GENERIC ABELIAN VARIETIES WITH REAL MULTIPLICATION ARE NOT JACOBIANS

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Section 1. Introduction

In this note everything is over the complex numbers $\mathbf{C}$. Let us first introduce some notation:

- g: a positive integer ≥ 4.
- A_g: the coarse moduli scheme of principally polarized abelian varieties of dimension g over $\mathbf{C}$.
- M_g: the coarse moduli scheme of nonsingular projective genus g curves.
- M_g^c: the Zariski closure of (the image) M_g in A_g. (It turns out that $M_g \to A_g$ is an immersion, see [OS79].)
- $A_g^{\rm dec} \subset A_g$: the locus of decomposable principal polarized abelian varieties; an $(A, \lambda : A \to A^t) \in A_g(\mathbb{C})$ is said to be *decomposable* if $A = A_1 \times A_2$ with $\lambda(A_1 \times 0) \subset A_1^t$.

Our main problem is to decide when a connected Shimura subvariety X of A_g is included in M_g^c so that the intersection of X with M_g is non-empty. Recall that X is by definition a connected component of a Shimura variety defined by a reductive subgroup G of GSp_g over $\mathbb{Q}$. Let $G^{\rm ad} = \prod_i G_i^{\rm ad}$ be the decomposition of $G^{\rm ad}$ into product of simple group and let G_i be the pre-image of $G_i^{\rm ad}$ in G. Then a finite covering X' will have decomposition $X' = \prod X_i'$ where X_i' are Shimura varieties associated to G_i. Thus replacing G by G_i we may assume that X is simple in the sense that $G^{\rm ad}$ is simple.

Our first main result is as follows:

Theorem 1.1. *Let X be a Shimura subvariety of A_g defined by a reductive subgroup G of $\mathrm{GSp_g}$ over $\mathbb{Q}$. Assume that G^{ad} is $\mathbb{Q}$-simple and that X is included in M_g^c such that $X \cap M_g$ is not empty. Then one of the following three conditions holds:*

(1) *the hermitian symmetric space covering X is a disc in some $\mathbb{C}^n$, or*
(2) *$X \cap A_g^{dec}$ has codimension ≤ 2, or*
(3) *the Baily-Borel-Satake compactification of X has boundary with codimension ≤ 2.*

It is complicated to list all Shimura varieties satisfying the conditions in the theorem. In the following we would like to apply our theorem to the Hilbert modular varieties. So we introduce the following notation:

- let F denote a totally real number field of degree g.
- let $\mathcal{O}$ denote an order in F.
- let $A_g^{\mathcal{O}}$ denote the subvariety of A_g corresponding to abelian varieties A such that $\mathrm{Hom}(\mathcal{O}, \mathrm{End}(A)) \neq 0$.

The above theorem gives the following:

Corollary 1.2. *Assume that $g \geq 4$. Let X be a component of $A_g^{\mathcal{O}}$. Then X is not included in M_g^c except the following possible case:*

(*) *F is a quadratic extension of a real quadratic field.*

Combining with an equidistribution theorem [Zha05] on CM-points on quaternion Shimura varieties, we obtain the following finiteness result on the CM-points in A_g:

Theorem 1.3. *Let K be an imaginary quadratic extension of F with a CM-type S_K such that the corresponding Mumford-Tate group $MT(K, S_K)$ is maximal. Then the set of curves $[C] \in M_g^c$ such that $\mathrm{Jac}(C)$ has a multiplication by an order K containing $\mathcal{O}$ with CM-type S_K is finite, except the the possible case (*) as in the above corollary.*

