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Heights of Heegner cycles and derivatives of L-series

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0. Introduction

In [18], Gross and Zagier proved an identity on modular curves between the height pairings of certain Heegner points and coefficients of certain cusp forms of weight 2. As a consequence, they showed that any modular elliptic curve over an imaginary quadratic field whose L-function has a simple zero at $s = 1$ contains a Heegner point of infinite order. This result plays a crucial rule in the solution of the Gauss class number problem by Goldfeld-Gross-Zagier [15, 18], and in the solution of the Birch and Swinnerton-Dyer conjecture [24] by Kolyvagin when the L-series of the modular elliptic curve over $\mathbb{Q}$ has order ≤ 1.

In this paper, we will extend Gross and Zagier's result to higher weights by using the arithmetic intersection theory. More precisely, we will define the (global) height pairing between CM-cycles in certain Kuga-Sato varieties, and show an identity between the height pairings of Heenger cycles and coefficients of certain cusp forms of higher weights.

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In the following, we will first describe our main result and applications, then summarize the main contents of the remaining sections.

0.1 Definitions of CM-cycles and height pairings. For an elliptic curve E with a CM by $\sqrt{D'}$, let $Z(E)$ denote the divisor class on $E \times E$ of $\Gamma - E \times \{0\} - D'\{0\} \times E$, where Γ is the graph of $\sqrt{D'}$. For k a positive integer, then $Z(E)^{k-1}$ is a cycle of codimension $k-1$ in E^{2k-2}. Let $S_k(E)$ denote the cycle

$$c \sum_{g \in G_{2k-2}} \operatorname{sgn} g^*(Z(E)^{k-1}),$$

where G_{2k-2} denotes the symmetric group of $2k-2$ letters which acts on E^{2k-2} by permuting the factors, and c is a real number such that the self-intersection of $S_k(E)$ on each fiber is $(-1)^{k-1}$.

For N a product of two relatively prime integers ≥ 3, one can show that the universal elliptic curve over the non-cuspidal locus of $X(N)_{\mathbb{Z}}$ can be extended uniquely to a regular semistable elliptic curve $\mathscr{E}(N)$ over whole $X(N)$. The Kuga-Sato variety $Y = Y_k(N)$ will be defined to be a canonical resolution of the $2k-2$-tuple fiber product of $\mathscr{E}(N)$ over $X(N)$. If y is a CM-point on $X(N)$, the CM-cycle $S_k(y)$ over x will be defined to be $S_k(\mathscr{E}_y)$ in Y.

If x a CM-divisor on $X_0(N)_{\mathbb{Z}}$, the CM-cycle $S_k(x)$ over x will be defined to be $\sum S_k(x_i)/\sqrt{\deg p}$, where p denotes the canonical morphism from $X(N)$ to $X_0(N)$, and $\sum x_i = p^* x$. One can show that $S_k(x)$ has zero intersection with any cycle of Y supported in the special fiber of $Y_{\mathbb{Z}}$, and that the class of $S_k(x)$ in $H^{2k}(Y(\mathbb{C}), \mathbb{C})$ is zero. So there is a green's current $g_k(x)$ on $Y(\mathbb{C})$ such that $\frac{\partial \bar{\partial}}{\pi i} g_k(x) = \delta_{S_k(x)}$. The arithmetic CM-cycle $\widehat{S}_k(x)$ over x, in the sense of Gillet and Soulé [13], is defined to be $(S_k(x), g_k(x))$.

If x and y are two CM-points on $X_0(N)$, then the height pairing of the CM-cycles $S_k(x)$ and $S_k(y)$ will be defined to be

$$\langle S_k(x), S_k(y) \rangle := (-1)^k \widehat{S}_k(x) \cdot \widehat{S}_k(y).$$

0.2. Main identity. Let K be an imaginary quadratic field with the discriminant D, such that every prime factor of N is split in K. Let H denote the Hilbert class field of K. Let σ be a fixed element $\mathrm{Gal}(H/K)$, and $\mathscr{A}$ the ideal class in $\mathcal{O}_K$ corresponding to σ via the Artin map. Let $f = \sum_{n \geq 1} a(n) e^{2\pi i n z}$ be a new form of weight $2k$ on $\Gamma_0(N)$. Define the L-series associated to f and $\mathscr{A}$ by

$$L_{\mathscr{A}}(f, s) = \sum_{\substack{n \geq 1 \\ (n, ND) = 1}} \left(\frac{D}{n}\right) n^{-2s+2k-1} \sum_{m \geq 1} a(m) r_{\mathscr{A}}(m) m^{-s},$$

where $r_{\mathscr{A}}(m)$ is the number of integral ideals in $\mathscr{A}$ with the norm m. Then Gross and Zagier proved that the function $L_{\mathscr{A}}(f, s)$ has analytical continuation to the entire complex plane, and satisfies a functional equation when s is replaced by $2k - s$, and vanishes at the point $s = k$. They have constructed explicitly an element $\Phi = \sum_{m\geq 1} a_{\mathscr{A}}(m)q^m \in S_{2k}^{\text{new}}(\Gamma_0(N))$ to represent the linear functional

$$f \longrightarrow \frac{(2k-2)!\sqrt{|D|}}{2^{4k-1}\pi^{2k}} L'_{\mathscr{A}}(f,k)$$

on the hermitian space $S_{2k}^{\text{new}}(\Gamma_0(N))$ with Petersson product. The main identity in this paper is as follows:

Theorem 0.2.1 *Let x be a Heegner point on $X_0(N)$ with discriminant D, and m an integer prime to N. Then*

$$\langle s_k(x), \mathrm{T}_m s_k(x^\sigma)\rangle = u^2 a_{\mathscr{A}}(m)$$

where $u = |\mathcal{O}_K^*|/2$.

Notice that when $k = 1$ Gross and Zagier showed that

$$\langle x - \infty,\ \mathrm{T}_m(x^\sigma - \infty)\rangle = u^2 a_{\mathscr{A}}(m),$$

where $\langle , \rangle$ is the Néron - Tate pairing on the Jacobian of $X_0(N)$.

0.3 Consequences. Let V be the subspace of Heegner cycles generated by

$$\left\{\mathrm{T}_m\, x^\sigma \middle| \sigma \in \text{ Gal } (H/K),\ (m,N) = 1\right\}.$$

Let V' be the quotient of V modulo the null subspace with respect to the height pairing. Then we will show the following

Theorem. 0.3.1. *The Hecke module V' is isomorphic to a sub-quotient module of $S_{2k}(\Gamma_0(N))^{\oplus h}$.*

Let χ a character of G, set $s'_\chi = \sum_{\sigma\in G} \chi^{-1}(\sigma) s'_k(x^\sigma)$ where $s'_k(x^\sigma)$ is the image of $s_k(x^\sigma)$ in V'. Let $f \in S_{2k}^{\text{new}}(\Gamma_0(N))$ be a normalized eigenform. Extend $\{f\}$ to an orthogonal basis $f_1 = f, \cdots,$ of $S_{2k}(\Gamma_0(N))$, then the cycle s'_χ can be written as a sum of f_i-isotropic components transform like that of f_j: $s'_\chi = \sum_{j=1}^d s'_\chi f_j$ with $\mathrm{T}_m s'_{\chi,f_j} = a_m(f_j) s'_{\chi,f_j}$. By the same reasoning as Gross-Zagier, we obtain the following corollaries:

Corollary 0.3.2.

$$L'(f,\chi,k) = \frac{2^{4k-1}\pi^{2k}(f,f)}{(2k-2)!u^2 h\sqrt{|D|}} \langle s'_{\chi,f}, s'_{\chi,f}\rangle.$$

Corollary 0.3.3. *If $L'(f,\chi,k) \neq 0$ then $s'_{\chi,f} \neq 0$.*

Corollary 0.3.4. *Let $f \in S_{2k}(\Gamma_0(N))$ be any newform and χ any character of* Gal(H/K). *Then either all conjugates $L(f^\alpha, \chi^\alpha,)$ $(\alpha \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}))$ have a simple zero at $s = k$ or else all have a zero of order ≥ 3.*

Corollary 0.3.5. *Let f be any new form of weight $2k$ and f^α* $(\alpha \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}))$ *any conjugate of f. Then*

$$\mathrm{ord}_{s=k}L(f,s) = 0 \Longleftrightarrow \mathrm{ord}_{s=k}L(f^\alpha,s) = 0$$

$$\mathrm{ord}_{s=k}L(f,s) = 1 \Longleftrightarrow \mathrm{ord}_{s=k}L(f^\alpha,s) = 1$$

$$\mathrm{ord}_{s=k}L(f,s) \geq 2 \Longleftrightarrow \mathrm{ord}_{s=k}L(f^\alpha,s) \geq 2$$

$$\mathrm{ord}_{s=k}L(f,s) \geq 3 \Longleftrightarrow \mathrm{ord}_{s=k}L(f^\alpha,s) \geq 3$$

Finally we give the following two consequences of the following index conjecture of Gillet-Soule and Beilinson-Bloch:

Corollary 0.3.6. *Assume the height pairing on $s_k(X)$ is positively definite. Then*

(a) *V is finitely dimensional.*
(b) *for any eigenform $f \in S_{2k}^{\mathrm{new}}(\Gamma_0(N))$ and any character χ of* $\mathrm{Gal}(H/K)$,

$$L'(f,\chi,k) \geq 0.$$

Notice that the inequality here is already predicted by the general Riemann hypothesis.

We will also obtain some conditional results about the algebraicity conjecture of Gross-Zagier, and the generalized Birch and Swinnerton-Dyer conjecture of Beilinson and Bloch. Since the statements of these results need some extra definitions, we postpone them until 5.

0.4. Plan of the proof. As in weight 2 case treated by Gross and Zagier, for m prime to N, we need to define the global pairing $\langle S_k(x), \mathrm{T}_m S_k(x^\sigma)\rangle$, and compute it as a sum of local pairings $\langle S_k(x), \mathrm{T}_m S_k(x^\sigma)\rangle_v$, even when $S_k(x)$ and $\mathrm{T}_m S_k(x^\sigma)$ are not disjoint.

In Sect. 1, we will define the global and local height pairings for general cycles Z_1, Z_2 on an arithmetic varieties over a number field, provided that they have good models over the ring of integers. Here we will use the arithmetic intersection theory introduced by Gillet and Soulé [13].

In Sect. 2, we will study Kuga-Sato varieties and CM-cycles. Here we will follow closely the work of Deligne-Rapoport [9], Katz-Mazur [23], Deligne [7], and Scholl [33].

In Sect. 3, we will study global and local height of CM-cycles. First we will define CM-cycles $S_k(x)$ and a height pairing on the group $S_k(X)$ of CM-cycles, and show that both Beilinson's index conjecture and Gillet-Soulé's index conjecture imply the positivity of the height pairing. Then we will give formulas for local heights. At Archimedean place, we will show that the local height pairings are given by certain standard green's functions. At the nonarchimedean place, the local heights are related to deformations of elliptic curves. The final formulas are similar to those given by Gross-Zagier [18] and Brylinski [5].

In Sect. 4, we will compute $\langle S_k(x), \mathrm{T}_m S_k(x^\sigma)\rangle$ for a Heegner point x on X. Here we will use our local formulas and some computations of Gross and Zagier.

In Sect. 5, we will first prove the main identity and its corollaries. Then we will give the three applications.

Acknowledgments. It should be mentioned that Perrin-Riou [32] has proved a p-adic version of Gross-Zagier's formula, and Nekovar [31] has extended Perrin-Riou's work to high weights.

It should be also mentioned that in [5], Brylinski worked some definitions of local heights suggested by Deligne. However, since the lack of the global theory of pairing, as well as the theory of self-pairing (e.g. adjunction formula), the results of Gross and Zagier seems difficult to be extended to higher weights by just using these definitions. At this point, our work should be considered a continuation of his work, even though our proof doesn't depend on any of his result.

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1 Intersections and heights

1.1. Intersections. Let us first review the arithmetic intersection theory of Gillet and Soulé [13]. Let Y be a regular arithmetic scheme of dimension d over Spec $\mathcal{O}_F$. This means that the morphism $Y \to \operatorname{Spec} \mathcal{O}_F$ is projective and flat and that Y is regular. For any integer $p \geq 0$, let $A^{p,p}(Y)$ (resp. $D^{p,p}(Y)$) denote the real vector space of real differential forms (resp. currents) α which are of type (p,p) on $Y(\mathbb{C})$ and such that $F_\infty^*\alpha = (-1)^p\alpha$, where $F_\infty : Y(\mathbb{C}) \to Y(\mathbb{C})$ denotes the complex conjugation.

A cycle of codimension p on Y with real coefficients is a finite formal sum $Z = \sum_\alpha r_\alpha Z_\alpha$, where $r_\alpha \in \mathbb{R}$, and Z_α are closed irreducible subvarieties of codimension p in Y. Such a cycle defines a current of integration $\delta_Z \in D^{p,p}(Y_\mathbb{R})$, whose value on a form η of complementary degree is

$$\delta_Z(\eta) = \sum_\alpha r_\alpha \int_{Z_\alpha(\mathbb{C})} \pi_\alpha^*(\eta).$$

A green current for Z is any current $g \in D^{p-1,p-1}(Y_\mathbb{R})$ such that the curvature

$$h_Z = \delta_Z - \frac{\partial\bar{\partial}}{\pi i} g$$

is a smooth form in $A^{p,p}(Y) \subset D^{p,p}(Y)$.

The (real) arithmetic group of codimension p is the real vector space $\widehat{\mathrm{Ch}}^p(Y)_{\mathbb{R}}$ generated by pairs (Z, g), where Z is a real cycle of codimension p on Y and g is a green current for Z, the addition being defined componentwise, with the following relation over $\mathbb{R}$. First any pair $(0, \partial u + \bar{\partial} v)$ is trivial in $\widehat{\mathrm{Ch}}^p(Y)_{\mathbb{R}}$. Second, if $Y' \subset Y$ is a irreducible subscheme of codimension $p-1$ on Y, $f \in k(Y')^*$ a nonzero rational function on Y, then the pair $(\mathrm{div}(f), -\log|f|\delta_{Y(\mathbb{C})})$ is zero in $\widehat{\mathrm{Ch}}^p(Y)_{\mathbb{R}}$.

Let $\widehat{Z}_1 = (Z_1, g_1)$ and $\widehat{Z}_2 = (Z_2, g_2)$ be two arithmetic cycles of Y of codimensions p and $d-p$. Assume both Z_1 and Z_2 are irreducible and intersect properly. Then we define the intersection of (Z_1, g_1) and (Z_2, g_2) as follows:

$$\widehat{Z}_1 \cdot \widehat{Z}_2 = \log|\Gamma(Z_1 \cdot Z_2, \mathcal{O})| + \int_{Z_2(\mathbb{C})} g_1 + \int_{Y(\mathbb{C})} g_2 h_Z.$$

For later use, we also define the intersection of Z_2 and $\widehat{Z}_1$:

$$Z_2 \cdot \widehat{Z}_1 = \log|\Gamma(Z_1 \cdot Z_2, \mathcal{O})| + \int_{Z_2(\mathbb{C})} g_1. \tag{1.1.1}$$

One can show that $\widehat{Z}_1 \cdot \widehat{Z}_2$ depends only on the classes of $\widehat{Z}_i$ in $\widehat{\mathrm{Ch}}^*(Y)_{\mathbb{R}}$, and $Z_1 \cdot \widehat{Z}_2$ depends only on Z_2 and the classes of $\widehat{Z}_1$ in $\widehat{\mathrm{Ch}}^*(Y)_{\mathbb{R}}$. Since Y is regular, for any class y in $\widehat{\mathrm{Ch}}^p(Y)_{\mathbb{R}}$ and any cycle Z_2, one can find a cycle $(Z_1, g) = \sum r_i(Z_i', g_i)$ (with real coefficients) representing y such that Z_i' is irreducible and intersects every irreducible component of Z_2 properly. In this way we may define $Z_2 \cdot y$ and $\widehat{Z}_2 \cdot y$ by linearity.

More generally, as showing in [13], there is an (associative and commutative) intersection product

$$\widehat{\mathrm{Ch}}^p(Y)_{\mathbb{R}} \otimes \widehat{\mathrm{Ch}}^q(Y)_{\mathbb{R}} \to \widehat{\mathrm{Ch}}^{p+q}(Y)_{\mathbb{R}}$$

such that if (Z_1, g_1) and (Z_2, g_2) are two cycle such that $\mathrm{cod}(Z_1 \cap Z_2) = p+q$ then

$$(Z_1, g_1) \cdot (Z_2, g_2) = (Z_1 \cdot Z_2, g_2 \delta_{Z_1(\mathbb{C})} + h_2 g_1).$$

If we identify $\widehat{\mathrm{Ch}}^d(Y)_{\mathbb{R}}$ with $\mathbb{R}$ by taking intersection with Y as (1.1.1), then the intersection product of cycles with complementary degrees gives the intersection pairing of these cycles.

We now state the following index conjecture of Gillet and Soulé. Given a line bundle $\mathscr{L}$ on Y, equipped with a smooth hermitian metric invariant under F_∞, one gets a first Chern class $\hat{c}_1(\bar{\mathscr{L}}) \in \widehat{\mathrm{Ch}}^1(Y)_{\mathbb{R}}$, defined as the class $(\mathrm{div}(s), -\log\|s\|)$, for any nonzero rational section s of $\mathscr{L}$ on Y. Denote by

$$L : \widehat{\mathrm{Ch}}^p(Y)_{\mathbb{R}} \to \widehat{\mathrm{Ch}}^{p+1}(Y)_{\mathbb{R}}$$

the product $\hat{c}(\bar{\mathscr{L}})$, i.e., $L(Y) = Y \cdot \hat{c}_1(\bar{\mathscr{L}})$.

We say that $\bar{\mathscr{L}}$ is positive if the following three conditions are satisfied:

(a) $\bar{\mathscr{L}}$ is ample on Y,
(b) the curvature $c_1(\bar{\mathscr{L}})$ is a positive 1–1 form on $Y(\mathbb{C})$,
(c) for any subvariety Y' of Y of dimension n and flat over Spec $\mathbb{Z}$, $c_1(\bar{\mathscr{L}})^n \cdot Y' > 0$.

Conjecture 1.1.1. (Gillet and Soulé [14]). *Let $\bar{\mathscr{L}}$ be a positive hermitian line bundle on Y and x an arithmetic Chow cycle of codimension p. If $2p \leq d$, $x \neq 0$, and $L^{d-2p+1}(x) = 0$, then*

$$(-1)^p \deg(xL^{d-2p}x) > 0$$

Remarks. (a) The case that Y is an arithmetic surface, the index conjecture is a theorem of Faltings [10] and Hriljac [21].

(b) Künnermann [25, 26] and Moriwaki [28] have proved the conjecture in several cases.

(c) In their original conjecture, instead of conditions above, Gillet and Soulé actually stated that for an ample line bundle $\mathscr{L}$ on Y, there is a metric on $\mathscr{L}$ such that Conjecture 1.1.1 is true.

1.2. Local decompositions. We would like to decompose $\widehat{Z}_1 \cdot \widehat{Z}_2$ into the local intersections $(\widehat{Z}_1 \cdot \widehat{Z}_2)_v$ for places v of F:

$$\widehat{Z}_1 \cdot \widehat{Z}_2 = \sum_v (\widehat{Z}_1 \cdot \widehat{Z}_2)_v \epsilon_v. \tag{1.2.1}$$

If Z_1 and Z_2 are disjoint at the generic fiber then the intersection $Z_1 \cdot Z_2$ with support defines an element in $\mathrm{Ch}^d_{|Z_1|\cap|Z_2|}(Y)$, (see 4.1.1 in [13]). Since $|Z_1|\cap|Z_2|$ is supported in special fibers, one has well defined $x_v \in \mathrm{Ch}^d_{|Y\otimes k(v)|}(Y)$ for each finite place v such that

$$Z_1 \cdot Z_2 = \sum_v x_v.$$

We define

$$(\widehat{Z}_1 \cdot \widehat{Z}_2)_v = \deg x_v$$

if v is finite, and

$$(\widehat{Z}_1 \cdot \widehat{Z}_2)_v = \int_{Z_{2v}(\mathbb{C})} g + \int_{Y_v(\mathbb{C})} g_{Z_2} h_1$$

if v is infinite, where Y_v denotes $Y \otimes_{\mathcal{O}_F,\sigma} \mathbb{C}$ for an embedding $\sigma : F \to \mathbb{C}$ inducing v and Z_{2v} is the pullback of Z_2 on Y_v. Notice that Y_v can be considered as a component of

$$Y \otimes_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\,\mathbb{C} = \coprod_{\sigma: F \to \mathbb{C}} X \otimes_{\mathcal{O}_F, \sigma} \mathbb{C},$$

and that the integrals do not depend on the choice of σ.

When $|Z_{1F}| \cap |Z_{2F}| \neq \emptyset$, we will try to define the local intersection in the following situation: there is a morphism $\pi : Y \to X$ from Y to a regular arithmetic surface X such that both Z_1 and Z_2 are contained in a fiber Y_D of Y over an integral divisor D of X, and that the morphism $\pi_D : Y_D \to D$ is smooth. Let $u : \tilde{D} \to D$ be the normalization of D, $i : D \to X$ the inclusion, and $v = iu$. We will use letters u_Y, i_Y and v_Y for their pullbacks under the base change $Y \to X$.

$$\begin{array}{ccccc} Y_{\tilde{D}} & \xrightarrow{u_Y} & Y_D & \xrightarrow{i_Y} & Y \\ \downarrow \pi_{\tilde{D}} & & \downarrow \pi_D & & \downarrow \pi \\ \tilde{D} & \xrightarrow{u} & D & \xrightarrow{i} & X \end{array}$$

Fix a local coordinate t for D_F in X_F by which we mean an element t in the algebraic closure of the function field $F(X)$ such that

(a) Some positive power t^e is in $F(X)$;
(b) The divisor $\frac{1}{e}\mathrm{div}(t^e) - D$ is disjoint with D on X_F.

