SMALL POINTS AND ADELIC METRICS

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Introduction

Consider following generalized Bogomolov conjecture: Let A be an abelian variety over $\bar{\mathbb{Q}}$, $h : A(\bar{\mathbb{Q}}) \to \mathbb{R}$ a Néron - Tate height function with respect to an ample and symmetric line bundle on A, Y a subvariety of A which is not a translate of an abelian subvariety by a torsion point; then there is a positive number ϵ such that the set

$$Y_\epsilon = \{x \in Y(\bar{\mathbb{Q}}) : h(x) \le \epsilon\}$$

is not Zariski dense in Y. Replacing A by an abelian subvariety, we may assume that $Y - Y = \{y_1 - y_2 : y_1, y_2 \in Y(\bar{\mathbb{Q}})\}$ generates A: A is the only abelian subvariety of A which contains $Y - Y$.

In this paper, we will prove that Y_ϵ is not Zariski dense if the map $\mathrm{NS}(A)_\mathbb{R} \to \mathrm{NS}(Y)_\mathbb{R}$ is not injective.

In spirit of Szpiro's paper [Sz], we will reduce the problem to the positivity of the height of Y with respect to certain metrized line bundle. To do this, we will first extend Gillet-Soulé's intersection theory of hermitian line bundles to certain limits of line bundles which are called integrable metrized line bundles, then for a dynamic system, construct certain special integrable metrized line bundles which are called admissible metrized line bundles, and finally prove the positivity of heights.

Integrable metrized line bundles. Consider a projective variety X over Spec $\mathbb{Q}$. For a line bundle $\mathcal{L}$ on X and an arithmetic model $(\widetilde{X}, \widetilde{\mathcal{L}})$ of $(X, \mathcal{L}^e)$ over Spec $\mathbb{Z}$, one can define an adelic metric $\|\cdot\|_{\tilde{\mathcal{L}}} = \big\{\|\cdot\|_p, p \in \mathcal{S}\big\}$ on $\mathcal{L}$, where e is a positive integer, $\mathcal{S}$ is the set of places of $\mathbb{Q}$, and $\|\cdot\|_p$ is a metric on $\mathcal{L} \otimes_\mathbb{Q} \overline{\mathbb{Q}}_p$ on $X(\overline{\mathbb{Q}}_p)$.

Let $\mathcal{L}_1, \cdots, \mathcal{L}_d$ $(d = \dim X + 1)$ be line bundles on X. For each positive integer n, let $(\widetilde{X}_n, \widetilde{\mathcal{L}}_{1,n}, \cdots, \widetilde{\mathcal{L}}_{d,n})$ be an arithmetic model of $(X, \mathcal{L}_1^{e_{1,n}}, \cdots \mathcal{L}_d^{e_{d,n}})$ on Spec $\mathbb{Z}$. Assume for each i that $(\mathcal{L}, \|\cdot\|_{\widetilde{\mathcal{L}}_{i,n}})$ converges to an adelic metrized line bundle $\overline{\mathcal{L}}_i$. One might ask whether the number

$$c_n = \frac{c_1(\widetilde{\mathcal{L}}_{1,n}) \cdots c_1(\widetilde{\mathcal{L}}_{d,n})}{e_{1,n} \quad \cdots \quad e_{d,n}}$$

in Gillet-Soulé's intersection theory converges or not.

We will show that c_n converges if all $\widetilde{\mathcal{L}}_{i,n}$ are relatively semipositive, and that $\lim_{n\to\infty} c_n$ depends only on $\overline{\mathcal{L}}_i$. Notice that some special case has been studied by Chinberg, Rumely, and Lau [CRL]. We say that an adelic line bundle $\overline{\mathcal{L}}$ is integrable if $\overline{\mathcal{L}} \cong \bar{\mathcal{L}}_1 \otimes \bar{\mathcal{L}}_2^{-1}$ with $\bar{\mathcal{L}}_i$ semipositive. It follows that Gillet-Soulé's theory can be extended to integrable metrized line bundles. Some theorems such as Hilbert-Samuel formula, Nakai-Moishezon theorem, and comparison inequality remain valid on integrable metrized line bundles.

Admissible metrized line bundles. Let $f : X \to X$ be a surjective endomorphism over Spec $\mathbb{Q}$, $\mathcal{L}$ a line bundle on X, and $\phi : \mathcal{L}^d \simeq f^*\mathcal{L}$ an isomorphism with $d > 1$. Using Tate's argument, we will construct a unique integrable metric $\|\cdot\|$ on $\mathcal{L}$ such that

$$\|\cdot\|^d = \phi^* f^* \|\cdot\|.$$

If $X = A$ is an abelian variety, and s is a section of $\mathcal{L} \otimes_\mathbb{Q} \overline{\mathbb{Q}}_p$, then $\log \|s\|_p$ is the Néron function for divisor div (s).

In case that $\mathcal{L}$ is ample, any effective cycle Y of X of pure dimension has an (absolute) height

$$h_{\mathcal{L}}(Y) = \frac{c_1(\overline{\mathcal{L}}\big|_Y)^{\dim Y+1}}{(\dim Y+1)\deg_{\mathcal{L}}(Y)}$$

which has property that

$$h_{\mathcal{L}}\big(f(Y)\big) = dh_{\mathcal{L}}(Y).$$

As Tate did, $h_{\mathcal{L}}$ can be defined without admissible metric. Some situations are studied by Philippon [P], Kramer [K], Call and Silverman [CS], and Gubler [G].

Assume $\mathcal{L}$ is ample as above. If Y is preperiodic: the orbit $\{Y, f(Y), f^2(Y), \cdots\}$ is finite, then $h_{\mathcal{L}}(Y) = 0$. We propose a generalized Bogomolov conjecture which claims that the converse is true: if $h(Y) = 0$ then Y is preperiodic. This is a theorem [Z2] for case of multiplicative group. A consequence is the generalized Lang's conjecture which claims that if Y is not preperiodic then the set of preperiodic points in Y is not Zariski dense. Lang's conjecture is proved by Laurent [L] and Sarnak [Sa] for multiplicative groups, and by Raynaud [R] for abelian varieties.

Positivity of heights of certain subvarieties. Let Y be a subvariety of an abelian variety A with a polarization $\mathcal{L}$. We prove the following special case of the generalized Bogomolov conjecture: if $Y - Y$ generates A, and the map $\mathrm{NS}(A)_{\mathbb{Q}} \to \mathrm{NS}(Y)_{\mathbb{Q}}$ is not injective, then $h_{\mathcal{L}}(Y) > 0$. The crucial facts used in the proof are comparison theorem of heights, Faltings index theorem, and nonvanishing of invariant $(1,1)$ forms on Y.

For a curve C of genus $g \geq 2$, let ω denote the admissible dualizing sheaf defined in [Z1]. The above positivity implies that $\omega^2 > 0$ if $\mathrm{End}\,\big(\mathrm{Jac}(C)\big)_{\mathbb{R}}$ is not isomorphic to $\mathbb{R}, \mathbb{C}$, and the quaternion divison algebra $\mathbb{D}$. This is the case when $\mathrm{Jac}(C)$ has a complex multiplication, or C has a finite morphism of deg > 1 to a nonrational curve. Notice that if C has good reduction everywhere, and $\mathrm{Jac}(C)$ has complex multiplication, the positivity of ω_{Ar}^2 is proved by Burnol [B] using Weierstrass points.

1. Integrable meterized line bundles

(1.1). For a line bundle $\mathcal{L}$ on a projective scheme X over an algebraically closed valuation field K, we define a K-metric $\|\cdot\|$ on $\mathcal{L}$ to be a collection of K-norms on each fiber $\mathcal{L}(x), x \in X(K)$.

For example when K is non-archimedean, if there is a projective scheme $\widetilde{X}$ on Spec R with generic fiber X, and a line bundle $\widetilde{\mathcal{L}}$ on $\widetilde{X}$ whose restriction on X is $\mathcal{L}^{\otimes n}$, where R is the valuation ring of K and $n > 0$ is an integer, we can define a metric $\|\cdot\|_{\tilde{\mathcal{L}}}$ as follows:

For an algebraic point $x \in X(K)$, denote by

$$\widetilde{x} : \text{Spec } R \longrightarrow \widetilde{X}$$

the section extending x: $x = \widetilde{x}\big|_{\text{Spec } K}$, then $\widetilde{x}^*\mathcal{L} \otimes_R K = x^*\mathcal{L}^n$. For any $\ell \in x^*(\mathcal{L})$, we define

$$\|\ell\|_{\tilde{\mathcal{L}}} = \inf_{a \in K} \left\{|a|^{\frac{1}{n}} : \quad \ell \in a\widetilde{x}^*(\mathcal{L})\right\}.$$

We say that $\|\cdot\|_{\tilde{\mathcal{L}}}$ is induced by the model $(\widetilde{X}, \widetilde{\mathcal{L}})$.

