



Calculus of the Exponent of Kurdyka–Łojasiewicz Inequality and Its Applications to Linear Convergence of First-Order Methods

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Abstract In this paper, we study the Kurdyka–Łojasiewicz (KL) exponent, an important quantity for analyzing the convergence rate of first-order methods. Specifically, we develop various calculus rules to deduce the KL exponent of new (possibly non-convex and nonsmooth) functions formed from functions with known KL exponents. In addition, we show that the well-studied Luo–Tseng error bound together with a mild assumption on the separation of stationary values implies that the KL exponent is $\frac{1}{2}$. The Luo–Tseng error bound is known to hold for a large class of concrete structured optimization problems, and thus we deduce the KL exponent of a large class of functions whose exponents were previously unknown. Building upon this and the calculus rules, we are then able to show that for many convex or nonconvex optimization models for applications such as sparse recovery, their objective function’s KL exponent is $\frac{1}{2}$. This includes the least squares problem with smoothly clipped absolute deviation regularization or minimax concave penalty regularization and the logistic regression problem with ℓ_1 regularization. Since many existing local convergence rate

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analysis for first-order methods in the nonconvex scenario relies on the KL exponent, our results enable us to obtain explicit convergence rate for various first-order methods when they are applied to a large variety of practical optimization models. Finally, we further illustrate how our results can be applied to establishing local linear convergence of the proximal gradient algorithm and the inertial proximal algorithm with constant step sizes for some specific models that arise in sparse recovery.

Keywords First-order methods · Convergence rate · Kurdyka–Łojasiewicz inequality · Linear convergence · Luo–Tseng error bound · Sparse optimization

Mathematics Subject Classification 90C05 · 90C25 · 90C26

1 Introduction

Large-scale nonsmooth and nonconvex optimization problems are ubiquitous in machine learning and data analysis. Tremendous efforts have thus been directed at designing efficient algorithms for solving these problems. One popular class of algorithms is the class of first-order methods. These methods are noted for their simplicity, ease-of-implementation and relatively (often surprisingly) good performance; some notable examples include the proximal gradient algorithm, the inertial proximal algorithms and the alternating direction method of multipliers. Due to the excellent performance and wide applicability of first-order methods, their convergence behaviors have been extensively studied in recent years, see, for example, [1–3, 10, 19, 20, 24, 26, 35] and references therein. Analyzing the convergence rate of first-order methods is an important step toward a better understanding of existing algorithms and is also crucial for developing new optimization models and numerical schemes.

As demonstrated in [2, Theorem 3.4], the convergence behavior of many first-order methods can be understood using the celebrated Kurdyka–Łojasiewicz (KL) property and its associated KL exponent, see Definitions 2.2 and 2.3. The KL property and its associated KL exponent have their roots in algebraic geometry, and they describe a qualitative relationship between the value of a suitable potential function (depending on the optimization model and the algorithm being considered) and some first-order information (gradient or subgradient) of the potential function. The KL property has been applied to analyzing local convergence rate of various first-order methods for a wide variety of problems by many researchers, see, for example, [2, 16, 24, 43]. In these studies, a prototypical theorem on convergence rate takes the following form:

Prototypical result on convergence rate. *For a certain algorithm of interest, consider a suitable potential function. Suppose that the potential function satisfies the KL property with an exponent of $\alpha \in [0, 1)$ and that $\{x^k\}$ is a bounded sequence generated by the algorithm. Then the following results hold.*

- (i) *If $\alpha = 0$, then $\{x^k\}$ converges finitely.*
- (ii) *If $\alpha \in (0, \frac{1}{2}]$, then $\{x^k\}$ converges locally linearly.*
- (iii) *If $\alpha \in (\frac{1}{2}, 1)$, then $\{x^k\}$ converges locally sublinearly.*

While this kind of convergence results is prominent and theoretically powerful, for the results to be fully informative, one has to be able to estimate the KL exponent.

Moreover, in order to guarantee a local linear convergence rate, it is desirable to be able to determine whether a given model has a KL exponent of at most $\frac{1}{2}$, or be able to construct a new model whose KL exponent is at most $\frac{1}{2}$ if the old one does not have the desired KL exponent.

However, as noted in [31, Page 63, Section 2.1], the KL exponent of a given function is often extremely hard to determine or estimate. There are only few results available in the literature concerning explicit KL exponent of a function. One scenario where an explicit estimate of the KL exponent is known is when the function can be expressed as the maximum of finitely many polynomials. In this case, it has been shown in [23, Theorem 3.3] that the KL exponent can be estimated explicitly in terms of the dimension of the underlying space and the maximum degree of the involved polynomials. However, the derived estimate grows rapidly with the dimension of the problem and so leads to rather weak sublinear convergence rate. It is only until recently that a dimension-independent KL exponent of convex piecewise linear-quadratic functions is known, thanks to [7, Theorem 5] that connects the KL property and the concept of error bound¹ for convex functions. In addition, a KL exponent of $\frac{1}{2}$ is only established in [27] very recently for a class of quadratic optimization problems with matrix variables satisfying orthogonality constraints. Nevertheless, the KL exponent of many common optimization models, such as the least squares problem with smoothly clipped absolute deviation (SCAD) regularization [14] or minimax concave penalty (MCP) regularization [45] and the logistic regression problem with ℓ_1 regularization [39], is still unknown to the best of our knowledge. In this paper, we attempt to further address the problem of determining the explicit KL exponents of optimization models, especially for those that arise in practical applications.

The main contributions of this paper are the rules for computing *explicitly* the KL exponent of many (convex or nonconvex) optimization models that arise in applications such as statistical machine learning. We accomplish this via two different means: studying calculus rules and building connections with the concept of Luo-Tseng error bound, see Definition 2.1. The Luo-Tseng error bound was used for establishing local linear convergence for various first-order methods, and was shown to hold for a wide range of problems, see, for example, [28–30, 40, 41, 46] for details. This concept is different from the error bound studied in [7] because the Luo-Tseng error bound is defined for specially structured optimization problems and involves first-order information, while the error bound studied in [7] does not explicitly involve any first-order information. The different nature of these two concepts was also noted in [7, Section 1], in which the Luo-Tseng error bound was referred as “first-order error bound.”

In this paper, we first study various calculus rules for the KL exponent. For example, we deduce the KL exponent of the minimum of finitely many KL functions, the KL exponent of the Moreau envelope of a convex KL function and the KL exponent of a convex objective from its Lagrangian relaxation, under suitable assumptions. This is the context of Sect. 3. These rules are useful in our subsequent analysis of the KL exponent of concrete optimization models that arise in applications. Next, we show that if the Luo-Tseng error bound holds and a mild assumption on the separation of

¹ This notion is different from the Luo-Tseng error bound to be discussed in Definition 2.1.

stationary values is satisfied, then the function is a KL function with an exponent of $\frac{1}{2}$. This is done in Sect. 4. Upon making this connection, we can now take advantage of the relatively better studied concept of Luo-Tseng error bound, which is known to hold for a wide range of concrete optimization problems, see, for example, [28–30,40,41,46]. Hence, in Sect. 5, building upon the calculus rules and the connection with Luo-Tseng error bound, we show that many optimization models that arise in applications such as sparse recovery have objectives whose KL exponent is $\frac{1}{2}$; this covers the least squares problem with smoothly clipped absolute deviation (SCAD) [14] or minimax concave penalty (MCP) [45] regularization, and the logistic regression problem with ℓ_1 regularization [39]. In addition, we also illustrate how our result can be used for establishing linear convergence of some first-order methods, such as the proximal gradient algorithm and the inertial proximal algorithm [35] with constant step sizes. Finally, we present some concluding remarks in Sect. 6.

2 Notation and Preliminaries

In this paper, we use \mathbb{R}^n to denote the n -dimensional Euclidean space, equipped with the standard inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$. The closed ball centered at $x \in \mathbb{R}^n$ with radius r is denoted by $B(x, r)$. We denote the nonnegative orthant by \mathbb{R}_+^n and the set of $n \times n$ symmetric matrices by \mathcal{S}^n . For a vector $x \in \mathbb{R}^n$, we use $\|x\|_1$ to denote the ℓ_1 norm and $\|x\|_0$ to denote the number of entries in x that are nonzero (“ ℓ_0 norm”). For a (nonempty) closed set $D \subseteq \mathbb{R}^n$, the indicator function δ_D is defined as

$$\delta_D(x) = \begin{cases} 0 & \text{if } x \in D, \\ \infty & \text{otherwise.} \end{cases}$$

In addition, we denote the distance from an $x \in \mathbb{R}^n$ to D by $\text{dist}(x, D) = \inf_{y \in D} \|x - y\|$, and the set of points in D that achieve this infimum (the projection of x onto D) is denoted by $\text{Proj}_D(x)$. The set $\text{Proj}_D(x)$ becomes a singleton if D is a closed convex set. Finally, we write $\text{ri } D$ to represent the relative interior of a closed convex set D .

For an extended-real-valued function $f : \mathbb{R}^n \rightarrow [-\infty, \infty]$, the domain is defined as $\text{dom } f = \{x : f(x) < \infty\}$. Such a function is called proper if it is never $-\infty$ and its domain is nonempty, and is called closed if it is lower semicontinuous. For a proper function $f : \mathbb{R}^n \rightarrow (-\infty, \infty]$, we let $z \xrightarrow{f} x$ denote $z \rightarrow x$ and $f(z) \rightarrow f(x)$. The regular subdifferential of a proper function f [38, Page 301, Definition 8.3(a)] at $x \in \text{dom } f$ is given by

$$\widehat{\partial}f(x) := \left\{ v \in \mathbb{R}^n : \liminf_{z \rightarrow x, z \neq x} \frac{f(z) - f(x) - \langle v, z - x \rangle}{\|z - x\|} \geq 0 \right\}.$$

The (limiting) *subdifferential* of a proper function f [38, Page 301, Definition 8.3(b)] at $x \in \text{dom } f$ is then defined by

$$\partial f(x) := \left\{ v \in \mathbb{R}^n : \exists x^t \xrightarrow{f} x, v^t \rightarrow v \text{ with } v^t \in \widehat{\partial}f(x^t) \text{ for each } t \right\}. \quad (1)$$

By convention, if $x \notin \text{dom } f$, then $\partial f(x) = \widehat{\partial} f(x) = \emptyset$. We also write $\text{dom } \partial f := \{x \in \mathbb{R}^n : \partial f(x) \neq \emptyset\}$. It is well known that when f is continuously differentiable, the subdifferential (1) reduces to the gradient of f denoted by ∇f , see, for example, [38, Exercise 8.8(b)]. Moreover, when f is convex, the subdifferential (1) reduces to the classical subdifferential in convex analysis, see, for example, [38, Proposition 8.12]. The limiting subdifferential enjoys rich and comprehensive calculus rules and has been widely used in nonsmooth and nonconvex optimization [33, 38]. We also define the limiting (resp. regular) normal cone of a closed set S at $x \in S$ as $N_S(x) = \partial \delta_S(x)$ (resp. $\widehat{N}_S(x) = \widehat{\partial} \delta_S(x)$) where δ_S is the indicator function of S . A closed set S is called regular at $x \in S$ if $N_S(x) = \widehat{N}_S(x)$ (see [38, Definition 6.4]), and a proper closed function f is called regular at $x \in \text{dom } f$ if its epigraph $\text{epi } f$ is regular at the point $(x, f(x)) \in \text{epi } f$ (see [38, Definition 7.25]). Finally, we say that x is a stationary point of proper closed function f if $0 \in \partial f(x)$. It is known that any local minimizer of f is a stationary point, see, for example, [38, Theorem 10.1].

For a proper closed convex function P , its proximal mapping at any z is defined as

$$\text{prox}_P(z) = \arg \min_{x \in \mathbb{R}^n} \left\{ P(x) + \frac{1}{2} \|x - z\|^2 \right\},$$

where $\arg \min$ denotes the unique minimizer of the optimization problem $\min_x \{P(x) + \frac{1}{2} \|x - z\|^2\}$.² This mapping is nonexpansive, i.e., for any y and z , we have

$$\|\text{prox}_P(y) - \text{prox}_P(z)\| \leq \|y - z\|; \tag{2}$$

see, for example, [36, Page 340]. Moreover, it is routine to show that $x = \text{prox}_P(z)$ if and only if $z \in x + \partial P(x)$.

The following property is defined for proper closed functions of the form $f = h + P$, where h is a proper closed function with an open domain and is continuously differentiable with a locally Lipschitz continuous gradient on $\text{dom } h$, and P is proper closed convex. Recall that for this class of functions, we have $\bar{x} \in \mathcal{X}$ if and only if $\bar{x} = \text{prox}_P(\bar{x} - \nabla h(\bar{x}))$, where \mathcal{X} denotes the set of stationary points of f . Indeed, we have

$$\begin{aligned} \bar{x} \in \mathcal{X} &\iff 0 \in \partial(h + P)(\bar{x}) \stackrel{(i)}{\iff} 0 \in \nabla h(\bar{x}) + \partial P(\bar{x}) \\ &\iff \bar{x} - \nabla h(\bar{x}) \in \bar{x} + \partial P(\bar{x}) \iff \bar{x} = \text{prox}_P(\bar{x} - \nabla h(\bar{x})), \end{aligned}$$

where (i) follows from [38, Exercise 8.8(c)].

