

GLOBAL WELL-POSEDNESS OF THE ONE-DIMENSIONAL COMPRESSIBLE NAVIER–STOKES EQUATIONS WITH CONSTANT HEAT CONDUCTIVITY AND NONNEGATIVE DENSITY*

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Abstract. In this paper we consider the initial-boundary value problem to the one-dimensional compressible Navier–Stokes equations for ideal gases. Both the viscous and heat conductive coefficients are assumed to be positive constants, and the initial density is allowed to have vacuum. Global existence and uniqueness of strong solutions is established for any H^2 initial data, which generalizes the well-known result of Kazhikhov and Shelukhin [*J. Appl. Math. Mech.*, 41 (1977), pp. 273–282] to the case that with nonnegative initial density. An observation to overcome the difficulty caused by the lack of the positive lower bound of the density is that the ratio of the density to its initial value is inversely proportional to the time integral of the upper bound of the temperature along the trajectory.

Key words. compressible Navier–Stokes equations, global well-posedness, heat conductivity

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1. Introduction.

1.1. The compressible Navier–Stokes equations. In this paper, we consider the following one-dimensional heat conductive compressible Navier–Stokes equations:

$$(1.1) \quad \partial_t \rho + \partial_x(\rho u) = 0,$$

$$(1.2) \quad \rho(\partial_t u + u \partial_x u) - \mu \partial_x^2 u + \partial_x p = 0,$$

$$(1.3) \quad c_v \rho(\partial_t \theta + u \partial_x \theta) + \partial_x u p - \kappa \partial_x^2 \theta = \mu(\partial_x u)^2,$$

where ρ, u, θ , and p , respectively, denote the density, velocity, absolute temperature, and pressure. The viscous coefficient μ and heat conductive coefficient κ are assumed to be positive constants. The state equation for the ideal gas reads as

$$p = R\rho\theta,$$

where R is a positive constant.

The compressible Navier–Stokes equations have been extensively studied. In the absence of vacuum, i.e., the case that the initial density is uniformly bounded away from zero, global well-posedness of strong solutions to the one dimensional compressible Navier–Stokes equations has been well-known since the pioneer works by Kazhikhov and Shelukhin [19] and Kazhikhov [18]. Inspired by these works, global existence and uniqueness of weak solutions were later established by Zlotnik and Amosov [40, 41] and Chen, Hoff, and Trivisa [2] for the initial boundary value problems, and

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by Jiang and Zlotnik [17] for the Cauchy problem. Large time behavior of solutions to the one dimensional compressible Navier–Stokes equations with large initial data was recently proved by Li and Liang [22]. The corresponding global well-posedness results for the multidimensional case were established only for small perturbed initial data around some nonvacuum equilibrium or for spherically symmetric large initial data; see, e.g., Matsumura and Nishida [28, 29, 30, 31], Ponce [33], Valli and Zajackowski [35], Deckelnick [8], Jiang [15], Hoff [12], Kobayashi and Shibata [20], Danchin [7], Chikami and Danchin [3], and the references therein.

In the presence of vacuum, that is, the density may vanish on some set or tends to zero at the far field, global existence of weak solutions to the isentropic compressible Navier–Stokes equations was first proved by Lions [26, 27], with adiabatic constant $\gamma \geq \frac{9}{5}$, and later generalized by Feireisl, Novotný, and Petzeltová [9] to $\gamma > \frac{3}{2}$, and further by Jiang and Zhang [16] to $\gamma > 1$ for the axisymmetric solutions. For the full compressible Navier–Stokes equations, global existence of the variational weak solutions was proved by Feireisl [10, 11], which however is not applicable for the ideal gases. Local well-posedness of strong solutions to the full compressible Navier–Stokes equations, in the presence of vacuum, was proved by Cho and Kim [6]; see also Salvi and Straškraba [34], Cho, Choe, and Kim [4], and Cho and Kim [5] for the isentropic case. The solutions in [34, 4, 5, 6] are established in the homogeneous Sobolev spaces, and, generally, one can not expect the solutions in the inhomogeneous Sobolev spaces, if the initial density has compact support, due to the recent nonexistence result by Li, Wang, and Xin [21]. Global existence of strong solutions to the compressible Navier–Stokes equations, with small initial data, in the presence of initial vacuum, was first proved by Huang, Li, and Xin [14] for the isentropic case (see also Li and Xin [25] for further developments), and later by Huang and Li [13] and Wen and Zhu [37] for the full case. Due to the finite blow-up results in [38, 39], the global solutions obtained in [13, 37] must have infinite entropy somewhere in the vacuum region, if the initial density has an isolated mass group; however, if the initial density is positive everywhere but tends to vacuum at the far field, one can expect the global existence of solutions with uniformly bounded entropy to the full compressible Navier–Stokes equations; see the recent work by the author and Xin [23, 24].

An interesting problem is to show the positivity of the temperature to the full compressible Navier–Stokes equations. Along this direction, some results have been proved by Mellet and Vasseur [32] and Baer and Vasseur [1], where under some appropriate constitutive assumptions on the internal energy, the pressure law, the viscous coefficients, and the heat conductive coefficient, they have shown that the temperature has a uniform positive lower bound, with the bound depending on the time, at each time after the initial time, for the initial-boundary value problem, under the nonheat flux boundary condition. However, due to the constitutive assumptions in [32, 1], the results there do not apply to the ideal gas with constant viscous and heat conductive coefficients. Besides, the positivity of the temperature inside the space time domain to the system considered in [32, 1] is also not clear if replacing the nonheat flux temperature boundary condition with the zero temperature boundary condition.

Note that in the global well-posedness results for system (1.1)–(1.3) in [19, 18, 40, 41, 2, 17, 22], the density was assumed to be uniformly away from vacuum. Global well-posedness of strong solutions to system (1.1)–(1.3) in the presence of vacuum was proved by Wen and Zhu [36]; however, due to the following assumption,

$$\kappa_0(1 + \theta^q) \leq \kappa(\theta) \leq \kappa_1(1 + \theta^q) \quad \text{for some } q > 0,$$

made on the heat conductive coefficient κ in [36], the case that $\kappa(\theta) \equiv \text{const.}$ was not included there. The aim of this paper is to study the global well-posedness of strong

solutions to system (1.1)–(1.3), with both constant viscosity and constant diffusivity, in the presence of vacuum. This result will be proven in the Lagrangian flow map coordinate; however, it can be equivalently translated back to the corresponding one in the Euler coordinate.

1.2. The Lagrangian coordinates and main result. Let y be the Lagrangian coordinate, and define the coordinate transform between the Lagrangian coordinate y and the Euler coordinate x as

$$x = \eta(y, t),$$

where $\eta(y, t)$ is the flow map determined by u , that is,

$$\begin{cases} \partial_t \eta(y, t) = u(\eta(y, t), t), \\ \eta(y, 0) = y. \end{cases}$$

Denote by ϱ, v, ϑ , and π the density, velocity, temperature, and pressure, respectively, in the Lagrangian coordinate; that is, we define

$$\begin{aligned} \varrho(y, t) &:= \rho(\eta(y, t), t), & v(y, t) &:= u(\eta(y, t), t), \\ \vartheta(y, t) &:= \theta(\eta(y, t), t), & \pi(y, t) &:= p(\eta(y, t), t). \end{aligned}$$

Recalling the definition of $\eta(y, t)$, by straightforward calculations, one can check that

$$\begin{aligned} (\partial_x u, \partial_x \theta, \partial_x p) &= \left(\frac{\partial_y v}{\partial_y \eta}, \frac{\partial_y \vartheta}{\partial_y \eta}, \frac{\partial_y \pi}{\partial_y \eta} \right), & (\partial_x^2 u, \partial_x^2 \vartheta) &= \left(\frac{1}{\partial_y \eta} \partial_y \left(\frac{\partial_y v}{\partial_y \eta} \right), \frac{1}{\partial_y \eta} \partial_y \left(\frac{\partial_y \vartheta}{\partial_y \eta} \right) \right), \\ \partial_t \rho + u \partial_x \rho &= \partial_t \varrho, & \partial_t u + u \partial_x u &= \partial_t v, & \partial_t \theta + u \partial_x \theta &= \partial_t \vartheta. \end{aligned}$$

Define a function $J = J(y, t)$ as

$$J(y, t) = \eta_y(y, t);$$

then it follows

$$(1.4) \quad \partial_t J = \partial_y v,$$

and system (1.1)–(1.3) can be rewritten in the Lagrangian coordinate as

$$(1.5) \quad \partial_t \varrho + \frac{\partial_y v}{J} \varrho = 0,$$

$$(1.6) \quad \varrho \partial_t v - \frac{\mu}{J} \partial_y \left(\frac{\partial_y v}{J} \right) + \frac{\partial_y \pi}{J} = 0,$$

$$(1.7) \quad c_v \varrho \partial_t \vartheta + \frac{\partial_y v}{J} \pi - \frac{\kappa}{J} \partial_y \left(\frac{\partial_y \vartheta}{J} \right) = \mu \left(\frac{\partial_y v}{J} \right)^2,$$

where $\pi = R \varrho \vartheta$.