A metric $\|\cdot\|$ on $\mathcal{L}$ is called continuous and bounded if there is a model $(\widetilde{X}, \widetilde{\mathcal{L}})$ such that $\log \frac{\|\cdot\|}{\|\cdot\|_{\tilde{\mathcal{L}}}}$ is bounded and continuous on $X(K)$ with respect to the K-topology.

(1.2). Denote by $\mathcal{S} = \{\infty, 2, 3, \cdots\}$ the set of all places of $\mathbb{Q}$. For each $p \in \mathcal{S}$, denote by $|\cdot|_p$ the valuation on $\mathbb{Q}$ such that $|p|_p = p^{-1}$ if $p \neq \infty$, by $|\cdot|_\infty$ the ordinary absolute value if $p = \infty$, by $\mathbb{Q}_p$ the completion of $\mathbb{Q}$ under $|\cdot|_p$, and by $\bar{\mathbb{Q}}_p$ a fixed algebraic closure of $\mathbb{Q}_p$. For an irreducible projective variety X over $\mathbb{Q}$, we define an adelic metrized line bundle $\widetilde{\mathcal{L}}$ to be a line bundle $\mathcal{L}$ on X and a collection of metrics $\|\cdot\| = \{\|\cdot\|_p, p \in \mathcal{S}\}$ such that the following conditions are verified.

(a) Each $\|\cdot\|_p$ is bounded, continuous, and Gal $\left(\bar{\mathbb{Q}}_p/\mathbb{Q}_p\right)$ invariant.

(b) There is a Zariski open subset $U = \text{Spec } \mathbb{Z}\left[\frac{1}{n}\right]$ of Spec $\mathbb{Z}$, a projective variety $\widetilde{X}$ on U with generic fiber X, and a line bundle $\widetilde{\mathcal{L}}$ on $\widetilde{X}$ extending $\mathcal{L}$ on X, such that for each $p \in U$, the metric $\|\cdot\|_p$ is induced by the model

$$\left(\widetilde{X}_p, \widetilde{\mathcal{L}}_p\right) = \left(\widetilde{X} \times_U \text{Spec } \bar{\mathbb{Z}}_p, \widetilde{\mathcal{L}} \otimes_{\mathbb{Z}[\frac{1}{n}]} \bar{\mathbb{Z}}_p\right),$$

where $\bar{\mathbb{Z}}_p$ denotes the valuation ring of $\bar{\mathbb{Q}}_p$.

For example, if there is a projective variety $\widetilde{X}$ on Spec $\mathbb{Z}$ with generic fiber X, and a hermitian line bundle $\widetilde{\mathcal{L}}$ on $\widetilde{X}$ whose restriction on X is $\mathcal{L}^{\otimes n}$ $(n \neq 0)$, then $(\widetilde{X}, \widetilde{\mathcal{L}})$ induces a metric $\|\cdot\|_{\tilde{\mathcal{L}}} = \left\{\|\cdot\|_p, p \in S\right\}$, where for $p \neq \infty$, $\|\cdot\|_p$ is induced by models $\left(\widetilde{X}_p, \widetilde{\mathcal{L}}_p\right)$, and $\|\cdot\|_\infty$ is the hermitian metric on $\mathcal{L}_{\mathbb{C}}$. The condition (a) is obviously verified. Since $\mathcal{L}$ is defined over the generic fiber of $\widetilde{X}$, there is an open subset U' of Spec $\mathbb{Z}$ such that $\mathcal{L}$ has an extention $\widetilde{\mathcal{L}}_1$ on $\widetilde{X}_{U'}$. Since $\widetilde{\mathcal{L}}_1^n\big|_X = \widetilde{\mathcal{L}}\big|_X$, there is an open subset U of U' such that $\widetilde{\mathcal{L}}_1^n\big|_U \simeq \widetilde{\mathcal{L}}\big|_U$. It follows that for $p \in U$, $\|\cdot\|_p$ is induced by $\widetilde{\mathcal{L}}_1$. The condition (b) is therefore verified.

A sequnce $\{\|\cdot\|_n : n = 1, 2, \cdots\}$ of adelic metrics is convergent to an adelic metric $\|\cdot\|$ if there is an open subset U of Spec $\mathbb{Z}$ such that for each $p \in U$, $\|\cdot\|_{n,p} = \|\cdot\|_p$ for all n, and that $\log \frac{\|\cdot\|_{n,p}}{\|\cdot\|}$ converges to 0 uniformly on $X(K)$.

(1.3). For a hermitian line bundle $\widetilde{\mathcal{L}}$ on an arithmetic variety X with a smooth metric at ∞, this means that for any holomorphic map

$$f : \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\} \to X(\mathbb{C})$$

the pullback metric on $f^*\mathcal{L}$ is smooth, we say that $\widetilde{\mathcal{L}}$ is relatively semipositive if $\widetilde{\mathcal{L}}$ has nonnegative degree on any curve in special fibers, and the curvature of $f^*\mathcal{L}_{\mathbb{C}}$ is semipositive for any holomorphic map $f : \mathbb{D} \to X(\mathbb{C})$; we say that $\widetilde{\mathcal{L}}$ is relatively ample if the associated algebraic bundle is relatively ample, and there is an embedding $\widetilde{X}(\mathbb{C}) \to Y$ into a projective complex manifold Y such that the hermitian line bundle $\widetilde{\mathcal{L}}(\mathbb{C})$ can be extended to a hermitian line bundle on Y with positive curvature.

For a projective variety X over Spec $\mathbb{Q}$, and a line bundle $\mathcal{L}$ on X, we say that an adelic metric $\|\cdot\|$ on $\mathcal{L}$ is ample (resp. semipositive) if it is the limit of a sequence $\|\cdot\|_n$ of adelic metrics induced by models $\left(\widetilde{X}_n, \widetilde{\mathcal{L}}_n\right)$ as in (1.2) such that $\widetilde{\mathcal{L}}_n$ are relatively ample (resp. relatively semipositive.)

Let $\bar{\mathcal{L}}_1, \cdots, \bar{\mathcal{L}}_d$ be metrized line bundles on X with semipositive metrics, $d = \dim X + 1$. Assume that $\|\cdot\|_i$ are approximated by metrics induced by models $\left(\widetilde{X}_{i,n}, \widetilde{\mathcal{L}}_{i,n}\right)$, where $\widetilde{\mathcal{L}}_{i,n}$ are semipositive such that $\widetilde{\mathcal{L}}_{i,n}\big|_X = \mathcal{L}^{e_{i,n}}$, with $e_{i,n} > 0$. For any d-tuple of positive integers $(n_1, \cdots, n_d)$, denote by $\widetilde{X}_{n_1,\cdots,n_d}$ the Zariski closure of $\Delta(X)$ in $\widetilde{X}_{n_1} \times_{\mathbb{Z}} \widetilde{X}_{n_2} \times_{\mathbb{Z}} \cdots \times_{\mathbb{Z}} \widetilde{X}_{n_d}$, where Δ is the diagonal map of X into the generic fiber $X \times_{\mathbb{Q}} X \times \cdots \times_{\mathbb{Q}} X$ of $\widetilde{X}_{n_1} \times_{\mathbb{Z}} \widetilde{X}_{n_2} \times_{\mathbb{Z}} \cdots \times_{\mathbb{Z}} \widetilde{X}_{n_d}$. We still denote the pullback of $\widetilde{\mathcal{L}}_{i,n_i}$ on $\widetilde{X}_{n_1,\cdots,n_d}$ by $\widetilde{\mathcal{L}}_{i,n_i}$.

Theorem (1.4). *(a) The intersection number*

$$c_{n_1,\cdots,n_d} = c_1(\widetilde{\mathcal{L}}_{n_1}) \cdots c_1(\widetilde{\mathcal{L}}_{n_d}) / e_{1,n_1} \cdots e_{d,n_d}$$

converges as $n_i \to \infty$. The limit does not depend on the choice of $(\widetilde{X}_{i,n}, \widetilde{\mathcal{L}}_{i,n})$.

(b) Denoted by $c_1(\bar{\mathcal{L}}_1) \cdots c_1(\bar{\mathcal{L}}_d)$ the limit, then $c_1(\bar{\mathcal{L}}_1) \cdots (\bar{\mathcal{L}}_d)$ is multi-linear in $\bar{\mathcal{L}}_1, \cdots, \bar{\mathcal{L}}_d$.

Proof. (a) Fix two d-tuples $(n_1, \cdots, n_d)$ and $(n'_1, \cdots, n'_d)$ of positive integers. Denote by $\widetilde{X}$ the Zariski closure of $\Delta(X)$ in $\widetilde{X}_{n_1,\cdots,n_d} \times \widetilde{X}_{n'_1,\cdots,n'_d}$. As before we use same notations for pullbacks of $\widetilde{\mathcal{L}}_{i,n_i}$ as themselves. Write $\widetilde{\mathcal{L}}_i = \widetilde{\mathcal{L}}_{i,n_i}^{e_{i,n'_i}}$, $\widetilde{\mathcal{L}}'_i = \widetilde{\mathcal{L}}_{i,n_i}^{e_{i,n_i}}$, and $e_i = e_{i,n_i} \cdot e_{i,n'_i}$. Then both $\widetilde{\mathcal{L}}_i$ and $\widetilde{\mathcal{L}}'_i$ have same restriction $\mathcal{L}^{e_i}$ on X. We need to show that

$$\frac{1}{e_1 \cdots e_d}\left(c_1(\widetilde{\mathcal{L}}_1) \cdots c_1(\widetilde{\mathcal{L}}_d) - c_1(\widetilde{\mathcal{L}}'_1) \cdots c_1(\widetilde{\mathcal{L}}'_d)\right)$$

approaches to 0 as $(n_1, \cdots, n_2, n'_1, \cdots, n'_d)$ approaches to ∞.