Definition 2.1 (Luo-Tseng error bound)³ Let \mathcal{X} be the set of stationary points of f . Suppose that $\mathcal{X} \neq \emptyset$. We say that the Luo-Tseng error bound⁴ holds if for any $\zeta \geq \inf f$, there exist $c, \epsilon > 0$ so that

² This problem has a unique minimizer because the objective is proper closed and strongly convex. For a general optimization problem $\min_x f(x)$, we use $\text{Arg min } f$ to denote the set of minimizers, which may be empty, may be a singleton or may contain more than one point.

³ We adapt the definition from [41, Assumption 2a].

⁴ This is referred as first-order error bound in [7, Section 1].

$$\text{dist}(x, \mathcal{X}) \leq c \|\text{prox}_P(x - \nabla h(x)) - x\| \tag{3}$$

whenever $\|\text{prox}_P(x - \nabla h(x)) - x\| < \epsilon$ and $f(x) \leq \zeta$.

It is known that this property is satisfied for many choices of h and P , and we refer to [28–30, 40, 41, 46] and references therein for more detailed discussions. This property was used for establishing local linear convergence of various first-order methods applied to minimizing $f = h + P$.

Recently, the following property was also used extensively for analyzing convergence rate of first-order methods, mainly for possibly nonconvex objective functions, see, for example, [2, 3].

Definition 2.2 (*KL property & KL function*) We say that a proper closed function f has the Kurdyka–Łojasiewicz (KL) property at $\bar{x} \in \text{dom } \partial f$ if there exist a neighborhood \mathcal{N} of \bar{x} , $\nu \in (0, \infty]$ and a continuous concave function $\psi : [0, \nu) \rightarrow \mathbb{R}_+$ with $\psi(0) = 0$ such that:

- (i) ψ is continuously differentiable on $(0, \nu)$ with $\psi' > 0$ over $(0, \nu)$;
- (ii) for all $x \in \mathcal{N}$ with $f(\bar{x}) < f(x) < f(\bar{x}) + \nu$, one has

$$\psi'(f(x) - f(\bar{x})) \text{dist}(0, \partial f(x)) \geq 1.$$

A proper closed function f satisfying the KL property at all points in $\text{dom } \partial f$ is called a KL function.

In this paper, we are interested in the KL exponent, which is defined [2, 3] as follows.

Definition 2.3 (*KL exponent*) For a proper closed function f satisfying the KL property at $\bar{x} \in \text{dom } \partial f$, if the corresponding function ψ can be chosen as $\psi(s) = \bar{c}s^{1-\alpha}$ for some $\bar{c} > 0$ and $\alpha \in [0, 1)$, i.e., there exist $c, \epsilon > 0$ and $\nu \in (0, \infty]$ so that

$$\text{dist}(0, \partial f(x)) \geq c(f(x) - f(\bar{x}))^\alpha \tag{4}$$

whenever $\|x - \bar{x}\| \leq \epsilon$ and $f(\bar{x}) < f(x) < f(\bar{x}) + \nu$, then we say that f has the KL property at \bar{x} with an exponent of α . If f is a KL function and has the same exponent α at any $\bar{x} \in \text{dom } \partial f$, then we say that f is a KL function with an exponent of α .⁵

This definition encompasses broad classes of function that arise in practical optimization problems. For example, it is known that if f is a proper closed semi-algebraic function [3], then f is a KL function with a suitable exponent $\alpha \in [0, 1)$. As established in [2, Theorem 3.4] and many subsequent work, KL exponent has a close relationship with the rate of convergence of many commonly used optimization methods.

Before ending this section, we state two auxiliary lemmas. The first result is an immediate consequence of the fact that the set-valued mapping $x \mapsto \partial f(x)$ is outer semicontinuous (with respect to the f -attentive convergence, i.e., $y \xrightarrow{f} x$, see [38,

⁵ In classical algebraic geometry, the exponent α is also referred as the Łojasiewicz exponent.

Proposition 8.7]), and can be found in [2, Remark 4(b)]. This result will be used repeatedly at various places in our discussion below. We include a proof for self-containedness.

Lemma 2.1 *Suppose that f is a proper closed function, $\bar{x} \in \text{dom } \partial f$ and $0 \notin \partial f(\bar{x})$. Then, for any $\alpha \in [0, 1)$, f satisfies the KL property at \bar{x} with an exponent of α .*

Proof Fix any $\alpha \in [0, 1)$. Since $0 \notin \partial f(\bar{x})$ and $\partial f(\bar{x})$ is nonempty and closed, it follows that $\text{dist}(0, \partial f(\bar{x}))$ is positive and finite. Define $\eta := \frac{1}{2} \text{dist}(0, \partial f(\bar{x})) > 0$. We claim that there exists $\epsilon \in (0, 1)$ so that $\text{dist}(0, \partial f(x)) > \eta$ whenever $\|x - \bar{x}\| \leq \epsilon$ and $f(\bar{x}) < f(x) < f(\bar{x}) + \epsilon$.

Suppose for the sake of contradiction that this is not true. Then there exists a sequence $\{x^k\}$ with $x^k \rightarrow \bar{x}$ and $f(x^k) \rightarrow f(\bar{x})$ so that

$$\text{dist}(0, \partial f(x^k)) \leq \eta.$$

In particular, there exists a sequence $\{\xi^k\}$ satisfying $\xi^k \in \partial f(x^k)$ and $\|\xi^k\| \leq \eta$. By passing to a subsequence if necessary, we may assume without loss of generality that $\xi^k \rightarrow \xi$ for some ξ , and we have $\xi \in \partial f(\bar{x})$, thanks to [38, Proposition 8.7]. But then we have $2\eta = \text{dist}(0, \partial f(\bar{x})) \leq \|\xi\| \leq \eta$, a contradiction. Thus, there exists $\epsilon \in (0, 1)$ so that $\text{dist}(0, \partial f(x)) > \eta$ whenever $\|x - \bar{x}\| \leq \epsilon$ and $f(\bar{x}) < f(x) < f(\bar{x}) + \epsilon$.

Using this, we see immediately that

$$\eta(f(x) - f(\bar{x}))^\alpha \leq \eta < \text{dist}(0, \partial f(x))$$

whenever $\|x - \bar{x}\| \leq \epsilon$ and $f(\bar{x}) < f(x) < f(\bar{x}) + \epsilon$, showing that f satisfies the KL property at \bar{x} with an exponent of α . This completes the proof. □

The second result concerns the equivalence of “norms,” whose proof is simple and is omitted.

Lemma 2.2 *Let $t > 0$. Then there exist $C_1 \geq C_2 > 0$ so that*

$$C_2 \|a\| \leq (a_1^t + \dots + a_m^t)^{\frac{1}{t}} \leq C_1 \|a\|$$

for any $a = (a_1, \dots, a_m) \in \mathbb{R}_+^m$.

3 Calculus of the KL Exponent

In this section, we discuss how the KL exponent behaves under various operations on KL functions. We briefly summarize our results below. The required assumptions will be made explicit in the respective theorems.

- (i) Exponent for $\min_{1 \leq i \leq r} f_i$ given the exponents of f_i for each i , see Theorem 3.1 and Corollary 3.1.
- (ii) Exponent for $g(F(x))$ when the Jacobian of F is surjective, given the exponent of g , see Theorem 3.2.

- (iii) Exponent for $\sum_{i=1}^m f_i(x_i)$ given the exponents of f_i for each i , see Theorem 3.3.
- (iv) Exponent for the Moreau envelope of a convex KL function, see Theorem 3.4.
- (v) Deducing the exponent from the Lagrangian relaxation for convex problems, see Theorem 3.5.
- (vi) Exponent for a potential function used in the convergence analysis of the inertial proximal algorithm in [35], see Theorem 3.6.
- (vii) Deducing the exponent of a partly smooth KL function by looking at its restriction on its active manifold, see Theorem 3.7.

We shall make use of some of these calculus rules in Sect. 5 to deduce the KL exponent of some concrete optimization models.

We start with our first result, which concerns the minimum of finitely many KL functions. This rule will prove to be useful in Sect. 5. Indeed, as we shall see there, many nonconvex optimization problems that arise in applications have objectives that can be written as the minimum of finitely many KL functions whose exponents can be deduced from our results in Sect. 4; this includes some prominent and widely used NP-hard optimization model problems, for example, the least squares problem with cardinality constraint [6].

Theorem 3.1 (Exponent for minimum of finitely many KL functions) *Let $f_i, 1 \leq i \leq r$, be proper closed functions, $f := \min_{1 \leq i \leq r} f_i$ be continuous on $\text{dom } \partial f$ and $\bar{x} \in \text{dom } \partial f \cap (\bigcap_{i \in I(\bar{x})} \text{dom } \partial f_i)$, where $I(\bar{x}) := \{i : f_i(\bar{x}) = f(\bar{x})\}$. Suppose further that each $f_i, i \in I(\bar{x})$, satisfies the KL property at \bar{x} with an exponent of $\alpha_i \in [0, 1)$. Then f satisfies the KL property at \bar{x} with an exponent of $\alpha = \max\{\alpha_i : i \in I(\bar{x})\}$.*

Proof From the definition of $I(\bar{x})$, we see that $\min_{i \notin I(\bar{x})} \{f_i(\bar{x})\} > f(\bar{x})$. Since $x \mapsto \min_{i \notin I(\bar{x})} \{f_i(x)\}$ is lower semicontinuous and the function f is continuous on $\text{dom } \partial f$, there exists $\eta > 0$ such that for all $x \in \text{dom } \partial f$ with $\|x - \bar{x}\| \leq \eta$, we have

$$\min_{i \notin I(\bar{x})} \{f_i(x)\} > f(x).$$

Thus, whenever $x \in \text{dom } \partial f$ and $\|x - \bar{x}\| \leq \eta$, we have $I(x) \subseteq I(\bar{x})$.

Next, using $f(x) = \min_{1 \leq i \leq r} \{f_i(x)\}$ and the subdifferential rule of the minimum of finitely many functions [32, Theorem 5.5], we obtain for all $x \in \text{dom } \partial f$ that

$$\partial f(x) \subseteq \bigcup_{i \in I(x)} \partial f_i(x). \tag{5}$$

On the other hand, by assumption, for each $i \in I(\bar{x})$, there exist $\epsilon_i, c_i, \nu_i > 0$ such that for all $x \in \text{dom } \partial f_i$ with $\|x - \bar{x}\| \leq \epsilon_i$ and $f_i(\bar{x}) < f_i(x) < f_i(\bar{x}) + \nu_i$, one has

$$\text{dist}(0, \partial f_i(x)) \geq c_i (f_i(x) - f_i(\bar{x}))^{\alpha_i}. \tag{6}$$

Let $c = \min_{i \in I(\bar{x})} c_i, \epsilon = \min\{\eta, \min_{i \in I(\bar{x})} \epsilon_i\}$ and $\nu = \min_{i \in I(\bar{x})} \{\nu_i\}$. Take any $x \in \text{dom } \partial f$ with $\|x - \bar{x}\| \leq \epsilon$ and $f(\bar{x}) < f(x) < f(\bar{x}) + \nu$. Then $I(x) \subseteq I(\bar{x})$ and we have

$$\begin{aligned} \text{dist}(0, \partial f(x)) &\geq \min_{i \in I(x)} \text{dist}(0, \partial f_i(x)) \\ &\geq c \min_{i \in I(x)} (f_i(x) - f_i(\bar{x}))^\alpha = c(f(x) - f(\bar{x}))^\alpha, \end{aligned}$$

where the first inequality follows from (5), the second inequality follows from (6), the construction of c, v, ϵ and α , as well as the facts that $f_i(x) = f(x)$ and $f_i(\bar{x}) = f(\bar{x})$ for $i \in I(x) \subseteq I(\bar{x})$; these facts also give the last equality. This completes the proof. \square

We have the following immediate corollary.

Corollary 3.1 *Let $f_i, 1 \leq i \leq r$, be proper closed functions with $\text{dom } f_i = \text{dom } \partial f_i$ for all i , and $f := \min_{1 \leq i \leq r} f_i$ be continuous on $\text{dom } \partial f$. Suppose further that each f_i is a KL function with an exponent of $\alpha_i \in [0, 1)$ for $1 \leq i \leq r$. Then f is a KL function with an exponent of $\alpha = \max\{\alpha_i : 1 \leq i \leq r\}$.*

Proof In view of Theorem 3.1, it suffices to show that for any $\bar{x} \in \text{dom } \partial f$, we have $\bar{x} \in \bigcap_{i \in I(\bar{x})} \text{dom } \partial f_i$. To this end, take any $\bar{x} \in \text{dom } \partial f$. Note that we have $\text{dom } \partial f \subseteq \text{dom } f$ by the definition, and hence $f(\bar{x}) < \infty$. In addition, from the definition of $I(\bar{x})$, we have $f(\bar{x}) = f_i(\bar{x})$ for all $i \in I(\bar{x})$. Hence, $f_i(\bar{x})$ is finite for all $i \in I(\bar{x})$. Thus, we conclude that $\bar{x} \in \text{dom } f_i$ for all $i \in I(\bar{x})$, which implies that $\bar{x} \in \bigcap_{i \in I(\bar{x})} \text{dom } \partial f_i$ because $\text{dom } f_i = \text{dom } \partial f_i$ for all i by assumption. \square

The next theorem concerns the composition of a KL function with a smooth function that has a surjective Jacobian mapping.