Due to (1.4) and (1.5), it holds that

$$\partial_t (J \varrho) = \partial_t J \varrho + J \partial_t \varrho = \partial_y v \varrho - J \frac{\partial_y v}{J} \varrho = 0,$$

from which, by setting $\varrho|_{t=0} = \varrho_0$ and noticing that $J|_{t=0} = 1$, we have

$$J \varrho = \varrho_0.$$

Therefore, one can replace (1.5) with (1.4), by setting $\varrho = \frac{\varrho_0}{J}$, and rewrite (1.6) and (1.7), respectively, as

$$\varrho_0 \partial_t v - \mu \partial_y \left(\frac{\partial_y v}{J} \right) + \partial_y \pi = 0$$

and

$$c_v \varrho_0 \partial_t \vartheta + \partial_y v \pi - \kappa \partial_y \left(\frac{\partial_y \vartheta}{J} \right) = \mu \frac{(\partial_y v)^2}{J}.$$

In summary, we only need to consider the following system:

$$(1.8) \quad \partial_t J = \partial_y v,$$

$$(1.9) \quad \varrho_0 \partial_t v - \mu \partial_y \left(\frac{\partial_y v}{J} \right) + \partial_y \pi = 0,$$

$$(1.10) \quad c_v \varrho_0 \partial_t \vartheta + \partial_y v \pi - \kappa \partial_y \left(\frac{\partial_y \vartheta}{J} \right) = \mu \frac{(\partial_y v)^2}{J},$$

where

$$\pi = R \frac{\varrho_0}{J} \vartheta.$$

We consider the initial-boundary value problem to system (1.8)–(1.10) on the interval $(0, L)$, with $L > 0$; that is, system (1.8)–(1.10) is defined in the space-time domain $(0, L) \times (0, \infty)$. We complement the system with the following boundary and initial conditions:

$$(1.11) \quad v(0, t) = v(L, t) = \partial_y \vartheta(0, t) = \partial_y \vartheta(L, t) = 0$$

and

$$(1.12) \quad (J, v, \vartheta)|_{t=0} = (1, v_0, \vartheta_0).$$

For $1 \leq q \leq \infty$ and positive integer m , we use $L^q = L^q((0, L))$ and $W^{m,q} = W^{m,q}((0, L))$ to denote the standard Lebesgue and Sobolev spaces, respectively, and in the case that $q = 2$, we use H^m instead of $W^{m,2}$. We always use $\|u\|_q$ to denote the L^q norm of u .

The main result of this paper is the following theorem.

THEOREM 1.1. *Given $(\varrho_0, v_0, \vartheta_0) \in H^2((0, L))$, satisfying*

$$\begin{aligned} \varrho_0(y) &\geq 0, \quad \vartheta(y) \geq 0 \quad \forall y \in [0, L], \\ v_0(0) &= v_0(L) = \partial_y \vartheta_0(0) = \partial_y \vartheta_0(L) = 0. \end{aligned}$$

Assume that the following compatibility conditions hold:

$$\begin{aligned} \mu v_0'' - R(\varrho_0 \vartheta_0)' &= \sqrt{\varrho_0} g_0, \\ \kappa \vartheta_0'' + \mu (v_0')^2 - R v_0' \varrho_0 \vartheta_0 &= \sqrt{\varrho_0} h_0 \end{aligned}$$

for two functions $g_0, h_0 \in L^2((0, L))$.

Then, there is a unique global solution (J, v, ϑ) to system (1.8)–(1.10), subject to (1.11)–(1.12), satisfying $J > 0$ and $\vartheta \geq 0$ on $[0, L] \times [0, \infty)$, and

$$\begin{aligned} J &\in C([0, T]; H^2), \quad \partial_t J \in L^2(0, T; H^2), \\ v &\in C([0, T]; H^2) \cap L^2(0, T; H^3), \quad \partial_t v \in L^2(0, T; H^1), \\ \vartheta &\in C([0, T]; H^2) \cap L^2(0, T; H^3), \quad \partial_t \vartheta \in L^2(0, T; H^1) \end{aligned}$$

for any $T \in (0, \infty)$.

Remark 1.1. The same result as in Theorem 1.1 still holds if replacing the boundary condition $\partial_y \vartheta(0, t) = \partial_y \vartheta(L, t) = 0$ by one of the following three,

$$\begin{aligned}\vartheta(0, t) &= \vartheta(L, t) = 0, \\ \vartheta(0, t) &= \partial_y \vartheta(L, t) = 0, \\ \partial_y \vartheta(0, t) &= \vartheta(L, t) = 0,\end{aligned}$$

and the proof is exactly the same as the one presented in this paper; the only difference is that the basic energy identity in Proposition 2.2 will then be an inequality.

Remark 1.2. The argument presented in this paper also works for the free boundary value problem to the same system. Because, if rewritten the system in the Lagrangian coordinates, the only difference between the initial boundary value problem and the free boundary value problem is the boundary conditions for v : in the free boundary problem, the boundary conditions for v in (1.11) are replaced by

$$\mu \frac{\partial_y v}{J} - \pi \Big|_{y=0, L} = 0.$$

Note that all the energy estimates obtained in this paper hold if replacing the boundary condition on v in (1.11) with the above ones, by slightly modifying the proof.

The argument used in Kazhikhov and Shelukhin [19], in which the nonvacuum case was considered, does not apply directly to the vacuum case. One main observation in [19] is the lower bound of the density is inversely proportional to the time integral of the upper bound of the temperature along the trajectory. Note that this only holds for the case that the density has a positive uniform lower bound. To overcome the difficulty caused by the lack of the positive lower bound of the density (it is of this case if the lower bound of the initial density is zero), our observation is the ratio of the density to its initial value is inversely proportional to the time integral of the temperature, along the trajectory, or, equivalently, the upper bound of J is proportional to the time integral of the upper bound of the temperature. This observation holds for both the vacuum and nonvacuum cases, which, in particular, reduces to the one in [19] for the nonvacuum case; this also indicates the advantage of taking J rather than ϱ as one of the unknowns.

The key issue of proving Theorem 1.1 is to establish the appropriate a priori energy estimates, up to any finite time, of the solutions to system (1.8)–(1.10), subject to (1.11)–(1.12). There are four main stages for carrying out the desired a priori energy estimates. In the first stage, we derive from (1.8)–(1.9) an identity

$$1 + \frac{R}{\mu} \varrho_0(y) \int_0^t \vartheta(y, \tau) H(\tau) B(y, \tau) d\tau = J(y, t) H(t) B(y, t)$$

for some functions $H(t)$ and $B(y, t)$. The temperature equation is not used at all in deriving the above identity, and this identity is in the spirit of the one in [19], but in different Lagrangian coordinates. The basic energy estimate implies that both H and B are uniformly away from zero and uniformly bounded, up to any finite time. As a direct corollary of the above identity, one can obtain the uniform positive lower bound of J , and the control of the upper bound of J in terms of the time integral of ϑ . By using the positive lower bound of J , one obtains a density-weighted embedding inequality (which can be viewed as the replacement of the Sobolev embedding inequality when the vacuum is involved) for ϑ ; see (ii) of Proposition 2.4, which implies that

the upper bound of $\sqrt{\varrho_0}\vartheta$ can be controlled by that of J , up to a small dependence on $\|\frac{\partial_y \vartheta}{\sqrt{J}}\|_2$, i.e., the term $\eta\|\frac{\partial_y \vartheta}{\sqrt{J}}\|_2$, with a small positive η . This combined with the above identity leads us to carry out the $L^\infty(L^2)$ type estimates on ϑ , or, more precisely, on $\sqrt{\varrho_0}\vartheta$. In the second stage, we carry out the $L^\infty(L^2)$ energy estimate on $\sqrt{\varrho_0}\vartheta$, and, at the same time, the $L^\infty(L^2)$ energy estimate on v will be involved naturally, due to the coupling structure between v and ϑ in the system; as a conclusion of this stage, by making use the control relationship between the upper bounds of $\sqrt{\varrho_0}\vartheta$ and J obtained in the first stage, we are able to obtain the a priori upper bound of J and the a priori $L^\infty(L^2) \cap L^2(H^1)$ type estimates on (v, ϑ) . In the third stage, by investigating the effective viscous flux $G := \mu \frac{\partial_y v}{J} - \pi$ and working on its $L^\infty(L^2) \cap L^2(H^1)$ type a priori estimate, we are able to get the a priori $L^\infty(H^1)$ estimate on (J, v) ; however, due to the presence of the term $\frac{\mu}{J}(\partial_y v)^2$ and the degeneracy of the leading term $\varrho_0 \partial_t \vartheta$ in the ϑ equation, we are not able to obtain the corresponding $L^\infty(H^1)$ estimate on ϑ , without appealing to higher order energy estimates than H^1 . In the fourth stage, we carry out the a priori $L^\infty(H^2)$ type estimates, which are achieved through performing the $L^\infty(L^2)$ type energy estimate on $\sqrt{\varrho_0} \partial_t \vartheta$ and $L^\infty(H^1)$ type estimate on G . It should be mentioned that the desired a priori $L^\infty(H^2)$ estimates on ϑ is obtained without knowing its a priori $L^\infty(H^1)$ bound in advance.

The rest of this paper is arranged as follows: in the next section, section 2, which is the main part of this paper, we consider the global existence and the a priori estimates to system (1.8)–(1.10), subject to (1.11)–(1.12), in the absence of vacuum, while Theorem 1.1 is proven in the last section.

Throughout this paper, we use C to denote a general positive constant which may differ from line to line.

2. Global existence and a priori estimates in the absence of vacuum.

We first recall the global existence results due to Kazhikohov and Shelukhin [19] stated in the following proposition. The original result in [19] was stated in the Lagrangian mass coordinates, rather than the Lagrangian map flow coordinates as here, and the initial data $(\varrho_0, v_0, \vartheta_0)$ was assumed in H^1 ; however, due to the sufficient regularities of the solutions established in [19], in particular the $L^1(0, T; W^{1,\infty})$ of v , the global existence result established there can be translated to the corresponding one in the Lagrangian map flow coordinates, and one can show that the solutions have correspondingly more regularities if the initial data has more regularities as stated in the following proposition.

PROPOSITION 2.1. *For any $(\varrho_0, v_0, \vartheta_0) \in H^2$, satisfying $v(0) = v(L) = 0$, and*

$$\min \left\{ \inf_{y \in (0, L)} \varrho_0(y), \inf_{y \in (0, L)} \vartheta_0(y) \right\} > 0,$$

there is a unique global solution (J, v, ϱ) to system (1.8)–(1.10), subject to (1.11)–(1.12), satisfying $J, \vartheta > 0$ on $(0, L) \times (0, \infty)$, and

$$\begin{aligned} J &\in C([0, T]; H^2), \partial_t J \in L^\infty(0, T; H^1) \cap L^2(0, T; H^2), \\ v &\in C([0, T]; H^2) \cap L^2(0, T; H^3), \partial_t v \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1), \\ \vartheta &\in C([0, T]; H^2) \cap L^2(0, T; H^3), \partial_t \vartheta \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1) \end{aligned}$$

for any $T \in (0, \infty)$.