Let $\mathrm{div}(d_D t)$ denote $(\frac{1}{e}\mathrm{div} t^e - D)|_D$. Define $\mathrm{ord}_v(d_D t)$ to be a rational number such that the pushforward of the 0-cycle $\mathrm{div}(d_D t)$ to $\mathrm{Spec}\,\mathcal{O}_F$ is $\sum_v \mathrm{ord}_v \mathrm{ord}(d_D t)[v]$. Notice that when D is a section of X over $\mathrm{Spec}\,\mathcal{O}_F$, $e\ \mathrm{div}(d_D t)$ is a section of $(\Omega^1_{X/\mathcal{O}_F})^{\otimes e}$.

By definitions, one has

$$\widehat{Z}_1 \cdot \widehat{Z}_2 = Z_2 \cdot \widehat{Z}_1 + \int_{Y(\mathbb{C})} g_2 h_1,$$

and

$$Z_2 \cdot \widehat{Z}_1 = u_Y^* Z_2 \cdot \tilde{v}_Y^* \widehat{Z}_1. \tag{1.2.2}$$

Now in $\mathrm{Ch}^*(Y_{\tilde{D}})$, one has

$$v^*(Z_1) = Z_1 \cdot v_Y^* \pi^* c_1(\mathcal{O}(D)) = -Z_1 \cdot u_Y^* \pi_D^*\ \mathrm{div}\ d_D t.$$

It follows that there is a current g' for cycle 0 such that

$$\begin{aligned} v_Y^*(Z_1, g_1) &= (-Z_1 \cdot u_Y^* \pi_D^*\ \mathrm{div}\ d_D t, g') \\ &= \Big(-\sum_{v|\infty} (\mathrm{ord}_v\ d_D t) Z_1 \otimes k(v), g' \Big), \end{aligned}$$

where $Z_1 \otimes k(v)$ denotes the pullback of Z_1 to the fiber $Y_D \otimes k(v)$ of Y_D on the closed point Spec $k(v)$ corresponding to v. Since Z_1 and Z_2 are flat over D, one has

$$u_Y^* Z_2 \cdot v_Y^*(Z_1, g_1) = -\sum_{v \nmid \infty} \epsilon_v(\mathrm{ord}_v\, d_D t)(Z_{1F} \cdot Z_{2F})_{Y_{D,F}} + \sum_{v \mid \infty} \epsilon_v \int_{Z_{2v}(\mathbf{C})} g',$$

where $(Z_{1F} \cdot Z_{2F})_{Y_{D,F}}$ is the intersection number being taking in $Y_{D,F}$.

To define the local intersections we have to describe $\int_{Z_{2v}(\mathbf{C})} g'$ explicitly for each fix each infinite place v. Let η be a differential form such that $\partial\eta = \bar{\partial}\eta = 0$ on each fiber $Y(\mathbf{C})_q$ when q is near $|D|(\mathbf{C})$, and that its restriction on $Y(\mathbf{C})_{D(\mathbf{C})}$ is $\delta_{Z_2(\mathbf{C})}$ modulo the images of ∂ and $\bar{\partial}$. Then the function $q \to \int_{Y_v(\mathbf{C})_q} g\eta$ is well defined on $X_v(\mathbf{C}) - \{p\}$. We hope this function gives some information for $\int_{Z_{2v}(\mathbf{C})} g' = \int_{Y_{D,v}(\mathbf{C})} \eta g'$. We would like to give the following

Conjecture 1.2.1. *With notations and assumptions as above, for any p in $|D_v|(\mathbf{C})$, one has*

$$\int_{Y_v(\mathbf{C})_q} g\eta + \log|t|(q) \int_{Z_v(\mathbf{C})} \eta = \int_{Y_v(\mathbf{C})_p} g'\eta + o(1)$$

as $q \to p$.

Suppose that the conjecture is true. We want to define the local intersection at a place v as follows. If v is finite, then define

$$(\widehat{Z}_1 \cdot \widehat{Z}_2)_v = -(Z_{1F} \cdot Z_{2F})_{Y_{D,F}} \mathrm{ord}_v\, d_D t,$$

If v is infinite, let G be a function on $|D|(\mathbf{C})$ defined by

$$G(p) = \lim_{q \to p} \left(\int_{Y(\mathbf{C})_q} g\eta + \log|t(q)|(Z_{1F} \cdot Z_{2F})_{Y_{D,F}} \right),$$

then we define

$$(\widehat{Z}_1 \cdot \widehat{Z}_2)_v = G(D_\sigma(\mathbf{C})) + \int_{Y_v(\mathbf{C})} g_2 h_1.$$

Remarks. (a) When Z_1 moves in a family, the local intersection at archimedean place has been defined by B. Harris and B. Wang [19]. It is not difficult to show that the above conjecture is true.

(b) In this paper we need to compute the intersection pairing of CM-cycles. In this case, the fiber of Y over D is an abelian scheme and Z_i are sums of subabelian schemes. However Z_i do not move in families. In 1.4, we will prove that above conjecture is true in this case. Moreover, we will prove

Theorem 1.2.2. *The conjecture is true in the following case: each irreducible component Z' of Z_1 is regular and $c_{n-1}(N_{Z'_F}(Y_{D,F}))$ is trivial as a Chow cycle.*

We now try to define the intersections over valuation fields. In non-archimedean case, we will consider a regular, projective, and flat scheme V over a discrete ring R. Let Z_1 and Z_2 be two irreducible cycles on V with complementary dimensions. If they are disjoint at the generic fiber, we can define the intersection $Z_1 \cdot Z_2$ as usual. Otherwise, we assume that V has a fiberation to a regular surface S over R, and that Z_1 and Z_2 are supported in the fiber V_D over an integral divisor D of S which is flat over R, and that every components of Z_1 and Z_2 are flat over D. Let t be a coordinate for D_η, where η is the generic point of Spec R. Then we can define the local intersection with respect to t by

$$(Z_1 \cdot Z_2) = -(Z_{1\eta} \cdot Z_{2\eta})_{Y_{D_\eta}} \operatorname{ord}_D t. \tag{1.2.3}$$

If v is a finite place of F, let W_v denote the completion of maximal unramified extension of $\mathcal{O}_v$. Then any two arithmetic cycles $\widehat{Z}_1$ and $\widehat{Z}_2$ on Y will induce two cycles Z_{1v} and Z_{2v} on $Y \otimes W_v$. One has

$$(\widehat{D}_1 \cdot \widehat{D}_2)_v = D_{1v} \cdot D_{2v}.$$

In archimedean case, we consider a regular and projective variety V over $\mathbb{C}$. By an arithmetic cycle of codimension p, we mean a pair $\widehat{Z} = (Z, g)$ of a cycles Z of codimension p on V and a Green's current g for Z as before. Let $\widehat{Z}_1 = (Z_1, g_1)$ and $\widehat{Z}_2 = (Z_2, g_2)$ be two arithmetic cycles on V with $\dim Z_1 + \dim Z_2 = \dim V - 1$. If Z_1 and Z_2 have disjoint support then we can define intersection as usual. Otherwise, we will assume that V has a fiberation to a regular curve C over $\mathbb{C}$ and Z_1 and Z_2 are contained in a fiber V_p over a point p. Let t be a local coordinate for p, let η be an differential form on $Y(\mathbb{C})$ which is ∂ and $\bar{\partial}$ closed over fibers $Y(\mathbb{C})_q$ for $q \in C(\mathbb{C})$ near p, and whose restriction on $Y(\mathbb{C})_p$ represents the cohomology class of $Z_2(\mathbb{C})$. Assume that the limit

$$G(p) := \lim_{q \to p} \left(\int_{Y(\mathbb{C})_q} g_1 \eta - (Z_{1p} \cdot Z_{2p}) \log |t| \right)$$

exists. Then we define

$$\widehat{Z}_1 \cdot \widehat{Z}_2 = G(p) + \int_{Y(\mathbb{C})} g_2 h_1.$$

If v is an archimedean place of F, then any two arithmetic cycles $\widehat{Z}_1$ and $\widehat{Z}_2$ in Y induce two arithmetic cycles $\widehat{Z}_{1v}$ and $\widehat{Z}_{2v}$ on Y_v. One has

$$(\widehat{Z}_1 \cdot \widehat{Z}_2)_v = Z_{1v} \cdot Z_{2v}.$$

For a morphisms of arithmetic varieties, one can define the pushforward and pullback maps on the groups of arithmetic cycles. One has projection formulas for intersections and local intersections. The formulas for cycles Z_1 and Z_2 with disjoint supports at the generic fiber are obvious. Otherwise, we will consider the case that cycles are contained in a fiber of a fiberation. For example we have the following projection formula for archimedean local intersections:

Proposition 1.2.3. *Consider the following diagram of morphisms of regular and projective varieties over* $\mathbb{C}$:

$$\begin{array}{ccc} Y' & \xrightarrow{\pi_Y} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{\pi_X} & X, \end{array}$$

where X and X' are curves. Let D and D' be irreducible divisors on X and X' respectively. Assume that the following induced diagrams are cartesian:

$$\begin{array}{ccc} Y'_{D'} & \longrightarrow & Y_{\pi D'} \\ {\scriptstyle f'_D}\downarrow & & \downarrow \\ D' & \longrightarrow & \pi D, \end{array} \qquad \begin{array}{ccc} Y'_{\pi^{-1}D} & \longrightarrow & Y_D \\ \downarrow & & \downarrow{\scriptstyle f_D} \\ \pi^{-1}D & \longrightarrow & D, \end{array}$$

and that the morphism $f_{D'}$ and f_D are smooth. Let $\widehat{Z} = (Z, g)$ be an arithmetic cycle of Y with Z supported in Y_D and flat over D, and $\widehat{Z}' = (Z', g')$ an arithmetic cycle of Y' with Z' supported in the fiber $Y'_{D'}$ and flat over D'. If $\pi D' = D$, let t be a local coordinate for D and $t' = (\pi^* t)^{\frac{1}{e}}$ be the local coordinate for D', where e is the ramification index of π_X along D'. One has

$$\widehat{Z} \cdot \pi_* \widehat{Z}' = \pi^* \widehat{Z} \cdot \widehat{Z}'.$$

Proof. By definitions. □

1.3. Heights. Now assume that the dimension of Y is $d = 2n$. Let $\mathscr{L}$ be an ample line bundle on Y. Let Z_{1F} and Z_{2F} be two cycles of Y_F of codimension n. Assume the following conditions:

(a) Z_{1F} has an integral model Z in Y which has zero intersection number with any cycle of Y of dimension n supported in special fibers; and

(b) the class of Z_1 vanishes in $H^{2n}(Y(\mathbb{C}), \mathbb{C})$. So there is a green's current g_1 on $Y(\mathbb{C})$ such that $\frac{\partial\bar{\partial}}{\pi i} g = \delta_{Z_1}$.

We define the global height pairing

$$\langle Z_{1F}, Z_{2F} \rangle = (-1)^n (Z_1, g_1) \cdot (Z_2, g_2)$$

where g_2 is any current for Z_2. It is not difficult to show that $\langle Z_{1F}, Z_{2F} \rangle$ depends only on the rationally equivalent classes of Z_{1F} and Z_{2F}. The commutativity of the height pairing follows from that of intersection pairing. Concerning the positivity of height pairing, one has the Gillet-Soulé's Conjecture 1.1.1 and the following

Conjecture 1.3.1 (Beilinson [2]). *Let* $\mathrm{Ch}^n(X)^{00}$ *denote the subgroup of Chow cycles satisfying the above two conditions and having 0 intersections with* $c_1(\mathscr{L})$. *Then the height pairing is positively definite on* $\mathrm{Ch}^n(X)^{00}$.

Remark. The conjecture implies in particular that the pairing is nondegenerate. This nondegeneracy is already conjectured by Bloch. Notice that Beilinson and Bloch independently defined their pairings by cohomological method. It is believed that their pairings are coincide with our pairing in Arakelov theory. See [2, 3, 34] for discussions.

One can also define the local height pairing by

$$\langle Z_{1F}, Z_{2F} \rangle_v = (-1)^n ((Z_1, g_1) \cdot (Z_2, g_2))_v,$$

when the right hand side is defined.

Let $Y \to X$ be a morphism over Spec $\mathcal{O}_F$ from an arithmetic variety Y to an arithmetic surface X. Assume that Z_1 and Z_2 satisfy the condition (a) and (b) in Sect. 1.3 and are contained in fibers over two irreducible divisors D_1 and D_2 of X over Spec $\mathcal{O}_F$. From the definition one can show

Proposition 1.3.2. *For* Z_1 *and* Z_2 *and* v *as above, the pairing* $\langle Z_{1F}, Z_{2F} \rangle_v$ *defined as above depends only on the classes of* Z_1 *and* Z_2 *in* $\mathrm{Ch}^*(Y_{D_1,F})$ *and* $\mathrm{Ch}^*(Y_{D_2,F})$.

Similarly, we can define height pairing for cycles over valuation fields using Sect. 1.2.

1.4. Proof of Theorem 1.2.2. The morphism $v_Y : Y_{\tilde{D}} \to Y$ is a composition of $\tilde{u} : \tilde{Y} = Y \otimes_D \tilde{D} \to Y$ and an embedding $\tilde{i} : Y_{\tilde{D}} \to \tilde{Y}$. One has a decomposition $\tilde{u}^*(Z, g) = (Z_\alpha, g_\alpha) + (Z_\beta, g_\beta)$ such that Z_α is supported in $Y_{\tilde{D}}$ and that $Z_{\beta,F}$ is disjoint with $Y_{\tilde{D},F}$. So we may assume that D is a section of X over Spec $\mathcal{O}_F$. By linearity, we may also assume that $Z := Z_1$ is integral.

In general, a direct computation for $i_T^*(Z, g)$ is difficult to find. We now first blow-up Y along Z to obtain the following diagrams:

$$\begin{array}{ccccccc} E & \overset{i_E}{\hookrightarrow} & Y' & & Z' & \overset{i_Z}{\hookrightarrow} & T' \\ f_E \downarrow & & \downarrow f & & f_{Z'} \downarrow & & \downarrow f_{T'} \\ Z & \overset{i_Z}{\hookrightarrow} & Y & & Z & \overset{i_Z}{\hookrightarrow} & T \end{array}$$

where E is the exceptional divisor, T' is the proper transformation of T, and Z' is the intersection of E and T'. We then show that $f^*(Z)$ has more space to move, more precisely, whose restriction on the generic fiber is a multiple of a divisor:

Lemma 1.4.1. *In the Chow group of Y', one has*

$$f^*(Z) = T' \cdot i_{E*}\{f_E^* c(N_Z(T))(1 + i_E^* E)^{-1}\}_{k-1} + i_{E*} f_E^* c_{k-1}(N_Z(T))$$

where k is the codimension of Z in Y and $\{\}_{k-1}$ denotes the part of degree $k-1$.

Proof. As Chow cycle, one has

$$f^*[Z] = i_{E*}(c_{k-1}(f_E^* N_Z(Y)/N_E(Y'))).$$

See [12] for a proof. From the exact sequence

$$0 \to N_Z(T) \to N_Z(Y) \to N_T(Y)|_Z \to 0,$$

we have

$$f^*(Z) = i_{E*}\{f_E^* c(N_Z(T))(1 + c(f_E^*(N_T(Y)|_Z)))(1 + c_1(N_E(Y')))^{-1}\}_{k-1}.$$

Since

$$f_E^*(N_T(Y)|_Z) = f_E^*(\mathcal{O}(T)|_Z) = \mathcal{O}(E)|_E \otimes \mathcal{O}(T')|_E$$

and $N_E(Y') = \mathcal{O}(E)$, we have

$$\begin{aligned} f^*(Z) &= i_{E*}\{f_E^* c(N_Z(T))(1 + i_E^* E + i_E^* T')(1 + i_E^* E)^{-1}\}_{k-1} \\ &= T' \cdot i_{E*}\{f_E^* c(N_Z(T))(1 + i_E^* E)^{-1}\}_{k-1} + i_{E*} f_E^* c_{k-1}(N_Z(T)) \end{aligned}$$

□

Let U be a cycle in Y representing

$$\{f_E^* c(N_Z(T))(1 + i_E^* E)^{-1}\}_{k-1},$$

and V be a cycle representing of $c_{k-1}(N_Z(T))$ with support in the special fibers. Choose currents g_U, g_E, $g_{T'}$ for U, E, T' such that $g_E + g_{T'} = \log|t|$ for points near T. Then there is a smooth form α such that

$$f^*(Z, g_Z) = (T', g_T) \cdot (U, g_U) + (i_{E*}f_E^*V, \alpha). \tag{1.4.1}$$

To get a formula for g_Z, we may assume that U is supported in E and properly intersect with Z' in E. Then $f^*(Z)$ is represented by $T' \cdot i_{E*}(U) + i_{E*}f_E^*V$. It is not difficult to show that as cycles,

$$f_*(T' \cdot i_{E*}U + i_{E*}f_E^*V) = Z.$$

It follows that

$$g_Z = f_*(g_U\delta_{T'} + h_U g_{T'} + \alpha). \tag{1.4.2}$$

To compute $i_T^*(Z, g_Z)$, we first pullback to T' and then pushforward to Y. Let U' be a cycle in Y representing

$$i_{E*}\{f_E^*c(N_Z(T))(1 + c_1(E))^{-1}\}_{k-1}$$

such that U' properly intersects with Z'. Write $\mathrm{div}(t) = T' + E + R$. Then by (1.4.1), one has

$$i_{T'}^*f_E^*(Z, g_Z) = i_{T'}^*(-E - R, g_{T'} + \log|t|) \cdot i_{T'}^*(U', g_{U'}) + (i_{Z'*}f_{Z'}^*V, \alpha|_{T'})$$

or simply

$$i_{T'}^*f_E^*(Z, g_Z) = (-Z' \cdot U' + i_{Z'*}f_{Z'}^*V - R \cdot U' \cdot T', -g_E * g_{U'} + \alpha|_{T'}) \tag{1.4.3}$$

We claim that $-Z' \cdot U' + i_{Z'*}f_{Z'}^*V$ can be moved away, more precisely, we have

Lemma 1.4.2. *The cycle $i_{Z'}^*(-U' + f_E^*V)$ vanishes in the Chow group of Z.*

Proof. Using $\mathcal{O}(E)|_{Z'} = N_{Z'}(T')$, one knows that $i_{Z'}^*(-U') + f_{Z'}^*V$ represents

$$\begin{aligned}
&\{f_{Z'}^*c(N_Z(T))(1 + c_1(N_{Z'}(T')))^{-1}(-c_1(N_{Z'}(T')) + f_{Z'}^*c(N_Z(T)))\}_{k-1} \\
&= \{f_{Z'}^*c(N_Z(T))(1 + c_1(N_{Z'}(T')))^{-1} \\
&\quad \cdot [-c_1(N_{Z'}(T')) + 1 + c_1(N_{Z'}(T'))]\}_{k-1} \\
&= \{f_{Z'}^*c(N_Z(T))(1 + c_1(N_{Z'}(T')))^{-1}\}_{k-1} \\
&= c_{k-1}\left(\frac{f_{Z'}^*N_Z(T)}{N_{Z'}(T')}\right) = 0,
\end{aligned}$$

where in the last equation, we use the fact that the bundle $\frac{f_{Z'}^*N_Z(T)}{N_{Z'}(T')}$ has rank $k-2$. □

This implies that there are subvarieties W_i $(1 \le i \le l)$ in Z' and functions f_i on W_i such that as cycles

$$-U' \cdot Z' + i_{Z'*} f_{Z'}^* V = \sum_{i=1}^{l} \mathrm{div}_{W_i}(f_i). \tag{1.4.4}$$

Now by (1.4.2) we have

$$i_{T'}^* f_E^*(Z, g_Z) = \left(-R \cdot U' \cdot T', -g_{Z'} * g_{U'}|_{T'} + \alpha|_T + \sum \log|f_i| \delta_{W_i} \right) \tag{1.4.5}$$

We pushforward this cycle to Y and get

$$\begin{aligned} i_T^*(Z, g_Z) =& (-f_*(R \cdot U' \cdot T), -f_{T'*}(g_E * g_U|_{T'}) \\ & + f_{T'*}(\alpha|_{T'}) + f_{T*}\left(\sum \log|f_i| \delta_{W_i} \right). \end{aligned} \tag{1.4.6}$$

We need to compute the current

$$f_{T*}\left(\sum \log|f_i| \delta_{W_i} \right) = f_*\left(\sum \log|f_i| \delta_{W_i} \right)$$

in terms of the given currents g_U, g_T and α.