Theorem 3.2 (Exponent for composition of KL functions) *Let $f(x) = (g \circ F)(x)$, where g is a proper closed function on \mathbb{R}^m and $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a continuously differentiable mapping. Suppose in addition that g is a KL function with an exponent of $\alpha \in [0, 1)$ and the Jacobian $JF(\bar{x}) \in \mathbb{R}^{m \times n}$ is a surjective mapping at some $\bar{x} \in \text{dom } \partial f$. Then f has the KL property at \bar{x} with an exponent of α .*

Proof Note from [38, Exercise 10.7] and $\bar{x} \in \text{dom } \partial f$ that $F(\bar{x}) \in \text{dom } \partial g$. As g is a KL function, there exist $\epsilon_0, c_0, \nu_0 > 0$ such that for all $u \in \text{dom } \partial g$ with $\|u - F(\bar{x})\| \leq \epsilon_0$ and $g(F(\bar{x})) < g(u) < g(F(\bar{x})) + \nu_0$, one has

$$\text{dist}(0, \partial g(u)) \geq c_0(g(u) - g(F(\bar{x})))^\alpha. \tag{7}$$

On the other hand, since the linear map $JF(\bar{x})$ is surjective and F is continuously differentiable, it follows from the classical Lyusternik–Graves theorem (see, e.g., [33, Theorem 1.57]) that there are numbers $l > 0$ and $\epsilon_1 \in (0, \epsilon_0)$ such that for all x with $\|x - \bar{x}\| \leq \epsilon_1$

$$l \mathbb{B}_{\mathbb{R}^m} \subseteq JF(x)(\mathbb{B}_{\mathbb{R}^n}),$$

where $\mathbb{B}_{\mathbb{R}^m}$ and $\mathbb{B}_{\mathbb{R}^n}$ are the closed unit balls in \mathbb{R}^m and \mathbb{R}^n , respectively. This implies that for all $z \in \mathbb{R}^m$ we have the following estimate:

$$\|JF(x)^* z\| \geq l \|z\|, \tag{8}$$

whenever $\|x - \bar{x}\| \leq \epsilon_1$. Moreover, from the chain rule of the limiting subdifferential for composite functions (see, e.g., [38, Exercise 10.7]), we have for all x with $\|x - \bar{x}\| \leq \epsilon_1$ that

$$\partial f(x) = \{JF(x)^*v : v \in \partial g(F(x))\},$$

because $JF(x)$ is a surjective mapping for all such x .

Now, let $\epsilon \in (0, \epsilon_1]$ be such that $\|F(x) - F(\bar{x})\| \leq \epsilon_0$ for all $\|x - \bar{x}\| \leq \epsilon$, $c = l c_0$ and $v = v_0$. Fix $x \in \text{dom } \partial f$ with $\|x - \bar{x}\| \leq \epsilon$ and $f(\bar{x}) < f(x) < f(\bar{x}) + v$. Let $a \in \partial f(x)$ be such that $\|a\| = \text{dist}(0, \partial f(x))$. Then, we have $a = JF(x)^*v$ for some $v \in \partial g(F(x))$. Hence, it follows from (8) that

$$\|a\| = \|JF(x)^*v\| \geq l\|v\|.$$

In addition, since $v \in \partial g(F(x))$, applying (7) with $u = F(x)$ gives us that

$$\|v\| \geq \text{dist}(0, \partial g(F(x))) \geq c_0(g(F(x)) - g(F(\bar{x})))^\alpha = c_0(f(x) - f(\bar{x}))^\alpha.$$

Therefore,

$$\text{dist}(0, \partial f(x)) = \|a\| \geq l\|v\| \geq c(f(x) - f(\bar{x}))^\alpha. \quad \square$$

Our next theorem concerns separable sums.

Theorem 3.3 (Exponent for block separable sums of KL functions) *Let $n_i, n \in \mathbb{N}$, $i = 1, \dots, m$ be such that $\sum_{i=1}^m n_i = n$. Let $f(x) = \sum_{i=1}^m f_i(x_i)$, where $f_i, 1 \leq i \leq r$, are proper closed functions on \mathbb{R}^{n_i} with $x = (x_1, \dots, x_m) \in \mathbb{R}^n$. Suppose further that each f_i is a KL function with an exponent of $\alpha_i \in (0, 1)$ and that each f_i is continuous on $\text{dom } \partial f_i, i = 1, \dots, r$. Then f is a KL function with an exponent of $\alpha = \max\{\alpha_i : 1 \leq i \leq r\}$.*

Proof Denote $\bar{x} = (\bar{x}_1, \dots, \bar{x}_m) \in \text{dom } \partial f$ with $\bar{x}_i \in \mathbb{R}^{n_i}, i = 1, \dots, m$. Then [38, Proposition 10.5] shows that $\bar{x}_i \in \text{dom } \partial f_i$ for each i . As each $f_i, i = 1, \dots, m$, is a KL function with exponent α_i , there exist $\epsilon_i, c_i, v_i > 0$ such that for all $x_i \in \text{dom } \partial f_i \subseteq \mathbb{R}^{n_i}$ with $\|x_i - \bar{x}_i\| \leq \epsilon_i$ and $f_i(\bar{x}_i) < f_i(x_i) < f_i(\bar{x}_i) + v_i$, one has

$$c_i \text{dist}^{\frac{1}{\alpha_i}}(0, \partial f_i(x_i)) \geq f_i(x_i) - f_i(\bar{x}_i). \quad (9)$$

Since the left-hand side of (9) is always nonnegative, the above relation holds trivially whenever $f_i(x_i) \leq f_i(\bar{x}_i)$. In addition, since f_i is continuous on $\text{dom } \partial f_i$ by assumption for each $i = 1, \dots, r$, by shrinking $\epsilon_i > 0$ if necessary, we conclude that $f_i(x_i) < f_i(\bar{x}_i) + v_i$ whenever $x_i \in \text{dom } \partial f_i$ and $\|x_i - \bar{x}_i\| \leq \epsilon_i$. Thus, we have from these two observations that for all $x_i \in \text{dom } \partial f_i \subseteq \mathbb{R}^{n_i}$ with $\|x_i - \bar{x}_i\| \leq \epsilon_i$,

$$c_i \text{dist}^{\frac{1}{\alpha_i}}(0, \partial f_i(x_i)) \geq f_i(x_i) - f_i(\bar{x}_i). \quad (10)$$

Let $\epsilon = \min_{1 \leq i \leq r} \epsilon_i$. Take any $x \in \text{dom } \partial f$ with $\|x - \bar{x}\| \leq \epsilon$ and $f(\bar{x}) < f(x) < f(\bar{x}) + 1$. We will now verify (4). To this end, let $z \in \partial f(x)$ be such that

$$\|z\| = \text{dist}(0, \partial f(x)).$$

If $\|z\| > 1$, then clearly

$$\text{dist}(0, \partial f(x)) = \|z\| \geq (f(x) - f(\bar{x}))^\alpha$$

since $0 < f(x) - f(\bar{x}) < 1$. Thus, we consider the case where $\|z\| \leq 1$. In this case, recall from [38, Proposition 10.5] that

$$\partial f(x) = \prod_{i=1}^m \partial f_i(x_i).$$

Hence, there exist $z_i \in \mathbb{R}^{n_i}$ with $z_i \in \partial f_i(x_i)$ and $\|z_i\| = \text{dist}(0, \partial f_i(x_i))$ such that

$$\sum_{i=1}^m \|z_i\|^2 = \|z\|^2.$$

This together with (10) implies that

$$\begin{aligned} C_0 \sum_{i=1}^m \|z_i\|^{\frac{1}{\alpha_i}} &\geq \sum_{i=1}^m c_i \|z_i\|^{\frac{1}{\alpha_i}} = \sum_{i=1}^m c_i \text{dist}^{\frac{1}{\alpha_i}}(0, \partial f_i(x_i)) \\ &\geq \sum_{i=1}^m [f_i(x_i) - f_i(\bar{x}_i)] = f(x) - f(\bar{x}) \end{aligned} \tag{11}$$

for $C_0 = \max_{1 \leq i \leq r} c_i$. Define $\alpha := \max\{\alpha_i : 1 \leq i \leq r\}$. Since $\|z_i\| \leq \|z\| \leq 1$ and $0 < \alpha_i \leq \alpha < 1$, it then follows from (11) that

$$(f(x) - f(\bar{x}))^\alpha \leq \left(C_0 \sum_{i=1}^m \|z_i\|^{\frac{1}{\alpha}} \right)^\alpha \leq C_1 C_0^\alpha \|z\| = C_1 C_0^\alpha \text{dist}(0, \partial f(x)),$$

where the second inequality follows from Lemma 2.2 with $t = \frac{1}{\alpha}$. This completes the proof. □

We now discuss the operation of taking Moreau envelope. This operation is a common operation for smoothing the objective function of convex optimization problems.

Theorem 3.4 (Exponent for Moreau envelope of convex KL functions) *Let f be a proper closed convex function that is a KL function with an exponent of $\alpha \in (0, \frac{2}{3})$. Suppose further that f is continuous on $\text{dom } \partial f$. Fix $\lambda > 0$ and consider*

$$F_\lambda(x) := \inf_{y \in \mathbb{R}^n} \left\{ f(y) + \frac{1}{2\lambda} \|y - x\|^2 \right\}.$$

Then F_λ is a KL function with an exponent of $\max\{\frac{1}{2}, \frac{\alpha}{2-2\alpha}\} < 1$.

Proof It suffices to consider the case $\text{Arg min } F_\lambda \neq \emptyset$ and show that F_λ has the KL property with an exponent of $\max\{\frac{1}{2}, \frac{\alpha}{2-2\alpha}\}$ at any fixed $\bar{x} \in \text{Arg min } F_\lambda$, in view of Lemma 2.1 and the convexity of F_λ .

To this end, recall from [4, Proposition 12.29] that, for all $x \in \mathbb{R}^n$,

$$\nabla F_\lambda(x) = \frac{1}{\lambda}(x - \text{prox}_{\lambda f}(x)), \tag{12}$$

and that ∇F_λ is Lipschitz continuous with a Lipschitz constant of $\frac{1}{\lambda}$. Consequently, we have for any y that

$$\begin{aligned} F_\lambda(y) - F_\lambda(\bar{x}) &= F_\lambda(y) - F_\lambda(\bar{y}) \\ &\leq \langle \nabla F_\lambda(\bar{y}), y - \bar{y} \rangle + \frac{1}{2\lambda} \|y - \bar{y}\|^2 = \frac{1}{2\lambda} \|y - \bar{y}\|^2, \end{aligned} \tag{13}$$

where \bar{y} is the projection of y onto $\text{Arg min } F_\lambda$, and the last equality holds because $\nabla F_\lambda(\bar{y}) = 0$.

Next, note that we have $\text{Arg min } f = \{x \in \mathbb{R}^n : x = \text{prox}_f(x)\}$ according to [4, Proposition 12.28], which implies $\text{Arg min } f = \{x \in \mathbb{R}^n : x = \text{prox}_{\lambda f}(x)\}$ as $\lambda > 0$. This together with (12) gives $\text{Arg min } F_\lambda = \text{Arg min } f$. Hence, $\bar{x} \in \text{Arg min } F_\lambda = \text{Arg min } f \subseteq \text{dom } \partial f$. Since f is a KL function with an exponent of α and $\bar{x} \in \text{dom } \partial f$, using the fact that f is continuous on $\text{dom } \partial f$, we obtain that there exist $c_0 > 0$ and $\epsilon > 0$ so that

$$\text{dist}(0, \partial f(y)) \geq c_0 (f(y) - f(\bar{x}))^\alpha = c_0 (f(y) - \inf f)^\alpha \tag{14}$$

whenever $\|y - \bar{x}\| \leq \epsilon$ and $y \in \text{dom } \partial f$; here, the condition on the bound on function values is waived by using the continuity of f on ∂f and choosing a smaller ϵ if necessary. Moreover, in view of [7, Theorem 5(i)], by shrinking ϵ if necessary, we conclude that there exists $c > 0$ so that

$$\text{dist}(y, \text{Arg min } f) \leq c (f(y) - \inf f)^{1-\alpha}, \tag{15}$$

whenever $\|y - \bar{x}\| \leq \epsilon$ and $y \in \text{dom } \partial f$; here, the condition on the bound on function values is waived similarly as before. Finally, since $\text{Arg min } F_\lambda = \text{Arg min } f$, we have $\|y - \bar{y}\| = \text{dist}(y, \text{Arg min } f)$. Combining this with (14) and (15) implies that for some $c_1 > 0$,

$$\|y - \bar{y}\| \leq c_1 [\text{dist}(0, \partial f(y))]^{\frac{1-\alpha}{\alpha}} \tag{16}$$

whenever $\|y - \bar{x}\| \leq \epsilon$ and $y \in \text{dom } \partial f$.

