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The arithmetic Hodge index theorem for adelic line bundles

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Abstract In this paper, we prove index theorems for integrable metrized line bundles on projective varieties over complete fields and number fields respectively. As applications, we prove a non-archimedean analogue of the Calabi theorem and a rigidity theorem about the preperiodic points of algebraic dynamical systems.

Contents

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1 Introduction

The Hodge index theorem for line bundles on arithmetic surfaces proved by Faltings [13] and Hriljac [21] is one of the fundamental results in Arakelov theory. In [29], this index theorem has been generalized by Moriwaki to higher-dimensional arithmetic varieties. In this paper, we will give further generalizations of the index theorem for adelic metrized line bundles in the setting of [40]. These adelic metrics naturally appear in algebraic dynamical systems and moduli spaces of varieties. As applications, we will prove a non-archimedean analogue of the Calabi theorem of [6] and a rigidity theorem about the preperiodic points of algebraic dynamical systems. In more details, our results are explained as follows.

1.1 Local Hodge index theorem

Let K be an algebraic closed field endowed with a nontrivial complete (either archimedean or non-archimedean) absolute value $|\cdot|$. Let X be a projective variety over K of dimension $n \geq 1$. Denote by X^{an} the associated Berkovich analytic space (resp. complex analytic spaces) introduced in [1] if K is non-archimedean (resp. archimedean). Then there is a theory of integrable metrized line bundles and their intersection numbers developed in [8,17,19,40]. We will briefly review the theory in "Appendix (Local intersections)", and will refer to "Appendix (Local intersections)" for our convention and terminology.

Let $\overline{M}$ be an integrable metrized line bundle on X with $M = \mathcal{O}_X$, and $\overline{L}_1, \ldots, \overline{L}_{n-1}$ be $n-1$ semipositive metrized line bundles on X. Then we show in Sect. 2 the following inequality:

$$\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1} \leq 0.$$

Moreover, if L_i is ample and $\overline{M}$ is $\overline{L}_i$*-bounded* for each i, then the equality holds if and only if $\overline{M} \in \pi^*\widehat{\mathrm{Pic}}(K)$. Here $\pi : X \to Spec K$ denotes the structure morphism.

When $K = \mathbb{C}$, the above inequality is written in terms of distributions as

$$-\int_{X(\mathbb{C})} f \frac{\partial\bar{\partial}}{\pi i} f \cdot c_1(\overline{L}_1) \cdots c_1(\overline{L}_{n-1}) \leq 0, \qquad f := -\log \|1\|_{\overline{M}}.$$

If X is smooth and the metrics of $\overline{M}$ and $\overline{L}_i$ are also smooth, this is a simple consequence of integration by parts.

In general, for the inequality part, it is easily reduced to the smooth case when K is archimedean or the model case when K is non-archimedean. For the equality part, we use a non-archimedean analogue of Błocki's method in [3], and the works of Gubler [19] and Chambert-Loir–Thuillier [10].

One application of our local Hodge index theorem is a non-archimedean analogue of the theorem of Calabi [6] on the uniqueness of semipositive metrics on an ample line bundle on X^{an} with a given volume form.

1.2 Arithmetic Hodge index theorem

Let K be a number field and X be a normal and geometrically integral projective variety over K of dimension $n \geq 1$. There is a theory of integrable metrized line bundles and their intersection numbers developed in [40]. We will briefly review the theory in "Appendix (Arithmetic intersections)", and will refer to "Appendix (Arithmetic intersections)" for our convention and terminology.

Let $\overline{M}$ be an integrable adelic $\mathbb{Q}$-line bundle on X, and $\overline{L}_1, \ldots, \overline{L}_{n-1}$ be $n-1$ nef adelic $\mathbb{Q}$-line bundles on X. Assume that each L_i is big and the equality $M \cdot L_1 \cdots L_{n-1} = 0$. Then in Sect. 3, we prove the following inequality:

$$\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1} \leq 0.$$

Moreover, if $\overline{L}_i$ is arithmetically positive and $\overline{M}$ is $\overline{L}_i$*-bounded* for each i, then the equality holds if and only if $\overline{M} \in \pi^* \widehat{\mathrm{Pic}}(K)_{\mathbb{Q}}$. Here $\pi : X \to \mathit{Spec} K$ denotes the structure morphism.

When X is regular with a projective and flat model $\mathcal{X}$ over O_K, and $(\overline{M}, \overline{L}_1, \ldots, \overline{L}_{n-1})$ comes from hermitian line bundles $(\overline{\mathcal{M}}, \overline{\mathcal{L}}, \ldots, \overline{\mathcal{L}})$ with $\overline{\mathcal{L}}$ arithmetically ample, this result is due to Faltings [13] and Hriljac [21] for $n = 1$ and Moriwaki [29] for $n > 1$. A slight extension of Moriwaki's result gives the inequality part in the adelic situation. For the equality part in adelic situation, we will use the local index theorem, the vanishing result of curvatures of flat metrics by Gubler [19] (cf. Proposition 5.17), the integration of Green's functions on Berkovich spaces of Chambert-Loir–Thuillier [10], and Lefschetz-type theorems for normal varieties (cf. "Appendix (Lefschetz theorems)").

1.2.1 Dynamical system

Let X be a projective variety over $\overline{\mathbb{Q}}$. A *polarizable algebraic dynamical system on* X is a morphism $f : X \to X$ such that there is an ample $\mathbb{Q}$-line bundle L satisfying $f^*L \simeq qL$ from some rational number $q > 1$. A central invariant for such a dynamical system is the set $\mathrm{Prep}(f)$ of *preperiodic points*, i.e., points of $X(\overline{\mathbb{Q}})$ with finite forward orbits under iterations of f. We show the following rigidity result about preperiodic points in Sect. 4. Let f and g be two polarized algebraic dynamical systems on X, and Z be the Zariski closure of $\mathrm{Prep}(f) \cap \mathrm{Prep}(g)$. Then

$$\mathrm{Prep}(f) \cap Z(\overline{\mathbb{Q}}) = \mathrm{Prep}(g) \cap Z(\overline{\mathbb{Q}}).$$

We prove the result by first constructing f-admissible metrics for *every line bundle on X* if X is normal, and then applying our arithmetic index theorem.

2 Local Hodge index theorem

In this section, we prove a local Hodge index theorem over complete fields. As a consequence, we will prove a version of the Calabi theorem. When the base field is $\mathbb{C}$, all of these are essentially due to the original work of Calabi [6] and the later extension by Błocki [3]. This local index theorem is a crucial ingredient in the proof of arithmetic Hodge index theorem in Sect. 3.

2.1 Statements

First let us recall the theory of integrable metrized line bundles developed in [8,17,19,40]. For more details, see "Appendix (Local intersections)".

Let K be an algebraically closed field endowed with a complete and nontrivial absolute value $|\cdot|$, which can be either archimedean or non-archimedean. Let X be a projective variety over K of dimension $n \geq 1$, and X^{an} the associated Berkovich analytic space (resp. complex analytic space if $K = \mathbb{C}$ is archimedean). Then there are a category $\widehat{\mathcal{P}\mathrm{ic}}(X)$ of integrable metrized line bundles $\overline{L} = (L, \|\cdot\|)$ on X and the group $\widehat{\mathrm{Pic}}(X)$ of isometry classes of these bundles. There are also notions of model metrics and semipositive metrics. In particular, we call a metric *semipositive* if it is a uniform limit of model metrics induced by relatively nef models over O_K. We call a metrized line bundle *semipositive* if the metric is semipositive, and call a metrized line bundle *integrable* if it is the difference of two semipositive metrized line bundles. When K was a discrete valuation field, these were defined in [40]. For general non-archimedean fields, they were defined by Gubler [17].

If $\overline{L}_0, \ldots, \overline{L}_n$ are integrable metrized line bundles with nonzero rational sections $\ell_0, \ldots, \ell_n$ such that $\cap|\mathrm{div}(\ell_i)| = \emptyset$, then there is a well-defined local intersection number $\widehat{\mathrm{div}}(\ell_0)\cdots\widehat{\mathrm{div}}(\ell_n)$ using a limit process. Assume that $L_0 \simeq \mathcal{O}_X$ and $\ell_0 = 1$ under this identity. Then the intersection does not depend on the choice of $\ell_1, \ldots, \ell_n$. Chambert-Loir [8] (when K contains a countable subfield) and Gubler [19] (in general) define a measure $c_1(\overline{L}_1)\cdots c_1(\overline{L}_n)$ on X^{an} so that

$$-\int_{X^{\mathrm{an}}} \log\|\ell_0\| c_1(\overline{L}_1)\cdots c_1(\overline{L}_n) = \widehat{\mathrm{div}}(\ell_0)\cdots\widehat{\mathrm{div}}(\ell_n).$$

Assuming further $c_1(L_1)\cdots c_1(L_n) = 0$, then the local intersection does not depend on the choice of ℓ_0, neither. This gives a well-defined intersection number

$$\overline{L}_0\cdots\overline{L}_n := -\int_{X^{\mathrm{an}}} \log\|\ell_0\| c_1(\overline{L}_1)\cdots c_1(\overline{L}_n).$$

For two integrable metrized line bundles $\overline{L}$ and $\overline{M}$ on X, say that $\overline{M}$ is *$\overline{L}$-bounded* if there is a positive integer m such that both $m\overline{L} + \overline{M}$ and $m\overline{L} - \overline{M}$ are semipositive.

Here we write tensor product of line bundles additively, so $m\overline{L}-\overline{M}$ means $\overline{L}^{\otimes m}\otimes\overline{M}^{\vee}$ (with the induced metric). We take this convention throughout this paper. The main result of this section is the following local index theorem.

Theorem 2.1 (Local Hodge index theorem) *Let $\pi: X\to \mathrm{Spec}K$ be an integral projective variety of dimension $n\geq 1$. Let $\overline{M}$ be an integrable metrized line bundle on X with $M\simeq \mathcal{O}_X$, and $\overline{L}_1,\ldots,\overline{L}_{n-1}$ be $n-1$ semipositive metrized line bundles on X. Then*

$$\overline{M}^2\cdot\overline{L}_1\cdots\overline{L}_{n-1}\leq 0.$$

Moreover, if L_i is ample and $\overline{M}$ is $\overline{L}_i$-bounded for each i, then the equality holds if and only if $\overline{M}\in\pi^*\widehat{\mathrm{Pic}}(K)$.

One application of our local index theorem is the following non-archimedean analogue of the Calabi theorem. We refer to Calabi [6] for the original theorem and to Kolodziej [23] and Błocki [3] for some regularity extensions.

Corollary 2.2 (Calabi Theorem) *Let X be an integral projective variety over K. Let L be an ample line bundle on X, and $\|\cdot\|_1$ and $\|\cdot\|_2$ be two semipositive metrics on L. Then*

$$c_1(L,\|\cdot\|_1)^{\dim X}=c_1(L,\|\cdot\|_2)^{\dim X}$$

if and only if $\dfrac{\|\cdot\|_1}{\|\cdot\|_2}$ is a constant function on X^{an}.

Remark 2.3 There is an open problem about the existence of a metric on a line bundle with a given volume form, i.e., the non-archimedean analogue of the theorem of Yau [37]. Some partial results have been obtained by Liu [26] and Boucksom–Favre–Jonsson [5].

We will prove Theorem 2.1 in the next three subsections. Here we see how it implies Corollary 2.2.

Proof of Corollary 2.2 Write $n=\dim X$. Set $f=-\log(\|\cdot\|_1/\|\cdot\|_2)$ as a continuous function on X^{an}. Then the assumption implies

$$\int_{X^{\mathrm{an}}} f\, c_1(L,\|\cdot\|_1)^n=\int_{X^{\mathrm{an}}} f\, c_1(L,\|\cdot\|_2)^n$$

We need to show that f is constant.

Denote $\overline{L}_1=(L,\|\cdot\|_1)$, $\overline{L}_2=(L,\|\cdot\|_2)$, and $\overline{M}=\overline{L}_1-\overline{L}_2$. Then the difference of two sides of the above equality is just

$$\sum_{i=0}^{n-1}\int_{X^{\mathrm{an}}} f\, c_1(\overline{M})c_1(\overline{L}_1)^i c_1(\overline{L}_2)^{n-1-i}=0.$$

Equivalently,

$$\sum_{i=0}^{n-1} \overline{M}^2 \cdot \overline{L}_1^i \cdot \overline{L}_2^{n-1-i} = 0.$$

By Theorem 2.1, every term in the sum is non-positive. It forces

$$\overline{M}^2 \cdot \overline{L}_1^i \cdot \overline{L}_2^{n-1-i} = 0, \quad \forall\, i = 0, \ldots, n-1.$$

It follows that

$$\overline{M}^2 \cdot (\overline{L}_1 + \overline{L}_2)^{n-1} = 0.$$

Note that $\overline{M}$ is $(\overline{L}_1 + \overline{L}_2)$-bounded. Theorem 2.1 implies $\overline{M} \in \pi^*\widehat{\mathrm{Pic}}(K)$, which is equivalent to the statement that f is a constant. □

2.2 Curves

In this section, we prove Theorem 2.1 when $\dim X = 1$. Then the line bundles $\overline{L}_i$ do not show up. Considering the pull-backs of the line bundles on the normalization of X, the problem is reduced to the case that X is smooth.

First, let us treat the case that K is non-archimedean. The result is obtained by combining many results of Gubler.

We start with the model case; i.e., (X, M) is induced by an integral model $(\mathcal{X}, \mathcal{M})$ over the valuation ring O_K of K. By the semistable reduction theorem of Bosch–Lütkebohmert [4], any integral model of X over K is dominated by a semistable model over O_K. Thus we only need to consider the case $\mathcal{X}$ is semistable. Then the result is proved by Gubler [18, Theorem 7.17] following the methods of Faltings and Hriljac.

Now we extend it to the non-model case. As the limit case of the model case, we have the inequality $\overline{M}^2 \leq 0$ for general vertical metrized line bundles, and a Cauchy–Schwarz inequality. To prove the equality part of the theorem, assume $\overline{M}^2 = 0$. Then for any vertical integrable line bundle $\overline{N}$ on X, we have

$$(\overline{M} \cdot \overline{N})^2 \leq (\overline{M}^2)(\overline{N}^2) = 0.$$

Thus for any vertical integrable line bundle $\overline{N}$,

$$\overline{M} \cdot \overline{N} = 0.$$

We first prove that the measure $c_1(\overline{M}) = 0$. Note that the generic fibers M and N are isomorphic to $\mathcal{O}_X$. Use the regular sections 1_M of M and 1_N of N to compute the intersection number $\overline{M} \cdot \overline{N}$. By the induction formula of Chambert-Loir, Gubler, and Chambert-Loir–Thuillier recalled in "Appendix (Local intersections)", we have

$$\begin{aligned} 0 = \overline{M} \cdot \overline{N} &= \widehat{\mathrm{div}}(1_M) \cdot \widehat{\mathrm{div}}(1_N) \\ &= \widehat{\mathrm{div}}(1_N) \cdot \widehat{\mathrm{div}}(1_M) = -\int_{X^{\mathrm{an}}} \log \|1_N\| c_1(\overline{M}). \end{aligned}$$

Here we have also used the symmetry of the intersection number in [17, Proposition 9.3]. Note that $\log \|1_N\|$ can be any model function on X^{an}. The space $C_{\mathrm{mod}}(X^{\mathrm{an}})$ of model functions on X^{an} is dense in the space $C(X^{\mathrm{an}})$ of continuous functions on X^{an} with respect to the supremum norm. This is part of [17, Theorem 7.12], which actually works in the current generality. Therefore, the measure $c_1(\overline{M}) = 0$.

It remains to prove that $\overline{M}$ is constant from the vanishing of the measure. The following argument is suggested by an anonymous referee of a previous version of this paper.

Take two points $P_1, P_2 \in X(K)$. Take $\overline{N}$ to be the line bundle $N = \mathcal{O}(P_1 - P_2)$ with a flat metric as in Definition 5.10 and Theorem 5.11. Denote by ℓ the canonical rational section of N with divisor $P_1 - P_2$. Use the rational sections 1 of M and ℓ of N to compute the intersection number $\overline{M} \cdot \overline{N}$. Apply the induction formula of Chambert-Loir, Gubler, and Chambert-Loir–Thuillier in "Appendix (Local intersections)" again. We have

$$\widehat{\mathrm{div}}(1) \cdot \widehat{\mathrm{div}}(\ell) = -\int_{X^{\mathrm{an}}} \log \|1\| c_1(\overline{N}) = 0,$$

and

$$\begin{aligned}\widehat{\mathrm{div}}(\ell) \cdot \widehat{\mathrm{div}}(1) &= \log \|1\|(P_2) - \log \|1\|(P_1) - \int_{X^{\mathrm{an}}} \log \|\ell\| c_1(\overline{M}) \\ &= \log \|1\|(P_2) - \log \|1\|(P_1).\end{aligned}$$

By the symmetry of the intersection number in [17, Proposition 9.3] again, we have

$$\log \|1\|(P_2) = \log \|1\|(P_1).$$

This is true for any $P_1, P_2 \in X(K)$. Note that $X(K)$ is dense in X^{an}. Hence, $\log \|1\|$ is constant on X^{an}. This finishes the proof of Theorem 2.1 in the non-archimedean case.

The theorem in the archimedean case follows from a similar argument. The theorem is clear in the case of smooth metrics by integration by parts. In fact, denoting $f = -\log \|1\|_{\overline{M}}$ in the smooth case, we have

$$\overline{M}^2 = -\int_{X(\mathbb{C})} f \frac{\partial \bar{\partial}}{\pi i} f = \int_{X(\mathbb{C})} \frac{1}{\pi i} \partial f \wedge \bar{\partial} f \leq 0$$

with equality holds only if f is constant. By taking limits, this implies that $\overline{M}^2 \leq 0$ in the case of integrable metrics. The proof of the remaining part is also similar.

2.3 Inequality

Now we prove the inequality part of Theorem 2.1 in the general case by induction on $n = \dim X$. We have already treated the case $n = 1$, so we assume $n \geq 2$ in the following discussion. First, we can make the following two assumptions:

(1) X is smooth by de Jong's alteration [11] and the projection formula in [8,17,40];
(2) each $\overline{L}_i$ is arithmetically positive by adding some multiple of an arithmetically positive integrable line bundle $\overline{A}$ on X as follows. Take a small rational number $\epsilon > 0$ and set $\overline{L}_i' = \overline{L}_i + \epsilon\overline{A}$. Then the inequality $\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1} \leq 0$ is the limit of the inequality $\overline{M}^2 \cdot \overline{L}_1' \cdots \overline{L}_{n-1}' \leq 0$.

When $K = \mathbb{C}$, by approximation, it suffices to prove the inequality under the assumption that all metrics are smooth, and the curvature forms of L_i are positive definite. Let $f = -\log \|1\|_{\overline{M}}$. Then $c_1(\overline{M}) = -\frac{\partial\bar{\partial}}{\pi i} f$ and

$$\begin{aligned}\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1} &= -\int_{X(\mathbb{C})} f \frac{\partial\bar{\partial}}{\pi i} f c_1(\overline{L}_1) \cdots c_1(\overline{L}_{n-1}) \\ &= \int_{X(\mathbb{C})} \frac{1}{\pi i} \partial f \wedge \bar{\partial} f \wedge c_1(\overline{L}_1) \cdots c_1(\overline{L}_{n-1}) \leq 0.\end{aligned}$$

When K is non-archimedean, by approximation, it is reduced to prove the inequality

$$\mathcal{M}^2 \cdot \mathcal{L}_1 \cdots \mathcal{L}_{n-1} \leq 0$$

under the following assumptions:

(1) X is smooth and $\mathcal{X}$ is a normal integral model of X over O_K;
(2) $\mathcal{M}, \mathcal{L}_1, \ldots, \mathcal{L}_{n-1}$ are line bundles on $\mathcal{X}$ with generic fibers $M, L_1, \cdots, L_{n-1}$;
(3) $\mathcal{L}_i$ is relatively ample on $\mathcal{X}$.