Fix a positive number ϵ and an open subset U of Spec $\mathbb{Z}$ such that for any $p \in U$ and any k, $\|\cdot\|_{p,\widetilde{\mathcal{L}}_k} = \|\cdot\|_{p,\widetilde{\mathcal{L}}'_k}$. Then for any sufficiently large n_k, n'_k and any p,

$$\left|\log \frac{\|\cdot\|_{p,\widetilde{\mathcal{L}}'_k}}{\|\cdot\|_{p,\widetilde{\mathcal{L}}_k}}\right| \le \epsilon \log p,$$

where $\log \infty$ is defined to be 1. Denote by s_k the rational section of $\widetilde{\mathcal{L}}_k \otimes \widetilde{\mathcal{L}'}_k^{(-1)}$ which gives 1 on X.

If $p \neq \infty$, then one has

$$p^{-\epsilon e_1 \cdots e_d} \leq \|s_k\|_p(x) \leq p^{\epsilon e_1 \cdots e_d}$$

for any x. Assume $[\text{div } (s_k)]_p = \sum n_{i,p} V_{i,p}$, where $[\text{div } (s_k)]_p$ is the cycle associated to div (s_k) supported in the special fiber $\widetilde{X}_p$ of $\widetilde{X}$ over p, and $V_{i,p}$'s are irreducible components of $\widetilde{X}_p$. It follows that $|n_p| \leq \epsilon e_1 \cdots e_d$, or in other words, that divisors

$$D_{1p} = [\text{div } (s_k)]_p + [\epsilon e_i \cdots e_n][\widetilde{X}_p]$$

and

$$D_{2p} = -[\text{div } (s_k)]_p + [\epsilon e_i \cdots e_d][\widetilde{X}_p]$$

are both effective, where $[\epsilon e_1 \cdots e_d]$ is the integral part of $\epsilon e_i \cdots e_d$. Therefore for $i = 1, 2$,

$$c_1(\widetilde{\mathcal{L}}'|_{D_{i,p}}) \cdots c_1(\widetilde{\mathcal{L}}'_{k-1}|_{D_{i,p}}) c_1(\widetilde{\mathcal{L}}_{k+1}|_{D_{i,p}}) \cdots c_1(\widetilde{\mathcal{L}}_d|_{D_{i,p}}) > 0,$$

or in other words, the contribution at p of

$$I_k = c_1(\widetilde{\mathcal{L}}') \cdots c_1(\widetilde{\mathcal{L}}'_{k-1}) c_1(\widetilde{\mathcal{L}}_{k+1}) \cdots c_1(\widetilde{\mathcal{L}}_d) \text{div } (s)$$

has absolute value bounded by

$$\epsilon e_1 \cdots e_d (\log p) c_1(\mathcal{L}_1) \cdots c_1(\mathcal{L}_{k-1}) c_1(\mathcal{L}_{k+1}) \cdots c_1(\mathcal{L}_d).$$

If $p = \infty$, the contribution at p of I_k is given by

$$\int \log \|s_k\|_\infty c'_1(\widetilde{\mathcal{L}}'_1) \cdots c'_1(\widetilde{\mathcal{L}}'_{k-1}) c'_1(\widetilde{\mathcal{L}}_{k+1}) \cdots c'_1(\widetilde{\mathcal{L}}_d)$$

where $c'_1(\widetilde{\mathcal{L}}_i)$ and $c'_1(\widetilde{\mathcal{L}}'_i)$ denote the curvatures of $\widetilde{\mathcal{L}}_i$ and $\widetilde{\mathcal{L}}'_i$. Since $|\log \|s_k\|_\infty| < \epsilon$ and $c'_1(\widetilde{\mathcal{L}}_i), c'_i(\widetilde{\mathcal{L}}'_i)$ are nonnegative, the above integral has absolute value bounded by

$$\epsilon e_1 \cdots e_d c_1(\mathcal{L}_1) \cdots c_1(\mathcal{L}_{k-1}) c_1(\mathcal{L}_{k+1}) \cdots c_1(\mathcal{L}_d)$$

It follows that for any $1 \leq k \leq d$,

$$|I_k| \leq \epsilon e_1 \cdots e_d c_1(\mathcal{L}_1) \cdots c_1(\mathcal{L}_{k-1}) c_1(\mathcal{L}_{k+1}) \cdots c_1(\mathcal{L}_d) \sum_{p \notin U} \log p.$$

Finally, for $(n_i, \cdots, n_d, n_i', \cdots, n_d')$ sufficiently large,

$$
\begin{aligned}
&\frac{1}{e_1\cdots e_d}\Big|c_1(\widetilde{\mathcal{L}}_1)\cdots c_1(\widetilde{\mathcal{L}}_d) - c_1(\widetilde{\mathcal{L}}_1')\cdots c_1(\widetilde{\mathcal{L}}_d')\Big| \\
\leq&\frac{1}{e_1\cdots e_d}\sum_{k=1}^{d}\Big|c_1(\widetilde{\mathcal{L}}_1')\cdots c_1(\widetilde{\mathcal{L}}_{k-1})c_1(\widetilde{\mathcal{L}}_k\otimes\widetilde{\mathcal{L}}_k'^{-1})c_1(\mathcal{L}_{k+1})\cdots c_1(\mathcal{L}_d)\Big| \\
\leq&\epsilon\cdot\sum_{k=1}^{d}c_1(L_1)\cdots c_1(\mathcal{L}_d)\sum_{p\notin U}\log p.
\end{aligned}
$$

This prove the first statement of (a).

If $\{(\widetilde{X}_{i,n}', \widetilde{\mathcal{L}}_{i,n}')\}$ is another sequence of models which induces metrics $\|\cdot\|_{\widetilde{\mathcal{L}}_{i,n'}}$ convergent to $\|\cdot\|$, then the alternating sequence

$$
\{(\widetilde{X}_{i,n}'', \widetilde{\mathcal{L}}_{i,n}'')\} = \{(\widetilde{X}_{i,1}, \widetilde{\mathcal{L}}_{i,1}), (\widetilde{X}_{i,1}', \widetilde{\mathcal{L}}_{i,1}'), (\widetilde{X}_{i,2}, \widetilde{\mathcal{L}}_{i,2}), (\widetilde{X}_{i,2}', \widetilde{\mathcal{L}}_{i,2}'), \cdots\}
$$

also induces metrics on $\mathcal{L}$ convergent to $\|\cdot\|$. By the first statement of (a), the intersection numbers induced by $\{(\widetilde{X}_{i,n}'', \widetilde{\mathcal{L}}_{i,n}'')\}$ are convergent. So limits defined by $\{(\widetilde{x}_{i,n}, \widetilde{\mathcal{L}}_{i,n})\}$ and $\{\widetilde{x}_{i,n}', \widetilde{\mathcal{L}}_{i,n}')\}$ are the same. This prove the second statement of (a).

The additivity of $c_1(\bar{\mathcal{L}}_1)\cdots c_1(\bar{\mathcal{L}}_d)$ in (b) is obvious from definition.

This completes the proof of the theorem.

(1.5). A metrized line bundle $\bar{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ is called integrable if there are two semipositive metrized line bundles $\bar{\mathcal{L}}_1, \bar{\mathcal{L}}_2$ such that $\bar{\mathcal{L}}$ is isometric to $\bar{\mathcal{L}}_1 \otimes \bar{\mathcal{L}}_2^{-1}$.

By theorem (1.4), for any integrable metrized line bundles $\bar{\mathcal{L}}_1, \cdots, \bar{\mathcal{L}}_d$, there is a uniquely defined intersection number $c_1(\bar{\mathcal{L}}_1)\cdots c_1(\bar{\mathcal{L}}_d)$ such that the following conditions are verified:

(a) $c_1(\bar{\mathcal{L}}_1)\cdots c_1(\bar{\mathcal{L}}_d)$ is multilinear

(b) $c_1(\bar{\mathcal{L}}_1)\cdots c_1(\bar{\mathcal{L}}_d)$ is the limit defined in (b) of (1.4) if the metrics on $\bar{\mathcal{L}}_1, \cdots, \bar{\mathcal{L}}_d$ are semipositive.

(1.6). Now we want to generalize results in [Z2] to integrable metrized line bundles. First of all, we need to define Hilbert function of a line bundle.