Now, using the definition of the proximal mapping as minimizer, we have by using the first-order optimality condition that for any u ,

$$0 \in \lambda \partial f(\text{prox}_{\lambda f}(u)) + (\text{prox}_{\lambda f}(u) - u).$$

In particular, $\text{prox}_{\lambda f}(u) \in \text{dom } \partial f$. In addition, using the above relation and (12), we deduce that

$$\text{dist}(0, \partial f(\text{prox}_{\lambda f}(u))) \leq \frac{1}{\lambda} \|u - \text{prox}_{\lambda f}(u)\| = \|\nabla F_\lambda(u)\|. \tag{17}$$

Fix an arbitrary u with $\|u - \bar{x}\| \leq \epsilon$. Then $\|\text{prox}_{\lambda f}(u) - \bar{x}\| = \|\text{prox}_{\lambda f}(u) - \text{prox}_{\lambda f}(\bar{x})\| \leq \|u - \bar{x}\|$, where the inequality is due to (2). Let $y = \text{prox}_{\lambda f}(u)$. Then $y \in \text{dom } \partial f$ and $\|y - \bar{x}\| \leq \epsilon$. Hence, the relations (16) and (17) imply that

$$\|y - \bar{y}\| \leq c_1 \|\nabla F_\lambda(u)\|^{\frac{1-\alpha}{\alpha}}.$$

Applying (13) with $y = \text{prox}_{\lambda f}(u)$ and combining this with the preceding relation, we obtain further that

$$F_\lambda(\text{prox}_{\lambda f}(u)) - F_\lambda(\bar{x}) \leq \frac{c_1^2}{2\lambda} \|\nabla F_\lambda(u)\|^{\frac{2-2\alpha}{\alpha}}. \tag{18}$$

whenever $\|u - \bar{x}\| \leq \epsilon$. Finally, from the convexity of F_λ , we have

$$F_\lambda(u) - F_\lambda(\text{prox}_{\lambda f}(u)) \leq \langle \nabla F_\lambda(u), u - \text{prox}_{\lambda f}(u) \rangle = \lambda \|\nabla F_\lambda(u)\|^2, \tag{19}$$

where the equality follows from (12).

Shrink ϵ further if necessary so that $\|\nabla F_\lambda(u)\| < 1$ whenever $\|u - \bar{x}\| \leq \epsilon$; this is possible since $\nabla F_\lambda(\bar{x}) = 0$. Summing (18) and (19), we obtain further that

$$\begin{aligned} F_\lambda(u) - F_\lambda(\bar{x}) &\leq \frac{c_1^2}{2\lambda} \|\nabla F_\lambda(u)\|^{\frac{2-2\alpha}{\alpha}} + \lambda \|\nabla F_\lambda(u)\|^2 \\ &\leq C \|\nabla F_\lambda(u)\|^{\min\{2, \frac{2-2\alpha}{\alpha}\}} \end{aligned}$$

for some $C > 0$, whenever $\|u - \bar{x}\| \leq \epsilon$. This completes the proof. □

Our next result concerns the Lagrangian relaxation. This result will be used later in Proposition 4.2 for a group LASSO model. In addition, the result, together with results in [23] that give the exponent of the maximum of finitely many convex polynomials, can be used to study the KL property of a large class of convex polynomial optimization problems with multiple convex polynomial constraints.

Theorem 3.5 (KL exponent from Lagrangian relaxation) *Let $h(x) = l(Ax)$ for some continuous strictly convex function l and $A \in \mathbb{R}^{m \times n}$, g be a continuous convex function and $\alpha \in (0, 1)$, D be a closed convex set. Let $C = \{x : g(x) \leq 0\}$. Suppose in addition that*

- (i) *there exists $x_0 \in D$ with $g(x_0) < 0$;*
- (ii) $\inf_{x \in D} h(x) < \inf_{x \in C \cap D} h(x)$;
- (iii) *for any $\lambda > 0$, the function $h(x) + \lambda g(x) + \delta_D(x)$ is a KL function with an exponent of α .*

Then $f(x) = h(x) + \delta_C(x) + \delta_D(x)$ has the KL property at any $\bar{x} \in C \cap D$, with an exponent of α .

Proof In view of Lemma 2.1 and the convexity of f , we only need to consider the case $\text{Arg min } f \neq \emptyset$ and look at those \bar{x} with $0 \in \partial f(\bar{x})$ (and so, $\bar{x} \in \text{Arg min } f$ by the convexity of f). From now on, fix any such \bar{x} .

First, from condition (i) and [36, Corollary 28.2.1], there exists $\lambda \geq 0$ so that

$$f(\bar{x}) = \inf_{x \in \mathbb{R}^n} f = h(\bar{x}) = \inf_{x \in C \cap D} h(x) = \inf_{x \in D} \{h(x) + \lambda g(x)\} \leq h(\bar{x}) + \lambda g(\bar{x}) \leq h(\bar{x}), \tag{20}$$

where the first inequality follows from $\bar{x} \in C \cap D$, while the second inequality follows from the fact that $\bar{x} \in C$ and hence $g(\bar{x}) \leq 0$. Thus, equality holds throughout (20); in particular, we have $h(\bar{x}) + \lambda g(\bar{x}) = h(\bar{x})$, which gives $\lambda g(\bar{x}) = 0$. Also, in view of the fourth equality in (20) and condition (ii), we must have $\lambda > 0$; consequently, we have $g(\bar{x}) = 0$. Fix any such $\lambda > 0$. Then, in view of [36, Theorem 28.1], we also have

$$\bar{x} \in \text{Arg min } f = \{x : g(x) = 0\} \cap \text{Arg min}(h + \lambda g + \delta_D). \tag{21}$$

Next, since $h(x) = l(Ax)$ for some strictly convex function l , it must hold true that Ax is constant for any $x \in \text{Arg min}(h + \lambda g + \delta_D)$. Since $\lambda > 0$, we deduce that $g(x)$ is constant over $\text{Arg min}(h + \lambda g + \delta_D)$. Hence, in view of (21), we have $g(x) = g(\bar{x}) = 0$ for any $x \in \text{Arg min}(h + \lambda g + \delta_D)$. Then we conclude further from (21) that

$$\text{Arg min } f = \{x : g(x) = 0\} \cap \text{Arg min}(h + \lambda g + \delta_D) = \text{Arg min}(h + \lambda g + \delta_D). \tag{22}$$

Now, using condition (iii) and [7, Theorem 5(i)], and noting that $\bar{x} \in \text{dom } \partial(h + \lambda g + \delta_D)$, we see that there exist ϵ, ν and $c > 0$ so that

$$\text{dist}(x, \text{Arg min}(h + \lambda g + \delta_D)) \leq c(h(x) + \lambda g(x) - h(\bar{x}))^{1-\alpha} \tag{23}$$

whenever $\|x - \bar{x}\| \leq \epsilon, x \in D$ and $h(\bar{x}) \leq h(x) + \lambda g(x) < h(\bar{x}) + \nu$, because $g(\bar{x}) = 0$. On the other hand, whenever $h(\bar{x}) < h(x) < h(\bar{x}) + \nu$ and $x \in C \cap D$, we have

$$h(\bar{x}) = \inf_{x \in D} \{h(x) + \lambda g(x)\} \leq h(x) + \lambda g(x) \leq h(x) < h(\bar{x}) + \nu,$$

where the equality follows from (20) and the second inequality follows from the definition of C . Combining this with (23), we see that whenever $x \in C \cap D, \|x - \bar{x}\| \leq \epsilon$ and $h(\bar{x}) < h(x) < h(\bar{x}) + \nu$, we have

$$\begin{aligned} \text{dist}(x, \text{Arg min } f) &= \text{dist}(x, \text{Arg min}(h + \lambda g + \delta_D)) \leq c(h(x) + \lambda g(x) - h(\bar{x}))^{1-\alpha} \\ &\leq c(h(x) - h(\bar{x}))^{1-\alpha} = c(f(x) - f(\bar{x}))^{1-\alpha}, \end{aligned}$$

where the first equality follows from (22) and the second inequality follows from the definition of C . The conclusion of the theorem now follows from this last relation and [7, Theorem 5(ii)]. □

In our next result, Theorem 3.6, we study the KL property of $F(x, y) := f(x) + \frac{\beta}{2}\|x - y\|^2$ for any $\beta > 0$ when f is a KL function. The function F is used in the convergence analysis of various first-order methods whose iterates involve *momentum* terms, see, for example, [9,35] for convergence analysis of some inertial proximal algorithms and [10,20] for convergence analysis of the proximal gradient algorithm with extrapolation. Theorem 3.6 will be used to analyze the convergence rate of an inertial proximal algorithm, the iPiano [35], with constant step sizes, in Theorem 5.1.

We start with the following simple lemma.

Lemma 3.1 *Let $\gamma \in (1, 2]$. Then there exist $\eta_1 > 0$ and $\eta_2 \in (0, 1)$ so that*

$$\|a + b\|^\gamma \geq \eta_1 \|a\|^\gamma - \eta_2 \|b\|^\gamma$$

for any $a, b \in \mathbb{R}^n$.

Proof Notice that $x \mapsto \|x\|^\gamma$ is convex because $\gamma \in (1, 2]$. Fix any $\lambda \in (0, \frac{1}{2})$. Then we have from convexity that

$$\|a\|^\gamma = \left\| \lambda \frac{a+b}{\lambda} + (1-\lambda) \frac{-b}{(1-\lambda)} \right\|^\gamma \leq \lambda^{1-\gamma} \|a+b\|^\gamma + (1-\lambda)^{1-\gamma} \|b\|^\gamma.$$

Rearranging terms, we obtain further that

$$\|a + b\|^\gamma \geq \lambda^{\gamma-1} \|a\|^\gamma - \left(\frac{\lambda}{1-\lambda} \right)^{\gamma-1} \|b\|^\gamma.$$

The proof is completed upon noting that $\frac{\lambda}{1-\lambda} \in (0, 1)$, since $\lambda \in (0, \frac{1}{2})$. □

Theorem 3.6 (Exponent for a potential function for iPiano) *Suppose that f is a proper closed function that has the KL property at $\bar{x} \in \text{dom } \partial f$ with an exponent of $\alpha \in [\frac{1}{2}, 1)$ and $\beta > 0$. Consider the function $F(x, y) := f(x) + \frac{\beta}{2}\|x - y\|^2$. Then the function F has the KL property at (\bar{x}, \bar{x}) with an exponent of α .*

Proof Since f has the KL property at \bar{x} with an exponent of $\alpha \in (0, 1)$, there exist c, ϵ and $\nu > 0$ so that

$$\text{dist}^{\frac{1}{\alpha}}(0, \partial f(x)) \geq c(f(x) - f(\bar{x})) \tag{24}$$

whenever $x \in \text{dom } \partial f, \|x - \bar{x}\| \leq \epsilon$ and $f(x) < f(\bar{x}) + \nu$, where the condition $f(\bar{x}) < f(x)$ is dropped because (24) holds trivially otherwise. By shrinking ϵ further if necessary, we assume that $\epsilon < \frac{1}{2}$.

Now, consider any (x, y) satisfying $x \in \text{dom } \partial f, \|x - \bar{x}\| \leq \epsilon, \|y - \bar{x}\| \leq \epsilon$ and $F(\bar{x}, \bar{x}) < F(x, y) < F(\bar{x}, \bar{x}) + \nu$. Clearly, any such (x, y) satisfies

$$f(x) \leq F(x, y) < F(\bar{x}, \bar{x}) + \nu = f(\bar{x}) + \nu$$

and thus (24) holds for these x . Then for some suitable positive constants C_0, C_1 and C_2 (to be specified below), we have for any such (x, y) that

$$\begin{aligned} \text{dist}_{\frac{1}{\alpha}}(0, \partial F(x, y)) &\geq C_0(\|\beta(y - x)\|_{\frac{1}{\alpha}} + \inf_{\xi \in \partial f(x)} \|\xi + \beta(x - y)\|_{\frac{1}{\alpha}}) \\ &\geq C_0(\|\beta(y - x)\|_{\frac{1}{\alpha}} + \inf_{\xi \in \partial f(x)} \eta_1 \|\xi\|_{\frac{1}{\alpha}} - \eta_2 \|\beta(x - y)\|_{\frac{1}{\alpha}}) \\ &\geq C_1(\inf_{\xi \in \partial f(x)} \|\xi\|_{\frac{1}{\alpha}} + \|\beta(y - x)\|_{\frac{1}{\alpha}}) \geq C_2 \left(\inf_{\xi \in \partial f(x)} \|\xi\|_{\frac{1}{\alpha}} + \frac{\beta c}{2} \|y - x\|_{\frac{1}{\alpha}} \right) \\ &\geq C_2 c \left(f(x) - f(\bar{x}) + \frac{\beta}{2} \|y - x\|_{\frac{1}{\alpha}} \right) \\ &\geq C_2 c \left(f(x) - f(\bar{x}) + \frac{\beta}{2} \|y - x\|^2 \right) = C_2 c (F(x, y) - F(\bar{x}, \bar{x})), \end{aligned}$$

where the existence of $C_0 > 0$ in the first inequality follows from Lemma 2.2; the second inequality follows from Lemma 3.1 applied to the term $\|\xi + \beta(x - y)\|_{\frac{1}{\alpha}}$, with η_1 and η_2 given by Lemma 3.1; the third inequality follows by setting $C_1 = C_0 \min\{\eta_1, 1 - \eta_2\}$; the fourth inequality follows by further shrinking C_1 , and c is the same number as in (24); the fifth inequality follows from (24), while the last inequality follows from the assumption on α and the observation that

$$\|x - y\| \leq \|x - \bar{x}\| + \|y - \bar{x}\| \leq 2\epsilon < 1.$$

This completes the proof. □

In our last theorem in this section, we examine KL property on a subset, more precisely, a manifold \mathfrak{M} . Roughly speaking, we show that under partial smoothness and some additional assumptions, one only needs to verify the KL property along \mathfrak{M} in order to establish the property at a point \bar{x} . Before stating the result, we recall some necessary definitions. First, following [38, Definition 13.27], a proper closed function f is said to be prox-regular at a point \bar{x} for a subgradient $\bar{v} \in \partial f(\bar{x})$ if $\bar{x} \in \text{dom } f$ and there exists $\rho \geq 0$ such that

$$f(x') \geq f(x) + \langle v, x' - x \rangle - \frac{\rho}{2} \|x' - x\|^2,$$

whenever x and x' are near \bar{x} with $f(x)$ near $f(\bar{x})$ and $v \in \partial f(x)$ is near \bar{v} . Furthermore, we say that f is prox-regular at \bar{x} if it is prox-regular at \bar{x} for every $\bar{v} \in \partial f(\bar{x})$.