We claim a Bertini-type result that, there is a positive integer m and a nonzero section $s \in H^0(\mathcal{X}, m\mathcal{L}_{n-1})$ such that $\mathcal{Y} := \mathrm{div}(s)$ is horizontal on $\mathcal{X}$ with a smooth generic fiber $\mathcal{Y}_K$.

Assuming the claim, then

$$\mathcal{M}^2 \cdot \mathcal{L}_1 \cdots \mathcal{L}_{n-1} = \frac{1}{m} \mathcal{M}|_{\mathcal{Y}}^2 \cdot \mathcal{L}_1|_{\mathcal{Y}} \cdots \mathcal{L}_{n-2}|_{\mathcal{Y}}.$$

Thus it is non-positive by induction.

It remains to prove the Bertini-type result. We use an argument from [28]. By our definition of models, there is a noetherian local subring $(R, \wp)$ of O_K with $\mathfrak{m} \cap R = \wp$, where $\mathfrak{m}$ denotes the maximal ideal of O_K, so that $(\mathcal{X}, \mathcal{M}, \mathcal{L}_1, \ldots, \mathcal{L}_{n-1})$ are base changes of models $(\mathcal{X}^0, \mathcal{M}^0, \mathcal{L}_1^0, \ldots, \mathcal{L}_{n-1}^0)$ over R. Write $\mathcal{L}$ for $\mathcal{L}_{n-1}^0$ for simplicity. We first find a section $s_0 \in H^0(\mathcal{X}^0, m\mathcal{L})$ for sufficiently large m such that $\mathrm{div}(s_0)$ is horizontal. In fact, denote by $V_1, \ldots, V_r$ the irreducible components of the special fiber of $\mathcal{X}^0$. Take a distinct closed point x_i in V_i^0 for every i, and denote by $k(x_i)$ the residue field of x_i, viewed as a skyscraper sheaf on $\mathcal{X}^0$. By the ampleness of $\mathcal{L}$, for sufficiently large m, the natural map

$$H^0(\mathcal{X}^0, m\mathcal{L}) \longrightarrow \bigoplus_i H^0(\mathcal{X}^0, (m\mathcal{L}) \otimes k(x_i))$$

is surjective. By lifting an element on the right-hand side which is non-zero at every component, we get a section $s_0 \in H^0(\mathcal{X}^0, m\mathcal{L})$ non-vanishing at any x_i. Thus $\mathrm{div}(s_0)$ is horizontal. Once we have s_0, consider the subset

$$\Gamma_{s_0} = s_0 + \wp \cdot H^0(\mathcal{X}^0, m\mathcal{L}) \subset H^0(X^0, mL).$$

Here $\wp$ denotes the maximal ideal of R as above, X^0 denotes the generic fiber of X, and $L = \mathcal{L}|_{X^0}$. Then any element of Γ_{s_0} has a horizontal zero locus. It is easy to check that any polynomial on $H^0(X^0, mL)$ which vanishes on Γ_{s_0} is identically 0. Hence, the image of Γ_{s_0} in $\mathbb{P}(H^0(X^0, mL))$ is Zariski dense. Assume that mL is very ample. By Bertini's theorem, there is a Zariski open subset of $\mathbb{P}(H^0(X^0, mL))$ whose elements correspond to smooth and irreducible hyperplane sections. This open set intersects the image of Γ_{s_0}. Hence, we can find an element $s^0 \in \Gamma_{s_0}$ with smooth and irreducible zero locus on X^0. The pullback of s^0 to $\mathcal{X}$ satisfies all the required properties.

The following Cauchy–Schwarz inequality is a direct consequence of the inequality part of Theorem 2.1.

Corollary 2.4 *Let $\overline{M}$ and $\overline{M}'$ be two vertical integrable line bundles on X, and $\overline{L}_1, \ldots, \overline{L}_{n-1}$ be $n-1$ semipositive line bundles on X. Then*

$$(\overline{M} \cdot \overline{M}' \cdot \overline{L}_1 \cdots \overline{L}_{n-1})^2 \leq (\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1})(\overline{M}'^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1}).$$

2.4 Equality

Now we prove the equality part of Theorem 2.1 in the general case by induction on $n = \dim X$. The case $n = 1$ is proved, so we assume $n \geq 2$.

Let $\overline{M}, \overline{L}_1, \ldots, \overline{L}_{n-1}$ be integrable line bundles on X such that the following conditions hold:

(1) M is trivial on X;
(2) $\overline{M}$ is $\overline{L}_i$-bounded for every i;
(3) $\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1} = 0$.

There are two key lemmas in our proof. The first one is inspired by Błocki [3].

Lemma 2.5 *For any semipositive integrable line bundles $\overline{L}_i^0$ on X with underlying bundle L_i^0 equal to L_i, and any integrable integrable line bundle $\overline{M}'$ with trivial underlying line bundle M', the following identities hold:*

$$\overline{M} \cdot \overline{M}' \cdot \overline{L}_1^0 \cdots \overline{L}_{n-1}^0 = 0,$$
$$\overline{M}^2 \cdot \overline{M}' \cdot \overline{L}_1^0 \cdots \overline{L}_{n-2}^0 = 0.$$

Proof The second equality follows from the first one by taking $\bar{L}_{n-1}^0 = \bar{L}_{n-1} \pm \epsilon \bar{M}$ with small ϵ. So it suffices to prove the first inequality. Note that condition (2) implies that every $\overline{L}_i$ is semipositive. Replacing $\overline{L}_i$ by a large multiple if necessary, we can assume that both $\overline{L}_i \pm \overline{M}$ are semipositive for each i. Denote $\overline{L}_i^{\pm} = \overline{L}_i \pm \overline{M}$.

First, we have the following equality

$$\overline{M}^2 \cdot \overline{L}_1^{\epsilon(1)} \cdots \overline{L}_{n-1}^{\epsilon(n-1)} = 0$$

for any sign function $\epsilon : \{1, \ldots, n-1\} \to \{+, -\}$. In fact, since $\overline{M}$ is $\overline{L}_i$-bounded, there is a constant $t > 0$ such that

$$\overline{L}_{it}^{\pm} := \overline{L}_i - t\overline{L}_i^{\pm} = (1-t)\overline{L}_i \mp t\overline{M}$$

is semipositive for any i. Write

$$\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1} = \overline{M}^2 \cdot t\overline{L}_1^{\epsilon(1)} \cdots t\overline{L}_{n-1}^{\epsilon(n-1)} + \sum_{\ell=1}^{n-1} \overline{M}^2 \cdot \prod_{i \le \ell} \overline{L}_{it}^{\epsilon(i)} \cdot \prod_{j > \ell} \overline{L}_i^{\epsilon(i)}.$$

Then the inequality part of Theorem 2.1 implies that every term on the right hand side is non-positive. It forces

$$\overline{M}^2 \cdot \overline{L}_1^{\epsilon(1)} \cdots \overline{L}_{n-1}^{\epsilon(n-1)} = 0.$$

Second, we claim that for any ℓ and any sign function ϵ on $\{\ell, \ldots, n-1\}$,

$$\overline{M}^2 \cdot \overline{L}_1^0 \cdots \overline{L}_{\ell-1}^0 \cdot \overline{L}_\ell^{\epsilon(\ell)} \cdots \overline{L}_{n-1}^{\epsilon(n-1)} = 0. \tag{1}$$

When $\ell = n$, it gives

$$\overline{M}^2 \cdot \overline{L}_1^0 \cdots \overline{L}_{n-1}^0 = 0.$$

By the Cauchy–Schwarz inequality in Corollary 2.4, it follows that

$$\overline{M} \cdot \overline{M}' \cdot \overline{L}_1^0 \cdots \overline{L}_{n-1}^0 = 0.$$

We now prove the claim by induction on ℓ. We already have the case $\ell = 1$. Assume the equality (1) for ℓ and for all sign functions on $\{\ell, \ldots, n-1\}$. Let ϵ be a sign function on $\{\ell+1, \ldots, n-1\}$ which has two extensions ϵ^+, ϵ^- on $\{\ell, \ldots, n-1\}$ by $\epsilon^{\pm}(\ell) = \pm$. Applying Corollary 2.4 again to $\overline{M}, \overline{M}_\ell^{\pm} = \overline{L}_\ell^{\pm} - \overline{L}_\ell^0$, and semipositive bundles $\bar{L}_1^0, \ldots, \bar{L}_{\ell-1}^0, \bar{L}_\ell^{\epsilon^{\pm}(\ell)}, \ldots, \bar{L}_{n-1}^{\epsilon(n-1)}$, we have four equalities:

$$\overline{M} \cdot \overline{M}_\ell^{\delta_1} \cdot \overline{L}_1^0 \cdots \overline{L}_{\ell-1}^0 \cdot \overline{L}_\ell^{\delta_2} \cdots \overline{L}_{n-1}^{\epsilon(n-1)} = 0, \qquad \delta_1 = \pm, \quad \delta_2 = \pm.$$

Take the difference for two equations with fixed δ_1 to obtain

$$\overline{M} \cdot \overline{M}_\ell^{\delta_1} \cdot \overline{L}_1^0 \cdots \overline{L}_{\ell-1}^0 \cdot \overline{M} \cdot \overline{L}_{\ell+1}^{\epsilon(\ell+1)} \cdots \overline{L}_{n-1}^{\epsilon(n-1)} = 0.$$

It is just

$$\overline{M}^2 \cdot \overline{L}_1^0 \cdots \overline{L}_{\ell-1}^0 \cdot (\overline{L}_\ell^{\delta_1} - \overline{L}_\ell^0) \cdot \overline{L}_{\ell+1}^{\epsilon(\ell+1)} \cdots \overline{L}_{n-1}^{\epsilon(n-1)} = 0.$$

The left-hand side splits to a difference of two terms. One term is zero by the induction assumption for ℓ. It follows that the other term is also zero, which gives

$$\overline{M}^2 \cdot \overline{L}_1^0 \cdots \overline{L}_{\ell-1}^0 \cdot \overline{L}_\ell^0 \cdot \overline{L}_{\ell+1}^{\epsilon(\ell+1)} \cdots \overline{L}_{n-1}^{\epsilon(n-1)} = 0.$$

It is exactly the case $\ell + 1$. The proof is complete. □

The second key lemma is the following induction result.

Lemma 2.6 *Assume that L_{n-1} is ample. For any closed subvariety Y of codimension one in X, we have*

$$\overline{M}|_Y^2 \cdot \overline{L}_1|_Y \cdots \overline{L}_{n-2}|_Y = 0.$$

Proof Replacing $\overline{L}_{n-1}$ by a multiple if necessary, we can assume that there is a global section s of L_{n-1} vanishing on Y. Write $\mathrm{div}(s) = \sum_i a_i Y_i$ with $a_i > 0$ and prime divisors Y_i. Use the non-archimedean induction formula developed by Chambert-Loir–Thuillier and Gubler reviewed in “Appendix (Local intersections)”. We have

$$\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1} = \sum_{i=1}^{t} a_i \overline{M}|_{Y_i}^2 \cdot \overline{L}_1|_{Y_i} \cdots \overline{L}_{n-2}|_{Y_i} - \int_{X^{\mathrm{an}}} \log \|s\| c_1(\overline{M})^2 c_1(\overline{L}_1) \cdots c_1(\overline{L}_{n-2}).$$

The integral vanishes by Lemma 2.5. In fact, take $\overline{M}'$ to be any vertical metrized line bundle on X and write $\phi = -\log \|1\|_{\overline{M}'}$ as a real-valued function on X^{an}. Then the lemma gives

$$\int_{X^{\mathrm{an}}} \phi \, c_1(\overline{M})^2 c_1(\overline{L}_1) \cdots c_1(\overline{L}_{n-2}) = 0.$$

It implies that the measure $c_1(\overline{M})^2 c_1(\overline{L}_1) \cdots c_1(\overline{L}_{n-2}) = 0$. Hence the integration of $-\log \|s\|$ is also zero. By the proved inequality on every Y_i, we have

$$\overline{M}|_{Y_i}^2 \cdot \overline{L}_1|_{Y_i} \cdots \overline{L}_{n-2}|_{Y_i} = 0.$$

Since one of Y_i is Y, the lemma is proved. □

Now are ready to prove the equality part of Theorem 2.1. Let $\overline{M}$ and $\overline{L}_i$ be as in the theorem. We need to prove that $\overline{M}$ is constant, or equivalently $\log \|1\|_{\overline{M}}$ is constant on X^{an}. By induction on n, Lemma 2.6 immediately gives the following result.

Lemma 2.7 *For any integral closed subvariety Y of codimension one in X, the restriction $\overline{M}|_Y$ is constant on Y.*

Go back to the line bundle $\overline{M}$ on X. Let $|X|_0$ be the set of closed points of X, which is naturally a dense subset of X^{an}. It suffices to prove that $\log \|1\|_{\overline{M}}$ is constant on $|X|_0$. By the lemma, it suffices to prove that any two points x_1, x_2 of $|X|_0$ are connected by integral subvarieties of codimension one in X. By [20, Chapter III, Corollary 7.9], any hyperplane section is connected. It follows that any hyperplane section passing through x_1 and x_2 connects them. This finishes the proof.

3 Arithmetic Hodge index theorem

In this section, we prove a Hodge index theorem for adelic metrized line bundles on projective varieties over number fields, generalizing the following theorem in the context of arithmetic intersection theory of Gillet–Soulé [15].

Theorem 3.1 ([13,21,29]) *Let K be a number field and $\pi : \mathcal{X} \to \mathrm{Spec}\, O_K$ be an arithmetic variety with a regular generic fiber of dimension $n \geq 1$. Let $\overline{\mathcal{L}}$ be an arithmetically ample hermitian $\mathbb{Q}$-line bundle on $\mathcal{X}$. Let $\overline{\mathcal{M}}$ be a hermitian $\mathbb{Q}$-line bundle on $\mathcal{X}$ such that $c_1(\mathcal{M}_K) \cdot c_1(\mathcal{L}_K)^{n-1} = 0$ on the generic fiber $\mathcal{X}_K$. Then*

$$\widehat{c}_1(\overline{\mathcal{M}})^2 \cdot \widehat{c}_1(\overline{\mathcal{L}})^{n-1} \leq 0,$$

and the equality holds if and only if $\overline{\mathcal{M}} = \pi^* \overline{\mathcal{M}}_0$ for some hermitian $\mathbb{Q}$-line bundle $\overline{\mathcal{M}}_0$ on $\mathrm{Spec}\, O_K$.

The theorem was due to Faltings [13] and Hriljac [21] for $n = 1$ and Moriwaki [29] for general n.

3.1 Statements

Let X a projective variety over $\bar{\mathbb{Q}}$. In [40] and as reviewed in "Appendix (Arithmetic intersections)", we have defined a category $\widehat{\mathcal{P}\mathrm{ic}}(X)$ of integrable metrized line bundles as certain limits of hermitian line bundles on models of X over O_K, the ring of integers of number fields K in $\bar{\mathbb{Q}}$, and their intersection pairing. Then $\widehat{\mathrm{Pic}}(X)$ denotes the group of isomorphism classes of objects of $\widehat{\mathcal{P}\mathrm{ic}}(X)$, and $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}} = \widehat{\mathrm{Pic}}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ is the group of integrable adelic $\mathbb{Q}$-line bundles. We also refer to "Appendix (Arithmetic intersections)" for the notions "arithmetically positive" and "$\overline{L}_i$-bounded." The main result of this section is the following Hodge index theorem for such metrized line bundles.

Theorem 3.2 *Let $\pi : X \to \mathrm{Spec}\bar{\mathbb{Q}}$ be a normal and integral projective variety of dimension $n \geq 1$. Let $\overline{M}$ be an integrable adelic $\mathbb{Q}$-line bundle on X, and $\overline{L}_1, \ldots, \overline{L}_{n-1}$ be $n-1$ nef adelic $\mathbb{Q}$-line bundles on X. Assume $M \cdot L_1 \cdots L_{n-1} = 0$ and that each L_i is big. Then*

$$\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1} \leq 0.$$

Moreover, if $\overline{L}_i$ is arithmetically positive and $\overline{M}$ is $\overline{L}_i$-bounded for each i, then the equality holds if and only if

$$\overline{M} \in \pi^*\widehat{\mathrm{Pic}}(\bar{\mathbb{Q}})_{\mathbb{Q}}.$$

Remark 3.3 The assumption "$\overline{M}$ is $\overline{L}_i$-bounded for each i" is necessary for the condition of the equality in Theorem 3.2. For a counter-example, assume $n = 2$ and $\bar{L}$ is induced by an arithmetically ample line bundle $\overline{\mathcal{L}}$ on an integral model $\mathcal{X}$ of X over O_K, the ring of integers in a number field. Let $\alpha : \mathcal{X}' \to \mathcal{X}$ be the blowing-up of a closed point in $\mathcal{X}$, and $\overline{\mathcal{M}}$ the (vertical) line bundle on $\mathcal{X}'$ associated to the exceptional divisor endowed with the trivial hermitian metric $\|1\| = 1$. Then $\mathcal{M}_K \cdot \mathcal{L}_K = 0$ and $\overline{\mathcal{M}}^2 \cdot \alpha^*\overline{\mathcal{L}} = 0$ by the projection formula. But $\overline{\mathcal{M}}$ is not coming from any line bundle on the base $Spec O_K$. In this case, $\overline{\mathcal{M}}$ is not $\overline{\mathcal{L}}$-bounded if we convert the objects to the adelic setting.

Remark 3.4 A similar proof of Theorem 3.2 should give a function field analogue. In fact, let K be an algebraic closure of the function field of a projective and smooth curve B over an algebraically closed field k. Fixing a point $x \in X(K)$ gives a morphism $i : X \to \mathrm{alb}(X)$ to the Albanese variety of X over K. See discussion in Sect. A.4. Let A be a K/k-trace of $\mathrm{alb}(X)$, i.e, A is an abelian variety defined over k such that A_K can be embedded into $\mathrm{alb}(X)$ as a maximal abelian subvariety defined over k. Then we have a surjective homomorphism $\mathrm{alb}(X) \to A \otimes_k K$ which induces a morphism $j : X \to A_K$ over K. Then the theorem should hold over K with the last inclusion $\overline{M} \in \pi^*\widehat{\mathrm{Pic}}(K)_{\mathbb{Q}}$ replaced by

$$\overline{M} \in \pi^*\widehat{\mathrm{Pic}}(K)_{\mathbb{Q}} + j^*\mathrm{Pic}^0(A)_{\mathbb{Q}},$$

where j^* denotes the following natural composition

$$\mathrm{Pic}^0(A)_{\mathbb{Q}} \longrightarrow \mathrm{Pic}(A)_{\mathbb{Q}} \longrightarrow \mathrm{Pic}(A \times_k B)_{\mathbb{Q}} \longrightarrow \widehat{\mathrm{Pic}}(A_K)_{\mathbb{Q}} \longrightarrow \widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}.$$

The second arrow is the pull-back via the projection $A \times_k B \to A$ to the first factor, and the last arrow is the pull-back via $j : X \to A_K$.

