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Gross–Schoen cycles and dualising sheaves

Shou-Wu Zhang

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S.-W. Zhang (✉)
Department of Mathematics, Columbia University, New York, NY 10027, USA
e-mail: szhang@math.columbia.edu

1 Introduction and statements of results

The aim of this paper is to study the modified diagonal cycle in the triple product of a curve over a global field defined by Gross and Schoen in [25]. Our main result is an identity between the height of this cycle and the self-intersection of the relative dualising sheaf. We have some applications to the following problems in number theory and algebraic geometry:

- *Hodge index and Bogomolov's conjectures for heights of cycles and points.* We will show that the arithmetic index conjecture of Beilinson and Gillet–Soulé [5, 21] gives a conjectural lower bound for the admissible self-dualising sheaf for arithmetic surfaces in term of local integrations. This gives an approach toward an effective version of Bogomolov conjecture [36, 41]. By applying Noether's formula, this will also give an alternative approach toward a slope inequality for Hodge bundles (or Faltings heights) on moduli space of curves, other than using stability in geometric invariant theory [10, 37].
- *Beilinson–Bloch's conjectures for special values of L-series and cycles.* By conjectures of Beilinson–Bloch, Swinnerton-Dyer, and Tate's conjectures [4–6], the non-triviality of Gross–Schoen cycles will imply the vanishing of the L-series for the triple product cohomology of a curve. We have a Northcott property for vanishing of L-series on moduli space of curves. In the case of function field, these are unconditional. Moreover, for non-isotrivial curve over function field of with good reduction, the Arakelov–Szpiro theorem implies the vanishing of the L-series of order ≥ 2.
- *Non-triviality of tautological classes in Jacobians.* We will show that the height of the canonical Gross–Schoen cycle Δ_ξ has the Northcott type property on the moduli spaces of curves. We will give an expression of this height in terms of the cycles X_1 and $\mathcal{F}(X_1)$ in Beauville's Fourier–Mukai transform [1–3] and Künnemann's height pairing [27]. This implies in particular that the Northcott property holds for Ceresa [8] cycles $X - [-1]^*X$. For a non-isogeny curve over function field with good reduction, these cycles are non-trivial by using a theorem of Arakelov–Szpiro [32].

In the following, we will describe in details the main results and applications, and a plan of proof.

1.1 Gross–Schoen cycles

Let us first review Gross and Shoen's construction of the modified diagonal cycles in [25] and definitions of heights of Bloch [6], Beilinson [4, 5], and Gillet–Soulé [21]. Let k be a field and let X be a smooth, projective, and geometrically connected curve over k. Let $Y = X^3$ be the triple product of X over k and let e be a divisor on X with rational coefficients of degree 1. Let $e = \sum a_i p_i$ be a decomposition over $\bar{k}$. Define the diagonal and the partially diagonal cycles with rational coefficients with respect to base e as follows:

$$\Delta_{123} = \{(x, x, x) : x \in X\},$$

$$\Delta_{12} = \sum_i a_i \{(x, x, p_i) : x \in X\},$$

$$\Delta_{23} = \sum a_i \{(p_i, x, x) : x \in X\},$$

$$\Delta_{31} = \sum a_i \{(x, p_i, x) : x \in X\},$$

$$\Delta_1 = \sum_{i,j} a_i a_j \{(x, p_i, p_j) : x \in X\},$$

$$\Delta_2 = \sum_{i,j} a_i a_j \{(p_i, x, p_j) : x \in X\},$$

$$\Delta_3 = \sum_{i,j} a_i a_j \{(p_i, p_j, x) : x \in X\}.$$

Then define the Gross–Schoen cycle associated to e to be

$$\Delta_e = \Delta_{123} - \Delta_{12} - \Delta_{23} - \Delta_{31} + \Delta_1 + \Delta_2 + \Delta_3 \in \mathrm{Ch}^2(X^3) \otimes \mathbb{Q}.$$

In this paper, Chow group always means the cycles with rational coefficients module rational equivalence. Gross and Gross only consider the cycle when the divisor e is a point. But results can translated to our more general cycles. In fact, if we let e_0 to be a point, then it is easy to check that $\Delta_e - \Delta_{e_0}$ is algebraically equivalent to 0. In particular by Gross and Schoen, Δ_e is homologous to 0 in general, and that Δ_e it is rationally equivalent to 0 if X is rational, or elliptic, or hyperelliptic when e is a Weierstrass point. A natural question is: *When is Δ_e non-zero in* $\mathrm{Ch}^2(X^3)$ *in non-hyperelliptic case*?

Over a global field k, a natural invariant of Δ_e to measure the non-triviality of a homologically trivial cycle is the height of Δ_e which was conditionally constructed by Beilinson–Bloch [4–6] and unconditionally by Gross–Schoen [25] for Δ_e. More precisely, assume that k is the fractional field of a discrete valuation ring R and that X has a regular, semi-stable model $\mathcal{X}$ over $S := \mathrm{Spec} R$. Then Gross–Schoen construct a regular model $\mathcal{Y}$ over S of $Y = X^3$ and show that the modified diagonal cycle Δ_e on Y can be extended to a codimension 2 cycle on $\mathcal{Y}$ which is numerically equivalent to 0 in the special fiber $\mathcal{Y}_s$.

If k is a function field of a smooth and projective curve B over a field over which X has a regular semistable model $\mathcal{X}$, then Gross and Schoen's construction gives a cycle $\hat{\Delta}_e$ with rational coefficients on a model $\mathcal{Y}$ of $Y = X^3$ over B. We can define the height of Δ_e as

$$\langle \Delta_e, \Delta_e \rangle = \hat{\Delta}_e \cdot \hat{\Delta}_e.$$

The right hand here is the intersection of cycles on $\mathcal{Y}$. This pairing does not depend on the choice of $\mathcal{Y}$ and the extension $\hat{\Delta}_e$ of Δ_e.

If k is a number field, then we use the same formula to define the height for the arithmetical cycle

$$\hat{\Delta} = (\widetilde{\Delta}_e, g)$$

in Gillet–Soulé's arithmetic intersection theory [18] where

- $\widetilde{\Delta}_e$ is the Gross–Schoen extension of Δ_e over a model $\mathcal{Y}$ over $\mathrm{Spec}\mathcal{O}_k$;

- g is a Green's current on the complex manifold $Y(\mathbb{C})$ of the complex variety $Y \otimes_{\mathbb{Q}} \mathbb{C}$ for the cycle Δ_e: g is a current on $Y(\mathbb{C})$ of degree $(1, 1)$ with singularity supported on $\Delta_e(\mathbb{C})$ such that the curvature equation holds:

$$\frac{\partial\bar{\partial}}{\pi i} g = \delta_{\Delta_e(\mathbb{C})}.$$

  Here the right hand side denotes the Dirac distribution on the cycle $\Delta_e(\mathbb{C})$ when integrate with forms of degree $(2, 2)$ on $Y(\mathbb{C})$.

Notice that this height can be also defined using Künnemann's results in [27], see Theorem 1.5.6 for more details. As the non-triviality of Δ_e follows from the nonvanishing of its height, a natural question is: *When is* $\langle\Delta_e, \Delta_e\rangle$ *non-zero*?

1.2 Admissible dualising sheaves

Our main result of this paper is an expression of the height $\langle\Delta_e, \Delta_e\rangle$ in terms of the self-intersection ω_a^2 of the relative dualising sheaf defined in our early paper [39] which we recall as follows. Let X be a curve over a field k of positive genus. We assume that k is either the fraction field of a smooth and projective curve B or a number field where we still set $B = \mathrm{Spec}\mathcal{O}_k$, and that X has a regular and semistable model $\mathcal{X}$ over B.

When B is a projective curve, then one has a usual intersection pairing of divisors on $\mathcal{X}$ and a usual relative dualising sheaf $\omega_{\mathcal{X}/B}$ which gives an adjunction formula for self-intersections of sections.

In number field case, Arakelov theory gives intersections on the arithmetic divisors of form $\hat{D} = (D, G)$ formed by a divisor D on $\mathcal{X}$ and an *admissible green's function* G on $X(\mathbb{C})$ in the sense that its curvature satisfies,

$$\delta_{D_{\mathbb{C}}} - \frac{\partial\bar{\partial}}{\pi i} G = \deg D \cdot d\mu$$

where $d\mu$ is the Arakelov measure on $X(\mathbb{C})$: on each connected component $X_v(\mathbb{C})$ corresponding to archimedean place v,

$$d\mu_v = \frac{i}{2g} \sum_{n=1}^{g} \omega_n \wedge \bar{\omega}_n$$

where g is the genus of X and ω_n are base of $\Gamma(X_v, \Omega_{X_v})$ normalized such that

$$\frac{i}{2} \int \omega_m \wedge \bar{\omega}_n = \delta_{m,n}.$$

Arakelov shows that there is a unique metric such that an adjunction formula is true for a dualising sheaf with admissible metric. By Faltings [15], we have a Hodge index theorem.

In [39], we construct an intersection theory (for function field case or number field case) on divisors of the form (D, G) formed by a divisor D of X and G an *adelic* green's function with *adelic* curvature $d\mu_a$. More precisely, G has a component G_v as a continuous and symmetric function on the reduction graph $R(X_v) \times R(X_v)$ of $X \otimes k_v$ [9] for each closed point v of B, and as a usual green's function on $X_v(\mathbb{C})$ for each archimedean place v in number field case. The admissible metric and the green's function on $R(X_v)$ are characterized in Sects. 3.1 and 3.2 in [39]:

$$\Delta_y G_v(x, y) = \delta_x - \mu_a, \qquad \int G_v(x, y) d\mu_a(y) = 0,$$

$$G_v(\delta_{K_{X_v}}, x) + G_v(x, x) = \text{constant}.$$

Here Δ_y is the Lapalacian operator on the graph with respect to the standard metric on the graph [39, Appendix], and K_{X_v} is the canonical divisor associate to the degree function of the usual relative dualising sheaf $\omega_{\mathcal{X}/B}$ [39, Sect. 2.1]. We show that in this intersection theory [39], we still have Hodge index theorem and an adjunction formula with admissible relative dualising sheaf ω_a whose component at archimedean place is the metrized line defined by Arakelov and at finite place is the usual relative dualising sheaf with a modification by $\exp(G_v(R(x), R(x))$ [39, Sect. 4], where $R : X_v(\bar{k}_v) \longrightarrow R(X_v)$ is the reduction map. We called such intersection pairing an *adelic* admissible pairing.

We have proved in [39], Theorem 4.4, the following inequalities:

$$\hat{\omega}^2_{\mathcal{X}/B} \geq \omega^2_a \geq 0.$$

Moreover the difference of the first two item is given by local integrations:

$$\omega^2_{\mathcal{X}/B} = \omega^2_a + \sum_v \epsilon(X_v) \deg(v) \tag{1.2.1}$$

where v runs through the set of non-archimedean places, and

$$\epsilon(X_v) := \int_{R(X_v)} G_v(x, x)(\delta_{K_{X_v}} + (2g-2)\, d\mu_v),$$

and $\deg(v)$ is the usual degree when B is a curve over field k_0 and is $\log \mathrm{N}(v)$ in the number field case with $\mathrm{N}(v)$ the cardinality of the residue field. The first inequality is strict unless X has genus 1 or X has good reductions at all non-archimedean place.

1.3 Main result and first consequences

The main result of this paper proved in Sect. 3.5 is an identity between the two canonical invariants:

Theorem 1.3.1 *Let X be a curve of genus $g > 1$ over a field k which is either a number field or the fraction field of a curve B. Assume that X has a semistable model $\mathcal{X}$ over B or* $\mathrm{Spec}\mathcal{O}_k$. *Then*

$$\langle \Delta_e, \Delta_e \rangle = \frac{2g+1}{2g-2}\omega^2_a + 6(g-1)\langle x_e, x_e \rangle - \sum_v \varphi(X_v) \deg(v).$$

Here $\langle x_e, x_e \rangle$ is the Neron–Tate height of the class $x_e := e - K_X/(2g-2)$ in $\mathrm{Pic}^0(X)_{\mathbb{Q}}$, and $\varphi(X_v)$ are some contribution from places v of K:

1. *If v is an archimedean place, then*

$$\varphi(X_v)=\sum_{\ell,m,n}\frac{2}{\lambda_\ell}\left|\int_{X_v}\phi_\ell\omega_m(x)\bar{\omega}_n(x)\right|^2$$

where ϕ_ℓ are normalized real eigenforms of the Arakelov Laplacian:

$$\frac{\partial\bar{\partial}}{\pi i}\phi_\ell=\lambda_\ell\cdot\phi_\ell\cdot d\mu_v,\qquad \int\phi_k\phi_\ell d\mu=\delta_{k,\ell},$$

and ω_i are basis of $\Gamma(X_v,\Omega_{X_v})$ normalized by

$$\frac{i}{2}\int\omega_m\bar{\omega}_n=\delta_{m,n}.$$

2. *If v is a nonarchimedean place, then*

$$\varphi(X_v)=-\frac{1}{4}\delta(X_v)+\frac{1}{4}\int_{R(X_v)}G_v(x,x)((10g+2)\,d\mu_a-\delta_{K_{X_v}})$$

where $\delta(X_v)$ is the number of singular points on the special fiber of the regular semistable model $\mathcal{X}$ over v, $G_v(x,y)$ is the admission Green function for the admissible metric $d\mu_v$, and the K_{X_v} is the canonical divisor on $R(X_v)$. In particular, $\varphi(X_v)=0$ if X has good reduction at v.

Replace k by an extension, we may fix a class $\xi\in\mathrm{Pic}(X)$ such that $(2g-2)\xi=K_X$. By the positivity of the Neron–Tate height pairing, $\langle\Delta_e,\Delta_e\rangle$ reaches its minimal value precisely when where

$$e=\xi+\text{torsion divisor}.$$

We call the cycle Δ_ξ a *canonical* Gross–Schoen cycle for X. Notice that the class of Δ_ξ in the Chow group with rational coefficients does not dependent of choice of ξ. The height of Δ_ξ does not depend on the choice of ξ.

Corollary 1.3.2

$$\omega_a^2=\frac{2g-2}{2g+1}\left(\langle\Delta_\xi,\Delta_\xi\rangle+\sum_v\varphi(X_v)\deg(v)\right).$$

If X is hyperelliptic, then we may take ξ to be a Weierstrass point. By Gross–Schoen, a positive multiple of Δ_ξ is rationally equivalent to 0. Thus we have the following identity:

Corollary 1.3.3 *Assume that X is a hyperelliptic curve, then*

$$\omega_a^2=\frac{2g-2}{2g+1}\sum_v\varphi(X_v)\deg(v).$$

Combining with (1.2.1), this also gives an identity for the self-intersection of the usual relative sheaf of hyperelliptic curve in term of bad reductions. Some explicit examples of such formulae have been given by Bost, Mestre, and Moret-Bailly in [7]. It is an interesting question to compare our formula with theirs.

It is a hard problem to check when the height $\langle \Delta_\xi, \Delta_\xi \rangle = 0$ even in the function field case. We have the following consequence of Theorem 1.3.1 in smooth case:

Corollary 1.3.4 *Assume that k is the function field a projective and smooth curve B, and that X can be extended to a non-isotrivial family $\mathcal{X} \longrightarrow B$ of smooth and projective curves of genus $g > 1$. Then*

$$\langle \Delta_\xi, \Delta_\xi \rangle = \frac{2g+1}{2g-2}\omega^2_{\mathcal{X}/B} > 0.$$

Proof The first equality follows from Corollary 1.3.2 and the formula (1.2.1) in Sect. 1.2. The second inequality is due to the ampleness of $\omega_{\mathcal{X}/B}$ by proved by Arakelov in case of characteristic 0 and by Szpiro in case of positive characteristic. □

We will show that $\langle \Delta_\xi, \Delta_\xi \rangle$ is essentially a height function in Sect. 4.2:

Theorem 1.3.5 *Let $Y \longrightarrow T$ be smooth and projective family of curves of genus $g \geq 3$ over a projective variety T over a number field k, or the function field of a curve over a finite field. Then the function*

$$t \in T(\bar{k}) \mapsto (2g-2)\langle \Delta_\xi(Y_t), \Delta_\xi(Y_t) \rangle$$

is a height function associate to Deligne's pairing

$$(2g+1)\langle \omega_{Y/T}, \omega_{Y/T} \rangle.$$

Moreover if the induced map $T \longrightarrow \mathcal{M}_g$ from T to the coarse moduli space of curves of genus g is finite, then we have a Northcott property: for any positive numbers D and H,

$$\#\left\{t \in T(\bar{k}) : \deg t \leq D,\ \langle \Delta(Y_t)_\xi, \Delta(Y_t)_\xi \rangle \leq H\right\} < \infty.$$

Remarks

We would like to give some remarks about the upper bound for $\langle \Delta_\xi, \Delta_\xi \rangle$.

When k is a function field of curve B of genus $q \geq 2$ a field of characteristic 0, the semi-stable model $\mathcal{X}$ is a surface of general type and one has the Bogomolov–Miyaoka–Yau inequality:

$$c_1(\mathcal{X})^2 \leq 3c_2(\mathcal{X}).$$

Equivalently, in term of relative data,

$$\omega^2_{\mathcal{X}/B} \leq (2g-2)(2q-2) + 3\sum_{b \in B} \delta(X_b).$$

See Moret–Bailly [28] for details. By Corollary 1.3.2, we have a bound for the height of Gross–Schoen cycle:

$$\langle \Delta_\xi, \Delta_\xi \rangle \leq (2g+1)(2q-2) + \sum_{b \in B}\left(\frac{6g+3}{2g-2}(\delta(X_b) + \epsilon(X_b)) - \varphi(X_b)\right).$$

When k is a function field of positive characteristic, then the Bogomolov–Miyaoka–Yau inequality is not true. Instead, one has a Szpiro [32, Theorem 3] inequality in which one needs to add some inseparableness of f. So we have a similar inequality for the height of Gross–Schoen cycle.

When k is a number field, then Parshin [30] and Moret–Bailly [28] have formulated an arithmetic Bogomolov–Miyaoka–Yau inequality. It has been proved that this conjecture is equivalent to the effective Mordell conjecture, Szpiro's discriminant conjecture, and the ABC conjecture. Conversely, Elkies [13] has proved that ABC-conjecture will imply the effective conjecture. By our main theorem, these are equivalent to an upper bound conjecture with ω^2 replaced by the height of Gross–Schoen cycle.

1.4 Hodge index and Bogomolov conjectures

By the construction, the cycle Δ_e has zero intersection in $\mathrm{Ch}^3(X^3)$ with $p_i^*\mathrm{Pic}(X)$ via the projections $p_i : X^3 \longrightarrow X$. Thus, it is primitive with respect an ample line bundle $\mathcal{L}$ on X^3 of the form $\sum p_i^*\mathcal{L}$ for an ample line bundle on X. In case where k is a function field of characteristic 0, by Hodge index theorem (which is called Hodge–Riemann bilinear relations in Griffiths–Harris [23, p. 123]), one can show that the height $\langle \Delta_\xi, \Delta_\xi\rangle$ is non-negative, and is vanishing precisely when $\hat{\Delta}_\xi$ is numerically equivalent to 0. There is nothing need to prove if $\hat{\Delta}_\xi$ is numerically equivalent to 0. Otherwise, without loss of generality, we may assume that k is a function field over $\mathbb{C}$ and then we may compute the intersection using class $[\hat{\Delta}_\xi]$ of $\hat{\Delta}_\xi$ in the de Rham cohomology $H^{2,2}(\mathcal{Y})$. By Hodge index theorem, the intersection $\langle [\hat{\Delta}_\xi], [\hat{\Delta}_\xi]\rangle$ is non-negative, and is vanishing precisely when $[\hat{\Delta}_\xi] = 0$. Since $[\hat{\Delta}_\xi] = 0$ would have implied that $\hat{\Delta}_\xi$ is numerically equivalent to 0, we must have $\langle \Delta_\xi, \Delta_\xi\rangle > 0$.

In cases that k is a number field or a function field over a finite field, the Hodge index theorem is part of the Standard Conjecture of Grothendieck, Beilinson, and Gillet–Soulé [5, 21]:

Conjecture 1.4.1 *Let k be a number field or a function field over a finite field, then*

$$\langle \Delta_\xi, \Delta_\xi\rangle \geq 0$$

and this height vanishes precisely when $\Delta_\xi = 0$ in $\mathrm{Ch}^2(X^3)$.

Granting the first part of this conjecture or assuming that k is a function field of characteristic 0, we then have a lower bound for ω_a^2:

$$\omega_a^2 \geq \frac{2g-2}{2g+1}\sum_v \varphi(X_v)\deg(v).$$

It is proved in [39] that $\omega_a^2 > 0$ is equivalent to the Bogomolov conjecture about the finiteness of points $x \in X(\bar{k})$ with small Neron–Tate height in the map

$$X \longrightarrow \mathrm{Jac}(X), \qquad x \mapsto [(2g-2)x - K_X] \in \mathrm{Jac}(X).$$

In number field case, the Bogomolov conjecture is proved by Ullmo [36, 41]. The conjecture of Gillet–Soulé thus implies an effective version of Bogomolov conjecture as $\varphi(X_v)$ can be computed effectively for any given graph. In view of the Bogomolov conjecture, we would to make the following:

Conjecture 1.4.2 *Let v be a finite place. Let $\delta_0(X_v), \dots, \delta_{[g/2]}(X_v)$ denote the numbers of singular points x in the special fiber $X_{k(v)}$ such the local normalization of $X_{k(v)}$ at x is connected when $i = 0$ or a disjoint union of two curves of genus i and $g - i$. Then*

$$\varphi(X_v) \geq c(g)\delta_0(X_v) + \sum_{i>0} \frac{2i(g-i)}{g}\delta_i(X_v)$$

where $c(g)$ is a positive function of $g > 1$.

The conjecture and Theorem 1.3.1 together imply that $\varphi(X_v) \geq 0$ and that $\varphi(X_v) = 0$ precisely when X has a good reduction at v. In Sect. 4.3, we will show that it suffices to show the conjecture when all $\delta_i = 0$ for $i > 0$. More precisely, we will give an explicit formula in Sect. 4.4 for φ_v for elementary graphs and prove the following:

Theorem 1.4.3 *Assume that the reduction graph $R(X_v)$ is elementary in the sense that every edge is included in at most one cycle. Then the conjecture is true with*

$$c(g) = \frac{g-1}{6g}.$$

Moreover, the equality in Conjecture 1.4.2 with above choice of $c(g)$ is true if and only if every circle has at most one vertex.

Recently, Xander Faber [14] has verified the conjecture for curves with small genera. For example, he shows for genus 2 and 3, we may take $c(2) = 1/27$ and that $c(3) = 2/81$. Thus he has a proof of the Bogomolov for all curves of genus 3.

The Bogomolov conjecture should hold for non-isotrivial curve over function field. Some partial results have been obtained by Moriwaki [29], Yamaki [38], and Gubler [26]. The work of Moriwaki and Yamaki are effective and follows from a slope inequality of Moriwaki [29] for general semistable fiberation $\pi : \mathcal{X} \longrightarrow B$:

$$\lambda(\mathcal{X}/B) := \deg \pi_*\omega_{\mathcal{X}/B} \geq \frac{g}{8g+4}\delta_0(X) + \sum_{i>0} \frac{i(g-i)}{2g+1}\delta_i(X)$$

where $\delta_i(X) = \sum_v \delta_i(X_v)\deg(v)$ is the intersection of B with i-the boundary component of the moduli space. This formula is a generalization of a work of Xiao [37] and Cornalba–Harris [10], and is proved based on the stability of the sheaf $\pi_*\omega_{\mathcal{X}/B}$ and by Noether's formula

$$\lambda(\mathcal{X}/B) = \frac{1}{12}\left(\omega^2_{\mathcal{X}/B} + \sum_v \delta(X_v)\right) \tag{1.4.1}$$

where $\delta(X_v) = \sum \delta_i(X_v)$ be the total number of singular points in the fiber over v. Thus, we have an equality

$$\lambda(\mathcal{X}/B) = \frac{2g-2}{2g+1}\langle \Delta_\xi, \Delta_\xi \rangle + \sum \lambda(X_v)\deg(v) \tag{1.4.2}$$

where

$$\lambda(X_v) = \frac{g-1}{6(2g+1)}\varphi(X_v) + \frac{1}{12}(\epsilon(X_v) + \delta(X_v)). \tag{1.4.3}$$

Thus the Hodge index theorem gives

Theorem 1.4.4 *If k is a function field of characteristic* 0, *then*

$$\lambda(\mathcal{X}/B) \geq \sum \lambda(X_v).$$

We believe that this is the sharpest slope inequality for fibred surfaces with given configuration of singular fibers. In particular, the Moriwaki's inequality should follows from the following

Conjecture 1.4.5 *If v is a non-archimedean place*, *then*

$$\lambda(X_v) \geq \frac{g}{8g+4}\delta_0(X_v) + \sum_{i>0} \frac{i(g-i)}{2g+1}\delta_i(X_v).$$

In Sect. 4.3, we will reduce this conjecture to the case where $\delta_i = 0$ and prove the conjecture for elementary graphs. Also Xander Faber [14] has verified the conjecture for curves with small genera.

In number field case, Faltings [15] defines a volume form on $\det \pi_* \omega_{\mathcal{X}/B}$ for each archimedean place v. The number $\lambda(\mathcal{X}/B)$ is called the Faltings height of $\mathcal{X}$. He also proves a Noether formula (1.4.1) with his $\delta(X_v)$. Thus we still have expression (1.4.2) with $\lambda(X_v)$ given in (1.4.3) when v is non-archimedean, and

$$\lambda(X_v) = \frac{g-1}{6(2g+1)}\varphi(X_v) + \frac{1}{12}\delta(X_v)$$

when v is archimedean, where $\varphi(X_v)$ is given in Theorem 1.3.1. Now Theorem 1.4.4 is a conjecture predicted by Hodge Index Conjecture 1.4.1:

Conjecture 1.4.6 *If k is a number field*, *then*

$$\lambda(\mathcal{X}/B) \geq \sum \lambda(X_v)\deg(v).$$

1.5 Beilinson–Bloch conjecture and tautological classes

Assume that k is a number field or a function field of a curve defined over a finite field and a prime ℓ-prime to the characteristic of k. For a smooth and projective variety Y defined over k, and an integer i between 0 and $\dim Y$, we should have a ℓ-adic cohomology $H^i(Y) := H^i(Y_{\bar{k}}, \mathbb{Q}_\ell)$ and a complete L-series $L(H^i(Y), s)$ with a conjectured holomorphic continuation and a function equation. For each n between 0 and $\dim Y$, we also have a Chow group $\mathrm{Ch}^n(Y)^0$ of codimension n-cycles on Y with $\mathbb{Q}$-coefficients and trivial classes in $H^{2n}(Y)(n)$, the twist of $H^{2n}(Y)$. The conjecture of Beilinson [4, 5] and Bloch [6] asserts that $\mathrm{Ch}^n(Y)^0$ is of finite rank and

$$\mathrm{rank}\mathrm{Ch}^n(Y)^0 = \mathrm{ord}_{s=0} L(H^{2n-1}(Y)(n), s). \tag{1.5.1}$$

If Y is a curve, then the above is the usual Birch and Swinnerton-Dyer conjecture for $\mathrm{Jac}(Y)$. If k is a function field, then the holomorphic continuation of the L-series and the functional equation are known. The Beilinson–Bloch conjecture in function field case is equivalent to Tate's conjecture, see Tate [34, 35] and Beilinson [5].