Lemma 1.4.3 *As cycles in* $\widehat{\mathrm{Ch}}^*(Z)$, *one has*

$$f_{E*}(g_{T'} * g_{U'}|_E + \alpha|_E) = f_*\left(\sum \log|f_i| \delta_{W_i} \right). \tag{1.4.7}$$

Proof. We will prove this by the equation $f_{E*} i_E^*(Z, g_Z) = 0$. From (1.4.1), one has

$$\begin{aligned} i_E^*(Z, g_Z) &= (T', g_{T'})|_E \cdot (U', g_{U'})|_E + (f_E^* V \cdot c_1(N_E(Y')), \alpha|_E) \\ &= (Z' \cdot U' + f_E^* V \cdot c_1(N_E(Y)), g_{T'} * g_{U'}|_E + \alpha|_E). \end{aligned}$$

By (1.4.4), this equals to

$$\left(f_E^* V \cdot Z' - \sum_{i=1}^{l} \mathrm{div}_{W_i}(f_i) + f_E^* V \cdot \cdot c_1(N_E(Y)), g_{T'} * g_{U'}|_E + \alpha|_E \right)$$

or

$$\left(f_E^* V(Z' + c_1(N_E(Y))), g_T * g_{U'}|_E + \alpha|_E - \sum \log|f_i| \delta_{W_i} \right).$$

□

Since $f_{E*} i_E^*(Z, g_Z) = 0$ in $\widehat{\mathrm{Ch}}^*(Z)$, and $f_{E*}(Z' + c_1(N_E(Y))) = 0$ as cycles in Z, we have

$$f_{E*}\left(g_{T'} * g_{U'}|_E + \alpha|_E - \sum \log|f_i|\delta_{W_i}\right) = 0.$$

From (1.4.6) and (1.4.7), it follows that

$$\begin{aligned} i_T^*(Z, g_Z) =&(-Z \cdot R, -f_{T'*}(g_E * g_{U'}|_{T'}) \\ &+ f_{T'*}(\alpha|_{T'}) + f_{E*}(g_{T'*} * g_{U'}|_E + \alpha|_E)) \end{aligned}$$

or simply

$$i_T^*(Z, g_Z) = (-Z \cdot \mathrm{div}(dt)), -h_U g_E|_{T'} + \alpha|_{T'} + f_{E*}(h_U g_{T'} + \alpha)|_E \delta_Z. \quad (1.4.8)$$

We are ready to give an asymptotic formula for $\int_{Y(\mathbb{C})_q} g\eta$. Define

$$\begin{aligned} Y_{E,q} &= \{y \in Y_q(\mathbb{C}), g_u(y) > g_{T'}(y)\} \\ Y_{T',q} &= \{y \in Y_q(\mathbb{C}), g_T(y) > g_u(y)\} \end{aligned}$$

for any $q \in X(\mathbb{C})$ near p. By (1.4.2),

$$\int_{Y(\mathbb{C})_q} (g_Z\eta) = \int_{Y(\mathbb{C})_q} (h_U g_{T'} + \alpha)\eta = \int_{Y(\mathbb{C})_q} (-g_E h_U + \alpha)\eta,$$

(since $\int_{Y_q} h_U\eta = 0$)

$$\begin{aligned} &= \int_{Y_{E,q}} (-g_E h_U + \alpha)\eta + \int_{Y_{T',q}} (-g_E h_U + \alpha)\eta \\ &= -\log|t| \int_{Y_{E,q}} (-h_U\eta) + \int_{Y_{E,q}} (g_{T'} h_U + \alpha)\eta + \int_{Y_{T',q}} (-g_E h_U + \alpha)\eta \\ &= -\log|t|\gamma(q) + \delta(q) \end{aligned}$$

where

$$\begin{aligned} \gamma(q) &= \int_{Y_{E,q}} -h_U\eta, \\ \delta(q) &= \int_{Y_{E,q}} (g_T h_U + \alpha)\eta + \int_{Y_{T',q}} (-g_E h_U + \alpha)\eta. \end{aligned}$$

We need a asymptotic formula for γ and δ as $q \to p$.

Lemma 1.4.4. *Let g' be the green current in* (1.4.8), *then*

$$\gamma(q) = \int_{Z(\mathbb{C})} \eta + O(|t| \log|t|)$$

$$\delta(p) = \int_{T'(\mathbb{C})} g'\eta + O(|t|(\log|t|)^2)$$

If we assume this lemma, then we have

$$\int_{Y_q} (g_Z\eta) = -\log|t| \int_Z \eta + \int_T g'\eta + O(|t|(\log|t|)^2). \qquad (1.4.9)$$

The theorem follows.

It remains to prove the lemma. We compute the leading terms first.

$$\begin{aligned}
\gamma(p) &= \int_{E(\mathbb{C})} -h_U\eta = \left(-\int_{E(\mathbb{C})/Z(\mathbb{C})} h_U\right)\int_{Z(\mathbb{C})}\eta \\
&= -f_{E*}c_1(\mathcal{O}(E)|_E)^{k-1}(-1)^{k-2}\int_{Z(\mathbb{C})}\eta \\
&= f_{E*}c_1(\mathcal{O}(1))^{k-1}\int_Z \eta \\
&= \int_{Z(\mathbb{C})}\eta,
\end{aligned}$$

$$\begin{aligned}
\delta(p) &= \int_{E(\mathbb{C})} (g_{T'}h_U + \alpha)\eta + \int_{T'(\mathbb{C})} (-g_Z h_U + \alpha)\eta \\
&= \int_{Z(\mathbb{C})} f_{E*}(g_{T'}h_U + \alpha)\eta + \int_{T'(\mathbb{C})} (-g_E h_U + \alpha)\eta.
\end{aligned}$$

Now we need to estimate the error term as $q \to p$. By a partition of the unit, we reduce the question to the same estimates for smooth η supported in any chosen neighborhood U of a given point p on Y. Here we drop the assumption on closeness of η on fibers. If $p \notin Z'$, we may choose U disjoint with U'. It is easy to show that both γ and δ are smooth. If $p \in Z'$, then locally in a neighborhood of p, Y' has the equation $z_1z_2 = t$ in $\mathbb{C}^{d+1}$ with coordinates $(t, z_1, z_2, \cdots z_d)$, E has the equation $z_1 = 0$, T' has the equation $z_2 = 0$, and Z' has the equations $z_1 = z_2 = 0$. We therefore reduce the question to the following lemma.

Lemma 1.4.5. *On* $\mathbb{C}^{d+1}$ $(d \geq 2)$, *let* ϕ *be a smooth real function and let* η *be a smooth form of degree* $2(d-1)$ *with compact support. For each* $t \in \mathbb{C}$, *define a subset* U_t *of* $\mathbb{C}^d$ *by*

$$U_t = \{(z_1, z_2, \cdots, z_d) : |z_1|e^{\phi} > |z_2|e^{-\phi}, z_1z_2 = t\},$$

and numbers

$$G(t) = \int_{U_t} \log|z_1|\eta,$$

$$H(t) = \int_{U_t} \eta.$$

Then as $t \to 0$, *one has*

$$G(t) = G(0) + O(|t|(\log|t|)^2),$$
$$H(t) = H(0) + O(|t|\log|t|).$$

Proof. We first reduce the problem to case $d = 2$ and $\phi = 0$. Write $\eta = \eta_1 + \eta_2$ such that the support of the restriction of η_1 on U_t is relatively compact, and that the points of the support of the restriction of η_2 on U_t have small coordinates z_1 and z_2. The functions G and H for $\eta = \eta_1$ are smooth. So we may assume $\eta = \eta_2$. We can change coordinates with z_1 replacing $z_1 e^{\phi}$ and z_2 replacing $z_2 e^{-\phi}$ and with z_i unchanged if $i > 2$. In these new coordinates we have $\phi = 0$. Let U_t' be in $\mathbb{C}^2$ defined by

$$U_t' = \{(z_1, z_2) : |z_1| > |z_2|, z_1 z_2 = t\}$$

and η' a 2-form on $\mathbb{C}^3$ defined by

$$\eta'(z_1, z_2) = \int_{\mathbb{C}^{d-2}} \eta,$$

where the integral is over the variables $z_3, \cdots, z_d$. Then η' is smooth and has compact support and

$$G(t) = \int_{U_t'} \log|z_1|\eta',$$

$$H(t) = \int_{U_t'} \eta'.$$

This reduces to the case $d = 2$ and $\phi = 0$.

Now we assume $d = 2$ and $\phi = 0$. The only nontrivial contribution to the integral is the (1, 1) part of η. We may assume that η is of type (1,1) of the form

$$\eta = a_{11} dz_1 \wedge d\bar{z}_1 + a_{12} dz_1 \wedge d\bar{z}_2 + a_{21} dz_2 \wedge d\bar{z}_1 + a_{22} dz_2 \wedge d\bar{z}_2,$$

where a_{ij}'s are smooth functions of (t, z_1, z_2) with compact supports. We want to change the integrals to one variable integrals by substituting $z_1 = z$ and $z_2 = t/z$. Then

$$U_t = \{(z, t/z) : |z| > \sqrt{|t|}\}$$

and

$$\begin{aligned}\eta &= \left(a_{11} - a_{12}\frac{\bar{t}}{\bar{z}^2} - a_{21}\frac{t}{z^2} + a_{22}\frac{|t|^2}{|z|^4}\right)dzd\bar{z} \\ &= h\left(z, \frac{t}{z^2}\right)dz \wedge d\bar{z}\end{aligned}$$

where h is a smooth function of two variables with compact support, for example we assume $h = 0$ if the norm of the first variable is $>A$.

The error term of G can be written as

$$\begin{aligned}G(t) - G(0) = &\int_{A>|z|>\sqrt{|t|}} \log|z|\left(h\left(z, \frac{t}{z^2}\right) - h(z, 0)\right)dz \wedge d\bar{z} \\ &- \int_{|z|<\sqrt{|t|}} \log|z| h(z, 0) dz \wedge d\bar{z}.\end{aligned}$$

Since $h(z, \frac{t}{z^2}) - h(z, 0)$ is dominated by $|\frac{t}{z^2}|$ and $h(z, 0)$ by 1, it follows that the first integral is dominated by

$$-\int_{A>|z|>\sqrt{|t|}} (\log|z|)\frac{|t|}{|z|^2}dz \wedge d\bar{z} = -2\pi i|t|[(\log A)^2 - (\log\sqrt{|t|})^2],$$

and the second integral is dominated by

$$\int_{|z|<\sqrt{|t|}} \log|z| dz \wedge d\bar{z} = -2\pi i|t|\left(\log\sqrt{|t|} - \frac{1}{2}\right).$$

We therefore have the following estimate:

$$G(t) - G(0) = O(|t|(\log|t|)^2).$$

Similarly we have the estimate:

$$H(t) - H(0) = O(|t|\log|t|). \qquad \square$$

2. Kuga-Sato varieties and CM-cycles

2.1. Universal semistable elliptic curves. Let $N \geq 3$ be a positive integer and ζ_N a primitive N-th root of the unity. Let $X(N)_{\mathbb{Q}[\zeta_N]}$ be the compactification of the moduli of elliptic curves E over a $\mathbb{Q}[\zeta_N]$-scheme S, with a canonical full level N structure, i.e., an isomorphism of groups

$$\phi : (\mathbb{Z}/N)^2 \to E[N]$$

over S, such that the Weil pairing of $\phi(1,0)$ and $\phi(0,1)$ is ζ_N. Let $\mathscr{E}(N)^0_{\mathbb{Q}[\zeta_N]}$ be the universal elliptic curve over the noncuspidal part $X(N)^0_{\mathbb{Q}[\zeta_N]}$ of $X(N)_{\mathbb{Q}[\zeta_N]}$, and $\mathscr{E}(N)_{\mathbb{Q}[\zeta_N]}$ be the Kodaira-Néron minimal compactification of $\mathscr{E}^0(N)_{\mathbb{Q}[\zeta_N]}$, which is a semistable elliptic curve over $X(N)_{\mathbb{Q}[\zeta_N]}$ with N-polygons at cusps. Let $\pi_{\mathbb{Q}[\zeta_N]}$ denote the structure morphism $\mathscr{E}(N)_{\mathbb{Q}[\zeta_N]} \to X(N)_{\mathbb{Q}[\zeta_N]}$. We want to construct certain model over $\mathbb{Z}[\zeta_N]$ for $\mathscr{E}(N)_{\mathbb{Q}[\zeta_N]}$ following [9] and [23].

Theorem 2.1.1. *Assume that N is a product of two relatively prime integers $N_1 \geq 3$. Then the morphism $\pi_{\mathbb{Q}[\zeta_N]}$ can be uniquely extended to a morphism $\pi = \pi_{\mathbb{Z}[\zeta_N]} : \mathscr{E}(N) \to X(N)$ of regular, flat, and projective $\mathbb{Z}[\zeta_N]$-schemes, such that π makes $\mathscr{E}(N)$ a semistable elliptic curves over $X(N)$.*

Proof. The uniqueness of π is clear. We need only show the existence. Write $N_0 = N = N_1 N_2$ with N_1 and N_2 relatively prime and ≥ 3. Over $\operatorname{Spec} \mathbb{Z}\left[\zeta_{N_i}, \frac{1}{N_i}\right]$ $(i = 0, 1, 2)$, Deligne and Rapoport have shown that $\mathscr{E}(N_i)_{\mathbb{Q}[\zeta_{N_i}]} \to X(N_i)_{\mathbb{Q}[\zeta_{N_i}]}$ have integral models

$$\mathscr{E}(N_i)_{\mathbb{Z}[\zeta_{N_i}, \frac{1}{N_i}]} \to X(N_i)_{\mathbb{Z}[\zeta_{N_i}, \frac{1}{N_i}]},$$

satisfying corresponding properties as stated in the theorem with the base $\mathbb{Z}[\zeta_{N_i}, \frac{1}{N_i}]$. One has the following commutative diagram:

$$\begin{array}{ccccc}
\mathscr{E}(N_1)_{\mathbb{Z}[\zeta_{N_1}, \frac{1}{N_1}]} & \leftarrow & \mathscr{E}(N)_{\mathbb{Z}[\zeta_N, \frac{1}{N}]} & \rightarrow & \mathscr{E}(N_2)_{\mathbb{Z}[\zeta_{N_2}, \frac{1}{N_2}]} \\
\downarrow & & \downarrow & & \downarrow \\
X(N_1)_{\mathbb{Z}[\zeta_{N_1}, \frac{1}{N_1}]} & \leftarrow & X(N_1)_{\mathbb{Z}[\zeta_N, \frac{1}{N}]} & \rightarrow & X(N_1)_{\mathbb{Z}[\zeta_{N_2}, \frac{1}{N_2}]} \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{Spec} \mathbb{Z}[\zeta_{N_1}, \frac{1}{N_1}] & \leftarrow & \operatorname{Spec} \mathbb{Z}[\zeta_N, \frac{1}{N}] & \rightarrow & \operatorname{Spec} \mathbb{Z}[\zeta_{N_2}, \frac{1}{N_2}].
\end{array}$$

Let $X(N)_{\mathbb{Z}[\zeta_N, \frac{1}{N_i}]}$ be the normalization of $X(N_i)_{\mathbb{Z}[\zeta_{N_i}, \frac{1}{N_i}]}$ in the function field of $X(N)_{\mathbb{Q}[\zeta_N]}$, and let $\mathscr{E}(N)'_{\mathbb{Z}[\zeta_N, \frac{1}{N_i}]}$ be the pullback of $\mathscr{E}(N_i)_{\mathbb{Z}[\zeta_{N_i}, \frac{1}{N_i}]}$ on $X(N)_{\mathbb{Z}[\zeta_N, \frac{1}{N_i}]}$, then we obtain the following diagram:

$$\begin{array}{ccccc}
\mathscr{E}(N)'_{\mathbb{Z}[\zeta_N, \frac{1}{N_1}]} & \leftarrow & \mathscr{E}(N)_{\mathbb{Z}[\zeta_N, \frac{1}{N}]} & \rightarrow & \mathscr{E}(N)'_{\mathbb{Z}[\zeta_N, \frac{1}{N_2}]} \\
\downarrow & & \downarrow & & \downarrow \\
X(N_1)_{\mathbb{Z}[\zeta_N, \frac{1}{N_1}]} & \leftarrow & X(N)_{\mathbb{Z}[\zeta_N, \frac{1}{N}]} & \rightarrow & X(N_2)_{\mathbb{Z}[\zeta_N, \frac{1}{N_2}]}
\end{array}$$

It is not difficult to see that the two lower arrows are open embeddings. So we can define scheme $X(N)$ over $\operatorname{Spec} \mathbb{Z}[\zeta_N]$ by gluing two schemes at two lower ends by the scheme at the lower middle.

The two upper arrows are open embedding over non cuspidal points of X's but not over cusps. At a cusp of $X(N)$, the fibers of $\mathscr{E}(N)'_{[\zeta_N, \frac{1}{N_i}]}$ have N_i sides. Locally near the intersection of two sides over a cusp of $X(N)$, where $X(N)$ is defined by an equation $t = 0$, the scheme $\mathscr{E}(N)'_{[\zeta_N, \frac{1}{N_i}]}$ has an equation $xy = t^{N/N_i}$. We blow-up $\mathscr{E}(N)'_{[\zeta_N, \frac{1}{N_i}]}$ along intersections several times, we will obtain a scheme $\mathscr{E}(N)_{[\zeta_N, \frac{1}{N_i}]}$ which has an equation $xy = t$ near the intersection of two sides near a fiber. It is obvious that over Spec $\mathbf{Z}[\zeta_N, \frac{1}{N}]$, these blow-ups are just $\mathscr{E}(N)_{\mathbf{Z}[\zeta_N, \frac{1}{N}]}$ so we obtain two open embeddings

$$\mathscr{E}(N)_{\mathbf{Z}[\zeta_N, \frac{1}{N_1}]} \leftarrow \mathscr{E}(N)_{\mathbf{Z}[\zeta_N, \frac{1}{N}]} \to \mathscr{E}(N)_{\mathbf{Z}[\zeta_N, \frac{1}{N_2}]}.$$

Gluing two schemes at sides by the middle one, we obtain a scheme $\mathscr{E}(N)$ over Spec $\mathbf{Z}[\zeta_N]$. The regularity of $X(N)$ can be found in [23]. The regularity of $\mathscr{E}(N)$ follows from the fact that the morphism $\mathscr{E}(N) \to X(N)$ is smooth at non cuspidal point, and that $\mathscr{E}(N)$ locally in étale topology has an equation $xy = t$ near a cuspidal point. □

Remark. Let $\mathscr{E}(N)^*$ be the smooth locus of the morphism $\mathscr{E}(N) \to X(N)$. The full level structure N on the generic fiber defines a homomorphism of group schemes:

$$\phi : (\mathbf{Z}/N)^2 \to \mathscr{E}(N)^*.$$

One can show that $\sum_{(a,b)\in(\mathbf{Z}/N)^2}[\phi(a,b)]$ equals to $\mathscr{E}(N)^*[N]$ as Cartier divisors. It is an interesting question to show that the morphism $\mathscr{E}(N) \to X(N)$ represents the moduli stack which assigns to a scheme S the category of semistable elliptic curves E over S with a morphism of group schemes over S

$$\phi : (\mathbf{Z}/N)^2 \to E^*$$

such that the following properties are verified:

(a) as a Cartier divisor, $\sum_{(a,b)\in(\mathbf{Z}/N)^2}[\phi(a,b)]$ equals to $E^*[N]$ and that
(b) im (ϕ) meets every irreducible component of each geometric fiber of $E \to S$.

Following [9] and [23], we can also describe the fibers of $X(N)$ and $\mathscr{E}(N)$ over a closed point Spec $k \to$ Spec $\mathbf{Z}[\zeta_N]$. Let p be the characteristic of k. If $p \nmid N$, then X_k is smooth and the cuspidal divisor is reduced, so $\mathscr{E}(N)_k$ is smooth. If $p|N$, say $N = p^n M$ with $(M, p) = 1$, then $X(N)$ is the disjoint union, with crossing at the supersingular points, of smooth curves $X(N)_{k,A}$ over k indexed by the set of cyclic subgroups A of $(\mathbf{Z}/p^n)^2$ of order p^n. For each cyclic subgroup A of $(\mathbf{Z}/p^n)^2$, the non-cuspidal and non-supersingular points of $X(N)_{k,A}$ parameterized elliptic curves E with a canonical full level M structure and a morphism $\psi : (\mathbf{Z}/p^n)^2 \to E$ of group scheme, such that the following conditions are verified:

(a) as Cartier divisors

$$\sum_{(a,b)\in(\mathbb{Z}/p^n)^2} [\psi(a,b)] = E[p^n],$$

(b) the Weil pairing of $\psi(1,0)$ and $\psi(0,1)$ is 1.
(c) the image of A is connected.

Proposition 2.1.2. *At each closed point* Spec $k \to$ Spec $\mathbb{Z}[\zeta_N]$, *each irreducible component C of $X_k(N)$ is a regular and projective curve over k and the fiber $\mathscr{E}_C$ of $\mathscr{E}(N)$ over C is a regular and semistable elliptic curve over C.*

2.2. Kuga-Sato varieties. Let X be a regular scheme and $\mathscr{E}$ be a regular and semistable elliptic curve over X. Assume that the cuspidal divisor of X (over which the morphism $\mathscr{E} \to X$ is not smooth) is smooth in X. Let w be an positive integer, then w-tuple fiber product scheme $\mathscr{E}^w$ over X is not regular, it has a regular resolution of singularities as follows [7].