Denote by $\mathrm{Pic}^\tau(X)$ the group of isomorphism classes of numerically trivial line bundles on X. Assume that $\overline{M}$ is a *flat* adelic line bundle on X in the sense of "Appendix (Flat metrics)", i.e., M is numerically trivial and the adelic metric is flat at every place. Then by Proposition 5.19, $\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1}$ does not depend on the choice of the flat metric of $\overline{M}$ and the metrics of $\overline{L}_1, \ldots, \overline{L}_{n-1}$. Then it is reasonable to write

$$\langle M, M\rangle_{L_1,\ldots,L_{n-1}} = \overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1}.$$

The definition extends to a quadratic form on $M \in \mathrm{Pic}^\tau(X)_{\mathbb{R}}$ by bilinearity. In this case, we present an $\mathbb{R}$-version of the theorem as follows.

Theorem 3.5 *Let X be a normal projective variety of dimension $n \geq 1$ over $\bar{\mathbb{Q}}$. Let $M \in \mathrm{Pic}^\tau(X)_\mathbb{R}$ and $L_1, \ldots, L_{n-1}$ be $n-1$ nef $\mathbb{Q}$-line bundles on X. Then*

$$\langle M, M\rangle_{L_1,\ldots,L_{n-1}} \leq 0.$$

Moreover, if each L_i is ample, then the equality holds if and only if $M = 0$.

The inequality part of Theorem 3.5 is implied by that of Theorem 3.2. In fact, it suffices to assume $M \in \mathrm{Pic}^\tau(X)_\mathbb{Q}$ for the inequality.

To apply Theorem 3.2, we need each L_i to be big. Thereafter, take an ample line bundle A and a positive rational number ϵ. Define

$$L_{i,\epsilon} = L_i + \epsilon A.$$

Let $\overline{L}_{1,\epsilon}, \ldots, \overline{L}_{n-1,\epsilon}$ be any nef adelic line bundles extending $L_{1,\epsilon}, \ldots, L_{n-1,\epsilon}$. Then we have

$$\langle M, M\rangle_{L_1,\ldots,L_{n-1}} = \lim_{\epsilon\to 0}\langle M, M\rangle_{L_{1,\epsilon},\ldots,L_{n-1,\epsilon}} = \lim_{\epsilon\to 0} \overline{M}^2 \cdot \overline{L}_{1,\epsilon} \cdots \overline{L}_{n-1,\epsilon} \leq 0.$$

3.2 Curves

In this section, we prove Theorems 3.2 and 3.5 when $\dim X = 1$. Then $\deg M = 0$ and the line bundles $\overline{L}_i$ do not show up.

We first consider Theorem 3.2. Then M has a flat metric structure $\overline{M}_0$ as defined in "Appendix (Flat metrics)". By Proposition 5.17(2)(3), $\overline{M}_0$ is actually induced by a hermitian line bundle $\overline{\mathcal{M}}$ on any regular integral model $\mathcal{X}/O_K$ of X such that $c_1(\overline{\mathcal{M}}) = 0$ on $X(\mathbb{C})$ and that $\overline{\mathcal{M}}$ is perpendicular to $\widehat{\mathrm{Pic}}(\mathcal{X})_{\mathrm{vert}}$. Define a vertical adelic $\mathbb{Q}$-line bundle $\overline{N}$ by

$$\overline{M} = \overline{M}_0 + \overline{N}.$$

Note $\overline{M}_0 \cdot \overline{N} = 0$ by the flatness of $\overline{M}_0$. It follows that

$$\overline{M}^2 = \overline{M}_0^2 + \overline{N}^2 = \overline{M}_0^2 + \sum_v \overline{N}_v^2,$$

where $\overline{N}_v$ be the restriction of $\overline{N}$ to $X \otimes_K K_v$ for each place v of K. Now Theorems 3.2 and 3.5 follow from the index theorem of Faltings and Hriljac for $\overline{M}_0$ and the local index theorem in Theorem 2.1 for $\overline{N}_v$ as follows:

(1) By the result Faltings and Hriljac, $\overline{M}_0^2 = \overline{\mathcal{M}}^2 \leq 0$ where the equality is attained if and only if $\overline{M}_0 \in \pi^*\widehat{\mathrm{Pic}}(K)_\mathbb{Q}$.
(2) By Theorem 2.1, $\overline{N}^2 \leq 0$ and the equality is attained if and only if $\overline{N}_v$ is constant for every v.

Now we consider Theorem 3.5. Then $M \in \mathrm{Pic}^0(X)_{\mathbb{R}}$. It still has a flat metric structure $\overline{M}_0$, as an $\mathbb{R}$-linear combinations of flat adelic line bundles. By linearity, the result of Faltings and Hriljac still gives

$$\overline{M}_0^2 = -2\, h_{\mathrm{NT}}(M).$$

Here the Neron–Tate height function $h_{\mathrm{NT}} : \mathrm{Pic}^0(X)_{\mathbb{R}} \to \mathbb{R}$ is extended from $\mathrm{Pic}^0(X)$ by bilinearity. By [34, §3.8], h_{NT} is positive definite on $\mathrm{Pic}^0(X)_{\mathbb{R}}$. This proves the theorem for $n = 1$.

3.3 Inequality

We already known that the inequality part of Theorem 3.5 is implied by that of Theorem 3.2. Now we prove the inequality of Theorem 3.2 by induction on $n = \dim X$. We have already treated the case $n = 1$, so we assume $n \geq 2$ in the following discussion. We make the following two further assumptions:

(1) X is smooth by the resolution of singularities of Hironaka and the projection formula in [40];
(2) each $\overline{L}_i$ is arithmetically positive in the sense of Definition 5.3.

For the second assumption, fix an arithmetically positive adelic line bundle $\overline{A}$ on X. Take a small rational number $\epsilon > 0$. Set $\overline{L}_i' = \overline{L}_i + \epsilon \overline{A}$ and $\overline{M}' = \overline{M} + \delta \overline{A}$. Here δ is a number such that

$$M' \cdot L_1' \cdots L_{n-1}' = (M + \delta A) \cdot L_1' \cdots L_{n-1}' = 0.$$

It determines

$$\delta = -\frac{M \cdot L_1' \cdots L_{n-1}'}{A \cdot L_1' \cdots L_{n-1}'}.$$

As $\epsilon \to 0$, we have $\delta \to 0$ since

$$M \cdot L_1' \cdots L_{n-1}' \to M \cdot L_1 \cdots L_{n-1} = 0,$$
$$A \cdot L_1' \cdots L_{n-1}' \to A \cdot L_1 \cdots L_{n-1} > 0.$$

Here we explain the last inequality, which uses the assumption that L_i is big and nef for each i. We refer to [25] for bigness and nefness in the geometric setting. In particular, the bigness of L_i implies that there are ample $\mathbb{Q}$-line bundles A_i such that $L_i - A_i$ is effective. It follows that

$$A \cdot L_1 \cdot L_2 \cdots L_{n-1} \geq A \cdot A_1 \cdot L_2 \cdots L_{n-1} \geq \cdots \geq A \cdot A_1 \cdots A_{n-1} > 0.$$

Therefore, the inequality $\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1} \leq 0$ is the limit of the inequality $\overline{M}'^2 \cdot \overline{L}_1' \cdots \overline{L}_{n-1}' \leq 0$. Here every $\overline{L}_i'$ is arithmetically positive. This shows that we may assume (2).

By approximation, it suffices to prove

$$\overline{\mathcal{M}}^2 \cdot \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{n-1} \le 0$$

for a model $(\mathcal{X}, \overline{\mathcal{M}}, \overline{\mathcal{L}}_1, \dots, \overline{\mathcal{L}}_{n-1})$ of $(X, \overline{M}, \overline{L}_1, \dots, \overline{L}_{n-1})$ under the following assumptions:

(3) X is smooth and $\mathcal{X}$ is normal;
(4) $\overline{\mathcal{L}}_i$ arithmetically ample on $\mathcal{X}$ in the sense of Lemma 5.4.

Then the inequality was proved by Moriwaki [29] in the case that all $\overline{\mathcal{L}}_i$ are equal. The current case is similar, but we still sketch a proof as follows.

Lemma 3.6 *There is a smooth metric $\|\cdot\|_0$ on $\mathcal{M}$ at archimedean places, unique up to scalars, such that the curvature form of $\overline{\mathcal{M}}' = (\mathcal{M}, \|\cdot\|_0)$ on $X(\mathbb{C})$ pointwise satisfies*

$$c_1(\overline{\mathcal{M}}')c_1(\overline{\mathcal{L}}_1)\cdots c_1(\overline{\mathcal{L}}_{n-1}) = 0,$$

where we omit the symbol for the wedge product of forms of degree 2. *Moreover, with this metric, one has pointwise*

$$c_1(\overline{\mathcal{M}}')^2 c_1(\overline{\mathcal{L}}_1)\cdots c_1(\overline{\mathcal{L}}_{n-2}) \le 0.$$

Proof For the first equality, write $\overline{\mathcal{M}} = (\mathcal{M}, \|\cdot\|)$ and $\|\cdot\|_0 = e^{\phi}\|\cdot\|$ for a real-valued C^∞-function ϕ on $X(\mathbb{C})$. Then the above equation becomes

$$\frac{1}{\pi i}\partial\bar{\partial}\phi \cdot c_1(\overline{\mathcal{L}}_1)\cdots c_1(\overline{\mathcal{L}}_{n-1}) = -c_1(\overline{\mathcal{M}})c_1(\overline{\mathcal{L}}_1)\cdots c_1(\overline{\mathcal{L}}_{n-1}).$$

Set $\Omega = c_1(\overline{\mathcal{L}}_1)\cdots c_1(\overline{\mathcal{L}}_{n-1})$. Then the left-hand side is a scalar multiple of the Laplacian $\Delta\phi$ with respect to Ω. The right-hand side is an exact form on every connected component $X_\sigma(\mathbb{C})$ of $X(\mathbb{C})$. In fact, its cohomology class is represented by $-M \cdot L_1 \cdots L_{n-1}$, which is zero in $H^{2n}(X_\sigma(\mathbb{C}), \mathbb{C}) = \mathbb{C}$. The solution ϕ exists by Gromov [14, Corollary 2.2 A2].

The inequality of the lemma is implied by the equality by Aleksandrov's lemma. See Gromov [14, Lemma 2.1 A] by setting $\Omega := c_1(\overline{\mathcal{L}}_1)\cdots c_1(\overline{\mathcal{L}}_{n-2})$ and $\omega_0 := c_1(\overline{\mathcal{L}}_{n-1})$. □

Write the original metric $\|\cdot\|$ of $\overline{\mathcal{M}}$ as $e^{-\phi}\|\cdot\|_0$. Then by the above lemma, we have

$$\overline{\mathcal{M}}^2 \cdot \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{n-1} = \overline{\mathcal{M}}'^2 \cdot \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{n-1} - \int_{X(\mathbb{C})} \phi \frac{1}{\pi i}\partial\bar{\partial}\phi \wedge c_1(\overline{\mathcal{L}}_1)\cdots c_1(\overline{\mathcal{L}}_{n-1}).$$

By integration by parts, the second term on the right

$$-\int_{X(\mathbb{C})} \phi \frac{1}{\pi i}\partial\bar{\partial}\phi \wedge c_1(\overline{\mathcal{L}}_1)\cdots c_1(\overline{\mathcal{L}}_{n-1}) = \int_{X(\mathbb{C})} \frac{1}{\pi i}\partial\phi \wedge \bar{\partial}\phi \wedge c_1(\overline{\mathcal{L}}_1)\cdots c_1(\overline{\mathcal{L}}_{n-1}).$$

It is non-positive since the volume form

$$\frac{1}{\pi i}\partial\phi \wedge \overline{\partial}\phi \wedge c_1(\overline{\mathcal{L}}_1)\cdots c_1(\overline{\mathcal{L}}_{n-1}) \leq 0.$$

Hence, it suffices to prove

$$\overline{\mathcal{M}}'^2 \cdot \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{n-1} \leq 0.$$

By Moriwaki's arithmetic Bertini theorem (cf. [28], Theorem 4.2 and Theorem 5.3), which is also stated at the beginning to [29, §1], replacing $\overline{\mathcal{L}}_{n-1}$ by a multiple if necessary, there is a nonzero section $s \in H^0(\mathcal{X}, \mathcal{L}_{n-1})$ satisfying the following conditions:

- The supremum norm $\|s\|_{\sup} = \sup_{x\in X(\mathbb{C})} \|s(x)\| < 1$;
- The horizontal part of $\mathrm{div}(s)$ on $\mathcal{X}$ is a generically smooth arithmetic variety $\mathcal{Y}$;
- The vertical part of $\mathrm{div}(s)$ on $\mathcal{X}$ is a positive linear combination $\sum_{\wp} a_{\wp}\mathcal{X}_{\wp}$ of smooth fibers $\mathcal{X}_{\wp}$ of $\mathcal{X}$ above (good) prime ideals $\wp$ of O_K.

Then

$$\overline{\mathcal{M}}'^2 \cdot \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{n-1} = \overline{\mathcal{M}}'|_{\mathcal{Y}}^2 \cdot \overline{\mathcal{L}}_1|_{\mathcal{Y}} \cdots \overline{\mathcal{L}}_{n-2}|_{\mathcal{Y}} + \sum_{\wp} a_{\wp}\, \overline{\mathcal{M}}'|_{\mathcal{X}_{\wp}}^2 \cdot \overline{\mathcal{L}}_1|_{\mathcal{X}_{\wp}} \cdots \overline{\mathcal{L}}_{n-2}|_{\mathcal{X}_{\wp}}$$
$$- \int_{X(\mathbb{C})} \log\|s\| c_1(\overline{\mathcal{M}}')^2 c_1(\overline{\mathcal{L}}_1)\cdots c_1(\overline{\mathcal{L}}_{n-2}).$$

It suffices to prove each term on the right-hand side is non-positive. The non positivity of the "main term"

$$\overline{\mathcal{M}}'|_{\mathcal{Y}}^2 \cdot \overline{\mathcal{L}}_1|_{\mathcal{Y}} \cdots \overline{\mathcal{L}}_{n-2}|_{\mathcal{Y}} \leq 0$$

follows from the induction hypothesis. By flatness of $\mathcal{X}/O_K$,

$$\overline{\mathcal{M}}'|_{\mathcal{X}_{\wp}}^2 \cdot \overline{\mathcal{L}}_1|_{\mathcal{X}_{\wp}} \cdots \overline{\mathcal{L}}_{n-2}|_{\mathcal{X}_{\wp}} = (M^2 \cdot L_1 \cdots L_{n-2}) \log(\# O_K/\wp) \leq 0.$$

Here we have used the Hodge index theorem in the geometric case as stated in Theorem 5.20. By the above lemma,

$$- \int_{X(\mathbb{C})} \log\|s\| c_1(\overline{\mathcal{M}}')^2 c_1(\overline{\mathcal{L}}_1)\cdots c_1(\overline{\mathcal{L}}_{n-2}) \leq 0.$$

3.4 Equality

In this section, we prove the equality part of Theorem 3.2 and Theorem 3.5. We have already treated the case $n = 1$, so we assume $n \geq 2$ in the following discussion.

3.4.1 Reduce to flat metrics

We first treat Theorem 3.2.

Lemma 3.7 *Under the conditions in the equality part of Theorem 3.2, M is numerically trivial on X.*

Proof Since $\overline{L}_{n-1}$ is arithmetically positive, $\overline{L}'_{n-1} := \overline{L}_{n-1} - \pi^*\overline{N}$ is nef for some $\overline{N} \in \widehat{\mathrm{Pic}}(K)$ with $c = \widehat{\deg}(\overline{N}) > 0$. Then

$$0 = \overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-2} \cdot \overline{L}_{n-1} = \overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-2} \cdot \overline{L}'_{n-1} + c\, M^2 \cdot L_1 \cdots L_{n-2}.$$

The two terms in the right hand are both non-positive:

(1) Apply the inequality of the theorem to $(\overline{M}, \overline{L}_1, \ldots, \overline{L}_{n-2}, \overline{L}'_{n-1})$ to obtain

$$\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-2} \cdot \overline{L}'_{n-1} \leq 0.$$

(2) Apply the geometric Hodge index theorem in Theorem 5.20 on X to obtain

$$M^2 \cdot L_1 \cdots L_{n-2} \leq 0.$$

It follows that

$$\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-2} \cdot \overline{L}'_{n-1} = M^2 \cdot L_1 \cdots L_{n-2} = 0.$$

By the geometric Hodge index theorem again, M is numerically trivial. □

In the following discussion, we assume that M is numerically trivial. Then we can extend M to a flat adelic $\mathbb{Q}$-line bundle $\overline{M}_0$. See "Appendix (Flat metrics)" for the definition of flat metrics again. The difference $\overline{N} := \overline{M} - \overline{M}_0$ is vertical. Then we have a decomposition

$$0 = \overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1} = \overline{M}_0^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1} + \overline{N}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1}.$$

The inequality part of Theorem 3.2 implies

$$\overline{M}_0^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1} = \overline{N}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1} = 0.$$

By the local Hodge index theorem in Theorem 2.1, $\overline{N} \in \widehat{\mathrm{Pic}}(K)_{\mathbb{Q}}$. Thus we only need to treat the case $\overline{M} = \overline{M}_0$.

3.4.2 Flat metric case

Here we prove the equality part of Theorem 3.2 assuming that $\overline{M}$ is flat. Let $\overline{M}, \overline{L}_1, \ldots, \overline{L}_{n-1}$ be as in the equality part of the corollary. The key is the following induction result.

Lemma 3.8 *For any closed integral subvariety Y of codimension one in X, we have*

$$\overline{M}|_Y^2 \cdot \overline{L}_1|_Y \cdots \overline{L}_{n-2}|_Y = 0.$$

Proof We prove this lemma by the same idea as in the proof of Lemma 2.6. Replacing $\overline{L}_{n-1}$ by a multiple if necessary, we can assume that there is a nonzero global section s of L_{n-1} vanishing on Y. Write

$$\operatorname{div}(s) = \sum_{i=1}^{t} a_i Y_i,$$

with $a_i > 0$, Y_i distinct, and $Y_1 = Y$. Use the section s to compute the intersection number $\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1}$. Then

$$\begin{aligned}\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1} = &\sum_{i=1}^{t} a_i \, \overline{M}|_{Y_i}^2 \cdot \overline{L}_1|_{Y_i} \cdots \overline{L}_{n-2}|_{Y_i} \\ &- \sum_v \int_{X_v^{\mathrm{an}}} \log \|s\|_v \, c_1(\overline{M})^2 c_1(\overline{L}_1) \cdots c_1(\overline{L}_{n-2}).\end{aligned}$$

By the flatness of $\overline{M}$, all the integrals above vanish. See Proposition 5.17 for more details on flat metrics. Hence,

$$\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1} = \sum_{i=1}^{t} a_i \, \overline{M}|_{Y_i}^2 \cdot \overline{L}_1|_{Y_i} \cdots \overline{L}_{n-2}|_{Y_i}.$$

Since every term on the right-hand side is non-positive, we obtain

$$\overline{M}|_Y^2 \cdot \overline{L}_1|_Y \cdots \overline{L}_{n-2}|_Y = 0.$$

This proves the lemma. □

Go back to the proof of Theorem 3.2. Still use induction on $n = \dim X$. We apply the lemma to a general hyperplane section Y of some very ample line bundle on X. Note that X is normal. By a Bertini-type result of Seidenberg [32], Y is normal (and projective). By Lemma 3.8 and the induction hypothesis, $\overline{M}|_Y$ is constant in the sense that it is a pull-back from a number field. In particular, $M = 0$ in $\mathrm{Pic}(Y)_{\mathbb{Q}}$. By Theorem 5.21. the natural map $\mathrm{Pic}^0(X) \to \mathrm{Pic}^0(Y)$ has a finite kernel. Hence, $M = 0$ in $\mathrm{Pic}(X)_{\mathbb{Q}}$. Replacing $\overline{M}$ by positive multiple, we may assume that $M = \mathcal{O}_X$. Now the constancy of $\overline{M}$ on X follows from the local Hodge index theorem in Theorem 2.1.