By a norm $\|\cdot\|$ on a vector space V of finite dimension over $\mathbb{Q}$, we mean a collection $\{\|\cdot\|_p, p\in\mathcal{S}\}$ of norms such that the following conditions are verified.

(a) For each p, $\|\cdot\|_p$ is a $\mathbb{Q}_p$-norm on $V_p = V\otimes_{\mathbb{Q}}\mathbb{Q}_p$, which is nonarchimedean if $p\neq\infty$, i.e. $\|x+y\|_p \leq \max(\|x\|_p, \|y\|_p)$ if $p\neq\infty$,

(b) There is a non zero integer n, a free module $\widetilde{V}$ over $\mathbb{Z}[\frac{1}{n}]$, and an isomorphism $V\simeq\widetilde{V}\otimes_{\mathbb{Z}[\frac{1}{n}]}\mathbb{Q}$ such that $\|\cdot\|_p$ is induced by $\widetilde{V}$ for all $p \not| n$.

Denote by $\mathbb{A}$ the ring of adeles of $\mathbb{Q}$, and by $V_{\mathbb{A}}$ the module $V\otimes_{\mathbb{Q}}\mathbb{A}$. There is unique invariant measure μ on $V_{\mathbb{A}}$ such that $\mu\big(\prod_p B_p\big) = 1$ where for each p, B_p is the unit ball in $V_p: B_p = \big\{x\in V_p, \|x\|_p\leq 1\big\}$. We define the Euler characteristic of $(V, \|\cdot\|)$ as follows:

$$
\chi_{\|\cdot\|}(V) = -\log \mathrm{volume}(V_{\mathbb{A}}/V).
$$

For a projective variety X over $\mathbb{Q}$, an ample line bundle $\mathcal{L}$ on X with a semipositive metric $\|\cdot\|$, and a place p of $\mathbb{Q}$, let $\|\cdot\|_p$ denote a norm on $\Gamma(\mathcal{L})\otimes_{\mathbb{Q}}\mathbb{Q}_p=\Gamma(\mathcal{L}_p)$ defined as follows: for each $\ell\in\Gamma(X_p,\mathcal{L}_p)$,

$$\|\ell\|_p=\sup_{x\in X(\overline{\mathbb{Q}}_p)}\|\ell\|(x).$$

In this way, $\|\cdot\|=\big\{\|\cdot\|_p, p\in\mathcal{S}\big\}$ defines an adelic norm on $\Gamma(\mathcal{L})$. Write $\chi\big(\Gamma(\bar{\mathcal{L}})\big)$ simply for $\chi_{\|\cdot\|}\big(\Gamma(\bar{\mathcal{L}})\big)$.

Theorem (1.7). *As n approaches ∞,*

$$\chi\big(\Gamma(\mathcal{L}^{\otimes n})\big)=\frac{n^d}{d!}c_1(\bar{\mathcal{L}})^d+o(n^d).$$

Proof. Assume that there is a sequence of models $(\widetilde{X}_m,\widetilde{\mathcal{L}}_m)$ of $(X,\mathcal{L})$ such that $\widetilde{\mathcal{L}}_m$'s are semipositive on $\widetilde{X}_m$ and that the induced adelic metrized line bundles $\bar{\mathcal{L}}_m$ converge to $\bar{\mathcal{L}}$, then the theorem is true for $\bar{\mathcal{L}}_m$ by (1.4) of [Z2]. Write

$$\chi_{m,n}=\frac{\chi\big(\Gamma(\bar{\mathcal{L}}_m^{\otimes n})\big)}{n\dim\Gamma(\mathcal{L}^{\otimes n})},$$

$$\chi_n=\frac{\chi\big(\Gamma(\bar{\mathcal{L}}^{\otimes n})\big)}{n\dim\Gamma(\mathcal{L}^{\otimes n})},$$

then $\chi_{n,m}\longrightarrow\chi_n$ uniformly in n as $m\to\infty$. Now the theorem for $\bar{\mathcal{L}}_m$ implies that $\lim_{n\to\infty}\chi_{m,n}=\frac{c_1(\bar{\mathcal{L}}_m)^d}{dc_1(\mathcal{L})^{d-1}}$. It follows that

$$\lim_{n\to\infty}\chi_n=\lim_{m\to\infty}\lim_{n\to\infty}\chi_{m,n}=\frac{c_1(\bar{\mathcal{L}})^d}{dc_1(\mathcal{L})^{d-1}}.$$

The theorem follows immediately.

Theorem (1.8). *Let $\bar{\mathcal{L}}$ be an ample metrized line bundle with an ample metric. Assume for each irreducible subvariety Y of X that $c_1(\bar{\mathcal{L}}|_Y)^{\dim Y+1}>0$. Then for $n\gg 0$, the $\mathbb{Q}$-vector space $\Gamma(\mathcal{L}^{\otimes n})$ has a basis $\{\ell_1,\cdots,\ell_N\}$ consisting of strictly effective elements: $\|\ell_i\|_p\le 1$ for $p\ne\infty$ and $\|\ell_i\|_\infty<1$.*

Proof. By (1.7) and the Minkowski theorem, for each Y of X there is a $n>0$, such that $\Gamma(\mathcal{L}\big|_Y^{\otimes n})$ has a section ℓ such that $\|\ell\|_p\le 1$ for all $p\ne\infty$ and $\|\ell\|_\infty<1$. The theorem follows from (4.2) of [Z2].

(1.9). For a projective variety X over $\operatorname{Spec}\mathbb{Q}$ of dimension $d-1$, and an integrable metrized ample line bundle $\bar{\mathcal{L}}$ on X, we define the height of X with respect to $\bar{\mathcal{L}}$ as follows:

$$h_{\bar{\mathcal{L}}}=\frac{c_1(\bar{\mathcal{L}})^d}{dc_1(\mathcal{L}_{\mathbb{Q}})^{d-1}}.$$

For $i=1,2,\cdots d$, define numbers

$$e_i(\bar{\mathcal{L}})=\sup_{\operatorname{cod}Y=i}\inf_{x\in X-Y}h_{\bar{\mathcal{L}}}(x)$$

where Y runs through the set of reduced subvarieties of X.

Theorem (1.10). *If $\bar{\mathcal{L}}$ is an ample metrized line bundle then*

$$e_1(\bar{\mathcal{L}}) \geq h_{\bar{\mathcal{L}}}(X) \geq \frac{e_1(\bar{\mathcal{L}}) + \cdots + e_d(\bar{\mathcal{L}})}{d}$$

Proof. Assume $\bar{\mathcal{L}}$ is approximated by metrized line bundles $\bar{\mathcal{L}}_n$ which are induced by models $(\widetilde{X}_n, \widetilde{\mathcal{L}}_n)$. Then the theorem is true for $\bar{\mathcal{L}}_n$ by (5.2) of [Z2]. Since $c_1(\bar{\mathcal{L}}_n)^d \to c_1(\bar{\mathcal{L}})^d$ and $e_i(\bar{\mathcal{L}}_n) \to e_i(\bar{\mathcal{L}})$, the theorem (1.10) is true for $\bar{\mathcal{L}}$.

Theorem (1.11). *If $\bar{\mathcal{L}}_1 \cdots \bar{\mathcal{L}}_d$ are ample metrized line bundles, then*

$$c_1(\bar{\mathcal{L}}_1) \cdots c_1(\bar{\mathcal{L}}_d) \geq \sum_{k=1}^{d} e_d(\bar{\mathcal{L}}_k) c_1(\mathcal{L}_1) \cdots c_1(\mathcal{L}_{k-1}) c_1(\mathcal{L}_{k+1}) \cdots c_1(\mathcal{L}_d)$$

Proof. By a limit argument, we may assume that $\bar{\mathcal{L}}_1, \cdots, \bar{\mathcal{L}}_d$ are exactly metrized line bundles associated to relatively ample hermitian line bundles $\widetilde{\mathcal{L}}_1, \cdots, \widetilde{\mathcal{L}}_d$ on a model $\widetilde{X}$ of X. By scaling metric at ∞, we may assume that $e_d(\widetilde{\mathcal{L}}_i) = 0$. We want to prove that $c_1(\widetilde{\mathcal{L}}_1) \cdots c_1(\widetilde{\mathcal{L}}_d) \geq 0$ by induction on d. It is obviously true for $d = 1$. Fix a $\epsilon > 0$ and denote by $\widetilde{\mathcal{L}}_d(\epsilon)$ the metrized line bundle which has $e^{-\epsilon} \cdot \|\cdot\|_{\widetilde{\mathcal{L}}_d}$ as metric at ∞. Now $\widetilde{\mathcal{L}}_d(\epsilon)$ has height $\geq \epsilon$ on $X(\overline{\mathbb{Q}})$. By (1.8), $\widetilde{\mathcal{L}}_d(\epsilon)$ is ample. In particular, some power $\widetilde{\mathcal{L}}_d(\epsilon)^m$ has an effective section ℓ. Write $Y = \operatorname{div}(\ell)$, then

$$\begin{aligned}
&c_1(\widetilde{\mathcal{L}}_1) \cdots c_1(\widetilde{\mathcal{L}}_d) \\
=&c_1(\widetilde{\mathcal{L}}_1) \cdots c_1(\widetilde{\mathcal{L}}_d(\epsilon)) - \epsilon c_1(\bar{\mathcal{L}}_1) \cdots c_1(\bar{\mathcal{L}}_{d-1}) \\
=&\frac{1}{m} c_1(\widetilde{\mathcal{L}}_1) \cdots c_1(\widetilde{\mathcal{L}}_{d-1})(Y, -\log \|\ell\|_\infty) - \epsilon c_1(\bar{\mathcal{L}}_1) \cdots c_1(\bar{\mathcal{L}}_{d-1}) \\
=&\frac{1}{m} c_1(\widetilde{\mathcal{L}}_1|_Y) \cdots c_1(\widetilde{\mathcal{L}}_{d-1}|_Y) + \frac{1}{m} \int_{X(\mathbb{C})} -\log \|\ell\|_\infty c_1'(\widetilde{\mathcal{L}}_1) \cdots c_1'(\widetilde{\mathcal{L}}_{d-1}) - \epsilon c_1(\bar{\mathcal{L}}_1) \cdots c_1(\bar{\mathcal{L}}_{d-1}).
\end{aligned}$$