Let e be a closed point in $\mathscr{E}$ over which the morphism $\phi : \mathscr{E} \to X$ is not smooth. Then $x = \phi(e)$ is in the cuspidal divisor of X. Let t be a parameter on $\mathscr{O}_{x,X}$ defining the cuspidal divisor. Let $B = \widehat{\mathscr{O}}_{e,\mathscr{E}}$, $A = \widehat{\mathscr{O}}_{x,X}$ be the completions of local rings, then one has

$$B = A[[u,v]]/(uv - t).$$

Now in étale topology, at each closed point $\mathscr{E}^w$ has the singularity like that of

$$V = \text{Spec}\left(\frac{A[s_1,\cdots,s_q,u_1,v_1,\cdots,u_r,v_r]}{(u_1v_1 - t, u_2v_2 - t,\cdots,u_rv_r - t)}\right)$$

where q,r are nonnegative integers.

Let I be the ideal of $\mathscr{O}_V$ generated by monomials $\prod_{i=1}^r \sigma u_i^{i-1}$ where σ is a permutation of coordinates which preserves the set of pairs $\{u_i, v_i\}$ $(1 \le i \le r)$. Then the variety $\widetilde{V}$ induced from V by blowing-up I is regular. To see this let U be the open affine subscheme of $\widetilde{V}$ over which $\prod_{i=1}^r u_i^{i-1}$ defines an invertible section of $\mathscr{O}(1)$. The structure algebra $\mathscr{O}(U)$ over $\mathscr{O}(V)$ is generated by elements

$$\frac{\prod \sigma {u_i}^{i-1}}{\prod u_i^{i-1}}$$

where σ are permutations of the set $\{u_1, v_1, \cdots, u_r, v_r\}$ preserving the set of pairs $\{(u_1,v_1),\cdots(u_r,v_r)\}$. The regularity will follows from the following lemma

Lemma 2.2.1 (Deligne [7]).

$$\mathscr{O}(U) = A[s_1,\cdots,s_w, v_1/u_2, u_1/u_2,\cdots,u_{r-1}/u_r, u_r].$$

Proof. We prove first that RHS $\subset$ LHS. For $0 \le j \le r-1$ we define permutations σ_j as follows. When $j = 0$,

$$\sigma_0(u_i) = \begin{cases} v_2 & \text{if } i = 1, \\ v_1 & \text{if } i = 2, \\ u_i & \text{otherwise.} \end{cases} \quad \sigma_0(v_i) = \begin{cases} u_2 & \text{if } i = 1, \\ u_1 & \text{if } i = 2, \\ v_i & \text{otherwise.} \end{cases}$$

When $j > 0$,

$$\sigma_j(u_i) = \begin{cases} u_{i+1} & \text{if } j = i, \\ u_{i-1} & \text{if } j = i+1, \\ u_i & \text{otherwise.} \end{cases} \sigma_j(v_i) = \begin{cases} v_{i+1} & \text{if } j = i, \\ v_{i-1} & \text{if } j = i+1, \\ v_i & \text{otherwise.} \end{cases}$$

It is easy to see that

$$\frac{\prod \sigma_j(u_i)^{i-1}}{\prod u_i^{i-1}} = \begin{cases} v_1/u_2 & \text{if } j = 0, \\ u_j/u_{j+1} & \text{if } j > 1. \end{cases}$$

This proves that RHS $\subset$ LHS.

Now we want to prove that LHS $\subset$ RHS. It is obvious that s_i's and u_i's are in RHS. For a permutation σ preserving the set of pairs, we need to show that $\prod \sigma(u_i)^{i-1} / \prod u_i^{i-1}$ is in RHS. If for some $j \ge 1$ and $k > 1$, $\sigma(u_i) = v_k$ then we may replace v_k by u_k, since $v_k = (v_1/u_k)(u_1/u_k)u_k$ and $(v_1/u_k)(u_1/u_k)$ is in RHS. Finally we may assume that σ takes u's to u's, or to v_1. We have a permutation α of $\{1, \cdots, r\}$ such that $\sigma(u_{\alpha(i)}) = u_i$ if $i > 1$ and $\sigma(u_{\alpha(1)}) = u_1$ or v_1 and we have

$$\frac{\prod \sigma u_i^{i-1}}{\prod u_i^{i-1}} = \left(\frac{u_1 \text{ or } v_1}{u_2}\right)^{\alpha^{-1}(1)-1} \left(\frac{u_2}{u_3}\right)^{\alpha^{-1}(1)+\alpha^{-1}(2)-1-2} \cdots \left(\frac{u_{r-1}}{u_r}\right)^{\sigma(1)+\cdots\sigma(r-1)-1-\cdots-(r-1)}.$$

Since the exponents are all nonnegative, this is in RHS. This proves that LHS $\subset$ RHS. $\square$

Now on $\mathscr{E}^w$, we may blow up the ideal $J = \mathcal{O}_{x,\mathscr{E}^w} \cap I$ and get a regular scheme Y.

Notation. For k a positive integer and N a product of two coprime integers ≥ 3, we let $Y_k(N)$ denote the canonical resolution of the $2k-2$-tuple product of $\mathscr{E}(N)$ over $X(N)$.

Let us now define the Hecke correspondences. Write $X := X(N)$ and $\mathscr{E} := \mathscr{E}(N)$. For m a positive integer prime to N, Let X_m^0 be the moduli scheme classifying elliptic curves E with a $\Gamma(N)$ structure and an isogeny $E \to E'$ of degree m. Write $\mathscr{E}_m^0$ for the universal curve over X_m^0 and $\mathscr{E}_m^0 \to \mathscr{E}_m'^0$

for the universal isogeny. Then $E_m^{0'}$ has a level $\Gamma(N)$ structure coming from that on $\mathcal{E}_m^0$. Consider the diagram

$$\begin{array}{ccccccc} (\mathcal{E}^0)^{2k-2} & \xleftarrow{\phi_1} & (\mathcal{E}_m^0)^{2k-2} & \xrightarrow{\psi} & (\mathcal{E}'^0_m)^{2k-2} & \xrightarrow{\phi_2} & (\mathcal{E}^0)^{2k-2} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X^0 & \leftarrow & X_m^0 & = & X_m^0 & \rightarrow & X^0 \end{array}$$

On $Y(M,N)$ we define Hecke correspondence T_m as the Zariski closure in $\mathcal{E}^{2k-2} \times \mathcal{E}^{2k-2}$ of the correspondence $\phi_{1*}\psi^*\phi_2^*$ on $(\mathcal{E}^0)^{2k-2}$.

2.3. ϵ-component of cohomology. In this subsection we will review some constructions and results of Scholl [33]. Let N be a product of two relatively prime integers ≥ 3. Let X be an irreducible component of a geometric fiber of $X(N) \to \operatorname{Spec} \mathbb{Z}[\zeta_N]$. Let $\mathcal{E}$ denote the fiber of $\mathcal{E}(N)$ on X and Y the fiber of the Kuga-Sato variety $Y(N)$ on X. By Proposition 2.1.2, X and Y are smooth over the base field k with characteristic p. As in [33], the full level N structure on $\mathcal{E}$ defines a homomorphism of group schemes $\phi : (\mathbb{Z}/N)^2 \to \mathcal{E}^*$, where $\mathcal{E}^*$ denote the smooth locus of the morphism $\mathcal{E} \to X$. Therefore, $(\mathbb{Z}/N)^2$ acts on $\mathcal{E}$ by translations. Combining with multiplication by ± 1, this also gives an action on $\mathcal{E}$ by the semiproduct $(\mathbb{Z}/N)^2 \times \mu_2$ on $\mathcal{E}$. The symmetric group G_w of w-letters acts on $\mathcal{E}^w$ by permuting the factors. Hence the semiproduct

$$\Delta_w := ((\mathbb{Z}/N)^2 \times \mu_2)^w \times G_w$$

acts on $\mathcal{E}^w$. As the resolution introduced in Sect. 2.2 is canonical, this semiproduct also acts on Y. Let $\epsilon : \Delta_w \to \{\pm 1\}$ be the homomorphism which is trivial on $(\mathbb{Z}/N)^w$, is the product map on μ_2^w, and is the sign character on G_w. For any $\mathbb{Q}[\Delta_w]$-module V, write $V(\epsilon)$ for ϵ - isotropic component of V.

Let π denote the projection $Y \to X$. Let $j : X^0 \to X$ denote the inclusion of the complement X^0 of cusps in X. Let H^* be either l-adic cohomology theory ($l \neq p$) or Betti cohomology (when $k = \mathbb{C}$). So the coefficient F of H^* is either $\mathbb{Q}_l$ or $\mathbb{Q}$. One main result proved in [33] is as follows.

Theorem 2.3.1 (Scholl [33]). *There are canonical isomorphisms*:

$$H^1(X, j_*\mathrm{Sym}^w R^w \pi_* F) \to H^*(Y, F)(\epsilon).$$

Moreover when characteristic $p \nmid N$, the actions of the Hecke correspondences on the right hand side are compatible with actions of Hecke correspondences on the left hand side defined in [7].

Strictly speaking, Scholl only stated his theorem when k has characteristic 0, but his proof is valid for our general case without any change, as in the case that k has positive characteristic, H^*_{et} is the cohomological part of a twisted Poincaré duality [4].

2.4. CM-cycles. Let E be a CM elliptic curve over an integral ring R whose generic point η has characteristic 0. The ring $\mathrm{End}_S(E) \times \mathbb{Q}$ which depends only on the isogeny class of E, is isomorphic to an imaginary quadratic extension. Fix an embedding $\tau_E : \mathrm{End}_S(E) \times \mathbb{Q} \to \mathbb{C}$. As the Néron-Sevri group $\mathrm{NS}(E_\eta \times E_\eta)$ of $E_\eta \times E_\eta$ has rank 4, there is a divisor of $E \times_S E$ whose image in $\mathrm{NS}(E_\eta \times E_\eta)$ is perpendicular to the diagonal Δ_η, $\{0\} \times E_\eta$, and $E_\eta \times \{0\}$. Let $\sqrt{-D'}$ $(D' > 0)$ be an element in $\mathrm{End}_S(E)$. Let Γ be the graph of the multiplication by $\sqrt{-D'}$ then we can choose this divisor as

$$Z(E) = \Gamma - E \times \{0\} - D'\{0\} \times E.$$

Let k be a positive integer, then $Z(E)^{k-1}$ defines a cycle of codimension $k-1$ in E^{2k-2}. Notice that the symmetric group G_{2k-2} has an action on E^{2k-2} by permuting the factors. We define

$$W_k(E) = \sum_{g \in G_{2k-2}} \mathrm{sgn}(g) g^*(Z(E)^{k-1})$$

and

$$S_k(E) = cW_k(E)$$

where c is a positive constant such that the self intersection of $S_k(E)_\eta$ in E_η^{2k-2} is $(-1)^{k-1}$. The existence of c follows from the fact that the cohomology class of $S_k(E)_\eta$ is nonzero and primitive with respect to the product polarization. By the Hodge index theorem the self intersection of $S_k(E)_\eta$ is non-zero and has signature $(-1)^{k-1}$. Actually, one can compute c by representing $Z(E)$ by differential forms on $E \times E$ as in Sect. 3.4. The following proposition follows from the definitions.

Proposition 2.4.1. (a) *The class* $[S_k(E)]$ *of* $S_k(E)$ *in* $\mathrm{Ch}^k(E^{2k-1})$ *does not depend on the choice of* $\sqrt{-D'}$ *when* τ *is fixed.*

(b) *If* τ *changed to its complex conjugate then in the Chow group,* $[S_k(E)]$ *is changed to* $(-1)^{k-1}[S_k(E)]$.

(c) *Let* $\phi : E_1 \to E_2$ *be an isogeny of CM-elliptic curves over two CM-divisors* x_1 *and* x_2 *of* X. *For a fixed embedding*

$$\tau : \mathrm{End}(E_1) \otimes \mathbb{Q} = \mathrm{End}(E_2) \otimes \mathbb{Q} \to \mathbb{C},$$

one has $\phi^*[S_k(E_2)] = (\deg \phi)^{k-1}[S_k(E_1)]$ *in the Chow group of* E_1^{2k-1}.

Let N be a product of two relatively prime integers ≥ 3. Write $\mathscr{E} := \mathscr{E}(N) \otimes R$ and $Y := Y_k(N) \otimes R$. So one has the following morphisms of schemes:

$$X \leftarrow \mathscr{E}^{2k-2} \leftarrow Y.$$

Let R be an integral and flat algebra over $\mathbb{Z}$ which is unramified over all primes dividing N. We want to define the following objects:

(a) the space $s_k(X)$ of CM-cycles over $\mathbb{Q}$ and
(b) for each CM-divisor x on X, the pair of elements $\pm s_k(x)$ in $s_k(X)_{\mathbb{R}}$.

Let x be an irreducible CM divisor in X, by this we mean that x is flat over Spec R and the corresponding elliptic curve $E := \mathscr{E}_x$ has a complex multiplication. Write $W_k(x)$ for $W_k(E)$ and $S_k(x)$ for $S_k(E)$. We call the class $s_k(x)$ of $S_k(x)$ in $\mathrm{Ch}^k(Y)$ the CM-Chow-cycles over x.

We define the space of CM-cycles $s_k(X)$ to be a subspace of $\mathrm{Ch}^k(Y)$ generated by the classes of $W_k(x)$ over all CM-divisors x of X. So $s_k(x) \in s_k(X) \otimes \mathbb{R}$. By the above proposition, $\pm s_k(x)$ depends only x.

Let N be a product of two relatively prime integers ≥ 3.

Proposition 2.4.2. *Let x be an irreducible CM-divisor of point X. One has*

$$\mathrm{T}_m([S_k(x)]) = \sum_i m^{k-1}[S_k(x_i)]$$

if $\mathrm{T}_m x = \sum_i x_i$

Proof. The actions of Hecke operators are compatible with map p_* and p^*. By functoriality, we may assume that N is a product of two integers ≥ 3. Then the proposition follows from Proposition 2.4.1 (c). □

Concerning action of Δ_{2k-2} one has

Lemma 2.4.3 *The action of Δ_{2k-2} on $s_k(X)$ has character ϵ.*

Proof. It is easy to see that $s_k(X)$ has signature character under G_{2k-2}. For $((\mathbb{Z}/N')^2 \times \mu_2)^{2k-2}$ part, we need only work on $Z(E)$ in $\mathrm{Ch}^1(E \times E)$ for an elliptic curve defined over an integral domain. But here the fact is obvious. □

3. Heights of CM-cycles

3.1. Global heights. Let N be a product of two relatively prime integers ≥ 3. Let F be a number field in which all prime factors of N are unramified. Write $X := X(N) \otimes \mathcal{O}_F$. We want to define the height pairing on $s_k(X_F)$. Let x and y be two CM-divisors on X_F. Let $\bar{x}$ and $\bar{y}$ be their Zariski closures on X. To define the global height pairing $\langle S_k(x), S_k(y)\rangle$ of CM-cycles in Y_F, we have to check two conditions in Sect. 1.3.1. For the first condition, we consider the integral CM-cycles $S_k(\bar{x})$ and show that the restriction of this cycle on any component W of a geometric fiber of Y over Spec $\mathcal{O}_F$ is numerically equivalent to 0. For this we need only consider the case that $F = \mathbb{Q}$. Let p be the characteristic of the ground field of W. Then for $l \neq p$, the restriction of $S_k(x)$ on W defines an element in $H^k_{\mathrm{et}}(W, \mathbb{Q}_l(k))$. This class is ϵ-isotropical by Lemma 2.4.3, so is 0 by Theorem 2.3.1. By the similar way, the second condition is also verified. So we can define global height pairing in $S_k(X_F)$.

More precisely, let $g_k(x)$ be the Greens current for the cycle $S_k(x)$ with the following properties:

(a) $\frac{\partial\bar{\partial}}{\pi i} g_k(x) = \delta_{S_k(x)}$, and

(b) the integration $\int g_k(x)\eta$ is 0 for any $\frac{\partial\bar{\partial}}{\pi i}$ closed form η on $Y(\mathbb{C})$. Such $g_k(x)$ is determined up to images of ∂ and $\bar{\partial}$.

We define the arithmetic CM-cycles over x as

$$\widehat{S}_k(x) = (S_k(\bar{x}), g_k(x)).$$

Now the height pairing of CM-cycles $S_k(x)$ and $S_k(y)$ is

$$\langle S_k(x), S_k(y)\rangle = (-1)^k \widehat{S}_k(x) \cdot \widehat{S}_k(y). \tag{3.1.1}$$

We are going to prove the positive definiteness of the height pairing on the groups of CM-cycles under Conjecture 1.2.1 of Gillet and Soulé, or under Conjecture 1.4.1 of Beilinson-Bloch. We need to show that all these arithmetic cycles are primitive for some fixed positive hermitian line bundle $\bar{\mathscr{L}} = (\mathscr{L}, \|\cdot\|)$.

We may choose $\bar{\mathscr{L}}$ as a sum of the following line bundles:

(a) $\bar{\mathscr{M}}$: the pull back of a line bundle on X with divisor supported at cusps, and

(b) $\bar{\mathscr{L}}_i$ $(1 \le i \le 2k-2)$: the pull-backs of a line bundle on $\mathscr{E}$ with a divisor supported on the unit section O, and

(c) $\bar{\mathscr{N}}$: a line bundle which has a divisor supported in the exceptional divisor of the blow-up $Y \to \mathscr{E}^{2k-2}$.

Replacing $\bar{\mathscr{L}}$ by $\sum_{\sigma\in\Delta_{2k-2}} \sigma^*\bar{\mathscr{L}}$ we may assume that $\bar{\mathscr{L}}$ is invariant under Δ_{2k-2}. Similarly, we may assume that $\widehat{S}_k(x)$ is ϵ-isotropic.

Proposition 3.1.1. *For any CM-cycle $S_k(\bar{x})$, the intersection $\hat{c}_1(\bar{\mathscr{L}}) \cdot S_k(\bar{x})$ in $\widehat{\mathrm{Ch}}(Y)$ is zero.*

Lemma 3.1.2. *For any CM-cycle $S_k(\bar{x})$, the intersection $c_1(\mathscr{L}) \cdot S_k(\bar{x})$ in* $\mathrm{Ch}(Y)$ *is* 0.

Let us first show that this lemma implies the proposition. Actually, this statement implies that the element $\hat{c}_1(\bar{\mathscr{L}}) \cdot \widehat{S}_k(x)$ in $\widehat{\mathrm{Ch}}^*(Y)$ is represented by $(0, g)$. Since $\frac{\partial\bar{\partial}}{\pi i} g$ is the product of the curvatures of $\bar{\mathscr{L}}$ and $\widehat{S}_k(x)$, so it must be 0. It follows that g defines an element in $H^k(Y, \mathbb{C})(\epsilon)$. By Theorem 2.3.1, this group is 0. So g is in the sum of images of ∂ and $\bar{\partial}$. Therefore $(0, g)$ is 0 in $\widehat{\mathrm{Ch}}(Y)_{\mathbb{R}}$.

To show the lemma, it suffice to show that $S_k(\bar{x})$ has 0 intersection in the Chow group of Y with all $c_1(\mathscr{L}_i)$, $c_1(\mathscr{M})$, and $c_1(\mathscr{N})$. It is easy for $c_1(\mathscr{M})$ and $c_1(\mathscr{N})$, as $\mathscr{M}$ and $\mathscr{N}$ are supported in the fibers of Y over the cusps of X which are disjoint with $S_k(\bar{x})$. To show that $c_1(\mathscr{L}_i) \cdot S_k(\bar{x}) = 0$, it suffice to show for each $\sigma \in G_{2k-2}$ that

$$Z(E)^{2k-2} \cdot \sigma^* c_1(\mathscr{L}_i) = 0,$$

where E is the elliptic curve corresponding to $\bar{x}$. Up to a permutation of factors, it is easy to see that the left hand side of this equation is

$$p_{12}^*(Z(E) \cdot \{0\} \times E) p_{34}^*(Z(E)) \cdots p_{2k-3,2k-2}^*(Z(E))$$

or

$$p_{12}^*(Z(E) \cdot E \times \{0\}) p_{34}^*(Z(E)) \cdots p_{2k-3,2k-2}^*(Z(E))$$

Now our lemma follows from the following lemma:

Lemma 3.1.3. *Let* $E \to \mathrm{Spec}\ \mathcal{O}_F$ *be a smooth elliptic curve with a complex multiplication by* $\sqrt{-D'}$. *Then in* $\mathrm{Ch}(E \times_{\mathrm{Spec}\ \mathcal{O}_F} E)_{\mathbb{Q}}$,

$$Z(E) \cdot \{0\} \times E = 0,$$

$$Z(E) \cdot E \times \{0\} = 0$$

where $Z(E) = \Gamma_{\sqrt{-D'}} - E \times \{0\} - D'\{0\} \times E$.

Proof. Indeed,

$$Z(E) \cdot \{0\} \times E = -D'(\{0\} \cdot \{0\}) \times E = 0,$$

as $\{0\} \cdot \{0\}$ in $\mathrm{Ch}(E)_{\mathbb{Q}}$ is 0. Similarly

$$Z(E) \cdot E \times \{0\} = -E \times (\{0\} \cdot \{0\}) = 0. \qquad \square$$

3.2. Local decompositions. The notations and assumptions are as above. We want to decompose global heights into local heights

$$\langle S_k(x), S_k(y) \rangle = \sum_v \langle S_k(x), S_k(y) \rangle_v \epsilon_v. \tag{3.2.1}$$

Let x and y be two irreducible CM-divisors of X_F. Then we can define as in Sect. 1.3.3,

$$\langle S_k(x), S_k(y) \rangle_v = (-1)^k (\widehat{S}_k(x) \cdot \widehat{S}_k(y))_v, \tag{3.2.2}$$

when the right hand side is defined. The right hand side is always defined and (3.2.1) is true when $x \neq y$. When $x = y$, we need to check the assumption of Theorem 1.2.2. The cycle $S_k(\bar{x}) = cW_k(E)$ is supported in the fiber E^{2k-2} of Y over $\bar{x}$, where E is the elliptic curve corresponding to $\bar{x}$. Notice that E^{2k-2} is an abelian variety and $W_k(E)$ is a sum of abelian subvarieties A_i. Since all A_i have trivial normal bundle in E^{2k-2} over F, it follows from Theorem 1.2.2

that the Conjecture 1.3.1 is true for A_i (with any current). It follows that Conjecture 1.3.1 is true for $S_k(x)$. Now we can define local intersection as in Sect. 1.2.