Now we assume that $Y = X^3$ is a power of a curve over k, $n = 2$. Then both sides of (1.5.1) has decomposition by correspondences defined by action of symmetric group $\mathcal{S}_3$

acting on X^3, projections and embeddings between X^i and X^j. In Sect. 5.1, we will show that Δ_ξ lies in the subgroup $\mathrm{Ch}(M)$ of $\mathrm{Ch}^2(X^3)^0$ of elements z satisfying the following conditions:

1. z is symmetric with respect to permutations on X^3;
2. the pushforward $p_{12*}z = 0$ with respect to the projection

$$p_{12}: \quad X^3 \longrightarrow X^2, \qquad (x, y, z) \mapsto (x, y);$$

3. let $i : X^2 \longrightarrow X^3$ be the embedding defined by $(x, y) \longrightarrow (x, x, y)$ and $p_2 : X^2 \longrightarrow X$ be the second projection. Then

$$p_{2*}i^*z = 0.$$

The operations induces some correspondences on X^3. The same condition apply to cohomology gives the kernel M of

$$\bigwedge^3 H^1(X)(2) \longrightarrow H^1(X)(1), \qquad a \wedge b \wedge c \mapsto a(b \cup c) + b(c \cup a) + c(a \cup b).$$

Here $\cup : H^1(X) \otimes H^1(X) \longrightarrow \mathbb{Q}(-1)$ is the Weil pairing on $H^1(X)$. The cohomology M is pure of weight -1 with an alternative pairing

$$M \otimes M \longrightarrow \mathbb{Q}_\ell(1).$$

Moreover, it can be shown that M is a Chow motive with projector given in Sect. 5.1. It is conjectured that the complete L-series of M has a holomorphic continuation to whole complex plane and satisfies a functional equation

$$L(M, s) = \pm c(M)^{-s} L(M, -s)$$

where $\epsilon(M) = \pm 1$ is the root number of M, and $c(M) \in \mathbb{N}$ is the conductor of M which is divisible only by places ramified in M. See Deligne [12] and Tate [33] for details. In our situation, the Beilinson and Bloch conjecture has a refinement:

Conjecture 1.5.1 (Beilinson–Bloch) *The height pairing on* $\mathrm{Ch}(M)$ *is positive definite and*

$$\mathrm{rank}\mathrm{Ch}(M) = \mathrm{ord}_{s=0} L(M, s).$$

If k is a number field, we don't know in general that $L(M, s)$ has a homomorphic continuation. But we attempt to guess that for most curve X over a field k, the L-series should has vanishing order ≤ 2. In other words, for general X,

$$\begin{aligned} \epsilon(M) = 1 \quad &\Longrightarrow \quad L(M, 0) \neq 0, \\ \epsilon(M) = -1 \quad &\Longrightarrow \quad L'(M, 0) \neq 0. \end{aligned}$$

The following are some formulae for computing epsilon factors proved in Sect. 5.2:

Theorem 1.5.2 *The epsilon factor has a decomposition*

$$\epsilon(M) = \prod_v \epsilon_v(M)$$

into a product of local epsilon factor give as follows.

1. *If v is a real place,*

$$\epsilon_v(M) = (-1)^{g(g-1)/2} = \begin{cases} 1, & \text{if } g \equiv 0, 1 \bmod 4 \\ -1, & \text{if } g = 2, 3 \bmod 4 \end{cases}$$

2. *If v is a complex place*

$$\epsilon_v(M) = (-1)^{g(g+1)(g+2)/6} = \begin{cases} 1 & \text{if } g \not\equiv 1 \bmod 4 \\ -1 & \text{if } g \equiv 1 \bmod 4 \end{cases}$$

3. *If v is a non-archimedean place, then*

$$\epsilon_v(M) = (-1)^{e(e-1)(e-2)/6+ge} \cdot \tau^{(e-1)(e-2)/2+g}$$

where e is the dimension of toric part T_v of the reduction of Néron model of $\mathrm{Jac}(X)$ *at v, and $\tau = \pm 1$ is the determinant of the Frobenius* Frob_v *acting on the character group $X^*(T_v)$.*

Let $\mathrm{Ch}_{num}(M)$ be the quotient of $\mathrm{Ch}(M)$ modulo the numerical equivalence. If k is a function field, we have an inequality

$$\mathrm{rank}\mathrm{Ch}_{num}(M) \le \mathrm{ord}_{s=0} L(M, s). \tag{1.5.2}$$

See Tate [35], Propositions 2.8, 2.9 and their proofs for details. If X is non-isotrivial and has good reduction everywhere over places of k, then by Theorem 1.3.4, Δ_ξ is non-zero. On the other hand, we can show that the sign of the functional equation is 1. Thus we must have

Theorem 1.5.3 *If X/k is a curve of over function field of a curve B over a finite field of genus $g \ge 3$. Assume that X can be extended into a non-isotrivial smooth family of curves over B. Then*

$$\mathrm{ord}_{s=0} L(M, s) \ge 2.$$

In view of Tate's conjecture, we have

$$\mathrm{rank}\mathrm{Ch}_{num}(M) = \mathrm{ord}_{s=0} L(M, s) \ge 2.$$

Thus we have a natural question: *how to find another cycle in* $\mathrm{Ch}^2(M)^0$ *which is linear independent of Δ_ξ?*

In general it is very difficult to compute the special values or derivatives of $L(M, s)$ at $s = 0$. However the following is a consequence of Theorem 1.3.5 and Beilinson–Bloch's conjecture, we conclude the following:

Conjecture 1.5.4 *Let $Y \longrightarrow T$ be a smooth and projective family of curves of genus $g \ge 3$ over a projective variety T over a number field k. Assume the induced map $T \longrightarrow \mathcal{M}_g$ from T to the coarse moduli space of curves of genus g is finite, then we have a Northcott property: for any positive numbers D,*

$$\#\left\{t \in T(\bar{k}) : \deg t \le D,\ L(M(Y_t), 0) \ne 0\right\} < \infty.$$

Over function field, this is a theorem induced from Theorem 1.3.5 and formula (1.5.2).

In the following, we want to apply our result to the tautological algebraic cycles in the Jacobian defined by Ceresa [8] and Beauville [3]. We will use Fourier–Mukai transform of Beauville [1, 2] and height pairing of Künnemann [27].

Let $f: X \longrightarrow J$ be an embedding given by taking x to the class of $x-\xi$. Then we define the tautological classes $\mathcal{R}$ to the smallest subspace of $\mathrm{Ch}^*(J)$ containing X closed under the following operations:

- intersection pairing "$\cdot$";
- Pontriajan's star operator "$*$";

$$x * y := m_*(p_1^* x \cdot p_2^* y)$$

  where p_1, p_2, m are projection and addition on J^2;
- Fourier–Mukai transform

$$\mathcal{F}: \mathrm{Ch}^*(J) \longrightarrow \mathrm{Ch}^*(J)$$

$$x \mapsto \mathcal{F}(x) := p_{2*}(p_1^* x \cdot e^\lambda)$$

  where λ is the Poincaré class:

$$\lambda = m^*\theta - p_1^*\theta - p_2^*\theta.$$

  Here θ is the theta divisor consisting of sums of $g-1$-points in $f(X)$.

Using Fourier–Mukai transform, we have spectrum decomposition

$$X = \sum_{s=0}^{g-1} X_s, \qquad X_s \in \mathrm{Ch}^{g-1}(J)$$

with $[k]_* X_s = k^{2+s} X_s$. By Beauville [3], the ring $\mathcal{R}$ under the intersection pairing is generated by $\mathcal{F}(X_s) \in \mathrm{Ch}^{1+s}(J)$. The pull-back of these cycles on X^3 under the morphism

$$f_3: X^3 \longrightarrow J, \qquad (x_1, x_2, x_3) \mapsto f(x_1) + f(x_2) + f(x_3)$$

can be computed explicitly. In particular, we can prove the following formulae proposed by Wei Zhang [42]:

Theorem 1.5.5 *Consider the addition morphism $f_3: X^3 \longrightarrow J$. Then*

$$f_3^* \mathcal{F}(X_1) = \Delta, \qquad f_{3*}\Delta_\xi = \sum_s (3^{2+s} - 3 \cdot 2^{2+s} + 3) X_s,$$

$$X_s = (3^{2+s} - 3 \cdot 2^{2+s} + 3)^{-1} \sum_{i+j+k=s-1} (X_i * X_j * X_k) \cdot \mathcal{F}(X_1), \qquad s > 0.$$

Moreover, the following are equivalent:

1. $\Delta_\xi = 0$ *in* $\mathrm{Ch}^2(X^3)$;
2. $X - [-1]^* X = 0$ *in* $\mathrm{Ch}^{g-1}(J)$;
3. $X_1 = 0$ *in* $\mathrm{Ch}^{g-1}(J)$;
4. $X_s = 0$ *in* $\mathrm{Ch}^{g-1}(J)$ *for all* $s > 0$.

By this theorem, under the operators $\cdot$ and $*$, the ring R is generated by X_0 and any one of three canonical classes X_1, Gross–Schoen cycle $f_{3*}\Delta_\xi$, and Ceresa cycle $X - [-1]^* X$. The following gives a more precise relation between the height of Δ_ξ and the height of class X_1 and $\mathcal{F}(X_1)$:

Theorem 1.5.6 *The cycle $\mathcal{F}(X_1)$ is primitive with respect to theta divisor θ, homologically trivial in* $\mathrm{Ch}^2(J)$, *and*

$$\langle \mathcal{F}(X_1), X_1\rangle_J = \frac{1}{6}\langle \Delta_\xi, \Delta_\xi\rangle_{X^3} = \frac{1}{(g-3)!}\langle \mathcal{F}(X_1), \theta^{g-3}\mathcal{F}(X_1)\rangle_J.$$

Plan of proof

The proof of Theorem 1.3.1 is proceeded in several steps in Sects. 2–3:

1. Reduction from X^3 to X^2: we express the height as a triple product on $X \times X$ of an adelic line bundles with generic fiber (Theorem 2.3.5):
$$\Delta - p_1^*\xi - p_2^*\xi.$$
2. Reduction form X^2 to X: we express the triple as the self-intersection of the canonical sheaf plus some local triple integrations (Theorem 2.3.5).
3. Local triple pairing: we develop an intersection theory on the reduction complex of the product $X \times X$ at a non-archimedean place (Theorem 3.4.2) and use this to complete the proof of Theorem 1.3.1.

The proof of other results about the estimate of the height follows form detailed calculation of constants ϕ and λ in Sect. 4. We first express these constants in terms of integration of resistance on metrized graph and reduce the computation to 2-edge connected graphs, and finitely compute everything for 1-edge graphs.

The last section is devoted to study the Beilinson–Bloch conjecture and the Beauville tautological cycles. We first define a minimal cohomology M so that its Chow group contains Δ_ξ. Then we compute the ϵ-constant of its L-series. Finally, we translate the statements to tautological cycles in the Jacobian varieties.

2 Gross–Schoen cycles and correspondences

The aim of this section is to prove some global formulae for the heights of Gross–Schoen cycles in terms of the self-intersections of the relative dualising sheaves and some local intersections:

$$\frac{2g+1}{2g-2}\omega^2 + \text{local contributions} \qquad \text{(Theorem 2.5.1).}$$

These local contributions will be computed in the next section. More generally, for any correspondences t_1, t_2, t_3 on $X \times X$, we compute the height pairing

$$\langle \Delta_e, (t_1 \otimes t_2 \otimes t_3)\Delta_e\rangle.$$

This pairing is positive if t_i are correspondence of positive type by Gillet–Soulé's Conjectures 2.4.1 and 2.4.2. We will show that this is equal to the intersection number $\hat{t}_1 \cdot \hat{t}_2 \cdot \hat{t}_3$ on $X \times X$ (Theorem 2.3.5).

2.1 Cycles and heights

In this subsection, we will review intersection theory of Gillet–Soulé and some adelic extensions with some variations. The basic references are Gillet–Soulé [18–20], Faltings [16], Deligne [11], and our previous paper [40].

Arithmetical intersection theory

Let k be a number field with the ring of integers $\mathcal{O}_k$. By an arithmetical variety over $\mathcal{O}_k$, we mean a flat and projective morphism $\mathcal{X} \longrightarrow \mathrm{Spec}\mathcal{O}_k$ such that $\mathcal{X}_k$ is regular. By (homological) arithmetic cycle, we mean a pair $\hat{Z} = (Z, g)$ where Z is a cycle on $\mathcal{X}$ with coefficients in $\mathbb{Q}$ and g is a current such that curvature

$$h(\hat{Z}) := \frac{\partial\bar{\partial}}{\pi i} g + \delta_Z$$

is smooth on $\mathcal{X}(\mathbb{C})$. The cycle is called irreducible if either Z is irreducible and horizontal (i.e., flat over $\mathcal{O}_k$), or Z is irreducible and vertical (i.e. including in a closed fiber of $\mathcal{X}$ over $\mathcal{O}_k$), or $Z = 0$. We define (homological) arithmetic Chow group as a combination of irreducible cycles with rational or real coefficients:

$$\sum_i a_i (Z_i, g_i)$$

where $a_i \in \mathbb{Q}$ if Z_i is horizontal, and $a_i \in \mathbb{R}$ if Z_i is vertical, modulo the relations:

- $(\mathrm{div}(f), -\log|f|) = 0$ for a rational function f on an integral subscheme $\mathcal{Y}$ of $\mathcal{X}$;
- $(0, \partial\alpha + \bar{\partial}\beta) = 0$;
- $a(0, g) = (0, ag)$.

For a morphism $\phi : \mathcal{X} \longrightarrow \mathcal{Y}$ of arithmetic varieties, one has push forward morphism $\phi_* : \widehat{\mathrm{Ch}}_*(\mathcal{X}) \longrightarrow \widehat{\mathrm{Ch}}_*(\mathcal{Y})$ if ϕ is proper and generically smooth, and pullback morphism $\phi^* : \widehat{\mathrm{Ch}}_*(\mathcal{Y}) \longrightarrow \widehat{\mathrm{Ch}}_*(\mathcal{X})$ if ϕ is flat.

One may define cohomological arithmetical Chow group $\widehat{\mathrm{Ch}}^*(\mathcal{X})$ as bivariant classes of morphisms of homological arithmetical groups as in Fulton book [17, Chapter 17]. In this paper, we only consider the classes defined by Chern classes of Hermitian bundles or by smooth forms. More precisely, let $\hat{K}(\mathcal{X})$ denote the arithmetic K-group of hermitan vector bundles and smooth forms module the usual secondary Chern class relation for an exact sequence. Then we have a Chern character:

$$\widehat{\mathrm{ch}} : \hat{K}(\mathcal{X}) \longrightarrow \mathrm{End}(\widehat{\mathrm{Ch}}_*(\mathcal{X})).$$

This is the usual Chern character for Hermitian vector bundles and the following formula for smooth forms: α on $\mathcal{X}(\mathbb{C})$:

$$\widehat{\mathrm{ch}}(\alpha)x = (0, \alpha \cdot h(x)).$$

See Gillet–Soulé [22]. In this paper, we define the cohomological group $\widehat{\mathrm{Ch}}^*(\mathcal{X})$ of arithmetical cycles as the quotient of $\hat{K}(\mathcal{X})$ modulo the subgroup of elements t such that

$$\widehat{\mathrm{ch}}(\phi^* t) = 0$$

for any morphism of arithmetic varieties $\phi : \mathcal{Y} \longrightarrow \mathcal{X}$. When $\mathcal{X}$ is regular, $\widehat{\mathrm{ch}}$ is an isomorphism, thus we have an isomorphism:

$$\hat{K}(\mathcal{X}) = \widehat{\mathrm{Ch}}^*(\mathcal{X}) = \widehat{\mathrm{Ch}}_*(\mathcal{X}).$$

For later use, we make two remarks. First one is a natural paring:

$$\widehat{\mathrm{Ch}}^p(\mathcal{X}) \times \widehat{\mathrm{Ch}}_p(\mathcal{X}) \longrightarrow \mathbb{R}, \qquad [\bar{\mathcal{E}}, \alpha] \times z \mapsto \widehat{\mathrm{ch}}(\bar{\mathcal{E}}, \alpha) \cdot z.$$

The second is that the Chern classes can be computed using first Chern classes of line bundles and smooth forms. More precisely, fix a hermitian vector bundle $\mathcal{E}$ on X of rank n as above. Let $\pi : \mathcal{P} \longrightarrow \mathcal{X}$ be the flag scheme over $\mathcal{P}$ and let

$$\mathcal{E}_1 := \pi^* E \supset \mathcal{E}_2 \supset \mathcal{E}_3 \supset \cdots \supset \mathcal{E}_n \supset \mathcal{E}_{n+1} = 0$$

be the universal filtration of $\pi^*\mathcal{E}$. Let $\mathcal{L}_i = \mathcal{E}_i/\mathcal{E}_{i+1}$ be the line bundle with subquotient metrics. Then for any $x \in \widehat{\mathrm{Ch}}_*(\mathcal{X})$ then we have

$$\widehat{\mathrm{ch}}(E) \cdot x = \pi_* \left(\sum_i \exp(\hat{c}_i(\mathcal{L}_i)) \cdot \prod_{j=1}^n \hat{c}_1(\mathcal{L}_j)^{n-j} \cdot \pi^* x \right) + (0, \alpha(E))x \tag{2.1.1}$$

where $\alpha(E)$ is a smooth form supported on $\mathcal{X}(\mathbb{C})$. This follows from two facts: for any $x \in \widehat{\mathrm{Ch}}(\mathcal{X})$,

$$x = \pi_* \left(\prod_{j=1}^n \hat{c}_1(\mathcal{L}_j)^{n-j} \cdot \pi^* x \right)$$

$$\pi^*(\widehat{\mathrm{ch}}(\mathcal{E})) = \sum_i \exp \hat{c}_1(\mathcal{L}_i) + (0, \widetilde{\mathrm{ch}}(\mathcal{E}))$$

where $\widetilde{\mathrm{ch}}(\mathcal{E})$ is the secondary Chern class associate to the filtration of $\mathcal{E}$.

Cycles homologous to zero

Let X be a smooth and projective variety of dimension n over a number field or a function field k. Let $\mathrm{Ch}(X)$ denote the Chow group of cycles with coefficient in $\mathbb{Q}$. Then we have a class map to ℓ-adic cohomology:

$$\mathrm{Ch}(X) \longrightarrow H^*(X)$$

where $H^*(X) = H^*(X \otimes \bar{k}, \mathbb{Q}_\ell)$ with ℓ a prime different than the characteristic of k. The kernel $\mathrm{Ch}(X)^0$ of this map is called the group of homologically trivial cycles. Beilinson [4, 5] and Bloch [6] have given a conditional definition of height pairing between cycles in $\mathrm{Ch}(X)^0$. We will focus on the case of number fields but all the results hold for case where k is the function field of a smooth and projective curve B over some field k_0, and where we have the same height pairing with $\mathrm{Spec}\mathcal{O}_k$ replaced by B and with the condition about green's function dropped.

Height pairing

One construction of this height pairing in number field case is based on Gillet and Soulé's intersection theory as follows. Assume that X has a regular model $\mathcal{X}$ over $\mathrm{Spec}\mathcal{O}_k$, and that every cycle $z \in \mathrm{Ch}(X)^0$ has an extension $\hat{z} = (\bar{z}, g_z)$ to an arithmetic cycle which has trivial intersection to vertical arithmetic cycles:

1. $\bar{z}$ is a cycle on $\mathcal{X}$ extending z;
2. g_z has curvature $h_z = 0$;
3. the restriction of $\bar{z}$ on each component in the special fibers of $\mathcal{X}$ is numerically trivial.

Then for any $z' \in \mathrm{Ch}(X)^0$ extended to an arithmetic cycle $\hat{z}'$ on $\mathcal{X}$, the height pairing is defined by

$$\langle z, z'\rangle := \hat{z} \cdot \hat{z}'.$$

It is clear that this definition does not depend on the choice of $\hat{z}'$, and that the pairing is linear and symmetric.

Let $C(X) = \mathrm{Ch}^n(X \times X)$ denote the ring of (degree 0) correspondences on X. Then $C(X)$ acts on $\mathrm{Ch}(X)$ and preserve $\mathrm{Ch}(X)^0$. Recall that the composition law is given by the intersection pairing on $X \times X \times X$ and various projections to $X \times X$:

$$t_2 \circ t_1 = p_{13*}(p_{12}^* t_1 \cdot p_{23}^* t_2), \quad t_1, t_2 \in C(X).$$

For any $t \in C(X)$, $z \in \mathrm{Ch}(X)$, the push-forward and pull-back of z under t are defined by

$$t_*(z) = p_{2*}(p_1^* z \cdot t), \qquad t^*(z) = p_{1*}(t \cdot p_2^* z).$$

Let $t \longrightarrow t^\vee$ be the involution defined by the permutation on X^2 then we have $t^* = (t^\vee)_*$. It can be shown that the involution operator is the adjoint operator for the height pairing:

Lemma 2.1.1 (Lemma 4.0.3 in Beilinson [5])

$$\langle t_* z, z'\rangle = \langle z, t^* z'\rangle = \langle z, t_*^\vee z'\rangle.$$

Adelic metrized line bundles

In the following, we want to review some facts about the adelic metrized bundles developed in [40, Sect. 1]. For a smooth variety X defined over a number field k, let us consider the category of arithmetic models $\mathcal{X}$ with generic fiber X, i.e. an arithmetic variety $\mathcal{X} \longrightarrow \mathrm{Spec}\mathcal{O}_k$ and an isomorphism $\mathcal{X}_k \simeq X$. As this category is partially ordered by morphisms, we can define the direct limit

$$\varinjlim \widehat{\mathrm{Pic}}(\mathcal{X}) \otimes \mathbb{Q}.$$

Every element in this group defines an algebraically metrized line bundle on X. The group $\overline{\mathrm{Pic}}(X)$ of integrable metrized line bundles are certain limits of these algebraically metrized line bundles.

The intersection pairing

$$c_1(\mathcal{L}_1) \cdots c_1(\mathcal{L}_n) \cdot \alpha \cdot [\mathcal{X}] \in \mathbb{R}, \qquad \mathcal{L}_i \in \widehat{\mathrm{Pic}}(\mathcal{X}),\ \alpha \in \widehat{\mathrm{Ch}}^{\dim \mathcal{X} - n}(\mathcal{X})$$

can be extend to a pairing with $\mathcal{L}_i \in \bar{\mathrm{Pic}}(X)$ with smooth metrics at infinite places. If α is the identity element then this is proved in Theorem 1.4 in [40] without even assuming the smoothness of metrics at infinity. In general case, we may use formula (2.1.1) to reduce to the case that α is the Chern classes of hermitian line bundles, or a smooth form. The intersection of line bundle case is already covered in [40], and form case is given by the product of α with curvature of $\mathcal{L}_i$. The following lemma shows that the pairing can be represented by a (homological) element on $\mathcal{X}$.

Lemma 2.1.2 *Let $\mathcal{X}$ be an arithmetical scheme and let $\bar{\mathcal{L}}_1, \dots, \bar{\mathcal{L}}_n$ be some adelic metrized line bundles on X. Assume that the metrics of $\mathcal{L}_i$ at infinite places are smooth. Then the functional*

$$\widehat{\mathrm{Ch}}^{\dim \mathcal{X}-n}(\mathcal{X}) \longrightarrow \mathbb{R}: \quad \alpha \mapsto \alpha \cdot c_1(\bar{\mathcal{L}}_1) \cdots c_1(\bar{\mathcal{L}}_n)$$

is represented by an element in $\widehat{\mathrm{Ch}}_{\dim X-n}(\mathcal{X})$ denoted by

$$c_1(\bar{\mathcal{L}}_1) \cdots c_1(\bar{\mathcal{L}}_n) \cdot [\mathcal{X}] \in \widehat{\mathrm{Ch}}_{\dim \mathcal{X}-n}(\mathcal{X}).$$

Moreover, this element has the following restriction on the generic fiber:

$$c_1(\mathcal{L}_1) \cdots c_1(\mathcal{L}_n)[X].$$

Proof It suffices to deal with the case where bundles are ample and are limits of some integral-ample models $(\mathcal{X}_i, \mathcal{M}_{i1}, \dots, \mathcal{M}_{in})$ of $(X_i, \mathcal{L}_{i1}^{e_i}, \dots, \mathcal{L}_{in}^{e_i})$. Without loss of generality, we may assume that $\mathcal{X}_i$ dominates $\mathcal{X}$, that $\mathcal{L}_{1k}, \dots, \mathcal{L}_{nk}$ have arithmetic modes $\mathcal{M}_{01}, \dots, \mathcal{M}_{0n}$, and that the metrics on the archimedean places induce the same metrics on each $\mathcal{L}_{ik}(\mathbb{C})$. Let π_i denote the projection $\mathcal{X}_i \longrightarrow \mathcal{X}$.

For any cohomological arithmetical cycles α on $\mathcal{X}$, we can define intersection pairings:

$$\left(\frac{1}{e_i^n} c_1(\mathcal{M}_{i1}) \cdots c_1(\mathcal{M}_{in})\right) \cdot \pi_i^* \alpha = \pi_{i*}\left(\frac{1}{e_i^n} c_1(\mathcal{M}_{i1}) \cdots c_1(\mathcal{M}_{in})\right) \cdot \alpha$$

which has a limit denoted by

$$c_1(\bar{\mathcal{L}}_1) \cdots c_1(\bar{\mathcal{L}}_n) \cdot \alpha.$$

We claim that the cycles

$$\pi_{i*}\left(\frac{1}{e_i^n} c_1(\mathcal{M}_{i1}) \cdots c_1(\mathcal{M}_{in} \cdot [\mathcal{X}_i])\right)$$

have a limit in $\widehat{\mathrm{Ch}}^*(\mathcal{X})_{\mathbb{R}}$ introduced in Gillet–Soulé [21]. Indeed, subtract them by

$$c_1(\mathcal{M}_{01}) \cdots c_1(\mathcal{M}_{0n})[\mathcal{X}],$$

we obtain vertical cycles

$$V_i := \pi_{i*}\left(\frac{1}{e_i^n} c_1(\mathcal{M}_{i1}) \cdots c_1(\mathcal{M}_{in}) \cdot [\mathcal{X}_i]\right) - c_1(\mathcal{M}_{01}) \cdots c_1(\mathcal{M}_{0n}) \cdot [\mathcal{X}]$$

supported in finitely many fibers of $\mathcal{X}$ over $\mathcal{O}_k$.

Let $\mathcal{F}$ be the union of these fibers as a closed subscheme of $\mathcal{X}$. If $\mathcal{F}$ is smooth, then we can use étale cohomology to compute the intersection. In general, we may use de Jong's alternation to work on a quasi-finite domination map $\pi : \mathcal{Y} \longrightarrow \mathcal{F}$ such that $\mathcal{Y}$ is projective and smooth. Let $f : \mathcal{Y} \longrightarrow \mathcal{X}$ denote the induced morphism. Then V_i can be written as $V_i = f_* W_i$ with W_i a cycle on $\mathcal{Y}$. In this way,

$$V_i \cdot \beta = W_i \cdot f^* \beta, \quad \beta \in \widehat{\mathrm{Ch}}^*(\mathcal{X}).$$

In other words, the intersection pairing of V_i with $\widehat{\mathrm{Ch}}^*(\mathcal{X})$ can be written as intersection of W_i with $f^*\widehat{\mathrm{Ch}}^*(\mathcal{X})$.