In the rest of this section, we always assume that (J, v, ϑ) is the unique global solution obtained in Proposition 2.1, and we will establish a series of a priori estimates of (J, v, ϑ) independent of the lower bound of the density.

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We start with the basic energy identity in the following proposition.

PROPOSITION 2.2. *It holds that*

$$\int_0^L J(y, t) dy = L$$

and

$$\left(\int_0^L \left(\frac{\varrho_0}{2} v^2 + c_v \varrho_0 \vartheta \right) dy \right) (t) = E_0$$

for any $t \in (0, \infty)$, where $E_0 := \int_0^L \left(\frac{\varrho_0}{2} v_0^2 + c_v \varrho_0 \vartheta_0 \right) dy$.

Proof. The first conclusion follows directly from integrating (1.8) with respect to y over $(0, L)$ and using the boundary condition (1.11). Multiplying (1.9) by v , integrating the resultant over $(0, L)$, one gets from integrating by parts that

$$\frac{1}{2} \frac{d}{dt} \int_0^L \varrho_0 v^2 dy + \mu \int_0^L \frac{(\partial_y v)^2}{J} dy = \int_0^L \partial_y v \pi dy.$$

Integrating (1.10) over $(0, L)$ yields

$$c_v \frac{d}{dt} \int_0^L \varrho_0 \vartheta dy + \int_0^L \partial_y v \pi dy = \mu \int_0^L \frac{(\partial_y v)^2}{J} dy,$$

which, summed with the previous equality, leads to

$$\frac{d}{dt} \int_0^L \left(\frac{\varrho_0}{2} v^2 + c_v \varrho_0 \vartheta \right) dy = 0;$$

the second conclusion follows. \square

2.1. A priori L^2 estimates. In this subsection, we will derive the uniform positive lower and upper bounds of J and the a priori $L^\infty(0, T; L^2) \cap L^2(0, T; H^1)$ estimates of v and ϑ .

Before carrying out the desired estimates, we first derive an equality, i.e., (2.3) below, in the spirit of [19]. As will be shown later, this equality leads to the positive lower bound of J , and it will be combined with the L^2 type energy inequalities to get the upper bound of J and the a priori $L^\infty(0, T; L^2) \cap L^2(0, T; H^1)$ estimates of v and ϑ . Due to (1.8), it follows from (1.9) that

$$\varrho_0 \partial_t v - \mu \partial_y \partial_t \log J + \partial_y \pi = 0.$$

Integrating the above equation with respect to t over $(0, t)$ yields

$$\varrho_0 (v - v_0) - \mu \partial_y \log J + \partial_y \left(\int_0^t \pi d\tau \right) = 0,$$

from which, integrating with respect to y over (z, y) , one obtains

$$\int_z^y \varrho_0 (v - v_0) d\xi - \mu (\log J(y, t) - \log J(z, t)) + \int_0^t (\pi(y, \tau) - \pi(z, \tau)) d\tau = 0$$

for any $y, z \in (0, L)$. Using

$$\int_z^y \varrho_0 (v - v_0) d\xi = \int_0^y \varrho_0 (v - v_0) d\xi - \int_0^z \varrho_0 (v - v_0) d\xi,$$

and rearranging the terms, we obtain

$$\begin{aligned} & \int_0^y \varrho_0(v - v_0)d\xi - \mu \log J(y, t) + \int_0^t \pi(y, \tau)d\tau \\ &= \int_0^z \varrho_0(v - v_0)d\xi - \mu \log J(z, t) + \int_0^t \pi(z, \tau)d\tau \quad \forall y, z \in (0, L). \end{aligned}$$

Therefore, the function

$$\int_0^y \varrho_0(v - v_0)d\xi - \mu \log J(y, t) + \int_0^t \pi(y, \tau)d\tau$$

is independent of y , and we denote it by $h(t)$, that is,

$$\int_0^y \varrho_0(v - v_0)d\xi - \mu \log J + \int_0^t \pi d\tau = h(t),$$

from which, recalling $\pi = R \frac{\varrho_0}{J} \vartheta$, one gets

$$(2.1) \quad \frac{1}{J} \exp \left\{ \frac{R}{\mu} \varrho_0 \int_0^t \frac{\vartheta}{J} d\tau \right\} = H(t)B(y, t),$$

or equivalently

$$(2.2) \quad \exp \left\{ \frac{R}{\mu} \varrho_0 \int_0^t \frac{\vartheta}{J} d\tau \right\} = J(y, t)H(t)B(y, t),$$

where

$$H(t) = \exp \left\{ \frac{h(t)}{\mu} \right\}, \quad B(y, t) = \exp \left\{ -\frac{1}{\mu} \int_0^y \varrho_0(v - v_0)d\xi \right\}.$$

Multiplying (2.1) by $\frac{R}{\mu} \varrho_0 \vartheta$ one obtains

$$\partial_t \left(\exp \left\{ \frac{R}{\mu} \varrho_0 \int_0^t \frac{\vartheta}{J} d\tau \right\} \right) = \frac{R}{\mu} \varrho_0 \vartheta HB,$$

which gives

$$\exp \left\{ \frac{R}{\mu} \varrho_0 \int_0^t \frac{\vartheta}{J} d\tau \right\} = 1 + \frac{R}{\mu} \varrho_0 \int_0^t \vartheta HB d\tau.$$

Combining this with (2.2), one gets

$$(2.3) \quad 1 + \frac{R}{\mu} \varrho_0(y) \int_0^t \vartheta(y, \tau)H(\tau)B(y, \tau)d\tau = J(y, t)H(t)B(y, t)$$

for any $y \in (0, L)$ and any $t \in [0, \infty)$.

A prior positive lower bound of J and the control of the upper bound of J in terms of $\varrho_0 \vartheta$ are stated in the following proposition.

PROPOSITION 2.3. *We have the following estimate:*

$$(m_1 f_1(t))^{-1} \leq J(y, t) \leq m_1^2 + \frac{R}{\mu} m_1^3 f_1(t) \int_0^t \varrho_0 \vartheta d\tau$$

for any $y \in (0, L)$ and any $t \in [0, \infty)$, where

$$m_1 = \exp \left\{ \frac{2}{\mu} \sqrt{2 \|\varrho_0\|_1 E_0} \right\}, \quad f_1(t) = m_1 \exp \left\{ \frac{R m_1^2 E_0 t}{\mu c_v L} \right\}.$$

Proof. By Proposition 2.2, it follows from the Hölder inequality that

$$\left| \int_0^y \varrho_0(v - v_0) d\xi \right| \leq \int_0^L (|\varrho_0 v| + |\varrho_0 v_0|) d\xi \leq 2 \|\varrho_0\|_1^{\frac{1}{2}} (2E_0)^{\frac{1}{2}},$$

and, thus,

$$(2.4) \quad \underline{m} := \exp \left\{ -\frac{2}{\mu} \sqrt{2\|\varrho_0\|_1 E_0} \right\} \leq B(y, t) \leq \exp \left\{ \frac{2}{\mu} \sqrt{2\|\varrho_0\|_1 E_0} \right\} =: m_1.$$

Integrating (2.3) over $(0, L)$, it follows from (2.4) and Proposition 2.2 that

$$\begin{aligned} L &\leq \int_0^L \left(1 + \frac{R}{\mu} \varrho_0 \int_0^t \vartheta H B d\tau \right) dy = H(t) \int_0^L J B dy \\ &\leq m_1 H(t) \int_0^L J dy = m_1 L H(t) \end{aligned}$$

and

$$\begin{aligned} \underline{m} L H(t) &= H(t) \underline{m} \int_0^L J dy \\ &\leq H(t) \int_0^L J B dy = \int_0^L \left(1 + \frac{R}{\mu} \varrho_0 \int_0^t \vartheta H B d\tau \right) dy \\ &\leq L + \frac{R m_1}{\mu} \int_0^t \int_0^L \varrho_0 \vartheta dy H d\tau \leq L + \frac{R m_1 E_0}{\mu c_v} \int_0^t H d\tau. \end{aligned}$$

Therefore, we have

$$m_1^{-1} \leq H(t) \leq \underline{m}^{-1} + \frac{R m_1 E_0}{\mu c_v \underline{m} L} \int_0^t H d\tau,$$

from which, by the Gronwall inequality, one further obtains

$$(2.5) \quad m_1^{-1} \leq H(t) \leq \underline{m}^{-1} \exp \left\{ \frac{R m_1 E_0 t}{\mu c_v \underline{m} L} \right\} =: f_1(t), \quad t \in [0, T].$$

Due to (2.4) and (2.5), it follows from (2.3) that

$$1 \leq 1 + \frac{R}{\mu} \varrho_0 \int_0^t \vartheta H B d\tau = J(y, t) H(t) B(y, t) \leq J(y, t) m_1 f_1(t),$$

and

$$\begin{aligned} m_1^{-1} \underline{m} J(y, t) &\leq H(t) B(y, t) J(y, t) = 1 + \frac{R}{\mu} \varrho_0 \int_0^t \vartheta H B d\tau \\ &\leq 1 + \frac{R}{\mu} m_1 f_1(t) \int_0^t \varrho_0 \vartheta d\tau. \end{aligned}$$

Therefore, we have

$$(m_1 f_1(t))^{-1} \leq J(y, t) \leq m_1 \underline{m}^{-1} + \frac{R m_1^2 f_1(t)}{\mu \underline{m}} \int_0^t \varrho_0 \vartheta d\tau$$

for any $y \in (0, L)$ and any $t \in [0, T]$, proving the conclusion. \square

As a preparation of deriving the a priori upper bound of J and the a priori $L^\infty(0, T; L^2) \cap L^2(0, T; H^1)$ type estimates on (v, ϑ) , we prove the following proposition, which, in particular, gives the density-weighted estimate of ϑ .