To choose a local coordinate t for x we introduce the differential

$$\omega = \left[q^{\frac{1}{24}}\prod_{n\geq 1}(1-q^n)\right]^4 \frac{dq}{q}.$$

We choose a local coordinate t at x such that

$$\omega = (1+O(t))t^{1-\frac{1}{u(x)}}\,dt, \tag{3.2.3}$$

in a neighborhood of x, where $u(x) = \mathrm{ord}_x\omega$. The pairing $\langle S_k(x), S_k(y)\rangle_v$ does not depend on the choice of t satisfying the above condition.

We may also formally define the heights for CM-cycles defined over valuation field, as local intersections can be defined in this case as in Sect. 1.2.

3.3. Nonarchimedean formulas. Let W be a complete discrete valuation ring. Assume that the field Q of fractions of W has characteristic 0, and the residue field W_0 of W has characteristic not dividing N and is algebraically closed. Write X for $X(N)\otimes W$. We want to compute the local height $\langle S_k(x), S_k(y)\rangle$ for two irreducible representable CM-divisors x and y on X.

Let W_x, W_y be the normalizations of the the structure rings of x and y. Let π be an uniformizer of W_x. For any integer $n\geq 0$ and any W_x scheme or algebra Z, write Z_n for $Z\otimes W_x/\pi^{n+1}$. We define $\mathrm{Hom}_n(x,y)_{\deg m}$ to be the set of pair (f,g) of an embedding $f:\mathrm{Spec}\, W_{xn}\to \mathrm{Spec}\, W_y$ over W, and a homomorphisms g of group schemes over W_{xn} which makes the following diagram commutative

$$\begin{array}{ccc}
(\mathbb{Z}/N)^2 & \xrightarrow{\alpha_{xn}} & E_{xn} \\
\mathrm{id}\big\downarrow & & g\big\downarrow \\
(\mathbb{Z}/N^2) & \xrightarrow{\alpha_{yn}} & E_{yn}
\end{array}$$

Where $E_{yn} = E_y\otimes_f W_{xn}$, $E'_{yn} = E'_y\otimes_f W_{xn}$, and α_{xn}, α_{yn} are full level structure representing x and y. We write $\mathrm{Isom}_n(x,y)$ for $\mathrm{Hom}_n(x,y)_{\deg 1}$, $\mathrm{Aut}_n(x,y)$ for $\mathrm{Isom}_n(x,x)$. If we drop subscript "n" in these definitions, we will obtain (old) sets $\mathrm{Hom}(x,y)_{\deg m}$, $\mathrm{Isom}(x,y)$, and $\mathrm{Aut}(x,y)$. These sets can be considered as subsets of previous ones with subscript "n". We will use supscript "new" to denote the complements of old subsets:

$$\mathrm{Hom}_n^{\mathrm{new}}(x,y)_{\deg m} = \mathrm{Hom}_n(x,y)_{\deg m}\backslash \mathrm{Hom}(x,y)_{\deg m}.$$

Proposition 3.3.1. (a) *If $x \neq y$ then*

$$\langle S_k(x), S_k(y)\rangle = \frac{(-1)^k}{2} \sum_{\substack{n\geq 0\\ \psi\in \mathrm{Isom}_n(x,y)}} S_k(y)_0 \cdot \psi_0^* S_k(x)_0.$$

Here for a morphism $\psi : x \to y$ over W_{xn} which induces particular a morphism $\psi_0 : E_{x0}^{2k-2} \to E_{y0}^{2k-2}$ over $W_{x0} = W_0$, the cycle $\psi_0^* S_k(x)_0$ is the pullback of $S_k(x)_0$. The intersection $S_k(y)_0 \cdot \psi_0^* S_k(x)_0$ is taking in E_{x0}^{2k-2}.

(b) *If $x = y$ then*

$$\langle S_k(x), S_k(x)\rangle = \mathrm{ord}\, d_x t.$$

The case $x = y$ follows from (3.2.2), (1.3.10), and the fact that the self intersection of $S_k(x)_{\mathbb{Q}}$ is $(-1)^{k-1}$. So we may assume $x \neq y$. We claim that the intersection cycle $S_k(x) \cdot S_k(y)$ has a flat presentation over $z = x \cap y$. Since $S_k(x)$ and $S_k(y)$ has presentations by abelian subvarieties we need only prove the following

Lemma 3.3.2. *Let R be a complete discrete valuation ring with algebraic closed residue field $k = R/m$. Let $A \to$ Spec R be an abelian scheme, and $A' \to$ Spec R be an abelian subscheme of A. Let n be a positive integer and $A'' \to$ Spec R/m^n be an abelian subscheme of $A \otimes_{\mathrm{Spec}\, R}$ Spec R/m^n. Assume that $\dim_R A' + \dim_{R/m^n} A'' = \dim_R A$.*

(a) *If $A' \otimes R/m + A'' \otimes R/m = A \otimes R/m$, then the schematic intersection of $A' \otimes R/m^n$ and A'' is flat over R/m^n.*

(b) *If $A' \otimes R/m + A'' \otimes R/m \neq A \otimes R/m$, then for any prime l not dividing char (R/m) and any element d in $A[l]$ whose restriction d_k on the special fiber of A is not in $A'[l](R/m) + A''[l](R/m)$, the scheme $A' + d$ is disjoint with A''.*

Proof. (a) The intersection of $A' \otimes R/m^n$ and A'' is the kernel of the projection

$$A'' \otimes R/m^n \to (A/A') \otimes R/m^n.$$

So it is flat over Spec R/m^n.

(b) We need to check the set theoretical intersection of schemes in $A \otimes k$. If there are $a' \in A'(k)$, $a'' \in A''(k)$ such that $d_k + a' = a''$ then $d_k = a'' - a' \in A' + A''$. Since $d \in A[l]$ and $A_k[l] \cap (A'_k + A''_k) = A'_k[l] + A''_k[l]$ so $d_k \in A'_k[l] + A''_k[l]$. This contradicts the property of d. □

By this lemma, we may represent U and V for $S_k(x)$ and $S_k(y)$ in the fibers Y_x and Y_y respectively such that U and V intersect properly and the intersection is flat over the z. So we have

$$S_k(x) \cdot S_k(y) = (x \cdot y)(S_k(x)_0 \cdot S_k(y)_0). \tag{3.3.1}$$

If x and y don't intersect then $x \cdot y = 0$. If x and y intersect and let h denote $(x \cdot y)$, then $\mathrm{Isom}_n(x, y)$ is empty if $n > h$ and has one element ϕ if $n \leq h$. The assertion of the proposition follows.

For an isogeny ϕ defined over W_0 between two elliptic curves E_1 and E_2 having CM by $\sqrt{-D'}$ over W, we want to give a formula for $S_k(E_1)_0 \cdot \phi^* S_k(E_2)_0$. For this, let $l \neq p$ be a prime. We want to compute the intersection number through the pairing of l-adic cohomology:

$$H^{2k-2}(E_1^{2k-2}, \mathbb{Q}_l(k-1)) \times H^{2k-2}(E_1^{2k-2}, \mathbb{Q}_l(k-1)) \to \mathbb{Q}_l.$$

For convenience we choose an l such that $\sqrt{D'}$ is in $\mathbb{Q}_l$ but $i = \sqrt{-1}$ is not in $\mathbb{Q}_l$. Write $F = \mathbb{Q}_l(\sqrt{-1})$. and $H_j = H^1_{\text{et}}(E_j, \mathbb{Q}_l) \otimes F$. Let $[i]$ be the endomorphism on H_j such that $\sqrt{D'}[i]$ is induced from the endomorphism $\sqrt{-D'}$ on E.

Let $H = FX \oplus FY$ be a vector space of dimension two over F with an alternate pairing

$$(\,,\,) : H \otimes H \to F$$

such that $(X, Y) = i$ and an endomorphism $[i]$ such that $[i]X = -iX$, $[i]Y = Y$. Then we can fix isomorphisms from H_j to H which is compatible with the pairings and the actions by $[i]$.

Let ϕ is an isogeny between E_1 and E_2 over W_0, then the induced endomorphism ϕ^* on H is given by a matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Proposition 3.3.3. *Let $P_{k-1}(t)$ denote a constant multiple of $\frac{d^{k-1}}{dt^{k-1}}(t^2 - 1)^{k-1}$ such that $P_{k-1}(1) = 1$ then*

$$S_k(E_1)_0 \cdot \phi^* S_k(E_2)_0 = (-\det M)^{k-1} P_{k-1}\left(\frac{bc + ad}{\det M}\right).$$

Proof. We first claim that the class $S_k(E_j)$ in $H^{2k-2}_{\text{et}}(E_j^{2k-2}, F)$ is a constant multiple of $X^{k-1}Y^{k-1}$ in $\mathrm{Sym}^{2k-2} H$. By construction of $S_k(E_j)$ we need only check that the class of $Z(E_j)$ is a multiple of XY. Also by construction $Z(E_j)$ is the projection of the class of the graph of $\sqrt{-D'}$ on $H \otimes H$. We need only check that the endomorphism by $[i]$ on H is represented by a multiple of XY. Indeed, the endomorphism $[i]$ is given by $X \otimes Y + Y \otimes X$ in $H \otimes H$.

Since the class $\phi^* S_k(E_2)$ is a constant multiple of $(aX + bY)^{k-1}$ $(cX + dY)^{k-1}$, the intersection $S_k(x^\sigma) \cdot \phi^* S_k(x)$ is a constant multiple of the coefficient of $X^{k-1}Y^{k-1}$ in $(aX + bY)^{k-1}(cX + dY)^{k-1}$. This coefficient is equal to

$$\frac{d^{k-1}}{dx^{k-1}}\bigg|_{x=0}\left[(ax+b)^{k-1}\cdot(cx+d)^{k-1}\right].$$

Write $u = x + \frac{b}{2a} + \frac{d}{2c}$. Then the last expression equals

$$(ac)^{k-1}\frac{d^{k-1}}{du^{k-1}}\bigg|_{u=\frac{b}{2a}+\frac{d}{2c}}\left[\left(u+\frac{b}{2a}-\frac{d}{2c}\right)^{k-1}\left(u+\frac{d}{2c}-\frac{b}{2a}\right)^{k-1}\right]$$

or

$$(ac)^{k-1}\frac{d^{k-1}}{du^{k-1}}\bigg|_{u=\frac{b}{2a}+\frac{d}{2c}}\left[\left(u-\frac{\det M}{2ac}\right)^{k-1}\left(u+\frac{\det M}{2ac}\right)^{k-1}\right].$$

Write $u = \frac{\det M}{2ac}t$. Then the last expression equals

$$\left(S_k(x^\sigma)\cdot\phi^* S_k(x)\right) = \text{const}\left(\frac{\det M}{2}\right)^{k-1}\frac{d^{k-1}}{dt^{k-1}}\bigg|_{t=\frac{bc+ad}{\det M}}\left(t^2-1\right)^{k-1}\Bigg)$$

$$= \text{const}(\det M)^{k-1}P_{k-1}\left(\frac{bc+ad}{\det M}\right).$$

Taking ϕ as the identity map, we obtain that the constant in the last line should be $(-1)^{k-1}$ as $P_{k-1}(1) = 1$ and the self-intersection of $S_k(x)$ is $(-1)^{k-1}$. The lemma follows. □

3.4. Archimedean formulas. Let x and y be two points on $X(\mathbb{C})$ where $X := X(N)\otimes_{\mathbb{Z}[\zeta_N]}\mathbb{C}$ where $\zeta_N = \exp(2\pi i/N)$. We want to identify $\langle x, y\rangle$ with a certain Green's function G_k constructed in page 238–239 in [18]. We recall the definition as follows. Write $X^0 = \mathscr{H}/\Gamma(N)$. Denote by $Q(t)$ the Legendre function of the second kind defined for $t > 1$ by

$$Q(t) = \int_0^\infty \left(t+\sqrt{t^2-1}\cosh u\right)^{-k}du,$$

and define

$$g_k(z, z') = -2Q\left(1+\frac{|z-z'|^2}{2\ \mathrm{im}\ z\ \mathrm{im}\ z'}\right),$$

where $1+\frac{|z-z'|^2}{2\ \mathrm{im}\ z\ \mathrm{im}\ z'}$ is the hyperbolic cosine of the distance between the points z, z' of $\mathscr{H}$. Then we define a function on $\mathscr{H}\times\mathscr{H}\backslash$"diagonal" by

$$G_k(z, z') = \sum_{\gamma\in\Gamma} g_k(z, \gamma z').$$

Proposition 3.4.1 *For two CM-points x and y on X, one has*

$$\langle S_k(x), S_k(y)\rangle = \frac{1}{2} G_k(x,y) \tag{3.4.1}$$

if $x \neq y$, and

$$\langle S_k(x), S_k(x)\rangle = \frac{1}{2}\lim_{y\to x}(G_k(x,y) - \log|t|^2(y)). \tag{3.4.2}$$

We want to extend the function $\langle S_k(x), S_k(y)\rangle$ to a continuous function of y on $X\backslash\{x\}$ when x is a fixed CM-point. Let E be the elliptic curve corresponding to y. Let β be a holomorphic form on E such that $\int \beta\bar{\beta} = 1$. If y is a CM-point, then the class of Z_E in $H^{1,1}(E\times E, \mathbf{C})$ is represented by a constant multiple of $\beta_1\bar{\beta}_2 + \bar{\beta}_1\beta_2$. It follows that the class of Z_E is actually included in $\mathrm{Sym}\, H^1(E,\mathbf{C}) \subset H^2(E\times E,\mathbf{C})$, and that the class of $W_k(E)$ is included in $\mathrm{Sym}^{\,k-1} H^1(E,\mathbf{C}) \subset H^{2k-2}(E^{2k-2},\mathbf{C})$.

Let us write $\beta^1 = \beta$ and $\beta^{-1} = \bar{\beta}$. For any $I \subset [1, 2k-2]$ with $|I| = k-1$, let $\epsilon_I : [1,2k] \to \{1,-1\}$ be a function such that $\epsilon_I(i) = 1$ iff $i \in I$. Then the class of $S_k(y)$ in $H^{2k-2}(E^{2k-2},\mathbf{C})$ is represented by

$$\eta_y = c_k \sum_I \beta_1^{\epsilon_I(1)} \cdots \beta_{2k-2}^{\epsilon_I(2k-2)},$$

where the constant c_k is a constant independent of y subject to the condition that the self intersection of $S_k(y)$ in E^{2k-2} is $(-1)^{k-1}$.

Since η_y is defined for all noncuspidal y in X, we define a function of noncuspidal points in $X' - \{x\}$ by

$$H_k(x,y) = (-1)^k \int_{Y_y} \eta_y g_k(x). \tag{3.4.3}$$

Since $g_k(x)$ is $\frac{\partial\bar{\partial}}{\pi i}$ closed, if y is a CM-point then

$$\int_{Y_y} \eta_y g_k(x) = \int_{S_k(y)} g_k(x)$$

or

$$\langle S_k(x), S_k(y)\rangle = H_k(x,y). \tag{3.4.4}$$

We claim that η_y for y near x is the restriction of a smooth form η on Y independent of y. Locally, on a neighborhood V of x, the universal elliptic curve $\mathscr{E}$ can be written as a quotient

$$\mathscr{E}_V = \mathbf{C}\times V/\mathbf{Z}^2,$$

where $(m,n)\in\mathbb{Z}^2$ acts as $(u,z)\to(u+m+nz,z)$. Then the form $du-\frac{t}{y}dz$ on $\mathbb{C}\times V$ descends to a differential form α on $\mathscr{E}$, where $u=s+\sqrt{-1}t$. The differential form we need can be defined as

$$\eta=c_k y^{1-k}\sum_I \alpha_1^{\epsilon_I(1)}\cdots\alpha_{2k-2}^{\epsilon_I(2k-2)}.$$

Since the self-intersection of $S_k(x)$ in $D=Y_x$ is $(-1)^{k-1}$, we have

$$\langle S_k(x),S_k(x)\rangle=\lim_{y\to x}(H_k(x,y)-\log|t|(y)). \tag{24}$$

By (3.4.4) and (3.4.5), to show Proposition 3.4.1, it suffices to show the equality

$$H_k(x,y)=\frac{1}{2}G_k(x,y).$$

It has been shown in [20], Chapter Six, that $G_k(z,z')$ on X has the following properties:

(a) $G_k(z,z')$ is killed by $\Delta-k(k-1)$ operating on the first variable when $\Delta=y^2(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})$, where $z=x+yi$.

(b) $G_k(z,z')=\log|z-z'|^2+O(1)$ as $z\to z'$.

(c) In a neighborhood of a cusp $\gamma^{-1}\infty$, the function $\mathrm{im}(\gamma z)^{k-1}G_k(z,z')$ extends to a continuous function.

We claim that these properties characterize $G_k(z,z')$ uniquely. Indeed, if $G_k'(z,z')$ is another function satisfying same properties, then for each z' the difference $f(z):=G_k(z,z')-G_k'(z,z')$ is a continuous function on X killed by $\Delta-k(k-1)$ for all noncuspidal point $z\neq z'$. Let g be a continuous function on a neighborhood U of z' such that $\Delta g=k(k-1)f$ as a distribution acting on the smooth forms with compact supports in U, then we have $\Delta(g-f)=0$ away from z'. It follows that $g-f$ is continuous at z' and harmonic away from z'. By a standard fact of harmonic functions, $g-f$ is harmonic at z'. So f is killed by $\Delta-k(k-1)$ at all points of $\mathscr{H}$. Since Δ is a negative operator on $L^2(X)$ it follows $f=0$. Our claim follows.

To show the equality $H_k(z,z')=\frac{1}{2}G_k(z,z')$ it suffice to check that $H_k(z,z')$ has corresponding properties. Notice that Property (b) is already shown in the formulas in Sect. 1.4.

3.5. Continuity at cusps. In this subsection, we want to show Property (c) in Sect. 3.4.4 of $H_k(x,y)$. Now fix a cusp of X. Choose a parameter q on X for the cusp such that the smooth locus of the universal elliptic curve near the cusp can be written as $\mathscr{E}^*=\mathbb{C}^*/q^{N\mathbb{Z}}$. The form $du=\frac{1}{2\pi i}\frac{dw}{w}$ gives a section of $\Gamma(\omega_{\mathscr{E}/X})$, where $u\in\mathbb{C}$ and $w=e^{2\pi iu}\in\mathbb{C}^*$, and $\omega_{\mathscr{E}/X}=\Omega_{\mathscr{E}/X}(\log\infty)$ is the relative dualizing sheaf. Now for a CM point x, when $q\neq 0$ we have

$$H_k(x,q)=\mathrm{const}(-\log|q|)^{1-k}\int_{Y_q}\sum_I\wedge_{i=1}^{2k-2}du^{\epsilon_I(i)}g_k(q).$$

By partition of the unit, to prove the asymptotic formula of $H_k(x,q)$ as $q \to 0$, we need only prove for each point $p \in Y(\mathbb{C})$, that the integral

$$G(q) = \int_{Y(\mathbb{C})_q} \sum_I \wedge_{i=1}^{2k-2} du_i^{\epsilon_I(i)} g$$

defines a continuous function in a neighborhood of 0, where g is any smooth form of Y supported in any neighborhood U of p.

Let p' be the image of p in $\mathscr{E}^{2k-2}$. Then locally near p', $\mathscr{E}^{2k-2}$ has equations $w_1 v_1 = w_2 v_2 = \cdots = w_r v_r = q$ in $\mathbb{C}^{2k-1+r}$ with coordinates

$$(q, w_1, \cdots, w_{2k-2}, v_1, \cdots, v_r)$$

and $du_i = \frac{dw_i}{w_i}$. By Lemma 2.2.1, after a possible permutation of indices, U is smooth with coordinates $z_0, \cdots, z_{2k-1}$ such that $z_0 = v_1/w_2$, $z_i = w_i/w_{i+1}$ if $1 \le i < 2k-2$, and $z_i = w_i$ otherwise. The fiber of U over q is defined by

$$U_q = \left\{ (z_1, \cdots, z_{2k-1}) : q = z_0 z_1 \prod_{i=2}^{r} z_i^2 \right\}.$$

Therefore,

$$\frac{dz_0}{z_0} + \frac{dz_1}{z_1} + 2\sum_{i=0}^{r} \frac{dz_i}{z_i} = 0.$$

For $0 \le i \le r$, let U_i be the subset of U defined by

$$U_i = \{(z_1, \cdots, z_{2k-2}) \in U : |z_i| \le |z_j| \text{ for any } i \ne j \le r\}.$$

Then over $U_{iq} = U_i \cap U_q$, one has

$$du_l = \frac{dw_i}{2\pi i w_i} = \sum_{j \ne i} t_{lj} \frac{dz_j}{z_j},$$

where t_{ij} are integers. In the following expansion over U_{iq}:

$$\sum_I \alpha_1^{\epsilon_I(1)} \cdots \alpha_{2k-2}^{\epsilon_I(2k-2)} = \sum_{I,J} a_{I,J} \wedge_{l \in I} \frac{dz_l}{z_l} \wedge_{j \in J} \frac{d\bar{z}_j}{\bar{z}_j},$$

one must have $a_{I,J} = 0$ if $I \cap J \ne \emptyset$, where I and J are sets of integers l such that $1 \le l \le 2k-2$ and $l \ne i$. It follows that

$$\int_{U_{iq}} \sum_I \alpha_1^{\epsilon_I(1)} \cdots \alpha_k^{\epsilon_I(2k)} g = \int_{U_{iq}} \sum_{I \cap J = \emptyset} g_{I,J} \wedge_{i \in I} \frac{dz_i}{z_i} \wedge_{j \in J} \frac{d\bar{z}_j}{\bar{z}_j},$$

where $g_{I,J}$ are smooth forms supported in U. The continuity of this function follows from the fact that the form $\frac{dz\wedge d\bar{z}}{z}$ and $\frac{dz\wedge d\bar{z}}{\bar{z}}$ are integrable in a neighborhood of 0 in $\mathbb{C}$.