Here are few words about the definition about the maximality of the Mumford-Tate group. Let A be a complex abelian variety of dimension g. Then the complex structure on $\mathrm{Lie}(A) \simeq H_1(A, \mathbb{R})$ defines a homomorphism $\mathbb{C}^\times \longrightarrow \mathrm{GL}(H_1(A, \mathbb{R}))$. The Mumford-Tate group $MT(A)$ of A is the minimal algebraic subgroup H of $\mathrm{GL}(H_1(A, \mathbb{Q}))$ defined over $\mathbb{Q}$ such that $MT(A)(\mathbb{R})$ contains the image of $\mathbb{C}^\times$. Assume that A has multiplication by an order in K, then $\mathrm{Lie}(A) \simeq \mathbb{C}^g$ is a K-module. The trace of an $x \in K$ is given by g-embeddings $\phi_i : K \longrightarrow \mathbb{C}$:

$$\mathrm{tr}(x|\mathrm{Lie}(A)) = \sum_i \phi_i(x).$$

The set of ϕ_i is called the CM-type. In this case, the Mumford-Tate group $MT(A)$ is an algebraic subgroup of $K^\times$ determined completely by the type S_K; so we may write it as $MT(S_K)$. We say that $MT(S_K)$ is maximal if $MT(S_K) \cdot F^\times$ generates $K^\times$ as an algebraic group over $\mathbb{Q}$.

Example 1.4. Assume that $\ell := 2g + 1$ is a prime number. For each integer a between 1 and g, let us define curve C_a of genus g as follows:

$$C_a : \qquad y^\ell = x^a(1 - x).$$

Then $\mathrm{Jac}(C_a)$ has CM by $K := \mathbb{Q}(\zeta_\ell)$. To describe the CM-type of $\mathrm{Jac}(C_a)$, we notice that all complex embeddings of K are indexed by $t \in \mathrm{Gal}(K/\mathbb{Q}) \simeq (\mathbb{Z}/p\mathbb{Z})^\times$

which bring ζ to ζ^t. The CM-type T_a of $\mathrm{Jac}(C_a)$ is the subset of $t \in (\mathbb{Z}/p\mathbb{Z})^\times$ such that

$$\left\langle \frac{at}{\ell} \right\rangle + \left\langle \frac{t}{\ell} \right\rangle < 1 \tag{1.1}$$

where $< x >$ denote the decimal part of a number $x \in \mathbb{R} \geq 0$. See [Wei76] for details. We will show that $MT(T_a)$ is maximal if g is not a multiple of 3. So our theorem 1.3 in some sense shows that there only finitely many curves isogenous $[C_a]$ with any fixed action by an order $\mathcal{O}$ of F.

To conclude this introduction, let us mention that the following recent work of M. Möller, E. Viehweg, and K. Zuo:

Theorem 1.5 (Möller-Viehweg-Zuo [MVZ])**.** *Let $g > 1$ be an integer and $\mathcal{M}_g$ be the moduli stack of curves of genus g. Then*

1. *$\mathcal{M}_g$ does not contain any compact Shimura curve;*
2. *For $g \neq 3$, $\mathcal{M}_g$ does not contains any non-compact Shimura curve;*
3. *For $g = 3$, there is essentially a unique Shimura curve C in $\mathcal{M}_3$ which is defined by the embedding $\mathbb{P}^1 \setminus \{0, 1, \infty\} \longrightarrow \mathcal{M}_g$ by parameterizing the curve*

$$y^4 = x(x-1)(x-t), \qquad t \in \mathbb{P}^1 \setminus \{0, 1, \infty\}.$$

Replacing $\mathcal{M}_g$ by $M_{g,n}$, we may reformulate the Theorem about non-existence of Shimura curves in any $M_{g,n}$ which *represents a family of curves*. As we see in the next section, any Shimura curve in $M_{g,n}$ is representable if it is either disjoint from the hyper-elliptic locus or included in the hyper-elliptic locus. Besides the representability problem, another difference with our formulation is that their result is about $M_{g,n}$ rather than its closure in $A_{g,n}$.