The first two terms in the last line are positive, since the first term is positive by induction, and the second term is positive by fact that $\|\ell\|_\infty < 1$ and $c_1'(\widetilde{\mathcal{L}}_i) \geq 0$. Letting $\epsilon \to 0$, the theorem follows.

2. Admissible metrized line bundles

(2.1). Let K be an algebraically closed valuation field, X a projective variety over Spec K, $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ a line bundle on X with a continuous and bounded metric, $f : X \to X$ a surjective morphism, and $\phi : \mathcal{L}^d \simeq f^*\mathcal{L}$ an isomorphism where $d > 1$ is an integer. We define $\|\cdot\|_n$ on $\mathcal{L}$ inductively as follows:

$$\|\cdot\|_1 = \|\cdot\|, \quad \|\cdot\|_n = \phi^* f^* \|\cdot\|_{n-1}^{\frac{1}{d}}.$$

Theorem (2.2). *(a) The metrics $\|\cdot\|_n$ on $\mathcal{L}$ converge uniformly to a metric $\|\cdot\|_0$ on $\mathcal{L}$, this means that the function $\log \frac{\|\cdot\|_n}{\|\cdot\|_1}$ converges uniformly on $X(K)$ to $\log \frac{\|\cdot\|_0}{\|\cdot\|_1}$.*

(b) $\|\cdot\|_0$ is the unique (continuous and bounded) metric on $\mathcal{L}$ satisfying the equation

$$\|\cdot\|_0 = \big(\phi^* f^* \|\cdot\|_0\big)^{\frac{1}{d}}.$$

(c) If ϕ changes to $\lambda\phi$ with $\lambda \in K^*$, then $\|\cdot\|_0$ changes to $|\lambda|^{\frac{1}{d-1}}\|\cdot\|_0$.

Proof. (a) Denote by h the continuous function $\frac{1}{d}\phi^* f^* \log \frac{\|\cdot\|_2}{\|\cdot\|_1}$ on X. Then

$$\begin{aligned}\log \|\cdot\|_n &= \big(\frac{1}{d}\phi^* f^*\big)^{n-2} \log \|\cdot\|_2 \\ &= \big(\frac{1}{d}\phi^* f^*\big)^{n-2} (h + \log \|\cdot\|_1) \\ &= \big(\frac{1}{d}\phi^* f^*\big)^{n-2} h + \log \|\cdot\|_{n-1}.\end{aligned}$$

Using induction on n, one has

$$\log \|\cdot\|_n = \log \|\cdot\|_1 + \sum_{k=0}^{n-2} \big(\frac{1}{d}\phi^* f^*\big)^k \cdot h.$$

Since $\big\|\big(\frac{1}{d}\phi^* f^*\big)^k \cdot h\big\|_{\sup} \le \frac{1}{d^k}\|h\|_{\sup}$, it follows that $\sum_{k=1}^{\infty} \big(\frac{1}{d}\phi^* f^*\big)^k \cdot h$ is absolutely and uniformly convergent to a bounded and continuous function h_0. Let $\|\cdot\|_0 = \|\cdot\|_1 e^{h_0}$, then $\|\cdot\|_n$ converges uniformly to $\|\cdot\|_0$.

(b) It is easy to see that $\|\cdot\|_0$ is continuous, bounded, and satisfies the equation

$$\|\cdot\|_0 = \big(\phi^* f^* \|\cdot\|_0\big)^{\frac{1}{d}}.$$

If $\|\cdot\|_0'$ be another continuous, bounded metric on $\mathcal{L}$ which satisfies the same equation, writing $g = \log \frac{\|\cdot\|_0}{\|\cdot\|_0'}$, then we have $g = \frac{\phi^* f^*}{d} g$, so $\|g\|_{\sup} = \|g\|_{\sup}/d$, or $g = 0$. This shows that $\|\cdot\|_0 = \|\cdot\|_0'$.

(c) If $\alpha\|\cdot\|_0$ is the metric corresponding to $\lambda\phi$ with α a function on $X(K)$, then for any $\ell \in \mathcal{L}(x)$, $x \in X$,

$$\alpha\|\ell\|_0(x) = \Big(\alpha\|\lambda\phi(\ell)\|_0\big(f(x)\big)\Big)^{\frac{1}{d}},$$

so $\alpha = (\alpha|\lambda|)^{\frac{1}{d}}$ or $\alpha = |\lambda|^{\frac{1}{d-1}}$.

The proof of the theorem is complete.

(2.3). Now let everything be defined over $\mathbb{Q}$: X is a projective variety over Spec $\mathbb{Q}$, $\mathcal{L}$ an ample line bundle on X, $f : X \to X$ a surjective morphism over $\mathbb{Q}$, and $\phi : \mathcal{L}^d \simeq f^*\mathcal{L}$ an isomorphism of line bundles. This implies that f is finite of degree $d^{\dim X}$. We fix a model $(\widetilde{X}, \widetilde{\mathcal{L}})$ of $(X, \mathcal{L}^e)$ on Spec $\mathbb{Z}$ with $e > 0$, such that $\widetilde{\mathcal{L}}$ is relatively ample. This induces an adelic metric $\|\cdot\|$ on $\mathcal{L}$.

There is an open subset U of Spec $\mathbb{Z}$ such that f and ϕ extend to an U-morphism $f_U : X_U \longrightarrow X_U$ and an isomorphism $\phi_U : \widetilde{\mathcal{L}}_U^{\otimes d} \longrightarrow f_U^*\widetilde{\mathcal{L}}_U$.

It follows for each $p \in U$ that

$$\|\cdot\|_p = (\phi^* f^* \|\cdot\|_p)^{\frac{1}{d}}.$$

We define the morphism $\widetilde{f}_n : \widetilde{X}_n \longrightarrow \widetilde{X}$ as the normalization of the composition of morphism

$$X_U \xrightarrow{f_U^n} X_U \hookrightarrow \widetilde{X}.$$

Denote by $\bar{\mathcal{L}}_n$ the metrized line bundles $(\mathcal{L}, \|\cdot\|_n)$ induced by model $\big(\widetilde{X}_n, \widetilde{f}_n^*\widetilde{\mathcal{L}}\big)$. Then for any $p \in U$ and any n, one has $\|\cdot\|_{n,p} = \|\cdot\|_p$. In general for any $p \in \mathcal{S}$, $\|\cdot\|_{n,p}$ is defined as in (2.1) from $\|\cdot\|_p$. By theorem (2.2.), $\|\cdot\|_{n,p}$ converges uniformly to a metric $\|\cdot\|_{0,p}$. So the adelic metric $\|\cdot\|_n$ of $\bar{\mathcal{L}}_n$ converges to an adelic metric $\|\cdot\|_0$ on $\mathcal{L}$. By (2.2), $\|\cdot\|_0$ doesn't depend on the choice of $(\widetilde{X}, \widetilde{\mathcal{L}})$. If ϕ changes to $\lambda\phi$ for $\lambda \in \Gamma(X, \mathcal{O}_X^*)$, then $\|\cdot\|_0$ changes to $\|\cdot\|_{0\lambda} = \{\|\cdot\|_p |\lambda|_p^{\frac{1}{d-1}}\}$. Therefore, if we write $\bar{\mathcal{L}}_0 = (\mathcal{L}, \|\cdot\|_0)$ then $\bar{\mathcal{L}}_0^{d-1}$ does not depend on the choice of ϕ. Since $\bar{\mathcal{L}}_n$ are all ample, $\bar{\mathcal{L}}_0$ is an ample metrized line bundle on X.

For any effective cycle Y of X of pure dimension, write $h_{f,\mathcal{L}}(Y)$ for $h_{\bar{\mathcal{L}}_0}(Y)$.