3.4.3 Proof of Theorem 3.5

The above proof also applies to the equality part of Theorem 3.5. Let $\overline{L}_1, \ldots, \overline{L}_{n-1}$ be any nef adelic line bundles extending $L_1, \ldots, L_{n-1}$. By Proposition 5.18, $M \in \mathrm{Pic}^\tau(X)_\mathbb{R}$ extends to an $\mathbb{R}$-linear combination $\overline{M}$ of flat adelic line bundles on X. Lemma 3.8 still works in the current setting. By induction, we can assume $M|_Y = 0$. Then the injectivity of $\mathrm{Pic}^0(X)_\mathbb{R} \to \mathrm{Pic}^0(Y)_\mathbb{R}$ implies $M = 0$.

Remark 3.9 As a consequence of the proof of Lemma 3.8, we have the interpretation

$$\langle M, M\rangle_{L_1,\ldots,L_{n-1}} = \langle M|_C, M|_C\rangle = -2\, h_{\mathrm{NT}}(M|_C).$$

Here C is any smooth projective curve in X representing the intersection $L_1 \cdot L_2 \cdots L_{n-1}$ in that $C = \mathrm{div}(s_1) \cdot \mathrm{div}(s_2) \cdots \mathrm{div}(s_{n-1})$ for a collection of sections $s_i \in H^0(X, L_i)$.

4 Algebraic dynamics

In this section, we present and prove the rigidity theorem (Theorem 4.1) about the preperiodic points of algebraic dynamical systems. After the statements, we first introduce a theory of admissible adelic line bundles and then prove the theorem using our arithmetic Hodge index theorem.

4.1 Statement of the main theorem

Let X be a projective variety over $\bar{\mathbb{Q}}$. A *polarizable algebraic dynamical system on* X is a morphism $f : X \to X$ such that there is an ample $\mathbb{Q}$-line bundle satisfying $f^*L \simeq qL$ for some rational number $q > 1$. We call L a *polarization of* f, and call the triple (X, f, L) with a fixed isomorphism $f^*L \simeq qL$ a *polarized algebraic dynamical system*.

Note that f is necessary finite, since the pull-back of the ample $\mathbb{Q}$-line bundle L by f is still ample. The projection formula for the top intersection $L^{\dim X}$ further gives that $\deg(f) = q^{\dim X}$.

For such an f, let $\mathrm{Prep}(f)$ denote the set of *preperiodic points*, i.e.,

$$\mathrm{Prep}(f) := \{x \in X(\overline{\mathbb{Q}}) \mid f^m(x) = f^n(x) \text{ for some } m, n \in \mathbb{N},\ m \neq n\}.$$

A well-known result of Fakhruddin [12, Theorem 5.1] asserts that $\mathrm{Prep}(f)$ is Zariski dense in X.

Denote by $\mathcal{DS}(X)$ the set of all polarizable algebraic dynamical systems f on X. Note that we do *not* require elements of $\mathcal{DS}(X)$ to be polarizable by the same ample line bundle or have the same dynamical degree q. The main theorem of this section is the following rigidity result about preperiodic points.

Theorem 4.1 *Let X be a projective variety over $\bar{\mathbb{Q}}$. Let $f, g \in \mathcal{DS}(X)$, and Z be the Zariski closure of* $\mathrm{Prep}(f) \cap \mathrm{Prep}(g)$ *in X. Then*

$$\mathrm{Prep}(f) \cap Z(\overline{\mathbb{Q}}) = \mathrm{Prep}(g) \cap Z(\overline{\mathbb{Q}}).$$

Remark 4.2 A similar proof as in the present paper should give an analogue of the theorem over $\overline{\mathbb{F}_p(t)}$.

Remark 4.3 When $X = \mathbb{P}^1$, the theorem was previously proved by Mimar [27] following the method of [38] for the power map on $\mathbb{P}^1$.

Remark 4.4 Let us reduce the theorem to the case that X is normal. Let $\psi : X' \to X$ be the normalization of X. For any finite morphism $f : X \to X$, the normalization of $f \circ \psi : X' \to X$ is a morphism $f' : X' \to X'$, which gives a (unique) lifting of $f : X \to X$. Moreover, if f is polarized by an ample $\mathbb{Q}$-line bundle L on X, then f' is polarized by the ample $\mathbb{Q}$-line bundle $L' = \psi^* L$ on X'. Since ψ is finite, we also have $\mathrm{Prep}(f') = \psi^{-1}\mathrm{Prep}(f)$. Hence, we can assume that X is normal by replacing (X, f, L) by (X', f', L') in the theorem.

Remark 4.5 In this paper, we will often use models of a dynamical system over a number field. More precisely, any polarized dynamical system (X, f, L) is the base change of a dynamical system (X_K, f_K, L_K) over some number field K.

4.2 Semisimplicity

Let (X, f, L) be a *polarized dynamical system* over $\overline{\mathbb{Q}}$. Assume that X is normal. We want to study the semisimplicity of f^* on $\mathrm{Pic}(X)_{\mathbb{Q}}$.

By definition, f^* preserves the exact sequence

$$0 \longrightarrow \mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{NS}(X) \longrightarrow 0.$$

See "Appendix (Lefschetz theorems)" for definitions of the terms in the exact sequence. It is clear that this exact sequence is the direct limit of the corresponding exact sequence

$$0 \longrightarrow \mathrm{Pic}^0(X_K) \longrightarrow \mathrm{Pic}(X_K) \longrightarrow \mathrm{NS}(X_K) \longrightarrow 0$$

for models (X_K, f_K, L_K) over a number field K. We further assume that $X(K)$ is non-empty in the following.

Lemma 4.6 *The groups* $\mathrm{NS}(X_K)$ *and* $\mathrm{Pic}^0(X_K)$ *are both finitely generated $\mathbb{Z}$-modules.*

Proof This is true for smooth curves, and for general normal varieties by induction using the hard Lefschetz theorem in Theorem 5.21. In fact, the Picard functor $\underline{\mathrm{Pic}}^0_{X/K}$ is represented by an abelian variety A over K. See [22, Theorem 9.5.4, Corollary 9.5.14]. Then $\mathrm{Pic}^0(X) = A(K)$ is just the Mordell–Weil group. □

Theorem 4.7 *Let (X, f, L) be a polarized algebraic dynamical system over $\overline{\mathbb{Q}}$. Assume that X is normal.*

(1) *The operator f^* is semisimple on* $\mathrm{Pic}^0(X)_{\mathbb{C}}$ *(resp.* $\mathrm{NS}(X)_{\mathbb{C}}$*) with eigenvalues of absolute values $q^{1/2}$ (resp. q).*
(2) *The operator f^* is semisimple on* $\mathrm{Pic}(X)_{\mathbb{C}}$ *with eigenvalues of absolute values $q^{1/2}$ or q.*

Proof When X is smooth, the result was proved by Serre [33] using the Hodge–Riemann bilinear relation. Here we give a variation of his proof for general X using both geometric and arithmetic Hodge index theorem. It suffices to prove (1), since (2) is a consequence of (1). It suffices to prove the corresponding assertion for models (X_K, f_K, L_K) over number fields where all the group are finite dimensional by the above lemma. By abuse of notation, we often neglect the subscipt K in f_K and L_K.

We first consider $\mathrm{NS}(X_K)_{\mathbb{C}}$. Write $n = \dim X_K$ as usual. Make the decomposition

$$\mathrm{NS}(X_K)_{\mathbb{R}} := \mathbb{R}L \oplus P(X_K), \quad P(X_K) = \{\xi \in \mathrm{NS}(X_K)_{\mathbb{R}} : \xi \cdot L^{n-1} = 0\}$$

and define a pairing

$$(\xi_1, \xi_2) := \xi_1 \cdot \xi_2 \cdot L^{n-2}$$

on $P(X_K)$. By Theorem 5.20, this pairing is negative definite. The projection formula gives

$$\langle f^*\xi_1, f^*\xi_2\rangle = q^2\langle \xi_1, \xi_2\rangle.$$

It follows that $q^{-1}f^*$ is an orthogonal transformation (with respect to the quadratic form). Then $q^{-1}f^*$ is diagonalizable on $\mathrm{NS}(X_K)_{\mathbb{C}}$ with eigenvalues of absolute values 1.

Next we consider $\mathrm{Pic}^0(X_K)_{\mathbb{C}}$. We will use an arithmetic intersection paring on $\mathrm{Pic}^0(X_K)$. By the notation introduced right before Theorem 3.5, we have a pairing on $\mathrm{Pic}^0(X_K)_{\mathbb{R}}$ defined by

$$\langle \xi_1, \xi_2\rangle = \overline{\xi}_1 \cdot \overline{\xi}_2 \cdot \overline{L}^{n-1}.$$

Here $\overline{\xi}_1$ and $\overline{\xi}_2$ are flat adelic line bundles extending ξ_1 and ξ_2, and $\overline{L}$ can be any integrable adelic line bundle extending L. Theorem 3.5 asserts that the pairing is negative definite on $\mathrm{Pic}^0(X_K)_{\mathbb{R}}$.

On the other hand, we claim

$$\langle f^*\xi_1, f^*\xi_2\rangle = q\,\langle \xi_1, \xi_2\rangle.$$

In fact, by the projection formula,

$$(f^*\overline{\xi}_1) \cdot (f^*\overline{\xi}_2) \cdot (f^*\overline{L})^{n-1} = (\deg f) \cdot (\overline{\xi}_1 \cdot \overline{\xi}_2 \cdot \overline{L}^{n-1}).$$

Here $f^*\overline{\xi}_1$ and $f^*\overline{\xi}_2$ are still flat. By Proposition 5.19, we can replace $f^*\overline{L}$ by $q\overline{L}_K$ on the left-hand side. Thus the formula gives the claim by $\deg f = q^n$.

Hence, $q^{-1/2}f^*$ is an orthogonal transformation on $\mathrm{Pic}^0(X_K)_{\mathbb{R}}$ (with respect to the negative of the pairing). Then $q^{-1/2}f^*$ is diagonalizable on $\mathrm{Pic}^0(X_K)_{\mathbb{C}}$ with eigenvalues of absolute values 1. The theorem is proved. □

By Theorem 4.7, the exact sequence

$$0 \longrightarrow \mathrm{Pic}^0(X)_{\mathbb{C}} \longrightarrow \mathrm{Pic}(X)_{\mathbb{C}} \longrightarrow \mathrm{NS}(X)_{\mathbb{C}} \longrightarrow 0.$$

has a unique splitting as f^*-modules by a section

$$\ell_f : \mathrm{NS}(X)_{\mathbb{C}} \longrightarrow \mathrm{Pic}(X)_{\mathbb{C}}.$$

We can make this splitting over $\mathbb{Q}$, i.e., it is the base change of a canonical linear map

$$\ell_f : \mathrm{NS}(X)_{\mathbb{Q}} \longrightarrow \mathrm{Pic}(X)_{\mathbb{Q}}.$$

Indeed, over a model (X_K, f_K, L_K), let $P \in \mathbb{Q}[T]$ and $Q \in \mathbb{Q}[T]$ be the minimal polynomials of f^* on $\mathrm{Pic}^0(X_K)_{\mathbb{Q}}$ and $\mathrm{NS}(X_K)_{\mathbb{Q}}$ respectively. Note that P and Q are coprime, since their roots have different absolute values. Then $R = PQ$ is the minimal polynomial for f^* on $\mathrm{Pic}(X_K)_{\mathbb{Q}}$. Moreover, $\mathrm{Pic}^0(X_K)_{\mathbb{Q}}$ has a complement in $\mathrm{Pic}(X_K)_{\mathbb{Q}}$ defined by

$$\mathrm{Pic}_f(X_K)_{\mathbb{Q}} := \ker Q(f^*)|_{\mathrm{Pic}(X_K)_{\mathbb{Q}}}.$$

Then the projection $\mathrm{Pic}_f(X_K)_{\mathbb{Q}} \to \mathrm{NS}(X_K)_{\mathbb{Q}}$ is an isomorphism of f^*-modules. If K is replaced by a finite extension K' in the construction, then $Q(f^*)$ is replaced by a multiple, and thus $\mathrm{Pic}_f(X_K)_{\mathbb{Q}}$ is naturally a subset of $\mathrm{Pic}_f(X_{K'})_{\mathbb{Q}}$.

Denote by $\mathrm{Pic}_f(X)_{\mathbb{Q}}$ the image of $\ell_f : \mathrm{NS}(X)_{\mathbb{Q}} \to \mathrm{Pic}(X)_{\mathbb{Q}}$, which is also the direct limit of $\mathrm{Pic}_f(X_K)_{\mathbb{Q}}$ as K varies.

Definition 4.8 We say an element of $\mathrm{Pic}(X)_{\mathbb{Q}}$ is f-pure of weight 1 (resp. 2) if it lies in $\mathrm{Pic}^0(X)_{\mathbb{Q}}$ (resp. $\mathrm{Pic}_f(X)_{\mathbb{Q}}$).

4.3 Admissible metrics

By [40], Tate's limiting argument gives a nef adelic $\mathbb{Q}$-line bundle $\overline{L}_f \in \widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$ extending L and satisfying $f^*\overline{L}_f = q\overline{L}_f$. We want to generalize the definition to any line bundle $M \in \mathrm{Pic}(X)_{\mathbb{Q}}$. The main theorem of this subsection is the following result.

Theorem 4.9 *Assume that X is normal. The projection*

$$\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}} \longrightarrow \mathrm{Pic}(X)_{\mathbb{Q}}$$

has a unique section

$$M \longmapsto \overline{M}_f$$

as f^*-modules. Moreover, the section satisfies the following properties:

(1) *If $M \in \mathrm{Pic}^0(X)_{\mathbb{Q}}$, then $\overline{M}_f$ is flat.*
(2) *If $M \in \mathrm{Pic}_f(X)_{\mathbb{Q}}$ is ample, then $\overline{M}_f$ is nef.*

Note that this theorem furnishes the construction for the canonical height functions for every line bundle $M \in \mathrm{Pic}(X)_{\mathbb{Q}}$, not only for the polarization L.

Definition 4.10 An element $\overline{M}$ of $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$ is called *f-admissible* if it is of the form $\overline{M}_f$. In this case, the adelic metric of $\overline{M}$ is also called *f-admissible*.

For any integrable adelic $\mathbb{Q}$-line bundle $\overline{N}$ on X with a model $(\overline{N}_K, X_K)$ over a number field K, recall that the height function $h_{\overline{N}} : X(\overline{\mathbb{Q}}) \to \mathbb{R}$ is defined by

$$h_{\overline{N}}(x) = \frac{1}{[K(\widetilde{x}) : \mathbb{Q}]}\overline{N} \cdot \tilde{x}, \quad x \in X(\overline{\mathbb{Q}}).$$

Here $\tilde{x}$ denotes the image of x in X_K, which is a closed point of X_K. The following simple consequences assert that the definition extends that of [40], and that preperiodic points have height zero under f-admissible adelic line bundles.

Corollary 4.11 *Let $M \in \mathrm{Pic}(X)_{\mathbb{Q}}$. The following are true:*

(1) *If $f^*M = \lambda M$ for some $\lambda \in \mathbb{Q}$, then $f^*\overline{M}_f = \lambda\overline{M}_f$ in $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$.*
(2) *For any $x \in \mathrm{Prep}(f)$, one has $\overline{M}_f|_{\tilde{x}} = 0$ in $\widehat{\mathrm{Pic}}(\tilde{x})_{\mathbb{Q}}$. Hence, the height function $h_{\overline{M}_f}$ is zero on $\mathrm{Prep}(f)$.*

Proof Part (1) follows from the first statement of the theorem. For (2), assume that $f^m(x) = f^n(x)$ with $m > n \geq 0$. Choose a number field K such that (X, f, L) has a model (X_K, f_K, L_K) over K as before, and such that x and M are defined over K. Start with the identity

$$\overline{M}_f|_{f^m(x)} = \overline{M}_f|_{f^n(x)}.$$

We have

$$(f^*)^m\overline{M}_f|_x = (f^*)^n\overline{M}_f|_x.$$

It follows that $\overline{N}_f|_x = 0$ for any N in the image of the linear map

$$(f_K^*)^m - (f_K^*)^n : \mathrm{Pic}(X_K)_{\mathbb{Q}} \longrightarrow \mathrm{Pic}(X_K)_{\mathbb{Q}}.$$

It suffices to prove that the map is surjective. By Lemma 4.6, $\mathrm{Pic}(X_K)_{\mathbb{Q}}$ is finite-dimensional. Thus we need only prove that the map $(f_K^*)^m - (f_K^*)^n$ is injective. This is true because the eigenvalues of f_K^* have absolute values q or $q^{\frac{1}{2}}$. $\square$

Now we prove Theorem 4.9 by the following steps, which takes up the rest of this subsection. It suffices to prove the corresponding statement over a model (X_K, f_K, L_K) over a sufficiently large number field K, with some extra care about the compatibility between different K. In the following discussion, we assume that f has a fixed point $x_0 \in X(K)$, and omit the subscript K in the following discussion by abuse of notation.

Step 1

Denote by $\widehat{\mathrm{Pic}}(X)'$ the group of adelic line bundles on X (with continuous metrics), and denote $\widehat{\mathrm{Pic}}(X)'_{\mathbb{Q}} = \widehat{\mathrm{Pic}}(X)' \otimes_{\mathbb{Z}} \mathbb{Q}$. In this step, we prove that the canonical projection

$$\widehat{\mathrm{Pic}}(X)'_{\mathbb{Q}} \longrightarrow \mathrm{Pic}(X)_{\mathbb{Q}}$$

has a unique section

$$M \longmapsto \overline{M}_f$$

as f^*-modules. Note that $\widehat{\mathrm{Pic}}(X)'$ contains the group $\widehat{\mathrm{Pic}}(X)$ of integrable adelic line bundles.

Define $D(X)$ by the exact sequence

$$0 \longrightarrow D(X) \longrightarrow \widehat{\mathrm{Pic}}(X)'_{\mathbb{Q}} \longrightarrow \mathrm{Pic}(X)_{\mathbb{Q}} \longrightarrow 0.$$

It is an exact sequence of f^*-modules. Recall that f^* has the minimal polynomial $R = PQ$ on $\mathrm{Pic}(X)_{\mathbb{Q}}$.

Lemma 4.12 *The operator $R(f^*)$ is invertible on $D(X)$.*

Assuming the lemma, then we have a unique f^*-equivariant splitting of the exact sequence. In fact, consider the composition

$$\widehat{\mathrm{Pic}}(X)'_{\mathbb{Q}} \stackrel{R(f^*)}{\longrightarrow} D(X) \stackrel{R(f^*)|_{D(X)}^{-1}}{\longrightarrow} D(X).$$

It becomes the identity map when restricted to $D(X)$, and thus gives a splitting of the exact sequence. More precisely, if we denote

$$E(X) = \ker(R(f^*) : \widehat{\mathrm{Pic}}(X)'_{\mathbb{Q}} \longrightarrow \widehat{\mathrm{Pic}}(X)'_{\mathbb{Q}}),$$

which is also the kernel of the above composition, then we have an f^*-equivariant decomposition

$$\widehat{\mathrm{Pic}}(X)'_{\mathbb{Q}} = D(X) \oplus E(X).$$

The projection $\widehat{\mathrm{Pic}}(X)'_{\mathbb{Q}} \to \mathrm{Pic}(X)_{\mathbb{Q}}$ induces an isomorphism $E(X) \to \mathrm{Pic}(X)_{\mathbb{Q}}$. The inverse of this isomorphism gives our desired f^*-equivariant splitting $\mathrm{Pic}(X)_{\mathbb{Q}} \to \widehat{\mathrm{Pic}}(X)'_{\mathbb{Q}}$. This is the existence of the splitting.