3.6. A differential equation. In this paragraph we will show that $H_k(z,z')$ satisfies the differential equation

$$\Delta_z H_k(z,z') = k(k-1)H_k(z,z').$$

As before, locally on a neighborhood V of z in $\mathscr{H}$, we may write

$$\mathscr{E}_V = \mathbb{C} \times U/\mathbb{Z}^2.$$

Let $u = t + \sqrt{-1}s$ be the coordinate of $\mathbb{C}$. Then $du - \frac{t}{y}dz$ descends to a differential form α on $\mathscr{E}_V$. Now on $Y_V = \mathscr{E}_V^{2k-2}$, $\{\alpha_1, \cdots, \alpha_{2k-2}, dz\}$ form a basis for $(1,0)$-form, where α_i is the pullback of α via the i-th projection from $\mathscr{E}^{2k-2} \to \mathscr{E}$. Now write

$$g_k(z') = \sum a_I\alpha_I + dz\sum b_J\alpha_J + d\bar{z}\sum_K c_K\alpha_K + dzd\bar{z}\sum d_L\alpha_L$$

where α_I (resp. $\alpha_J, \alpha_K, \alpha_L$) runs through the set of (p,p)-type [resp. $(\mathrm{p}-1,\mathrm{p})$-type, $(\mathrm{p},\mathrm{p}-1)$-type, $(\mathrm{p}-1,\mathrm{p}-1)$-type] monomials of 1-forms

$$\alpha_1, \cdots, \alpha_{2k-2}, \bar{\alpha}_1, \cdots, \bar{\alpha}_{2k-2}.$$

Since $\frac{\partial\bar{\partial}}{\pi i}g_k(z') = 0$, on fibers $\mathscr{E}_z^{2k-2}$, $\sum a_I\alpha_I|_{\mathscr{E}_z^{2k-2}}$ is $\frac{\partial\bar{\partial}}{\pi i}$-closed. Since $\alpha_I|_{\mathscr{E}_z^{2k-2}}$ are monomials from $\{du_1, \cdots, du_{2k-2}\}$, replacing g_z by $g_z + \partial\alpha + \bar{\partial}\beta$, we may assume that a_I are constants on fibers. In other words, a_I are functions of z only. Since $\sigma^* g_k(z') = \mathrm{sgn}(\sigma)g_k(z')$, we may write

$$\begin{aligned} g_k(p) =& f(z)\sum_I \alpha_1^{\epsilon_I(1)} \cdots \alpha_{2k-2}^{\epsilon_I(2k-2)} + \sum a_{I'}\alpha_{I'} \\ &+ dz\sum b_J\alpha_J + d\bar{z}\sum c_K\alpha_K + dzd\bar{z}\sum g_L\alpha_L \end{aligned} \tag{3.6.1}$$

where $\alpha_{I'}$ is a multiple of $\alpha_i \wedge \bar{\alpha}_i$ for some i. Now $\frac{\partial\bar{\partial}}{\pi i}g_z = 0$ implies that the coefficients M of $dzd\bar{z}du_1d\bar{u}_1 \cdots du_{2k-2}d\bar{u}_{2k-2}$ is zero in $\frac{\partial\bar{\partial}}{\pi i}g_z$ and in particular, $\int M du = 0$, where du is a volume form on $\mathscr{E}_z^{2k-2}$.

Lemma 3.6.1. *The only non-trivial contribution to* $\int M$ *is from the first term of the above expression* (3.6.1).

Proof. (a) Contribution of $\frac{\partial\bar{\partial}}{\pi i}a_{I'}\alpha_{I'}$ to M: Write $\alpha_{I'} = \wedge_{i_1\in I_1}\alpha_{i_1} \wedge \wedge_{i_2\in I_2}\bar{\alpha}_{i_2}$, then

$$\begin{aligned} \alpha_{I'}a_{I'} =& du_{I'} - dz\sum_{i\in I_1} du_{I-\{i\}}\frac{s_i}{y}a_{I'} - d\bar{z}\sum_{j\in I_2} du_{I-\{j\}}\frac{s_j}{y}a_{I'} \\ &+ dzd\bar{z}\sum_{i\in I_1, j\in I_2} du_{I-\{i,j\}}\frac{s_i}{y}\frac{s_j}{y}a_I. \end{aligned}$$

Since $a_{I'}$ is constant on fibers, the contribution to M is the sum of terms like

$$\frac{\partial}{\partial\bar{z}}\left(\frac{a_I}{y}\right)\frac{\partial(s_i)}{\partial u_{i'}}, \text{ where } i,i'\in I_1,\ \ i\neq i',$$
$$\frac{\partial}{\partial z}\left(\frac{a_I}{y}\right)\frac{\partial(s_j)}{\partial u_{j'}}, \text{ where } j,j'\in I_2,\ \ j\neq j',$$

and

$$\frac{\partial^2}{\partial u_i\partial u_j}(s_is_j)\cdot a_{I'}, \text{ where } (i',j')\neq(i,j).$$

They are all 0.

(b) Contribution of $\frac{\partial\bar{\partial}}{\pi i}dzb_J\alpha_J$ to $\int M$: Write $\alpha_J=\wedge_{j_1\in J_1}\alpha_{j_1}\wedge\wedge_{j_2\in J_2}\bar{\alpha}_{j_2}$, then

$$dzb_J\alpha_J=dzb_Jdu_J-dzd\bar{z}\sum_{j\in J_2}du_{J-\{j\}}\frac{s_j}{y}b_J.$$

The contribution to $\int M$ is the sum of the terms like

$$\int\frac{\partial^2}{\partial\bar{z}\partial u}b_J,$$
$$\int\frac{\partial^2}{\partial\bar{u}_i\partial\bar{u}_k}\frac{b_Js_j}{y}, \text{ where } i\neq k.$$

Since $\frac{\partial b_J}{\partial\bar{z}}$ is a function on $\mathscr{E}^{2k-2}$, the integral vanishes. Now if $i\neq j$, then $\frac{\partial b_Js_j/y}{\partial\bar{u}_k}$ is a function of the i-th factor of $\mathscr{E}^{2k-2}$, so

$$\int\frac{\partial^2}{\partial u_i\partial\bar{u}_k}\frac{b_Js_j}{y}=\int\frac{\partial}{\partial u_i}\left(\frac{\partial}{\partial\bar{u}_k}\frac{b_Js_j}{y}\right)=0.$$

Same for $k\neq j$. So the contribution of $\frac{\partial\bar{\partial}}{\pi i}dzb_z\alpha_k$ to $\int M$ is 0.

Similarly the contribution of $\frac{\partial\bar{\partial}}{\pi i}d\bar{z}dc_K\alpha_K$ is 0.

(c) Contribution of $\frac{\partial\bar{\partial}}{\pi i}dzd\bar{z}\alpha_L$. Write $\alpha_L=\wedge_{l_1\in L_1}\wedge\wedge_{l_2\in L_2}\bar{\alpha}_{l_2}$, then

$$g_Ldzd\bar{z}\alpha_L=g_Ldzd\bar{z}du_L.$$

So the contribution is $\int\frac{\partial^2}{\partial u_i\partial u_{\bar{j}}}g_L$. This is 0. □

Lemma 3.6.2.

$$\frac{\partial^2}{\partial z\partial\bar{z}}\frac{f(z)}{y^{k-1}}+\frac{k-1}{2iy}\frac{\partial}{\partial\bar{z}}\frac{f(z)}{y^{k-1}}-\frac{k-1}{2i}\frac{\partial}{\partial z}\frac{f(z)}{y^k}-\frac{k-1}{4}\frac{f(z)}{y^{k+1}}=0.$$

Proof. By Lemma 3.6.1 and the fact that $\frac{\partial\bar{\partial}}{\pi i}g = 0$ we have

$$\partial\bar{\partial}\frac{f(z)}{y^{k-1}}\sum_I \alpha_1^{\epsilon_I(1)}\cdots\alpha_{2k-2}^{\epsilon_I(2k-2)} = 0. \tag{3.6.2}$$

Notice that for $\alpha = du - \frac{s}{y}dz$,

$$\partial\alpha = -\frac{1}{2iy}\alpha dz,\ \bar{\partial}\alpha = \frac{1}{2iy}\bar{\alpha}dz,$$
$$\partial\bar{\alpha} = -\frac{1}{2iy}\alpha d\bar{z},\ \bar{\partial}\bar{\alpha} = \frac{1}{2iy}\bar{\alpha}d\bar{z}.$$

Using these formulas to compute $\bar{\partial}$ part of (3.6.2), we obtain

$$\partial\frac{\partial}{\partial\bar{z}}\frac{f(z)}{y^{k-1}}\sum_I \alpha_1^{\epsilon_I(1)}\cdots\alpha_{2k-2}^{\epsilon_I(2k-2)}d\bar{z} \tag{3.6.3}$$

$$+\frac{-1}{2\sqrt{-1}}\partial\frac{f(z)}{y^k}\sum_I\sum_i \alpha_1^{\epsilon_I(1)}\cdots\alpha_{i-1}^{\epsilon_I(i-1)}\alpha^{-1}\alpha_{i+1}^{\epsilon_I(i+1)}\cdots dz^{\epsilon_I(i)} \tag{3.6.4}$$

Computing ∂ part of (3.6.3), we obtain

$$\left(\frac{\partial^2}{\partial z\partial\bar{z}}\frac{f(z)}{y^{k-1}} + \frac{k-1}{2iy}\frac{\partial}{\partial\bar{z}}\frac{f(z)}{y^{k-1}}\right)\sum_I \alpha_1^{\epsilon_I(1)}\cdots\alpha_{2k-2}^{\epsilon_I(2k-2)}dzd\bar{z} \tag{3.6.5}$$

Similarly, computing ∂ part of (3.6.4) we obtain

$$\left(-\frac{k-1}{2i}\frac{\partial}{\partial z}\frac{f(z)}{y^k} - \frac{k-1}{4}\frac{f(z)}{y^{k+1}}\right)\sum_I \alpha_1^{\epsilon_I(1)}\cdots\alpha_{2k-2}^{\epsilon_I(2k-2)}dzd\bar{z} \tag{3.6.6}$$

The lemma follows from (3.6.5) and (3.6.6). □

By lemma one has

$$y^2 f_{z\bar{z}} = \frac{k(k-1)}{4}f.$$

Since

$$\Delta = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = 4y^2\frac{\partial^2}{\partial z\partial\bar{z}}$$
i

it follows that

$$\Delta f = k(k-1)f.$$

4. Heights of Heegner cycles

4.1. Heights of CM-cycles over $\mathbf{X_0(N)}$. Let N be a positive integer. In this subsection we want to define the space $s_k(X_0(N))$ cycles and the height pairing on this space. For this let N' be any multiple of N such that N' is the product of two relatively prime integers ≥ 3. Let p denote the natural morphism from $X(N')$ to $X_0(N)$. If x is a CM-point, then $p^*x = \sum x_i$ is a sum of CM-points x_i. Then the elliptic scheme E over $p^{-1}x$ has an endomorphism by some $\sqrt{-D'}$. We use this endomorphism to define $W_k(x_i)$, $W_k(p^*x) = \sum W_k(x_i)$, and $S_k(p^*x) = \sum S_k(x_i)$. Let $s_k(X)$ be the subspace of $s_k(X')$ generated by $W_k(p^*x)$ for all CM-divisors x on X. This space does not depend on the choice of N' up to canonical isomorphisms. Finally we define $S_k(x)$ to be the image in $s_k(X)$ of $S_k(x) = S_k(p^*x)/\sqrt{\deg p}$. Then the restriction of the height papring on CM-cycles over $X(N')$ gives a height papring on $s_k(X)$. In the following we want to find formulas for these local intersections.

In the nonarchimedean case, let W be a complete discrete valuation ring, such that the field Q of fractions of W has characteristic 0, and the residue field W_0 of W has characteristic not dividing N and is algebraically closed. Write X for $X(N) \otimes W$. We say an irreducible CM-divisor x on X is representable, if x represents an a pair (E_x, β_x) of an elliptic curve E_x, and a cyclic isogeny $\beta_x : E_x \to E'_x$ of degree N. We want to compute the local height $\langle S_k(x), S_k(y)\rangle$ for two irreducible representable CM-divisors x and y on X.

Let W_x, W_y be the normalizations of the the structure rings of x and y. Let π be an uniformizer of W_x. For any integer $n \geq 0$ and any W_x scheme or algebra Z, write Z_n for $Z \otimes W_x/\pi^{n+1}$. We define $\mathrm{Hom}_n(x,y)_{\deg m}$ to be the set of triples (f, g, h) of an embedding $f : \mathrm{Spec}\, W_{xn} \to \mathrm{Spec}\, W_y$ over W, and homomorphisms g, h of group schemes over W_{xn} which makes the following diagram commutative

$$\begin{array}{ccc} E_{xn} & \xrightarrow{\beta_{xn}} & E'_{xn} \\ g\big\downarrow & & h\big\downarrow \\ E_{yn} & \xrightarrow{\beta_{yn}} & E'_{yn} \end{array}$$

Where $E_{yn} = E_y \otimes_f W_{xn}$ and $E'_{yn} = E'_y \otimes_f W_{xn}$. We write $\mathrm{Isom}_n(x,y)$ for $\mathrm{Hom}_n(x,y)_{\deg 1}$, $\mathrm{Aut}_n(x,y)$ for $\mathrm{Isom}_n(x,x)$. If we drop subscript "n" in these definitions, we will obtain (old) sets $\mathrm{Hom}(x,y)_{\deg m}$, $\mathrm{Isom}(x,y)$, and $\mathrm{Aut}(x,y)$. These sets can be considered as subsets of previous ones with subscript "n". We will use supscript "new" to denote the complements of old subsets such as

$$\mathrm{Hom}_n^{\mathrm{new}}(x,y)_{\deg m} = \mathrm{Hom}_n(x,y)_{\deg m} \backslash \mathrm{Hom}(x,y)_{\deg m}.$$

Proposition 4.1.1. (a) If $x \neq y$ then

$$\langle S_k(x), S_k(y)\rangle = \frac{(-1)^k}{2} \sum_{\substack{n\geq 0 \\ \psi\in \mathrm{Isom}_n(x,y)}} S_k(y)_0 \cdot \psi_0^* S_k(x)_0.$$

Here for a morphism $\psi : x \to y$ over W_{xn} which induces particular a morphism $\psi_0 : E_{x0}^{2k-2} \to E_{y0}^{2k-2}$ over $W_{x0} = W_0$, the cycle $\psi_0^* S_k(x)_0$ is the pullback of $S_k(x)_0$. The intersection $S_k(y)_0 \cdot \psi_0^* S_k(x)_0$ is taking in E_{x0}^{2k-2}.
(b) If $x = y$ then

$$\langle S_k(x), S_k(x)\rangle = \mathrm{ord}\ d_x t.$$

Proof. Choose N' as above such that N' is prime to the characteristic of W_0. Write X' for $X(N') \otimes W$ and let $p : X' \to X$ be the canonical unramifield morphism.

Then $p^* x = u(x) \sum_i x_i$ and $p^* y = u(y) \sum_j y_j$, where $u(x) = |\mathrm{Aut}(x)/\pm 1|$, $u(y) = |\mathrm{Aut}(y)/\pm 1|$, and x_i and y_j are distinct components of $p^* x$ and $p^* y$ respectively. They are defined over W_x and W_y respectively, and represent all different $\Gamma(N')$ structures on E_x and E_y with fixed $\Gamma_0(N)$ structures respectively. By definition we have

$$\begin{aligned}\langle S_k(x), S_k(y)\rangle &= \left\langle \frac{u(x)}{\sqrt{\deg p}} \sum_i S_k(x_i), \frac{u(y)}{\sqrt{\deg p}} \sum_j S_k(y_j) \right\rangle \\ &= u(x) \sum_i \langle S_k(x_i), S_k(y_1)\rangle,\end{aligned}$$

as $\Gamma(N')$ acts transitively on x_i's.

If $x \neq y$, the proposition in the case of X' implies

$$\langle S_k(x_1), S_k(y)\rangle = \frac{u(x)}{2} \sum_{\substack{n\geq 0 \\ \phi\in \mathrm{Isom}_n(x_i,y_1)}} S_k(x_i)_0 \cdot \phi_0^* S_k(y_1)_0.$$

For any $\phi \in \mathrm{Isom}_n(x, y)$, the $\Gamma(N')$ structure of E_y corresponding to y_1 gives a $\Gamma(N')$ structure on x. This induces a map

$$\mathrm{Isom}_n(x, y)/\mathrm{Aut}(x) \to \coprod_i \mathrm{Isom}_n(x_i, y_1).$$

It is easy to see that this map is bijective. The first part of the proposition follows.

If $x = y$, we can choose a local coordinate $t_1 = t^{\frac{1}{u(x)}}$ for x_1. Write $\mathrm{div}\, t = x + x'$. Then $\mathrm{div}\, t_1 = \sum_i x_i + p^* x'/u(x)$. It follows that

$$\begin{aligned}\langle S_k(x), S_k(x)\rangle &= (-1)^k u(x)\Big[(S_k(x_1)\cdot S_k(x_1)) \\ &\qquad + \sum_{i>1}(x_i\cdot x_1)(S_k(x_i)_0\cdot S_k(x_1)_0)\Big] \\ &= u(x)\left(ord d_{x_1} t_1 - \sum_{i>1} x_i\cdot x_1\right) \\ &= u(x)x_1\cdot\left(\text{div } t_1 - x_1 - \sum_{i>1} x_i\right) \\ &= x_1\cdot p^*x' = x\cdot x' \\ &= \text{ord } (d_x t).\end{aligned}$$

The second part of the proposition follows. □

For archimedean place, we can also define $G_k(x,y)$ as in $X(N)$ case. We will also have

Proposition 4.1.2. *For two CM-points x and y on X, one has*

$$\langle S_k(x), S_k(y)\rangle = \frac{1}{2}G_k(x,y) \tag{4.1.1}$$

if $x\neq y$, and

$$\langle S_k(x), S_k(x)\rangle = \frac{1}{2}\lim_{y\to x}(G_k(x,y) - \log|t|^2(y)). \tag{4.1.2}$$

Proof. Let $N'\geq 3$ be a multiple of N. Let X' denote the modular curve $X(N')\otimes_{\mathbb{Z}[\zeta_N]}\mathbb{C}$. Consider the map $p: X'\to X$. For two CM-points x and y on X, write $p^*x = u(x)\sum_i x_i$ and $p^*y = u(y)\sum_j y_j$. Where $u(x)$ and $u(y)$ are the cardinalities of the stabilizers of Γ at x and y, and x_i and y_j are distinct components of p^*x and p^*y respectively. By definition, we have

$$\begin{aligned}\langle S_k(x), S_k(y)\rangle &= \left\langle \frac{u(x)}{\sqrt{\deg p}}\sum_i S_k(x_i), \frac{u(y)}{\sqrt{\deg p}}\sum_j S_k(y_j)\right\rangle \\ &= u(x)\sum_i\langle S_k(x_i), S_k(y_1)\rangle.\end{aligned}$$

We write Γ' for $\Gamma(N')$ and $G'_k(x,y)$ for green's function defined in X', and choose points z and z' in $\mathscr{H}$ projecting to x_1 and y_1. If $x\neq y$ one has

$$\begin{aligned}\langle S_k(x), S_k(y)\rangle &= \frac{1}{2}u(x)\sum_i G'_k(x_i, y_1) \\ &= \frac{1}{2}\sum_{\gamma''\in\Gamma'\backslash\Gamma}\sum_{\gamma'\in\Gamma'} g_k(\gamma'\gamma''z, z')\end{aligned}$$

$$= \frac{1}{2} \sum_{\gamma \in \Gamma} g_k(\gamma z, z')$$
$$= \frac{1}{2} G_k(x, y).$$

If $x = y$, then $t_1 = t^{\frac{1}{u(x)}}$ gives a local coordinate for x_1, and one has

$$\begin{aligned}
\langle S_k(x), S_k(x) \rangle &= \frac{1}{2} u(x) \left[\left\langle S_k(x_1), S_k(x_1) \rangle + \sum_{i \geq 2} \langle S_k(x_i), S_k(x_1) \right\rangle \right] \\
&= \frac{1}{2} u(x) \lim_{y \to x_1} \left(G_k'(x_1, y) - \log |t_1|^2(y) \right) \\
&\quad + \frac{1}{2} u(x) \sum_{i \geq 2} G_k'(x_i, x_1) \\
&= \frac{1}{2} \lim_{y \to x_1} \left[u(x) \sum_i G_k'(x_i, y) - \log |t_1|^{2u(x)} \right] \\
&= \frac{1}{2} \lim_{y \to x} \left[G_k(x, y) - \log |t|^2(y) \right]
\end{aligned}$$

The proposition follows. □

4.2. Heights of Heegner cycles. Let N be a positive integer and K an imaginary quadratic field with discriminant D, such that every prime factor of N splits in K. Let x be a Heegner point on $X_0(N) \otimes \bar{\mathbb{Q}}$ of discriminant D. This means that in the corresponding isogeny $E \to E'$, both E and E' have complex multiplications by the full ring of integers of K. Then x is rational over the Hilbert class field H of K. Let m be a integer prime to N and σ be an element in $\mathrm{Gal}(H/K)$.