Next, let \mathfrak{M} be a C^2 manifold about $\bar{x} \in \mathfrak{M}$.⁶ For a function f that is C^2 around $\bar{x} \in \mathfrak{M}$, the covariant derivative $\nabla_{\mathfrak{M}} f(\bar{x})$ is defined as the unique vector in $T_{\bar{x}}(\mathfrak{M})$ with

$$\langle \nabla_{\mathfrak{M}} f(\bar{x}), \xi \rangle = \left. \frac{d}{dt} f(P_{\mathfrak{M}}(\bar{x} + t\xi)) \right|_{t=0}$$

⁶ Following [18], this notion means that locally \mathfrak{M} can be expressed as the solution set of a collection of C^2 equations with linearly independent gradients.

for any $\xi \in T_{\bar{x}}(\mathfrak{M})$, where $T_{\bar{x}}(\mathfrak{M})$ is the tangent space of \mathfrak{M} at \bar{x} , and $P_{\mathfrak{M}}$ is the projection onto \mathfrak{M} ; this latter operation is well defined in a sufficiently small neighborhood (in \mathbb{R}^n) of $\bar{x} \in \mathfrak{M}$.

Finally, we recall the notion of partial smoothness. From [18, Definition 2.3] (see also [22, Definition 2.7]), a function f is \mathcal{C}^2 -partly smooth at \bar{x} relative to \mathfrak{M} if the following four properties hold:

- (i) (restricted smoothness) the restriction of f on \mathfrak{M} is a \mathcal{C}^2 function near \bar{x} ;
- (ii) (regularity) at every point in \mathfrak{M} close to \bar{x} , the function f is regular and $\partial f(\bar{x}) \neq \emptyset$;
- (iii) (normal sharpness) the affine span of $\partial f(\bar{x})$ is a translate of the limiting normal cone $N_{\mathfrak{M}}(\bar{x})$;
- (iv) (subgradient continuity) the subdifferential mapping ∂f restricted to \mathfrak{M} is continuous at \bar{x} .

The class of prox-regular and \mathcal{C}^2 -partly smooth function is a broad class of functions. For example, as pointed out in [18, Example 2.5], any function f that can be expressed as $f(x) = \max\{f_i(x) : 1 \leq i \leq p\}$ for some $p \in \mathbb{N}$ and \mathcal{C}^2 functions f_i is prox-regular. Moreover, in the same example, the authors also demonstrated that if $\{\nabla f_i(\bar{x})\}_{1 \leq i \leq p}$ is linearly independent, then f is \mathcal{C}^2 -partly smooth at \bar{x} relative to $\mathfrak{M} = \{x : I_f(x) = I_f(\bar{x})\}$ where $I_f(x) = \{i \in \{1, \dots, p\} : f(x) = f_i(x)\}$.

Theorem 3.7 (Exponent for partly smooth KL functions) *Suppose that f is a proper closed function that is prox-regular at \bar{x} and \mathcal{C}^2 -partly smooth at \bar{x} relative to a manifold \mathfrak{M} . Suppose further that $0 \in \text{ri } \partial f(\bar{x})$ and that there exist $c, \nu, \epsilon > 0$ and $\alpha \in [0, 1)$ so that*

$$\|\nabla_{\mathfrak{M}} f(x)\| \geq c(f(x) - f(\bar{x}))^\alpha$$

whenever $x \in \mathfrak{M}$ with $\|x - \bar{x}\| \leq \epsilon$ and $f(\bar{x}) < f(x) < f(\bar{x}) + \nu$. Then f has the KL property at \bar{x} with an exponent of α .

Proof Our proof proceeds by contradiction. Suppose the contrary holds. Then there exists $\{x^k\} \subseteq \text{dom } \partial f$ with $x^k \rightarrow \bar{x}$ and $f(x^k) > f(\bar{x})$, $f(x^k) \rightarrow f(\bar{x})$ such that

$$\text{dist}(0, \partial f(x^k)) < \frac{1}{k}(f(x^k) - f(\bar{x}))^\alpha. \tag{25}$$

In particular, $\text{dist}(0, \partial f(x^k)) \rightarrow 0$. This together with the assumptions on prox-regularity, partial smoothness, $0 \in \text{ri } \partial f(\bar{x})$ and [18, Theorem 5.3] shows that $x^k \in \mathfrak{M}$ for all sufficiently large k . Thus, by considering even larger k if necessary so that $\|x^k - \bar{x}\| \leq \epsilon$ and $f(\bar{x}) < f(x^k) < f(\bar{x}) + \nu$, we conclude from the assumption that

$$\|\nabla_{\mathfrak{M}} f(x^k)\| \geq c(f(x^k) - f(\bar{x}))^\alpha$$

for all sufficiently large k . Since $\text{dist}(0, \partial f(x^k)) = \|\nabla_{\mathfrak{M}} f(x^k)\|$ for all sufficiently large k as a consequence of [11, Proposition 23], we have obtained a contradiction to (25). This completes the proof. □

4 Structured Problems: Luo–Tseng Error Bound and KL Property

In this section, we examine structured optimization problems where the objective functions f are proper closed functions taking the following form:

$$f(x) := h(x) + P(x). \quad (26)$$

Here, h is a proper closed (possibly nonconvex) function with an open domain and is continuously differentiable with a locally Lipschitz continuous gradient on $\text{dom } h$, and P is proper closed convex. Since f is proper closed, we must have $\text{dom } h \cap \text{dom } P \neq \emptyset$. For this class of functions, the Luo–Tseng error bound (3) is commonly used in the literature for establishing local linear convergence of various first-order methods applied to minimizing f , see, for example, [28–30, 40, 41]. In this section, we study the relationship between the Luo–Tseng error bound and the KL property for the class of functions (26), under the following assumption concerning separation of stationary values. Recall that \mathcal{X} is the set of stationary points of f .

Assumption 4.1 For any $\bar{x} \in \mathcal{X}$, there exists $\delta > 0$ so that $f(y) = f(\bar{x})$ whenever $y \in \mathcal{X}$ and $\|y - \bar{x}\| \leq \delta$.

This assumption is trivially satisfied if h is, in addition, convex. Moreover, a global version of this assumption is commonly used together with the Luo–Tseng error bound for convergence analysis in the literature.

Before proving our main result of this section under Assumption 4.1, we first establish the following auxiliary lemma. This lemma is a generalization of [13, Proposition 1.5.14], where we have a general proper closed convex function P instead of just the indicator function of a closed convex set.

Lemma 4.1 Consider the function f given in (26). For any $x \in \text{dom } \partial f$, it holds that

$$\|\text{prox}_P(x - \nabla h(x)) - x\| \leq \text{dist}(0, \partial f(x)). \quad (27)$$

Proof For any $x \in \text{dom } \partial f$, we have $\partial f(x) \neq \emptyset$. Since $\partial f(x) = \nabla h(x) + \partial P(x)$ thanks to [38, Exercise 8.8(c)], we must then have $\partial P(x) \neq \emptyset$. Consequently, the set $x + \partial P(x)$ is nonempty. Using the definition of proximal mapping, this means that there exists z so that $x = \text{prox}_P(z)$.

Now, for any $x \in \text{dom } \partial f$, let z be such that $x = \text{prox}_P(z)$. Then

$$\begin{aligned} \|\text{prox}_P(x - \nabla h(x)) - x\| &= \|\text{prox}_P(\text{prox}_P(z) - \nabla h(\text{prox}_P(z))) - \text{prox}_P(z)\| \\ &\leq \|\text{prox}_P(z) - \nabla h(\text{prox}_P(z)) - z\|, \end{aligned}$$

where the inequality follows from (2). Consequently, we have

$$\|\text{prox}_P(x - \nabla h(x)) - x\| \leq \inf_z \{\|\text{prox}_P(z) - \nabla h(\text{prox}_P(z)) - z\| : x = \text{prox}_P(z)\}. \quad (28)$$

On the other hand, observe that we have

$$\begin{aligned}
 \text{dist}(0, \partial f(x)) &= \inf_{\xi} \{ \|\nabla h(x) + \xi\| : \xi \in \partial P(x) \} \\
 &= \inf_{\xi} \{ \|\nabla h(x) - x + \underbrace{x + \xi}_z\| : \xi \in \partial P(x) \} \\
 &= \inf_z \{ \|\nabla h(x) - x + z\| : z \in x + \partial P(x) \} \\
 &= \inf_z \{ \|\nabla h(x) - x + z\| : x = \text{prox}_P(z) \} \\
 &= \inf_z \{ \|\nabla h(\text{prox}_P(z)) - \text{prox}_P(z) + z\| : x = \text{prox}_P(z) \},
 \end{aligned}
 \tag{29}$$

where the fourth equality follows from the definition of the proximal mapping. The desired inequality (27) now follows by combining (28) and (29). \square

Using the above lemma, we can now prove the following result, which states that if the Luo-Tseng error bound and Assumption 4.1 hold, then f is a KL function with an exponent of $\frac{1}{2}$.

Theorem 4.1 (Luo-Tseng error bound implies KL) *Suppose that $\mathcal{X} \neq \emptyset$, and that Assumption 4.1 and the Luo-Tseng error bound hold. Then f in (26) is a KL function with an exponent of $\frac{1}{2}$.*

Proof From Lemma 2.1, we only need to show that f has the KL property at any $\bar{x} \in \mathcal{X}$ with an exponent of $\frac{1}{2}$. To see this, fix any $\bar{x} \in \mathcal{X}$. Let $\delta > 0$ be defined as in Assumption 4.1 and take any $\epsilon \in (0, \frac{\delta}{4})$ such that $B(\bar{x}, 2\epsilon) \subset \text{dom } h$ (this is possible as $\text{dom } h$ is open and $\bar{x} \in \text{dom } h$). Then for any x with $\|x - \bar{x}\| \leq \epsilon$, we have

$$\|u - \bar{x}\| \leq \|u - x\| + \|x - \bar{x}\| = \text{dist}(x, \mathcal{X}) + \|x - \bar{x}\| \leq 2\|x - \bar{x}\| \leq 2\epsilon < \frac{\delta}{2} < \delta$$

whenever $u \in \text{Proj}_{\mathcal{X}}(x)$. Thus, it holds that $f(u) = f(\bar{x})$ for any $u \in \text{Proj}_{\mathcal{X}}(x)$ whenever $\|x - \bar{x}\| \leq \epsilon$. On the other hand, notice that $B(\bar{x}, 2\epsilon)$ is a compact subset of $\text{dom } h$ and ∇h is locally Lipschitz on $\text{dom } h$. Consequently, ∇h is globally Lipschitz continuous on $B(\bar{x}, 2\epsilon)$; we denote its Lipschitz constant by $L > 0$.

Next, from the assumption on Luo-Tseng error bound, we see that for $\zeta = f(\bar{x}) + 1$, there exist $c_1, \epsilon_1 > 0$ so that

$$\text{dist}(x, \mathcal{X}) \leq c_1 \|\text{prox}_P(x - \nabla h(x)) - x\|
 \tag{30}$$

whenever $\|\text{prox}_P(x - \nabla h(x)) - x\| < \epsilon_1$ and $f(x) \leq f(\bar{x}) + 1$. Since $\|\text{prox}_P(x - \nabla h(x)) - x\| < \epsilon_1$ is a neighborhood of \mathcal{X} containing \bar{x} , by shrinking ϵ if necessary, we may assume that (30) holds whenever $\|x - \bar{x}\| \leq \epsilon$ and $f(x) \leq f(\bar{x}) + 1$.