For the uniqueness, note that $\mathrm{Pic}(X)_{\mathbb{Q}}$ is killed by $R(f^*)$, so any f^*-equivariant splitting $j : \mathrm{Pic}(X)_{\mathbb{Q}} \hookrightarrow \widehat{\mathrm{Pic}}(X)'_{\mathbb{Q}}$ has an image killed by $R(f^*)$. In other words, the image of j should lie in $E(X)$. But $E(X) \to \mathrm{Pic}(X)_{\mathbb{Q}}$ is already an isomorphism, so j is unique.

The splitting $\mathrm{Pic}(X)_{\mathbb{Q}} \hookrightarrow \widehat{\mathrm{Pic}}(X)'_{\mathbb{Q}}$ can be written down explicitly. In fact, for any $M \in \mathrm{Pic}(X)_{\mathbb{Q}}$, let $\overline{M}$ be any lifting of M in $\widehat{\mathrm{Pic}}(X)'_{\mathbb{Q}}$. Then the f^*-equivariant lifting is given by

$$\overline{M}_f = \overline{M} - R(f^*)|_{D(X)}^{-1} R(f^*)\overline{M}.$$

One checks that this expression lies in $E(X)$ and maps to M in $\mathrm{Pic}(X)_{\mathbb{Q}}$.

Note that our triple (X, f, L) is an abbreviation of a model (X_K, f_K, L_K) over a number field K. The compatibility between different K is as follows. If we replace K by a finite extension, then the minimal polynomial R of f^* on $\mathrm{Pic}(X)_{\mathbb{Q}}$ becomes a multiple of it, and $E(X)$ becomes a space naturally including it.

Step 2

Proof of Lemma 4.12 Recall that we have assumed that the base field is a number field K, and $x_0 \in X(K)$ is a rational point. Denote by $\mathrm{Pic}(X, x_0)$ and $\widehat{\mathrm{Pic}}(X, x_0)'$ the isomorphism classes of bundles with a rigidification at x_0. More precisely, an element of $\mathrm{Pic}(X, x_0)$ consists of an element $M \in \mathrm{Pic}(X)$ and an isomorphism $M(x_0) \to K$; an element of $\widehat{\mathrm{Pic}}(X, x_0)'$ consists of an element $\overline{M} \in \widehat{\mathrm{Pic}}(X)'$ and an isometry $M(x_0) \to K$, where K is endowed with the trivial adelic metric at every place. Then we have identifications:

$$\mathrm{Pic}(X)_{\mathbb{Q}} = \mathrm{Pic}(X, x_0)_{\mathbb{Q}}, \qquad \widehat{\mathrm{Pic}}(X)'_{\mathbb{Q}} = \widehat{\mathrm{Pic}}(X, x_0)'_{\mathbb{Q}} \oplus \widehat{\mathrm{Pic}}(K)_{\mathbb{Q}},$$

where $\widehat{\mathrm{Pic}}(K)_{\mathbb{Q}}$ is embedded into $\widehat{\mathrm{Pic}}(X)'_{\mathbb{Q}}$ by pull-back via the structure morphism $X \to \mathit{Spec} K$. The kernel of the projection $\widehat{\mathrm{Pic}}(X, x_0)'_{\mathbb{Q}} \to \mathrm{Pic}(X)_{\mathbb{Q}}$ can be identified with metrics on $\mathcal{O}_X$ rigidified at x_0. By taking $-\log \|1\|$, it is identified with

$$C(X, x_0) := \oplus_v C(X_v^{\mathrm{an}}, x_0),$$

where the sum is over all places v of K, and $C(X_v^{\mathrm{an}}, x_0)$ is the space of continuous functions on X_v^{an} vanishing at x_0.

Thus we have shown the equality

$$D(X) = C(X, x_0) \oplus \widehat{\mathrm{Pic}}(K)_{\mathbb{Q}}.$$

This decomposition is stable under f^* since x_0 is a fixed point of f. On $\widehat{\mathrm{Pic}}(K)_{\mathbb{Q}}$, the operator f^* acts as the identity map. Thus $R(f^*)$ acts as $R(1)$ on $\mathrm{Pic}(K)_{\mathbb{Q}}$, which is non-zero since all eigenvalues of f^* have absolute values $q^{1/2}$ or q.

It remains to show that $R(f^*)$ is invertible over $C(X, x_0)$. By the action of the complex conjugation, it suffices to show that $R(f^*)$ is invertible over $C(X, x_0)_{\mathbb{C}} = C(X, x_0) \otimes_{\mathbb{R}} \mathbb{C}$. Decompose $R(T)$ over the complex numbers as follows:

$$R(T)=a\prod_i\left(1-\frac{T}{\lambda_i}\right),\qquad a\neq 0,\qquad |\lambda_i|=q^{1/2},\ q.$$

It suffices to show that $1-f^*/\lambda_i$ is invertible for each i. For that, it suffices to show that the operator series

$$\sum_{k=0}^{\infty}\frac{(f^*)^k}{\lambda_i^k}$$

is absolutely convergent on each Banach space $C(X_v^{\mathrm{an}},x_0)_{\mathbb{C}}$ defined by the norm

$$\|\phi_v\|_{\sup},\quad \forall\,\phi_v\in C(X_v^{\mathrm{an}},x_0)_{\mathbb{C}}.$$

By definition, f^* does not change this norm. Thus the operator norm $\|f^*\|=1$. Then

$$\left\|\frac{(f^*)^k}{\lambda_i^k}\right\|=\frac{1}{|\lambda_i|^k}\leq\frac{1}{q^{k/2}}.$$

The convergence is clear. This finishes the proof of the lemma. □

Step 3

The previous steps have constructed a unique lifting $\overline{M}_f\in\widehat{\mathrm{Pic}}(X)'_{\mathbb{Q}}$ for any $M\in\mathrm{Pic}(X)_{\mathbb{Q}}$. To prove that $\overline{M}_f$ actually lies in $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$, which means that $\overline{M}_f$ is integrable, it suffices to prove statements (1) and (2) of the theorem since $\mathrm{Pic}^0(X)_{\mathbb{Q}}$ and the ample elements of $\mathrm{Pic}_f(X)_{\mathbb{Q}}$ generate $\mathrm{Pic}(X)_{\mathbb{Q}}$.

In this step, we prove (1); namely, if $M\in\mathrm{Pic}^0(X)_{\mathbb{Q}}$, then $\overline{M}_f$ is flat. We refer the construction of flat adelic line bundles to "Appendix (Flat metrics)".

We first reduce X to abelian varieties. As above, by extending K and replacing f by an iteration, we can assume that f has a fixed point $x_0\in X(K)$. As in "Appendix (Flat metrics)", let $i:X\to A^\vee$ be the Albanese map sending x_0 to 0. Here $A=\underline{\mathrm{Pic}}^0_{X/K}$ is the Picard variety. Then the dynamical system $f:X\to X$ extends to an endomorphism $f':A^\vee\to A^\vee$. In fact, f' is just the dual of the endomorphism $A\to A$ given by $f^*:\mathrm{Pic}^0(X)\to\mathrm{Pic}^0(X)$.

The map $i:X\to A^\vee$ induces an isomorphism $i^*:\mathrm{Pic}^0(A^\vee)\to\mathrm{Pic}^0(X)$. By the expression of $\overline{M}_f$ at the end of Step 1, this isomorphism carries f'^*-admissible adelic metrics on elements of $\mathrm{Pic}^0(A^\vee)_{\mathbb{Q}}$ to f^*-admissible adelic metrics on elements of $\mathrm{Pic}^0(X)_{\mathbb{Q}}$. Hence, it suffices to prove that any f'^*-admissible adelic metrics on $\mathrm{Pic}^0(A^\vee)_{\mathbb{Q}}$ is flat. In other word, we only need to treat abelian varieties.

Now assume that X is an abelian variety, and $f:X\to X$ is an endomorphism (fixing the origin x_0). Let $M\in\mathrm{Pic}^0(X)_{\mathbb{Q}}$ be a general element. Note that f commutes with $[2]:X\to X$, and thus $[2]^*$ carries f^*-admissible adelic metrics to f^*-admissible adelic metrics. It follows that

$$[2]^*\overline{M}_f=\overline{[2]^*M}_f=2\overline{M}_f.$$

Then the adelic metric of $\overline{M}_f$ is invariant under the dynamical system $[2] : X \to X$. It is flat by Theorem 5.16

Step 4

It remains to prove (2) of the theorem. Let $M \in \mathrm{Pic}_f(X)_{\mathbb{Q}}$ be ample (and f-pure of weight 2). We need to prove that $\overline{M}_f$ is nef.

Let $\overline{M}^0$ be any nef adelic $\mathbb{Q}$-line bundle extending M and consider the sequence $\overline{M}_m = (q^{-1}f^*)^m\overline{M}^0$ of nef bundles. We will pick a subsequence "convergent" to $\overline{M}_f$.

First, denote by $\overline{M}_{m,f}$ the admissible adelic line bundle extending M_m, and denote

$$C_m := \overline{M}_m - \overline{M}_{m,f}.$$

Then C_m lies in $C(X) := \oplus C(X_v^{\mathrm{an}})$, the direct sum of the spaces of continuous functions on X_v^{an}. The relation $M_m = (q^{-1}f^*)^m M$ gives $\overline{M}_{m,f} = (q^{-1}f^*)^m\overline{M}_f$. It follows that

$$C_m = (q^{-1}f^*)^m(\overline{M}^0 - \overline{M}_f).$$

Here $\overline{M}^0 - \overline{M}_f$ also lies in $C(X)$. Since f^* does not change the norm of $C(X)$, the sequence C_m converges to 0.

Second, take a basis N_i $(i = 1, \ldots, r)$ of $\mathrm{Pic}_f(X)_{\mathbb{Q}}$. Write

$$M_m - M = \sum_{i=1}^r a_{i,m}N_i, \quad a_{i,m} \in \mathbb{Q}.$$

Then

$$\overline{M}_{m,f} - \overline{M}_f = \sum_{i=1}^r a_{i,m}\overline{N}_{i,f}.$$

It follows that

$$\overline{M}_m - \overline{M}_f = C_m + \sum_{i=1}^r a_{i,m}\overline{N}_{i,f}.$$

This is a relation in $\widehat{\mathrm{Pic}}(X)'_{\mathbb{Q}}$.

Third, notice that the orthogonal transformation $q^{-1}f^*$ has eigenvalues μ_i with absolute values 1 on $\mathrm{Pic}_f(X)_{\mathbb{Q}}$. We can find an infinite sequence $\{m_k\}_k$ of integers such that $\mu_i^{m_k} \to 1$ for every i. Thus the operator $(q^{-1}f^*)^{m_k}$ converges to the identity operator on $\mathrm{Pic}_f(X)_{\mathbb{Q}}$. In other words, $a_{i,m_k} \to 0$ for all i. Then the theorem follows from the nefness of the limits of nef bundles in a finite-dimensional space in Proposition 5.8.

4.4 Preperiodic points

We prove Theorem 4.1 in this subsection. We start by summarizing the canonical height theory to be used.

Let (X, f, L) be a polarized dynamical system over $\overline{\mathbb{Q}}$ with $f^*L \simeq qL$ for some $q > 1$. Assume that X is normal. Then the construction of [40] gives a nef adelic $\mathbb{Q}$-line bundle $\overline{L}_f \in \widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$ extending L and satisfying $f^*\overline{L}_f = q\overline{L}_f$. The height function $h_{\overline{L}_f} : X(\overline{\mathbb{Q}}) \to \mathbb{R}$ coincides with the canonical height of Call–Silverman [9]. In particular, it has the following properties:

(1) $h_{\overline{L}_f}(f(x)) = q\, h_{\overline{L}_f}(x), \ \forall x \in X(\overline{\mathbb{Q}})$.
(2) $h_{\overline{L}_f}(x) \geq 0, \ \forall x \in X(\overline{\mathbb{Q}})$.
(3) $h_{\overline{L}_f}(x) = 0$ if and only if $x \in \mathrm{Prep}(f)$.

Theorem 4.9 extends any $M \in \mathrm{Pic}(X)_{\mathbb{Q}}$ to a unique f-admissible integrable adelic $\mathbb{Q}$-line bundle $\overline{M}_f \in \widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$, and thus defines a canonical height function $h_{\overline{M}_f} : X(\overline{K}) \to \mathbb{R}$. The height function satisfies some partial properties of $h_{\overline{L}_f}$ as described in Corollary 4.11.

Furthermore, as a consequence of Theorem 4.7, we also have another f^*-equivariant linear map

$$\ell_f : \mathrm{NS}(X)_{\mathbb{Q}} \to \mathrm{Pic}(X)_{\mathbb{Q}}$$

Denote by

$$\widehat{\ell}_f : \mathrm{NS}(X)_{\mathbb{Q}} \to \widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$$

the composition of the two liftings; i.e.,

$$\widehat{\ell}_f(\xi) := \overline{\ell_f(\xi)}_f, \quad \forall \xi \in \mathrm{NS}(X)_{\mathbb{Q}}.$$

To prove Theorem 4.1, we present the following refinement.

Theorem 4.13 *Let X be a normal projective variety over $\overline{\mathbb{Q}}$. Let $f, g \in \mathcal{DS}(X)$ be two algebraic dynamical systems on X (with possibly different polarizations and weights), and Y_0 be an irreducible component of the Zariski closure of* $\mathrm{Prep}(f) \cap \mathrm{Prep}(g)$ *in X. Let Y be the normalization of Y_0. Then for any element $\xi \in \mathrm{NS}(X)_{\mathbb{Q}}$,*

$$\widehat{\ell}_f(\xi)|_Y = \widehat{\ell}_g(\xi)|_Y.$$

In the theorem, we take the convention that $\overline{M}|_Y$ denotes the image of an element $\overline{M} \in \widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$ under the pull-back map $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}} \to \widehat{\mathrm{Pic}}(Y)_{\mathbb{Q}}$.

Proof of "Theorem 4.13 $\Rightarrow$ Theorem 4.1". As in Remark 4.4, we can reduce Theorem 4.1 to the case that X is normal. It suffices to prove $\mathrm{Prep}(g)\cap Y_0(\overline{\mathbb{Q}}) = \mathrm{Prep}(f)\cap Y_0(\overline{\mathbb{Q}})$ for any irreducible component Y_0 of the Zariski closure of $\mathrm{Prep}(f) \cap \mathrm{Prep}(g)$ in X.

Denote by Y the normalization of Y_0. Let L be an ample line bundle on X polarizing f. Let ξ be the image of L in $\mathrm{NS}(X)_{\mathbb{Q}}$, and M be the lifting of ξ in $\mathrm{Pic}_g(X)_{\mathbb{Q}}$. By Theorem 4.13, $\overline{L}_f|_Y = \overline{M}_g|_Y$. For any $x_0 \in \mathrm{Prep}(g) \cap Y_0(\overline{\mathbb{Q}})$, let $x \in Y(\overline{\mathbb{Q}})$ be a preimage of x_0 in $Y(\overline{\mathbb{Q}})$. We have

$$h_{\overline{L}_f}(x_0) = h_{\overline{L}_f|_Y}(x) = h_{\overline{M}_g|_Y}(x) = h_{\overline{M}_g}(x_0) = 0$$

by Corollary 4.11. It follows that $x_0 \in \mathrm{Prep}(f) \cap Y_0(\overline{\mathbb{Q}})$. This proves

$$\mathrm{Prep}(g) \cap Y_0(\overline{\mathbb{Q}}) \subset \mathrm{Prep}(f) \cap Y_0(\overline{\mathbb{Q}}).$$

By symmetry, we have the other direction and thus the equality. □

Proof of Theorem 4.13 We need to prove $\widehat{\ell}_f(\xi)|_Y = \widehat{\ell}_g(\xi)|_Y$ for any $\xi \in \mathrm{NS}(X)_{\mathbb{Q}}$. By linearity, it suffices to assume that ξ is ample. Denote $L = \ell_f(\xi)$ and $M = \ell_g(\xi)$. They are ample $\mathbb{Q}$-line bundles on X. Then $\overline{L}_f = \widehat{\ell}_f(\xi)$ and $\overline{M}_g = \widehat{\ell}_g(\xi)$ are nef by Theorem 4.9.

Consider the sum $\overline{N} = \overline{L}_f + \overline{M}_g$ and the associated height function $h_{\overline{N}}$. By Remark 5.2, this is an ample metrized line bundle. By the successive minima of Zhang [40, Theorem 1.10] for $\overline{N}|_Y$ on Y,

$$\lambda_1(Y, \overline{N}) \geq h_{\overline{N}}(Y) \geq 0.$$

Here $h_{\overline{N}}(Y)$ and $\lambda_1(Y, \overline{N})$ denote the height and the essential minimum of Y with respect to $\overline{N}$:

$$h_{\overline{N}}(Y) = \frac{(\overline{N}|_Y)^{d+1}}{(n+1)\deg_N(Y)}, \qquad d = \dim Y,$$
$$\lambda_1(Y, \overline{N}) = \sup_{U \subset Y} \inf_{x \in U(\overline{\mathbb{Q}})} h_{\overline{N}|_Y}(x),$$

where the supremum is taken over all dense Zariski open subsets U of Y.

Note that $h_{\overline{N}}$ is zero on $\mathrm{Prep}(f) \cap \mathrm{Prep}(g)$, which is Zariski dense in Y_0 by definition. Hence, $\lambda_1(Y, \overline{N}) = 0$. It forces $h_{\overline{N}}(Y) = 0$.

Writing in terms of intersections, we have

$$(\overline{L}_f|_Y + \overline{M}_g|_Y)^{d+1} = 0.$$

Note that in the binomial expansion of the left hand side, every term is non-negative by Proposition 5.6. It follows that

$$\overline{L}_f|_Y^i \cdot \overline{M}_g|_Y^{d+1-i} = 0, \quad \forall\, i = 0, 1, \ldots, d+1.$$

It particularly gives

$$(\overline{L}_f|_Y - \overline{M}_g|_Y)^2 \cdot (\overline{L}_f|_Y + \overline{M}_g|_Y)^{d-1} = 0.$$

Note that

$$(L|_Y - M|_Y)\cdot(L|_Y + M|_Y)^{d-1} = 0$$

since $L - M \in \mathrm{Pic}^0(X)_{\mathbb{Q}}$ is numerically trivial. We are in a situation to apply Theorem 3.2 to

$$(\overline{L}_f|_Y - \overline{M}_g|_Y,\ \overline{L}_f|_Y + \overline{M}_g|_Y).$$

It is immediate that $(\overline{L}_f - \overline{M}_g)$ is $(\overline{L}_f + \overline{M}_g)$-bounded. The only condition that does not match the theorem is that $\overline{L}_f|_Y + \overline{M}_g|_Y$ is not arithmetically positive. However, since $L - M$ is numerically trivial, we can take any $\overline{C} \in \widehat{\mathrm{Pic}}(\bar{\mathbb{Q}})$ with $\widehat{\deg}(\overline{C}) > 0$, and replace

$$(\overline{L}_f - \overline{M}_g,\ \overline{L}_f + \overline{M}_g)$$

by

$$(\overline{L}_f - \overline{M}_g,\ \overline{L}_f + \overline{M}_g + \pi^*\overline{C}).$$

Then all the conditions are satisfied. The theorem implies that

$$\overline{L}_f|_Y - \overline{M}_g|_Y = \pi_Y^*\overline{N}, \quad \overline{N} \in \widehat{\mathrm{Pic}}(\bar{\mathbb{Q}})_{\mathbb{Q}}.$$

To finish the proof, we need to show $\overline{N} = 0$ in $\widehat{\mathrm{Pic}}(\bar{\mathbb{Q}})_{\mathbb{Q}}$. Take any point x in $\mathrm{Prep}(f) \cap \mathrm{Prep}(g)$. By Corollary 4.11, $\overline{N} = x^*\pi^*\overline{N}$ in $\widehat{\mathrm{Pic}}(x)_{\mathbb{Q}}$ is zero This finishes the proof. □

4.5 Consequences and questions

Now we consider some consequences and questions related to Theorem 4.1. The first result concerns the case $Y = X$.