Proposition 4.2.1. *Over* $X := X_0(N) \otimes H$, *with the notations are adopted from* [18], *one has*

$$\begin{aligned}
\langle S_k(x), \mathrm{T}_m S_k(x^\sigma) \rangle &= m^{k-1} \gamma_{N,k}^m(\mathscr{A}) + hur_{\mathscr{A}}(m) m^{k-1} \log \frac{N}{m} \\
&\quad - m^{k-1} u^2 \sum_{0 < n < \frac{m|D|}{N}} \sigma'_{\mathscr{A}}(n) r_{\mathscr{A}}(m|D| - nN) P_{k-1} \left(1 - \frac{2nN}{m|D|} \right).
\end{aligned}$$

As in weight 2 case [18], we will deduce this equality from the total local heights $\langle S_k(x), \mathrm{T}_m S_k(x^\sigma) \rangle_p$ over each place p of $\mathbb{Q}$, where

$$\langle S_k(x), \mathrm{T}_m S_k(x^\sigma) \rangle_p = \sum_{v | p} \langle S_k(x), \mathrm{T}_m S_k(x^\sigma) \rangle_v d_v,$$

where v runs through the set of places of H, and $d_v = 2$ if v is complex, and $d_v = [k(v) : \mathbb{Z}/p]$ if v is finite.

Proposition 4.2.2.

$$\langle x, \mathrm{T}_m x^\sigma \rangle_\infty = m^{k-1} \gamma_{N,k}^m(\mathscr{A}).$$

Proposition 4.2.3. *Let p be a prime.*

(a) *If p splits in K then*

$$\langle x, \mathrm{T}_m x^\sigma \rangle_p = -u r_{\mathscr{A}}(m) h \, \mathrm{ord}_p(m/N) m^{k-1}.$$

(b) *If p is inert in K then*

$$\begin{aligned}\langle x, \mathrm{T}_m x^\sigma \rangle_p = & - r_{\mathscr{A}}(m) hu \, \mathrm{ord}_p(m) m^{k-1} \\ & - u^2 m^{k-1} \sum_{\substack{0<n<\frac{m|D|}{N} \\ n \equiv 0 \bmod p}} \mathrm{ord}_p(pn) r_{\mathscr{A}}(m|D| - nN) \delta(n) \\ & \cdot R_{\{\mathscr{A}\mathfrak{q}\mathfrak{n}\}}(n/p) P_{k-1}\left(1 - \frac{2nN}{m|D|}\right),\end{aligned}$$

where $\mathfrak{q}$ *is an ideal of* $\mathcal{O}_K$ *whose norm is a prime q with property that* $\left(\frac{q}{l}\right) = \left(\frac{-p}{l}\right)$ *for all* $l|D$.

(c) *If p is ramified in K, then*

$$\begin{aligned}\langle x, \mathrm{T}_m x^\sigma \rangle_p = & - r_{\mathscr{A}}(m) hu \, \mathrm{ord}_p(m) m^{k-1} \\ & - u^2 m^{k-1} \sum_{\substack{0<n<\frac{m|D|}{N} \\ n \equiv 0 \bmod p}} \mathrm{ord}_p(pn) r_{\mathscr{A}}(m|D| - nN) \delta(n) \\ & \cdot R_{\{\mathscr{A}\mathfrak{q}\mathfrak{p}\mathfrak{n}\}}(n/p) P_{k-1}\left(1 - \frac{2nN}{m|D|}\right),\end{aligned}$$

where $\mathfrak{p}^2 = p$ *and* $\mathfrak{q}$ *is an ideal whose norm is a prime q such that* $\left(\frac{q}{l}\right) = \left(\frac{-1}{l}\right)$ *for all prime factor* $l \neq p$ *of D, and that* $\left(\frac{-q}{p}\right) = -1$.

Let

$$\langle S_k(x), \mathrm{T}_m S_k(x^\sigma) \rangle_{\text{finite}} := \sum_{p:\text{prime}} \langle S_k(x), \mathrm{T}_m S_k(x^\sigma) \rangle_p \log p.$$

As in [18], from the second part of Proposition (4.6) in p. 285 in [18], one can show that Proposition 4.1.3 implies

$$\begin{aligned}\langle S_k(x), \mathrm{T}_m S_k(x^\sigma) \rangle_{\text{finite}} = & \, hur_{\mathscr{A}}(m) m^{k-1} \log \frac{N}{m} \\ & - m^{k-1} u^2 \sum_{0<n<\frac{m|D|}{N}} \sigma'_{\mathscr{A}}(n) r_{\mathscr{A}}(m|D| - nN) P_{k-1}\left(1 - \frac{2nN}{m|D|}\right).\end{aligned}$$

Now Proposition 4.1.1 follows from this equality, Proposition 4.2.2, and the fact

$$\langle S_k(x), \mathrm{T}_m S_k(x^\sigma)\rangle = \langle S_k(x), \mathrm{T}_m S_k(x^\sigma)\rangle_\infty + \langle S_k(x), \mathrm{T}_m S_k(x^\sigma)\rangle_{\mathrm{finite}}.$$

4.3. Proof of Proposition 4.2.2. We identify the noncuspidal complex points of $X_0(N)$ with $\Gamma_0(N)\backslash\mathscr{H}$. As in (1.2) in [18] p. 235, the Hecke correspondence T_m ($m \in \mathbb{N}$, $(m,N) = 1$) acts on $X_0(N)$ by

$$\mathrm{T}_m(z) = \sum_{\substack{\gamma\in\Gamma_0(N)\backslash R_N \\ \det\gamma=m}} \gamma z$$

where $R_N = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & \mathbb{Z} \end{pmatrix}$. By Proposition 2.4.2 and Proposition 4.1.2, for any two point z and z' in $\mathscr{H}$, one has

$$\begin{aligned}\langle S_k(z), \mathrm{T}_m S_k(z')\rangle &= \frac{m^{k-1}}{2} G_k(z,z')\Big|_{z'} \mathrm{T}_m \\ &= \frac{m^{k-1}}{2} \sum_{\substack{\Gamma_0(N)\backslash R_N \\ \det\gamma=m}} G_k(z,\gamma z') \\ &= \frac{m^{k-1}}{2} \sum_{\substack{\gamma\in R_N/\pm 1 \\ \det r=m}} g_k(z,\gamma z').\end{aligned}$$

By (2.24) in [18] p. 242, we have

$$\langle S_k(z), \mathrm{T}_m S_k(z')\rangle = \frac{m^{k-1}}{2} G^m_{N,k}.$$

The set of Heegner points in $X(\mathbb{C})$ is parameterized by the set of pairs $(\mathscr{A},\mathfrak{n})$ with $\mathscr{A}$ an ideal class of $\mathcal{O}_K$ and $\mathfrak{n}$ an integral ideal of $\mathcal{O}_K$ of norm N in the way that each pair $(\mathscr{A},\mathfrak{n})$ corresponds to the point $\tau_{\mathscr{A},\mathfrak{n}}$ which represents the point $(\mathbb{C}/\mathfrak{a}, \mathfrak{a}\mathfrak{n}^{-1}/\mathfrak{a})$, where $\mathfrak{a}$ is an ideal in $\mathscr{A}$. If we identify $\mathrm{Gal}(H/K)$ with the group of ideal class via Artin map, then $\mathrm{Gal}(H/K)$ acts on the set of Heegner points by multiplication on $\mathscr{A}$ and trivially on $\mathfrak{n}$.

Assume $r_{\mathscr{A}}(m) = 0$ and $x = \tau_{\mathscr{A},\mathfrak{n}}$ for one embedding $H \subset \mathbb{C}$. Then

$$\begin{aligned}\langle S_k(x), \mathrm{T}_m S_k(x^\sigma)\rangle_\infty &= 2\sum_{v|\infty}\langle S_k(x), \mathrm{T}_m S_k(x^\sigma)\rangle_v \\ &= 2\sum_{\alpha\in\mathrm{Gal}(H/K)} \langle S_k(x^\alpha), \mathrm{T}_m S_k(x^{\alpha\sigma})\rangle \\ &= 2\sum_{\mathscr{C}\in\mathrm{Pic}(\mathcal{O}_K)} \langle S_k(\tau_{\mathscr{A}_0\mathscr{C},\mathfrak{n}}), \mathrm{T}_m S_k(\tau_{\mathscr{A}_0\mathscr{C}\mathscr{A},\mathfrak{n}})\rangle \\ &= 2\sum_{\substack{\mathscr{A}_1,\mathscr{A}_2\in\mathrm{Pic}(\mathcal{O}_K) \\ \mathscr{A}_1\mathscr{A}_2^{-1}=\mathscr{A}}} \langle S_k(\tau_{\mathscr{A}_1,\mathfrak{n}}), \mathrm{T}_m S_k(\tau_{\mathscr{A}_2,\mathfrak{n}})\rangle.\end{aligned}$$

By (3.3) in [18] p. 242, we have

$$\langle S_k(x), \mathrm{T}_m S_k(x^\sigma)\rangle_\infty = m^{k-1}\gamma_{N,k}^m.$$

Proposition 4.2.1 therefore follows from (3.17) in [18] p. 247.

Now we assume that $r_{\mathscr{A}}(m) \neq 0$. Here we observe that

$$\log|t(y)|_v - u\log|2\pi i\eta^4(z)(w-z)|_v \to 0$$

as $y \to x$, where z and w are points in the upper half-plane which map to x and y on $X_0(N)(\mathbf{C})$.

By Proposition 4.4.2, we find

$$\langle S_k(z), \mathrm{T}_m S_k(z')\rangle = \frac{m^{k-1}}{2}\left\{\sum_{\substack{\gamma\in R_N/\pm 1\\ \det\gamma=m\\ \gamma z'\neq z}} g_k(z,\gamma z') + ur_{\mathscr{A}}(m)\lim_{w\to z}\left[g_k(z,w) - \log|2\pi i\eta^4(z)(w-z)|^2\right]\right\}$$

By (5.7) in [18] p. 251, this is again $\frac{m^{k-1}}{2}G_{N,k}^m(z,z')$. Also by Proposition 5.8 in [18] p. 252, the same computation gives

$$\langle S_k(x), \mathrm{T}_m S_k(x^\sigma)\rangle_\infty = m^{k-1}\gamma_{N,k}^m.$$

4.4. Intersections and homomorphisms. Let v be a finite place of H, W the completion of the maximal unramified extension of H_v, and π a prime of W. In this subsection, we want to show the following formula for $\langle S_k(x), \mathrm{T}_m S_k(x^\sigma)\rangle_v$.

Proposition 4.4.1. *Let p denote the characteristic of the residue field of v.*

(a) *If p is inert in K then*

$$\langle S_k(x), \mathrm{T}_m S_k(x^\sigma)\rangle_v = \frac{(-1)^k}{2}\sum_{\substack{n\geq 0\\ \phi\in\mathrm{Hom}^{\mathrm{new}}_{W_n}(x^\sigma,x)_{\deg m}}} S_k(x^\sigma)_0\cdot\phi_0^* S_k(x)_0 - \frac{1}{2}ur_{\mathscr{A}}(m)\mathrm{ord}_p(m)m^{k-1}.$$

(b) *If p is ramified in K then*

$$\langle S_k(x), \mathrm{T}_m S_k(x^\sigma)\rangle_v = \frac{(-1)^k}{2}\sum_{\substack{n\geq 0\\ \phi\in\mathrm{Hom}^{\mathrm{new}}_{W/\pi^n}(x^\sigma,x)_{\deg m}}} S_k(x^\sigma)_0\cdot\phi_0^* S_k(x)_0 - ur_{\mathscr{A}}(m)\mathrm{ord}_p(m)m^{k-1}.$$

(c) *If* $p = \mathfrak{p}\bar{\mathfrak{p}}$ *is split in* K *and* $v|\mathfrak{p}$ *then*

$$\langle S_k(x), \mathrm{T}_m S_k(x^\sigma)\rangle_v = -ur_{\mathscr{A}}(m)\ \mathrm{ord}_p(m) j_{\mathfrak{p}} m^{k-1}.$$

where $j_{\mathfrak{p}}$ are integers depending on $\mathfrak{p}$ such that $j_{\mathfrak{p}} + j_{\bar{\mathfrak{P}}} = \mathrm{ord}_p(m/N)$.

Proof. We first assume that $r_{\mathscr{A}}(m) = 0$. If m is prime to p, then components of $\mathrm{T}_m(x^\sigma)$ are defined over W, and then for any $n \geq 0$, the canonical map

$$\coprod_{y \in \mathrm{T}_m(x^\sigma)} \mathrm{Isom}_n(y, x) \to \mathrm{Hom}_n(x^\sigma, x)_{\deg m}$$

is bijective. The proposition therefore follows from Proposition 4.1.1.

Now we write $m = p^t r$ with r prime to p. The points z in the divisor $\mathrm{T}_r x^\sigma$ are rational over W, but the points y in the divisor $\mathrm{T}_m x^\sigma = \sum_z \mathrm{T}_{p^t}(z)$ are rational over ramified extensions W_y of W. These y are the quasi-canonical liftings of their reductions. Let $y(s)$ be the divisor over W obtaining by taking the sum of a point of level s with all of its conjugates over W. When p is split in K the proposition is true since x and $\mathrm{T}_m(x^\sigma)$ has no intersection and $\mathrm{Hom}_n(x^\sigma, x)$ is empty for every n, as explained in [18].

Now we assume that p is inert in K. The number $\langle S_k(x), \mathrm{T}_m S_k(x^\sigma)\rangle_v$ is the sum of $r^{k-1}\langle S_k(x),\ \mathrm{T}_{p^t} S_k(z)\rangle_v$, and therefor the sum of $m^{k-1}\langle S_k(x), S_k(y(s))\rangle_v$, for all z in $\mathrm{T}_m(x^\sigma)$ and all irreducible components $y(s)$ of $T_{p^t} z$. If $s > 0$ and $n > 0$, then $\mathrm{Isom}_{W_n}(x, y(s))$ is empty. Since all $y(s)$ are congruent to a fixed y_0 of level 0, by Proposition 4.1.1, it follows that

$$\langle S_k(x), S_k(y(s))\rangle_v = \frac{(-m)^k}{2} \sum_{\substack{n \geq 1 \\ \psi \in \mathrm{Isom}_{W/\pi^n}(y_0, x)}} S_k(y_0)_0 \cdot \psi_0^* S_k(x)_0$$

if $s = 0$, and

$$\langle S_k(x), S_k(y(s))\rangle_v = \frac{(-m)^k}{2} \sum_{\psi \in \mathrm{Isom}_0(y_0, x)} S_k(y_0) \cdot \psi^* S_k(x)$$

if $s > 0$. Therefore $\langle S_k(x), \mathrm{T}_m S_k(x^\sigma)\rangle_v$ is the sum of $\frac{(-m)^k}{2}(S_k(y_0)_0 \cdot \psi_0^* S_k(x)_0)$ with ψ runs through a disjoint union of sets $\mathrm{Isom}_n(y_0, x)$ with certain multiplicities. The arguments in [18] pp. 260–261 yields a bijective map from this disjoint union to $\mathrm{Hom}(x^\sigma, x)_{\deg m}$, by composing each ψ in the disjoint union with a certain homomorphism ψ' from x^σ to y_0 defined over W. Since

$$m^{k-1}(S_k(y_0)_0 \cdot \psi_0^* S_k(x)) = S_k(x^\sigma) \cdot \psi'^* \psi^* S_k(x),$$

the proposition therefore follows.

Now we drop the assumption $r_{\mathscr{A}}(m) = 0$. The proof of the case $v \nmid mN$ is similar to the case $r_{\mathscr{A}}(m) = 0$. When $v \nmid N$ but $v|m$, the formula is changed slightly, because $ur_{\mathscr{A}}(m)$ elements in $\mathrm{Hom}_W(x^\sigma, x)_{\deg m}/(\pm 1)$ contribute to the isomorphism over W/π between quasi-canonical liftings in $\mathrm{T}_m x^\sigma$ to x. For any homomorphism ϕ from x^σ to x of degree m, $\phi^* S_k(x) = m^{k-1} S_k(x^\sigma)$.

The proposition follows in this case. □

4.5. Proof of Proposition 4.2.3. If p has a unique prime factor $\mathfrak{p}$ in K, then x and x^σ have supersingular reductions (mod π) and $\mathrm{End}_{W/\pi}(\bar{x}) = R$ is an order in the quaternian algebra B over $\mathbb{Q}$ which is ramified at ∞ and p. Then the embedding $\mathcal{O} = \mathrm{End}_W(\bar{x}) \to R = \mathrm{End}_{W/\pi}(\bar{x})$, given by the reduction of the endomorphisms, extends to a linear map $K \to B$. This in turn yields a decomposition

$$B = B_+ + B_- = K + Kj,$$

where j is an element in the nontrivial coset of $\mathrm{N}_{B*}(K^*)/K^*$. The decomposition is represented by the reduced norm

$$\mathrm{N}(b) = \mathrm{N}(b_+) + \mathrm{N}(b_-).$$

Proposition 4.5.1. *With notation as above, one has*

(a) *If p is inert in K then*

$$\langle S_k(x), \mathrm{T}_m S_k(x^\sigma)\rangle_v = -\sum_{\substack{b \in R\mathfrak{a}/\pm \\ \mathrm{N}b = m\mathrm{N}\mathfrak{a} \\ b_- \neq 0}} m^{k-1} P_{k-1}\left(\frac{\mathrm{N}b_+ - \mathrm{N}b_-}{\mathrm{N}b}\right)\frac{1}{2}\ \mathrm{ord}_p(p\mathrm{N}b_-)$$
$$-\frac{1}{2} ur_{\mathscr{A}}(m)\ \mathrm{ord}_p(m) m^{k-1}$$

(b) *If p is ramified in K then*

$$\langle S_k(x), \mathrm{T}_m S_k(x^\sigma)\rangle_v = -\sum_{\substack{b \in R\mathfrak{a}/\pm \\ \mathrm{N}b = m\mathrm{N}\mathfrak{a} \\ b_- \neq 0}} m^{k-1} P_{k-1}\left(\frac{\mathrm{N}b_+ - \mathrm{N}b_-}{\mathrm{N}b}\right)\mathrm{ord}_p(D\mathrm{N}b_-)$$
$$- ur_{\mathscr{A}}(m)\ \mathrm{ord}_p(m) m^{k-1}.$$

(c) *If $p = \mathfrak{p}\bar{\mathfrak{p}}$ is split in K as $v|\mathfrak{p}$, then*

$$\langle S_k(x), \mathrm{T}_m S_k(x^\sigma)\rangle_v = -ur_{\mathscr{A}}(m) j_{\mathfrak{p}} m^{k-1}.$$

where $j_{\mathfrak{p}} + j_{\bar{\mathfrak{p}}} = \mathrm{ord}_p(m/N)$.

To prove the proposition we need the explicit descriptions of

$$\mathrm{Hom}_n^{\mathrm{new}}(x^\sigma, x)_{\deg m} \text{ and } (S_k(x^\sigma)_0 \cdot \phi_0^* S_k(x)_0).$$

The first one is already given in [18]:

Proposition 4.5.2. (Gross-Zagier [18]). (a)

$$\mathrm{End}_{W/\pi^n}(x) = \left\{ b \in R, D\mathrm{N}b = 0 \bmod p(Np)^{n-1} \right\}$$

(b)

$$\mathrm{Hom}_{W/\pi^n}(x^\sigma, x) \simeq \mathrm{End}_{W/\pi^n}(x)\mathfrak{a}$$

in B*, where* $\mathfrak{a}$ *is any ideal in the class* $\mathscr{A}$*. If an isogeny* $\phi : x^\sigma \to x$ *corresponds to* $b \in B$*, then* $\deg \phi = \mathrm{N}b/\mathrm{N}\mathfrak{a}$.