Section 2. Mapping class groups and hyperelliptic locus

In this section, we want to prove Theorem 1.1 and its Corollary 1.2. We will use ideas of Hain in [Ric99]. More precisely, we want to show that under all three conditions of the theorem and after taking a covering $X_n \longrightarrow X$ classifying level structures, there is a closed subset Y of X_n with dimension ≤ 2 such that its complement $X_n \setminus Y$ parameterizes curves $\mathcal{C}$. Let $x \in U$ then the monodromy action induces homomorphisms

$$\pi_1(X_n, x) = \pi_1(U, x) \longrightarrow \mathrm{Aut}(\pi_1(\mathcal{C}_x)) \longrightarrow \mathrm{Mod}(\mathcal{C}_x)$$

where $\mathrm{Mod}(\mathcal{C}_x)$ is the mapping class group, i.e., the quotient of $\mathrm{Aut}(\mathcal{C}_x)$ modulo the inner automorphisms. Notice that $\pi_1(X_n, x)$ is a lattice in a reductive group of rank ≥ 2 under the assumptions of the theorem. It follows from a theorem of Faber and Masur [FM98] that the image of the composition of the above morphisms is finite. Thus the family is isotrivial and we then have a contradiction. The main obstruction for the existence of the family $\mathcal{C}$ is the possible presence of the hyper-elliptic locus of codimension 1. We prove the non-existence of such locus using the fact that the hyper-elliptic locus is always affine. This is probably the only new (but trivial) idea not included in Hain's paper.

Subsection 2.1. **Mapping class groups.** We will use some ideas in Hain's paper [Ric99] which we explain here. First, there is a reference to the paper [FM98] of Farb and Masur. Theorem 1.1 of [FM98] implies that if Γ is an irreducible lattice in a semisimple Lie group G of real rank ≥ 2 then any homomorphism $\Gamma \to \mathrm{Mod}(S_g)$ has finite image. Here $\mathrm{Mod}(S_g)$ is the mapping class group of a compact Riemann surface of genus g. In particular, if X is a nonsingular complex algebraic variety with $\pi_1(X) = \Gamma$ then any family of smooth projective curves $\mathcal{C} \to X$ has to be topologically isotrivial.

Take an integer n and consider the moduli space $A_{g,n}$ of principally polarized abelian varieties with symplectic level n structure. There is a finite morphism $A_{g,n} \to A_g$. Similarly we have the moduli space $M_{g,n}$, and a finite morphism $M_{g,n} \to M_g$. Whence the diagram:

$$\begin{array}{ccc} M_{g,n} & \longrightarrow & A_{g,n} \\ \downarrow & & \downarrow \\ M_g & \longrightarrow & A_g \end{array}$$

It is no longer true that $M_{g,n} \to A_{g,n}$ is an immersion, namely it ramifies exactly along the hyperelliptic locus $H_{g,n} \subset M_{g,n}$. See [OS79]. Finally, let $M^c_{g,n}$ denote the total inverse image of M^c_g.

Let $M^b_g \subset M_g$ denote the locus of "good" stable curves (sometimes called "compact type"); these are the stable curves C so that $\mathrm{Pic}^{00}(C)$ is an abelian variety of dimension g. Similarly there is a moduli space $M^b_{g,n}$ of good curves of genus g with a symplectic level n system; this is a smooth quasi-projective variety. There is a surjective projective morphism

$$M^b_{g,n} \longrightarrow M^c_{g,n}.$$

This morphism has positive dimensional fibres over the points in the image corresponding to decomposable principally polarized abelian varieties.

The remarks above imply that $M_{g,n} \to A_{g,n}$ is an immersion over the complement of $H_{g,n}$.

Now, let's go back to the situation of our theorem, and suppose that we have $X \subset M^c_g$ with real rank ≥ 2. Consider an irreducible component $X_n \subset A_{g,n}$ of the full inverse image of X. The first idea of the paper of Hain is that if X misses the locus of decomposable polarized abelian varieties and if it misses the locus of hyperelliptic Jacobians then actually there is a universal family of curves $\mathcal{C} \to X_n$. This means we can apply Theorem of Farb and Masur to conclude (see first paragraph of this subsection). Namely, the assumption of X implies that X_n is a Shimura variety. More precisely, $X_n \cong D/\Gamma$ where D denotes a hermitian symmetric domain, and Γ is a congruence subgroup of $G(\mathbb{Q})$.