Theorem (2.4). *(a) Denote by $f(Y)$ the push-forward of Y under f, then $h_{f,\mathcal{L}}(fY) = dh_{f,\mathcal{L}}(Y)$.*

(b) $h_{f,\mathcal{L}}(Y) \geq 0$.

(c) If the orbit $\{Y, f(Y), \cdots, f^n(Y), \cdots\}$ is finite then $h_{f,\mathcal{L}}(Y) = 0$.

(d) If $y \in X(\overline{\mathbb{Q}})$ is a point and $h_{f,\mathcal{L}}(y) = 0$ then the orbit $\{y, f(y), \cdots, f^n(y), \cdots\}$ is finite.

Proof. (a)

$$\begin{aligned} h_{f,\mathcal{L}}(f(Y)) &= c_1\big(\bar{\mathcal{L}}_0\big|_{f(Y)}\big)^{\dim Y+1} \Big/ (\dim Y+1) c_1\big(\mathcal{L}\big|_{f(Y)}\big)^{\dim Y} \\ &= c_1\big(f^*\bar{\mathcal{L}}_0\big|_Y\big)^{\dim Y+1} \Big/ (\dim Y+1) c_1(f^*\mathcal{L})^{\dim Y} \\ &= c_1\big(\bar{\mathcal{L}}_0^d\big|_Y\big)^{\dim Y+1} \Big/ (\dim Y+1) c_1\big(\mathcal{L}^d\big|_Y\big)^{\dim Y} \\ &= dc_1\big(\bar{\mathcal{L}}_0\big|_Y\big)^{\dim Y+1} \Big/ (\dim Y+1) c_1\big(\mathcal{L}\big|_Y\big)^{\dim Y} \\ &= dh_{f,\mathcal{L}}(f(Y)). \end{aligned}$$

(b) Applying (1.10) to Y, we have that

$$h_{f,l}(Y) \geq e_d(\mathcal{L}\big|_Y) \cdot \big(\dim Y + 1\big) \geq e_d\big(\mathcal{L}\big)\big(\dim Y + 1\big).$$

But

$$e_d(\mathcal{L}) = \inf_x h_{\mathcal{L}}(x) = \inf_x h_{\mathcal{L}}\big(f(x)\big) = d \inf_x h_{\mathcal{L}}(x) = d e_d(\mathcal{L}).$$

Therefore $e_d(\mathcal{L}) = 0$

(c) The finiteness of the orbit $\{Y, f(Y), \cdots, f^n(Y), \cdots\}$ implies the finiteness of the orbit

$$\{h_{f,\mathcal{L}}(Y), d h_{f,\mathcal{L}}(Y), \cdots, d^n h_{f,\mathcal{L}}(Y) \cdots\}.$$

So we must have $h_{f,\mathcal{L}}(Y) = 0$.

(d) If $h_{f,\mathcal{L}}(y) = 0$ then the orbit $\{y, f(y), \cdots\}$ has bounded degree $[\mathbb{Q}(y) : \mathbb{Q}]$ and bounded height $(= 0)$, so must be a finite set.

The proof of the theorem is complete.

Conjecture (2.5). *If Y is an effective cycle of X of positive dimension and $h_{f,\mathcal{L}}(Y) = 0$ then the orbit of Y under f is finite.*

This conjecture is a converse of (c) in (2.4). A subvariety Z of X is called a preperiodic subvariety if the orbit of Z under f is finite. A preperiodic subvariety Z contained in Y is called maximal preperiodic if no other preperiodic subvariety of Y contains Z.

If $h_{f,\mathcal{L}}(Y) = 0$, by conjecture (2.5), Y is a preperiodic variety, of course a maximal preperiodic subvariety of Y. If $h_{f,\mathcal{L}}(Y) \neq 0$, by theorem (1.10), there is a Zariski open set U of Y such that $h_{f,\mathcal{L}}$ on $U(\overline{\mathbb{Q}})$ has a positive lower bound, and any preperiodic subvariety Z of Y will be contained in $X - U$. This shows that (2.5) implies the following conjecture:

Conjecture (2.6). *Any subvariety Y of X contains at most finitely many maximal preperiodic subvarieties.*

(2.7). Let $f_0, \cdots, f_n$ be $n+1$-homogeneous polynomial of degree of $d > 1$ in $n+1$ variables $z_0, \cdots, z_n$ such that the only common zero of $f_0, \cdots, f_n$ is 0. Then

$$f : (z_0, \cdots, z_n) \longrightarrow \big(f_0(z_0, \cdots, z_n), \cdots, f_n(z_0, \cdots, z_n)\big)$$

defines a morphism $\mathbb{P}^n \longrightarrow \mathbb{P}^n$. One has a unique homomorphism $\phi : \mathcal{O}(d) \simeq f^*\mathcal{O}(1)$ such that $\phi(f_i) = f^*(z_i)$, where we consider z_i as sections of $\mathcal{O}(1)$.

When $f_i = z_i^d$, the preperiodic subvarieties of $\mathbb{P}^n$ are Zariski closure of translates of subgroups by torsion points of $\mathbb{G}_m^n = \big\{(z_0 \cdots z_n) = z_0 \cdots z_n \neq 0\big\}$. In this case, (2.6) is a theorem of Laurent [L], Sarnak [Sa], while (2.5) is a theorem in [Z2].

(2.8). As in (2.1), let $f : X \longrightarrow X$ be a surjective morphism, d a positive integer, and $\mathrm{Pic}(X)_{f,d}$ the subgroup of $\mathrm{Pic}(X)$ consisting of line bundles $\mathcal{L}$ such that $\mathcal{L}^{\otimes d} \simeq f^*\mathcal{L}$. Assume that $\mathrm{Pic}(X)_{f,d}$ contains an ample line bundle of X. Then any line bundle $\mathcal{L}$ in $\mathrm{Pic}(X)_{f,d}$ can be written as $\mathcal{L}_1 \otimes \mathcal{L}_2^{-1}$ for two ample line bundles $\mathcal{L}_1, \mathcal{L}_2$ in $\mathrm{Pic}(X)_{f,d}$. By (2.3), there are ample metrized line bundles $\bar{\mathcal{L}}_1, \bar{\mathcal{L}}_2$ whose generic fibers are $\mathcal{L}_1, \mathcal{L}_2$ and $\bar{\mathcal{L}}_i^{\otimes d} \simeq f^*\bar{\mathcal{L}}_i$. Now $\bar{\mathcal{L}} = \bar{\mathcal{L}}_1 \otimes \bar{\mathcal{L}}_2^{-1}$ is an integrable metrized line bundle on X, and

$\bar{\mathcal{L}}^{\otimes d} \simeq f^*\bar{\mathcal{L}}$. By theorem (2.2), $\bar{\mathcal{L}}^{(d-1)}$ does not depend on the choice of $\bar{\mathcal{L}}_1, \bar{\mathcal{L}}_2$. Let $\overline{\text{Pic}(x)}_{f,d}$ denote the group of integrable metrized line bundles $\bar{\mathcal{L}}$ such that $\bar{\mathcal{L}}^{\otimes d} \simeq f^*\bar{\mathcal{L}}$. Then we have shown that $\overline{\text{Pic}(X)}_{f,d}$ is generated by ample metrized elements. We call elements in $\overline{\text{Pic}(X)}_{f,d}$ admissible metrized line bundles. The following theorem is useful in the next section.

Theorem (2.9). *Let $Y \hookrightarrow X$ be a subvariety of dimension n, and $\bar{\mathcal{L}} \in Pic(X)_{f,d}$ an ample metrized line bundle such that $h_{f,\mathcal{L}}(Y) = 0$, then*

$$c_1(\bar{\mathcal{L}}_1\big|_Y) \cdots c_1(\bar{\mathcal{L}}_{n+1}\big|_Y) = 0$$

for any $\bar{\mathcal{L}}_1, \cdots, \bar{\mathcal{L}}_{n+1}$ in $\overline{Pic(X)}_{f,d}$, where $n = \dim Y$.

Proof. Since $\overline{\text{Pic}(X)}_{f,d}$ is generated by ample metrized line bundles, we may assume $\bar{\mathcal{L}}_1, \cdots, \bar{\mathcal{L}}_{n+1}$ are ample metrized. Since $\mathcal{L}$ is ample, there is a positive integer m, and an ample line bundle $\mathcal{L}_0$ such that $\mathcal{L}^m \simeq \mathcal{L}_0 \otimes \mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_{n+1}$. Put a metric on $\mathcal{L}_0$ such that $\bar{\mathcal{L}}^m \simeq \bar{\mathcal{L}}_0 \otimes \bar{\mathcal{L}}_1 \otimes \cdots \otimes \bar{\mathcal{L}}_{n+1}$ then $\bar{\mathcal{L}}_0^{\otimes d} \simeq f^*\bar{\mathcal{L}}_0$. By theorem (2.4)(b), $\bar{\mathcal{L}}, \bar{\mathcal{L}}_0, \cdots, \bar{\mathcal{L}}_{n+1}$ are all semipositive. For any $n+1$ integers $i_1, \cdots, i_{n+1}$ between 0 and $n+1$, the number $c_1(\bar{\mathcal{L}}_{i_1}) \cdots c_1(\bar{\mathcal{L}}_{i_{n+1}})$ is nonnegative by (1.11). Since

$$\begin{aligned} 0 &= m^{n+1}\left(c_1(\bar{\mathcal{L}}\big|_Y)^{n+1}\right) = c_1(\bar{\mathcal{L}}^m\big|_Y)^{n+1} \\ &= \sum_{0 \le i_1 \le i_2 \le \cdots \le i_{n+1} \le n+1} c_1(\bar{\mathcal{L}}_{i_1}) \cdots c_1(\bar{\mathcal{L}}_{i_{n+1}}), \end{aligned}$$

we must have $c_1(\bar{\mathcal{L}}_1) \cdots c_1(\bar{\mathcal{L}}_{n+1}) = 0$.