So we may work on the intersections of cycles on the smooth variety $\mathcal{Y}$. Let $N_*(\mathcal{Y})$ denote the group of group of cycles modulo numerical equivalence. Let M be the image of $f^*\widehat{\mathrm{Ch}}^*(\mathcal{X})$ in $N^*(\mathcal{Y})$. The elements W_i thus define a sequence of convergent functionals on M. Then $N_*(\mathcal{Y})$ is finite dimensional (see Tate [35]), this sequence will converge to a functional represented by an element W of $\mathrm{Ch}_*(\mathcal{Y}) \otimes \mathbb{R}$. Let $V = f_*(W)$. Thus we have shown that

$$\lim_i \pi_{i*}\left(\frac{1}{e_i^n} c_1(\mathcal{M}_{i1}) \cdots c_1(\mathcal{M}_{in}) \cdot [\mathcal{X}_i]\right) = c_1(\mathcal{M}_{01}) \cdots c_1(\mathcal{M}_{0n}) \cdot [\mathcal{X}] + V.$$

In this way we define a correspondence

$$c_1(\bar{\mathcal{L}}_1) \cdots c_1(\bar{\mathcal{L}}_n) \cdot [\mathcal{X}] = \lim_i \pi_{i*}\left(\frac{1}{e_i^n} c_1(\mathcal{M}_{i1}) \cdots c_1(\mathcal{M}_{in}) \cdot [\mathcal{X}_n]\right). \qquad \square$$

Deligne pairing

In the following, we want to construct Deligne pairing of metrized line bundles. Let $f : X \longrightarrow Y$ be a flat and projective morphism of two smooth varieties over valuation field k of relative dimension n. Let $\bar{\mathcal{L}}_0 = (\mathcal{L}_0, \|\cdot\|_0)$, $\bar{\mathcal{L}}_1 = (\mathcal{L}_1, \|\cdot\|), \ldots, \bar{\mathcal{L}}_n = (\mathcal{L}_n, \|\cdot\|_n)$ be $n+1$ integrable metrized line bundles over X. We want to define a Deligne paring

$$\langle \bar{\mathcal{L}}_0, \bar{\mathcal{L}}_1, \ldots, \bar{\mathcal{L}}_n \rangle = (\langle \mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_n \rangle, \|\cdot\|)$$

as an adelic metrized line bundle over Y. Recall that $\bar{\mathcal{L}}_i$ can be approximated by models over $\mathcal{O}_k$:

$$(\mathcal{X}_i, \mathcal{M}_{i0}, \ldots, \mathcal{M}_{in})$$

of $(X, \mathcal{L}_0^{e_i}, \ldots, \mathcal{L}_n^{e_i})$ for some $e_i \in \mathbb{N}$. Without loss of generality, we may assume that $\mathcal{X}_i$ is flat and projective over a model $\mathcal{Y}_i$ over $\mathcal{O}_k$. Indeed, for a model $\mathcal{Y}$ of Y, we may replace $\mathcal{X}_i$ by Zarisk closure of X in $\mathcal{X}_i \times_{\mathrm{Spec}\mathcal{O}_k} \mathcal{Y}$, we may assume that $\mathcal{X}_i$ has a morphism to $\mathcal{Y}$. Then we apply Raynaud's flattening [31], Theorem 1, chapter 4, to blow up $\mathcal{Y}$, and to replace $\mathcal{X}_i$ by its pure-transform to get a flat family $\mathcal{X}_i \longrightarrow \mathcal{Y}_i$. In this way, we have a Deligne's pairing:

$$\langle \mathcal{M}_{i0}, \ldots, \mathcal{M}_{in} \rangle \in \mathrm{Pic}(\mathcal{Y}_i).$$

It is easy to see that this bundle is semiample if all $\mathcal{L}_i$ is semiample, and that the induced metrics on $\langle \mathcal{L}_0, \ldots, \mathcal{L}_n \rangle$ is convergent using estimate in the proof in [40], Theorem 1.4. Thus this sequence of bundles on models $\mathcal{M}_i$ defines an adelic metrized line bundle on Y.

In the following, we would like to describe a formula for computing norm of a section of Deligne's pairing. Let $\ell_0, \ell_1, \ldots, \ell_n$ be non-zero sections of $\mathcal{L}_i$ on X. By writing $\mathcal{L}_i$ as linear combination of very ample line bundles and applying Bertini's theorem, we may assume that any intersection of any subset of $\mathrm{div}(\ell_i)$'s is a linear combination of subvarieties which are smooth over Y. Then the pairing $\langle \ell_0, \ldots, \ell_n\rangle$ is well-defined as a section of $\langle \mathcal{L}_0, \ldots, \mathcal{L}_n\rangle$. The norm of this section can be defined by the following induction formula:

$$\log \|\langle \ell_0, \ldots, \ell_n\rangle\| = \log \|\langle \ell_0|_{\mathrm{div}\ell_n}, \ldots, \ell_{n-1}|_{\mathrm{div}\ell_n}\rangle\| + \int_{X/Y} \log \|\ell_n\| c_1(\bar{\mathcal{L}}_0)\cdots c_1(\bar{\mathcal{L}}_{n-1}). \tag{2.1.2}$$

We need to explain the integration in the above formula in terms of models $(\mathcal{X}_i, \mathcal{M}_{i0}, \ldots, \mathcal{M}_{in})$ as above. In this case $\ell_n^{e_i}$ extends to a rational section m of $\mathcal{M}_n$. The divisor $\mathrm{div}(m)$ has a decomposition of Weil divisor:

$$\mathrm{div}(m) = e_i \overline{\mathrm{div}(\ell_n)} + V_i$$

where $\overline{\mathrm{div}(\ell_n)}$ is the Zariski closure of $\mathrm{div}(\ell_n)$ on $\mathcal{X}_i$ and V_i is a divisor in the special fiber of $\mathcal{X}_i$ over $\mathrm{Spec}\mathcal{O}_k$. Then the integral is defined as

$$\int_{X/Y} \log \|\ell_n\| c_1(\bar{\mathcal{L}}_0)\cdots c_1(\bar{\mathcal{L}}_{n-1}) = \lim_{i\to\infty} \frac{1}{e_i^n} V_i \cdot c_1(\mathcal{M}_{i0})\cdots c_1(\mathcal{M}_{in-1}).$$

2.2 Correspondences on a curve

In this subsection we want to construct arithmetic classes for divisors on a product without using regular models.

Decompositions

Lemma 2.2.1 *Let X_1 and X_2 be two varieties over a field k with product $Y = X_1 \times_k X_2$. Let e_1, e_2 be two rational points on X_1 and X_2, and let* $\mathrm{Pic}^-(Y)$ *be the subgroup of line bundles which are trivial when restrict on $\{e_1\} \times X_2$ and $X_1 \times \{e_2\}$. Then we have a decompositions of line bundles on Y:*

$$\mathrm{Pic}(Y) \simeq p_1^*\mathrm{Pic}(X_1) \oplus p_2^*\mathrm{Pic}(X_2) \oplus \mathrm{Pic}^-(Y). \tag{2.2.1}$$

Proof For any class $t \in \mathrm{Pic}(Y)$, the equation

$$t = p_1^*\alpha_1 + p_2^*\alpha_2 + s, \quad \alpha_i \in \mathrm{Pic}(X_i),\ s \in \mathrm{Pic}^-(Y)$$

is equivalent to

$$\alpha_2 = t|_{\{e_1\}\times X_2}, \qquad \alpha_1 = t|_{X\times\{e_2\}}.$$

□

For any line bundle $\mathcal{L} \in \mathrm{Pic}^-(Y)$, we can define a homomorphism $\mathrm{Alb}(X_1)$ to the picard variety $\underline{\mathrm{Pic}}^0(X_2)$ which sends a zero cycle $\sum_i n_i(x_i)$ to the bundle $\prod_i \mathcal{L}^{n_i}|_{\{x_i\}\times X_2}$. In this way we can get an isomorphism of groups:

$$\mathrm{Pic}^-(Y) = \mathrm{Hom}(\mathrm{Alb}(X_1), \underline{\mathrm{Pic}}^0(X_2)). \tag{2.2.2}$$

Now we assume that X_i are curves over a number field. Consider an embedding

$$Y = X_1 \times X_2 \longrightarrow A := \mathrm{Alb}(Y) = \mathrm{Jac}(X_1) \times \mathrm{Jac}(X_2)$$

$$(x_1, x_2) \mapsto (x_1 - e_1, x_2 - e_2).$$

This induces a homomorphism of groups of line bundles:

$$\mathrm{Pic}(A) \longrightarrow \mathrm{Pic}(Y).$$

We also have a decomposition for line bundles on A with respect to the base points $(0, 0)$ on A:

$$\mathrm{Pic}(A) = p_1^*\mathrm{Pic}(\mathrm{Jac}(X_1)) \oplus p_2^*\mathrm{Pic}(\mathrm{Jac}(X_2)) \oplus \mathrm{Pic}^-(A).$$

Lemma 2.2.2 *The morphism* $\mathrm{Pic}(A) \longrightarrow \mathrm{Pic}(Y)$ *is surjective. More precisely, it induces the following*:

1. *An isomorphism* $\mathrm{Pic}^0(A) \simeq \mathrm{Pic}^0(Y)$;
2. *An isomorphism* $\mathrm{Pic}^-(A) \simeq \mathrm{Pic}^-(Y)$.

Proof The first statement is the duality between $\mathrm{Alb}(Y)$ and $\underline{\mathrm{Pic}}^0(Y)$. The second statement follows from (2.2.2) applying to $X_1 \times X_2$ and $\mathrm{Jac}(X_1) \times \mathrm{Jac}(X_2)$ identities

$$\mathrm{Alb}(X_1) = \mathrm{Jac}(X_1) = \mathrm{Alb}(X_1), \qquad \underline{\mathrm{Pic}}^0(X_1) = \underline{\mathrm{Pic}}^0(\mathrm{Jac}(X_2)). \qquad \square$$

Admissible metrics

Since the class map gives an embedding from $\mathrm{Pic}^-(A) \otimes \mathbb{Q}$ to $H^2(A)$, every line bundle in $\mathrm{Pic}^-(A) \otimes \mathbb{Q}$ is even under action by $[-1]^*$ and thus have eigenvalues n^2 under action $[n]^*$. In this way, we may construct admissible, integral, and adelic metrics $\|\cdot\|$ on each line bundle in $\mathrm{Pic}^-(A)$. In other words, after a positive power, each $\mathcal{L}$ in $\mathrm{Pic}^-(A)$ can be extended into an integrable metrized line bundle $\widehat{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ such that

$$[n]^*\widehat{\mathcal{L}} \simeq \widehat{\mathcal{L}}^{n^2}.$$

See our previous paper [40] for details.

The abelian variety A has an action by $\mathbb{Z}^2$ by double multiplications: for $m, n \in \mathbb{Z}$,

$$[m, n] : A = \mathrm{Jac}(X_1) \times \mathrm{Jac}(X_2) \longrightarrow A$$

$$(x, y) \mapsto (mx, ny).$$

In this notation the multiplication on A by $\mathbb{Z}$ is diagonally embedded into $\mathbb{Z}^2$. In particular these action are commutative. By the uniqueness of the admissible metrics, the admissible metrics are admissible with respect to the multiplication by $\mathbb{Z}^2$:

$$[m, n]^*\widehat{\mathcal{L}} \simeq \widehat{\mathcal{L}}^{mn}.$$

This shows that the bundle $\widehat{\mathcal{L}}$ is admissible in each fiber of $A^2 \longrightarrow A$ of two projection. Thus we have shown the following:

Lemma 2.2.3 *For any class $t \in \mathrm{Pic}^-(A)$, the restriction on $Y = X_1 \times X_2$ with admissible metric gives an adelic metrized line bundle $\hat{t}$ satisfies the following conditions*

- *for any closed points $p_i \in X_i$, the restrictions of $\hat{t}$ on $\{p_1\} \times X_1$ and $X_1 \times \{p_2\}$ defines admissible bundles $\widehat{\mathcal{L}_i}$ on X_i over some number fields k_i of degree* 0. *In other words, over some regular models $\mathcal{X}_i$, some positive multiples of $\widehat{\mathcal{L}_i}$ is induced by line bundles $\mathcal{M}_i$ on $\mathcal{X}_i$ which has zero degree on any vertical curve in $\mathcal{X}_i$ and curvature* 0 *at archimedean places.*
- *$\hat{t}$ is trivial on $\{e_1\} \times X_2$ and on $X_1 \times \{e_1\}$.*

Moreover, such an adelic structure over t is unique.

Proof The difference of two different adelic structures on t satisfying the above conditions will give an adelic structure $\hat{t}_0$ on the trivial bundle $t_0 = \mathcal{O}_Y$ satisfying the condition in the lemma. This is certainly trivial by checking on the curves $\{p\} \times X_1$ and $X_1 \times \{p_2\}$ on closed points p_i on X_i. □

Our method above also shows that the line bundles in $\mathrm{Pic}^0(A)$ (which is odd) on any abelian variety A also have integrable, admissible, integrable metrics. Indeed, let $\mathcal{P}$ be the Poincaré universal bundle on $A \times \mathrm{Pic}^0(A)$ with trivial restriction on $\{0\} \times \mathrm{Pic}^0(A)$ and $A \times \{0\}$. Then $\mathcal{P}$ is an even line bundle thus admits an integrable metrized bundles. The action by $\mathbb{Z}^2$ shows that this admissible metric is admissible fiber-wise. The following are some expressions for bundles on $\mathrm{Pic}^-(A)$ and $\mathrm{Pic}(Y)$:

Lemma 2.2.4 *Assume $X_1 = X_2 =: X$.*

1. *Any line bundle $\mathcal{L} \in \mathrm{Pic}^-(A)$ is induced from a unique endomorphism $\alpha \in \mathrm{End}(\mathrm{Jac}(X))$ by the following way:*
$$\mathcal{L} = (\alpha, 1)^*\mathcal{P}.$$
Moreover $\mathcal{L}$ is symmetric if and only if α is symmetric with respect to Rosati involution;
2. *A bundle $\mathcal{L}$ in $\mathrm{Pic}^-(Y)$ is symmetric (with respect to involution on $Y = X \times X$) if and only if there is a symmetric line bundle $\mathcal{M}$ on $\mathrm{Pic}(X)$ such that*
$$\mathcal{L}^2 \simeq s(\mathcal{M}) := m^*\mathcal{M} \otimes p_1^*\mathcal{M}^{-1} \otimes p_2^*\mathcal{M}^{-1} \otimes 0^*\mathcal{M}.$$
Moreover such an $\mathcal{M}$ is isomorphic to $\Delta^*\mathcal{L}$ where Δ is the diagonal embedding $\mathrm{Jac}(X) \longrightarrow A$.

Proof Indeed, any such $\mathcal{L}$ induces an endomorphism
$$\alpha : \mathrm{Jac}(X) \longrightarrow \mathrm{Pic}(\mathrm{Jac}(X)) = \mathrm{Jac}(X), \quad x \mapsto \mathcal{L}|_{x \times \mathrm{Jac}(X)}.$$
By universality of the Poincaré bundle we have that
$$\mathcal{L} = (\alpha, 1)^*\mathcal{P}.$$
The rest of statements in (1) is clear. If $\mathcal{L}$ is symmetric, we take
$$\mathcal{M} = \Delta^*\mathcal{L} = \Delta^*(\alpha, 1)^*\mathcal{P}.$$
Then we can show that
$$s(\mathcal{M}) = \mathcal{L}^2.$$
□

Example

Let e be a class in $\mathrm{Pic}^1(X)$. The class $s(\Delta) := \Delta - p_1^*e - p_2^*e$ on $X \times X$ is the pull-back of Poincaré bundle via the embedding $X \longrightarrow A$ via e. It is also induced from the theta divisor:

$$2s(\Delta) = -s(\Theta)|_{X\times X}.$$

Composition of arithmetic correspondences

Let X_i $(i = 1, 2, 3)$ be three curves over a number field. Let $(\bar{\mathcal{L}}, \|\cdot\|)$ and $\bar{\mathcal{M}} = (\mathcal{M}, \|\cdot\|)$ be integrable metrized line bundles on $X_1 \times X_2$ and $X_2 \times X_3$ respectively. We can define a composition $\bar{\mathcal{L}} \circ \bar{\mathcal{M}}$ by Deligne pairing for the projection

$$\pi_{13*}: \quad X_1 \times X_2 \times X_3 \longrightarrow X_1 \times X_3$$

$$\bar{\mathcal{L}} \circ \bar{\mathcal{M}} := \langle \pi_{12}^*\mathcal{M}, \pi_{23}^*\mathcal{L}\rangle.$$

It is easy to see that the composition is compatible with the induced action on Chow groups of models of X_i.

If $\mathcal{L}$ and $\mathcal{M}$ are in $\mathrm{Pic}^-(X_1 \times X_2)$ and $\mathrm{Pic}^-(X_2 \times X_3)$ with respect to some base points e_i and the metrics are admissible then the composition is also an admissible class in $\mathrm{Pic}^-(X_1 \times X_3)$.

2.3 Gross–Schoen cycles

In this subsection, we will study the height of Gross–Schoen cycles. We will deduce a formula between the height of Gross–Schoen cycles and the triple pairing of correspondences in Theorem 2.3.5.

Let X^3 be a triple product of a smooth and projective curve X over k. Let e be a rational point on X. For each subset T of $\{1, 2, 3\}$ define an embedding from X to X^3 which takes x to (x_1, x_2, x_3) where $x_i = x$ if $i \in T$ and $x_i = e$ otherwise. Let Δ_T be the image of X under i_T. Then we define the modified diagonal by

$$\Delta_e = \sum_{T\neq\emptyset}(-1)^{\#T-1}\Delta_T.$$

We may extend this cycle for case where e is a divisor on X of degree 1 as in Introduction:

Lemma 2.3.1 (Gross–Schoen) *The cycle Δ_e is cohomologically trivial. In other words, its class in $H^4(X^3)$ has zero cup product with elements in $H^2(X^3)$.*

Proof As

$$H^2(X^3) = \bigoplus_{i+j+k=2} H^i(X) \otimes H^j(X) \otimes H^k(X),$$

any element in the above group is a sum of elements of the form $p_{ij}^*\alpha$ where p_{ij} is the projection to (i, j) factors $X \times X$ and $\alpha \in H^2(X^2)$. For such form, the pairing is given by

$$\langle \Delta_e, p_{ij}^*\alpha\rangle = \langle p_{ij*}\Delta_e, \alpha\rangle.$$

It is easy to show that $p_{ij*}\Delta_e = 0$. Thus Δ_e is homologically trivial. □

Arithmetical Gross–Schoen cycles and heights

Gross and Schoen have constructed a vertical 2-cycle V in certain regular model of X^3 such that $\bar{\Delta}_e - V$ is numerically trivial on each fiber, where $\bar{\Delta}_e$ is the Zariski closure of Δ_e. One may further extend this to an arithmetic cycle $\hat{\Delta}_e = (\bar{\Delta}_e - V, g)$ by adding a green current g for Δ_e with curvature 0. Thus we have a well define pairing. More generally, let t_1, t_2, t_3 be three correspondence, then $t = t_1 \otimes t_2 \otimes t_3 \in \mathrm{Ch}^3(X^3 \times X^3)$ is a correspondence of X^3, and we have a pairing

$$\langle \Delta_e, t^* \Delta_e \rangle. \tag{2.3.1}$$

Triple pairing on correspondences

In the following we want to sketch a process to relate this pairing to some intersection numbers of cycles t_i on X^2.

First let δ denote an idempotent correspondence on X defined by the cycle

$$\delta := \Delta_{12} - p_1^* e.$$

Let $\delta^3 = \delta \otimes \delta \otimes \delta \in \mathrm{Ch}^3(X^3 \times X^3)$ denote the corresponding correspondence on X^3. Then it is not difficult to show that

$$\Delta_e = (\delta^3)^* (\Delta_{123}). \tag{2.3.2}$$

Indeed, the pull-back of the cycle $p_1^* e \in C(X)$ takes every point to e on X.

The projection in Lemma 2.2.1 is given by the idempotent δ:

$$t \mapsto t_e := \delta \circ t \circ \delta^\vee = t - p_1^*(t^* e) - p_2^*(t_* e) \in \mathrm{Pic}^-(Y).$$

Since $\delta \circ \delta = \delta$, $(\delta^3)^* \Delta_e = \Delta_e$,

$$\langle \Delta_e, t^* \Delta_e \rangle = \langle (\delta^3)^* \Delta_e, t^* (\delta^3)^* \Delta_e \rangle = \langle \Delta_e, (\delta_e^3)_* t^* (\delta^3)^* \Delta_e \rangle = \langle \Delta_e, t_e^* \Delta_e \rangle.$$

If follows that in the expression (2.3.1) we may assume that $t_i \in \mathrm{Pic}^-(X \times X)$.

Notice that the cycle δ in X^2 has degree 0 for the second projection. Thus we can construct arithmetic class $\widehat{\delta}$ extending δ as an integrable adelic metrized line bundles so that it is numerically zero on fibers of X^2 via the second projection. In other words, for any point $p \in X$ and vertical divisor V on a regular model of X, the intersection

$$V \cdot i_p^*(\hat{\delta}) = 0$$

where i_p is the embedding $x \longrightarrow (x, p)$. For example, we may construct such a metric by decomposition

$$\delta = (\Delta_{12} - p_1^* e - p_2^* e) + p_2^* e$$

and put the admissible metric on the first class as in the last subsection, and put any pull-back metric on $p_2^* e$. We may further assume that $\hat{\delta}$ has trivial restriction on $X \times \{e\}$.

Lemma 2.3.2

$$\hat{\delta} \circ \hat{\delta} = \hat{\delta}.$$

Proof Let $\bar{\mathcal{L}}$ be the adelic metrized line bundle corresponding to $\hat{\delta}$. Let ℓ be a rational section of $\mathcal{L}$ with divisor δ. By definition, $\delta \circ \delta$ is a divisor of a rational section

$$\langle \pi_{12}^*\ell, \pi_{23}^*\ell\rangle$$

of the line bundle $\langle \pi_{12}^*\mathcal{L}, \pi_{23}^*\mathcal{L}\rangle$ for the projection $\pi_{13} : X^3 \longrightarrow X^2$. By formula (2.1.1), the norm $\langle \pi_{12}^*\ell, \pi_{23}^*\ell\rangle$ at a place v can be written as

$$\log \|\langle \pi_{12}^*\ell, \pi_{23}^*\ell\rangle\| = \log \|\langle \pi_{12}^*\ell|_{\mathrm{div}\pi_{23}^*\ell}\rangle\| + \pi_{13*}(\log \|\pi_{12}^*\ell\| c_1(\pi_{23}^*\bar{\mathcal{L}})).$$

For the first term, notice that

$$\mathrm{div}\pi_{23}^*\ell = \pi_{23}^*\delta = X \times \Delta - X \times \{e\} \times X.$$

Both terms are isomorphic to $X \times X$ via projection π_{13}. Thus the Deligne's pairing is given by inversion of π_{13}

$$\langle \pi_{12}^*\mathcal{L}|_{\mathrm{div}\pi_{23}^*\ell}\rangle = \mathcal{L} \otimes \alpha^*\mathcal{L}^{-1} = \mathcal{L}$$

where α is the morphism

$$\alpha : \quad X^2 \longrightarrow X^2, \qquad (x, y) \mapsto (x, e).$$

The second equality is given by the assumption that $\mathcal{L}$ has trivial restriction on $X \times \{e\}$. It is easy to check that this isomorphism takes $\langle \pi_{12}^*\mathcal{L}|_{\mathrm{div}\pi_{23}^*\ell}\rangle$ to ℓ.

For the second term, the restriction of the integration on a point $(q, p) \in X^2$ is given by integration

$$\int_X \log \|j_q^*\ell\| c_1(\bar{i}_p^*\bar{\mathcal{L}})$$

where $j_q : X \longrightarrow X^2$ is a morphism sending x to (q, x). This integral is a limit of intersection of $\bar{\mathcal{L}}$ with some vertical divisors on models of X. Thus it is zero by assumption of $\hat{\delta}$. □

We want to apply the above Lemma to construct an extension $\hat{\Delta}_e$ on Δ_e on some model which are numerically trivial on special fiber. We will use regular models X^3 constructed in Gross–Schoen [25]. Let $\mathcal{X} \longrightarrow B$ be a good model $\mathcal{X}$ in the sense that the morphism has only ordinary double points as singular point, and that every component of fiber is smooth. Then we can get a good model $\widetilde{\mathcal{X}^3}$ of X^3 by blowing up all components in fiber product $\mathcal{X}^3$ in any fixed order of components. Let $\hat{\Delta}_{123}$ be any arithmetical cycle on $\widetilde{\mathcal{X}^3}$ extending Δ_e. By Lemma 2.1.2, the divisors $\hat{\delta}^3$ defines a correspondence on $\widetilde{\mathcal{X}^3}$. Thus we have well defined arithmetical cycle $(\hat{\delta}^3)^*\hat{\Delta}_e$.

Lemma 2.3.3 *The cycle $(\hat{\delta}^3)^*\hat{\Delta}_{123}$ is numerically zero on every fiber of $\widetilde{\mathcal{X}^3}$ over* $\mathrm{Spec}\,\mathcal{O}_k$.

Proof In other words, we want to show that for any vertical cycle V

$$0 = (\hat{\delta}^3)^*\hat{\Delta}_{123} \cdot V = \hat{\Delta}_{123} \cdot \hat{\delta}_*^3 V = 0.$$

Actually we will to show the following

$$\hat{\delta}_*^3 V = 0. \tag{2.3.3}$$

First let us consider an archimedean place. The curvature of $\hat{\delta}$ is zero on each fiber of p_2. Thus it has a class in

$$p_2^* H^2(X) + p_1^* H^1(X) \otimes p_2^* H^1(X).$$

In particular it is represented by a form $\omega(x, y)$ of degree 2 whose degree on x is at most 1. It follows that the curvature of $\hat{\delta}^3$ is represented by a form

$$\omega(x_1, y_1)\omega(x_2, y_2)\omega(x_3, y_3)$$

on $X^3 \times X^3$ whose total degree in x_i's is at most 3. It follows that for any smooth form ϕ on the first three variable (x_1, x_2, x_3) of degree 2 the integral on x-variable

$$p_{456*}(\omega(x_1, y_1)\omega(x_2, y_2)\omega(x_3, y_3)\phi(x_1, x_2, x_3)) = 0.$$

Now let us consider finite places. Now let V be an irreducible vertical 2-cycle on $\widetilde{\mathcal{X}^3}$ over a prime v of $\mathcal{O}_k$. Then there are three components A_1, A_2, A_3 of $\mathcal{X}$ over v such that V is included in the proper transformation $\widetilde{A_1 A_2 A_3}$ of the product $A_1 \times A_2 \times A_3$ in $\mathcal{X}^3$. Notice that $\widetilde{A_1 A_2 A_3}$ is obtained from $A_1 \times A_2 \times A_3$ by blowing up from some curves of the form $A_1 \times \{p\} \times \{q\}$, etc. Thus V is a linear combination of exceptional divisor and pull-back divisors from $A_1 \times A_2 \times A_3$. By the theorem of cube, V is linear equivalent to a sum of pull-back of divisors $V_{i,j}$ via the (i, j)-projection:

$$V \equiv p_{12}^* V_{12} + p_{23}^* V_{23} + p_{31}^* V_{31}.$$

We may assume that V is one of this term in the right, say

$$V = (p_{12}^* V_{12})_{A_1 \times A_2 \times A_3} = (p_{12}^* V_{12})_{\mathcal{X}^3} \cdot p_3^* A_3.$$

Now the intersection con be computed as follows:

$$\hat{\delta}_*^3 V = (\hat{\delta}_*^2 V_{12})(\hat{\delta}_* A_3).$$

By definition,

$$\hat{\delta}_* A_3 = p_{2*}(p_1^* A_3 \cdot \hat{\delta}).$$

The cycle $p_1^* A_3 \cdot \hat{\delta}$ in $\mathcal{X}_v^2$ over each point y of $\mathcal{X}_v$ is a divisor $A_3 \times \{y\} \cdot \hat{\delta}$. This is zero by assumption on $\hat{\delta}$. Thus we have shown (2.3.3). □

Now we go back to the intersection number in (2.3.1) for $t_i \in \mathrm{Pic}^-(X \times X)$. Let $\hat{t}_i$ be any arithmetic model of t_i. There product $\hat{t}$ is an arithmetic extension of the product t of t_i. By our construction, we see that

$$\langle \Delta_e, t^* \Delta_e \rangle = (\hat{\delta}^3)^* \hat{\Delta}_{123} \cdot \hat{t}^* (\hat{\delta}^3)^* \hat{\Delta}_{123} = \hat{\Delta}_{123} \cdot \hat{\delta}_*^3 \hat{t}^* (\hat{\delta}^3)^* \hat{\Delta}_{123}.$$

Recall that $t \in \mathrm{Pic}^-(X \times X)_e^3$, $\delta^3 \circ t \circ (\delta^\vee)^3 = t$. We may replace $\hat{t}_i$ by $\hat{\delta} \circ \hat{t}_i \circ \hat{\delta}^\vee$ to assume that

$$\hat{t}_i = \hat{\delta} \circ \hat{t}_i = \hat{t}_i \circ \hat{\delta}^\vee. \tag{2.3.4}$$

Under this assumption, the height pairing is given by

$$\begin{aligned}\langle \Delta_e, t^*\Delta_e\rangle &= \hat{\Delta}_{123}\cdot(\hat{t}_1\otimes\hat{t}_2\otimes\hat{t}_3)^*\hat{\Delta}_{123}\\ &= p_{123}^*\hat{\Delta}_{123}\cdot(\hat{t}_1\otimes\hat{t}_2\otimes\hat{t}_3)\cdot p_{456}^*\hat{\Delta}_{123}.\end{aligned}$$

Here the intersection is taken X^6.

