its potential SOS structure. Furthermore, we present two interesting applications: examining the copositivity of symmetric extended Z-tensors with even or odd orders and computing the maximum H-eigenvalues of Laplacian tensors of hyperstars and hypertrees.

Our results point out some interesting future directions. For example, note that the W-tensors considered in this paper are with even order. Can we compute the maximum H-eigenvalues of odd-order symmetric W-tensors? Moreover, note that the second smallest eigenvalue and its associated eigenvector are of great interest for graph clustering. Can we also compute other H-eigenvalues (other than the maximum H-eigenvalues) as well as their associated eigenvectors of W-tensors? In addition, how do we extend the definition of W-tensors and our algorithms to complex tensors? These will be considered in a future work.

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