3. Positivity of heights of certain subvarieties of an abelian variety

(3.1). Consider an abelian variety A over a number field K. For any integer n, let $[n]$ denote the endomorphism of A defined as the multiplication of n. Then for any symmetric line bundle $\mathcal{L}$ of $A, \mathcal{L}^{\otimes n^2} \simeq [n]^*\mathcal{L}$. If $X = A, f = [n]$ for a $n > 1$, then $h_{\mathcal{L}} = h_{f,\mathcal{L}}$ is the usual Néron - Tate height function studied by Philippon [P], Kramer [K], and Gubler [G]. In this case, conjecture (2.5) is a theorem of Raynaud [R], and (2.6) is a conjecture of Bogomolov [B] if $\dim Y = 1$.

Theorem (3.2). *Let $\mathcal{L}$ be a symmetric ample line bundle on A, and $Y \hookrightarrow A$ a subvariety of positive dimension such that $Y - Y$ generates A. This means that A is the only abelian subvariety of A which contains $Y - Y$. Assume that the induced map*

$$NS(A)_{\mathbb{Q}} \longrightarrow NS(Y)_{\mathbb{Q}}$$

is not injective, where $NS(A) = Pic(A)/Pic^0(A)$ and $NS(Y) = Pic(Y)/Pic^0(Y)$. Then $h_{\mathcal{L}}(Y) > 0$.

(3.3). The crucial facts used in the proof of the theorem are theorem (2.9), a variant form (3.4) of Faltings' index theorem [F1], and a nonvanishing theorem (3.5) for restriction on Y of an invariant 1 - 1 form of $A(\mathbb{C})$.

Lemma (3.4). *Let X be a variety over $\mathbb{Q}$, and $\bar{\mathcal{L}}$ and $\bar{\mathcal{M}}$ two integrable line bundles on X with smooth metrics at ∞. Assume that $\bar{\mathcal{L}}$ is semipositive and $\omega = c'(\bar{\mathcal{L}})$ is positive on a dense subset of the regular part $X_r(\mathbb{C})$ of $X(\mathbb{C})$, and that $\mathcal{M}$ is in $Pic^0(X)$. Then $c_1(\bar{\mathcal{M}})^2 c_1(\bar{\mathcal{L}})^{d-1} \leq 0$, and the equality $c_1(\bar{\mathcal{M}})^2 c_1(\bar{\mathcal{L}})^{d-1} = 0$ implies that the metric on $\bar{\mathcal{M}}$ has curvature 0 on $X_r(\mathbb{C})$.*

Proof. Let $f : X' \to X$ be a resolution of singularities. Replacing X by X', $\bar{\mathcal{L}}$ by $f^*\bar{\mathcal{L}}$, and $\bar{\mathcal{M}}$ by $f^*\bar{\mathcal{M}}$ we may assume that X is regular. Choose a metric $\|\cdot\|'_{\mathcal{M}}$ on $\mathcal{M}$ such that its curvature is 0, let $\varphi = \log \frac{\|\cdot\|'_{\mathcal{M}}}{\|\cdot\|_{\bar{\mathcal{M}}}}$.

Fix a positive number ϵ. By approximation, there is a model $(\widetilde{X}, \widetilde{\mathcal{L}}, \widetilde{\mathcal{M}})$ such that

(a) $\widetilde{\mathcal{L}}$ is a relatively semipositive line bundle on $\widetilde{X}$ whose restriction on X is $\mathcal{L}^{e_1}, e_1 > 0$, and whose metric at ∞ is the e_1-th power of the metric of $\bar{\mathcal{L}}$;

(b) $\widetilde{\mathcal{M}}$ is a line bundle on $\widetilde{X}$, whose restriction on X is $\mathcal{M}^{e_2}$, $e_2 > 0$, and whose metric at ∞ is e_2-th power of the metric of $\bar{\mathcal{M}}$;

(c) $c_1(\bar{\mathcal{M}})^2 c_1(\bar{\mathcal{L}})^{d-1} \leq \frac{1}{e_1^{d-1} e_2^2} c_1(\widetilde{\mathcal{M}}) c_1(\widetilde{\mathcal{L}})^{d-1} + \epsilon$.

Denote by $\widetilde{\mathcal{M}}'$ the metrized line bundle on $\widetilde{X}$ which has same finite part as $\widetilde{\mathcal{M}}$ on $\widetilde{X}$, and which has metric $\|\cdot\|'_{\mathcal{M}}$. Then

(d) $c_1(\widetilde{\mathcal{M}})^2 = c_1(\widetilde{\mathcal{M}}')^2 + \left(0, -\varphi \frac{\partial\bar{\partial}}{\pi i} \varphi\right)$ as cycles on $\widetilde{X}$.

We claim that

(e) $c_1(\widetilde{\mathcal{M}}')^2 c_1(\widetilde{\mathcal{L}})^{d-1} \leq 0$.

Fix a relatively ample line bundle $\widetilde{\mathcal{L}}'$ on $\widetilde{X}$, then

$$\lim_{n\to\infty} n^{1-d} c_1(\widetilde{\mathcal{M}}')^2 c_1(\widetilde{\mathcal{L}}^n \otimes \widetilde{\mathcal{L}}')^{d-1} = c_1(\widetilde{\mathcal{M}}')^2 c_1(\widetilde{\mathcal{L}})^{d-1}.$$

Replacing $\widetilde{\mathcal{L}}$ by $\widetilde{\mathcal{L}}^n \otimes \widetilde{\mathcal{L}}'$ for $n = 1, 2, \cdots$, we may assume that $\widetilde{\mathcal{L}}$ is relatively ample. Now, $c_1(\widetilde{\mathcal{L}})^{d-1}$ is represented by $\frac{1}{m}(Z, g_Z)$, where Z is an integral subvariety of X with a regular generic fiber, $m > 0$ an integer. Since $\widetilde{\mathcal{M}'}$ has curvature 0, one has

$$c_1(\widetilde{\mathcal{M}'})^2 c_1(\widetilde{\mathcal{L}})^{d-1} = \frac{1}{m} c_1(\widetilde{\mathcal{M}'}\big|_Z)^2.$$

Now $c_1(\widetilde{\mathcal{M}'}\big|_Z)^2 \leq 0$ by the Faltings-Hodge index theorem. The claim is proved.

Combining (a)-(e) we have that

$$c_1(\bar{\mathcal{M}})^2 c_1(\bar{\mathcal{L}})^{d-1} \leq - \int\limits_{X(\mathbb{C})} \varphi \frac{\partial\bar{\partial}}{\pi i} \varphi \omega^{d-1} + \epsilon.$$

Since $\omega^{d-1} \geq 0$ and $\omega^{d-1} > 0$ on a dense subset of $X(\mathbb{C})$, by letting $\epsilon \to 0$ it follows that

$$c_1(\bar{\mathcal{M}})^2 c_1(\bar{\mathcal{L}})^{d-1} \leq - \int \varphi \frac{\partial\bar{\partial}}{\pi i} \varphi \omega^{d-1} \leq 0,$$

and that $\int \varphi \frac{\partial\bar{\partial}}{\pi i} \varphi \omega^{d-1} = 0$ if and only if φ is locally constant.

Lemma (3.5). *Let A be a complex abelian variety, and $Y \hookrightarrow A$ a subvariety such that $\{y_1 - y_2 \big| y_1, y_2 \in Y(\mathbb{C})\}$ generates A. This means that A is the only abelian subvariety of A which contains $Y - Y$. If ω is a 1-1 form on A which is invariant under translation and $\omega\big|_Y = 0$, then $\omega = 0$.*

Proof. We write $A = \mathbb{C}^n \big/ \Lambda$ and $\omega = \sum a_{ij} dz_i \Lambda d\bar{z}_j$. After a translation, we may assume that 0 is a smooth point of Y. Fix points $y_1, \cdots, y_m$ on Y such that $\{y_1, \cdots, y_m\}$ generates A. One can find a complex curve $C \hookrightarrow Y$, such that $y_i \in C$ and 0 is a regular point of C. Fix any holomorphic map $\varphi : \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\} \hookrightarrow \mathbb{C}^n \big/ \Lambda$ such that $\varphi(\mathbb{D}) \hookrightarrow C$. Write $\varphi(z) = \big(f_1(z), \cdots, f_n(z)\big)$, then $f_1'(z), \cdots, f_n'(z)$ are linearly independent over $\mathbb{C}$.