Theorem 4.14 *Let X be a normal projective variety over $\overline{\mathbb{Q}}$. For any $f, g \in \mathcal{DS}(X)$, the following are equivalent:*

(1) $\mathrm{Prep}(f) = \mathrm{Prep}(g)$*;*
(2) $g\mathrm{Prep}(f) \subset \mathrm{Prep}(f)$*;*
(3) $\mathrm{Prep}(f) \cap \mathrm{Prep}(g)$ *is Zariski dense in X;*
(4) $\widehat{\ell}_f = \widehat{\ell}_g$ *as maps from* $\mathrm{NS}(X)_{\mathbb{Q}}$ *to* $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$.

Proof We will prove (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4)$\Rightarrow$ (1).

First, (1) $\Rightarrow$ (2) is trivial.

Second, (2) $\Rightarrow$ (3). Let K be a number field such that (X, f, g) has a model (X_K, f_K, g_K) over K, and such that both f_K and g_K have polarizations over X_K.

By abuse of notation, abbreviate (X_K, f_K, g_K) as (X, f, g). For any integer $d > 0$, denote

$$\mathrm{Prep}(f, d) := \{x \in \mathrm{Prep}(f) \mid \deg(x) < d\}.$$

Here the degree is with respect to the base field K. By Northcott's property, $\mathrm{Prep}(f, d)$ is a finite set since its points have trivial canonical heights with respect to a polarization of f. Assuming (2), then g stabilizes

$$\mathrm{Prep}(f) = \bigcup_{d>0} \mathrm{Prep}(f, d).$$

Since g is also defined over K, it stabilizes the set $\mathrm{Prep}(f, d)$. By the finiteness, we obtain that

$$\mathrm{Prep}(f, d) \subset \mathrm{Prep}(g), \quad \forall\, d.$$

Hence,

$$\mathrm{Prep}(f) \subset \mathrm{Prep}(g).$$

Then (3) is true since $\mathrm{Prep}(f)$ is Zariski dense in X by Fakhruddin [12, Theorem 5.1].

Third, (3) $\Rightarrow$ (4) follows from Theorem 4.13 by setting $Y = X$.

Finally, we prove (4) $\Rightarrow$ (1). Let L be an ample line bundle on X polarizing f. By (4), $\overline{L}_f = \overline{L}_g$. For any $x \in \mathrm{Prep}(g)$, we have $h_{\overline{L}_f}(x) = h_{\overline{L}_g}(x) = 0$ by Corollary 4.11. It follows that $x \in \mathrm{Prep}(f)$. This proves $\mathrm{Prep}(g) \subset \mathrm{Prep}(f)$. By symmetry, we have the other direction and thus the equality. □

Semigroup

For any subset P of $X(\overline{\mathbb{Q}})$, denote

$$\mathcal{DS}(X, P) := \{g \in \mathcal{DS}(X) \mid \mathrm{Prep}(g) = P\}.$$

We say that that P is a *special set of* X if $\mathcal{DS}(X, P)$ is non-empty.

Question 4.15 *Let P be a special set of X. Is the set $\mathcal{DS}(X, P)$ a semigroup?*

The question asks whether $g \circ h \in \mathcal{DS}(X, P)$ for any $g, h \in \mathcal{DS}(X, P)$. By Theorem 4.14, we can write:

$$\mathcal{DS}(X, P) = \{g \in \mathcal{DS}(X) \mid gP \subset P\}.$$

Then we have $g \circ h \in \mathcal{DS}(X, P)$ if $g \circ h$ is polarizable.

The polarizability is automatically true if $X = \mathbb{P}^n$. Hence, in this case $\mathcal{DS}(X, P)$ is a semigroup.

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Appendix 1

For any scheme S, denote by $\mathcal{P}\mathrm{ic}(S)$ the category of line bundles on S, and by $\mathrm{Pic}(S)$ the group of isomorphism classes of line bundles on S. We may also work on the groupoid $\mathcal{P}\mathrm{ic}(S)_{\mathbb{Q}}$ of $\mathbb{Q}$-line bundles on S. The objects of $\mathcal{P}\mathrm{ic}(S)_{\mathbb{Q}}$ are of the form aL with $a \in \mathbb{Q}$ and $L \in \mathcal{P}\mathrm{ic}(S)$. The isomorphism of two such objects is defined to be

$$\mathrm{Isom}(aL, bM) := \varinjlim_N \mathrm{Isom}(L^{\otimes aN}, M^{\otimes bN}),$$

where N runs through positive integers such that aN and bN are both integers. The group of isomorphism classes of such $\mathbb{Q}$-line bundles is isomorphic to $\mathrm{Pic}(S)_{\mathbb{Q}} := \mathrm{Pic}(S) \otimes \mathbb{Q}$. We will usually write the tensor products of line bundles additively.

Local intersections

Let K be an algebraic closed field with a a complete and non-trivial absolute value $|\cdot|$. In this subsection, we describe a local intersection theory for varieties over a K. If K is archimedean, it is standard in complex geometry. If K is non-archimedean, it is due to Gubler [17,19] based on Berkovich's analytic spaces introduced in [1]. If the valuation of K was discrete, this intersection theory was also constructed in [40] without using Berkovich spaces. A measure-theoretic interpretation of local intersection numbers has been introduced by Chambert-Loir [8], Gubler [19] and Chambert-Loir–Thuillier [10].

Integrable metrized line bundles

Let us first consider the case K is non-archimedean. Denote by O_K *the ring of integers* consisting of elements $a \in K$ with norm $|a| \leq 1$, by $\mathfrak{m}$ the *maximal ideal* of O_K consisting of elements $a \in O_K$ with norm $|a| < 1$, and by k *the residue field* $O_K/\mathfrak{m}$.

Let X be a projective variety over K, and let X^{an} be the analytification of X in the sense of Berkovich [1]. By an *integral model* (or a *model*) of X over O_K, we mean a projective, flat, and finitely presented, integral scheme $\mathcal{X}$ over O_K with generic fiber X. Note that Gubler has used *formal models* $\mathfrak{X}$ with generic fiber X^{an}. Since all formal models are dominated by the completions $\widehat{\mathcal{X}}$ of projective models $\mathcal{X}$ along their special fibers, the constructions using (projective) integral models are equivalent to those using formal models. See [18, Proposition 10.5] for more details.

Let L be a line bundle on X, and let L^{an} be the analytification of L as a line bundle on X^{an}. By a *continuous metric* on L we mean a collection of $K(x)$-metrics $\|\cdot\|$ on $L^{\mathrm{an}}(x)$ indexed by $x \in X^{\mathrm{an}}$, where $K(x)$ is the residue field of x; the collection is required to be continuous in sense that for every rational section s of L, the map $x \mapsto \|s(x)\|$ is continuous away from the poles of s on X^{an}. A *metrized line bundle* $\overline{L}$ on X is a pair $(L, \|\cdot\|)$ consisting of a line bundle L on X and a continuous metric $\|\cdot\|$ on L.

Let e be a positive integer, and let $(\mathcal{X}, \mathcal{L})$ be an integral model of (X, eL). Namely, $\mathcal{X}$ is an integral model of X and $\mathcal{L}$ is a line bundle on $\mathcal{X}$ with restriction $\mathcal{L}_K = eL$ on X. Then one can associate a metric $\|\cdot\|$ on L so that on any open set $\mathcal{U}$ of $\mathcal{X}$ with a trivialization of $\mathcal{L}|_{\mathcal{U}} \simeq \mathcal{O}_{\mathcal{U}}$ by a section ℓ of $\mathcal{L}|_{\mathcal{U}}$, the induced metric on eL satisfies $\|\ell\|(x) = 1$ for any $x \in X^{\mathrm{an}}$ reducing to $\mathcal{U}_k$. It gives the metric on L by taking the e-th root. Such a metric is called a *model metric*.

A model metric on L is called *semipositive* if it is induced by an integral model $(\mathcal{X}, \mathcal{L})$ of (X, eL) (for some positive integer e) with $\mathcal{L}$ relatively nef on $\mathcal{X}$, that is, $\mathcal{L}_k$ has a non-negative degree on any closed sub-curve in the special fiber $\mathcal{X}_k$. Note that it implies that L is nef on the generic fiber X.

Definition 5.1 Let $\overline{L} = (L, \|\cdot\|)$ and $\overline{M}$ be two metrized line bundles on X.

(1) $\|\cdot\|$ is *semipositive* if it is equal to a uniform limit of semipositive model metrics.
(2) $\overline{L}$ is *semipositive* if $\|\cdot\|$ is semipositive, which implies that L is also nef.
(3) $\overline{L}$ is *arithmetically positive* if L is ample and $\|\cdot\|$ is semipositive.
(4) $\overline{L}$ is *integrable* if $\overline{L} = \overline{L}_1 - \overline{L}_2$ with $\overline{L}_i$ semipositive.
(5) $\overline{M}$ is *$\overline{L}$-bounded* if there is a positive integer m such that both $m\overline{L} + \overline{M}$ and $m\overline{L} - \overline{M}$ are semipositive.
(6) $\overline{M}$ is *vertical* if it is integrable and its generic fiber M is isomorphic to the trivial bundle $\mathcal{O}_X$.
(7) $\overline{M}$ is *constant* if it is isometric to the pull-back of a metrized line bundle on $Spec K$.

Let $\widehat{\mathcal{P}\mathrm{ic}}(X)$ denote the category of integrable metrized line bundles on X with morphisms given by isometries, and $\widehat{\mathrm{Pic}}(X)$ the group of isometry classes of integrable metrized line bundles.

Consider the case $L = \mathcal{O}_X$. The group of continuous metrics $\|\cdot\|$ on $\mathcal{O}_X$ is isomorphic to the space $C(X^{\mathrm{an}})$ of real-valued continuous functions f on X^{an} by the relation $f = -\log \|1\|$. Under this relation, we say that f is an *integrable function* (resp. *model function*) if $\|\cdot\|$ is an integrable metric (resp. model metric). Denote by $C_{\mathrm{int}}(X^{\mathrm{an}})$ (resp. $C_{\mathrm{mod}}(X^{\mathrm{an}})$) the $\mathbb{Q}$-vector space of integrable functions (resp. model functions) on X^{an}. By definition, and using that on a projective variety over the valuation ring any line bundle is isomorphic to the difference of two very ample ones, we have

$$C_{\mathrm{mod}}(X^{\mathrm{an}}) \subset C_{\mathrm{int}}(X^{\mathrm{an}}) \subset C(X^{\mathrm{an}}).$$

By [17, Theorem 7.12], $C_{\mathrm{mod}}(X^{\mathrm{an}})$ is dense in $C(X^{\mathrm{an}})$ under the topology induced by the supremum norm.

For a morphism $f : X \to Y$ of projective varieties, we have an obvious notion of pull-back functor $f^* : \widehat{\mathcal{P}\mathrm{ic}}(X) \to \widehat{\mathcal{P}\mathrm{ic}}(Y)$ which respect properties in Definition 5.1 except arithmetic positivity.

Intersection theory

Let Z be a cycle on X of dimension d, $\overline{L}_0, \ldots, \overline{L}_d$ be integrable metrized line bundles on X, and $\ell_0, \ldots, \ell_d$ be rational sections of $L_0, \ldots, L_d$ respectively such that the common support $\cap_i |\mathrm{div}(\ell_i)|$ has no intersection with $|Z|$. Gubler [17] and Chambert-Loir [8] have defined the local height $\widehat{\mathrm{div}}(\ell_0) \cdots \widehat{\mathrm{div}}(\ell_d) \cdot [Z]$ of Z with respect to $\ell_0, \ldots, \ell_d$. It is characterized by the following properties:

(1) It is linear in the variables $\widehat{\mathrm{div}}(\ell_i)$ and Z.
(2) For fixed $\ell_1, \ldots, \ell_d$, it is continuous with respect to the metrics.
(3) When every $\overline{L}_i$ is induced by a model $\mathcal{L}_i$ on a common model $\mathcal{X}$ of X,

$$\widehat{\mathrm{div}}(\ell_0) \cdots \widehat{\mathrm{div}}(\ell_d) \cdot [Z] = \mathrm{div}_{\mathcal{X}}(\ell_0) \cdots \mathrm{div}_{\mathcal{X}}(\ell_d) \cdot [\mathcal{Z}],$$

where $\mathcal{Z}$ is the Zariski closure of Z on $\mathcal{X}$.

By Chambert-Loir [8], Gubler [19] and Chambert-Loir–Thuillier [10], if the support of $\mathrm{div}(\ell_0)$ does not contain any irreducible component of Z, the intersection number can be computed using a measure $c_1(\overline{L}_1) \cdots c_1(\overline{L}_d)\delta_Z$ of X^{an} supported on Z^{an} inductively by

$$\begin{aligned}\widehat{\mathrm{div}}(\ell_0) \cdots \widehat{\mathrm{div}}(\ell_d) \cdot [Z] &= \widehat{\mathrm{div}}(\ell_1) \cdots \widehat{\mathrm{div}}(\ell_d) \cdot [\mathrm{div}(\ell_0) \cdot Z] \\ &\quad - \int_{X^{\mathrm{an}}} \log \|\ell_0\| c_1(\overline{L}_1) \cdots c_1(\overline{L}_d)\delta_Z.\end{aligned}$$

In [10, Theorem 4.1], the above formula is proved over $\mathbb{C}$ or finite extensions of $\mathbb{Q}_p$, but the proof for general K goes through, with minor adaptations as in Gubler [19].

Let $\overline{L}_0, \cdots \overline{L}_d$ be integrable metrized line bundles on X such that the underlying line bundles L_i satisfy the following properties:

(1) $L_0|_{Z_j} \simeq \mathcal{O}_{Z_j}$ on every irreducible component Z_j of Z;
(2) $c_1(L_1) \cdots c_1(L_d) \cdot [Z_j] = 0$ on X for every irreducible component Z_j of Z.

We further assume that ℓ_0 does not vanish on any Z_j. Then the local pairing $\widehat{\mathrm{div}}(\ell_0) \cdots \widehat{\mathrm{div}}(\ell_d) \cdot [Z]$ does not depend on the choice of the rational sections ℓ_i. Thus we can define a pairing

$$\overline{L}_0 \cdot \overline{L}_1 \cdots \overline{L}_d \cdot Z := \widehat{\mathrm{div}}(\ell_0) \cdots \widehat{\mathrm{div}}(\ell_d) \cdot [Z].$$

If $Z = X$, we will omit X in the notation.

Now we extend the above definitions to the archimedean case. If $K = \mathbb{C}$, then $X^{\mathrm{an}} = X(\mathbb{C})$ is the complex analytic space associated to X. For a metrized line bundle $\overline{L}$ on X, the metric is called *smooth* if for any analytic map $\phi : B \to X(\mathbb{C})$ from any complex open ball B of any dimension to $X(\mathbb{C})$, the metric of the pull-back $\phi^*\overline{L}$ is a smooth hermitian metric. In that case, we say that $\overline{L}$ is *semipositive* if the curvature form $c_1(\phi^*\overline{L})$ is positive semi-definite pointwise on B for any such ϕ. If $K = \mathbb{R}$, then X^{an} is the quotient topological space of $X(\mathbb{C})$ by the action of the complex conjugate.

A metric on X^{an} is called *smooth* if its pull-back to $X(\mathbb{C})$ is smooth, and a smooth metric on X^{an} is called *semipositive* if its pull-back to $X(\mathbb{C})$ is semipositive.

Replacing model metrics by *smooth metrics*, Definition 5.1 works in the archimedean case, and the intersection theory follows the same way.

Remark 5.2 By [39, Theorem 3.5, (2)] for non-archimedean K and [30, Theorem 2] for $K = \mathbb{C}$, the notion "arithmetically positive" is equivalent to the notion of "semi-ample metrized" in the sense that for any $\epsilon > 0$ and any closed point $x \in X$, there is a section ℓ of a positive power nL such that $\|\ell\|_{\sup} \le \|\ell(x)\| \cdot e^{n\epsilon}$.

Arithmetic intersections

Bundles with smooth metrics

We first review the theory of arithmetic line bundles in Gillet–Soulé [15] with some extensions in [39] to arithmetic varieties with singular generic fibers.

Let K be a number field with the ring O_K of integers. Let $\pi : \mathcal{X} \to Spec O_K$ be an arithmetic variety. This means that $\mathcal{X}$ is an integral scheme and the morphism π is projective and flat. By a *hermitian line bundle* on $\mathcal{X}$, we mean a pair $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ consisting of a line bundle $\mathcal{L}$ on $\mathcal{X}$ and a smooth hermitian metric $\|\cdot\|$ on $\mathcal{L}(\mathbb{C})$ over the complex analytic space $\mathcal{X}(\mathbb{C}) = \coprod_{v|\infty} X_v(\mathbb{C})$, invariant under the action of the complex conjugation. Here the smoothness of the metric is as in "Appendix (Local intersections)". Let $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{X})$ denote the category of hermitian line bundles and $\widehat{\mathrm{Pic}}(\mathcal{X})$ the group of isometry classes of hermitian line bundles on $\mathcal{X}$. We can define the category $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{X})_{\mathbb{Q}}$ of hermitian $\mathbb{Q}$-line bundles on $\mathcal{X}$. The group of isometry classes of such bundles is again given by $\widehat{\mathrm{Pic}}(\mathcal{X})_{\mathbb{Q}}$.

For any integral subvariety $\mathcal{Y}$ of $\mathcal{X}$ of dimension $d+1$, and hermitian line bundles $\overline{\mathcal{L}}_0, \ldots, \overline{\mathcal{L}}_d$ with rational sections $\ell_0, \ldots, \ell_d$ such that $\cap|\mathrm{div}(\ell_i)|$ has no common point with the generic fiber $\mathcal{Y}_K$ of $\mathcal{Y}$, the intersection number $\widehat{\mathrm{div}}(\ell_0) \cdots \widehat{\mathrm{div}}(\ell_d) \cdot [\mathcal{Y}]$ can be defined as the sum of local intersection numbers over all places v of K by

$$\widehat{\mathrm{div}}(\ell_0) \cdots \widehat{\mathrm{div}}(\ell_d) \cdot [\mathcal{Y}] = \sum_v (\widehat{\mathrm{div}}(\ell_0) \cdots \widehat{\mathrm{div}}(\ell_d) \cdot [\mathcal{Y}])_v.$$

This number does not depend on the choices of ℓ_i. Thus we have a pairing $\widehat{\mathrm{Pic}}(\mathcal{X})^{d+1} \times Z_{d+1}(\mathcal{X}) \to \mathbb{R}$ by

$$\overline{\mathcal{L}}_0 \cdots \overline{\mathcal{L}}_d \cdot \mathcal{Z} := \widehat{\mathrm{div}}(\ell_0) \cdots \widehat{\mathrm{div}}(\ell_d) \cdot [\mathcal{Z}].$$

If $\mathcal{Z} = \mathcal{X}$, then we usually omit $\mathcal{X}$ in the above notation. If furthermore $\dim \mathcal{X} = 1$, then it is common to write $\widehat{\deg}(\overline{\mathcal{L}}_0)$ for the number $\overline{\mathcal{L}}_0 \cdot \mathcal{X}$, which is called *the arithmetic degree*.