PROPOSITION 2.4. *We have the following two items.*

(i) *It holds that*

$$\begin{aligned} \|\varrho_0^2 \vartheta\|_\infty^2 &\leq \left(\frac{E_0}{c_v}\right)^2 \left(\frac{8\bar{\varrho}^2}{L^2} + 32\|\varrho'_0\|_\infty^2\right) + 6\bar{\varrho}^{\frac{10}{3}} \left(\frac{E_0}{c_v}\right)^{\frac{2}{3}} \left\|\frac{\partial_y \vartheta}{\sqrt{J}}\right\|_2^{\frac{4}{3}} \|J\|_\infty^{\frac{2}{3}}, \\ \|\vartheta\|_\infty &\leq \sqrt{L} \left\|\frac{\partial_y \vartheta}{\sqrt{J}}\right\|_2 + \frac{2E_0}{c_v \omega_0 \bar{\varrho}}, \end{aligned}$$

where

$$\bar{\varrho} = \|\varrho_0\|_\infty, \quad \omega_0 = |\Omega_0|, \quad \Omega_0 := \left\{y \in (0, L) \mid \varrho_0(y) \geq \frac{\bar{\varrho}}{2}\right\}.$$

(ii) *As a consequence of (i), we have*

$$\|\sqrt{\varrho_0} \vartheta\|_\infty^2 \leq \eta \left\|\frac{\partial_y \vartheta}{\sqrt{J}}\right\|_2^2 + C_\eta (\|J\|_\infty^2 + 1)$$

for any $\eta \in (0, \infty)$, where C_η is a positive constant depending only on η and N_1 , where

$$N_1 := \frac{E_0}{c_v} + \bar{\varrho} + \frac{1}{\bar{\varrho}} + L + \frac{1}{L} + \frac{1}{\omega_0} + \|\varrho'_0\|_\infty.$$

Proof. (i) Choose $y_0 \in (0, L)$ such that

$$\varrho_0^2(y_0) \vartheta(y_0, t) \leq \frac{2}{L} \int_0^L \varrho_0^2 \vartheta d\xi \leq \frac{2\bar{\varrho}}{L} \|\varrho_0 \vartheta\|_1.$$

By the Hölder and Young inequalities, we deduce

$$\begin{aligned} (\varrho_0^2 \vartheta)^2(y, t) &\leq (\varrho_0^2 \vartheta)^2(y_0, t) + 2 \int_0^L \varrho_0^2 \vartheta |\partial_y(\varrho_0^2 \vartheta)| d\xi \\ &\leq \left(\frac{2\bar{\varrho}}{L}\right)^2 \|\varrho_0 \vartheta\|_1^2 + 2 \int_0^L (2\varrho_0^2 \vartheta \varrho_0 \vartheta |\varrho'_0| + \varrho_0^2 \vartheta \varrho_0^2 |\partial_y \vartheta|) d\xi \\ &\leq \left(\frac{2\bar{\varrho}}{L}\right)^2 \|\varrho_0 \vartheta\|_1^2 + 4\|\varrho_0^2 \vartheta\|_\infty \|\varrho_0 \vartheta\|_1 \|\varrho'_0\|_\infty + 2\bar{\varrho}^{\frac{5}{2}} \left\|\frac{\partial_y \vartheta}{\sqrt{J}}\right\|_2 \|J\|_\infty^{\frac{1}{2}} \|\varrho_0 \vartheta\|_1^{\frac{1}{2}} \|\varrho_0^2 \vartheta\|_\infty^{\frac{1}{2}} \\ &\leq \frac{1}{2} \|\varrho_0^2 \vartheta\|_\infty^2 + \left(\frac{2\bar{\varrho}}{L}\right)^2 \|\varrho_0 \vartheta\|_1^2 + 16\|\varrho_0 \vartheta\|_1^2 \|\varrho'_0\|_\infty^2 + 3\bar{\varrho}^{\frac{10}{3}} \|\varrho_0 \vartheta\|_1^{\frac{2}{3}} \left\|\frac{\partial_y \vartheta}{\sqrt{J}}\right\|_2^{\frac{4}{3}} \|J\|_\infty^{\frac{2}{3}} \end{aligned}$$

for any $y \in (0, L)$, and, thus,

$$\|\varrho_0^2 \vartheta\|_\infty^2 \leq 2 \left(\frac{2\bar{\varrho}}{L}\right)^2 \|\varrho_0 \vartheta\|_1^2 + 32\|\varrho_0 \vartheta\|_1^2 \|\varrho'_0\|_\infty^2 + 6\bar{\varrho}^{\frac{10}{3}} \|\varrho_0 \vartheta\|_1^{\frac{2}{3}} \left\|\frac{\partial_y \vartheta}{\sqrt{J}}\right\|_2^{\frac{4}{3}} \|J\|_\infty^{\frac{2}{3}},$$

from which, by Proposition 2.2, we have

$$\|\varrho_0^2 \vartheta\|_\infty^2 \leq \left(\frac{E_0}{c_v}\right)^2 \left(\frac{8\bar{\varrho}^2}{L^2} + 32\|\varrho'_0\|_\infty^2\right) + 6\bar{\varrho}^{\frac{10}{3}} \left(\frac{E_0}{c_v}\right)^{\frac{2}{3}} \left\|\frac{\partial_y \vartheta}{\sqrt{J}}\right\|_2^{\frac{4}{3}} \|J\|_\infty^{\frac{2}{3}}.$$

Noticing that

$$\vartheta(y, t) = \frac{1}{\omega_0} \int_{\Omega_0} \vartheta dz + \frac{1}{\omega_0} \int_{\Omega_0} \int_z^y \partial_y \vartheta d\xi dz,$$

we deduce, by the Hölder inequality, that

$$\begin{aligned} \|\vartheta\|_\infty &\leq \frac{1}{\omega_0} \int_{\Omega_0} \frac{\varrho_0 \vartheta}{\varrho_0} dz + \int_0^L |\partial_y \vartheta| dz \\ &\leq \frac{2}{\omega_0 \bar{\varrho}} \|\varrho_0 \vartheta\|_1 + \left(\int_0^L \left| \frac{\partial_y \vartheta}{\sqrt{J}} \right|^2 dz \right)^{\frac{1}{2}} \left(\int_0^L J dz \right)^{\frac{1}{2}}, \end{aligned}$$

from which, by Proposition 2.2, one obtains

$$\|\vartheta\|_\infty \leq \sqrt{L} \left\| \frac{\partial_y \vartheta}{\sqrt{J}} \right\|_2 + \frac{2E_0}{c_v \omega_0 \bar{\varrho}}.$$

(ii) Thanks to (i), one has

$$\|\varrho_0^2 \vartheta\|_\infty^2 \leq C \left(1 + \left\| \frac{\partial_y \vartheta}{\sqrt{J}} \right\|_2^{\frac{4}{3}} \|J\|_\infty^{\frac{2}{3}} \right), \quad \|\vartheta\|_\infty \leq C \left(\left\| \frac{\partial_y \vartheta}{\sqrt{J}} \right\|_2 + 1 \right),$$

for a positive constant C depending only on N_1 . Therefore, we have

$$\begin{aligned} \|\sqrt{\varrho_0} \vartheta\|_\infty^2 &= \left\| (\varrho_0^2 \vartheta)^{\frac{1}{4}} \vartheta^{\frac{3}{4}} \right\|_\infty^2 \leq \|\varrho_0^2 \vartheta\|_\infty^{\frac{1}{2}} \|\vartheta\|_\infty^{\frac{3}{2}} \\ &\leq C \left(1 + \left\| \frac{\partial_y \vartheta}{\sqrt{J}} \right\|_2^{\frac{4}{3}} \|J\|_\infty^{\frac{2}{3}} \right)^{\frac{1}{4}} \left(\left\| \frac{\partial_y \vartheta}{\sqrt{J}} \right\|_2 + 1 \right)^{\frac{3}{2}}, \end{aligned}$$

for a positive constant C depending only on N_1 , from which, by the Young inequality, (ii) follows. \square

We can now prove the desired a priori $L^\infty(0, T; L^2) \cap L^2(0, T; H^1)$ estimates on (v, ϑ) as in the next proposition.

PROPOSITION 2.5. *Given $T \in (0, \infty)$, it holds that*

$$\begin{aligned} \sup_{0 \leq t \leq T} (\|(\sqrt{\varrho_0} v^2, \sqrt{\varrho_0} \vartheta)\|_2^2 + \|J\|_\infty^2) + \int_0^T (\|\vartheta\|_\infty^2 + \|(\partial_y \vartheta, v \partial_y v)\|_2^2) dt \\ \leq C(1 + \|(\sqrt{\varrho_0} \vartheta_0, \sqrt{\varrho_0} v_0^2)\|_2^2) \end{aligned}$$

for a positive constant C depending only on $R, c_v, \mu, \kappa, m_1, N_1$, and T , where m_1 and N_1 are the numbers in Proposition 2.3 and Proposition 2.4, respectively.

Proof. Denote $\mathcal{E} := \frac{v^2}{2} + c_v \vartheta$. Then one can derive from (1.9) and (1.10) that

$$(2.6) \quad \varrho_0 \partial_t \mathcal{E} + \partial_y(v\pi) - \kappa \partial_y \left(\frac{\partial_y \vartheta}{J} \right) = \mu \partial_y \left(\frac{1}{J} \partial_y \left(\frac{v^2}{2} \right) \right).$$

Multiplying (2.6) by \mathcal{E} and integrating the resultant over $(0, L)$, one get from integration by parts that

$$(2.7) \quad \frac{1}{2} \frac{d}{dt} \int_0^L \varrho_0 \mathcal{E}^2 dy + \int_0^L \frac{1}{J} (\kappa \partial_y \vartheta + \mu v \partial_y v) \partial_y \mathcal{E} dy = \int_0^L v \pi \partial_y \mathcal{E} dy.$$