The second idea of Hain is that it suffices in the argument above that the intersection $X \cap (H_g \cup A^{\mathrm{dec}}_g)$ has codimension ≥ 2 inside X. Namely, in this case we apply the argument to the complement of this locus in X_n which won't change the fundamental group.

The third idea of [Ric99] is that X cannot be contained inside the hyperelliptic locus H_g. The proof of this statement is hidden in the proof of Theorem 2 of that paper and consists of one sentence. We elaborate. By the above we may assume that away from codimension 2, say over an open V, the points of X correspond to Jacobians of smooth curves. To reach a contradiction, assume all of these curves are hyperelliptic. For a hyperelliptic curve C we have

(1) $\mathrm{Aut}(C) = \mathrm{Aut}(J(C), \lambda)$ and
(2) the multiplication $\mathrm{Sym}^2(H^0(C, \Omega^1_C)) \to H^0(C, (\Omega^1_C)^{\otimes 2})$ maps onto the invariant part (under the hyperelliptic involution).

These two facts, plus Torelli, imply that $X \to A_{g,n}$ is an immersion, see [OS79]. So now, over some the open $V_{g,n} \subset X_n$, with $\pi_1(V_{g,n}) = \pi_1(X_n) = \Gamma$, there is a universal family of hyperelliptic curves, and we win as before.

Subsection 2.2. **Hyperelliptic locus: proof of Theorem 1.1.** With what we have discussed in the previous section, we may assume that $H_g \cap X$ is a divisor of X. Let $\bar{X}$ be the Baily-Borel-Satake compactification of X. Then $\bar{X}$ is a projective variety and the codimension of $\bar{X} \setminus X$ is ≥ 3. By cutting by hyper-plane sections, we obtain a surface S in $\bar{X}$ such that

(1) S is included in $X \setminus A_g^{\mathrm{dec}}$;
(2) $S \cap H$ is nonempty and 1 dimensional.

In this case $S \cap H$ is a projective curve parameterizing smooth hyper-elliptic curves. This is impossible.

Subsection 2.3. **Proof of Corollary 1.2.** For a Hilbert modular variety X defined by a totally real number field F, the three conditions in Theorem 1.1 are easy to check:

- the symmetric space of X are given by a product of g-unit discs in $\mathbb{C}$;
- the generic point represents an non-decomposable abelian variety and subvarieties corresponding to decomposable abelian variety are defined by subfields F' of F which codimension $g - g' \geq 3$ except case (*).
- The Baily-Borel-Sataki compactification is given by adding finitely many cusps. Thus the boundary has codimension g.

Section 3. Finiteness of CM-points

In this section, we want to prove Theorem 1.3 and Example 1.4. The main problem is to check the maximality of the Mumford-Tate group. We will show that the maximality is equivalent to the non-vanishing of some generalized Bernoulli numbers, which by the class number formula is equivalent to the standard (but highly nontrivial) non-vanishing of some Dirichlet L-series at $s = 1$.

Subsection 3.1. **Proof of Theorem 1.3.** Let $CM(K, S_K)$ be the set of CM-points on $A_g^{\mathcal{O}}$. By Corollary 2.7 in [Zha05], any infinite subset of CM-points in $CM(K, S_K)$ is Zariski dense. By Corollary 1.2, $M_g^c \cap A_g^{\mathcal{O}}$ is a proper subvariety of $A_g^{\mathcal{O}}$. It follows that $M_g^c \cap CM(K, S_K)$ as a subset in $CM(K, S_K)$ must be a finite set.

Subsection 3.2. **Proof of Example 1.4.** In the following we want to show that the CM-type defined in the example has maximal CM-type. First we use Proposition 6.1 in [Zha05] to describe the character group of $MT(S_K)F^\times/F^\times$. Let us identify the group of algebraic characters of $K^\times := \mathrm{Res}_{K/\mathbb{Q}}\mathbb{G}_m$ as $\mathbb{Z}[\mathrm{Gal}(K/\mathbb{Q})]$, then the quotient group $K^\times/F^\times$ has character group $\mathbb{Z}[\mathrm{Gal}(K/\mathbb{Q})]^-$ of elements annihilated by $1+c$ where $c \in \mathrm{Gal}(K/\mathbb{Q})$ is the complex conjugation. By part 1 and 2 of Proposition 6.1 in [Zha05], the character group of $MT(S_K)F^\times/F^\times$ is the quotient of $\mathbb{Z}[\mathrm{Gal}(K/\mathbb{Q})]^-$ modulo the sub-modulo Φ consisting of $\phi \in \mathbb{Z}[\mathrm{Gal}(K/\mathbb{Q})]^-$ such that