As the product of the operators $\hat{t}_i$ annihilated any vertical cycles, the above intersection number is equal to the following expression on $\mathcal{X}\times\mathcal{X}$ via embedding

$$\mathcal{X}^2 \longrightarrow \mathcal{X}^6, \qquad (x,y)\mapsto(x,x,x,y,y,y).$$

This is simply the intersection product of $\hat{t}_i$ since the tensor product of cycles $\hat{t}_i$ are the pull-back via $p_{i,3+i}$. Thus we have shown the following identity:

$$\langle \Delta_e, (t_1\otimes t_2\otimes t_3)^*\Delta_e\rangle = \hat{t}_1\cdot\hat{t}_2\cdot\hat{t}_3 \tag{2.3.5}$$

for cycle $t_i\in \mathrm{Pic}^-(X\times X)$ and its extension satisfying equation (2.3.4).

In the following we describe the arithmetic class $\hat{t}_i$ satisfying (2.3.4).

Lemma 2.3.4 *The arithmetic divisors $\hat{t}$ on $\mathcal{X}^2$ satisfying* (2.3.4) *are exactly the arithmetic divisors $t\in \mathrm{Pic}^-(X\times X)$ with admissible metrics.*

Proof By Lemma 2.2.3, we need only check conditions in Lemma 2.2.3. Assume that $\hat{t}$ satisfies (2.3.4). From the definition of $\hat{\delta}$ we see that for any vertical component

$$\hat{\delta}_*(v)=0, \qquad \hat{\delta}^*(\bar{e})=0.$$

From the expression $\hat{t}=\hat{\delta}\circ\hat{t}$ we see that

$$\hat{t}_*(v)=\hat{\delta}_*(\hat{t}_*v)=0, \qquad \hat{t}^*(\bar{e})=\hat{t}^*\delta^*\bar{e}=0.$$

Similarly we can prove other two equalities by expression $\hat{t}=\hat{t}\circ\hat{\delta}$.

Now assume that $\hat{t}$ satisfies the condition in the Lemma. Consider the divisor

$$\hat{s}:=\hat{t}-\hat{\delta}\circ\hat{t}\circ\hat{\delta}^\vee.$$

By what we have proved, $\hat{s}$ is trivial on fibers over closed points and divisor $\{\bar{e}\}$ for both projection. This divisor must be trivial. Thus we must have $\hat{t}=\hat{\delta}\circ\hat{t}\circ\hat{\delta}^\vee$. Then the property (2.3.4) follows immediately. □

In summary, we have shown the following:

Theorem 2.3.5 *For any correspondences t_1,t_2,t_3 in* $\mathrm{Pic}^-(X\times X)$ *we have*

$$\langle \Delta_e, (t_1\otimes t_2\otimes t_3)^*\Delta_e\rangle = \hat{t}_1\cdot\hat{t}_2\cdot\hat{t}_3$$

where $\hat{t}_i$ are arithmetic cycles on some model of X^2 extending t_i and satisfying conditions in Lemma 2.2.3.

2.4 Gillet–Soulé's conjectures

By the standard conjecture of Gillet–Soulé [21] the pairing should be positively on the primitive cohomologically trivial cycles. This implies the following

Conjecture 2.4.1 *The following triple pairing is semi-positive definite*:

$$\mathrm{Pic}^{-}(X\times X)^{\otimes 3}\times \mathrm{Pic}^{-}(X\times X)^{\otimes 3}\longrightarrow \mathbb{R}$$

$$\begin{aligned}(t_1\otimes t_2\otimes t_3, s_1\otimes s_2\otimes s_3)\ \mapsto\ & \widehat{s_1\circ t_1^\vee}\cdot\widehat{s_2\circ t_2^\vee}\cdot\widehat{s_3\circ t_3^\vee}\\ &= \langle (t_1\otimes t_2\otimes t_3)^*\Delta_e, (s_1\otimes s_2\otimes s_3)^*\Delta_e\rangle.\end{aligned}$$

Indeed, Δ_ξ is perpendicular to $\sum p_i^*\mathrm{Pic}(X)$ for projections $p_i: X^3\longrightarrow X$, so is $(t_1\otimes t_2\otimes t_3)^*\Delta_\xi$ by translation. Thus $(t_1\otimes t_2\otimes t_3)^*\Delta_\xi$ is primitive respect to an ample line bundle on X^3 of the form $\sum p_i^*\mathcal{L}$ with $\mathcal{L}$ ample on X.

Notice that for any $t\in \mathrm{Pic}^-(X\times X)$, the correspondence $t\circ t^\vee$ is a symmetric and positive correspondence in $\mathrm{Pic}^-(X\times X)$ in the sense that there is a morphism $\phi: X\longrightarrow A$ from X to an abelian variety A with ample and symmetric lines bundle $\mathcal{L}$ such that $-t\circ t^\vee$ (up to a positive multiple) is the restriction on $X\times X$ of the Chern class of the following Poincaré bundle on $A\times A$:

$$-t\circ t^\vee = s(\mathcal{L}) := m^*\mathcal{L}\otimes p_1^*\mathcal{L}^{-1}\otimes p_2^*\mathcal{L}^{-1}\otimes 0^*\mathcal{L}$$

where $m: A^2\longrightarrow A$ is the addition map.

Based on the conjectured positivity of height pairing of zero cycles, we make the following:

Conjecture 2.4.2 *Let X be a curve in abelian variety A passing through* 0. *Let $\mathcal{L}_i$ be three semipositive and symmetric line bundle on A and let $s(\mathcal{L}_i)$ be the induced Poincaré bundles*:

$$s(\mathcal{L}_i) := m^*\mathcal{L}_i\otimes p_1^*\mathcal{L}_i^{-1}\otimes p_2^*\mathcal{L}_i^{-1}\otimes 0^*\mathcal{L}_i.$$

Let t_i be the correspondence induced by the restriction of $s(\mathcal{L}_i)$ in $X\times X$. Then

$$s(\widehat{\mathcal{L}}_1)\cdot s(\widehat{\mathcal{L}}_2)\cdot s(\widehat{\mathcal{L}}_3)|_{X\times X}\le 0,$$

where $\widehat{\mathcal{L}}_i$ is the admissible adelic metric on $\mathcal{L}_i$. Then this number vanishes if and only if cycle

$$(t_1\otimes t_2\otimes t_3)^*\Delta_e$$

is trivial.

2.5 Height pairing and relative dualising sheaf

In this subsection we want to give a formula for the self-intersection of Δ_e in terms of intersection theory of admissible metrized line bundles in our previous paper [39]. Recall that in this theory, an adelic line bundle $\mathcal{O}(\hat{\Delta}) := (\mathcal{O}(\Delta), \|\cdot\|)$ has been constructed for a curve over a global field. More precisely, for an archimedean place v, $-\log\|1\|_v$ is the usual Arakelov function on the Riemann surface $X_v(\mathbb{C})$. For non-archimedean place v,

$$-\log\|1\|(x,y)=i_v(x,y)+G_v(x,y),$$

where $i_v(x,y)$ is the local intersection index and $G_v(x,y)$ is a green's function on the metrized graph $R(X_v)$. We will prove in Sect. 3.5 that this adelic metric line bundle is actually integrable in sense of [40]. In the following we assume this fact and try to prove a formula for height of Gross–Schoen cycle.

Now fix a divisor e on X of degree 1 and put a metric on it by restriction of $\mathcal{O}(\hat{\Delta})$ on $X\times\{e\}$. Then we have the admissible (adelic) divisor on $X\times X$ satisfying conditions in Lemma 2.2.3:

$$\hat{t}_e=\hat{\Delta}-p_1^*\hat{e}-p_2^*\hat{e}+\hat{e}^2\cdot F.$$

Here the last number $\hat{e}^2\cdot F$ means $\hat{e}^2$ multiple of a vertical fiber F.

Theorem 2.5.1 *Assume that $g\geq 2$ and that the adelic metric line bundle $\mathcal{O}(\hat{\Delta})$ is integrable. Then with notation as above*

$$\begin{aligned}\langle\Delta_e,\Delta_e\rangle=\hat{t}_e^3=&\frac{2g+1}{2g-2}\hat{\omega}^2+6(g-1)\|x_e\|^2\\&-\log\|1_\Delta\|\cdot(\hat{\Delta}^2-6\hat{\Delta}\cdot p_1^*\hat{e}+6p_1^*\hat{e}\cdot p_2^*\hat{e}).\end{aligned}$$

Here the last term is an abbreviation for the adelic integration in (2.1.1) of $-\log\|1_\Delta\|$ against the product of the first Chern class of various arithmetic divisors involved.

Proof By Theorem 2.3.5, we have a formula

$$\begin{aligned}\langle\Delta_e,\Delta_e\rangle=\hat{t}_e^3&=(\hat{\Delta}-p_1^*\hat{e}-p_2^*\hat{e})^3+3\hat{e}^2\cdot(\hat{\Delta}-p_1^*\hat{e}-p_2^*\hat{e})^2\\&=\hat{\Delta}^3-3\hat{\Delta}^2\cdot(p_1^*\hat{e}+p_2^*\hat{e})+3\hat{\Delta}\cdot(p_1^*\hat{e}+p_2^*\hat{e})^2\\&\quad-(p_1^*\hat{e}+p_2^*\hat{e})^3+3\hat{e}^2F\cdot(\hat{\Delta}-p_1^*\hat{e}-p_2^*\hat{e})^2.\end{aligned}$$

The last four terms can be simplified as follows:

$$-3\hat{\Delta}^2\cdot(p_1^*\hat{e}+p_2^*\hat{e})=-6\hat{\Delta}^2\cdot p_1^*\hat{e},$$

$$3\hat{\Delta}\cdot(p_1^*\hat{e}+p_2^*\hat{e})^2=3\hat{\Delta}\cdot(p_1^*\hat{e}^2+p_2\hat{e}_2^2+2p_1^*\hat{e}\cdot p_2^*\hat{e})=6\hat{e}^2+6\hat{\Delta}\cdot p_1^*\hat{e}\cdot p_2^*\hat{e},$$

$$3\hat{e}^2F\cdot(\hat{\Delta}-p_1^*\hat{e}-p_2^*\hat{e})^2=3\hat{e}^2(2-2g-2-2+2)=-6g\hat{e}^2,$$

$$-(p_1^*\hat{e}+p_2^*\hat{e})^3=-(p_1^*\hat{e}^3+p_2^*\hat{e}^3+3p_1^*\hat{e}^2\cdot p_2^*\hat{e}+3p_1^*\hat{e}\cdot p_2^*\hat{e}^2)=-6\hat{e}^2.$$

In this way we have the following expression:

$$\begin{aligned}\langle\Delta_e,\Delta_e\rangle&=\hat{\Delta}^3-6\hat{\Delta}^2\cdot p_1^*\hat{e}+6\hat{\Delta}\cdot p_1^*\hat{e}\cdot p_2^*\hat{e}-6g\hat{e}^2\\&=-6g\hat{e}^2+\hat{\Delta}\cdot(\hat{\Delta}^2-6\hat{\Delta}\cdot p_1^*\hat{e}+6p_1^*\hat{e}\cdot p_2^*\hat{e}).\end{aligned}$$

The last term can be written as a sum of the restriction on $\hat{\Delta}$, and an intersection of $-\log\|1_\Delta\|$ against other cycles:

$$\begin{aligned}\langle\Delta_e,\Delta_e\rangle &= -6g\hat{e}^2+\hat{\omega}^2-6\hat{\omega}\cdot\hat{e}+6\hat{e}^2\\ &\quad -\log\|1_\Delta\|\cdot(\hat{\Delta}^2-6\hat{\Delta}\cdot p_1^*\hat{e}+6p_1^*\hat{e}\cdot p_2^*\hat{e}).\end{aligned}\qquad\square$$

When the genus of X is one this formula gives $\hat{t}_e^3=0$. Assume that $g>1$. Then we can get a formula in terms of the class of $x_e := e-\frac{1}{2g-2}\omega$ in $\mathrm{Pic}^0(X)_{\mathbb{Q}}$ using the formula for the Neron–Take height:

$$\begin{aligned}-\|x_e\|^2 &= \left(\frac{\hat{\omega}}{2g-2}-\hat{e}\right)^2=\frac{\hat{\omega}^2}{4(g-1)^2}-\frac{\hat{\omega}\hat{e}}{g-1}+\hat{e}^2\\ &= \frac{\hat{\omega}^2}{4(g-1)^2}-\left(\frac{\omega\hat{e}}{g-1}-\hat{e}^2\right).\end{aligned}$$

Corollary 2.5.2 *The pairing $\langle\Delta_e,\Delta_e\rangle$ gets its minimum when $e=\xi$. More precisely, we have*

$$\langle\Delta_e,\Delta_e\rangle=\langle\Delta_\xi,\Delta_\xi\rangle+6(g-1)\|x_e\|^2.$$

The last term in Theorem 2.5.1 is a sum of local contribution over places of k. The contributions from archimedean place is easy to compute:

Proposition 2.5.3 *At an archimedean place, the contribution in the last term of Theorem 2.5.1 is given by*

$$-2\sum_{i,j,\ell}\frac{1}{\lambda_\ell}\left|\int_X\phi_\ell(x)\bar{\omega}_i(x)\omega_j(x)\right|^2.$$

Proof At an archimedean place, $-\log\|1_\Delta\|=G(x,y)$ is the usual Arakelov Green's function, and $\mathcal{O}(\hat{\Delta})$ has curvature

$$h_\Delta(x,y)=d\mu(x)+d\mu(y)-\sqrt{-1}\sum_i(\omega_i(x)\bar{\omega}_i(y)+\omega_i(y)\bar{\omega}_i(y))$$

where ω_i is a bases of $\Gamma(X,\Omega)$ such that

$$\sqrt{-1}\int\omega_i\bar{\omega}_j=\delta_{i,j}.$$

It follows that

$$h_\Delta\,d\mu(x)=h_\Delta\,d\mu(y)=d\mu(x)\,d\mu(y)$$

also $\int G(x,y)\,d\mu(x)\,d\mu(y)=0$. Thus we get the formula

$$\int G\cdot[h_\Delta^2-3h_\Delta\cdot(p_1^*\,d\mu+p_2^*\,d\mu)+3(p_1^*\,d\mu+p_2^*\,d\mu)^2]=\int Gh_\Delta^2.$$

Let ϕ_ℓ be the real eigen function on X of the Laplacian for the Arakelov metric with eigenvalue $\lambda_\ell > 0$ then

$$G(x, y) = \sum_\ell \frac{\phi_\ell(x)\phi_\ell(y)}{\lambda_\ell}.$$

Since $\int_X G(x, y)\, d\mu(x) = \int_X g(x, y)\, d\mu(y) = 0$, it follows that

$$\begin{aligned}
\int Gh_\Delta^2 &= \int_{X^2} G(x, y)\Big[d\mu(x) + d\mu(y) - \sqrt{-1}\sum(\omega_i(x)\bar{\omega}_i(y) + \omega_i(y)\bar{\omega}(x)\Big]^2 \\
&= -\int_{X^2} G(x, y)\Big(\sum(\omega_i(x)\bar{\omega}_i(y) + \omega_i(y)\bar{\omega}(x)\Big)^2 \\
&= -\int G(x, y)\sum_{i,j}[\omega_i(x)\bar{\omega}_j(x)\bar{\omega}_i(y)\omega_j(y) + \bar{\omega}_i(x)\omega_j(x)\omega_i(y)\bar{\omega}_j(y)] \\
&= -\sum_{i,j,\ell}\frac{1}{\lambda_\ell}\Big[\int \phi_\ell(x)\omega_i(x)\bar{\omega}_j(x)\int \phi_\ell(y)\bar{\omega}_i(y)\omega_j(y) \\
&\quad + \int_X \phi_\ell(x)\bar{\omega}_i(x)\omega_j(x)\int_X \phi_\ell(y)\omega_i(y)\bar{\omega}_j(y)\Big].
\end{aligned}$$

Since ϕ_ℓ are all real, it follows that

$$\int Gh_\Delta^2 = -2\sum_{i,j,\ell}\frac{1}{\lambda_\ell}\left|\int_X \phi_\ell(x)\bar{\omega}_i(x)\omega_j(x)\right|^2. \qquad \square$$

Remark 1 The quality in the proposition is negative. Indeed it vanishes only when $\bar{\omega}_i\omega_j$ is perpendicular to all ϕ_ℓ. As $d\mu$ is the only measure satisfying this property, $\omega_i\bar{\omega}_j$ are all proportional to each other. Thus we must have $g = 1$, and thus a contradiction. This leads to a conjecture that all local contributions at bad place are all negative.

Remark 2 When X is hyperelliptic, Gross and Schoen have shown that Δ_ξ is rationally equivalent to 0. It follows that $\hat{t}_\xi^3 = 0$. Our conjecture thus gives a formula for $\hat{\omega}^2$ in terms of local contributions.

3 Intersections on reduction complex

The aim of this section is to describe an intersection theory on the product Z of two curves X and Y over a local field k and use this to finish the proof of Main Theorem 1.3.1. The reduction map on the usual curves over local field gives a reduction map

$$Z(\bar{k}) \longrightarrow R(Z) := R(X) \times R(Y)$$

where the right hand side is the product of the reduction graphs for X and Y. The space $R(X)$ has a simplicial structure with triangulation given by sides corresponding to vertices in $R(X)$ and $R(Y)$ and the diagonals D. The semistable models $\mathcal{X}$ and $\mathcal{Y}$ over finite extensions k' gives a model $\mathcal{X} \times_{\mathcal{O}_{k'}} \mathcal{Y}$. Blow-up these models at its singular points to get regular (but not semistable) models $\mathcal{Z}$ for Z. We will show that the vertical divisors of $\mathcal{Z}$ can be naturally identified with piece-wise linear functions on $R(Z)$. The intersection pairing on vertical divisors can be extended into a pairing on continuous functions f_i $(i=1,2,3)$ on $R(Z)$ which are piece-wise smooth except on sides:

$$(f_1, f_2, f_3) = \int_{R(X)\times R(Y)} (\Delta_x(f_1)(f_{2y}f_{3y}) + \Delta_x(f_2)(f_{3y}f_{1y}) + \Delta_x(f_3)(f_{1y}f_{2y}))\,dx\,dy + \frac{1}{4}\int_D \delta(f_1)\delta(f_2)\delta(f_3)\,dx,$$

where Δ_x and Δ_y are Laplacian operators on piece-wise smooth functions, and $\delta(f_i)$ are some invariants of f_i on the diagonal to measure the difference of two limits of first derivatives defined in [39, Appendix].

3.1 Regular models

In this subsection, we will study local intersection theory on a product of two curves with semi-stable reduction. We first blow-up the singular points in the special fiber to get a regular model. This model has non-reduced exceptional divisors isomorphic to $\mathbb{P}^1\times\mathbb{P}^1$ but is canonical in the sense that it does not depend on the order of blowing-ups. Also one can get semistable models by blowing-down exceptional divisor to one of two factors $\mathbb{P}^1$. Then we give an explicit description of the intersections of curves and surfaces in this threefold. Finally, we show that the inverse of the relative dualising sheaf on a semistable model can be written as in a similar way as the restriction of the ideal sheaf on the proper transformation of the diagonal.

Let R be a discrete valuation ring with fraction field K and algebraically closed residue field k. Let X and Y be two smooth, absolute connected, and projective curves over K, and $Z=X\times Y$ their fiber product over R. Assume that X and Y have regular and semistable models X_R and Y_R with no self intersections. Then Z has a model $X_R\times_R Y_R$ which is singular at products of two singular points on special fibers X_k and Y_k. Blowing-up these singular points we obtain a regular model Z_R over R.

Covering charts

The special fiber of Z_R consists of proper transformations $\widetilde{AB}$ of the products of components A and B of X_R and Y_R and exceptional divisors $E_{p,q}$ indexed by singular points p and q of X_k and Y_k. To see this, we cover X_R and Y_R formally near their singular points by local completions of the open affine schemes of the form:

$$V = \operatorname{Spec} R[x_0,x_1]/(x_0x_1-\pi), \qquad W = \operatorname{Spec} R[y_0,y_1]/(y_0y_1-\pi).$$

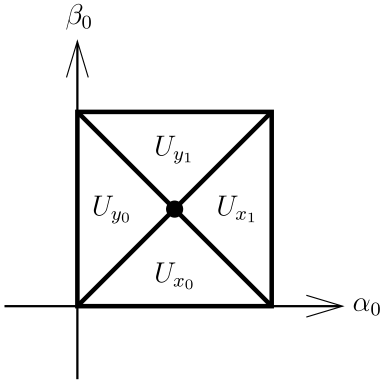
Then Z_R is covered by blow-up at the singular point (x_0,x_1,y_0,y_1) of

$$V\times_R W = \operatorname{Spec} R[x_0,x_1,y_0,y_1]/(x_0x_1-\pi, y_0y_1-\pi).$$

It is clear that Z_R is covered by four charts of spectra of subrings of the fraction field $K(x_0,y_0)$ of $V\times W$:

$$U_{x_0} = \operatorname{Spec} R[x_0, y_0/x_0, y_1/x_0]/((y_0/x_0)(y_1/x_0)x_0^2-\pi),$$
$$U_{x_1} = \operatorname{Spec} R[x_1, y_0/x_1, y_1/x_1]/((y_0/x_1)(y_1/x_1)x_1^2-\pi),$$

Fig. 1 Reduction complex



$$U_{y_0} = \mathrm{Spec} R[y_0, x_0/y_0, x_1/y_0]/((x_0/y_0)(x_1/y_0)y_0^2 - \pi),$$
$$U_{y_1} = \mathrm{Spec} R[y_1, x_0/y_1, x_1/y_1]/((x_0/y_1)(x_1/y_1)y_1^2 - \pi).$$

We use ord to denote the valuation (or order as called in this paper) on K and extend it to an algebraic closure $\bar{K}$ so that $\mathrm{ord}(\pi) = 1$. Let $\bar{R}$ be the valuation ring of elements of non-negative order. Then $\bar{R}$ points in V and W has coordinates $x_i, y_i \in \mathbb{R}$ with orders $\alpha_i := \mathrm{ord}(x_i)$, $\beta_i := \mathrm{ord}(y_i)$ non-negative rational numbers satisfying

$$\alpha_0 + \alpha_1 = \beta_0 + \beta_1 = 1.$$

We consider the reduction map:

$$Z_R(\bar{R}) \longrightarrow [0, 1]^2, \qquad p \mapsto (\mathrm{ord} x_0(p), \mathrm{ord} y_0(p)).$$

The $\bar{R}$-points in these charts are can be described by the following inequalities:

$$\begin{aligned} U_{x_0}: &\quad \min(\beta_0, 1 - \beta_0) \geq \alpha_0, \\ U_{x_1}: &\quad \min(\beta_0, 1 - \beta_0) \geq 1 - \alpha_0, \\ U_{y_0}: &\quad \min(\alpha_0, 1 - \alpha_0) \geq \beta_0, \\ U_{y_1}: &\quad \min(\alpha_0, 1 - \alpha_0) \geq 1 - \beta_0. \end{aligned}$$

These are exactly four domains in the unit square divided by two diagonals. We call the square $[0, 1]^2$ with triangulation by diagonals *the reduction complex of* Z_R.

Figure 1 shows the reduction complex associated to Z_R placed on (α_0, β_0)-coordinate axes. The four corners and the center point of the square correspond to the four product components and the exceptional divisor of Z_R, respectively. The eight segments in the square correspond to the curves of intersection of the components in Z_R. The four 2-cells labeled $U_{x_0}, U_{x_1}, U_{y_0}, U_{y_1}$ correspond to the four points of Z_R where three components meet transversally.

Let A_0, A_1, B_0, B_1 be divisors in V and W defined by x_1, x_0, y_1, y_0 respectively. Then the special fiber of Z_R is a union of five divisors

$$\widetilde{A_0 B_0}, \quad \widetilde{A_0 B_1}, \quad \widetilde{A_1 B_0}, \quad \widetilde{A_1 B_1}, \quad E.$$

Here the first four terms are proper transforms of the products of curves in $V \times_R W$ and E is the exceptional divisor. Each divisor is defined by an element in each of the above charts.

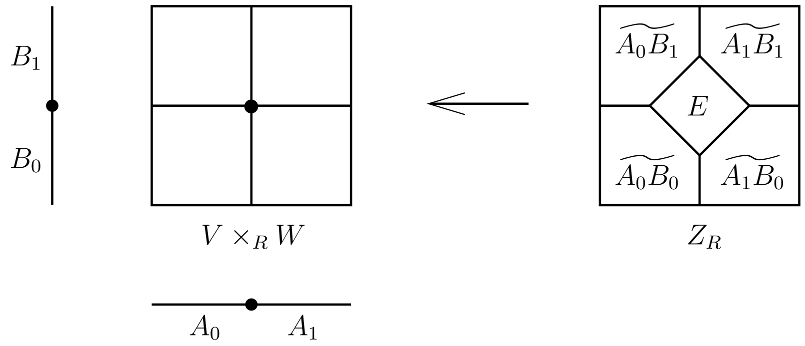


Fig. 2 Special fibers of blow-ups

For example, E is defined by equations x_0, x_1, y_0, y_1 in the above four charts, and $\widetilde{A_0B_0}$ is defined by $y_1/x_0, 1, x_1/y_0, 1$ respectively.

Figure 2 shows the configuration of special fibers before and after blow-ups. On the left is a diagram representing the various product components in the special fiber of $V \times_R W$ and how they project onto each factor. On the right is a diagram representing Z_R, which is the blow-up of $V \times_R W$. The components $\widetilde{A_iB_j}$ are strict transforms of product components in $V \times_R W$. The component shaped like a diamond in the diagram on the right collapses to the singular point at the center of the cross in the left diagram.