If $\omega\big|_Y = 0$ then $\varphi^*\omega = 0$, it follows that

$$\sum a_{ij} \frac{\partial f_i}{\partial z} \frac{\partial \bar{f}_j}{\partial \bar{z}} = 0$$

for all $z \in \mathbb{D}$. Comparing coefficients of power series in z and $\bar{z}$, since $\frac{\partial f_i}{\partial z}$ are linearly independent in $\mathbb{C}$, for any i we must have

$$\sum_j a_{ij} \frac{\partial \bar{f}_j}{\partial \bar{z}} = 0,$$

or

$$\sum_j \bar{a}_{ij} \frac{\partial f_j}{\partial z} = 0.$$

It follows that $a_{ij} = 0$ for all i, j. So $\omega = 0$. The proof of the lemma is complete.

(3.6) Proof of (3.2). Assume $h_{\mathcal{L}}(Y) = 0$, then by theorem (2.9) for any admissible line bundles $\bar{\mathcal{L}}_1, \cdots, \bar{\mathcal{L}}_n, n = \dim Y + 1$, one has

$$c_1\big(\bar{\mathcal{L}}_1\big|_Y\big) \cdots c_1\big(\bar{\mathcal{L}}_n\big|_Y\big) = 0.$$

By assumption there is a line bundle $\mathcal{M} \in \mathrm{Pic}(A) \setminus \mathrm{Pic}^0(A)$ whose restriction on Y is in $\mathrm{Pic}^0(Y)$. Replacing $\mathcal{M}$ by $\mathcal{M} \otimes [-1] * \mathcal{M}$, we may assume $\mathcal{M}$ is symmetric. Put an admissible metric on $\mathcal{M}$, then we have

$$c_1(\bar{\mathcal{M}}\big|_Y)^2 c_1(\bar{\mathcal{L}})^{n-1} = 0.$$

By lemma (3.4), $c_1'(\bar{\mathcal{M}}\big|_Y) \equiv 0$. Let $\omega = c_1'(\bar{\mathcal{M}})$, then ω is an invariant 1 - 1 form on A, and $\omega \neq 0$. Since $Y - Y$ generates A, this contradicts lemma (3.5).

Theorem (3.7). *Let $\mathcal{L}$ be a symmetric ample line bundle on A, and $C \hookrightarrow A$ a curve such that $C - C$ generates A. Assume that the ring End $(A) \otimes_{\mathbb{Z}} \mathbb{R}$ is not isomorphic to $\mathbb{R}, \mathbb{C}$, and $\mathbb{D}$, where $\mathbb{D}$ is the division quaternion algebra. Then $h_{\mathcal{L}}(C) > 0$.*

Proof. By theorem (3.2), since $\mathrm{NS}(C)_{\mathbb{Q}} \simeq \mathbb{Q}$, we need only show that $\mathrm{NS}(A)_{\mathbb{Q}}$ or $\mathrm{NS}(A)_{\mathbb{R}}$ has rank ≥ 2. Fix a polarization on A. Decompose End $(A) \otimes_{\mathbb{Z}} \mathbb{R}$ into a product of copies of matrix algebras of $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{D}$, such that the involution of End (A) induced by the given polarization is identified with the involution on matrix algebras. Then $\mathrm{NS}(A) \otimes_{\mathbb{Z}} \mathbb{R}$ is isomorphic to the set of fixed endomorphisms under the involution. So $\mathrm{NS}(A) \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}$ implies that End $(A)_{\mathbb{R}} \simeq \mathbb{R}, \mathbb{C}$, or $\mathbb{D}$.

(3.8). Let C be a curve of genus ≥ 2, and c a divisor of degree 1. Define the morphism $\phi_c : C \hookrightarrow \mathrm{Jac}(C)$ such that $\phi(x) =$ the class of $x - c$. Denote by Θ be the divisor on $\mathrm{Jac}(C)$ which is the translate of the theta divisor on $\mathrm{Jac}^{g-1}(C)$ by $-(g-1)c$, and by $\mathrm{Pic}_{\Theta}(\mathrm{Jac}(C))$ the admissible metrized line bundles on $\mathrm{Jac}(C)$ with respect to the endomorphism [2], whose classes in $\mathrm{NS}(\mathrm{Jac}(C))$ are multiples of the class of Θ. Then $\phi^*(\mathrm{Pic}_{\Theta}(\mathrm{Jac}(C))$ is the group of admissible metrized line bundles defined in [Z1]. Denote by ω the admissible metrized relative dualizing sheaf on C, and by $O(D)$ the admissible line bundle associated to a divisor D. We want to show the following theorem:

Theorem (3.9). *If c_0 is a divisor of degree 1 on C such that $(2g-2)c_0$ is in the canonical divisor class on C, then*

$$h_{\mathcal{L}}\big(\phi_c(C)\big) = \frac{1}{8(g-1)}\omega^2 + (1 - \frac{1}{g})h_{\bar{\mathcal{L}}}(c - c_0).$$

Proof. For any divisor D of degree 0 on C, the Faltings-Hodge index theorem shows that

$$(D, D) = -2h_{\mathcal{L}}(D)$$

where (D, D) denotes the admissible pairing on divisions of C. In particular, for any $x \in C(\bar{\mathbb{Q}})$,

$$(x - c, x - c) = -2h_{\phi^*\bar{\mathcal{L}}}(x).$$

Applying the adjunction formula: $(x,x) = -(x,\omega)$, one has

$$\begin{aligned}(x-c,x-c) &= (x,x) - 2(x,c) + (c,,c)\\ &= -(x,\omega) - 2(x,c) + (c,c)\\ &= \big(-(\omega+2c) + (c,c), x\big).\end{aligned}$$

It follows that

$$-2\phi^*\bar{\mathcal{L}} + (\omega+2c) - (c,c)$$

has height 0 at every point. Consider this as a line bundle. Then one may prove that this bundle has curvatures 0 at all places of K, see 4.7 of [Z1]. Therefore it is numerically equivalent to 0.

Now

$$\begin{aligned}4c_1(\phi^*\bar{\mathcal{L}})^2 &= \big[c_1(\omega) + 2c_1\big(\mathcal{O}(c)\big)\big]^2 - 2\cdot 2g(c,c)\\ &= \omega^2 + 4(1-g)c^2 + 4\omega c\\ &= \omega^2 + 4(1-g)\big(c - \frac{\omega}{2g-2}\big)^2 + \frac{\omega^2}{g-1}\\ &= \frac{g}{g-1}\omega^2 + 8(g-1)h_{\bar{\mathcal{L}}}(c-c_0).\end{aligned}$$

Since $\deg\phi^*\mathcal{L} = g$, one has

$$\begin{aligned}h_{\bar{\mathcal{L}}}\big(\phi(C)\big) &= \frac{c_1(\bar{\mathcal{L}}\big|_{\phi(C)})^2}{2\deg\big(\mathcal{L}\big|_{\phi(C)}\big)} = \frac{c_1\big(\phi_C^*(\bar{\mathcal{L}})\big)^2}{2g}\\ &= \frac{\omega^2}{8(g-1)} + (1-\frac{1}{g})h_{\bar{\mathcal{L}}}(c-c_0).\end{aligned}$$

Corollary (3.10). *(a) If $(2g-2)c-\omega$ is not a torsion point of $\mathrm{Jac}(C)$ then $h_{\bar{\mathcal{L}}}\big(\phi(C)\big) > 0$.*

(b) If End $\big(\mathrm{Jac}(C)\big)_{\mathbb{R}}$ is not isomorphic to $\mathbb{R}, \mathbb{C}$, and $\mathbb{D}$ then $(\omega,\omega) > 0$.

Proof. Combine (3.7), (3.8), and the fact that $(\omega,\omega) \geq 0$ in [Z1].

Remarks (3.11). (a) The first part of (3.10) implies that the Bogomolov's conjecture is true if $c - c_0$ is not torsion. This fact has been proven in [Z1]. The second part shows Bogomolov's conjecture if $\mathrm{Jac}(C)$ has a nondivision endomorphism ring End $\big(\mathrm{Jac}(C)\big)_{\mathbb{R}}$.

(b) If C has good reductions at all finite places of a number field, one can prove that $\big(\omega,\omega\big) = \big(\omega_{Ar},\omega_{Ar}\big)$, where ω_{Ar} is the Arakelov dualizing sheaf. In this case, Bost told me he has proved (3.9).

(c) If C has good reduction at all finite places of a number field and $\mathrm{Jac}(C)$ has a complex multiplication, then End $(\mathrm{Jac}(C)_{\mathbb{R}}$ contains a subring isomorphic to $\mathbb{C}^g$. It follows from (3.10)(b) that $(\omega_{Ar},\omega_{Ar}) > 0$. This has been already proved by Burnol [Bu] using Weierstrass points.

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