By the Young inequality, we have

$$\begin{aligned} \int_0^L \frac{1}{J} (\kappa \partial_y \vartheta + \mu v \partial_y v) \partial_y \mathcal{E} dy &= \int_0^L \frac{1}{J} (\kappa \partial_y \vartheta + \mu v \partial_y v) (v \partial_y v + c_v \partial_y \vartheta) dy \\ &\geq \frac{3\kappa c_v}{4} \int_0^L \left| \frac{\partial_y \vartheta}{\sqrt{J}} \right|^2 dy - C \int_0^L \left| \frac{v \partial_y v}{\sqrt{J}} \right|^2 dy \end{aligned}$$

and

$$\begin{aligned} \int_0^L v \pi \partial_y \mathcal{E} dy &= R \int_0^L v \frac{\varrho_0}{J} \vartheta (v \partial_y v + c_v \partial_y \vartheta) dy \\ &\leq \frac{\kappa c_v}{4} \int_0^L \left| \frac{\partial_y \vartheta}{\sqrt{J}} \right|^2 dy + C \int_0^L \frac{1}{J} [\varrho_0^2 v^2 \vartheta^2 + (v \partial_y v)^2] dy \end{aligned}$$

for a positive constant C depending only on $R, c_v, \mu,$ and κ . Substituting the above two inequalities into (2.7) and applying Proposition 2.2 and Proposition 2.3, one obtains

$$\begin{aligned} \frac{d}{dt} \|\sqrt{\varrho_0} \mathcal{E}\|_2^2 + \kappa c_v \left\| \frac{\partial_y \vartheta}{\sqrt{J}} \right\|_2^2 &\leq C \left(\left\| \frac{v \partial_y v}{\sqrt{J}} \right\|_2^2 + \int_0^L \frac{\varrho_0 v^2}{J} \varrho_0 \vartheta^2 dy \right) \\ &\leq C \left(\left\| \frac{v \partial_y v}{\sqrt{J}} \right\|_2^2 + \|\sqrt{\varrho_0} v\|_2^2 \|\sqrt{\varrho_0} \vartheta\|_\infty^2 m_1 f_1(t) \right) \\ &\leq C \left(\left\| \frac{v \partial_y v}{\sqrt{J}} \right\|_2^2 + E_0 m_1 f_1(t) \|\sqrt{\varrho_0} \vartheta\|_\infty^2 \right) \end{aligned}$$

for a positive constant C depending only on $R, c_v, \mu,$ and κ , and, thus,

$$(2.8) \quad \frac{d}{dt} \|\sqrt{\varrho_0} \mathcal{E}\|_2^2 + \kappa c_v \left\| \frac{\partial_y \vartheta}{\sqrt{J}} \right\|_2^2 \leq A_1 \left(\left\| \frac{v \partial_y v}{\sqrt{J}} \right\|_2^2 + E_0 m_1 f_1(t) \|\sqrt{\varrho_0} \vartheta\|_\infty^2 \right)$$

for a positive constant A_1 depending only on $R, c_v, \mu,$ and κ .

Multiplying (1.9) by $4v^3$ and integrating the resultant over $(0, L)$, one gets from integration by parts and the Young inequality that

$$\begin{aligned} &\frac{d}{dt} \int_0^L \varrho_0 v^4 dy + 12\mu \int_0^L \left| \frac{v \partial_y v}{\sqrt{J}} \right|^2 dy \\ &= 12 \int_0^L \pi v^2 \partial_y v dy = 12R \int_0^L \frac{\varrho_0}{J} \vartheta v^2 \partial_y v dy \\ &\leq 6\mu \int_0^L \left| \frac{v \partial_y v}{\sqrt{J}} \right|^2 dy + \frac{6R^2}{\mu} \int_0^L \frac{\varrho_0 v^2}{J} \varrho_0 \vartheta^2 dy, \end{aligned}$$

from which, by Proposition 2.3, the Hölder inequality, and Proposition 2.2, one obtains

$$\begin{aligned} &\frac{d}{dt} \int_0^L \varrho_0 v^4 dy + 6\mu \int_0^L \left| \frac{v \partial_y v}{\sqrt{J}} \right|^2 dy \leq \frac{6R^2}{\mu} \int_0^L \frac{\varrho_0 v^2}{J} \varrho_0 \vartheta^2 dy \\ &\leq \frac{6R^2}{\mu} m_1 f_1(t) \|\sqrt{\varrho_0} v\|_2^2 \|\sqrt{\varrho_0} \vartheta\|_\infty^2 \leq \frac{12R^2 m_1 E_0}{\mu} f_1(t) \|\sqrt{\varrho_0} \vartheta\|_\infty^2; \end{aligned}$$

that is,

$$(2.9) \quad \frac{d}{dt} \|\sqrt{\varrho_0} v^2\|_2^2 + 6\mu \left\| \frac{v \partial_y v}{\sqrt{J}} \right\|_2^2 \leq \frac{12R^2}{\mu} E_0 m_1 f_1(t) \|\sqrt{\varrho_0} \vartheta\|_\infty^2.$$

Multiplying (2.9) by $\frac{A_1}{\mu}$, adding the resultant to (2.8), and noticing that $f_1(t)$ is nondecreasing in t , one gets

$$(2.10) \quad \begin{aligned} & \frac{d}{dt} \left(\|\sqrt{\varrho_0} \mathcal{E}\|_2^2 + \frac{A_1}{\mu} \|\sqrt{\varrho_0} v^2\|_2^2 \right) + 5A_1 \left\| \frac{v \partial_y v}{\sqrt{J}} \right\|_2^2 + \kappa c_v \left\| \frac{\partial_y \vartheta}{\sqrt{J}} \right\|_2^2 \\ & \leq \left(1 + \frac{12R^2}{\mu^2} \right) A_1 E_0 m_1 f_1(t) \|\sqrt{\varrho_0} \vartheta\|_\infty^2 \leq A_2 E_0 m_1 f_1(T) \|\sqrt{\varrho_0} \vartheta\|_\infty^2(t) \end{aligned}$$

for any $t \in (0, T)$, where $A_2 = (1 + \frac{12R^2}{\mu^2}) A_1$.

By Proposition 2.3 and (ii) of Proposition 2.4, we have

$$\|\sqrt{\varrho_0} \vartheta\|_\infty^2(t) \leq \eta \left\| \frac{\partial_y \vartheta}{\sqrt{J}} \right\|_2^2(t) + C_\eta \left(1 + \int_0^t \|\sqrt{\varrho_0} \vartheta\|_\infty^2 d\tau \right)$$

for any $\eta \in (0, \infty)$, and any $t \in (0, T)$, where C_η is a positive constant depending only on $R, c_v, \mu, \kappa, m_1, N_1, T$, and η . Multiplying both sides of the above inequality by $2A_2 E_0 m_1 f_1(T)$, choosing $\eta = \frac{\kappa c_v}{4A_2 E_0 m_1 f_1(T)}$, and summing the resultant with (2.10), one obtains

$$\begin{aligned} & \frac{d}{dt} \left(\|\sqrt{\varrho_0} \mathcal{E}\|_2^2 + \frac{A_1}{\mu} \|\sqrt{\varrho_0} v^2\|_2^2 + A_2 E_0 m_1 f_1(T) \int_0^t \|\sqrt{\varrho_0} \vartheta\|_\infty^2 d\tau \right) \\ & + 5A_1 \left\| \frac{v \partial_y v}{\sqrt{J}} \right\|_2^2 + \frac{\kappa c_v}{2} \left\| \frac{\partial_y \vartheta}{\sqrt{J}} \right\|_2^2 \leq C A_2 E_0 m_1 f_1(T) \left(1 + \int_0^t \|\sqrt{\varrho_0} \vartheta\|_\infty^2 d\tau \right) \end{aligned}$$

for any $t \in (0, T)$, where C is a positive constant depending only on $R, c_v, \mu, \kappa, m_1, N_1$, and T . Applying the Gronwall inequality to the above inequality, one gets

$$(2.11) \quad \begin{aligned} & \sup_{0 \leq t \leq T} \|(\sqrt{\varrho_0} v^2, \sqrt{\varrho_0} \vartheta)\|_2^2 + \int_0^T (\|\sqrt{\varrho_0} \vartheta\|_\infty^2 + \|(\partial_y \vartheta, v \partial_y v)\|_2^2) dt \\ & \leq C(1 + \|(\sqrt{\varrho_0} \vartheta_0, \sqrt{\varrho_0} v_0^2)\|_2^2) \end{aligned}$$

for a positive constant C depending only on $R, c_v, \mu, \kappa, m_1, N_1$, and T . The desired estimate

$$\sup_{0 \leq t \leq T} \|J\|_\infty^2 + \int_0^T \|\vartheta\|_\infty^2 dt \leq C(1 + \|(\sqrt{\varrho_0} \vartheta_0, \sqrt{\varrho_0} v_0^2)\|_2^2)$$

follows from (2.11), by applying Proposition 2.3 and (i) of Proposition 2.4. \square

As a corollary of Proposition 2.3 and Proposition 2.5, we have the following corollary.

COROLLARY 2.1. *Given $T \in (0, \infty)$, it holds that*

$$0 < \underline{J} \leq J(y, t) \leq C(1 + \|(\sqrt{\varrho_0} v_0, \sqrt{\varrho_0} v_0^2, \sqrt{\varrho_0} \vartheta_0)\|_2^2)$$

for all $(y, t) \in (0, L) \times (0, T)$, and

$$\begin{aligned} \sup_{0 \leq t \leq T} \|(\sqrt{\varrho_0}v, \sqrt{\varrho_0}\vartheta)\|_2^2 + \int_0^T (\|\vartheta\|_\infty^2 + \|(\partial_y\vartheta, \partial_yv)\|_2^2) dt \\ \leq C(1 + \|(\sqrt{\varrho_0}v_0, \sqrt{\varrho_0}v_0^2, \sqrt{\varrho_0}\vartheta_0)\|_2^2) \end{aligned}$$

for positive constants \mathcal{J} and C depending only on $R, c_v, \mu, \kappa, m_1, N_1$, and T , where m_1 and N_1 are the numbers in Proposition 2.3 and Proposition 2.4, respectively.