Intersections

Back to the global situation. The following properties are easy to verified:

- each component $\widetilde{AB}$ is obtained from $A \times B$ by blowing at the singular points of $X_R \times Y_R$ on $A \times B$, and has multiplicity one in divisor (π);
- two different components $\widetilde{A_0B_0}$ and $\widetilde{A_1B_1}$ intersect if and only if either $A_0 = A_1$ and $B_0 \cap B_1 \neq \emptyset$, or $B_0 = B_1$ and $A_0 \cap A_1 \neq \emptyset$;
- each exceptional divisor is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and has multiplicity 2 in (π);
- two component $\widetilde{AB}$ and $E_{p,q}$ intersect if and only if $p \in A$ and $q \in B$;
- two different exceptional divisors do not intersect.

In the following we want to compute the intersection numbers more precisely. Assume that $p = A_0 \cdot A_1$, $q = B_0 \cdot B_1$. The intersection can be described as follows:

- $\widetilde{A_0B_0} \cdot \widetilde{A_0B_1}$ is given by the proper transformations $\widetilde{A_0q}$ of $A_0 \times q$ in $\widetilde{A_0B_0}$ and $\widetilde{A_0B_1}$;
- one may choose an isomorphism $E_{p,q} \simeq \mathbb{P}^1 \times \mathbb{P}^1$ so that the following hold:

$$\widetilde{A_0B_0} \cdot E_{p,q} = \mathbb{P}^1 \times 0, \qquad \widetilde{A_1B_1} = \mathbb{P}^1 \times \infty,$$

$$\widetilde{A_0B_1} \cdot E_{p,q} = 0 \times \mathbb{P}^1, \qquad \widetilde{A_1B_0} = \infty \times \mathbb{P}^1.$$

We may also compute the self intersection of vertical divisors in Chow group using the following equation: for any vertical divisor F in Z_k,

$$0 = F \cdot (\pi) = F \cdot \left(\sum_{A,B} \widetilde{AB} + 2 \sum_{p,q} E_{p,q} \right)$$

where the sums are over components A, B and singular points p and q of X_k, Y_k. It follows that

$$\widetilde{AB}^2 = -\sum_q \widetilde{Aq} - \sum_p \widetilde{pB} - 2\sum_{p,q} \text{exceptional divisors over } (p,q),$$

$$E_{p,q}^2 = -\mathbb{P}^1 \times 0 - 0 \times \mathbb{P}^1.$$

Here the sums are over singular points p, q in A, B.

In the following we want to compute the intersection numbers between a curve C and a surface F included in the special fiber $\mathcal{Z}_k$ of $\mathcal{Z}$. Assume that C is included in a surface G then

$$C \cdot F = (C \cdot F_G)_G$$

where F_G is the pull-back of F in G via the inclusion $G \longrightarrow Z_R$, and the right hand is an intersection in G. Thus to study intersection pairing it suffices to study the intersection pairing of Z with the subgroup $B(G)$ of divisors of G generated by $F_G = F \cdot G$ in $\mathrm{NS}(G)$, the Néron–Severi group of G.

Lemma 3.1.1 *The intersection pairing on $B(G)$ is non-degenerate.*

Proof If $G = \widetilde{AB}$, this group is generated by $\mathrm{NS}(A)$, $\mathrm{NS}(B)$ via projections and the exceptional divisors. It is clear that the intersection pairing on $B(G)$ is non-degenerate. If $G = \mathbb{P}^1 \times \mathbb{P}^1$, then $B(G) = \mathrm{NS}(G)$ and the intersection pairing is clearly non-degenerate. □

By this lemma, we may replace C by its projection $B(C)$ in $B(G)$. As all $B(G)$'s are generated by intersections $F \cdot G$, we need only describe the intersection of three surfaces in Z_k. Let F_1, F_2, F_3 be three components.

- If they are all distinct, then the intersection is non-zero only if they have the following forms after an reordering

$$F_1 = \widetilde{AB_0}, \qquad F_1 = \widetilde{AB_1}, \qquad F_3 = E_{p,q}$$

  where p is a singular point on A and $q = B_0 \cdot B_1$. In this case the intersection is 1:

$$F_1 \cdot F_2 \cdot F_3 = 1.$$

- If $F_1 = F_2 \neq F_3$ then

$$F_1 \cdot F_2 \cdot F_3 = i^*(F_1)^2, \quad i : F_3 \longrightarrow X.$$

  Furthermore if $F_1 = \widetilde{AB_0}$, $F_3 = \widetilde{AB_1}$, then $i^* F_1 = \widetilde{Aq}$ and

$$F_1^2 \cdot F_3 = i^*(F_1)^2 = -s(A) := -\text{number of singularity of } X_k \text{ on } A.$$

- If $F_1 = \widetilde{AB}$, $F_3 = E_{p,q}$ with $p \in A$ and $q \in B$, then $i^* F_1$ is a $\mathbb{P}^1$ on $F_3 \simeq \mathbb{P}^1 \times \mathbb{P}^1$ of degree $(1,0)$ or $(0,1)$. It follows that

$$F_1^2 \cdot F_3 = 0.$$

- If $F_1 = E_{p,q}$, $F_3 = \widetilde{AB}$ with $p \in A$, $q \in B$, then i^*F_1 is one exceptional divisor on F_3 and then

$$F_1^2 \cdot F_3 = -1.$$

- Finally if $F_1 = F_2 = F_3$ then

$$0 = F^2 \cdot (\pi) = F^2 \cdot \left(\sum_{A,B} \widetilde{AB} + 2 \sum_{p,q} E_{pq} \right).$$

It follows that

$$\widetilde{AB}^3 = 2s(A)s(B), \qquad \widetilde{E}_{p,q}^3 = 2.$$

Relative dualising sheaf

Now assume that $X = Y$. Let $\widetilde{\Delta}_R \subset Z_R$ be the Zariski closure of the diagonal in Z_R. Then $\widetilde{\Delta}_R$ is the blowing-up of X_R at its double points in the special fiber. Let $i : \widetilde{\Delta}_R \longrightarrow Z_R$ and $f : \widetilde{\Delta}_R \longrightarrow X_R$ be the induced morphisms and let ω be the relative dualising sheaf on X_R.

Lemma 3.1.2

$$f^*\omega = i^*\mathcal{O}_{Z_R}(-\widetilde{\Delta}_R).$$

Proof Since the question is local, we may assume that X_R is given by

$$\mathrm{Spec} R[x_0, x_1]/(x_0 x_1 - \pi)$$

then the relative dualising sheaf is given by the subsheaf of $\Omega^1_{X_R/R} \otimes K(X)$ generated by $dx_0/x_0 = -dx_1/x_1$. The scheme Z_R is obtained by blowing up the singular point on $X_R \times_R X_R$ and is covered by

$$U_{x_0}, \quad U_{x_1}, \quad U_{y_0}, \quad U_{y_1}.$$

The subscheme $\widetilde{\Delta}_R$ is defined by an ideal I generated by $y_0/x_0 - 1$, $y_1/x_1 - 1$ in these charts and has coverings given by

$$V_{x_0} = \mathrm{Spec} R[x_0, x_1/x_0]/((x_1/x_0)x_0^2 - \pi),$$
$$V_{x_1} = \mathrm{Spec} R[x_1, x_0/x_1]/((x_0/x_1)x_1^2 - \pi).$$

As dx_0 and dx_1 are the image of $y_0 - x_0$ and $y_1 - x_1$ on $\widetilde{\Delta}_R$, we see that I/I^2 is generated by $dx_0/x_0 = -dy_0/y_0$. □

3.2 Bas changes and reduction complex

In this subsection, we describe the pull-back of vertical divisors respect to base changes. The direct limit of vertical divisors can be identified with piece-wise linear functions on the reduction complex which is the product of metrized graphs.

Let S be a ramified extension of R of degree n with fraction field L. Let Z_S be the model of Z_L obtained by the same way as Z_R. In the following we want to describe the morphism

$Z_S \longrightarrow Z_R$ in terms of charts. As this question is local, we may assume that X_R and Y_R are given by

$$X_R = \operatorname{Spec} R[x_0, x_1]/(x_0x_1 - \pi), \qquad Y_R = \operatorname{Spec} R[y_0, y_1]/(y_0y_1 - \pi).$$

Then Z_R is obtained by blowing up at the singular point (x_0, x_1, y_0, y_1) of $X_R \times_R Y_R$ and is covered by four charts of spectra of subrings of the fraction field $K(x_0, y_0)$ of $X_R \times Y_R$:

$$U_{x_0}, \quad U_{x_1}, \quad U_{y_0}, \quad U_{y_1}.$$

Let t be a local parameter of S such that $\pi = t^n$. For integers $a, b \in [0, n-1]$, set

$$x_{0a} = x_0/t^a, \quad x_{1a} = x_1/t^{n-1-a}, \quad y_{0b} = y_0/t^b, \quad y_{1b} = y_1/t^{n-1-b}.$$

Then X_S and Y_S are unions of the following spectra:

$$V_a = \operatorname{Spec} R[x_{0,a}, x_{1,a}]/(x_{0,a} \cdot x_{1,a} - t), \quad 0 \le a \le n-1,$$
$$W_b = \operatorname{Spec} R[y_{0b}, y_{1b}]/(y_{0b} \cdot y_{1b} - t), \quad 0 \le b \le n-1.$$

In terms of valuations on $\bar{R}$-points, $\alpha = \operatorname{ord}_\pi(x_0)$, $\beta = \operatorname{ord}_\pi(y_0)$, V_α and W_β are defined by inequalities

$$a/n \le \alpha \le (a+1)/n, \qquad b/n \le \beta \le (b+1)/n.$$

The special component of X_S has n-singular points with one on V_a each and is the union of $n+1$-components $A_{n,a}$ ($a = 0 \dots n$) defined by x_{1a} in U_a if $a \le n-1$, by $x_{0,a-1}$ in U_{a-1} if $a \ge 1$, and by 1 on other components. Similarly, we have components $B_{n,b}$ for Y_S. The product $V_a \times W_b$ has the special fiber to be a union of

$$A_{n,a} \times B_{n,b}, \quad A_{n,a} \times B_{n,b+1}, \quad A_{n,a+1} \times B_{n,b}, \quad A_{n,a+1} \times B_{n,b+1}.$$

The scheme Z_S is covered by the blowing up $U_{a,b}$ of $V_a \times W_b$ at its singular point. Here we have diagrams of the schemes $X_S \times_S Y_S$ and the blow-up along the singular points, denoted Z_S (for the case $n = 3$). We have labeled a general product component $A_{n,a} \times B_{n,b}$ as well as its strict transforms $\widetilde{AB}_{n,a,b}$. We have also labeled one of the exceptional divisors $E_{a,b}$ that arises from the blow-up

The scheme $U_{a,b}$ is covered by four affine schemes with equations:

$$U_{x_0,a,b}: \quad \left(\frac{y_{0b}}{x_{0a}}\right)\left(\frac{y_{1b}}{x_{0a}}\right)(x_{0a})^2 = t,$$

$$U_{x_1,a,b}: \quad \left(\frac{y_{0b}}{x_{1a}}\right)\left(\frac{y_{1b}}{x_{1a}}\right)(x_{1a})^2 = t,$$

$$U_{y_0,a,b}: \quad \left(\frac{x_{0a}}{y_{0b}}\right)\left(\frac{x_{1a}}{y_{0b}}\right)(y_{0b})^2 = t,$$

$$U_{y_1,a,b}: \quad \left(\frac{x_{0a}}{y_{1b}}\right)\left(\frac{x_{1a}}{y_{1b}}\right)(y_{1b})^2 = t.$$

The divisor (t) has five components over $V_a \times W_b$:

$$\widetilde{AB}_{n,a,b}, \quad \widetilde{AB}_{n,a,b+1}, \quad \widetilde{AB}_{n,a+1,b}, \quad \widetilde{AB}_{n,a+1,b+1}, \quad E_{a,b}.$$

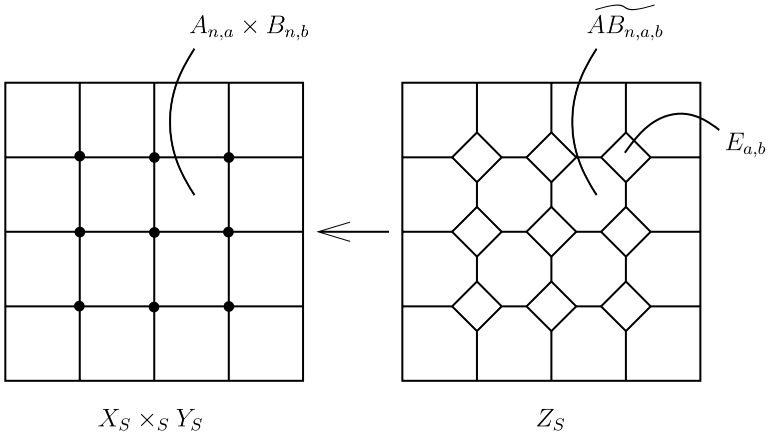
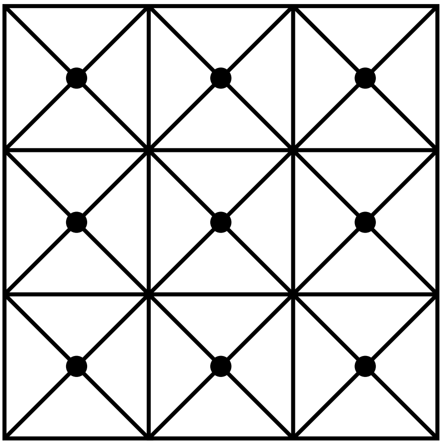


Fig. 3 Special fibers of base changes

Fig. 4 Reduction complex of base changes



In terms of valuations $\alpha = \mathrm{ord}_\pi(x_0)$, $\beta = \mathrm{ord}_\pi(y_0)$, $U_{a,b}$ are defined by following inequalities:

$$\min(\alpha_0 - a/n, 1/n - (\alpha_0 - a/n)) \geq \beta_0 - b/n,$$

$$\min(\beta_0 - b/n, 1/n - (\beta_0 - b/n)) \geq 1/n - (\alpha_0 - a/n),$$

$$\min(\beta_0 - b/n, 1/n - (\beta_0 - b/n)) \geq (\alpha_0 - a/n),$$

$$\min(\alpha_0 - a/n, 1/n - (\alpha_0 - a/n)) \geq 1/n - (\beta_0 - b/n).$$

These are parts divided by diagonals in the square $[a/n, (a+1)/n] \times [b/n, (b+1)/n]$. Thus the morphism from Z_S to Z_R is given by the inclusion of the parts in $[0, 1]^2$.

Here is a diagram of the reduction complex associated to the special fiber in Fig. 3. The vertices of the complex correspond to the components of Z_S, with the nine fat points corresponding to the 9 exceptional divisors. (They have multiplicity 2 in the special fiber.)

Pull-back of vertical divisors

In the following we want to compute the pull-back of vertical divisors Z_R in Z_S. Let $\varphi : Z_S \longrightarrow Z_R$ denote the morphism. Let us index divisors using set Λ_n on $[0, 1]^2$ of the form $(\frac{a}{2n}, \frac{b}{2n})$:

$$D_{\frac{a}{n},\frac{b}{n}} := \widetilde{AB_{n,a,b}}, \quad a, b \in \mathbb{Z},$$

$$D_{\frac{a}{n},\frac{b}{n}} := 2E_{\frac{a-1}{2},\frac{b-1}{2}}, \quad a, b \in \mathbb{Z} + \frac{1}{2},$$

$$D_{\frac{a}{n},\frac{b}{n}} := 0, \quad a + b \in \mathbb{Z} + \frac{1}{2}.$$

Lemma 3.2.1

$$\varphi^*\widetilde{AB} = n \sum_{(a,b)\in\Lambda_n} \max(1 - a - b, 0)D_{a,b},$$

$$\varphi^*(E) = n \sum_{(a,b)\in\Lambda_n} \min(a, 1 - a, b, 1 - b)D_{a,b}.$$

Proof First we notice that $\varphi^*\widetilde{A_0B_0}$ is defined as zeros of $x_1/y_0 = y_1/x_0$ on U_{x_0} and U_{y_0}, and 1 on U_{x_1} and U_{y_1}. Thus it is defined by 1 on $U_{x_i,a,b}$ and $U_{y_i,a,b}$ if $a + b \geq n$. It follows that the multiplicity of $\widetilde{AB_{n,a,b}}$ and $E_{a,b}$ are zero if $a + b \geq n$. Now we assume that $a + b < n$. Then $x_1/y_0 = y_1/x_0$ on U_{x_0} has the following expressions in the charts $U_{x_0,a,b}$ and $U_{y_0,a,b}$:

$$\frac{y_1}{x_0} = \frac{y_{1b}}{x_{0a}}t^{n-1-a-b} = \left(\frac{y_{0b}}{x_{0a}}\right)^{n-a-b} \left(\frac{y_{1b}}{x_{0a}}\right)^{n-1-a-b} (x_{0a})^{2(n-1-a-b)},$$

$$\frac{x_1}{y_0} = \frac{x_{1a}}{y_{0b}}t^{n-1-a-b} = \left(\frac{x_{0a}}{y_{0b}}\right)^{n-1-a-b} \left(\frac{x_{1a}}{y_{0b}}\right)^{n-a-b} (y_{0b})^{2(n-1-a-b)}.$$

Either one of these formulae shows that the pull-back of $\widetilde{A_0B_0}$ has multiplicity $n - a - b$ at $\widetilde{AB_{n,a,b}}$, and $2(n - 1 - a - b)$ at $E_{a,b}$. This proves the first formula in lemma.

For exceptional divisor, we may using the following decompositions

$$\mathrm{div}(\pi) = \sum_{i,j=0}^{1} \widetilde{A_iB_j} + 2E,$$

$$\mathrm{div}(t) = \sum_{(a,b)\in\Lambda} D_{a,b}.$$

The fact $\varphi^*\mathrm{div}(\pi) = n\mathrm{div}(t)$ implies that

$$\varphi^*(E) = n \sum_{(a,b)\in\Lambda} \min(a, 1 - a, b, 1 - b)D_{a,b}.$$

□

Remarks

Alternatively, we may compute pull-back of an exceptional divisor directly by using charts:

$$x_0 = \left(\frac{y_{0b}}{x_{0a}}\right)^a \left(\frac{y_{1b}}{x_{0a}}\right)^a (x_{0a})^{2a+1},$$

$$x_1 = \left(\frac{y_{0b}}{x_{1a}}\right)^{n-a-1} \left(\frac{y_{1b}}{x_{1a}}\right)^{n-a-1} (x_{1a})^{2n-2a-1},$$

$$y_0 = \left(\frac{x_{0a}}{y_{0b}}\right)^b \left(\frac{x_{1a}}{y_{0b}}\right)^b (y_{0b})^{2b+1},$$

$$y_1 = \left(\frac{x_{0a}}{y_{1b}}\right)^{n-b-1} \left(\frac{x_{1a}}{y_{1b}}\right)^{n-b-1} (y_{1b})^{2n-b-1}.$$

Reduction complex

Let $R(X)$ and $R(Y)$ be the reduction graphs of X and Y respectively with reduction morphisms

$$r_X: \quad X(\bar{K}) \longrightarrow R(X), \qquad r_Y: \quad Y(\bar{K}) \longrightarrow R(Y).$$

Recall that $R(X)$ and $R(Y)$ are metrized graphs with edges of lengths 1 parameterized by irreducible components and singular points in special fibers of X_R and Y_R. The reduction map is given as follows. An edge $E \simeq [0, 1]$ corresponds to a singular point near which X_R has local structure

$$R[x_0, x_1]/(x_0 x_1 - \pi)$$

such that 0 and 1 correspond to $x_1 = 0$ and $x_0 = 0$ respectively. Then the reduction morphism is given by

$$(x_0, y_0) \longrightarrow \mathrm{ord}(x_0).$$

Here $\mathrm{ord}(x_0)$ is a valuation on $\bar{K}$ such that $\mathrm{ord}(\pi) = 1$. The reduction map of a point is a vertex (resp. a smooth point in a edge) if and only if the reduction modulo π of this point is in a corresponding smooth point in a component (resp. a singular point). After a base change L/K of degree n, the dual graph is unchanged if we change the lengths of edges to be $1/n$. In other words, the irreducible components of X_L and singular points corresponding to rational points on $R(X)$ with denominator n and intervals between them.

Let us define the reduction complex of $Z = X \times Y$ over R to be

$$R(Z_R) := R(X) \times R(Y)$$

with a triangulation by adding diagonals. We have induced reduction map:

$$r : Z(\bar{K}) \longrightarrow R(Z_R).$$

The vertices in the complex correspond to irreducible components in Z_R; the edges correspond to intersection of two components; the triangle correspond to intersection of three components. For an irreducible component C of in the special component of Z_R, we let $r(C) \in R(Z)$ to denote the corresponding vertex.

The reduction complex of base change $[L : K] = n$ after a change of size coincides with the same complex with an n-subdivision of squares and then an triangulation on it. Thus we may define $R(Z)$ to be the underline metric space of the complex $R(Z_R)$ without triangulation.

Let $V(Z_R)$ denote the group of divisors with real coefficients supported in the special fiber. Let $\mathbb{R}(R(Z))$ denote the space of continuous real functions on $R(Z)$. Then we can define a map

$$V(Z_R) \longrightarrow \mathbb{R}(R(Z)), \quad F \mapsto f_{R,F}$$

with following properties: write

$$F = \sum_C a_C C + \sum_E 2b_E E$$

where C runs through all non-exceptional components of Z_R, and E all exceptional components, then

- $f_{R,F}$ is linear on all triangles,
- $f_{R,F}(r(C)) = a_C$,
- $f_{R,F}(r(E)) = b_E$.

Let $V(Z)$ denote the direct limit of $V(Z_S)$ via pull-back map in the projective system X_S defined by finite extensions of R in $\bar{R}$. The main result in the last subsection gives the following:

Lemma 3.2.2 *The map* $[S : R]^{-1} f_S$ *induces a map*

$$\phi : \quad V(Z) = \lim V(Z_S) \longrightarrow \mathbb{R}(R(Z)).$$

Moreover the image of this map are continuous function which are linear on some n-triangulation. So it is dense in the space of continuous functions.

3.3 Triple pairing

In this section we are try to define a triple pairing for functions on $R(Z)$. More precisely, let f_1, f_2, f_3 be three continuous functions on $R(Z)$. Then for any positive integer n, let us define piece wise linear functions $f_{i,n}$ such that $f_{i,n}$ is linear on each triangle of the n-triangulation, and has the same values as f_i at vertices of triangles. Then $f_{i,n}$ will correspond to vertical divisors $F_{i,n}$ in $V(Z_{R_n})$ where R_n is a ramified extension of degree n. Lets define the triple pairing

$$(f_{1,n}, f_{2,n}, f_{3,n}) = n^2(F_{1,n} \cdot F_{2,n} \cdot F_{3,n})$$

where the right hand side is the intersection pairing on Z_{R_n}. Notice that if $f_i = f_{i,1}$, then $f_{i,n} = f_i$ and $F_{i,n} = n^{-1}\varphi_n^* D_{i,1}$ by Sect. 3.2, where φ_n is the projection $Z_{R_n} \longrightarrow Z_R$. It follows that

$$n^2(F_{1,n} \cdot F_{i,n} \cdot F_{i,n}) = n^{-1}\varphi_n^* F_{1,1} \cdot \varphi_n^* F_{2,1} \cdot \varphi_n^* F_{3,1} = F_{1,1} \cdot F_{2,1} \cdot F_{3,1}.$$

Thus the above pairing does not depend on the choice of n if every $f_i = f_{i1}$. We want to examine when the limit does exist and what expression we can get for this limit.

Proposition 3.3.1 *Assume that the functions f_1, f_2, f_3 on $R(Z)$ are smooth on each square with bounded first and second derivatives. Then the intersection pairing on vertical divisors induces a trilinear pairing*

$$(f_1, f_2, f_3) = \int_{R(Z)} (f_{1x} f_{2y} f_{3xy} + permutations)\, dx\, dy,$$

where the integrations are taken on the smooth part of the complex and f_{1x}, f_{2y}, etc. are partial derivatives for any directions on edges of $R(X)$ and $R(Y)$.

Proof Our first remark is that from the formulae given in Sect. 3.1, the computation can be taken as a sum of intersections on squares. In other words, we may assume that both X_R and Y_R have one singular point. Then the complex $R(Z)$ can be identified with the square $[0, 1]^2$. By calculation in Sects. 3.1 and 3.2, we have the following expression of divisors:

$$F_{i,n} = \sum_{(a,b)\in\Lambda_n} f_i(a,b) D_{a,b}.$$

Again the intersection can be taken on sum of small squares:

$$\begin{aligned} &n^2(F_{1n}\cdot F_{2n}\cdot F_{3n}) \\ &= n^2 \sum_{(a,b)\in\Lambda_n} \prod_{i=1}^{3} \Bigg(f_i(a,b)D_{a,b} + f_i\left(a+\frac{1}{n}, b\right) D_{a+\frac{1}{n},b} + f_i\left(a, b+\frac{1}{n}\right) D_{a,b+\frac{1}{n}} \\ &\quad + f_i\left(a+\frac{1}{n}, b+\frac{1}{n}\right) D_{a+\frac{1}{n},b+\frac{1}{n}} + f_i\left(a+\frac{1}{2n}, b+\frac{1}{2n}\right) D_{a+\frac{1}{2n},b+\frac{1}{2n}} \Bigg)_{a,b} \end{aligned}$$

where the last product is the intersection on the square starting at (a, b). As the sum

$$D_{a,b} + D_{a+1/n,b} + D_{a,b+1/n} + Da + 1/n, b + 1/n + D_{a+1/2n,b+1/2n}$$

has zero intersection with products, we subtract each coefficient by $D_{a+1/2n}$. Thus the last product has the form

$$\prod_{i=1}^{3} (a_i D_{a,b} + b_i D_{a+1/n,b} + c_i D_{a,b+1/n} + d_i D_{a+1/n,b+1/n})$$

with

$$a_i = f_i(a,b) - f_i(a + 1/2n, b + 1/2n),$$

$$b_i = f_i(a + 1/n, b) - f_i(a + 1/2n, b + 1/2n),$$

$$c_i = f_i(a, b + 1/n) - f_i(a + 1/2n, b + 1/2n),$$

$$d_i = f_i(a + 1/n, b + 1/n) - f_i(a + 1/2n, b + 1/2n).$$

We use the following facts to compute this product among the divisors in the sum:

- the product of three distinct element will be 0;
- the product of the square of one divisor with another divisor is -1 if they intersect, and 0 otherwise;
- the cube of any element is 2.