From the above discussions, for any $x \in \text{dom } \partial f$ with $f(\bar{x}) < f(x) < f(\bar{x}) + 1$ and $\|x - \bar{x}\| \leq \epsilon$, we have $f(\bar{x}) = f(u)$ for any $u \in \text{Proj}_{\mathcal{X}}(x)$, and

$$\begin{aligned} f(x) - f(\bar{x}) &= f(x) - f(u) = h(x) - h(u) + P(x) - P(u) \\ &\leq \langle \nabla h(u), x - u \rangle + \frac{L}{2} \|x - u\|^2 + \langle \xi, x - u \rangle \\ &= \langle \nabla h(u) - \nabla h(x), x - u \rangle + \frac{L}{2} \|x - u\|^2 + \langle \nabla h(x) + \xi, x - u \rangle \\ &\leq \frac{3L}{2} \|x - u\|^2 + \|\nabla h(x) + \xi\| \cdot \|x - u\| \\ &= \frac{3L}{2} \text{dist}^2(x, \mathcal{X}) + \|\nabla h(x) + \xi\| \cdot \text{dist}(x, \mathcal{X}) \\ &\leq C_0(\|\text{prox}_P(x - \nabla h(x)) - x\|^2 + \|\nabla h(x) + \xi\| \cdot \|\text{prox}_P(x - \nabla h(x)) - x\|) \end{aligned}$$

for any $\xi \in \partial P(x)$, where the first inequality is a consequence of the subgradient inequality applied to P and the fact that ∇h is Lipschitz continuous with a Lipschitz constant of L on $B(\bar{x}, 2\epsilon)$, which contains both x and u , the last equality follows from the definition of u , while the last inequality follows from (30) for some suitable $C_0 > 0$. Taking infimum over all possible $\xi \in \partial P(x)$ and invoking Lemma 4.1, we see further that

$$f(x) - f(\bar{x}) \leq C_1[\text{dist}(0, \partial f(x))]^2$$

for some $C_1 > 0$. This completes the proof. □

Before ending this section, we discuss some immediate applications of Theorem 4.1. The first application involves piecewise linear-quadratic (PLQ) functions⁷

$$f(x) = l(Ax) + P(x), \tag{31}$$

where l is strongly convex on any compact convex set and is twice continuously differentiable, $A \in \mathbb{R}^{m \times n}$ and P is a convex PLQ function on \mathbb{R}^n . We show in the next proposition that the f in (31) is a KL function with an exponent of $\frac{1}{2}$. We would also like to point out that, in the special case of ℓ_1 -regularized least squares function (i.e., $f(x) = l(Ax) + P(x)$ where $l(y) = \frac{1}{2}\|y - b\|^2$ and $P(x) = \mu \sum_{i=1}^n |x_i|$ for some $\mu > 0$), it was known in [7, Section 3.2.1] that f is a KL function with an exponent of $\frac{1}{2}$.

To achieve our promised result, we will need the notions of metric subregularity and calmness for set-valued mappings. Recall that a set-valued mapping $G : \mathbb{R}^n \rightarrow 2^{\mathbb{R}^m}$ is said to be metrically subregular [12, Page 183] at \bar{x} for \bar{y} if $\bar{y} \in G(\bar{x})$, and there exist a constant $\kappa > 0$, a neighborhood U of \bar{x} and a neighborhood V of \bar{y} such that

$$\text{dist}(x, G^{-1}(\bar{y})) \leq \kappa \text{dist}(\bar{y}, G(x) \cap V) \quad \text{for all } x \in U,$$

⁷ Recall that a proper closed function F is called piecewise linear-quadratic [38, Definition 10.20] if $\text{dom } F$ can be represented as the union of finitely many polyhedrons, relative to each of which $F(x)$ is given by the form $\frac{1}{2}x^T Mx + a^T x + \alpha$, where $M \in S^n$, $a \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$.

A property closely related to metric subregularity is the notion of calmness (see [12, Page 182]). A mapping $S : \mathbb{R}^m \rightarrow 2^{\mathbb{R}^n}$ is said to be calm at \bar{y} for \bar{x} if $\bar{x} \in S(\bar{y})$, and there is a constant $r > 0$ along with neighborhoods U of \bar{x} and V of \bar{y} such that

$$S(y) \cap U \subseteq S(\bar{y}) + r\|y - \bar{y}\|\mathbb{B}_{\mathbb{R}^n} \text{ for all } y \in V,$$

where $\mathbb{B}_{\mathbb{R}^n}$ is the closed unit ball in \mathbb{R}^n . It is known that a set-valued mapping is calm if and only if its inverse mapping is metrically subregular [12, Theorem 3H.3].

Proposition 4.1 (Convex problems with convex piecewise linear-quadratic regularizers) *Consider a convex function f given as in (31). Suppose that $\text{Arg min } f$ is a nonempty compact set. Then f is a KL function with an exponent of $\frac{1}{2}$.*

Proof Fix any $\bar{x} \in \text{Arg min } f$. Denote $\bar{y} = A\bar{x}$ and $\bar{g} = A^*\nabla l(A\bar{x})$. Then $-\bar{g} \in \partial P(\bar{x})$ from the first-order optimality condition and [38, Exercise 8.8(c)]. Notice that $Ax \equiv \bar{y}$ over $\text{Arg min } f$ due to the strong convexity of l on compact convex sets. Consequently, we also have $A^*\nabla l(Ax) \equiv A^*\nabla l(\bar{y}) = \bar{g}$ over $\text{Arg min } f$. Define

$$\Gamma_f(\bar{y}) := A^{-1}\bar{y} \text{ and } \Gamma_P(\bar{g}) := \partial P^*(-\bar{g})$$

as in [46, Section 3.3]. We will subsequently show that

- (1) The pair of sets $\{\Gamma_f(\bar{y}), \Gamma_P(\bar{g})\}$ is boundedly linearly regular.
- (2) The subdifferential mapping ∂P is metrically subregular at \bar{x} for $-\bar{g}$.

Granting these, since $\bar{x} \in \text{Arg min } f$ is arbitrary, we conclude using [46, Theorem 2] and [46, Corollary 1] that the Luo-Tseng error bound holds. Since f is convex so that Assumption 4.1 is trivially satisfied, the desired conclusion follows from Theorem 4.1.

Now it remains to prove the two claims above.

For (1), note that P^* is convex piecewise linear-quadratic according to [38, Theorem 11.14], which implies that $\partial P^*(-\bar{g})$ is a polyhedral set [38, Proposition 10.21]. Consequently, it follows from [5, Corollary 5.2.6] that the pair of sets $\{\Gamma_f(\bar{y}), \Gamma_P(\bar{g})\}$ is linearly regular in the sense that there exists $c > 0$ such that

$$\text{dist}(x, \Gamma_f(\bar{y}) \cap \Gamma_P(\bar{g})) \leq c \max\{\text{dist}(x, \Gamma_f(\bar{y})), \text{dist}(x, \Gamma_P(\bar{g}))\} \text{ for all } x \in \mathbb{R}^n.$$

Hence, this pair of set is in particular boundedly linearly regular, see [5, Definition 5.6].

For (2), recall that $\partial P^* = (\partial P)^{-1}$ is a piecewise polyhedral set-valued mapping [38, Proposition 12.30]. Thus, the set-valued mapping $(\partial P)^{-1}$ is calm by Robinson's theorem on calmness of piecewise affine mappings [37] (see also [38, Example 9.57]). Using this and the fact that a set-valued mapping is calm if and only if its inverse mapping is metrically subregular [12, Theorem 3H.3], we conclude further that ∂P is metrically subregular at \bar{x} for $-\bar{g} \in \partial P(\bar{x})$. This completes the proof. \square

In our second application, we consider the following model

$$f(x) = l(Ax) + \delta_C(x), \tag{32}$$

where l is strongly convex on any compact convex set and is continuously differentiable with Lipschitz gradient, and $A \in \mathbb{R}^{m \times n}$. In addition, the set C is defined as

$$C := \left\{ x : \sum_{i=1}^m w_i \|x_i\|_p \leq \sigma \right\},$$

where $\sigma > 0$, $p \in [1, 2]$, and for each $i = 1, \dots, m$, $x_i \in \mathbb{R}^{n_i}$ with $\sum_{i=1}^m n_i = n$, $w_i > 0$, and

$$\|x_i\|_p := \left(\sum_{j=1}^{n_i} |(x_i)_j|^p \right)^{\frac{1}{p}}.$$

When $p = 2$ and $l(y) = \frac{1}{2}\|y - b\|^2$ for some b , the model (32) can be viewed as a variant of the group LASSO problem [44]. In the next proposition, we show that the function in (32) has the KL property with an exponent of $\frac{1}{2}$, under mild assumptions.

Proposition 4.2 *Consider a convex function f given as in (32). Suppose that $\inf_{x \in \mathbb{R}^n} f(x) > \inf_{x \in \mathbb{R}^n} l(Ax)$. Then f is a KL function with an exponent of $\frac{1}{2}$.*

Proof Let $g(x) := \sum_{i=1}^m w_i \|x_i\|_p - \sigma$. Then $g(0) = -\sigma < 0$. Now, fix any $\lambda > 0$ and consider $h_\lambda(x) = l(Ax) + \lambda g(x)$. We first consider the case $\text{Arg min}(h_\lambda) \neq \emptyset$. In this case, in view of [47, Proposition 1], the set $\text{Arg min}(h_\lambda)$ is compact. Thus, if $p = 1$, we conclude from Proposition 4.1 that the Luo-Tseng error bound holds for h_λ . On the other hand, if $p \in (1, 2]$, then it follows from [47, Corollary 2] that the Luo-Tseng error bound also holds for h_λ . Since h_λ is convex so that Assumption 4.1 is trivially satisfied, we see further from Theorem 4.1 that h_λ is a KL function with an exponent of $\frac{1}{2}$ in this case. Note that the same conclusion holds when $\text{Arg min}(h_\lambda) = \emptyset$ due to Lemma 2.1. Consequently, the assumptions in Theorem 3.5 are satisfied with $D = \mathbb{R}^n$ and $h = l \circ A$, and the desired conclusion follows immediately from this theorem. \square

5 Applications

In this section, we apply our results in the previous sections to deducing the KL exponent of some specific functions that arise in various applications. We then discuss how our results can be applied to establishing the linear convergence of some first-order methods.

5.1 Piecewise Linear Regularizers

In this subsection, we demonstrate how our results can be applied to some problems with possibly nonconvex piecewise linear regularizers, i.e., their epigraphs are unions of polyhedrons. In particular, we have the following corollary.

Corollary 5.1 *Suppose that f is a proper closed function taking the form*

$$f(x) = l(Ax) + \min_{1 \leq i \leq r} P_i(x),$$

where $A \in \mathbb{R}^{m \times n}$, P_i are proper closed polyhedral functions for $i = 1, \dots, r$ and l satisfies either one of the following two conditions:

- (i) l is a proper closed convex function with an open domain, is strongly convex on any compact convex subset of $\text{dom } l$ and is twice continuously differentiable on $\text{dom } l$.
- (ii) $l(y) = \max_{u \in D} \{ \langle y, u \rangle - q(u) \}$ for all $y \in \mathbb{R}^m$, with D being a polyhedron and q being a strongly convex differentiable function with a Lipschitz continuous gradient.

Suppose in addition that f is continuous on $\text{dom } \partial f$. Then f is a KL function with an exponent of $\frac{1}{2}$.

Proof Since f is proper, we can assume without loss of generality that $\text{dom } l \cap \text{Adom } P_i \neq \emptyset$ for $i = 1, \dots, r$. Let $f_i(x) = l(Ax) + P_i(x)$ for $i = 1, \dots, r$. Consider those f_i with $\text{Arg min } f_i \neq \emptyset$. Suppose first that condition (i) holds. Our arguments follow the proof of [41, Lemma 7]. Define

$$g_i(x, s) = \underbrace{l(Ax) + s}_{h(x,s)} + \delta_{K_i}(x, s) = l\left(\begin{bmatrix} A & 0 \end{bmatrix} \begin{bmatrix} x \\ s \end{bmatrix}\right) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \begin{bmatrix} x \\ s \end{bmatrix} + \delta_{K_i}(x, s).$$

where $K_i = \{(x, s) : P_i(x) \leq s\}$. In particular, h takes the form of [28, Eq 1.1]. Also, one can check that l satisfies the assumptions 1.1 and 1.2 in [28].⁸ Thus, in view of [28, Theorem 2.1], we conclude that the Luo-Tseng error bound holds for g_i , i.e., for any $\zeta \geq \inf g_i = \inf f_i$, there exist C and $\epsilon > 0$ so that

$$\text{dist}((x, s), \text{Arg min } g_i) \leq C \|\text{Proj}_{K_i}[\begin{bmatrix} x \\ s \end{bmatrix} - \nabla h(x, s)] - (x, s)\| \tag{33}$$

whenever $\|\text{Proj}_{K_i}[\begin{bmatrix} x \\ s \end{bmatrix} - \nabla h(x, s)] - (x, s)\| < \epsilon$ and $g_i(x, s) \leq \zeta$. Moreover, note that we have

$$\begin{aligned} & \|\text{Proj}_{K_i}[\begin{bmatrix} x \\ P_i(x) \end{bmatrix} - \nabla h(x, P_i(x))] - (x, P_i(x))\| \\ & \leq \kappa \|\text{prox}_{P_i}(x - \nabla(l \circ A)(x)) - x\| \end{aligned}$$

⁸ Assumption 1.1(a) in [28] holds because $\text{dom } l$ is open and l is proper. Assumption 1.1(b) and Assumption 1.2(b) in [28] hold as l is strongly convex on any compact convex subset of $\text{dom } l$ and is twice continuously differentiable on $\text{dom } l$. Assumption 1.2(a) in [28] holds because we are considering the case that $\text{Arg min } f_i \neq \emptyset$ and so $\text{Arg min } g_i \neq \emptyset$. Finally, assumption 1.1(c) in [28] holds because l is lower semicontinuous with an open domain, so that for any \bar{y} in the boundary of the domain, one has $\liminf_{y \rightarrow \bar{y}} l(y) \geq l(\bar{y}) = \infty$.

whenever $x \in \text{dom } f_i$ for some $\kappa > 0$, thanks to [41, Lemma 6],⁹ and that for these x , it is easy to show that

$$\text{dist}(x, \text{Arg min } f_i) \leq \text{dist}((x, P_i(x)), \text{Arg min } g_i).$$

Combining these two observations with (33), we conclude that the Luo-Tseng error bound holds for f_i whenever $\text{Arg min } f_i \neq \emptyset$, i.e., for any $\zeta \geq \inf f_i$, there exist C_1 and $\epsilon_1 > 0$ so that

$$\text{dist}(x, \text{Arg min } f_i) \leq C_1 \|\text{prox}_{P_i}(x - \nabla(l \circ A)(x)) - x\|$$

whenever $\|\text{prox}_{P_i}(x - \nabla(l \circ A)(x)) - x\| < \epsilon_1$ and $f_i(x) \leq \zeta$. On the other hand, suppose that condition (ii) holds. Then it follows from [41, Theorem 4] that the Luo-Tseng error bound also holds for f_i . Next, observe that Assumption 4.1 is trivially satisfied for all f_i because f_i is convex for all i . Using these, Theorem 4.1 and Lemma 2.1 (for the case $\text{Arg min } f_i = \emptyset$), we conclude that the functions $f_i(x) = l(Ax) + P_i(x)$, $1 \leq i \leq r$ are all KL functions with exponents $\frac{1}{2}$.