Proof. All estimates except that for $\int_0^T \|\partial_yv\|_2^2 dt$ follow directly from Proposition 2.3 and Proposition 2.5. Multiplying (1.9) by v , integrating the resultant over $(0, L)$, one gets from integration by parts that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\sqrt{\varrho_0}v\|_2^2 + \mu \left\| \frac{\partial_yv}{\sqrt{J}} \right\|_2^2 &= \int_0^L \pi \partial_yv dy \\ &= R \int_0^L \frac{\varrho_0}{J} \vartheta \partial_yv dy \leq \frac{\mu}{2} \left\| \frac{\partial_yv}{\sqrt{J}} \right\|_2^2 + \frac{R}{2\mu} \int_0^L \frac{\varrho_0^2}{J} \vartheta^2 dy, \end{aligned}$$

and, thus, by Proposition 2.3,

$$\frac{d}{dt} \|\sqrt{\varrho_0}v\|_2^2 + \mu \left\| \frac{\partial_yv}{\sqrt{J}} \right\|_2^2 \leq \frac{R}{\mu} m_1 f_1(t) \bar{\varrho} \|\sqrt{\varrho_0}\vartheta\|_\infty^2,$$

from which, by Proposition 2.5, the conclusion follows. □

2.2. A priori H^1 estimates. This section is devoted to the a priori H^1 type estimates on (J, v, ϑ) . Precisely, we will carry out the a priori $L^\infty(0, T; H^1) \cap L^2(0, T; H^2)$ estimate on v and the $L^\infty(0, T; H^1)$ estimate on J ; however, due to the presence of the term $\frac{\mu}{J}(\partial_yv)^2$ on the right-hand side of the equation for ϑ , (1.10), one can not get the desired a priori H^1 estimate of ϑ independent of the lower bound of the density without appealing to the higher than H^1 energy estimates.

Define the effective viscous flux G as

$$G := \mu \frac{\partial_yv}{J} - \pi = \mu \frac{\partial_yv}{J} - R \frac{\varrho_0}{J} \vartheta.$$

Then, one can derive from (1.8)–(1.10) that

$$(2.12) \quad \partial_t G - \frac{\mu}{J} \partial_y \left(\frac{\partial_y G}{\varrho_0} \right) = - \frac{\kappa(\gamma - 1)}{J} \partial_y \left(\frac{\partial_y \vartheta}{J} \right) - \gamma \frac{\partial_y v}{J} G.$$

Moreover, by (1.9), one has $\partial_y G = \varrho_0 \partial_t v$, from which, recalling the boundary condition (1.11), we have

$$\partial_y G(0, t) = \partial_y G(1, t) = 0, \quad t \in (0, \infty).$$

We have the a priori L^2 estimates on G sated in the following proposition.

PROPOSITION 2.6. *Given $T \in (0, \infty)$, it holds that*

$$\sup_{0 \leq t \leq T} \|G\|_2^2 + \int_0^T \left\| \frac{\partial_y G}{\sqrt{\varrho_0}} \right\|_2^2 dt \leq C,$$

for a positive constant C depending only on $R, c_v, \mu, \kappa, m_1, N_1, N_2$, and T , where

$$N_2 := \|\sqrt{\varrho_0}v_0^2\|_2 + \|\sqrt{\varrho_0}\vartheta_0\|_2 + \|v'_0\|_2,$$

and m_1 and N_1 are the numbers in Proposition 2.3 and Proposition 2.4, respectively.

Proof. Multiplying (2.12) by JG , integrating the resultant over $(0, L)$, and recalling $\partial_y G|_{y=0,L} = 0$, one gets from integration by parts that

$$\int_0^L \partial_t G J G dy + \mu \int_0^L \left| \frac{\partial_y G}{\sqrt{\varrho_0}} \right|^2 dy = \kappa(\gamma - 1) \int_0^L \frac{\partial_y \vartheta \partial_y G}{J} dy - \gamma \int_0^L \partial_y v G^2 dy.$$

Using (1.8), one has

$$\begin{aligned} \int_0^L \partial_t G J G dy &= \frac{1}{2} \frac{d}{dt} \int_0^L J G^2 dy - \frac{1}{2} \int_0^L \partial_t J G^2 dy \\ &= \frac{1}{2} \frac{d}{dt} \int_0^L J G^2 dy - \frac{1}{2} \int_0^L \partial_y v G^2 dy. \end{aligned}$$

Therefore, it follows from the Hölder, Young, and Gagliardo–Nirenberg inequalities and Corollary 2.1 that

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_0^L J G^2 dy + \mu \int_0^L \left| \frac{\partial_y G}{\sqrt{\varrho_0}} \right|^2 dy \\ &= \kappa(\gamma - 1) \int_0^L \frac{\partial_y \vartheta \partial_y G}{J} dy + \left(\frac{1}{2} - \gamma \right) \int_0^L \partial_y v G^2 dy \\ &\leq \kappa \gamma \sqrt{\varrho} \left\| \frac{\partial_y G}{\sqrt{\varrho_0}} \right\|_2 \left\| \frac{\partial_y \vartheta}{\sqrt{J}} \right\|_2 + \gamma \|\partial_y v\|_2 \|G\|_2 \|G\|_\infty \\ &\leq C \left(\left\| \frac{\partial_y G}{\sqrt{\varrho_0}} \right\|_2 \|\partial_y \vartheta\|_2 + \|\partial_y v\|_2 \|G\|_2^{\frac{3}{2}} (\|G\|_2 + \|\partial_y G\|_2)^{\frac{1}{2}} \right) \\ &\leq \frac{\mu}{2} \left\| \frac{\partial_y G}{\sqrt{\varrho_0}} \right\|_2^2 + C \left[(1 + \|\partial_y v\|_2^2) \|G\|_2^2 + \|\partial_y \vartheta\|_2^2 \right], \end{aligned}$$

that is,

$$(2.13) \quad \frac{d}{dt} \|\sqrt{J}G\|_2^2 + \mu \left\| \frac{\partial_y G}{\sqrt{\varrho_0}} \right\|_2^2 \leq C \left[(1 + \|\partial_y v\|_2^2) \|G\|_2^2 + \|\partial_y \vartheta\|_2^2 \right]$$

for any $t \in (0, T)$, where C is a positive constant depending only on $R, c_v, \mu, \kappa, m_1, N_1$, and T . Applying the Gronwall inequality to (2.13) and using Corollary 2.1, the conclusion follows. \square

Based on Proposition 2.6, we can obtain the desired H^1 type estimates on J and v as stated in the next proposition.

PROPOSITION 2.7. *Given $T \in (0, \infty)$, it holds that*

$$\sup_{0 \leq t \leq T} (\|\partial_y J\|_2^2 + \|\partial_y v\|_2^2) + \int_0^T (\|\sqrt{\varrho_0} \partial_t v\|_2^2 + \|\partial_y^2 v\|_2^2) dt \leq C$$

for a positive constant C depending only on $R, c_v, \mu, \kappa, m_1, N_1, N_2$, and T , where m_1, N_1 and N_2 are the numbers in Propositions 2.3, 2.4, and 2.6, respectively.

Proof. The estimate $\sup_{0 \leq t \leq T} \|\partial_y v\|_2^2 + \int_0^T \|\sqrt{\varrho_0} \partial_t v\|_2^2 dt \leq C$ is straightforward from Corollary 2.1 and Proposition 2.6, by the definition of G and noticing that $\varrho_0 \partial_t v = \partial_y G$. Note that, by the Sobolev embedding inequality, it follows from Proposition 2.6 that

$$(2.14) \quad \int_0^T \|G\|_\infty^2 dt \leq C \int_0^T \|G\|_{H^1}^2 dt \leq C$$

for a positive constant C depending only on $R, c_v, \mu, \kappa, m_1, N_1, N_2$, and T . Rewrite (1.8) in terms of G as

$$\partial_t J = \frac{1}{\mu}(JG + R\rho_0\vartheta).$$

Differentiating the above equations in y , multiplying the resultant by $\partial_y J$, and integrating over $(0, L)$, it follows from the Hölder and Young inequalities that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\partial_y J\|_2^2 &= \frac{1}{\mu} \int_0^L [G|\partial_y J|^2 + \partial_y G J \partial_y J + R(\rho'_0 \vartheta + \rho_0 \partial_y \vartheta) \partial_y J] dy \\ &\leq \frac{1}{\mu} [\|G\|_\infty \|\partial_y J\|_2^2 + \|J\|_\infty \|\partial_y G\|_2 \|\partial_y J\|_2 + R(\|\vartheta\|_\infty \|\rho'_0\|_2 + \bar{\rho} \|\partial_y \vartheta\|_2) \|\partial_y J\|_2] \\ &\leq C(\|G\|_\infty^2 + 1) \|\partial_y J\|_2^2 + C(\|J\|_\infty^2 \|\partial_y G\|_2^2 + \|\rho'_0\|_2^2 \|\vartheta\|_\infty^2 + \bar{\rho}^2 \|\partial_y \vartheta\|_2^2) \end{aligned}$$

for a positive constant C depending only on R and μ . Applying the Gronwall inequality to the above inequality, it follows from (2.14), Corollary 2.1, and Proposition 2.6 that

$$(2.15) \quad \sup_{0 \leq t \leq T} \|\partial_y J\|_2^2 \leq C$$

for a positive constant C depending only on $R, c_v, \mu, \kappa, m_1, N_1, N_2$, and T . Noticing that $\partial_y v = \frac{1}{\mu}(JG + R\rho_0\vartheta)$, one has

$$\partial_y^2 v = \frac{1}{\mu}(\partial_y JG + J\partial_y G + R\rho'_0\vartheta + R\rho_0\partial_y \vartheta),$$

and, thus, by the Hölder inequality, (2.14), and (2.15), it follows from Corollary 2.1 and Proposition 2.6 that

$$\begin{aligned} \int_0^T \|\partial_y^2 v\|_2^2 dt &\leq C \int_0^T (\|\partial_y J\|_2^2 \|G\|_\infty^2 + \|J\|_\infty^2 \|\partial_y G\|_2^2 + \|\rho'_0\|_2^2 \|\vartheta\|_\infty^2 + \|\partial_y \vartheta\|_2^2) dt \\ &\leq C \int_0^T (\|G\|_\infty^2 + \|J\|_\infty^2 \|\partial_y G\|_2^2 + \|\vartheta\|_\infty^2 + \|\partial_y \vartheta\|_2^2) dt \leq C \end{aligned}$$

for a positive constant C depending only on $R, c_v, \mu, \kappa, m_1, N_1, N_2$, and T , proving the conclusion. \square

We summarize the estimates obtained in this subsection in the following corollary.