Then we have

$$\begin{aligned}&\prod_{i=1}^{3}(a_i D_{a,b}+b_i D_{a+1/n,b}+c_i D_{a,b+1/n}+d_i D_{a+1/n,b+1/n})\\&\quad=2(a_1a_2a_3+b_1b_2b_3+c_1c_2c_3+d_1d_2d_3)\\&\qquad-(a_1a_2+d_1d_2)(b_3+c_3)-(b_1b_2+c_1c_2)(a_3+d_3)+\text{permutations}.\end{aligned}$$

Write Taylor expansions for a_i, b_i, c_i, d_i at $a'=a+1/2n, b'=b+1/2n$:

$$\alpha_i=\frac{1}{2n}(f_{ix}(a',b')+f_{iy}(a',b')),\qquad \beta_i=\frac{1}{8n^2}(\Delta f_i(a',b')+2f_{ixy}(a',b')),$$
$$\gamma_i=\frac{1}{2n}(f_{ix}(a',b')-f_{iy}(a',b')),\qquad \delta_i=\frac{1}{8n^2}(\Delta f_i(a',b')-2f_{ixy}(a',b')).$$

Then

$$a_i=-\alpha_i+\beta_i+O(1/n^3),$$

$$b_i=\gamma_i+\delta_i+O(1/n^3),$$

$$c_i=-\gamma_i+\delta_i+O(1/n^3),$$

$$d_i=\alpha_i+\beta_i+O(1/n^3).$$

It is clear that the product is an even function in α_i and γ_i. It follows that their appearance in the product have the even total degree. Also we have neglected term $O(1/n^5)$. Thus we can write

$$\begin{aligned}&\prod_{i=1}^{3}(a_i D_{a,b}+b_i D_{a+1/n,b}+c_i D_{a,b+1/n}+d_i D_{a+1/n,b+1/n})\\&\quad=4(\alpha_1\alpha_2\beta_3+\gamma_1\gamma_2\delta_3)-4\alpha_1\alpha_2\delta_3-4\gamma_1\gamma_2\beta_3+\text{permutations}\\&\quad=4(\alpha_1\alpha_2-\gamma_1\gamma_2)(\beta_3-\delta_3)+\text{permutations}.\end{aligned}$$

By a direct computation, we see that

$$\alpha_1\alpha_2-\gamma_1\gamma_2=\frac{1}{2n^2}(f_{1x}(a',b')f_{2y}(a',b')+f_{1y}(a',b')f_{2x}(a',b')),$$

$$\beta_3-\delta_3=\frac{1}{2n^2}f_{ixy}(a',b').$$

It follows that

$$\prod_{i=1}^{3}(a_i D_{a,b} + b_i D_{a+1/n,b} + c_i D_{a,b+1/n} + d_i D_{a+1/n,b+1/n})$$
$$= \frac{1}{n^4} f_{1x}(a', b') f_{2y}(a', b') f_{ixy}(a', b') + \text{permutations} + O(1/n^5).$$

Put everything together to obtain

$$n^2(F_{1n} \cdot F_{2n} \cdot F_{3n}) = \frac{1}{n^2} \sum_{a,b}^{n-1} (f_{1x}(a', b') f_{2y}(a', b') f_{ixy}(a', b') + \text{permutations}) + O(1/n).$$

This is of course convergent to

$$\int_0^1 \int_0^1 (f_{1x} f_{2y} f_{3xy} + \text{permutations})\, dx\, dy.$$

Adding all integrals over squares we obtain the identity in lemma. □

3.4 Intersection of functions in diagonals

Now we want to treat case where f_i has some singularity lying on edges of some n-triangulation. For this we decompose the intersection into smooth and singular parts:

$$(f_1, f_2, f_3) = (f_1, f_2, f_3)_{\text{smooth}} + (f_1, f_2, f_3)_{\text{singular}}$$

with smooth contribution given by the formula in Proposition 3.3.1:

$$(f_1, f_2, f_3)_{\text{smooth}} = \int_{R(Z)} (f_{1x} f_{2y} f_{3xy} + \text{permutations})\, dx\, dy.$$

As the additive property stated in the last subsection, we need only consider the diagonal in a square. More precisely, we consider functions f_i on the square $[0, 1]^2$ with following properties:

- f_i is continuous on $[0, 1]^2$, and smooth in the interior part of two triangles divided by diagonal $y = x$;
- the first and second derivatives of f_i has continuous limits at sides;
- the restriction of the function on the diagonal is smooth.

Let us write

$$\delta f_i = f_{ix}^{+} - f_{ix}^{-} = f_{iy}^{-} - f_{iy}^{+}$$

where we use super script $\pm$ to denote the limits of derivatives in upper and down triangles on the diagonals. The second identity follows from the fact that the restriction of f on the diagonal is smooth.

Proposition 3.4.1 *If f_i has only singularity on some union $\mathcal{D}$ of diagonals in n-triangulation, the singular contribution is given by*

$$(f_1, f_2, f_3)_{\text{singular}} = \int_{\mathcal{D}} \left(-\frac{1}{2} \delta f_1 \delta f_2 \delta f_3 + f_{1x} f_{2y} \delta f_3 + \textit{permutations} \right) dx.$$

Proof The total contribution in the diagonal is given by

$$\begin{aligned}
&n^2(F_{1n}\cdot F_{2n}\cdot F_{3n})_{\mathcal{D}}\\
&\quad = n^2\sum_{a,0}^{n-1}\prod_{i=1}^{3}(f_i(a,a)D_{a,a}+f_i(a+1/n,a)D_{a+1/n,a}+f_i(a,a+1/n)D_{a,a+1/n}\\
&\qquad + f_i(a+1/n,a+1/n)D_{a+1/n,a+1/n}+f_i(a+1/2n,a+1/2n)D_{a+1/2n,a+1/2n})_{a,a}.
\end{aligned}$$

Again we may replace the last product by

$$\begin{aligned}
&\prod_{i=1}^{3}(a_iD_{a,a}+b_iD_{a+1/n,a}+c_iD_{a,a+1/n}+d_iD_{a+1/n,a+1/n})\\
&\quad = 2(a_1a_2a_3+b_1b_2b_3+c_1c_2c_3+d_1d_2d_3)\\
&\qquad -(a_1a_2+d_1d_2)(b_3+c_3)-(b_1b_2+c_1c_2)(a_3+d_3)+\text{permutations}
\end{aligned}$$

with

$$a_i = f_i(a,a)-f_i(a+1/2n,a+1/2n),$$

$$b_i = f_i(a+1/n,b)-f_i(a+1/2n,a+1/2n),$$

$$c_i = f_i(a,a+1/n)-f_i(a+1/2n,a+1/2n),$$

$$d_i = f_i(a+1/n,a+1/n)-f_i(a+1/2n,a+1/2n).$$

We have Taylor expansions for a_i, b_i, c_i, d_i at (a',a') with $a'=a+1/2n$:

$$a_i = -\frac{1}{2n}(f_{ix}^- + f_{iy}^-)(a',a')+O(1/n^2),$$

$$b_i = \frac{1}{2n}(f_{ix}^- - f_{iy}^-)(a',a')+O(1/n^2),$$

$$c_i = \frac{1}{2n}(-f_{ix}^+ + f_{iy}^+)(a',a')+O(1/n^2),$$

$$d_i = \frac{1}{2n}(f_{ix}^+ + f_{iy}^+)(a',a')+O(1/n^2).$$

Write

$$f_{ix}=\frac{1}{2}(f_{ix}^+ + f_{ix}^-),\qquad f_{iy}=\frac{1}{2}(f_{iy}^+ + f_{iy}^-),$$

then,

$$f_{ix}^{\pm}=f_{ix}\pm\frac{1}{2}\delta f_i,\qquad f_{iy}^{\pm}=f_{iy}\mp\frac{1}{2}\delta f_i.$$

Then we have the following expressions:

$$a_i = -\frac{1}{2n}(f_{ix}+f_{iy})(a')+O(1/n^2),$$

$$b_i = \frac{1}{2n}(f_{ix} - f_{iy} - \delta f_i)(a') + O(1/n^2),$$

$$c_i = \frac{1}{2n}(-f_{ix} + f_{iy} - \delta f_i)(a') + O(1/n^2),$$

$$d_i = \frac{1}{2n}(f_{ix} + f_{iy})(a') + O(1/n^2).$$

As $a_i + d_i = O(1/n^2)$, it follows that

$$\begin{aligned}
&\prod_{i=1}^{3}(a_i D_{a,a} + b_i D_{a+1/n,a} + c_i D_{a,a+1/n} + d_i D_{a+1/n,a+1/n}) \\
&\quad = 2(b_1b_2b_3 + c_1c_2c_3) - 2a_1a_2(b_3 + c_3) + \text{permutations} + O(1/n^4) \\
&\quad = \frac{-1}{2n^3}\delta f_1 \delta f_2 \delta f_3(a') + \frac{1}{n^3} f_{1x} f_{2y} \delta f_3(a') + \text{permutation} + O(1/n^4).
\end{aligned}$$

The total diagonal intersection is

$$\begin{aligned}
&n^2(F_{1n} \cdot F_{2n} \cdot F_{3n})_{\mathcal{D}} \\
&\quad = \frac{1}{n}\sum_{a=0}^{n-1}\left(\frac{-1}{2}\delta f_1 \delta f_2 \delta f_3(a') + f_{1x} f_{2y} \delta f_3(a') + \text{permutation}\right) + O(1/n).
\end{aligned}$$

Taking limits of sum over all, we get singular contribution:

$$\int_0^1 \left(-\frac{1}{2}\delta f_1 \delta f_2 \delta f_3 + f_{1x} f_{2y} \delta f_3 + \text{permutations}\right) dx. \qquad \square$$

In the following we want to apply integration by parts to the smooth formulae in Propositions 3.3.1 and 3.4.1. For simplicity, we will only consider the space $\mathcal{F}(R(Z))$ of continuous functions f on $R(Z)$ with following properties for its restriction on a triangle T for a triangulation $R(Z_S)$ defined in Sect. 3.2:

1. f is smooth in the interior T^0 of T;
2. f_x and f_y has continuous limits on the boundary ∂T;
3. the restriction of f on the interior part $(\partial T)^0$ is smooth.

We need to recall the definition of Laplacian operator in [39, Appendix]. For a metrized graph G and piece-wise smooth function f on G, we define the Laplacian operator Δ on the space $\mathcal{F}(G)$ of continuous and smooth functions on G by the formula:

$$(\Delta(f), g)_G = (f', g')_G.$$

For each f, $\Delta(f)$ is sum of piece-wise continuous and smooth function and a Dirac distribution:

$$\Delta(f) = -f'' + \sum_p \delta(f)(p) \cdot \delta_p.$$

Here for each $p \in G$, δ_p denotes the Dirac distribution attached to p, and

$$\delta(f)(p) = -\sum_i \lim_{x_i} f'(x_i)$$

where the sum runs through the edges E_i of G with vertices p, and x_i are local coordinates on E_i at p.

Theorem 3.4.2 *Assume $f_i \in \mathcal{F}(R(Z))$. Then the intersection is given by*

$$(f_1, f_2, f_3) = \int_{R(Z)} (\Delta_x(f_1)(f_{2y} f_{3y}) + \Delta_x(f_2)(f_{3y} f_{1y}) + \Delta_x(f_3)(f_{1y} f_{2y}))\, dx\, dy$$
$$+ \frac{1}{4} \int_{\mathcal{D}} \delta(f_1)\delta(f_2)\delta(f_3)\, dx$$

where $\mathcal{D}$ is the union of diagonal in a triangulation $R(Z_S)$.

Proof Let us first group the smooth contribution as product of derivatives of x:

$$(f_1, f_2, f_3)_{\text{smooth}} = \int_{R(Z)} [f_{1x}(f_{2y} f_{3y})_x + f_{2x}(f_{3y} f_{1y})_x + f_{3x}(f_{1y} f_{2y})_x]\, dx\, dy.$$

Now we apply integration by parts to obtain

$$(f_1, f_2, f_3)_{\text{smooth}} = \int_{R(Z)\setminus\mathcal{D}} \Delta'_x f_1(f_{2y} f_{3y})\, dx\, dy + \int_{\mathcal{D}} (f_{1x}^+ f_{2y}^+ f_{3y}^+ - f_{1x}^- f_{2y}^- f_{3y}^-) dy + \cdots .$$

Here $\mathcal{D}$ is the union of diagonals in an n-triangulation, and the coordinates in each square are chosen such that $\mathcal{D}$ is given by $x = y$,

$$\Delta'_x f_i = \Delta_x f_i - \delta(f_i) \cdot \delta_{\mathcal{D}}$$

which is the Laplacian of f_i calculated on the complement of $\mathcal{D}$, and f_i^+ and f_i^- are restrictions of f_i in the upper and lower triangles respectively. Define the derivatives of f_i at the diagonal as the average of two limits of derivatives at two sides separated by diagonals. Then we have the formulae:

$$f_{ix}^{\pm} = f_{ix} \pm \frac{1}{2}\delta f_i, \qquad f_{iy}^{\pm} = f_{iy} \mp \frac{1}{2}\delta f_i.$$

The integrand in the diagonal on $\mathcal{D}$ has the following expression:

$$\left(f_{1x} + \frac{1}{2}\delta f_1\right)\left(f_{2y} - \frac{1}{2}\delta f_2\right)\left(f_{3y} - \frac{1}{2}\delta f_3\right) - \left(f_{1x} - \frac{1}{2}\delta f_1\right)\left(f_{2y} + \frac{1}{2}\delta f_2\right)\left(f_{3y} + \frac{1}{2}\delta f_3\right).$$

It is clear that the above expression is odd in δ; thus it has an expression

$$\delta(f_1) f_{2y} f_{3y} + \frac{1}{4}\delta(f_1)\delta(f_2)\delta(f_3) - f_{1x} f_{2y}\delta(f_3) - f_{1x} f_{3y}\delta(f_2).$$

As the restriction of $\Delta_x f_i$ on $\mathcal{D}$ is $\delta(f_i)$,

$$(f_1, f_2, f_3)_{\text{smooth}} = \int_{R(Z)} \Delta_x f_1(f_{2y} f_{3y})\, dx\, dy + \cdots + \int_{\mathcal{D}} \left(\frac{3}{4}\delta(f_1)\delta(f_2)\delta(f_3) - f_{1x} f_{2y}\delta(f_3) - \cdots\right) dy.$$

Combined with singular contribution, we have the following expression for the total pairing:

$$(f_1, f_2, f_3) = \int_{R(X)^2} \big(\Delta_x(f_1)(f_{2y} f_{3y}) + \Delta_x(f_2)(f_{3y} f_{1y}) + \Delta_x(f_3)(f_{1y} f_{2y})\big)\, dx\, dy + \frac{1}{4}\int_{\mathcal{D}} \delta(f_1)\delta(f_2)\delta(f_3)\, dx.$$

□

Example

In the following we would like to use Theorem 3.4.2 to compute the self-intersection of an exceptional divisor E. We need only consider the case where X and Y both have only one singularity. Thus $R(Z)$ is a unit square and the function f corresponding to E has the graph like a pyramid over $R(Z)$ of height $\frac{1}{2}$. If we parameterize $R(Z)$ using coordinates $(x, y) \in [-1/2, 1/2]^2$, then

$$f(x, y) = \frac{1}{2} - \max(|x|, |y|).$$

The function f is smooth on the 4 open triangles divided by 4 sides and two diagonals. The first derivatives are given by

$$f_x(x, y) = \begin{cases} -1, & x > |y|, \\ 1, & x < -|y|, \\ 0, & |x| < |y|, \end{cases} \qquad f_y(x, y) = \begin{cases} -1, & y > |x|, \\ 1, & y < -|x|, \\ 0, & |y| < |x|. \end{cases}$$

From this, the x-Laplacian of f on $R(Z)$ and the average y-derivative of f on the diagonal $|x| = |y|$ are given by

$$\Delta_x f = -\delta_{|x|=\frac{1}{2}} + \delta_{|x|=|y|}, \qquad f_y(x, y) = \begin{cases} -\frac{1}{2}, & y = |x|, \\ \frac{1}{2}, & y = -|x|. \end{cases}$$

Here the Dirac measures are push-forwards of the usual measure dx on the sides or diagonals. Bring this to the formula in Theorem 3.4.2, we obtain

$$E^3 = (f, f, f) = 3\int_{|x|=|y|} \frac{1}{4}\, dx + \frac{1}{4}\int_{|x|=|y|} dx = \frac{3}{2} + \frac{1}{2} = 2.$$

This agrees with formula computed in Sect. 3.1.

3.5 Completing proof of main theorem

In this subsection, we will complete the proof of Main Theorem 1.3.1. By Theorem 2.5.1 and Proposition 2.5.3, it remains to compute the quantity in Theorem 2.5.1 in the local setting.

More precisely, let $X = Y$ be a curve of genus $g \geq 2$ on over a local field K. Let G be the admissible green's function on the reduction graph constructed in our inventions paper [39]. Then we have a metrized line bundle $\mathcal{O}(\hat{\Delta})$ with norm $\|\cdot\|$ given by

$$-\log \|1_\Delta\| = i(x, y) + G(R(x), R(y))$$

where $R : X(\bar{K}) \longrightarrow R(X)$ is the reduction map. We want to show the following

Proposition 3.5.1 *The adelic metrized bundle $\mathcal{O}(\hat{\Delta})$ is integrable and*

$$\begin{aligned} & -\log \|1_\Delta\| \cdot (\hat{\Delta}^2 - 6\hat{\Delta} \cdot p_1^*\hat{e} + 6p_1^*\hat{e} \cdot p_2^*\hat{e}) \\ & = -\frac{1}{4}\delta(X) + \frac{1}{4}\int_{R(X)} G(x, x)((10g + 2)\, d\mu_a - \delta_{K_X}). \end{aligned}$$

To see that $\mathcal{O}(\hat{\Delta})$ is integrable, we let ξ denote a class of degree 1 such that $(2g - 2)\xi = \omega_X$ and put an admissible metric on it. Then we have seen that the class

$$\hat{t} := \hat{\Delta} - p_1^*\hat{\xi} - p_2^*\hat{\xi}$$

is integrable. On the other hand the adjunction formula gives

$$\Delta^*\xi = -\hat{\omega}_X - 2\hat{\xi} = -2g\hat{\xi}.$$

Thus $\hat{\xi}$ is integrable and then $\hat{\Delta}$ is integrable.

In the following, let us give precise models of Δ over $\mathcal{O}_K$ which converges to $\hat{\Delta}$. We consider integral models $\mathcal{Z}_{\mathcal{O}_L}$ of $X_L \times X_L$ for finite Galois extension L of K. The special fiber $\mathcal{Z}_w$ of $\mathcal{Z}_{\mathcal{O}_L}$ over a finite place w over a place v of K has components parameterized by some $e(w)$-division points in the reduction complex $R(Z_w)$ where $e(w)$ is the ramification index of w over v. Let G_w be the restriction of the Green's function G on these points. Then we get a vertical divisor V_w in $\mathcal{Z}_w$ with rational coefficients. The divisor $\widetilde{\Delta} + \sum_w V_w$ with Green's function at archimedean place defines an arithmetic divisor $\hat{\Delta}_L$.

We claim that this divisor is the pull-back of some divisor on some model $\mathcal{Z}_{\mathcal{O}_K}^L$ over $\mathcal{O}_K$. Indeed, let $\mathcal{L}$ be an ample line bundle on $\mathcal{Z}_{\mathcal{O}_L}$ invariant under $\mathrm{Gal}(L/K)$; for example, we may take

$$\mathcal{L} = \mathcal{O}\left(\sum \text{-Exceptional divisors}\right) \otimes \pi_1^*\omega^n \otimes \pi_2^*\omega^n,$$

where n is some big positive number. In this way, we may write

$$\mathcal{Z}_{\mathcal{O}_L} = \mathrm{Proj} \bigoplus_{m \geq 0} \pi_*\mathcal{L}^m$$

where π is the projection $\mathcal{Z}_{\mathcal{O}_L} \longrightarrow \mathrm{Spec}\mathcal{O}_K$. It is well known that the algebra $(\bigoplus_{m\geq 0} \pi_*\mathcal{L}^m)^{\mathrm{Gal}(L/K)}$ of $\mathrm{Gal}(L/K)$ invariants is finitely generated and thus defines a $\mathcal{O}_K$-scheme:

$$\mathcal{Z}_{\mathcal{O}_K}^L = \mathrm{Proj}\left(\bigoplus_{m\geq 0} \pi_*\mathcal{L}^m\right)^{\mathrm{Gal}(L/K)}.$$

It is well known that the inclusion of rings defines a morphism:

$$\phi_{L/K}: \quad \mathcal{Z}_{\mathcal{O}_L} \longrightarrow \mathcal{Z}^L_{\mathcal{O}_K}.$$

In this way, we get a divisor

$$\hat{\Delta}^L := [L:K]^{-1}\phi_{L/K*}(\hat{\Delta}_L).$$

To get a metrized line bundle on $\mathcal{Z}^L_{\mathcal{O}_K}$, we may take positive integer t such that $t\hat{\Delta}$ has integral coefficients, and then take a norm

$$\mathrm{N}_{\phi_{L/K}}(\mathcal{O}(t\hat{\Delta}_L))$$

which is an arithmetical model of $t\Delta$. By our definition the integration of $-\log\|1_\Delta\|$ against curvatures of line bundles is the limit of intersection of V_i with arithmetic divisors. Thus we may replace $-\log\|1_\Delta\|$ in the proposition by Green's function G on the reduction complex. We will finish the proof of the Proposition by computing the triple pairings one by one in the following three lemmas.

Lemma 3.5.2

$$(G, \hat{\Delta}, p_1^*\hat{e}) = 0.$$

Proof Using formulae $\hat{\Delta} = \widetilde{\Delta} + G$ and $\hat{e} = \bar{e} + G_e$, we have decomposition

$$\begin{aligned}(G, \hat{\Delta}, p_1^*\hat{e}) &= (G, p_1^*\hat{e})_{\widetilde{\Delta}} + (G, G, p_1^*\hat{e})\\ &= (G, p_1^*\hat{e})_{\widetilde{\Delta}} + (G, G)_{p_1^*e} + (G, G, p_1^*G_e).\end{aligned}$$

Let us to compute each of these of term:

$$(G, p_1^*\hat{e})_{\widetilde{\Delta}} = \int_{R(X)} G(x,x)\,d\mu,$$

$$\begin{aligned}(G, G)_{p_1^*e} &= -\int_{R(X)} G(e,y)\Delta_y G(e,y)\,dy\\ &= -\int_{R(X)} G(e,y)(\delta_e(y) - d\mu(y)) = -G(e,e),\end{aligned}$$

$$\begin{aligned}(G, G, p_1^*G_e) &= \int \Delta_x p_1^* G_e (G_y)^2\,dx\,dy\\ &= \int_{R(X)^2} (\delta_e(x) - d\mu(x))(G_y(x,y)^2)\,dy\\ &= \int_{R(X)} G_y(e,y)^2\,dy - \int_{R(X)^2} G_y(x,y)^2\,dy\,d\mu(x)\\ &= \int_{R(X)} \Delta_y G_y(e,y)\cdot G_y(e,y)\,dy - \int_{R(X)^2} \Delta_y G_y(x,y)\cdot G_y(x,y)\,dy\,d\mu(x)\end{aligned}$$

$$= G(e,e) - \int_{R(X)^2} (\delta_x(y) - d\mu(y)) G_y(x,y)\, dy\, d\mu(x)$$

$$= G(e,e) - \int_{R(X)^2} G_y(x,x)\, d\mu(x).$$

The lemma follows from the above three formulae. □

Lemma 3.5.3

$$(G, p_1^*\hat{e}, p_2^*\hat{e}) = 0.$$

Proof Using the decomposition of cycles, we have the following expression:

$$(G, p_1^*\hat{e}, p_2^*\hat{e}) = (G, p_2^*\hat{e})_{p_1^* e} + (G, p_1^* G_e)_{p_2^* e} + (G, p_1^* G_e, p_2^* G_e).$$

We compute each term as follows:

$$(G, p_2^*\hat{e})_{p_1^* e} = \int G(e,y)\, d\mu(y) = 0,$$

$$(G, p_1^* G_e)_{p_2^* e} = -\int G(x,e) \Delta_x G(x,e) = -G(e,e),$$

$$\begin{aligned}(G, p_1^* G_e, p_2^* G_e) &= \int \Delta_x G_e(x,e) \cdot (G_y(x,y) G_y(e,y))\, dx\, dy \\ &= \int (\delta_e(x) - d\mu(x) \cdot (G_y(x,y) G_y(e,y))\, dy \\ &= \int_{R(X)} G_y(e,y) G_y(e,y)\, dy - \int G_y(x,y) G_y(e,y)\, d\mu(x)\, dy \\ &= \int \Delta_y G(e,y) \cdot G(e,y)\, dy = G(e,e).\end{aligned}$$

The lemma follows from the above three computations. □

Lemma 3.5.4 *Let g be the genus of the curve, then*

$$(G, \hat{\Delta}, \hat{\Delta}) = \frac{1}{4}\delta(X) + \frac{1}{4}\int G(x,x)(K_X - (10g+2)\, d\mu).$$

Proof Using the formula, $\hat{\Delta} = \widetilde{\Delta} + G$, the left hand side can be decomposed as follows:

$$\begin{aligned}&(G, \hat{\delta})_\Delta + (G, G)_\Delta + (G, G, G) \\ &\quad = \int_{R(X)} -G(x,x) c_1(\omega) - \int_{R(X)} G(x,x) \Delta G(x,x) + (G, G, G).\end{aligned}$$

The curvature $c(\omega)$ is $(2g-2)d\mu$ where $d\mu$ is the admissible metric. To compute Laplacian of $G(x,x)$ we use the following formula

$$c + G_a(K_X, x) + G(x,x) = 0.$$

It follows that the curvature of $\Delta(G(x,x))$ is given by

$$-\Delta G(K_X, x) = (2g-2)d\mu - K_X.$$

Here K_X is the canonical divisor on $R(X)$. Thus we have the formula

$$(G, \hat{\Delta}, \hat{\Delta}) = \int_{R(X)} G(x,x)(K_X - 4(g-1)\, d\mu) + (G, G, G).$$

It remains to compute the triple pairing (G, G, G). Notice that G_x (resp. G_y) are continuous in y (resp. x) except on diagonal. By the main formula in the last section, we have

$$(G, G, G) = \frac{1}{4}\int_D (\delta G)^3\, dx + 3\int \Delta_x G G_y^2\, dx\, dy.$$

By definition,

$$(\Delta_x G)dx = \delta_y(x) - d\mu.$$

It follows that $\delta(G) = 1$ on the diagonal and thus the first integral is

$$\frac{1}{4}\ell(R(X)) = \frac{1}{4}\delta(X).$$

The second integral is given by

$$3\int_{R(X)} G_y^2(y,y)\, dy - 3\int_{R(X)^2} G_y^2(x,y)\, d\mu(x)\, dy.$$

Recall that $G_y(y,y)$ is defined to be

$$\frac{1}{2}(G_y^+(y,y) + G_y^-(y,y)) = \frac{1}{2}(G_y^+(y,y) + G_x^+(y,y)) = \frac{1}{2}G(y,y)_y.$$

It follows that the above integral is given by

$$\begin{aligned}
&\frac{3}{4}\int G^2(y,y)_y\, dy - 3\int_{R(X)^2} \Delta_y G(x,y)G(x,y)\, d\mu(x)\, dy \\
&\quad = \frac{3}{4}\int \Delta_y G(y,y)G(y,y)\, dy - 3\int_{R(X)^2} \Delta_y G(x,y)G(x,y)\, d\mu(x)\, dy \\
&\quad = \frac{3}{4}\int G(y,y)((2g-2)\, d\mu - K_X) - 3\int_{R(X)^2} G(x,y)(\delta_x(y) - d\mu(y))\, d\mu(x) \\
&\quad = \frac{3}{4}\int G(y,y)((2g-2)\, d\mu - K_X) - 3\int_{R(X)} G(x,x)\, d\mu \\
&\quad = \int_{R(X)} G(x,x)\left(\frac{3}{2}(g-3)\, d\mu - \frac{3}{4}K_X\right).
\end{aligned}$$

The lemma follows from the above computations. $\square$

4 Integrations on metrized graph

In this section, we reformulate Conjectures 1.4.2 and 1.4.5 in terms of metrized graphs. We will verify the conjectures in the elementary graphs where every edge is included in at most one circle. We will conclude the section by reducing the conjecture to the case that the graph is 2-edge connected in the sense that the complement of any point is still connected.

4.1 Some conjectures on metrized graphs

In this subsection we want to reformulate Conjectures 1.4.2 and 1.4.5 in terms of metrized graphs. We will also give some trivial formula which can be used to prove Theorem 1.3.5.