Let $\overline{\mathcal{L}}$ be a hermitian line bundle on an arithmetic variety $\mathcal{X}$ over O_K. If $x \in \mathcal{X}(\overline{K})$, we define the height of x with respect to $\overline{\mathcal{L}}$ by

$$h_{\overline{\mathcal{L}}}(x) := \frac{\overline{\mathcal{L}} \cdot \bar{x}}{[K(x) : K]},$$

where $\bar{x}$ denotes the Zariski closure of the image of x in $\mathcal{X}$.

Definition 5.3 Let $\overline{\mathcal{L}}$ be a hermitian line bundle on an arithmetic variety $\pi : \mathcal{X} \to Spec O_K$.

(1) We say that $\overline{\mathcal{L}}$ is *nef* if $\overline{\mathcal{L}}$ has a non-negative (arithmetic) degree on any integral subscheme on $\mathcal{X}$, and the curvature $c_1(\overline{\mathcal{L}})$ is positive semi-definite on the smooth locus of $\mathcal{X}(\mathbb{C})$.
(2) We say that $\overline{\mathcal{L}}$ is *arithmetically positive* if the generic fiber $\mathcal{L}_K$ is ample on $\mathcal{X}_K$, and $\overline{\mathcal{L}}-\pi^*\overline{\mathcal{M}}$ is nef for some hermitian line bundle $\overline{\mathcal{M}}$ of $Spec O_K$ with $\widehat{\deg}(\overline{\mathcal{M}}) > 0$.

The intersection pairing extends to hermitian $\mathbb{Q}$-line bundles by linearity. We say that an hermitian $\mathbb{Q}$-line bundle $\overline{\mathcal{L}}$ is *nef* (resp. *arithmetically positive*) if there is a positive integer a such that $a\overline{\mathcal{L}}$ is a hermitian line bundle which is nef (resp. arithmetically positive).

The above notion of "arithmetic positivity" is slightly weaker that the notion of "arithmetic ampleness" used by Zhang [39] and [29] in the case $\mathcal{X}_K$ is regular.

Lemma 5.4 *Let $\mathcal{X} \to Spec O_K$ be an arithmetic variety such that $\mathcal{X}_K$ is regular. Let $\overline{\mathcal{L}}$ be a hermitian $\mathbb{Q}$-line bundle on $\mathcal{X}$. Then $\overline{\mathcal{L}}$ is arithmetically positive if it is arithmetically ample in the following sense:*

(a) *The line bundle $\mathcal{L}$ is relatively ample on $\mathcal{X}$;*
(b) *The curvature of $\overline{\mathcal{L}}$ is positive definite on $\mathcal{X}(\mathbb{C})$;*
(c) *$\overline{\mathcal{L}}$ is horizontally positive; i.e., the intersection number $\overline{\mathcal{L}}^{\dim \mathcal{Y}} \cdot \mathcal{Y} > 0$ for any horizontal irreducible closed subscheme $\mathcal{Y}$ of $\mathcal{X}$.*

Proof Let $\overline{\mathcal{M}}$ be a hermitian line bundle on $Spec O_K$ with $\widehat{\deg}(\overline{\mathcal{M}}) > 0$. It suffices to prove that $\overline{\mathcal{N}}_m = m\overline{\mathcal{L}}-\pi^*\overline{\mathcal{M}}$ is nef for some integer $m > 0$. By the arithmetic Nakai–Moishezon criterion of [39, Corollary 4.8], for sufficiently large m, the hermitian line bundle $\Gamma(\mathcal{X}, \mathcal{N}_m)$ has a $\mathbb{Z}$-basis $s_1, \ldots, s_r$ with every $\|s_i\|_{\sup} < 1$. Using this basis to compute intersection numbers, we see that $\overline{\mathcal{N}}_m$ is nef. ◻

Bundles with adelic metrics

Now let us review the intersection theory of adelic line bundles in [40] and the extensions by Chambert-Loir [8] and Chambert-Loir–Thuillier [10].

Let K be a number field, $\pi : X \to Spec K$ a projective variety, and L a line bundle on X. An *integral model* (or a *model*) $\mathcal{X}$ of X over O_K means a projective and flat integral scheme $\mathcal{X}$ over O_K with generic fiber X. If furthermore $\overline{\mathcal{L}}$ is a hermitian line bundle on $\mathcal{X}$ with generic fiber $\mathcal{L}_K = L$, we say that $(\mathcal{X}, \overline{\mathcal{L}})$ is an *integral model* (or a *model*) of (X, L). An integral model of (X, eL) (for some positive integer e) induces a collection $\|\cdot\|_{\overline{\mathcal{L}},\mathbb{A}} = \{\|\cdot\|_{\overline{\mathcal{L}},v}\}_v$ of continuous metrics $\|\cdot\|_{\overline{\mathcal{L}},v}$ of $L^{\mathrm{an}}_{K_v}$ on $X^{\mathrm{an}}_v = X^{\mathrm{an}}_{K_v}$ over all places v of K. The collection is called a *model adelic metric* on L. In general, by an *adelic metric* $\|\cdot\|_{\mathbb{A}} = \{\|\cdot\|_v\}_v$ on L, we mean a collection of continuous metrics

$\|\cdot\|_v$ of $L_{K_v}^{\mathrm{an}}$ on X_v^{an} over all places v of K, which is assumed to be coherent in the sense that it agrees with some model adelic metric at all but finitely many places v.

An adelic metric $\|\cdot\|_{\mathbb{A}}$ is called *semipositive* if it is semipositive at every place in the sense of Definition 5.1(1). An adelic metric $\|\cdot\|_{\mathbb{A}}$ on L is *integrable* if it is the difference of two semipositive metrized line bundles. Let $\widehat{\mathcal{P}\mathrm{ic}}(X)$ denote the category of integrable adelic metrized line bundles on X, and let $\widehat{\mathrm{Pic}}(X)$ the isometry classes of such bundles. We can define $\widehat{\mathcal{P}\mathrm{ic}}(X)_{\mathbb{Q}}$ and $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$ for $\mathbb{Q}$-line bundles.

Note that the definition of adelic metrics in [40] used the space $X(\bar{K}_v)$, while we use the Berkovich space $X_{K_v}^{\mathrm{an}}$. But they give the same classes of model adelic metrics, semipositive adelic metrics and integrable adelic metrics. We make some new definitions not included in [40] in the following.

Definition 5.5 Let $\overline{L}, \overline{M}$ be integrable metrized line bundles on X.

(1) We say that $\overline{L}$ is *nef* if the adelic metric is a uniform limit of model adelic metrics induced by nef hermitian line bundles on integral models of X;
(2) We say that $\overline{L}$ is *arithmetically positive* if L is ample, and $\overline{L} - \pi^*\overline{N}$ is nef for some adelic line bundle $\overline{N}$ on $Spec K$ with $\widehat{\deg}(\overline{N}) > 0$.
(3) We say that $\overline{M}$ is $\overline{L}$*-bounded* if there is an integer $m > 0$ such that both $m\overline{L} + \overline{M}$ and $m\overline{L} - \overline{M}$ are nef.
(4) We say that $\overline{L}$ is *vertical* if L is trivial on X.
(5) We say that $\overline{L}$ is *constant* if L is isometric to the pull-back of an adelic line bundle on $Spec K$.

By the above definitions, $\widehat{\mathrm{Pic}}(X)$ has sub-semigroups $\widehat{\mathrm{Pic}}(X)_{\mathrm{nef}}$, $\widehat{\mathrm{Pic}}(X)_{\mathrm{pos}}$ and a subgroup $\widehat{\mathrm{Pic}}(X)_{\mathrm{vert}}$. We extend the definitions to adelic $\mathbb{Q}$-line bundles as in the case of hermitian $\mathbb{Q}$-line bundles.

Consider the case $L = \mathcal{O}_X$. Similar to the local case, the group of adelic metrics $\|\cdot\|_{\mathbb{A}}$ on $\mathcal{O}_X$ is isomorphic to the space

$$C(X) = \oplus_v C(X_v^{\mathrm{an}});$$

the group of integrable metrics $\|\cdot\|_{\mathbb{A}}$ on $\mathcal{O}_X$ is isomorphic to the space

$$C_{\mathrm{int}}(X) = \oplus_v C_{\mathrm{int}}(X_v^{\mathrm{an}}).$$

The isomorphisms are given by

$$\|\cdot\|_{\mathbb{A}} \longmapsto \oplus_v(-\log \|1\|_v).$$

The space $C(X)$ has a supremum norm

$$\|f\| := \max_v \|f_v\|_{\sup}, \qquad f = \oplus f_v.$$

By abuse of notations, we will write $C(X)$ for the corresponding group of adelic line bundles.

Let Y be an integral subvariety of X of dimension d and let $\overline{L}_0, \ldots, \overline{L}_d$ be integrable metrized line bundles with sections $\ell_0, \ldots, \ell_d$ such that $\cap|\mathrm{div}(\ell_i)|$ has no intersection with Y. Then we can define intersection pairing by summing up local intersections:

$$\widehat{\mathrm{div}}(\ell_0)\cdots\widehat{\mathrm{div}}(\ell_d)\cdot[Y] = \sum_v (\widehat{\mathrm{div}}(\ell_0)\cdots\widehat{\mathrm{div}}(\ell_d)\cdot[Y])_v.$$

This pairing does not depend on the choice of ℓ_i. Thus it defines a pairing $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}^{d+1} \times Z_d(X) \to \mathbb{R}$ by

$$\overline{L}_0\cdot\overline{L}_1\cdots\overline{L}_d\cdot Z := \widehat{\mathrm{div}}(\ell_0)\cdots\widehat{\mathrm{div}}(\ell_d)\cdot[Z].$$

We will omit Z if $Z = X$, and write it as $\widehat{\deg}(\overline{L}_0)$ if $Z = X$ and $\dim X = 0$.

The local induction formula of Chambert-Loir–Thuillier recalled in "Appendix (Local intersections)" gives a global induction formula as follows. Let ℓ_0 be a rational section of L_0 on X such that $\mathrm{div}(\ell_0)$ has a proper intersection with Z, then

$$\begin{aligned}\overline{L}_0\cdot\overline{L}_1\cdots\overline{L}_d\cdot Z &= \overline{L}_1\cdots\overline{L}_d\cdot(Z\cdot\mathrm{div}(\ell_0))\\ &\quad -\sum_v\int_{X_v^{\mathrm{an}}}\log\|\ell_0\|_v c_1(\overline{L}_{1,v})\cdots c_1(\overline{L}_{d,v})\delta_Z.\end{aligned}$$

Notice that the summation has only finitely many non-zero terms.

In this paper, we will often study adelic metrics of line bundles on a projective variety X over $\bar{\mathbb{Q}}$. Using base change, there is an obvious way to give the following definitions,

$$\widehat{\mathcal{P}\mathrm{ic}}(X) = \lim_{X_K}\widehat{\mathcal{P}\mathrm{ic}}(X_K), \qquad \widehat{\mathrm{Pic}}(X) = \lim_{X_K}\widehat{\mathrm{Pic}}(X_K)$$

where limit is over models X_K over number fields K, i.e., X_K is a projective variety over a number field K equipped with an isomorphism $X_K \otimes_K \bar{\mathbb{Q}} \simeq X$. Similarly we can define the sub-semigroups $\widehat{\mathrm{Pic}}(X)_{\mathrm{nef}}$, $\widehat{\mathrm{Pic}}(X)_{\mathrm{pos}}$, and subgroups $\widehat{\mathrm{Pic}}(X)_{\mathrm{vert}}$, $C(X)$, $C_{\mathrm{int}}(X)$, etc. The intersection numbers on X_K with normalization factor $1/[K:\mathbb{Q}]$ will give intersection numbers on X.

Arithmetic positivity

In this section, we collect some facts about arithmetic positivity needed in this paper. The first one is about the positivity of intersections of nef bundles.

Proposition 5.6 *Let X be a projective variety over a number field K. Let $\overline{L}_0, \overline{L}_1, \ldots, \overline{L}_n$ be nef adelic line bundles on X. Then*

$$\overline{L}_0\cdot\overline{L}_1\cdots\overline{L}_n \geq 0.$$

Proof We prove it by induction on n. By resolution of singularity, we may assume that X is regular. By approximation, we may assume that $(\overline{L}_0, \ldots, \overline{L}_n)$ is induced by a model $(\overline{\mathcal{L}}_0, \ldots, \overline{\mathcal{L}}_n)$ on an integral model $\mathcal{X}$ of X over O_K. Adding a small positive multiple of an arithmetically ample line bundle on $\mathcal{X}$ to each $\overline{\mathcal{L}}_i$, we may assume that each $\overline{\mathcal{L}}_i$ is arithmetically ample. By [39, Corollary 5.7(2)], replacing $\overline{\mathcal{L}}_i$ by a positive multiple if necessary, $\overline{\mathcal{L}}_i$ is a hermitian line bundle such that $H^0(\mathcal{X}, \mathcal{L}_i)$ has a $\mathbb{Z}$-basis consisting of small sections. Let ℓ_0 be such a small section of $\mathcal{L}_0$. Then $\|\ell_0\|_{\sup} < 1$. We have

$$\begin{aligned}\overline{\mathcal{L}}_0 \cdot \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_n &= \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_n \cdot [\mathrm{div}(\ell_0)] - \int_{\mathcal{X}(\mathbb{C})} \log \|\ell_0\| c_1(\overline{\mathcal{L}}_1) \cdots c_1(\overline{\mathcal{L}}_n) \\ &\geq \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_n \cdot [\mathrm{div}(\ell_0)].\end{aligned}$$

Write $\mathrm{div}(\ell_0)$ as a linear combinations of integral subvarieties of $\mathcal{X}$. The problem is reduced to a lower dimension. □

The second one is the openness of arithmetic ampleness.

Proposition 5.7 *Let $\mathcal{X} \to \mathrm{Spec}\, O_K$ be an arithmetic variety, and $\overline{\mathcal{M}}, \overline{\mathcal{N}}_1, \ldots, \overline{\mathcal{N}}_r$ be hermitian $\mathbb{Q}$-line bundles on $\mathcal{X}$. Assume that $\mathcal{X}_K$ is regular and $\overline{\mathcal{M}}$ is arithmetically ample. Then the hermitian $\mathbb{Q}$-line bundle*

$$\overline{\mathcal{M}} + \sum_i \epsilon_i \overline{\mathcal{N}}_i, \qquad \epsilon_i \in \mathbb{Q}$$

is arithmetically ample when $\max |\epsilon_i|$ *is sufficiently small.*

Proof This can be proved by the same argument as in the geometric case in [25, Proposition 1.3.7]. First, we claim that there is a positive integer m such that $m\overline{\mathcal{M}} \pm \overline{\mathcal{N}}_i$ are all arithmetically ample. Then for each r-tuple $(\epsilon_1, \ldots \epsilon_r)$ of sufficiently small ϵ_i, the sum $\overline{\mathcal{M}} + \sum_i \epsilon_i \overline{\mathcal{N}}_i$ can be written as positive linear combinations of $\overline{\mathcal{M}} \pm \frac{1}{m}\mathcal{N}_i$.

Now we prove the claim. To ease the notation, let $\overline{\mathcal{N}}$ be a general hermitian line bundle on $\mathcal{X}$, and it suffices to prove that $m\overline{\mathcal{M}} + \overline{\mathcal{N}}$ is arithmetically ample for some positive integer m. Note that for sufficiently large m, $m\mathcal{M} + \mathcal{N}$ is ample and its curvature is positive definite. By Lemma 5.4, it suffices to check the positivity of the intersection numbers with horizontal subschemes. The rest of the proof is similar to that of Lemma 5.4. In fact, by the arithmetic Nakai–Moishezon criterion of [39, Corollary 4.8], for sufficiently large m, the hermitian line bundle $\Gamma(\mathcal{X}, m\mathcal{M} + \mathcal{N})$ has a $\mathbb{Z}$-basis with supremum norms less than 1. Using this basis to compute intersection numbers, we finish the proof. □

The third one is the nefness of certain limit of nef bundles.

Proposition 5.8 *Let X be a projective variety over a number field K. Suppose for each $m \in \mathbb{N}$, there is an equality of adelic $\mathbb{Q}$-line bundles as follows:*

$$\overline{M}_m - \overline{M} = C_m + \sum_{i=1}^r a_{i,m} \overline{N}_i, \quad C_m \in C(X), \quad a_{i,m} \in \mathbb{Q}.$$

Assume the following conditions:

(1) *M is ample, and every $\overline{M}_m$ is nef;*
(2) *$C_m \to 0$ and $a_{i,m} \to 0$ as $m \to \infty$.*

Then $\overline{M}$ is nef.

Proof We need to construct a sequence $\overline{M}'_m$ of nef adelic $\mathbb{Q}$-line bundles with the same underlying bundle M such that $\overline{M}'_m$ converges to $\overline{M}$ (in the sense that the metric of $\overline{M}'_m$ converges to that of $\overline{M}$).

Let $\overline{M}^0$ and $\overline{N}_i^0$ be any integrable metrized $\mathbb{Q}$-line bundles with underlying $\mathbb{Q}$-line bundles M and N_i. Let $\{\epsilon_m\}_m$ be a sequence of positive rational numbers convergent to 0, with some conditions to be imposed later. Denote

$$\overline{M}'_m := (1-\epsilon_m)\overline{M}_m + \epsilon_m \overline{M}^0 - (1-\epsilon_m)\sum_{i=1}^{r} a_{i,m}\overline{N}_i^0.$$

We see that the underlying line bundle of $\overline{M}'_m$ is exactly M, and that $\overline{M}'_m$ converges to $\overline{M}$ (in the metric sense). It remains to pick $\overline{M}^0$, $\overline{N}_i^0$ and ϵ_m so that $\overline{M}'_m$ is nef. Since $\overline{M}_m$ is nef, it suffices to find ϵ_m such that

$$\epsilon_m \overline{M}^0 - (1-\epsilon_m)\sum_{i=1}^{r} a_{i,m}\overline{N}_i^0$$

is nef for sufficiently large m. Equivalently, we need

$$\overline{M}''_m := \overline{M}^0 - (\epsilon_m^{-1}-1)\sum a_{i,m}\overline{N}_i^0 = \overline{M}^0 + \sum b_{i,m}\overline{N}_i^0$$

to be nef for sufficiently large m, where

$$b_{i,m} = -(\epsilon_m^{-1}-1)a_{i,m} = -\frac{a_{i,m}}{\epsilon_m}(1-\epsilon_m).$$

Take $\epsilon_m = \max_i |a_{i,m}|^{1/2}$, then $b_{i,m} \to 0$. Thus $\overline{M}''_m$ is in a small neighborhood of $\overline{M}_0$ in a finite-dimensional vector space of hermitian $\mathbb{Q}$-line bundles. We have reduced the problem to the following lemma. □

Lemma 5.9 *Let X be a projective variety over a number field K. Let $M, N_1, \ldots, N_r$ be line bundles on X with M ample. Then there are integrable adelic metric structures $\overline{M}, \overline{N}_1, \ldots, \overline{N}_r$ of $M, N_1, \ldots, N_r$ such that the adelic $\mathbb{Q}$-line bundle*

$$\overline{M} + \sum_i b_i \overline{N}_i, \quad b_i \in \mathbb{Q}$$

is nef when $\max |b_i|$ *is sufficiently small.*

Proof We first assume that X is regular. Take an integral model $\mathcal{X}$ of X over O_K. Extend $M, N_1, \ldots, N_r$ to hermitian line bundles $\overline{\mathcal{M}}, \overline{\mathcal{N}}_1, \ldots, \overline{\mathcal{N}}_r$ on $\mathcal{X}$. We can further assume that $\overline{\mathcal{M}}$ is arithmetically ample.