COROLLARY 2.2. *Given $T \in (0, \infty)$, it holds that*

$$\sup_{0 \leq t \leq T} \|(G, \partial_y J, \partial_y v)\|_2^2 + \int_0^T \left\| \left(\frac{\partial_y G}{\sqrt{\rho_0}}, \partial_y^2 v, \sqrt{\rho_0} \partial_t v \right) \right\|_2^2 dt \leq C$$

for a positive constant C depending only on $R, c_v, \mu, \kappa, m_1, N_1, N_2$, and T , where m_1, N_1 and N_2 are the numbers in Propositions 2.3, 2.4, and 2.6, respectively.

2.3. A priori H^2 estimates. This subsection is devoted to the a priori H^2 estimates on (J, v, ϑ) . As will be shown in this subsection that one can get the desired a priori $L^\infty(0, T; H^2)$ estimate of ϑ without using the a priori $L^\infty(0, T; H^1)$ bound of it.

As a preparation, we first give some estimates on $\|\partial_y \vartheta\|_2$ and $\|\partial_t \vartheta\|_\infty$, in terms of $\|\sqrt{\rho_0} \partial_t \vartheta\|_2$ and $\|\partial_y \partial_t \vartheta\|_2$, stated in the following proposition.

PROPOSITION 2.8. Given $T \in (0, \infty)$.

(i) It holds that

$$\|\partial_y \vartheta\|_2^2 \leq C(1 + \|\sqrt{\varrho_0} \partial_t \vartheta\|_2)$$

for a positive constant C depending only on $R, c_v, \mu, \kappa, m_1, N_1, N_2$, and T , where m_1, N_1 and N_2 are the numbers in Propositions 2.3, 2.4, and 2.6, respectively.

(ii) It holds that

$$\begin{aligned} \|\partial_t \vartheta\|_\infty &\leq \sqrt{\frac{2}{\omega_0 \varrho}} \|\sqrt{\varrho_0} \partial_t \vartheta\|_2 + \sqrt{L} \left\| \frac{\partial_y \partial_t \vartheta}{\sqrt{J}} \right\|_2, \\ \|\partial_t v\|_\infty &\leq \sqrt{\frac{2}{\omega_0 \varrho}} \|\sqrt{\varrho_0} \partial_t v\|_2 + \sqrt{L} \left\| \frac{\partial_y \partial_t v}{\sqrt{J}} \right\|_2, \end{aligned}$$

where ω_0 is the number in Proposition 2.4.

Proof. (i) Multiplying (2.16) by ϑ , integrating the resultant over $(0, L)$, and integrating by parts, it follows from the Hölder inequality that

$$\begin{aligned} \kappa \int_0^L \left| \frac{\partial_y \vartheta}{\sqrt{J}} \right|^2 dy &= \int_0^L (\partial_y v G - c_v \varrho_0 \partial_t \vartheta) \vartheta dy \\ &\leq \|\partial_y v\|_2 \|G\|_2 \|\vartheta\|_\infty + c_v \|\sqrt{\varrho_0} \partial_t \vartheta\|_2 \|\sqrt{\varrho_0} \vartheta\|_2, \end{aligned}$$

from which, by Corollaries 2.1–2.2 and (i) of Proposition 2.4 that

$$\begin{aligned} \|\partial_y \vartheta\|_2^2 &\leq C \int_0^L \left| \frac{\partial_y \vartheta}{\sqrt{J}} \right|^2 dy \leq C(\|\vartheta\|_\infty + \|\sqrt{\varrho_0} \partial_t \vartheta\|_2) \\ &\leq C(\|\partial_y \vartheta\|_2 + 1 + \|\sqrt{\varrho_0} \partial_t \vartheta\|_2) \leq \frac{1}{2} \|\partial_y \vartheta\|_2^2 + C(1 + \|\sqrt{\varrho_0} \partial_t \vartheta\|_2) \end{aligned}$$

and, thus,

$$\|\partial_y \vartheta\|_2^2 \leq C(1 + \|\sqrt{\varrho_0} \partial_t \vartheta\|_2)$$

for a positive constant C depending only on $R, c_v, \mu, \kappa, m_1, N_1, N_2$, and T .

(ii) Recall that $\Omega_0 := \{y \in (0, L) \mid \varrho_0(y) \geq \frac{\varrho}{2}\}$ and $|\omega_0| = |\Omega_0| > 0$. Noticing

$$\partial_t \vartheta(y, t) = \frac{1}{\omega_0} \int_{\Omega_0} \partial_t \vartheta(z, t) dz + \frac{1}{\omega_0} \int_{\Omega_0} \int_z^y \partial_y \partial_t \vartheta(\xi, t) d\xi dz,$$

it follows from the Hölder inequality and Proposition 2.2 that

$$\begin{aligned} |\partial_t \vartheta(y, t)| &\leq \frac{1}{\omega_0} \left| \int_{\Omega_0} \frac{\sqrt{\varrho_0} \partial_t \vartheta}{\sqrt{\varrho_0}} dz \right| + \int_0^L |\partial_y \partial_t \vartheta(\xi, t)| d\xi \\ &\leq \sqrt{\frac{2}{\omega_0 \varrho}} \|\sqrt{\varrho_0} \partial_t \vartheta\|_2 + \left(\int_0^L \left| \frac{\partial_y \partial_t \vartheta}{\sqrt{J}} \right|^2 d\xi \right)^{\frac{1}{2}} \left(\int_0^L J d\xi \right)^{\frac{1}{2}} \\ &= \sqrt{\frac{2}{\omega_0 \varrho}} \|\sqrt{\varrho_0} \partial_t \vartheta\|_2 + \sqrt{L} \left\| \frac{\partial_y \partial_t \vartheta}{\sqrt{J}} \right\|_2, \end{aligned}$$

which implies

$$\|\partial_t \vartheta\|_\infty \leq \sqrt{\frac{2}{\omega_0 \varrho}} \|\sqrt{\varrho_0} \partial_t \vartheta\|_2 + \sqrt{L} \left\| \frac{\partial_y \partial_t \vartheta}{\sqrt{J}} \right\|_2.$$

In the same way as above, the same conclusion holds for $\partial_t v$. \square

PROPOSITION 2.9. *Given $T \in (0, \infty)$, it holds that*

$$\sup_{0 \leq t \leq T} \left\| \left(\sqrt{\varrho_0} \partial_t \vartheta, \frac{\partial_y G}{\sqrt{\varrho_0}} \right) \right\|_2^2 + \int_0^T \|(\partial_t G, \partial_y \partial_t \vartheta)\|_2^2 dt \leq C(\|g_0\|_2 + \|h_0\|_2)$$

for a positive constant C depending only on $R, c_v, \mu, \kappa, m_1, N_1, N_2$, and T , where

$$g_0 := \frac{\mu v_0'' - R(\varrho_0 \theta_0)'}{\sqrt{\varrho_0}}, \quad h_0 := \frac{1}{\sqrt{\varrho_0}} [\mu(v_0')^2 + \kappa \vartheta_0'' - R v_0' \varrho_0 \vartheta_0],$$

and m_1, N_1 , and N_2 are the numbers in Propositions 2.3, 2.4, and 2.6, respectively.

Proof. Rewrite (1.10) as

$$(2.16) \quad c_v \varrho_0 \partial_t \vartheta - \kappa \partial_y \left(\frac{\partial_y \vartheta}{J} \right) = \partial_y v G,$$

or, equivalently,

$$c_v \varrho_0 \partial_t \vartheta - \kappa \partial_y \left(\frac{\partial_y \vartheta}{J} \right) = \frac{1}{\mu} (JG + R \varrho_0 \vartheta) G,$$

from which, differentiating in t and using (1.8), one has

$$\begin{aligned} & c_v \varrho_0 \partial_t^2 \vartheta - \kappa \partial_y \left(\frac{\partial_y \partial_t \vartheta}{J} - \frac{\partial_y v \partial_y \vartheta}{J^2} \right) \\ &= \frac{1}{\mu} (\partial_y v G^2 + 2JG \partial_t G) + \frac{R \varrho_0}{\mu} (\partial_t \vartheta G + \vartheta \partial_t G) \\ &= \frac{\partial_y v}{\mu} G^2 + \frac{1}{\mu} (2JG + R \varrho_0 \vartheta) \partial_t G + \frac{R}{\mu} \varrho_0 \partial_t \vartheta G. \end{aligned}$$

Multiplying the above equation by $\partial_t \vartheta$, integrating the resultant over $(0, L)$, one gets from integration by parts that

$$(2.17) \quad \begin{aligned} & \frac{c_v}{2} \frac{d}{dt} \int_0^L \varrho_0 |\partial_t \vartheta|^2 dy + \kappa \int_0^L \left| \frac{\partial_y \partial_t \vartheta}{\sqrt{J}} \right|^2 dy \\ &= \kappa \int_0^L \frac{\partial_y v \partial_y \vartheta}{J^2} \partial_y \partial_t \vartheta dy + \frac{R}{\mu} \int_0^L \varrho_0 G (\partial_t \vartheta)^2 dy \\ & \quad + \frac{1}{\mu} \int_0^L [(2JG + R \varrho_0 \vartheta) \partial_t G + \partial_y v G^2] \partial_t \vartheta dy. \end{aligned}$$