Let Γ be a connected metrized graph and let q be a function on Γ with a finite support. We define the canonical divisor of (Γ, q) by

$$K := \sum_{x\in\Gamma}(v(x) + 2q(x) - 2)x.$$

The genus of the metrized graph is defined to be

$$g = 1 + \frac{1}{2}\deg K = \sum_x q(x) + b(\Gamma)$$

where $b(\Gamma)$ is the first Betti number of the (topological) graph Γ without metric. We say that the pair (Γ, q) is a *polarized metrized graph* if the following conditions hold

- q is non-negative;
- K is effective.

Notice that the reduction graph $R(X)$ of any semistable curve X of genus g over a discrete valuation ring is a polarized metrized graph of genus g.

Let $G(x, y)$ and $d\mu$ be the admissible green's function and metric associate to the pair (Γ, q). We are interested in the following constants:

$$\varphi(\Gamma) := -\frac{1}{4}\ell(\Gamma) + \frac{1}{4}\int_\Gamma G(x, x)((10g + 2)\,d\mu - \delta_K),$$

$$\lambda(\Gamma) := \frac{g-1}{6(2g+1)}\varphi(\Gamma) + \frac{1}{12}(\epsilon(\Gamma) + \ell(\Gamma))$$

where $\ell(\Gamma)$ is the total length of Γ and

$$\epsilon(\Gamma) := \int G(x, x)[(2g - 2)\,d\mu + \delta_K].$$

When $\Gamma = R(X)$ is the reduction graph for a curve, then notation of invariants here coincides with the invariant defined in the introduction except we use $\ell(\Gamma)$ for $\delta(X)$ there.

A point $p \in \Gamma$ is called *a smooth point* if it is not in the support of K. For such a smooth point p, let Γ_p be the subgraph obtained from Γ by removing p and attached two points p_1, p_2. More precisely, Γ_p is a metrized graph with a surjective map to Γ which is injective and isometric over $\Gamma \setminus \{p\}$ and two-to-one over p. The function q defines a function on Γ_p. We call p of *type* 0 if Γ_p is connected. In this case Γ_p has genus $g - 1$. If p is not of type 0, then Γ_p is a union of two connected graphs of genus i and $g - i$ for some $i \in (0, g/2]$. In

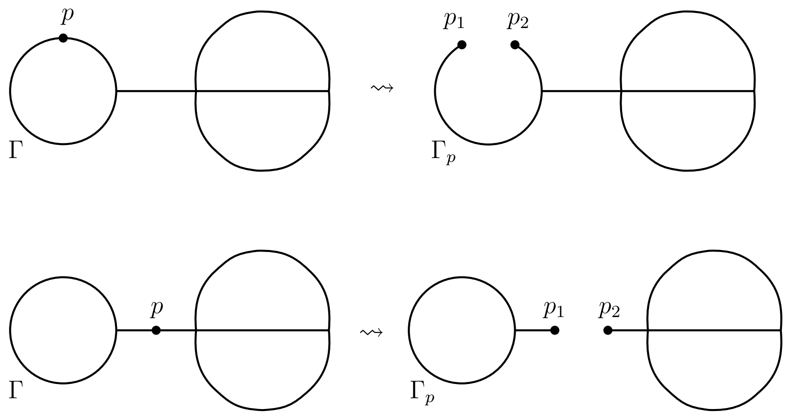


Fig. 5 Type of smooth points

this case, we say that p is of *type* i. For each number i in the interval $[0, g/2]$ let Γ_i be the subgraph of Γ of points of type i. Let $\ell_i(\Gamma)$ denote the length of Γ_i. It is easy to see that there are only finitely many $i \in [0, g/2]$ with non-zero $\ell_i(\Gamma)$.

Here is a diagram illustrating the definition of the type of a point for a graph with $q = 0$. In the left figures, p is a smooth point of Γ. In the top, p is of type 0 because Γ_p is connected. In the bottom, p is of type 1 because the minimum genus of the two connected components of Γ_p is 1.

Conjecture 4.1.1 *There is positive function $c(g)$ of $g > 1$ such that*

$$\varphi(\Gamma) \geq c(g)\ell_0(\Gamma) + \sum_{i \in (0, g/2]} \frac{2i(g-i)}{g} \ell_i(\Gamma),$$

$$\lambda(\Gamma) \geq \frac{g}{8g+4}\ell_0(\Gamma) + \sum_{i \in (0, g/2]} \frac{i(g-i)}{2g+1} \ell_i(\Gamma).$$

Formulae for Green's functions and admissible metrics

We need to have a formula for $G(x, x)$ in terms of resistance $r(x, y)$. Recall that we always have a formula like

$$r(x, y) = G(x, x) - 2G(x, y) + G(y, y). \tag{4.1.1}$$

See formula (3.5.1) in [39]. Double integrations gives

$$\tau(\Gamma) := \int G(x, x)\, d\mu(x) = \frac{1}{2} \int r(x, y)\, d\mu(x)\, d\mu(y). \tag{4.1.2}$$

One integral with $d\mu(y)$ gives

$$G(x, x) = \int r(x, y)\, d\mu(y) - \frac{1}{2} \int r(x, y)\, d\mu(x)\, d\mu(y). \tag{4.1.3}$$

Bring this to the definition of $\epsilon(\Gamma)$ to obtain

$$\epsilon(\Gamma) = \int r(x, y)\delta_K(x)\,d\mu(y). \tag{4.1.4}$$

The constants $\varphi(\Gamma)$ and $\lambda(\Gamma)$ can be expressed in terms of $\ell(\Gamma)$, $\tau(\Gamma)$ and $\epsilon(\Gamma)$:

$$\varphi(\Gamma) = 3g\tau(\Gamma) - \frac{1}{4}(\epsilon(\Gamma) + \ell(\Gamma)) \tag{4.1.5}$$

$$\lambda(\Gamma) = \frac{g(g-1)}{2(2g+1)}\tau(\Gamma) + \frac{g+1}{8(2g+1)}(\ell(\Gamma) + \epsilon(\Gamma)). \tag{4.1.6}$$

Recall form Lemma 3.7 in [39] that $d\mu$ has an expression

$$d\mu = \frac{1}{g}\left(\sum q(x)\delta_x + \sum \frac{dx_e}{\ell_e + r_e}\right). \tag{4.1.7}$$

We will reduce Conjecture 4.1.1 to the case where Γ is 2-edge connected. In this case, the conjecture is equivalent to the following

Conjecture 4.1.2 *Assume that Γ is 2-edge connected. Then the following two inequalities hold*:

$$\frac{g-1}{g+1}(\ell(\Gamma) - 4g\tau(\Gamma)) \le \epsilon(\Gamma) \le 12g\tau(\Gamma) - (1 + c(g))\ell(\Gamma),$$

here $c(g)$ is a positive number for each $g > 1$.

4.2 Proof of Theorem 1.3.5

In this subsection, we give a trivial bound for $\varphi(\Gamma)$ and use it to complete the proof of Theorem 1.3.5.

Lemma 4.2.1

$$-\frac{2g-1}{4}\ell(\Gamma) \le \varphi(\Gamma) \le \frac{3g}{2}\ell(\Gamma).$$

Proof From formulae (4.1.5) and (4.1.2), we obtain

$$\varphi(\Gamma) \le 3g\tau(\Gamma) \le \frac{3g\ell(\Gamma)}{2}$$

where we use an inequality $r(x, y) \le \ell(\Gamma)$ for any points $x, y \in \Gamma$. Similarly, we can get a lower bound:

$$\varphi(\Gamma) \ge -\frac{1}{4}(\epsilon(\Gamma) + \ell(\Gamma)) \ge -\frac{1}{4}(2g - 2 + 1)\ell(\Gamma).$$ □

Proof of Theorem 1.3.5 To prove Theorem 1.4.4, we need only to prove that the following difference function is bounded, for all closed point $t \in T$:

$$f(t) = \frac{1}{\deg t}((2g-2)\langle\Delta_\xi(Y_t), \Delta_\xi(Y_t)\rangle - (2g+1)\langle\omega_{Y/T}, \omega_{Y/T}\rangle).$$

Replace T by a finite covering, we may assume that the family can be extended into an semi-stable family $\mathcal{Y} \longrightarrow \mathcal{T}$ of integral schemes over $\mathcal{O}_K$. Let $\delta_{\mathcal{T}}$ be the boundary divisor induced from the morphism $\mathcal{T} \longrightarrow \bar{\mathcal{M}}_g$ and the boundary divisor $\bar{\mathcal{M}}_g \setminus \mathcal{M}_g$. Then $\delta_{\mathcal{T}}$ is supported over finitely many closed fibers of $\mathcal{T} \longrightarrow \mathrm{Spec}\mathcal{O}_k$, say over points in a finite subset S of $\mathrm{Spec}\mathcal{O}_k$. Now by Theorem 1.3.1, the function is given by

$$f(t) = \frac{2g-2}{\deg t} \sum_w \varphi_w(Y_t)$$

where the sum is over all places of $K(t)$. When w is archimedean over an archimedean place v of K, $\phi_w(Y_t)$ is a continuous function on $t \in T_v(\mathbb{C})$ thus it is bounded by a constant C_v depends only on place v.

If w is archimedean, then by Lemma 4.2.1, $\varphi_w(Y_t)$ is bounded by a constant multiple of the length $\ell(\Gamma)$ of the reduction graph of Y_t at w. We notice that this length is equal to the number of singular points on Y_t over w and can be computed by divisor $\delta_{\mathcal{T}}$:

$$\ell(\Gamma) = (\delta_{\mathcal{T}} \cdot \bar{t})_w$$

where the right hand side is the local intersection number of $\delta_{\mathcal{T}}$ and the Zariski closure $\bar{t}$ of t over w. This number is also bounded by a number C_v as $\delta_{\mathcal{T}}$ is a vertical divisor. In summary we have shown that

$$|f(t)| \le \frac{1}{\deg t} \sum_w C_v = \sum C_v$$

where C_v are some constant which is zero at all but finitely many places of k. Thus this is a finite number. This shows the boundedness of $f(t)$. □

4.3 Additivity of constants

In this section we want to reduce Conjecture 4.1.1 to the case where Γ is either a line segment or a 2-edge connected in the case that for any smooth point $p \in \Gamma$, the complement Γ_p is still connected. If Γ_p is not connected, then it is the union of two graphs Γ_1 and Γ_2 and Γ is a pointed sum of Γ_1 and Γ_2.

Lemma 4.3.1 *Any metrized graph Γ is a successive pointed sum of graphs Γ_i such that each Γ_i is either 2-edge connected or an edge with all inner points smooth.*

Proof Let Γ_+ be closure of the subgraph of points p such that Γ_p is not connected. Then Γ_+ is a finite disjoint union of trees, and the closed complement Γ_0 of Γ_+ in Γ is a finite disjoint union of the maximal 2-edge connected subgraphs. The graph Γ_+ can be further decomposed to edges with smooth inner points. We let Γ_i be the components of these 2-edge connected points or edges with smooth inner points. □

Assume that we have a decomposition of Γ into a pointed sum of connected subgraphs Γ_i as in Lemma 4.3.1. For each i and each $A \in \Gamma_i$, let Γ_A be the closure of the connected component of A in complement the $\Gamma_i \setminus \{A\}$. Then for all but finitely many A, $\Gamma_A = A$. We have a map $\pi_i : \Gamma \longrightarrow \Gamma_i$ with fiber Γ_A over $A \in \Gamma_i$. Let $q_i(A)$ be the genus of the polarized graph $(\Gamma_A, q|_{\Gamma_A})$.

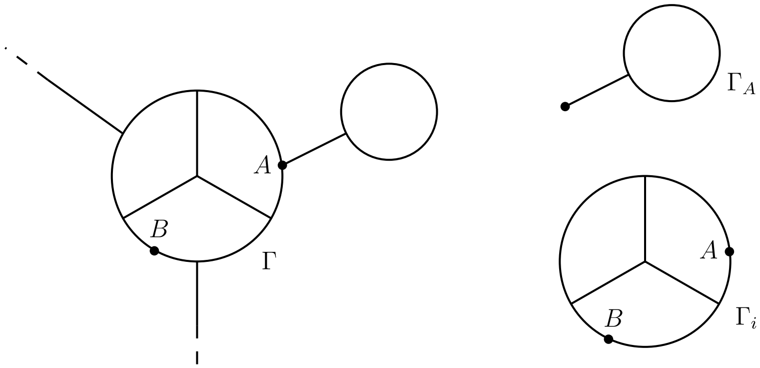


Fig. 6 Quotient graphs

Here is a figure of a graph Γ that draws attention to one of its 2-edge connected components Γ_i. The point $A \in \Gamma_i$ gives rise to the graph Γ_A, which is the fiber over A of the projection map $\pi_i : \Gamma \to \Gamma_i$. The point B satisfies $\pi_i^{-1}(B) = \{B\}$.

Our main result is as follows:

Theorem 4.3.2 *Each pair* (Γ_i, q_i) *is a polarized metrized graph with the same genus* g *as* (Γ, q). *Moreover all invariants have the additivity*:

$$\epsilon(\Gamma) = \sum_i \epsilon(\Gamma_i, q_i), \qquad \tau(\Gamma) = \sum_i \tau(\Gamma_i, q_i),$$

$$\varphi(\Gamma) = \sum_i \varphi(\Gamma_i, q_i), \qquad \lambda(\Gamma) = \sum_i \lambda(\Gamma_i, q_i).$$

Proof By definition, we need to show that q_i is non-negative and K_{Γ_i} is effective. By definition, the genus of Γ_A with restriction genus function $q(x)$ is given by

$$q(\Gamma_A) = \sum_{x \in \Gamma_A} q(x) + b(\Gamma_A)$$

where $b(\Gamma_A)$ is the first Betti number of the topological space Γ_A. It is clear that $q(\Gamma_A) \geq 0$. We need to compute the degree of the canonical divisor K_i of (Γ_i, q_i). Notice that the canonical divisor K_A of $(\Gamma_A, q|_{\Gamma_A})$ and K on a point $x \in \Gamma_A$ have the same multiplicities respectively:

$$2q(x) - 2 + v_{\Gamma_A}(x), \qquad 2q(x) - 2 + v(x).$$

These two numbers are equal except at $x = A$ where the difference is $v_{\Gamma_i}(A)$. It follows that

$$\mathrm{ord}_A K_i = 2q(A) - 2 + v_{\Gamma_i}(A) = \sum_{x \in \Gamma_A} \mathrm{ord}_x K.$$

This implies that K_i is effective and thus (Γ_i, q_i) is polarized. Take sum over A to obtain that

$$2g(\Gamma_i, q_i) - 2 = 2g(\Gamma) - 2.$$

It follows that $g(\Gamma_i)=g(\Gamma)$.

For four identities, by (4.1.5) and (4.1.6), it suffices to prove the first two. For any two points $x, y\in\Gamma$, the resistance $r(x,y)$ can be computed using Γ_i:

$$r(x,y)=\sum_i r(\pi_i(x),\pi_i(y))$$

where right side is the resistance on Γ_i which is the same as the resistance on Γ. We have a decomposition

$$\tau(\Gamma)=\sum_i \tau_i(\Gamma),\qquad \epsilon(\Gamma)=\sum \epsilon_i(\Gamma), \tag{4.3.1}$$

where

$$\tau_i(\Gamma)=\frac{1}{2}\int r(\pi_i(x),\pi_i(y))\,d\mu(x)\,d\mu(y),$$

$$\epsilon_i(\Gamma)=\int r(\pi_i(x),\pi_i(y))\delta_K(x)\,d\mu(y).$$

We may compute these last two integrations over fibers of $\pi_i:\Gamma\longrightarrow\Gamma_i$:

$$\tau_i(\Gamma)=\frac{1}{2}\int_{\Gamma_i^2} r(x,y)\,d\mu_i(x)\,d\mu_i(y),$$

$$\epsilon_i(\Gamma)=\int r(x,y)\delta_{K,i}(x)\,d\mu_i(y)$$

where $d\mu_i(x)$ is the sum of smooth part of $d\mu(x)$ supported on Γ_i plus the Dirac measure A in Γ_i with mass

$$\int_{\Gamma_A} d\mu(x)=\frac{q_i(A)}{g}.$$

Similarly, $\delta_{K,i}(x)$ is the Dirac measure with mass

$$\int_{\Gamma_A}\delta_K=2q_i(A)-2+v_{\Gamma_i}(x)=\deg_A K_i.$$

It follows that

$$\tau_i(\Gamma)=\tau(\Gamma_i,q_i),\qquad \epsilon_i(\Gamma)=\epsilon(\Gamma_i,q_i).$$

The formulae (4.3.1) thus finishes the proof. □

4.4 Reduction and elementary graphs

In this section, we want to reduce Conjecture 4.1.1 to the case where G is 2-edge connected. Then we prove the conjecture for elementary graphs.

Proposition 4.4.1 *Let $D_1,\dots,D_m$ be the set of maximal 2-edge connected subgraphs of Γ_i. Then*

$$\varphi(\Gamma)=\sum_{i\in(0,g/2]}\frac{2i(g-i)}{g}\ell_i(\Gamma)+\sum_D\varphi(D,q_D),$$

$$\lambda(\Gamma) = \sum_{i\in(0,g/2]} \frac{i(g-i)}{8g+4}\ell_i(\Gamma) + \sum_D \lambda(D, q_D).$$

Proof By Lemma 4.3.1 and Theorem 4.3.2, we need only prove the proposition when Γ is an edge with smooth inner points. Let i and $g-i$ be values of genus function at two ends a and b. Then

$$K = (2i-1)a + (2g-2i-1)b, \qquad d\mu = \frac{1}{g}(i\delta_a + (g-i)\delta_b).$$

As $r(x, y)$ is the distance between x and y, it follows that

$$\tau(\Gamma) = \frac{1}{2}\int r(x, y)\, d\mu(x)\, d\mu(y) = \frac{i(g-i)}{g^2}\ell(\Gamma),$$

$$\epsilon(\Gamma) = \int r(x, y)\delta_K(x)\, d\mu(y) = \left(4\frac{i(g-i)}{g} - 1\right)\ell(\Gamma).$$

The formulae in the proposition follows from (4.1.5) and (4.1.6). □

Corollary 4.4.2 *Conjecture 4.1.1 in general case follows from the case where Γ is 2-edge connected.*

A graph is called *elementary* if every edge is included in at most one circle. In the following, we give some explicit formulae for $\varphi(\Gamma)$ and $\lambda(\Gamma)$ for elementary graphs and then deduce Conjecture 4.1.1. For each circle C in Γ, let V_C be the set of points on C such that $q(x) > 0$, and write $C^0 = C \setminus V_C$. Then $\Gamma \setminus C^0$ is a union of subgraphs Γ_A for $A \in V_C$. Let g_A denote the genus of Γ_A for the restriction of genus function g_A, and let $r_C(A, B)$ denote the resistance between two points A and B on the circle C. We want to prove Conjecture 4.1.1 for elementary graph:

Proposition 4.4.3

$$\varphi(\Gamma) = \frac{g-1}{6g}\ell_0(\Gamma) + \sum_{i\in(0,g/2]} \frac{2i(g-i)}{g}\ell_i(\Gamma) + \sum_{c\in C}\sum_{A,B\in V_C} \frac{g_A g_B}{g} r_C(A, B),$$

$$\lambda(\Gamma) = \frac{g}{8g+4}\ell_0(\Gamma) + \sum_{i\in(0,g/2]} \frac{i(g-i)}{8g+4}\ell_i(\Gamma) + \sum_{c\in C}\sum_{A,B\in V_C} \frac{g_A g_B}{4g+2} r_C(A, B).$$

Proof By Proposition 4.4.1, it suffices to prove the proposition for case where Γ is a circle. Let us compute the integrals $\epsilon(\Gamma)$ and $\tau(\Gamma)$:

$$\tau(\Gamma) = \frac{1}{2}\int r(x, y)\, d\mu(x)\, d\mu(y), \qquad \epsilon(\Gamma) = \int r(x, y)\delta_K(x)\, d\mu(y).$$

For $A, B \in \Gamma$ the resistance $r(A, B)$ is given by $\ell(A, B)\ell'(A, B)/\ell$ where $\ell(A, B)$ and $\ell'(A, B)$ are the lengths of two segments of in the complement of A, B in Γ. The measures in the integrals are given by

$$d\mu = \frac{1}{g}\left(\sum_A q(A)\delta_A + \ell^{-1}dx\right), \qquad \delta_K = \sum_A 2q(A)\delta_A.$$

Let Γ^0 be the complement of the support of q. Then we have discrete contribution when both x and y are not in C^0. The contributions in this case are given by

$$\tau_{A,B}(\Gamma) = \frac{q(A)q(B)}{2g^2} r(A, B), \qquad \epsilon_{A,B}(\Gamma) = 2\frac{q(A)q(B)}{g} r(A, B).$$

Next we consider the case where $x \notin C^0$, $y \in C^0$. We assume that $x = A$. Let us choose coordinate t on C such that $t(A) = 0$. Then we have contributions:

$$\tau_A^1(\Gamma) := \frac{q(A)}{2g} \int_{\Gamma^0} \frac{t(\ell_c - t)}{\ell_c} d\mu = \frac{q(A)}{2g^2} \int_0^\ell \frac{t(\ell - t)}{\ell} \frac{dt}{\ell} = \frac{q(A)}{6g^2} \ell,$$

$$\epsilon_A^1(\Gamma) := 2q(A) \int_{\Gamma^0} \frac{t(\ell - t)}{\ell} d\mu = \frac{q(A)}{3g} \ell.$$

Now let us consider the case $x \in \Gamma^0$, $y = A$. Then we have contribution:

$$\tau_A^2(\Gamma) := \frac{1}{2} \int_{\Gamma^0} \frac{t(x)(\ell - t(x))}{\ell} d\mu(x) \cdot \frac{q(A)}{g} = \frac{g_A}{12g^2} \ell,$$

$$\epsilon_A^2(\Gamma) := \int_{\Gamma^0} \frac{t(\ell - t)}{\ell} \delta_K \cdot \frac{q(A)}{g} = 0.$$

Finally, lets us consider the case where both x and y are in Γ^0. Then we have contribution:

$$\begin{aligned}\tau^0(\Gamma) &:= \frac{1}{2} \int_{\Gamma^0} \int_{\Gamma^0} \frac{|t(x) - t(y)|(\ell - |t(x) - t(y)|)}{\ell} d\mu(x)\, d\mu(y) \\ &= \frac{1}{2g^2} \int_0^\ell \int_0^\ell \frac{|t(x) - t(y)|(\ell_e - |t(x) - t(y)|)}{\ell} \frac{dx\, dy}{\ell^2} = \frac{\ell}{12g^2}.\end{aligned}$$

Thus a total contribution from a circle is

$$\begin{aligned}\tau(\Gamma) &= \sum_{A,B} \tau_{A,B}(\Gamma) + \sum_A (\tau_A^1(\Gamma) + \tau_A^2(\Gamma)) + \tau^0(\Gamma) \\ &= \sum_{A,B} \frac{q(A)q(B)}{2g^2} \cdot r(A, B) + \frac{1}{6g^2} \left(\sum_A q(A) \right) \ell + \frac{\ell}{12g^2},\end{aligned}$$

$$\begin{aligned}\epsilon(\Gamma) &= \sum_{A,B} \epsilon_{A,B}(\Gamma) + \sum_A (\epsilon_A^1(\Gamma) + \epsilon_A^2(\Gamma)) + \epsilon^0(\Gamma) \\ &= \sum_{A,B} 2\frac{q(A)q(B)}{g} \cdot r(A, B) + \frac{1}{3g} \left(\sum_A q(A) \right) \ell.\end{aligned}$$

It is easy to verify that $\sum q(A) + 1 = g$. Thus we have formulae

$$\tau_C(\Gamma) = \sum_{A,B} \frac{q(A)q(B)}{g^2} \cdot r(A, B) + \frac{2g - 1}{6g^2} \ell,$$

$$\epsilon_C(\Gamma) = \sum_{A,B} 2\frac{q(A)q(B)}{g} \cdot r(A,B) + \frac{g-1}{3g}\ell.$$

By formulae (4.1.5) and (4.1.6), we obtain the formulae in proposition.

$$\varphi(\Gamma) = \left(\frac{g-1}{6g} + \sum_{A,B\in\Gamma} \frac{q(A)q(B)}{g} r(A,B)\right)\ell_c,$$

$$\lambda(\Gamma) = \left(\frac{g}{4(2g+1)} + \frac{1}{2}\sum_{A,B\in V_C} \frac{q(A)q(B)}{2g+1} r(A,B)\right)\ell_c. \qquad \square$$

5 Triple product L-series and tautological cycles

In this section, we define a subgroup containing the Gross–Schoen cycle of homologous to zero cycles of codimension 2 on the triple product X^3 of a curve X. The Beilinson–Bloch conjecture relates the rank of this group and the order of vanishing of L-series at $s = 0$ associated to the cohomology M defined as the kernel

$$\bigwedge^3 H^2(X)\,(2) \longrightarrow H^1(X)(1).$$

We will list some formulae for L-series and root numbers in the semistable case. At the end of this section, we want to rewrite heights of Δ_ξ in terms of Künnemann's height pairing of tautological cycles X_1 and $\mathcal{F}(X_1)$ in the Beauville–Fourier–Mukai theory. In particular, we can show that the non-vanishing of height of Δ_ξ will implies the non-vanishing of the Ceresa cycle $X - [-1]^*X$ in the Jacobian.

5.1 Beilinson–Bloch's conjectures

In this subsection, we define some groups of cycles homologous to 0 of codimension 2 on a product of three curves and state the Beilinson–Bloch's conjectures for corresponding cohomologies.

Let X_i $(i = 1, 2, 3)$ be three curves over a number field with three fixed points e_i. We consider the triple product $Y = X_1 \times X_2 \times X_3$, the group $\mathrm{Ch}^2(Y)$ of cycles of dimension 1 on the Y, and the class map

$$\mathrm{Ch}^2(Y) \longrightarrow H^4(Y)(2).$$

The kernel of this map is called the group of cycles homologous to 0 and is denoted by $\mathrm{Ch}^2(Y)^0$. We have the following Beilinson–Bloch's conjecture [4–6]:

Conjecture 5.1.1 (Beilinson–Bloch) *The rank of* $\mathrm{Ch}^2(Y)^0$ *is finite and is equal to the order of vanishing of* $L(H^3(Y), 2)$.

By the Künneth formula, we have a decomposition:

$$H^3(Y) = H^1(X_1) \otimes H^1(X_2) \otimes H^1(X_3) \oplus \oplus_i H^1(X_i)(-1)^{\oplus 2}. \qquad (5.1.1)$$

Thus, the right hand side is the product of L-series corresponding to the decomposition. We would like to decompose the group $\mathrm{Ch}^2(Y)^0$ into a sum of subgroups and formulate a conjecture for these subgroups. For this, we need only find correspondence decomposition of the identity correspondence which gives decomposition. In the following we want to describe the group $\mathrm{Ch}^2(Y)^0$ in terms of projections and embeddings.

Lemma 5.1.2 *Let* $\mathrm{Ch}^2(Y)_0$ *be the subgroup of elements with trivial projection onto* $X_i \times X_j$ *and* $\mathrm{Ch}^1(X_i)^0$ *be the group of zero cycles on* X_i *of degree* 0. *Then*

$$\mathrm{Ch}^2(Y)^0 = \mathrm{Ch}^2(Y)_0 \oplus \bigoplus_i (\mathrm{Ch}^1(X_i)^0)^{\oplus 2}.$$

Moreover, this decomposition is compatible with Künneth decomposition in the sense that they are given by same correspondences on Y.