Now we want to give a formula for $(S_k(x^\sigma)_0 \cdot \phi_0^* S_k(x)_0)$:

Proposition 4.5.3. *If* $b = b_+ + b_-$ *then*

$$(S_k(x)_0 \cdot b^* S_k(x)_0) = \left(\frac{-\mathrm{N}b}{\mathrm{N}\mathfrak{a}} \right)^{k-1} P_{k-1}\left(\frac{\mathrm{N}b_+ - \mathrm{N}b_-}{\mathrm{N}b} \right).$$

Proof. We want to apply Proposition 3.3.3. First of all we make identifications from H_{E_x} and $H_{E_{x^\sigma}}$ to H. Then there are u and v in F such that for any $\alpha \in \mathfrak{a}$ as an morphism from E_{x^σ} to E_x, one has

$$\alpha^*(X) = \alpha u X, \; \alpha^*(Y) = \bar{\alpha} v Y$$

Taking care of the pairings one has

$$(\alpha^*(X), \alpha^*(Y)) = \deg \alpha (X, Y),$$

so $\deg \alpha = \mathrm{N}(\alpha) uv$. It follows that the isogeny $b = b_+ + b_-$ has matrix

$$\begin{pmatrix} b_+/\sqrt{\mathrm{N}\mathfrak{a}} & b_-/\sqrt{\mathrm{N}\mathfrak{a}} \\ -\bar{b}_-/\sqrt{\mathrm{N}\mathfrak{a}} & \bar{b}_+/\sqrt{\mathrm{N}\mathfrak{a}} \end{pmatrix}.$$

The proposition follows from Proposition 3.3.3. □

When p is split in K, Proposition 4.5.1 follows from Proposition 4.4.1.(c). Assume p is inert or ramified in K. The first term of the expression is easy to obtain. We compute the sum of the second term in Proposition 4.5.1 over all places v with weights $d_v = [k(v) : \mathbb{Z}/p]$ as follows.

Fix a place v of H over a prime p. In the proof of proposition (9.2) in [18], p. 265–266, Gross and Zagier gave the following description for the set

$$S_v = \{ b \in R\mathfrak{a}/\pm 1, \mathrm{N}b = m\mathrm{N}\mathfrak{a}, b_- \neq 0 \}.$$

First of all, S_v is a disjoint union of subsets

$$S_{v,n} = \left\{ b \in R\mathfrak{a}/\pm 1, \mathrm{N}b_+ = \frac{m|D| - nN}{|D|}\mathfrak{a}, \mathrm{N}b_- = \frac{nN}{|D|}\mathrm{N}\mathfrak{a} \right\}$$

indexed by the set

$$I = \left\{ n \in \mathbb{Z}, 0 < n < \frac{m|D|}{N}, n \equiv 0(p) \right\}.$$

It follows that the sum we want to compute is

$$\sum_{n \in I} m^{k-1} P_{k-1}\left(1 - \frac{2nN}{|D|}\right) \frac{1}{2} \operatorname{ord}_p(pn) \sum |S_{v,n}| d_v \tag{4.5.1}$$

when p is inert, and

$$\sum_{n \in I} m^{k-1} P_{k-1}\left(1 - \frac{2nN}{|D|}\right) \frac{1}{2} \operatorname{ord}_p(n) \sum |S_{v,n}| d_v \tag{4.5.2}$$

when p is ramified.

For a choosing $\mathfrak{q}$ as in the proposition, Gross and Zagier have shown

$$\sum |S_{v,n}| d_v = 2u^2 \delta(n) r_{\mathscr{A}}(m|D| - nN) R_{\mathscr{A}\mathfrak{q}\mathfrak{p}\mathfrak{n}}(n/p). \tag{4.5.3}$$

Proposition 4.5.1 follows from (4.5.1), (4.5.2), and (4.5.3).

5. Proof of the main identity and the consequences

5.1. Main identity. Let N be a positive integer, K an imaginary quadratic field, such that every prime divisor of N splits in K. Let D be the discriminant of K and H the Hilbert class field of K. Let $x \in X_0(N)(H)$ be one of the Heegner points associated to K, $\mathscr{A}$ an ideal class of K, and σ the corresponding element of $G = \mathrm{Gal}(H/K)$.

As in weight 2, let $f = \sum_{n \geq 1} a_n e^{2\pi i n z}$ be an element in the space of new form of weight $2k$ on $\Gamma_0(N)$. Define the L-series associated to f and $\mathscr{A}$ by

$$L_{\mathscr{A}}(f, s) = \sum_{\substack{n \geq 1 \\ (n, ND) = 1}} \epsilon(n) n^{-2s+2k-1} \sum_{m \geq 1} a(m) r_{\mathscr{A}}(m) m^{-s}$$

where $\epsilon(n) := \left(\frac{D}{n}\right)$ the associated quadratic character of K.

Theorem 5.1.1 (Gross-Zagier [18]). (a) *The function $L_{\mathscr{A}}(f, s)$ has an analytical continuation to the entire complex plane, satisfies the functional equation*

$$L^*_{\mathscr{A}} := (2\pi)^{-2s} N^s |D|^s \Gamma(s)^2 L_{\mathscr{A}}(f, s) = -L^*_{\mathscr{A}}(f, 2k - s).$$

In particular, $L_{\mathscr{A}}(f, k) = 0$.

(b) *There is a holomorphic cusp form $\Phi = \sum_{m \geq 1} a_{m,\mathscr{A}} q^m \in S_{2k}(\Gamma_0(N))$ satisfying*

$$(f,\Phi) = \frac{(2k-2)!\sqrt{|D|}}{2^{4k-1}\pi^{2k}} L'_{\mathscr{A}}(f,k)$$

for all new form $f \in S_{2k}^{\text{new}}(\Gamma_0(N))$ and with $a_{m,\mathscr{A}}$ (m prime to N) given by

$$\begin{aligned} a_{m,\mathscr{A}} = & \frac{m^{k-1}}{u^2} r_{N,k}^m(\mathscr{A}) + \frac{h}{u} r_{\mathscr{A}}(m) m^{k-1} \log \frac{N}{m} \\ & - m^{k-1} \sum_{0<n<\frac{m|D|}{N}} \sigma_{\mathscr{A}}(m|D| - nN) P_{k-1}\left(1 - \frac{2nN}{m|D|}\right). \end{aligned}$$

By Proposition 4.2.1, we therefore prove Theorem 0.2.1 in Introduction. Now we want to prove Theorem 0.3.1. By Theorem 0.2.1, it suffices to show the following.

Proposition 5.1.2. *Let $v_1, \cdots, v_h$ be CM-cycles of weight $2k$ for $\Gamma_0(N)$. Assume for each i,j there is an element $g_{i,j} \in S_{2k}(\Gamma_0(N))$ such that for any positive integer m prime to N the number $\langle v_i, \mathrm{T}_m(v_j)\rangle$ is the m-th coefficient of $g_{i,j}$. Let V be the subspace of Heegner cycles generated by*

$$\{\mathrm{T}_m v_j | 1 \le j \le h, (m,N) = 1\}.$$

Let V' be the quotient of V modulo the null subspace with respect to the height pairing on $V \times V$. Then the Hecke module V' is isomorphic to a sub-quotient module of $S_{2k}(\Gamma_0(N))^{\oplus h}$.

Proof. Define the action of Hecke algebra $\mathbb{T}$ (generated by correspondences T_m, $(m,N) = 1$) on $\mathbb{C}[[q]]$ by the usual formula:

$$\mathrm{T}_m\left(\sum_n a(n)q^n\right) = \sum_n \left[\sum_{d|(m,n)} d^{k-1} a\left(\frac{mn}{d^2}\right)\right] q^n.$$

Let S' be the image of $S_{2k}(\Gamma_0(N))$ of the map

$$\sum_n a(n)e^{2\pi nz} \to \sum_{(n,N)=1} a(n)q^n.$$

Define h map ϕ_i $(1 \le i \le h)$ from V to $\mathbb{C}[[q]]$ by

$$v \to \phi_i(v) = \sum_{\substack{m \ge 1 \\ (m,N)=1}} \langle v, \mathrm{T}_m(v_i)\rangle q^m.$$

It is easy to see that these maps induce an embedding of V' into $\mathbb{C}[[q]]^{\oplus h}$ as $\mathbb{C}$-vector space. To prove the theorem it suffices to show that ϕ_i is a morphism of Hecke modules and the image of ϕ_i is in S'.

Now for any $(m, N) = 1$ and any $v \in V$, since T_m is self-adjoint on V with respect to the height pairing, it follows that

$$\phi_i(\mathrm{T}_m(v)) = \sum_{(n,N)=1} \langle \mathrm{T}_m(v),\, \mathrm{T}_n(v_i)\rangle q^n = \sum_{(n,N)=1} \langle v, \mathrm{T}_m \mathrm{T}_n(v_i)\rangle q^n.$$

To show that ϕ_i is a morphism of Hecke modules it suffices to show that the action of Hecke algebra on V satisfies the property

$$\mathrm{T}_m \mathrm{T}_n = \sum_{d|(m,n)} d^{k-1} \mathrm{T}_{\frac{mn}{d^2}}.$$

But this follows from the corresponding property of the Hecke correspondences on $X_0(N)$, and from Proposition 2.4.2, where we have shown that for any CM-point on $X_0(N)$,

$$\mathrm{T}_m s_k(x) = m^{k-1} s_k(\mathrm{T}_m(x)).$$

This finishes the proof of that ϕ is a morphism of Hecke modules.

By the assumption of the theorem $\phi_i(v_j)$ $(1 \le j \le h)$ is in S'. It follows that $\phi_i(V)$ is in S' as $\phi_i(V)$ is generated by $\{\phi_i(x_j) : j = 1, \cdots, h\}$ as a Hecke module. This completes the proof of the theorem. □

The proof of corrolaries in Introduction uses the same argument as Gross-Zagier. We omit all details.

5.2. Algebraicity conjecture. Recall from [18] that a relation of $S_{2k}^{\mathrm{new}}(\Gamma_0(N))$ is a sequence of integers $\lambda = (\lambda_m)_{m \ge 1}$ satisfying the following conditions:

(a) $\lambda_m \in \mathbb{Z}$, $\lambda_m = 0$ for all but finitely many m.

(b) $\sum_{m \ge 1} \lambda_m a_m = 0$ for any cusp form $\sum a_m q^m \in S_{2k}(\Gamma_0(N))$.

(c) $\lambda_m = 0$ for m not prime to N or $r_{\mathscr{A}}(m) \ne 0$.

We write

$$G_{N,k,\lambda}(z, w) = \sum_{m=1}^{\infty} \lambda_m m^{k-1} G_{N,k}^m(z, w).$$

Conjecture 5.2.1 (Gross and Zagier) *Let $\lambda = (\lambda_m)_{m \ge 1}$ be a relation for $S_{2k}^{\mathrm{new}}(\Gamma_0(N))$. Fix a Heegner point x and an embedding $H \subset \mathbb{C}$. Then there exists an element $\alpha \in H$ such that*

$$G_{N,k,\lambda}(x^\tau, x^{\sigma\tau}) = u^2 D^{1-k} \log |\alpha^\tau|$$

for all $\tau \in G = \mathrm{Gal}(H/K)$.

Theorem 5.2.2. *Assume that the height pairing on CM-cycles is non-degenerate. Fix a Heegner point x and an embedding $H \subset \mathbb{C}$. Then there exist a rational number r and an element $\alpha \in H$ such that*

$$G_{N,k,\lambda}(x^{\tau}, x^{\sigma\tau}) = r \log |\alpha^{\tau}|$$

for all $\tau \in G = \mathrm{Gal}(H/K)$.

Proof. The condition on λ implies that $\sum_m \lambda_m s_k'(x) = 0$ in $\mathrm{Ch}^*(Y)$. It follows that there are subvarieties W_i of Y and functions f_i on W_i such that

$$\sum \lambda_m \mathrm{T}_m S_k(x) = c \sum_i \mathrm{div}(f_i | Wi)$$

where c is a rational number. Now for any embedding $H \hookrightarrow \mathbb{C}$ inducing an infinite place v of H, and any $\sigma \in \mathrm{Gal}(H/K)$, the local height pairing is given by

$$\begin{aligned} \left\langle \sum_{m \geq 1} \lambda_m \mathrm{T}_m s_k(x), s_k(x^{\sigma}) \right\rangle_v &= c \sum_i \log | f_i(S_k(x^{\sigma}) \cdot W_i)|_v \\ &= c \log \left| \prod_i f_i(S_k(x^{\sigma}) \cdot W_i) \right|_v . \end{aligned}$$ □

5.3. Beilinson-Bloch conjecture. We will combine our result with a result of Nekovář to give some application to Beilinson-Bloch conjecture. We start with a review of a result of Nekovář.

Let $f = \sum_{n \geq 1} a_n q^n \in S_{2k}^{\mathrm{new}}(\Gamma_0(N))$ be a normalized eigenform of weigh $2k$ on $\Gamma_0(N)$ with coefficients in $\mathbb{Q}$. Let $M = M(f)$ be the Grothendieck motive over $\mathbb{Q}$ constructed by U. Jannsen and A.J. Scholl. The l-adic realization M_l is a two dimensional representation of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ corresponding to f and appearing as a factor in the cohomology group $H_{\text{et}}^{2k-1}(Y \otimes \bar{\mathbb{Q}}, \mathbb{Q}_l)$, where $Y = Y(N') \otimes \mathbb{Q}$ for multiple N' of N such that N' is a product of two relatively prime integers ≥ 3. For any number field F, let $\mathrm{Ch}^k(Y_F)_0$ be the group of homologically trivial cycles of codimension k in Y_F modulo the rational equivalence. The l-adic Abel-Jacobi map

$$\Phi_F : \mathrm{Ch}^k(Y_F)_0 \otimes \mathbb{Q}_l \to H^1_{\mathrm{cont}}(F, H_{\text{et}}^{2k-1}(Y \otimes \bar{\mathbb{Q}}, \mathbb{Q}_l)(k))$$

induces a map

$$\Phi_{F,f} : \mathrm{Ch}^k(Y_F)_0 \otimes \mathbb{Q}_l \to H^1_{\mathrm{cont}}(F, M_l(k)).$$

Conjecture 5.3.1. (Beilinson and Bloch) *The dimension of* $\mathrm{im}\,\Phi_{F,f}$ *is equal to the order of* $L(f \otimes F, s)$ *at* $s = k$.

If $F = K$, x a Heegner point on X as before then

$$s_K = \sum_{\sigma \in \mathrm{Gal}(H/K)} s_k(x^\sigma)$$

defines a Heegner cycle in $\mathrm{Ch}^k(Y_K)_0 \otimes \mathbb{Q}_l$.

Theorem 5.3.2. (Nekovář [30] [31]). *Suppose that l does not divide $2N$. If $y_0 = \Phi_f(s_K)$ is nonzero then* im $\Phi_f = \mathbb{Q}_l y_0$, *and an analogue of the l-primary part of the Tate-Shafarevich group is finite.*

By a theorem of Scholl, we know that $s_k(X_K)$ is included on $\mathrm{Ch}^k(Y_K)_0 \otimes \mathbb{R}$.

Theorem 5.3.3. *Assume that $k > 1$. If Φ_K is injective on the subgroup $s_k(X_K)$ of CM-cycles for every imaginary quadratic field K, then the equality*

$$\mathrm{rank}_{\mathbb{Q}_l}\ \mathrm{im}(\Phi_{f,\mathbb{Q}}) = \ \mathrm{ord}_{s=k} L(f,s)$$

holds if $\mathrm{ord}_{s=k} L(f,s) \leq 1$ *and l does not divide $2N$.*

Proof. If $\mathrm{ord}_{s=k} L(f,s) \leq 1$ then this number is 0 or 1 depending totally on the sign of the functional equation of $L(f,s)$. By the theorems of Waldspurger [36], Murty-Murty [29], and Bump-Friedberg-Hoffstein [6], one has infinitely many imaginary quadratic fields K in which every prime factor of N is split, such that

$$\mathrm{ord}_{s=k} L_\epsilon(f,s) + \mathrm{ord}_{s=k} L(f,s) = 1,$$

where $L_\epsilon(f,s) = \sum \epsilon(n) a_n n^{-s}$ is the L-function of f twisted by the character $\epsilon(n) = \left(\frac{D}{n}\right)$ associated to K. It follows that $\mathrm{ord}_{s=k} L(f,\ \mathrm{Id}, s) = 1$ as

$$L(f, \mathrm{Id}, s) = L(f \otimes K, s) = L_\epsilon(f,s) L(f,s).$$

By Corollary 5.1.6, the Heegner point $s_{K,f}$ is not zero, so is $y_0 = \Phi_{K,f}(s_{K,f})$. Then by Nekovář's theorem, im $\Phi_{f,K} = \mathbb{Q}_l y_0$. Let τ be the nontrivial element in $\mathrm{Gal}(K/\mathbb{Q})$, then $\tau y_0 = \pm y_0$ with opposite sign as the functional equation for $L(f,s)$. It follows that $\dim \mathrm{im}\ \Phi_{f,\mathbb{Q}} = \mathrm{ord}_{s=k} L(f,s)$. □

References

1. S. J. Arakelov, Intersection theory of divisors on an arithmetic surfaces, Math. U.S.S.R. Izvestija **8**, pp. 1167–1180 (1974)
2. A. A. Beilinson, Heights pairing between algebraic cycles, Comtemp. Math. **67**, pp 1–24 (1987)
3. S. Bloch, Height paring for algebraic cycles, J. Pure Appl. Algebra **34**, pp. 119–145 (1984)
4. S. A. Bloch and O. Ogus, Gerstein's conjecture and homology of schemes, Ann. Sci. Éc. Norm. Super., IV. Ser. **7**, pp. 181–202 (1974)
5. J.-L. Brylinski, Heights for local system on curves, Duke Math. J. **59**, pp. 1–26 (1989)
6. D. Bump, S. Friedberg, and J. Hoffstein, Non-vanishing theorems for L-functions of modular forms and their derivatives, Inventiones Math. **102**, 543–618 (1990)

7. P. Deligne, Formes modulaire et représentation l-adiques, Séminaire Bourbaki 1968/69 exp. 355. (Lect. Notes Math., vol 179, pp. 139–172) Berlin-Heidelberg-New York: Springer 1971
8. P. Deligne, Le déterminant de la cohomologie, Comtemporary Mathematics, vol 67, pp. 93–178
9. P. Deligne and M. Rapoport, Les schémas de modules de courbes elliptiques, in: Modular function of one variable II (ed. P. Deligne, W. Kuyk, Lect. Notes Math. vol 349, pp. 143–316), Berlin-Heidelberg-New York: Springer 1973
10. G. Faltings, Calculus on arithmetic surfaces, Ann. of Math. **119**, pp. 387–424 (1984)
11 G. Faltings, Lectures on arithmetic Riemann-Roch theorem Ann. Math. studies **127** (1992)
12 Fulton, Intersection theory Ergebn. Math. Grenzgeb. (3), Band 2, Springer-Verlag, Berlin, Heidelberg, and New York (1984)
13 H. Gillet and C. Soulé, Arithmetic intersection theory Publ. Math. I.H.E.S **72**, pp. 94–174 (1990)
14. H. Gillet and C. Soulé, Arithmetic analog of the standard conjectures Proceedings of Symposia in mathematics, vol 55, part 1, (1994)
15. D. Goldfeld, The class numbers of quadratic fields and the conjectures of Birch and Swinnerton-Dyer, Ann. Sc. Norm. Super. Pisa **3**, pp. 623–663 (1976)
16. B. H. Gross, On canonical and quasi-canonical liftings, Invent. Math. **84**, pp. 321–326 (1986)
17. B. H. Gross, Local heights on curves in: Arithmetic Geometry (ed. Cornell and Silverman) Springer-Verlag
18. B. H. Gross and D. B. Zagier, Heegner points and derivatives of L-series, Invent. Math. **84**, pp. 225–320 (1986)
19. B. Harris and B. Wang, Archimedean height pairing of intersection cycles, Internat. Math. Res. Notices **4**, pp. 107–111 (1993)
20. D. A. Hejhal,The Selberg trace formula for $PSL(2,\mathbb{R})$, Lect. Notes Math. **1001**. Berlin-Heidelberg-New York-Tokyo: Springer 1983
21. P. Hriljac, Heights and Arakelov's intersection theory, Amer. J. math. **107**, pp. 23–38
22. U. Jannsen, Mixed Motives and algebraic K-theory. Lect. Notes Math, vol. 1400 Berlin-Heidelberg-New York: Springer 1990
23. N. Katz and B. Mazur, Arithmetic moduli of elliptic curves, Ann. Math. Study. **108**. Princeton University Press 1985
24. V. A. Kolyvagin, Euler system, in : The Grothendieck Festschrift, vol. II. (Prog. Math), vol. 87, pp. 435-483) Boston Basel Berlin: Birkhäuser 1990
25. K. Künnermann, Some remarks on the arithmetic Hodge index theorem, Preprint Münster (1994)
26. K. Künnermann, Higher picard varieties and the height pairing, Preprint (1995)
27. S. Lang, Introduction to Arakelov theory, Springer-Verlag (1988)
28. A. Moriwaki, Remarks on arithmetic analogues of Grothendieck's standard conjectures, Preprint UCLA (1994)
29. M. R. Murty and V. K. Murty, Mean values of derivatives of modular L-series. Ann. Math. **133**, 447–475 (1991)
30. J. Nekovar, Kolyvagin's method for Chow groups of Kuga-Sato varieties, Invent. Math. **107**, pp. 99–125 (1992)
31. J. Nekovar, On the p-adic heights of Heegner cycles, Math. Ann. **302**, 609–686 (1995)
32 B. Perrin-Riou, Points de Heegner et dérivées de fonction L p-adiques, Invent. Math. **89**, 455–510 (1987)
33. A. J. Scholl, Motives for modular forms, Invent. Math. **100**, pp. 419–430 (1990)
34. A. J. Scholl, Height pairings and special values of L-functions, Proc. Sym. AMS. 55, part 1, 571–598 (1994)
35. L. Szpiro, Degrés, intersections, hauteurs, in Séminaire sur les pinceaux arithmétiques: la conjecture de Mordell, Astérisque **127**, pp. 11–28 (1985)
36. J.-L. Waldspurger, Sur les values de certaines fonctions L automorphes en leur centre de symmetre, Comp. math. **54**, pp. 173–242 (1985)