In fact, replacing M by a positive multiple if necessary, we can assume that global sections of M give a closed embedding $i : X \to \mathbb{P}^m_K$ with $M \simeq i^*\mathcal{O}(1)$. Take $\mathcal{X}$ to be the Zariski closure of X in $\mathbb{P}^m_{O_K}$, and take $\overline{\mathcal{M}}$ to be the restriction of the standard arithmetically ample line bundle $\overline{\mathcal{O}}_{\mathbb{P}^m_{O_K}}(1)$ (endowed with the Fubini–Study metric).

Take $\overline{M}, \overline{N}_1, \ldots, \overline{N}_r$ to be the adelic line bundles induced by $\overline{\mathcal{M}}, \overline{\mathcal{N}}_1, \ldots, \overline{\mathcal{N}}_r$. Then the conclusion follows from the openness of arithmetic ampleness in Proposition 5.7.

If X is singular, it suffices to prove that there is a closed embedding from X to a regular projective variety P such that some multiple of N_i can be extended to a line bundle on P, and some positive multiple of M can be extended to an ample line bundle on P.

In fact, take a positive integer t such that $N_{0,t} := tM$ and $N_{\pm i,t} := tM \pm N_i$ are all very ample, and form projective embeddings $X \to \mathbb{P}_i$ using sections of $N_{i,t}$ for $i \in \{-r, r\}$. Define P to be the product of all $\mathbb{P}_i$ and denote by $\pi_i : P \to \mathbb{P}_i$ the projections. Then $N_{i,t}$ extends to the line bundle $N'_{i,t} = \pi_i^*\mathcal{O}_{\mathbb{P}_i}(1)$. Hence $2N_i$ extends to the line bundle $N'_{i,t} - N'_{-i,t}$, and $(2r+1)tM$ extends to the ample bundle $\sum_{i=-r}^{r} N'_{i,t}$. □

Flat metrics

In subsection we review a construction of flat metrics for numerically trivial line bundles. Most of the subsection will treat the local case, except that in the end we extend the definition to the global case. Let K be an algebraically closed field equipped with a complete and non-trivial absolute value.

Definition 5.10 Let X be a projective variety over K, and $\overline{L} = (L, \|\cdot\|)$ an integrable metrized line bundle on X. We say that $\overline{L}$ (resp. $\|\cdot\|$) is flat if for any morphism $f : C \to X$ from a projective curve C over K, the measure $c_1(f^*\overline{L}) = 0$ on C^{an}.

If $\overline{L}$ is flat, then L is *numerically trivial* in the sense that $\deg L|_C = 0$ for any curve C in X. This follows from the identity

$$\deg(L|_C) = \int_{C^{\mathrm{an}}} c_1(\overline{L}|_C) = 0.$$

The main result of this section is the converse:

Theorem 5.11 *Let K be a complete field, X a geometrically connected and geometrically normal projective variety over K, and L a numerically trivial line bundle on X. Then L has a flat metric. Moreover such a metric is unique up to constant multiple.*

The uniqueness part of the theorem is equivalent to that any flat metric $\|\cdot\|$ on O_X gives the constant function $\|1\|$. Any two closed points x_1, x_2 on X can be connected by a curve Y. Let C be the normalization of Y, it is reduced to prove the constancy of $\|1\|$ on C. This is one-dimension case of our local Hodge index theorem proved in Sect.1.2.

Remark 5.12 If $K = \mathbb{C}$ and X is smooth, then $c_1(L) = 0$ in $H^2(X, \mathbb{Q})$, and the existence of the flat metric follows from complex algebraic geometry. See [16].

Remark 5.13 If K is a discrete valuation field and X is a smooth variety over K, the existence of flat metric has been proved by Künnemann [24] by constructing regular models of albanese $\mathrm{alb}(X)$. If $\dim X = 1$ with a regular (projective and flat) model $\mathcal{X}$, then the existence of flat metric follows from a model $\mathcal{M}$ for L^e $(e \geq 1)$ which has zero degree on every fiber of $\mathcal{X}$, see [13].

Before we give a construction of the flat metric in the general case, we formulate some functorial properties under finite and flat morphisms.

Proposition 5.14 *Let $f : X \to Y$ be a morphism of projective varieties over K. Let $\overline{L}$, $\overline{M}$ be integrable metrized line bundles on X and Y respectively.*

(1) If $\overline{M}$ is flat, then $f^*\overline{M}$ is flat.
(2) If f is surjective and $f^*\overline{M}$ is flat, then $\overline{M}$ is flat on X.

Proof The first part is trivial. For the second part, we notice that any curve $C \to Y$ is dominated by a curve $D \to X$ in $C \times_Y X$ with a finite morphism $\pi : D \to C$. Thus

$$c_1(L|_C) = \frac{1}{\deg \pi} \pi_* c_1(f^* L|_D).$$

□

In the following, we prove the existence of flat metrics in Theorem 5.11 by construction using algebraic dynamical systems.

Abelian varieties

Proposition 5.15 *Let A be an abelian variety over K. For any algebraically trivial line bundle L on A, there is an integrable metric $\|\cdot\|$ on $\overline{L}$, unique up to constant multiples, called the admissible metric, satisfying the isometry $([2]^*\overline{L}) \otimes \overline{L}(0) \simeq 2\overline{L}$. Here $\overline{L}(0)$ is the restriction of $\overline{L}$ via the identity of A, and viewed as a metrized line bundles on A via pull-back by the structure morphism.*

Proof Replacing $\overline{L}$ by $\overline{L} \otimes \overline{L}(0)^{-1}$, we have a canonical isomorphism $[2]^*\overline{L} = 2\overline{L}$. We may use Tate's limiting argument on $\overline{L}$ as in [40] to define the metric. We refer to Chambert-Loir [7, §2] and Gubler [19, 10.4] for discussions of the integrability. □

Theorem 5.16 *Let A be an abelian variety over K and L an algebraically trivial line bundle on A. Then any admissible metric $\|\cdot\|$ is flat.*

Proof Let $\overline{L} = (L, \|\cdot\|)$ denote the corresponding integrable metrized line bundle and let $C \to A$ be a smooth projective curve. We want to show that $c_1(\overline{L}|_C) = 0$.

If $K = \mathbb{C}$, L has a smooth metric $\|\cdot\|'$ with $c_1(L, \|\cdot\|') = 0$. It is clear that this metrized line bundle is admissible. By the uniqueness of admissible metrics, $\|\cdot\|/\|\cdot\|'$ is constant. Then $c_1(\overline{L}) = 0$. Thus $\overline{L}$ is flat.

Assume that K is non-archimedean. Fix a point $x_0 \in C(K)$ mapping to $0 \in A$. Denote the Jacobian variety of C by J, and the canonical embedding by

$$i_C : C \longrightarrow \mathrm{Jac}(C), \quad i_C(x) = O_C(x - x_0).$$

Then the morphism $C \to A$ factorizes through i_C. Let $\overline{M}$ denote the pull-back of $\overline{L}$ on J. Then it is easy to check that $\overline{M}$ is admissible. Apply [19, Remark 3.14]. We obtain $c_1(\overline{M}|_C) = 0$. This finishes the proof. □

General case

We are ready to prove Theorem 5.11. It is well-known that some multiple eL of L is algebraically trivial (cf. [22, Theorem 9.6.3]). Thus we may assume that L is algebraically trivial. The plan is to map X to its "Albanese variety".

By [22, Proposition 9.5.3, Theorem 9.5.4], the Picard functor $\underline{\mathrm{Pic}}^0_{X/K}$ is represented by a projective and irreducible group scheme $\mathrm{Pic}^0_{X/K}$ over K. Denote by $A = \mathrm{Pic}^0_{X/K,\mathrm{red}}$ the maximal reduced subscheme of $\mathrm{Pic}_{X/K}$. Then A is is an abelian variety over K. Fix an element $x_0 \in X(K)$.

Under the assumptions, there is a universal line bundle Q on $X \times A$ with rigidifications on $x_0 \times A$ and $X \times \{0\}$ which represents the functor $\underline{\mathrm{Pic}}^{0,x_0}_{X/K,\mathrm{red}}$ over the category of reduced schemes over K, which takes a reduced scheme S over K to the set of the isomorphism classes of line bundles M on $X \times S$ with a rigidification on $\{x_0\} \times S$ such that M is algebraically trivial over any geometric fiber of $X \times S \to S$. Let $\xi \in A(K)$ denote the point corresponding to $L \otimes L(0)^{-1}$.

Notice that every $x \in X(K)$ also defines an algebraically trivial line bundle $Q|_{x \times A}$. Thus we have a morphism $i : X \to A^\vee$, where $A^\vee$ is the dual of A. This morphism is known as *the Albanese map of* X. It is easy to see that the bundle Q is the pull-back of the Poincaré bundle P on $A^\vee \times A$ via the morphism

$$i \times 1_A : \quad X \times A \longrightarrow A^\vee \times A.$$

In particular, we have the following identity

$$L \otimes L(0)^{-1} = Q|_{X \times \{\xi\}} = i^*(P|_{A \times \{\xi\}}).$$

In other words, $L = i^* M$ where M is an algebraically trivial bundle on $A^\vee$ defined by $M = P_{A \times \{\xi\}} \otimes L(0)$. By Theorem 5.16, M has a flat metric. It follows from Proposition 5.14 that L has a flat metric.

Curvatures and intersections

Here we introduce a proposition which justifies the term "flat" in terms of intersections.

Proposition 5.17 *Let K be a complete field, and X be a geometrically connected and geometrically normal projective variety over K. Let $\overline{M}$ be a flat metrized line bundle on X. Then for any integrable metrized line bundles $\overline{L}_1, \ldots, \overline{L}_{n-1}$ on X,*

$$c_1(\overline{M})c_1(\overline{L}_1)\cdots c_1(\overline{L}_{n-1}) = 0$$

as a measure on X^{an}. Here $n = \dim X$.

Proof By the proof of Theorem 5.11, there is a finite extension K' of K so that $\overline{M}_{K'}$ is the pull-back of an admissible metrized line bundle $\overline{N}$ on the Albanese $A^\vee$. If K is archimedean, then $c_1(\overline{N}) = 0$. The above identity is trivial. If K is non-archimedean, this is just [19, Remark 3.14]. Note that the statement in [19] assumes that X is smooth, but it quotes back to [17], which does not require the smoothness (for our purpose). Thus the result holds in the above generality. (Alternatively, one can prove the theorem directly by induction on n using Bertini's theorem on integral models of X.) □

Flat adelic line bundles

In the end, we sketch a global theory of flat metrics.

Let X be a normal projective variety over $\bar{\mathbb{Q}}$. An adelic line bundle $\overline{M}$ on X is called *flat* if M is numerically trivial and the adelic metric is flat at every place. In this case, we also call the adelic metric *flat*. Globally, we have the following result.

Proposition 5.18 *Let M be a numerically trivial line bundle on X. There M has a flat adelic metric, unique up to constant metrics in $\widehat{\mathrm{Pic}}(\bar{\mathbb{Q}})$.*

Proof As in the local case, we can assume that $\overline{M}$ is induced by a flat adelic line bundle on $A^\vee$ via the Albanese map $i : X \to A^\vee$. Then $\overline{M}$ is integrable by [7, Corollary 2.2]. □

As a dilation, we remark that $\overline{M}$ is usually not nef (in spit of the above properties). In fact, Theorem 3.2 in the main body of this paper simply implies that $\overline{M}$ is nef only if it is constant.

We will need the following result.

Proposition 5.19 *Let X be as above, and write $n = \dim X$. Let*

$$\overline{M}, \overline{N}, \overline{L}_1, \ldots, \overline{L}_n, \quad \overline{M}', \overline{N}', \overline{L}'_1, \ldots, \overline{L}'_n$$

be integrable adelic line bundles on X. Assume that $\overline{M}, \overline{N}, \overline{M}', \overline{N}'$ are flat, and the underlying line bundles $M = M'$, $N = N'$ and $L_i = L'_i$ for every $i = 1, \ldots, n$. Then the following are true:

(1)

$$\overline{M} \cdot \overline{L}_1 \cdots \overline{L}_n = \overline{M} \cdot \overline{L}'_1 \cdots \overline{L}'_n.$$

(2)

$$\overline{M} \cdot \overline{N} \cdot \overline{L}_1 \cdots \overline{L}_{n-1} = \overline{M}' \cdot \overline{N}' \cdot \overline{L}'_1 \cdots \overline{L}'_{n-1}.$$

Proof We first prove (1). By induction, it suffices to prove

$$\overline{M} \cdot \overline{L}_1 \cdots \overline{L}_{n-1} \cdot \overline{L}_n = \overline{M} \cdot \overline{L}_1 \cdots \overline{L}_{n-1} \cdot \overline{L}'_n.$$

Assume that everything is defined over a model X_K of X over a number field K. For any place v of K, the quotient of the metrics of $\overline{L}'_n$ and $\overline{L}_n$ at v is of the form e^{-f_v} for some continuous function $f_v : X^{\mathrm{an}}_{K_v} \to \mathbb{C}$. The function $f_v = 0$ for all but finitely many v. Then

$$\begin{aligned}\overline{M} \cdot \overline{L}_1 \cdots \overline{L}_{n-1} \cdot (\overline{L}'_n - \overline{L}_n) &= \sum_v \int_{X^{\mathrm{an}}_v} f_v\, c_1(\overline{M}_v) c_1(\overline{L}_{1,v}) \cdots c_1(\overline{L}_{n-1,v}) \\ &= 0.\end{aligned}$$

Here the first equality follows from the induction formula of Chambert-Loir–Thuillier in Sect. "Appendix (Arithmetic intersections)", and the second equality follows from Gubler's result in Proposition 5.17 (1).

Now we treat (2). Applying (1) twice, we have

$$\overline{M} \cdot \overline{N} \cdot \overline{L}_1 \cdots \overline{L}_{n-1} = \overline{M} \cdot \overline{N}' \cdot \overline{L}'_1 \cdots \overline{L}'_{n-1} = \overline{M}' \cdot \overline{N}' \cdot \overline{L}'_1 \cdots \overline{L}'_{n-1}.$$

□

In the situation of (2), it is reasonable to denote

$$\langle M, N \rangle_{L_1, \cdots, L_{n-1}} = \overline{M} \cdot \overline{N} \cdot \overline{L}_1 \cdots \overline{L}_{n-1}.$$

For fixed $L_1, \ldots, L_{n-1}$, it is viewed as a pairing on the group of numerically trivial line bundles.

Lefschetz theorems

We list several classical results on the Picard functor and two classical Lefschetz-type results applicable to normal projective varieties over any characteristic.

Let X be a geometrically integral projective variety of dimension $n \geq 2$ over a field k. Consider the exact sequence

$$0 \longrightarrow \mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{NS}(X) \longrightarrow 0.$$

Here $\mathrm{Pic}^0(X)$ denotes the subgroup of algebraically trivial line bundles (cf. [22, Definition 9.5.9]), and $\mathrm{NS}(X)$ denotes the quotient group.

By [22, Theorem 9.4.8], the Picard functor $\underline{\mathrm{Pic}}_{X/k}$ is represented by a separated group scheme, locally of finite type over k. Denote by $\underline{\mathrm{Pic}}^0_{X/k}$ the identity component of (the group scheme representing) $\underline{\mathrm{Pic}}_{X/k}$, which is a group scheme of finite type over k.

Assume furthermore that $X(k)$ is non-empty. By [22, Theorem 9.2.5], $\underline{\mathrm{Pic}}_{X/k}(k) = \mathrm{Pic}(X)$. By [22, Proposition 9.5.10], $\underline{\mathrm{Pic}}^0_{X/k}(k) = \mathrm{Pic}^0(X)$.

By [36, Exp. XIII, Theorem 5.1], $\mathrm{NS}(X_{\bar{k}})$ is a finitely generated abelian group. The result can be passed to X. In fact, $\underline{\mathrm{Pic}}^0_{X/k}$ is a connected scheme with a rational point, so $\underline{\mathrm{Pic}}^0_{X/k} \times_{Speck} Spec\bar{k}$ is also connected and thus isomorphic to $\underline{\mathrm{Pic}}^0_{X_{\bar{k}}/\bar{k}}$. This implies that a line bundle L on X is algebraically trivial if and only if the base change $L_{\bar{k}}$ on $X_{\bar{k}}$ is algebraically trivial. Consequently, the natural map $\mathrm{NS}(X) \to \mathrm{NS}(X_{\bar{k}})$ is injective. Then $\mathrm{NS}(X)$ is also finitely generated.

Recall that a line bundle L on X is numerically trivial if $L \cdot C = 0$ for any closed curve C in X. It is known that a line bundle L is numerically trivial if and only if the multiple mL is algebraically trivial for some nonzero integer m. See [22, Theorem 9.6.3] for the case that k is algebraically closed. To pass from $X_{\bar{k}}$ to X, it suffices to note that a line bundle L on X is numerically trivial if and only if the base change $L_{\bar{k}}$ on $X_{\bar{k}}$ is numerically trivial, which can be checked by the projection formula of intersection numbers.

The first Lefschetz-type theorem which we introduce here is the Hodge index theorem applicable to any projective variety.

Theorem 5.20 ([36], Exposé XIII, Corollary 7.4) *Let $L_1, \ldots, L_{n-1}$ be ample line bundles on an integral projective variety X of dimension $n \geq 2$ over a field k. For any $M \in \mathrm{Pic}(X)$ with $M \cdot L_1 \cdots L_{n-1} = 0$, one has*

$$M^2 \cdot L_1 \cdots L_{n-2} \leq 0.$$

Moreover, the equality holds if and only if M is numerically trivial.

Note that the original result is only stated in the case $L_1 = \cdots = L_{n-1}$, but the proof works in the current case without much more effort. It also assumes that k is algebraically closed, but it is easy to pass from $\bar{k}$ to k as above.

The second Lefschetz-type theorem is the Lefschetz hyperplane theorem for normal varieties. Let X be a normal projective variety over an infinite field k with a very ample line bundle L. By a general hyperplane section Y of L in X, we mean the divisor of an element in a Zariski open and dense subset of $\mathbb{P}(\Gamma(X, L))$. By the Bertini-type result of Seidenberg [32, Theorem 7', p. 376], if X is normal, then a general hyperplane section Y is also normal. The following is the Lefschetz hyperplane theorem in the current setting.

Theorem 5.21 *Let X be a geometrically integral and geometrically normal projective variety of dimension n over a field k. Assume that $X(k)$ is non-empty. Let L be a very ample line bundle on X. Let Y be a general hyperplane section of L.*

(1) *The natural map* $\mathrm{Pic}^0(X) \to \mathrm{Pic}^0(Y)$ *has a finite kernel if* $n \geq 2$.
(2) *The natural map* $\mathrm{NS}(X) \to \mathrm{NS}(Y)$ *has a finite kernel if* $n \geq 3$.
(3) *The natural map* $\mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ *has a finite kernel if* $n \geq 3$.

Proof Part (3) is a consequence of (1) and (2). For (1), we refer to [22, Remark 9.5.8] for a historical account of the finiteness of the morphism $\underline{\mathrm{Pic}}^0_{X/k} \to \underline{\mathrm{Pic}}^0_{Y/k}$. It gives what we need by taking k-points. Part (2) is a consequence of Theorem 5.20. In fact, assume that M lies in the kernel of $\mathrm{NS}(X) \to \mathrm{NS}(Y)$. In Theorem 5.20, set $L_1 = \cdots = L_{n-1} = \mathcal{O}(Y)$. We see that M is numerically trivial on X. Then some integer multiple of M lies in $\mathrm{Pic}^0(X)$. Hence, the kernel of $\mathrm{NS}(X) \to \mathrm{NS}(Y)$ is a torsion subgroup. It must be finite since $\mathrm{NS}(X)$ is a finitely generated abelian group. □

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