$$\sum_{s \in S_K} \phi(gs) = 0, \qquad \forall g \in \mathrm{Gal}(K/\mathbb{Q}). \tag{3.1}$$

Thus the maximality of $MT(S_K)$ is equivalent to $\Phi = 0$. It is equivalent to show $\Phi \otimes \mathbb{C} = 0$. As $\mathrm{Gal}(K/\mathbb{Q})$ is commutative, $\Phi \otimes \mathbb{C}$ is generated by characters χ of $\mathrm{Gal}(K/\mathbb{Q})$. Thus we need to show that there is no non-trivial characters χ of $(\mathbb{Z}/\ell\mathbb{Z})^\times$ such that $\chi(c) = -1$ and that

$$\sum_{t \in T_a} \chi(t) = 0. \tag{3.2}$$

Let $b = \ell - 1 - a$. Then for any $t \in (\mathbb{Z}/\ell)^\times$, then

$$< t/\ell > + < at/\ell > + < bt/\ell >= 1 \quad \text{or} \quad 2.$$

Thus the equation (3.2) is equivalent to

$$\sum_t (< t/\ell > + < at/\ell > + < bt/\ell > -2)\chi(t) = 0.$$

Let B_χ be the generalized Bernoulli number defined by

$$B_{1,\chi} = \sum_t (< t/\ell > -1/2)\chi(t).$$

As $\sum_t \chi(t) = 0$, then the above equation can be written as

$$B_{1,\chi}(1 + \chi^{-1}(a) + \chi^{-1}(b)) = 0.$$

By the class number formula,

$$L(1, \chi) = \frac{\pi i}{p} \tau(\chi) B_{1,\chi}$$

where $\tau(\chi) = \sum_t \chi(t) e^{2\pi i t/p}$ is the Gauss sum. Thus the nonvanishing of $L(1, \chi)$ implies that $B_{1,\chi} \neq 0$; see Theorem 4.9 in [Was82]. Thus the equation (3.2) is equivalent to

$$1 + \chi^{-1}(a) + \chi^{-1}(b) = 0. \tag{3.3}$$

As all three terms here are roots of unity, the only possible solution is when they are 3 cubic roots of unity. It follows that $\chi(a^3) = 1$. Since $g = (\ell - 1)/2$ is prime to 3, the order of χ is prime to 3, thus $\chi(a) = 1$. This is impossible.

References

[FM98] B. Farb and H. Masur. Superrigidity and mapping class groups. *Topology*, 37(6):1169–1176, 1998.

[MVZ] M. Möller, E. Viehweg, and K. Zuo. Special families of curves, of abelian varieties, and of certain minimal manifolds over curves. arXiv: math.AG/051254v1 7 Dec 2005.

[OS79] F. Oort and J. Steenbrink. The local Torelli problem for algebraic curves. *Journées de Géometrie Algébrique d'Angers*, pages 157–204, 1979.

[Ric99] H. Richard. Locally symmetric families of curves and jacobians. In Carel Faber and Eduard Looijenga, editors, *Moduli of Curves and Abelian Varieties*, Aspects of Mathematics, pages 91–108. Vieweg, Wiesbaden, 1999.

[Was82] L. C. Washington. *Introduction to cyclotomic fields*. Springer-Verlag, 1982.

[Wei76] A. Weil. Sur les périodes des intégrales. *Communications between pure and applied mathematics*, 29:391–397, 1976.

[Zha05] S.-W. Zhang. Equidistribution of CM-points on quaternion Shimura varieties. *International Mathematics Research Notices*, (59):3657–3689, 2005.

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