Proof Let i, j, k be a reordering of $1, 2, 3$. For any factor X_k, we have an injection $\iota_k : X_k \longrightarrow Y$ by putting e_i, e_j for other factors; similarly we have an embedding $\iota_{i,j} : X_i \times X_j \longrightarrow Y$ be the inclusion by putting component e_k on X_k. Then we have an inclusion $\mathrm{Ch}^0(X_k) \longrightarrow \mathrm{Ch}^2(Y)$ and $\mathrm{Ch}^1(X_i \times X_j) \longrightarrow \mathrm{Ch}^2(Y)$ by push-push forward. Let π_k and $\pi_{i,j}$ denote the projections to X_k and $X_i \times X_j$.

For any cycle Z on Y, let $Z_{i,j}$ and Z_k denote push forwards of Z under $\iota_{i,j} \circ \pi_{i,j}$ and $\iota_k \circ \pi_k$ respectively. We define the following combinations:

$$Z^0 = Z - \sum_{i,j} Z_{i,j} + \sum_k Z_k,$$

$$Z^0_{i,j} = Z_{i,j} - Z_i - Z_j.$$

Then we have a decomposition

$$Z = Z^0 + \sum_{i,j} Z^0_{i,j} + \sum_k Z_k. \tag{5.1.2}$$

It can be proved that Z^0 has the trivial projection to $X_i \times X_j$, and that $Z_{i,j}$ has trivial projection on X_i and X_j. These imply that Z^0 is cohomologically trivial, and that $Z^0_{i,j}$ and Z_k have cohomological classes in the following groups respectively:

$$H^1(X_i) \otimes H^1(X_j) \otimes H^2(X_k), \qquad H^2(X_i) \otimes H^2(X_j) \otimes H^0(X_k).$$

Assume now that Z is homologically trivial. Then $Z_k = 0$ (as it is a multiple of $\{e_i\} \times \{e_j\} \times X_k$) and the class $Z_{i,j}$ are cohomologically trivial with decomposition by the theorem of square of line bundles algebraically equivalent to 0 on $X_i \times X_j$:

$$Z_{i,j} = A \times X_j \times \{e_k\} + X_i \times B \times \{e_k\}$$

where A and B are divisors on X_i and X_j with degree 0 respectively. □

The group $\mathrm{Ch}^1(X_i)^0$ is nothing other than the Mordell–Weil group of $\mathrm{Jac}(X_i)$. The Birch and Swinnerton–Dyer conjecture gives

$$\mathrm{ord}_{s=1} L(H^1(X_i), s) = \mathrm{rank}\mathrm{Ch}^1(X_i)^0.$$

Thus Conjecture 5.1.1 is equivalent to the following:

Conjecture 5.1.3 *The rank of* $\mathrm{Ch}^2(Y)_0$ *is finite and is equal to*

$$\mathrm{ord}_{s=2} L(H^1(X_1) \otimes H^1(X_2) \otimes H^1(X_3), s).$$

In the following we try to discuss the conjecture in the spacial case $X_1 = X_2 = X_3 = X$, where X is a general curve of genus $g \geq 2$. In this case, we have more correspondences to decompose the cohomology $H^1(X)^{\otimes 3}$. We will decide a subgroup whose Chow group containing the modified diagonal.

First of all, we notice that the modified diagonal is invariant under the symmetric group S_3. Thus it corresponds to the component of $H^1(X)^{\otimes 3}$ under the action of S_3. Notice that the action of S_3 on this group is given by the following: for $\alpha_i \in H^1(X)$ then it defines an element $\alpha_1(x_1) \wedge \alpha_2(x_2) \wedge \alpha_3(x_3)$ in $H^1(X)^{\otimes 3}$. The group S_3 acts by the permutations of x_i's. Thus the invariant under S_3 is exactly the subspace $\bigwedge^3 H^1(X)$ of $H^1(X)^{\otimes 3}$. Indeed, invariant space are generated by elements of the form

$$\sum_{\sigma \in \mathcal{S}_3} \alpha_1(x_{\sigma(1)}) \wedge \alpha_2(x_{\sigma(2)}) \wedge \alpha_3(x_{\sigma(3)}) = \sum_{\sigma \in \mathcal{S}_3} \mathrm{sgn}(\sigma) \alpha_{\sigma(1)}(x_1) \wedge \alpha_{\sigma(2)}(x_2) \wedge \alpha_{\sigma(3)}(x_3).$$

Thus the Beilinson–Bloch conjecture gives

$$\mathrm{ord}_{s=2} L\Big(s, \bigwedge^3 H^1(X)\Big) = \dim \mathrm{Ch}^2(Y)_0^{S_3}.$$

Here $\mathrm{Ch}^2(Y)_0^{S_3}$ is the group of cohomologically trivial cycles with trivial projection under $\pi_{i,j}$ and invariant under permutation. Both sides are nontrivial only if $g \geq 2$.

Using the alternating paring on $H^1(X)$, we can define a surjective morphism

$$\bigwedge^3 H^1(X)(2) \longrightarrow H^1(X)(1), \qquad a \wedge b \wedge c \mapsto a(b \cup c) + b(c \cup a) + c(a \cup b),$$

where $a \cup b$ is the canonical alternating pairing of $a, b \in H^1(X)$ with values in $\mathbb{Q}_\ell(-1)$. This morphism is defined by a correspondence between X^3 and X as follows:

$$X^2 \longrightarrow (X^3) \times (X): \quad (x, y) \mapsto (x, x, y) \times (y).$$

Thus the kernel M is fitted in a splitting:

$$\bigwedge^3 H^1(X)(2) = M \oplus H^1(X)(1),$$

with embedding

$$H^1(X)(1) \longrightarrow \bigwedge^3 H^1(X)(2)$$

given by

$$\alpha \mapsto \frac{1}{6} \sum_{\sigma \in \mathcal{S}_3} \sigma^*(\alpha \wedge \delta)$$

where $\delta \in \bigwedge^2 H^1(X)\,(1)$ is the projection of class of the diagonal which gives the alternative pairing of $H_1(X)$.

The corresponding decomposition is given by

$$\mathrm{Ch}^2(Y)_0^{S_3} = \mathrm{Ch}(M) \oplus \mathrm{Pic}^0(X)(K)$$

where $\mathrm{Ch}(M)$ is a subgroup of $\mathrm{Ch}^2(Y^3)^0$ consists of elements z satisfying the following conditions:

1. z is symmetric with respect to permutations on X^3;
2. the pushforward $p_{12*}z = 0$ with respect to the projection

$$p_{12}: \quad X^3 \longrightarrow X^2, \qquad (x, y, z) \mapsto (x, y);$$

3. let $i : X^2 \longrightarrow X^3$ be the embedding defined by $(x, y) \longrightarrow (x, x, y)$ and $p_2 : X^2 \longrightarrow X$ be the second projection. Then

$$p_{2*}i^*z = 0.$$

For any $\eta \in \mathrm{Jac}(X)(K)$, the corresponding element in $\mathrm{Ch}^2(Y)_0^{S_3}$ is given by

$$\alpha(\eta) = \sum_{i,j,k} \Delta_{i,j}^0 \times \eta_k$$

where $\eta_k \in X_k$ is corresponding to η.

The Beilinson–Bloch conjecture gives the following

Conjecture 5.1.4 *The group* $\mathrm{Ch}(M)$ *has finite rank and*

$$\mathrm{ord}_{s=0} L(s, M) = \dim \mathrm{Ch}(M).$$

Let us check if the modified diagonal is in the above group:

Lemma 5.1.5

$$\Delta_\xi \in \mathrm{Ch}(M).$$

Proof Indeed, it is easy to show that

$$i^*\Delta_\xi = (2 - 2g(X))\xi_\Delta - (2 - 2g)\xi \times \xi - 2\xi_\Delta + 2\xi \times \xi.$$

It is clear that

$$p_{2*}i^*\Delta_\xi = (2 - 2g)\xi - (2 - 2g)\xi - 2\xi + 2\xi = 0.$$

□

5.2 L-series and root numbers

In this section we want to compute L-series and the epsilon factor for $L(s, M)$ when the curve has semi-stable reduction. Our reference for definitions is Deligne [12]. For convenience, we will work on homology $H_1(X) = H^1(X)(1)$. Recall that M is the kernel of a canonical surjective morphism on homology groups:

$$\bigwedge^3 H_1(X)(-1) \longrightarrow H_1(X).$$

It follows that the cohomology M is of weight -1 with a non-degenerate alternative pairing

$$M \otimes M \longrightarrow \mathbb{Q}(1).$$

It is conjectured that the L-series $L(s, M)$ should be entire and satisfies a functional equation

$$L(s, M) = \pm f(M)^{-s} L(s, M)$$

where $f(M) \geq 1$ is the conductor of M (an integer divisible only by finite places ramify in M).

Local L-functors

By definition, the L-series is defined by an Euler product:

$$L(s, M) = \prod_v L_v(s, M)$$

where v runs through the set of places of K, and $L_v(s, M)$ is a local L-factor of M at v. For v an archimedean place, the local L-factor is determined by the Hodge weights. Notice that we have a decomposition

$$H_1(X, \mathbb{C}) = H^{-1,0}(X, \mathbb{C}) \oplus H^{0,-1}(X, \mathbb{C})$$

of Hodge structure into two spaces of dimension g, and that $\mathbb{C}(-1)$ has Hodge weight $(1, 1)$. As M is the kernel of a surjective morphism of Hodge structure

$$\bigwedge^3 H_1(X, \mathbb{C})(-1) \longrightarrow H_1(X, \mathbb{C}),$$

it follows that M has Hodge numbers given by

$$h^{1,-2} = h^{-2,1} = \frac{g(g-1)(g-2)}{6}, \qquad h^{0,-1} = h^{-1,0} = \frac{g(g-2)(g+1)}{2}.$$

The L-factor then is given by

$$L_v(s, M) = \Gamma_{\mathbb{C}}(s+2)^{h^{-2,1}} \Gamma_{\mathbb{C}}(s+1)^{h^{-1,0}}, \qquad \Gamma_{\mathbb{C}} = 2 \cdot (2\pi)^{-s} \Gamma(s). \tag{5.2.1}$$

For v a finite place with inertia group I_v, residue field $\mathbb{F}_{q_v}$, and geometric Frobenius F_v, the L-series is given by

$$L_v(s, M) = \det(1 - q_v^{-s} F_v; M_\ell^{I_v})^{-1} \tag{5.2.2}$$

where M_ℓ is the ℓ-adic realization of M at a prime $\ell \nmid q_v$. For v unramified, the L-series can be computed simply by Weil numbers. For v a ramified place, then $\mathrm{Jac}(X)$ has a semi-abelian reduction: the connected component J of the Neron model of $\mathrm{Jac}(X)$ is an extension of an abelian variety A by a torus

$$0 \longrightarrow T \longrightarrow J \longrightarrow A \longrightarrow 0.$$

Here A is the product of Jacobians of the irreducible components in the semistable reduction of X and T is a torus determined by homology group in the reduction graph of X. We have a filtration of $V := H_1(X_{\bar{K}}, \mathbb{Q}_\ell)$:

$$H_1(\bar{T}, \mathbb{Q}_\ell) \subset H_1(\bar{J}, \mathbb{Q}_\ell) \subset H_1(\bar{X}, \mathbb{Q}_\ell).$$

This filtration is compatible with action of the decomposition group D_v. By Serre–Tate, we have an identity:

$$H_1(\bar{J}, \mathbb{Q}_\ell) = H_1(\bar{X}, \mathbb{Q}_\ell)^{I_v},$$

and $H_1(\bar{T}, \mathbb{Q}_\ell)$ is the orthogonal complement of $H_1(\bar{J}, \mathbb{Q}_\ell)$ with respect to the Weil pairing on $H_1(\bar{X}, \mathbb{Q}_\ell)$. In particular, the action of F_v on these space are semiample with eigenvalues of absolute value q^{-1}, $q^{1/2}$, and 1. Thus F_v on $H_1(\bar{X}, \mathbb{Q}_\ell)$ is semi-simple.

By Grothendieck, the action of I_v on $H_1(\bar{X}, \mathbb{Q}_\ell)$ is given by

$$\sigma x = x + t_\ell(\sigma) N x, \quad x \in H_1(\bar{X}, \mathbb{Q}_\ell),\ N \in \mathrm{End}(H_1(\bar{X}, \mathbb{Q}_\ell)),$$

where $t_\ell : I_v \longrightarrow \mathbb{Q}_\ell$ is a nonzero homomorphism. We may decompose $V := H^1(\bar{X}, \mathbb{Q}_\ell)$ into an orthogonal sum of two dimensional spaces V_i $(i = 1, \ldots, g)$ invariant under D_v. The $\bigwedge^3 H_1(\bar{X}, \mathbb{Q}_\ell)$ is then a direct sum of tensors

$$\bigwedge^3 H_1(\bar{X}, \mathbb{Q}_\ell) = \bigoplus_{n_1+n_2\cdots=3} \bigwedge^{n_1} V_1 \otimes \bigwedge^{n_2} V_2 \otimes \cdots.$$

The invariants of I_v must have decomposition:

$$\bigwedge^3 H_1(\bar{X}, \mathbb{Q}_\ell)^{I_v} = \bigoplus_{n_0+n_1+n_2\cdots=3} \left(\bigwedge^{n_0} V_0\right)^{I_v} \otimes \left(\bigwedge^{n_1} V_1\right)^{I_v} \otimes \cdots.$$

Thus $M(1)$ has the following orthogonal decomposition of D_v-modules:

$$M(1) = \sum_{i<j<k} V_i \otimes V_j \otimes V_k + \sum_i V_i \otimes \left(\sum_{j\neq i}\left(\bigwedge^2 V_i\right)\right)^0 \tag{5.2.3}$$

where superscript 0 means kernel in the Weil pairing. The space M^{I_v} has a decomposition

$$M^{I_v}(1) = \sum_{i<j<k} V_i^{I_v} \otimes V_j^{I_v} \otimes V_k^{I_v} + \sum_i V_i^{I_v} \otimes \left(\sum_{j\neq i} \bigwedge^2 V_i\right)^0. \tag{5.2.4}$$

In this way, we have a precise description of the Galois action on M and therefore a formula for L-factor.

Local root numbers

In the following we want to compute the root numbers of the functional equation. Recall that the root number ϵ is the product of local root numbers ϵ_v.

Lemma 5.2.1 *For v complex we have*

$$\epsilon_v = i^{6h^{-2,1}+2h^{-1,0}} = \begin{cases} 1, & g \equiv 0, 1 \pmod 4, \\ -1, & g \equiv 2, 3 \pmod 4). \end{cases}$$

For v a real place

$$\epsilon_v = i^{4h^{-2,1}+2h^{-1,0}} = \begin{cases} 1, & g \not\equiv 1 \pmod 4, \\ -1, & g \equiv 1 \pmod 4). \end{cases}$$

Lemma 5.2.2 *Let v be a non-archimedean place. Let $\tau = \pm 1$ be the product of α_i. Then the root number is given by*

$$\begin{aligned}\epsilon_v &:= (-1)^{e(e-1)(e-2)/6}\tau^{(e-1)(e-2)/2}(-1)^{e(g-2)}\tau^{(g-2)} \\ &= (-1)^{e(e-1)(e-2)/6+ge}\tau^{(e-1)(e-2)/2+g}.\end{aligned}$$

Here e is the rank of the first homology group of the reduction graph of X at v, and τ is the determinant of F_v acts on the character group of (e*-dimensional*) *toric part of the reduction of* $\mathrm{Jac}(X)$.

Proof If v is finite unramified place, then $\epsilon_v = 1$. It remains to compute the root number at a ramified finite place. It is given by

$$\epsilon_v = \frac{\det(-F_v|_{M_\ell})}{\det(-F_v|_{M_\ell^{I_v}})}.$$

Now we want to compute ϵ_v using decompositions (5.2.3) and (5.2.4). Notice that on each V_i, $-F_v$ has determinant $-q^{-1}$, and on $V_i^{I_v}$, it has eigenvalues q^{-1} for $i \le e$. We assume that $V_i \neq V_i^{I_v}$ exactly for the first e V_i's. Let F_v have eigenvalues α_i on $V_i / V_i^{I_v}$ which has absolute value 1. The contribution to root number from each term is given as follows:

$$\begin{array}{lll}
V_i \otimes V_j \otimes V_k: & 1, & \\
V_i \otimes \left(\sum_{j\neq i} \bigwedge^2 V_i\right): & 1, & \\
V_i^{I_v} \otimes V_j^{I_v} \otimes V_k^{I_v}: & -\alpha_i\alpha_j\alpha_k, & i<j<k\le e, \\
V_i^{I_v} \otimes V_j^{I_v} \otimes V_k^{I_v}: & \alpha_i^2\alpha_j^2, & i<j\le e<k, \\
V_i^{I_v} \otimes V_j^{I_v} \otimes V_k^{I_v}: & \alpha_i^4, & i\le e<j<k, \\
V_i^{I_v} \otimes \left(\sum_{j\neq i} \bigwedge^2 V_i\right): & (-\alpha_i)^{g-2}, & i\le e.
\end{array}$$

□

5.3 Tautological classes in Jacobians

In this subsection, we would like to study tautological algebraic cycles in the Jacobian defined by Ceresa [8] and Beauville [3]. We will use Fourier–Mukai transform of Beauville [1, 2] and height pairing of Künnemann [27].

Let A be an abelian variety of dimension $g \ge 3$ over a global field k with a fixed symmetric and ample line bundle $\mathcal{L}$. Let L be the operator on cohomology $h(A)$ induced by intersecting with $c_1(\mathcal{L})$ which thus induces operator on Chow group and cohomology group. For each integer p in the interval $[0, (g+1)/2]$, it is conjectured that the map

$$L^{g+1-2p}: \quad \mathrm{Ch}^p(A)^0 \longrightarrow \mathrm{Ch}^{g+1-p}(A)^0$$

is an isomorphism of two vector spaces of finite dimensional. Let $\mathrm{Ch}^p(A)^{00}$ denote the kernel of L^{g+2-2p} which is called *the group of primitive class of degree* p. By the same way, we can define the primitive cohomology classes $H^{2p-1}(A)^{00}$. Then the Beilinson–Bloch conjecture says that

$$\mathrm{rank}\mathrm{Ch}^p(A)^{00} = \mathrm{ord}_{s=p} L(H^{2p-1}(A)^{00}, s).$$

Moreover, Künnemann has constructed a height pairing on $\mathrm{Ch}^*(A)^0$:

$$\langle \cdot, \cdot \rangle : \quad \mathrm{Ch}^p(A)^0 \otimes \mathrm{Ch}^{g-p+1}(A)^0 \longrightarrow \mathbb{R}.$$

The index conjecture of Gillet–Soulé says

$$(-1)^p \langle x, L^{g+1-2p} x \rangle > 0, \quad 0 \neq x \in \mathrm{Ch}^p(X)^{00}.$$

Using Mukai–Fourier transform, we may decompose the group $\mathrm{Ch}^p(A)$ into a direct sum of eigen spaces under multiplications:

$$\mathrm{Ch}^p(A) = \sum_s \mathrm{Ch}^p_s(A)$$

where s are integers and $\mathrm{Ch}^p_s(A)$ is the subgroup of cycles $x \in \mathrm{Ch}^p(A)$ with the property

$$[k]^* x = k^{2p-s} x, \quad \forall k \in \mathbb{Z},$$

where $[k]$ is the multiplication on A by k. It has been conjectured that $\mathrm{Ch}^p_s(A) = 0$ if $s \neq 0, 1$. By the projection formula

$$\langle k^* x, y \rangle = \langle x, k_* y \rangle$$

we see that $\mathrm{Ch}^p_s(A)^0$ are perpendicular to $\mathrm{Ch}^q_t(A)^0$ unless

$$p + q = g + 1, \qquad s + t = 2.$$

Let X be a curve over a global field k with Jacobian J. For an integer $n \in [0, g]$, we can define morphism $f_n : X^n \longrightarrow J$ by sending $(x_1, \ldots, x_n)$ to the class of $\sum (x_i - \xi)$. Notice that the image does not depend on the choice of ξ. We view X as a subvariety of J via embedding f_1 and define the theta divisor θ as the image of f_{g-1}. We use θ for the primitive decomposition and Fourier–Mukai transform:

$$\mathcal{F} : \mathrm{Ch}^*(J) \longrightarrow \mathrm{Ch}^*(J)$$

$$x \mapsto \mathcal{F}(x) := p_{2*}(p_1^* x \cdot e^{\lambda})$$

where λ is the Poincaré class:

$$\lambda = p_1^* \theta + p_2^* \theta - m^* \theta.$$

The decomposition into s-space can be made explicit as follows: Define a decomposition $\mathcal{F} = \sum \mathcal{F}_s$ by

$$\mathcal{F}_s(x) = p_{2*}\left(p_1^* x \cdot \frac{\lambda^{2g-2p+s}}{(2g-2p+s)!}\right).$$

Then we have decomposition $x = \sum x_s$ with

$$x_s = \mathcal{F}^{-1}(\mathcal{F}_s(x)) \in \mathrm{Ch}_s^p(J),$$

where $\mathcal{F}^{-1}$ is the inverse of $\mathcal{F}$ which has an expression:

$$\mathcal{F}^{-1} = (-1)^g[-1]^* \circ \mathcal{F}.$$

Following Beauville, we define the ring $\mathcal{R}$ of tautological cycles of $\mathrm{Ch}^*(J)$ as the smallest $\mathbb{Q}$-vector generated by X under the following operations: the intersection, the star operator, and the Fourier–Mukai transform. By Beauville, in the decomposition $\mathcal{R} = \bigoplus_s \mathcal{R}_s$, $\mathcal{R}_s = 0$ if $s < 0$ and $\mathcal{R}_0$ is generated by θ. Thus $\mathcal{R}_0$ maps injectively into cohomology group. Thus cohomological trivial cycles have components $s > 0$. The height intersection on these cycles factors through the first component:

$$\langle x, y\rangle = \langle x_1, y_1\rangle.$$

The key to prove Theorem 1.5.5 is the following pull-back formula:

Theorem 5.3.1 *Consider the morphism $f_3 : X^3 \longrightarrow J$. Then*

$$f_3^* \mathcal{F}(X) = -g \sum_i p_i^* \xi - \sum_{ij} p_{ij*}\delta_\xi + \Delta_\xi,$$

where δ_ξ is the class

$$\delta_\xi = p_1^*\xi + p_2^*\xi - \Delta \in \mathrm{Ch}^1(X^2).$$

Proof By discussion above,

$$\mathcal{F}(X) = p_{2*}(p_1^* X \cdot e^\lambda).$$

Consider the morphism

$$g : X^4 \longrightarrow J \times J, \qquad (x_i) \mapsto (x_0 - \xi, x_1 + x_2 + x_3 - 3\xi).$$

Then it is easy to see that

$$f_3^* \mathcal{F}(X) = p_{123*} g^* e^\lambda = p_{123*} \exp g^*\lambda.$$

Let us compute the class $g^*\lambda$:

$$g^*\lambda = p_0^*\theta + p_{123}^* f_3^*\theta - f_4^*\theta.$$

We want to use the theorem of cube to decompose this bundle into a sum of pull-backs of bundles of a face X^2 of X^4. More conveniently, we may consider this bundle as pull-back of bundle on A^4 of the following bundle:

$$m_0^*\theta + m_{123}^*\theta - m_{0123}^*\theta$$

where for a subset I of $\{0, 1, 2, 3, 4\}$, m_I is the sum of elements in I. By the theorem of cube, this bundle has an expression

$$\sum_{ij} \mathcal{L}_{ij} + \sum_i \mathcal{M}_i$$

where $\mathcal{L}_{ij}$ are line bundles on J^2 with trivial restriction on $\{0\} \times J$ and $J \times \{0\}$ and $\mathcal{M}_i$ are line bundles on J. Now lets us restrict the bundle on ij-factors with 0 on other factors to obtain:

$$\mathcal{L}_{0i} = \lambda, \quad \mathcal{L}_{ij} = 0, \quad \forall i, j > 0.$$

Similarly, restrict on a single factor to get $\mathcal{M}_i = 0$. In summary, we have shown that

$$g^*\lambda = \sum_{i=0}^{3} f_{0i}^*\lambda$$

where f_{0i} is the projection $X^3 \longrightarrow A^2$. To compute the bundle $f_{0i}\lambda$ we consider the embedding $X^2 \longrightarrow A^2$. It is easy to see that the restriction of λ is given by δ_ξ. It follows that

$$g^*\lambda = \sum_i p_{0i}^*\delta_\xi.$$

Thus we have

$$f_3^*\mathcal{F}(X) = p_{123*} \exp g^*\lambda = \sum_{ijk} \frac{1}{i!j!k!} p_{123*}(p_{01}^*\delta_\xi^i \cdot p_{02}^*\delta_\xi^j \cdot p_{03}^*\delta_\xi^k).$$

The identity in Theorem follows from a direct computation. □

Proof of Theorem 1.5.5 The first formula follows from Theorem 5.3.1. The second follows form the identity

$$f_{3*}\Delta_\xi = [3]_*X - 3[2]_*X + 3X, \quad X = \sum X_s.$$

For the third formula, we notice the star operator and intersection operator respect to the s-graduation. Push the first formula in the Theorem to J to obtain:

$$X^{*3} \cdot \mathcal{F}(X_1) = [3]_*X - 3[2]_*X + 3X.$$

Decompose this into s-components to obtain:

$$\mathcal{F}(X_1) \cdot \sum_{i+j+k=s-1} X_i * X_j * X_k = (3^{2+s} - 3 \cdot 2^{2+s} + 3)X_s.$$

This proves the identity in the third formula. The list of equivalence is clear by three identities and the following expression for Ceresa cycle:

$$X - [-1]_*X = 2\sum_{s \text{ odd}} X_s.$$

□

Proof of Theorem 1.5.6 By Theorem 1.5.5, $f_3^*\mathcal{F}(X_1) = \Delta_\xi$. The first inequality follows from the projection formula:

$$\langle \Delta_\xi, \Delta_\xi \rangle_{X^3} = \langle \mathcal{F}(X_1), (f_{3*}\Delta_\xi)_1 \rangle_{X^3}.$$

Now we use the identity

$$f_{3*}\Delta_\xi = [3]_* X - 3[2]_* X + 3X = 6X_1 + \cdots.$$

For the second inequality, we use another projection formula

$$\begin{aligned}\langle \Delta_\xi, \Delta_\xi \rangle_{X^3} = \langle f_3^*\mathcal{F}(X_1), f_3^*\mathcal{F}(X_1) \rangle_{X^3} &= \langle \mathcal{F}(X_1), f_{3*}f_3^*\mathcal{F}(X_1) \rangle_J \\ &= \langle \mathcal{F}(X_1), X^{*3} \cdot \mathcal{F}(X_1) \rangle.\end{aligned}$$

As the intersection pairing depends only on the $s = 1$ component, we may replace X^{*3} by

$$X_0^{*3} = \frac{6}{(g-3)!}\theta^{g-3}.$$

Here for a subvariety Y of X, Y^{*d} denote d-th star product power of Y. This proves the identity in the Theorem. To show that $\mathcal{F}(X_1)$ is primitive, we use the following identity:

$$L \cdot L^{g-3}\mathcal{F}(X_1) = \frac{(g-3)!}{6}(\theta \cdot X^{*3}\mathcal{F}(X_1))_1 = \frac{(g-3)!}{6} f_{3*}(f_3^*\theta \cdot \Delta_\xi).$$

Thus it suffices to prove

$$f_3^*\theta \cdot \Delta_\xi = 0.$$

By Theorem 5.3.1,

$$f_3^*\theta = -f_3^*\mathcal{F}(X_0) = g \sum i p_i^* \xi + \sum_{ij} p_{ij}^* \delta_\xi.$$

It is easy to show all of these terms have zero intersection with Δ_ξ. □

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