Finally, notice that l has an open domain and is continuously differentiable on it under either condition (i) or (ii); indeed, under condition (ii), l is the conjugate of the strongly convex function $u \mapsto q(u) + \delta_D(u)$ and is thus continuously differentiable everywhere thanks to [4, Theorem 18.15]. Consequently, in view of [38, Exercise 8.8(c)], we have $\partial f_i(x) = \nabla(l \circ A)(x) + \partial P_i(x)$ for any x . From this, we deduce further that, for each i ,

$$\begin{aligned} \text{dom } f_i &= \text{dom}(l \circ A) \cap \text{dom } P_i = \text{dom}(l \circ A) \cap \text{dom } \partial P_i \\ &= \text{dom} \nabla(l \circ A) \cap \text{dom } \partial P_i = \text{dom } \partial f_i, \end{aligned}$$

where the second equality is a consequence of the polyhedricity of P_i : Polyhedral functions are piecewise affine on their domains, see [8, Proposition 5.1.1]. The desired conclusion now follows from Corollary 3.1. \square

With this corollary in mind, we can show that the following classes of proper closed functions have KL exponents of $\frac{1}{2}$:

$$\begin{aligned} f_1(x) &= l(Ax) + \mu \|x\|_1 - \mu \gamma \sum_{i=1}^k |x_{[i]}|, \\ f_2(x) &= l(Ax) + \delta_{\|\cdot\|_0 \leq r}(x), \\ f_3(x) &= l(Ax) + \delta_{\|\cdot\|_0 \leq r}(x) + \delta_{\Delta}(x), \end{aligned} \tag{34}$$

where $A \in \mathbb{R}^{m \times n}$ is a measurement matrix and l is defined as in Corollary 5.1, $|x_{[i]}|$ is the i th largest (in magnitude) entry in x , $k \leq n$, $\mu > 0$, $\gamma \in (0, 1]$, $r > 0$ and

⁹ The statement of [41, Lemma 6] is proved under the assumption that $x \mapsto l(Ax)$ is smooth on an open set containing $\text{dom } P_i$, but it is not hard to see that the proof is valid also in our settings, i.e., when $\text{dom } l \cap \text{Adom } P_i \neq \emptyset$ and $\text{dom } l$ is open. For the convenience of the readers, we include a proof in ‘‘Appendix.’’

$\Delta := \{x \in \mathbb{R}^n : e^T x = 1, x \geq 0\}$. Indeed, it is clear that the regularizers on the right-hand side in (34) are representable as the minimum of finitely many proper closed polyhedral functions. Specifically,

$$\begin{aligned} \mu \|x\|_1 - \mu\gamma \sum_{i=1}^k |x_{[i]}| &= \min_{I \in \mathcal{I}_k} \left\{ \mu \|x\|_1 - \mu\gamma \sum_{i \in I} |x_i| \right\}, \\ \delta_{\|\cdot\|_0 \leq r}(x) &= \min_{I \in \mathcal{I}_{n-r}} \{ \delta_{H_I}(x) \}, \\ \delta_{\|\cdot\|_0 \leq r}(x) + \delta_{\Delta}(x) &= \min_{I \in \mathcal{I}_{n-r}} \{ \delta_{H_I \cap \Delta}(x) \}, \end{aligned}$$

where \mathcal{I}_k is the collection of all subsets of $\{1, \dots, n\}$ of size k and $H_I = \{x : x_i = 0 \ \forall i \in I\}$. Thus, the conclusions on the KL exponents follow immediately from Corollary 5.1.

The functions in (34) arise as models for sparse recovery. Specifically, the function f_1 with $l(y) = \frac{1}{2} \|y - b\|^2$ was considered in [42], the function f_2 with $l(y) = \frac{1}{2} \|y - b\|^2$ was considered in [6] and many subsequent work, while the function f_3 with $l(y) = \frac{1}{2} \|y - b\|^2$ arises in the problem of sparse index tracking in portfolio optimization, see, for example, [21, Section 8]. In addition, we would like to point out that our assumptions on l are general enough to cover the loss functions for logistic and Poisson regression, where $l(y) = \sum_{i=1}^m \log(1 + \exp(b_i y_i))$ and $l(y) = \sum_{i=1}^m (-b_i y_i + \exp(y_i))$, respectively, for some suitable $b \in \mathbb{R}^m$, see [46, Table 1]. Moreover, one can also consider functions of the form $l(y) = -\sum_{i=1}^m \log(y_i)$ that are not defined on the whole space. This makes it possible to analyze the local convergence rate for first-order methods applied to some models that involve these functions.

5.2 Some Nonconvex Minimum-of-Quadratic Regularizers

In this subsection, we demonstrate how our results can be applied to some least squares problems with regularizers that can be written as the minimum of finitely many functions, each of which is the sum of a quadratic function (not necessarily convex) and a proper closed polyhedral function.

Specifically, we consider the following class of functions:

$$f(x) = \min_{1 \leq i \leq r} \left\{ \frac{1}{2} x^T M_i x + u_i^T x + \beta_i + P_i(x) \right\}, \tag{35}$$

where P_i are proper closed polyhedral functions, $M_i \in \mathcal{S}^n$, $u_i \in \mathbb{R}^n$ and $\beta_i \in \mathbb{R}$, for $i = 1, \dots, r$. This class of functions covers nonconvex functions which need not be piecewise linear-quadratic,¹⁰ and encompasses a lot of problems that arise in applications, as we shall see later. We now state the following result.

¹⁰ For a simple example, consider $f(x) = -|x_1^2 + x_2^2 - 1|$. Clearly, f can be written as a form of (35), while f is not piecewise linear-quadratic because the pieces of this function cannot be arranged as a polyhedral union.

Corollary 5.2 *Let f be defined as in (35). Suppose in addition that f is continuous on $\text{dom } \partial f$. Then f is a KL function with an exponent of $\frac{1}{2}$.*

Proof Let $f_i(x) := \frac{1}{2}x^T M_i x + u_i^T x + \beta_i + P_i(x)$ for each $i = 1, \dots, r$. When the set of stationary points of f_i is nonempty, it follows from [41, Theorem 4] that the Luo-Tseng error bound holds. In addition, Assumption 4.1 can be seen to hold by applying [29, Lemma 3.1] to the problem $\min\{\frac{1}{2}x^T M_i x + u_i^T x + \zeta : P_i(x) \leq \zeta\}$. Thus, for these f_i , we have from Theorem 4.1 that they are KL functions with an exponent of $\frac{1}{2}$. On the other hand, if f_i does not have stationary points, then we see from Lemma 2.1 that f_i is a KL function with an exponent of $\frac{1}{2}$. Finally, note from [38, Exercise 8.8(c)] that $\text{dom } \partial f_i = \text{dom } \partial P_i$. Thus, it holds that, for each i ,

$$\text{dom } \partial f_i = \text{dom } \partial P_i = \text{dom } P_i = \text{dom } f_i,$$

where the second equality follows from the polyhedricity of P_i , as these functions are piecewise affine in their domains, see [8, Proposition 5.1.1]. The conclusion of this corollary now follows from Corollary 3.1. \square

We would like to point out that in the special case where $r = 1$ and $P_1(x) = \delta_C(x)$ with $C = \{x : Ax \leq b\}$, it was established in [15, Theorem 1] that the KL inequality holds with an exponent of $\frac{1}{2}$ under the *additional assumption* that C is compact and contains the origin in its interior. As we will see below, our extension here allows us to cover many popular optimization problems with nonconvex regularizers (such as the SCAD and MCP regularizers), while [15, Theorem 1] cannot be applied due to the additional compactness assumption of the domain.

In details, we consider the least squares problems with SCAD [14] or MCP [45] regularization functions. Recall that the objective function of these problems takes the form

$$f(x) = \frac{1}{2} \|Ax - b\|^2 + \sum_{i=1}^n r(x_i), \quad (36)$$

where $A \in \mathbb{R}^{m \times n}$ is a measurement matrix and $r : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function. For SCAD, the continuous function r takes the form

$$r(t) = \begin{cases} \lambda|t| & \text{if } |t| \leq \lambda, \\ \frac{-t^2 + 2\theta\lambda|t| - \lambda^2}{2(\theta-1)} & \text{if } \lambda < |t| \leq \theta\lambda, \\ \frac{(\theta+1)\lambda^2}{2} & \text{if } |t| > \theta\lambda, \end{cases}$$

where $\lambda > 0$ and $\theta > 2$, while for MCP, the function r is given by

$$r(t) = \begin{cases} \lambda|t| - \frac{t^2}{2\theta} & \text{if } |t| \leq \theta\lambda, \\ \frac{\theta\lambda^2}{2} & \text{if } |t| > \theta\lambda, \end{cases}$$

where $\lambda > 0$ and $\theta > 0$. In view of the definitions of SCAD and MCP, one can see that the regularization functions $\sum_{i=1}^n r(x_i)$ are continuous functions taking the form

$$\sum_{i=1}^n \min_{1 \leq l \leq m_i} \{f_{i,l}(x_i) + \delta_{C_{i,l}}(x_i)\}$$

for some closed intervals $C_{i,l}$ and one-dimensional quadratic (or linear) functions $f_{i,l}$, for each $1 \leq l \leq m_i$ and $i = 1, \dots, n$. Since this is a sum of functions with independent variables, it can be equivalently written as

$$\min_{j \in \mathfrak{J}} \left\{ \sum_{i=1}^n [f_{i,j_i}(x_i) + \delta_{C_{i,j_i}}(x_i)] \right\},$$

where $\mathfrak{J} = \{(j_1, \dots, j_n) \in \mathbb{N}^n : 1 \leq j_i \leq m_i \forall i\}$ and \mathbb{N} is the set of positive integers. Hence, the objective in the corresponding regularized least squares problems can be written as

$$\min_{j \in \mathfrak{J}} \left\{ \frac{1}{2} \|Ax - b\|^2 + \sum_{i=1}^n [f_{i,j_i}(x_i) + \delta_{C_{i,j_i}}(x_i)] \right\}$$

to which Corollary 5.2 is applicable. Thus, the function f in (36) with SCAD or MCP regularizers has a KL exponent of $\frac{1}{2}$.

5.3 Local Linear Convergence of Some First-Order Methods

In this subsection, we discuss local linear convergence of two common first-order methods, the proximal gradient algorithm and the inertial proximal algorithm, based on the KL exponent.

We first discuss the proximal gradient algorithm, also known as the forward-backward splitting algorithm. This algorithm has been studied extensively in recent years, see, for example, [3] and references therein. The algorithm, in its general form, is applicable to the following optimization problem

$$\min_{x \in \mathbb{R}^n} f(x) := h(x) + g(x),$$

where h is a smooth function whose gradient is Lipschitz continuous with constant L and g is a proper closed function. The proximal gradient algorithm for this problem can be formulated as

$$x^{k+1} \in \text{Arg min}_{x \in \mathbb{R}^n} \left\{ \langle \nabla h(x^k), x - x^k \rangle + \frac{1}{2\gamma_k} \|x - x^k\|^2 + g(x) \right\}. \tag{37}$$

where $0 < \inf_{k \geq 0} \gamma_k \leq \sup_{k \geq 0} \gamma_k < \frac{1}{L}$. The global convergence of this algorithm has been discussed in [3, Section 5.2]. We state below the local linear convergence of

the method under an explicit assumption on the KL exponent, which is an immediate consequence of [16, Theorem 3.4]. This result will be used together with results in Sects. 5.1 and 5.2 for deriving new local convergence results when the proximal gradient algorithm is applied to some concrete optimization problems, see Remark 5.1.

Proposition 5.1 *Suppose that $\inf f > -\infty$ and that f is a KL function with an exponent of $\frac{1}{2}$. Let $\{x^k\}$ be the sequence generated by the proximal gradient method given in (37). Suppose that $\{x^k\}$ is bounded. Then $\{x^k\}$ converges locally linearly to a stationary point of f .*

The next algorithm we consider is the inertial proximal algorithm, known as iPiano, proposed and studied in [35]. The algorithm can be applied to minimizing (26) when $\text{dom } h = \mathbb{R}^n$ and ∇h is globally Lipschitz continuous with Lipschitz constant L , and its algorithmic framework can be described below: set $x^{-1} = x^0$ and update

$$x^{k+1} = \arg \min_{x \in \mathbb{R}^n} \left\{ \left\langle \nabla h(x^k) - \frac{\beta_k}{\alpha_k}(x^k - x^{k-1}), x - x^k \right\rangle + \frac{1}{2\alpha_k} \|x - x^k\|^2 + P(x) \right\}. \tag{38}$$

where $\{\beta_k\}$ and $\{\alpha_k\}$ are suitable choices of sequence specified in [35, Algorithm 5] and [35, Theorem 4.9]. In particular, the global convergence of the iPiano has been established in [35, Theorem 4.9] by assuming that $\{\beta_k\} \subset [0, 1)$, $\delta_k := \frac{2-\beta_k}{2\alpha_k} - \frac{L}{2} \equiv \delta > 0$ for some $\delta > 0$, $\inf_k \left\{ \frac{1-\beta_k}{\alpha_k} - \frac{L}{2} \right\} > 0$ and $\inf_k \alpha_k > 0$. Here, we focus on the local linear convergence of (38). For simplicity, we only consider a version of iPiano with constant step sizes, presented as [35, Algorithm 2].

Theorem 5.1 *Suppose that f is a coercive KL function with an exponent of $\frac{1}{2}$. Let $\beta_k \equiv \beta \in [0, 1)$, $\alpha_k \equiv \alpha \in (0, \frac{2(1-\beta)}{L})$ and $\{x^k\}$ be a sequence generated by (38). Then $\{x^k\}$ converges locally linearly to a stationary point of f .*

Proof Define $F_\delta(x, y) := f(x) + \delta \|x - y\|^2$, where $\delta := \frac{2-\beta}{2\alpha} - \frac{L}{2} > 0$. Since f is a KL function with an exponent of $\frac{1}{2}$, we see from Theorem 3.6 that F_δ has the KL property with an exponent of $\frac{1}{2}$ at any points of the form $(a, a) \in \text{dom } \partial F_\delta$. Thus, by [35, Theorem 4.9], the whole sequence $\{x^k\}$ converges to a stationary point \bar{x} of f . It remains to establish local linear convergence.

To see this, recall from [35, Theorem 4.9] that properties **H1**, **H2** and **H3** in [35, Section 3.2] are satisfied, i.e., there are positive numbers c_1 and c_2 such that for all $k \geq 0$,

$$\begin{aligned} c_1 \|x^k - x^{k-1}\|^2 &\leq F_\delta(x^k, x^{k-1}) - F_\delta(x^{k+1}, x^k), \\ \text{dist}(0, \partial F_\delta(x^{k+1}, x^k)) &\leq c_2 (\|x^k - x^{k-1}\| + \|x^{k+1} - x^k\|), \\ F_\delta(x^{k_i+1}, x^{k_i}) &\rightarrow F_\delta(\bar{x}, \bar{x}) \text{ for some subsequence } \{x^{k_i}\}. \end{aligned} \tag{39}$$

The first and the third properties together imply that $\lim_{k \rightarrow \infty} F_\delta(x^{k+1}, x^k) = F_\delta(\bar{x}, \bar{x})$. Using this, the KL property of F_δ at (\bar{x}, \bar{x}) and the fact that $x^k \rightarrow \bar{x}$, we see the existence of $c_3 > 0$ and $k_0 \in \mathbb{N}$ so that

$$\text{dist}(0, \partial F_\delta(x^{k+1}, x^k)) \geq c_3 \sqrt{F_\delta(x^{k+1}, x^k) - F_\delta(\bar{x}, \bar{x})}$$

whenever $k \geq k_0$. Combining this relation with (39), we have

$$\begin{aligned} F_\delta(x^{k+2}, x^{k+1}) - F_\delta(\bar{x}, \bar{x}) &\leq F_\delta(x^{k+1}, x^k) - F_\delta(\bar{x}, \bar{x}) \\ &\leq \frac{1}{c_3^2} \text{dist}^2(0, \partial F_\delta(x^{k+1}, x^k)) \\ &\leq 2 \left(\frac{c_2}{c_3}\right)^2 (\|x^k - x^{k-1}\|^2 + \|x^{k+1} - x^k\|^2) \\ &\leq \frac{2}{c_1} \left(\frac{c_2}{c_3}\right)^2 [F_\delta(x^k, x^{k-1}) - F_\delta(x^{k+2}, x^{k+1})] \end{aligned}$$

for $k \geq k_0$, where the first inequality follows from the first relation in (39). Rearranging terms, we see that

$$0 \leq F_\delta(x^{k+2}, x^{k+1}) - F_\delta(\bar{x}, \bar{x}) \leq \frac{C}{1+C} [F_\delta(x^k, x^{k-1}) - F_\delta(\bar{x}, \bar{x})],$$

where $C = \frac{2}{c_1} \left(\frac{c_2}{c_3}\right)^2$. This shows that the sequences $\{F_\delta(x^{2k+1}, x^{2k})\}$ and $\{F_\delta(x^{2k}, x^{2k-1})\}$ are both Q -linearly convergent. This implies that the whole sequence $\{F_\delta(x^{k+1}, x^k)\}$ is R -linearly convergent in the sense that

$$\limsup_{k \rightarrow \infty} \sqrt[k]{|F_\delta(x^{k+1}, x^k) - F_\delta(\bar{x}, \bar{x})|} < 1.$$

Combining this observation with the first relation in (39), we conclude further that there exist $M > 0$ and $k_1 \in \mathbb{N}$ and $c \in (0, 1)$ so that for $k \geq k_1$, $\|x^{k+1} - x^k\| \leq M c^k$. Consequently, whenever $k \geq k_1$, we have

$$\|x^k - \bar{x}\| \leq \sum_{i=k}^{\infty} \|x^i - x^{i+1}\| \leq \frac{M}{1-c} c^k,$$

showing that $\{x^k\}$ is R -linearly convergent, that is, $\limsup_{k \rightarrow \infty} \sqrt[k]{\|x^k - \bar{x}\|} < 1$. This completes the proof. □

Remark 5.1 (New local linear convergence result for existing first-order methods) As applications of Proposition 5.1 and Theorem 5.1, we derive new local linear convergence results for two existing first-order methods applied to some concrete optimization problems:

- (i) Combining Proposition 5.1 with the discussions in Sects. 5.1 and 5.2, it is immediate that the proximal gradient algorithm exhibits local linear convergence when it is applied to $\min_{x \in \mathbb{R}^n} h(x) + P(x)$, where
- (a) $h(x) = l(Ax)$ and $P(x) = \min_{1 \leq i \leq r} P_i(x)$. Here, l is strongly convex on any compact convex set and is twice continuously differentiable with a Lipschitz continuous gradient, $A \in \mathbb{R}^{m \times n}$ and P_i are proper closed polyhedral functions for $i = 1, \dots, r$. This, in particular, covers the ℓ_1 regularized logistic regression problem; or
 - (b) $h(x) = \frac{1}{2} \|Ax - b\|^2$ and P is the SCAD or MCP regularizers; assuming that $\inf f > -\infty$ and the sequence generated is bounded for both cases.
- (ii) Similarly, one can also study local convergence rate when applying the iPiano to minimizing functions in the form of (26) for which the Luo-Tseng error bound holds using Theorems 4.1 and 5.1. For example, as a consequence of Proposition 4.1, local linear convergence is guaranteed when applying the iPiano to $\min_{x \in \mathbb{R}^n} h(x) + P(x)$ with coercive objectives, where $h(x) = l(Ax)$, l and A are given as in item (i) case (a), and P is a convex piecewise linear-quadratic regularizer.

6 Concluding Remarks

In this paper, we studied the KL exponent by developing calculus rules for the exponent and relating the exponent to the concept of Luo-Tseng error bound. Consequently, many convex or nonconvex optimization models that arise in practical applications can be shown to have a KL exponent of $\frac{1}{2}$. We also discuss how our results can be applied to establishing local linear convergence of some first-order methods.

One future research direction is to develop calculus rules for deducing the exponent of potential functions such as the augmented Lagrangian function $L_\beta(x, y, z) = f(x) + g(y) + z^T(x - y) + \frac{\beta}{2} \|x - y\|^2$ used in the convergence analysis of the alternating direction method of multiplier in a nonconvex setting, see, for example, [1, 19, 25]. Another direction is to derive the exponent for least squares models with some other popular nonconvex regularizers P such as the logistic penalty function, $P(x) = \lambda \sum_{i=1}^n \log(1 + \alpha|x_i|)$ [34], and the fraction penalty function, $P(x) = \lambda \sum_{i=1}^n \frac{\alpha|x_i|}{1 + \alpha|x_i|}$ [17].

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Appendix: An Auxiliary Lemma

In this appendix, we prove a version of [41, Lemma 6] for a class of proper closed functions taking the form $f := \ell + P$, where ℓ is a proper closed function with an open domain and is continuously differentiable on $\text{dom } \ell$, and P is a proper closed polyhedral function. Our proof follows exactly the same line of arguments as [41, Lemma 6] and is only included here for the sake of completeness.

In what follows, we let $K := \{(x, s) : s \geq P(x)\}$ and define

$$g(x, s) = \underbrace{\ell(x) + s}_{h(x,s)} + \delta_K(x, s).$$

Then we have the following result.

Lemma A.1 *There exists $C > 0$ so that for any $x \in \text{dom } f$, we have*

$$\|\text{Proj}_K[(x, P(x)) - \nabla h(x, P(x))] - (x, P(x))\| \leq C \|\text{prox}_P(x - \nabla \ell(x)) - x\|.$$

Proof For notational simplicity, let

$$(y, \mu) := \text{Proj}_K[(x, P(x)) - \nabla h(x, P(x))],$$

$$w := \text{prox}_P(x - \nabla \ell(x)).$$

Note that $\nabla h(x, P(x)) = (\nabla \ell(x), 1)$. Using these and the definitions of proximal mapping and projection, we have

$$(y, \mu) = \arg \min_{(u,s) \in K} \left\{ \langle \nabla \ell(x), u - x \rangle + (s - P(x)) + \frac{1}{2} \|u - x\|^2 + \frac{1}{2} (s - P(x))^2 \right\}, \tag{40}$$

$$w = \arg \min_u \left\{ \langle \nabla \ell(x), u - x \rangle + \frac{1}{2} \|u - x\|^2 + P(u) \right\}. \tag{41}$$

Now, using the strong convexity of the objective function in (40) and comparing its function values at the points (y, μ) and $(w, P(w))$, we have

$$\begin{aligned} & \langle \nabla \ell(x), y - x \rangle + (\mu - P(x)) + \frac{1}{2} \|y - x\|^2 + \frac{1}{2} (\mu - P(x))^2 \\ & \leq \langle \nabla \ell(x), w - x \rangle + (P(w) - P(x)) + \frac{1}{2} \|w - x\|^2 + \frac{1}{2} (P(w) - P(x))^2 \tag{42} \\ & \quad - \frac{1}{2} \|y - w\|^2 - \frac{1}{2} (\mu - P(w))^2. \end{aligned}$$

Similarly, using the strong convexity of the objective function in (41) and comparing its function values at the points w and y , we have

$$\begin{aligned} & \langle \nabla \ell(x), w - x \rangle + \frac{1}{2} \|w - x\|^2 + P(w) \\ & \leq \langle \nabla \ell(x), y - x \rangle + \frac{1}{2} \|y - x\|^2 + P(y) - \frac{1}{2} \|y - w\|^2 \tag{43} \\ & \leq \langle \nabla \ell(x), y - x \rangle + \frac{1}{2} \|y - x\|^2 + \mu - \frac{1}{2} \|y - w\|^2, \end{aligned}$$

where the last inequality follows from the fact that $(y, \mu) \in K$. Summing the inequalities (42) and (43) and rearranging terms, we see further that

$$\frac{1}{2}(\mu - P(x))^2 + \|y - w\|^2 \leq \frac{1}{2}(P(w) - P(x))^2 - \frac{1}{2}(\mu - P(w))^2. \quad (44)$$

Since P is a proper closed polyhedral function, it is piecewise linear on its domain (see, e.g., [8, Proposition 5.1.1]) and hence is Lipschitz continuous on its domain. Thus, it follows from this and (44) that there exists $M > 0$ so that

$$\begin{aligned} |\mu - P(x)| &\leq |P(w) - P(x)| \leq M\|w - x\| \quad \text{and} \\ \|y - w\| &\leq |P(w) - P(x)| \leq M\|w - x\|. \end{aligned} \quad (45)$$

Moreover, we can deduce further from the second relation in (45) that

$$\|y - x\| \leq \|y - w\| + \|w - x\| \leq (M + 1)\|w - x\|.$$

This together with the first relation in (45) and the definitions of (y, μ) and w completes the proof. \square

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