The terms on the right-hand side of (2.17) are estimated as follows. By Corollary 2.1, it follows from the Young inequality and (i) of Proposition 2.8 that

$$\begin{aligned} \kappa \int_0^L \frac{\partial_y v \partial_y \vartheta}{J^2} \partial_y \partial_t \vartheta dy &\leq \frac{\kappa}{4} \left\| \frac{\partial_y \partial_t \vartheta}{\sqrt{J}} \right\|_2^2 + C \|\partial_y v\|_\infty^2 \|\partial_y \vartheta\|_2^2 \\ &\leq \frac{\kappa}{4} \left\| \frac{\partial_y \partial_t \vartheta}{\sqrt{J}} \right\|_2^2 + C(\|G\|_\infty^2 + \|\vartheta\|_\infty^2) \|\partial_y \vartheta\|_2^2 \\ &\leq \frac{\kappa}{4} \left\| \frac{\partial_y \partial_t \vartheta}{\sqrt{J}} \right\|_2^2 + C(\|G\|_\infty^2 + \|\vartheta\|_\infty^2) (1 + \|\sqrt{\varrho_0} \partial_t \vartheta\|_2) \end{aligned}$$

for a positive constant C depending only on $R, c_v, \mu, \kappa, m_1, N_1$, and T . By Corollary 2.1, Corollary 2.2, and (ii) of Proposition 2.8, it follows from the Hölder and Young inequalities that

$$\begin{aligned} & \frac{1}{\mu} \int_0^L [(2JG + R\varrho_0\vartheta)\partial_t G + \partial_y v G^2] \partial_t \vartheta dy \\ & \leq C(\|J\|_\infty \|G\|_2 + \|\sqrt{\varrho_0}\vartheta\|_2) \|\partial_t G\|_2 + \|\partial_y v\|_2 \|G\|_2 \|G\|_\infty \|\partial_t \vartheta\|_\infty \\ & \leq C(\|\partial_t G\|_2 + \|G\|_\infty) \left(\|\sqrt{\varrho_0}\partial_t \vartheta\|_2 + \left\| \frac{\partial_y \partial_t \vartheta}{\sqrt{J}} \right\|_2 \right) \\ & \leq \frac{\kappa}{4} \left\| \frac{\partial_y \partial_t \vartheta}{\sqrt{J}} \right\|_2^2 + C \left(\|\sqrt{J}\partial_t G\|_2^2 + \|\sqrt{\varrho_0}\partial_t \vartheta\|_2^2 + \|G\|_\infty^2 \right) \end{aligned}$$

for a positive constant C depending only on $R, c_v, \mu, \kappa, m_1, N_1, N_2$, and T . Therefore, one obtains from (2.17) that

$$(2.18) \quad \begin{aligned} & c_v \frac{d}{dt} \|\sqrt{\varrho_0}\partial_t \vartheta\|_2^2 + \kappa \left\| \frac{\partial_y \partial_t \vartheta}{\sqrt{J}} \right\|_2^2 \\ & \leq A_3 \left[\|\sqrt{J}\partial_t G\|_2^2 + (1 + \|G\|_\infty^2 + \|\vartheta\|_\infty^2) (\|\sqrt{\varrho_0}\partial_t \vartheta\|_2^2 + 1) \right] \end{aligned}$$

for a positive constant A_3 depending only on $R, c_v, \mu, \kappa, m_1, N_1, N_2$, and T .

Using (2.16), one can rewrite (2.12) as

$$\partial_t G - \frac{\mu}{J} \partial_y \left(\frac{\partial_y G}{\varrho_0} \right) = -R \frac{\varrho_0}{J} \partial_t \vartheta - \frac{\partial_y v}{J} G.$$

Multiplying the above equation by $J\partial_t G$, integrating the resultant over $(0, L)$, and integrating by parts, it follows from the Hölder and Young inequalities, Corollary 2.1, and Corollary 2.2 that

$$\begin{aligned} & \frac{\mu}{2} \frac{d}{dt} \int_0^L \left| \frac{\partial_y G}{\sqrt{\varrho_0}} \right|^2 dy + \int_0^L J |\partial_t G|^2 dy \\ & = -R \int_0^L \varrho_0 \partial_t \vartheta \partial_t G dy - \int_0^L \partial_y v G \partial_t G dy \\ & \leq \frac{1}{2} \|\sqrt{J}\partial_t G\|_2^2 + C(\|\sqrt{\varrho_0}\partial_t \vartheta\|_2^2 + \|\partial_y v\|_2^2 \|G\|_\infty^2) \\ & \leq \frac{1}{2} \|\sqrt{J}\partial_t G\|_2^2 + C(\|\sqrt{\varrho_0}\partial_t \vartheta\|_2^2 + \|G\|_\infty^2) \end{aligned}$$

and, thus,

$$(2.19) \quad \mu \frac{d}{dt} \left\| \frac{\partial_y G}{\sqrt{\varrho_0}} \right\|_2^2 + \|\sqrt{J}\partial_t G\|_2^2 \leq C(\|\sqrt{\varrho_0}\partial_t \vartheta\|_2^2 + \|G\|_\infty^2)$$

for a positive constant C depending only on $R, c_v, \mu, \kappa, m_1, N_1, N_2$, and T .

Multiplying (2.19) by $2A_3$ and summing the resultant with (2.18), one obtains

$$\begin{aligned} & \frac{d}{dt} \left(c_v \|\sqrt{\varrho_0}\partial_t \vartheta\|_2^2 + 2A_3 \mu \left\| \frac{\partial_y G}{\sqrt{\varrho_0}} \right\|_2^2 \right) + \kappa \left\| \frac{\partial_y \partial_t \vartheta}{\sqrt{J}} \right\|_2^2 + A_3 \|\sqrt{J}\partial_t G\|_2^2 \\ & \leq C(1 + \|G\|_\infty^2 + \|\vartheta\|_\infty^2) (\|\sqrt{\varrho_0}\partial_t \vartheta\|_2^2 + 1) \end{aligned}$$

for a positive constant C depending only on $R, c_v, \mu, \kappa, m_1, N_1, N_2$, and T . Applying the Gronwall inequality to the above inequality, by Corollary 2.1, and using (2.14), the conclusion follows. \square

PROPOSITION 2.10. *Given $T \in (0, \infty)$, it holds that*

$$\sup_{0 \leq t \leq T} \left(\|(\partial_y^2 J, \partial_y^2 v, \partial_y \vartheta, \partial_y^2 \vartheta)\|_2^2 + \|\vartheta\|_\infty \right) + \int_0^T \|(\partial_y^3 v, \partial_y \partial_t v, \partial_y^3 \vartheta)\|_2^2 \leq C$$

for a positive constant C depending only on $R, c_v, \mu, \kappa, m_1, N_1, N_2, N_3$, and T , where

$$N_3 := \|\varrho'_0\|_2 + \|g_0\|_2 + \|h_0\|_2,$$

and m_1, N_1 , and N_2 are the numbers in Propositions 2.3, 2.4, and 2.6, respectively.

Proof. Combining (i) of Proposition 2.8 and Proposition 2.9, one gets

$$(2.20) \quad \sup_{0 \leq t \leq T} \|\partial_y \vartheta\|_2^2 \leq C \sup_{0 \leq t \leq T} (1 + \|\sqrt{\varrho_0} \partial_t \vartheta\|_2^2) \leq C(1 + \|g_0\|_2^2 + \|h_0\|_2^2)$$

and, thus, by (i) of Proposition 2.4 and Corollary 2.1 that

$$(2.21) \quad \sup_{0 \leq t \leq T} \|\vartheta\|_\infty \leq C \sup_{0 \leq t \leq T} (\|\partial_y \vartheta\|_2 + 1) \leq C(1 + \|g_0\|_2 + \|h_0\|_2)$$

for a positive constant C depending only on $R, c_v, \mu, \kappa, m_1, N_1, N_2$, and T . Using (2.20)–(2.21), it follows from the Hölder inequality and Corollaries 2.1–2.2 that

$$(2.22) \quad \begin{aligned} \|\partial_y \pi\|_2 &= R \left\| \frac{\varrho'_0}{J} \vartheta + \frac{\varrho_0}{J} \partial_y \vartheta - \frac{\varrho_0}{J^2} \partial_y J \vartheta \right\|_2 \\ &\leq C(\|\varrho'_0\|_2 \|\vartheta\|_\infty + \|\varrho_0\|_\infty \|\partial_y \vartheta\|_2 + \|\varrho_0\|_\infty \|\partial_y J\|_2 \|\vartheta\|_\infty) \\ &\leq C(1 + \|g_0\|_2 + \|h_0\|_2) \end{aligned}$$

for a positive constant C depending only on $R, c_v, \mu, \kappa, m_1, N_1, N_2$, and T .

Noticing that $\partial_y v = \frac{1}{\mu}(JG + R\varrho_0 \vartheta)$, and using (1.8), one has

$$\begin{aligned} \partial_t \partial_y v &= \frac{1}{\mu}(\partial_y v G + J \partial_t G + R \varrho_0 \partial_t \vartheta), \\ \partial_y^2 v &= \frac{1}{\mu}(\partial_y J G + J \partial_y G + R \varrho'_0 \vartheta + R \varrho_0 \partial_y \vartheta), \end{aligned}$$

and, thus, by the Hölder and Sobolev embedding inequalities, and using (2.20)–(2.21), it follows from Corollaries 2.1–2.2 and Proposition 2.9 that

$$(2.23) \quad \begin{aligned} \int_0^T \|\partial_t \partial_y v\|_2^2 dt &\leq C \int_0^T (\|\partial_y v\|_2^2 \|G\|_\infty^2 + \|J\|_\infty^2 \|\partial_t G\|_2^2 + \|\sqrt{\varrho_0} \partial_t \vartheta\|_2^2) dt \\ &\leq C \int_0^T (\|\partial_y v\|_2^2 \|G\|_{H^1}^2 + \|J\|_\infty^2 \|\partial_t G\|_2^2 + \|\sqrt{\varrho_0} \partial_t \vartheta\|_2^2) dt \\ &\leq C(1 + \|g_0\|_2^2 + \|h_0\|_2^2), \end{aligned}$$

and

$$\sup_{0 \leq t \leq T} \|\partial_y^2 v\|_2^2 \leq C \sup_{0 \leq t \leq T} (\|\partial_y J\|_2^2 \|G\|_\infty^2 + \|J\|_\infty^2 \|\partial_y G\|_2^2 + \|\varrho'_0\|_2^2 \|\vartheta\|_\infty^2 + \|\partial_y \vartheta\|_2^2)$$

$$\begin{aligned}
&\leq C \sup_{0 \leq t \leq T} (\|\partial_y J\|_2^2 \|G\|_{H^1}^2 + \|J\|_\infty^2 \|\partial_y G\|_2^2 + \|\vartheta\|_\infty^2 + \|\partial_y \vartheta\|_2^2) \\
(2.24) \quad &\leq C(1 + \|g_0\|_2^2 + \|h_0\|_2^2)
\end{aligned}$$

for a positive constant C depending only on $R, c_v, \mu, \kappa, m_1, N_1, N_2$, and T .

Using (2.16), we have

$$\partial_y^2 \vartheta = J \partial_y \left(\frac{\partial_y \vartheta}{J} \right) + \partial_y J \frac{\partial_y \vartheta}{J} = \frac{J}{\kappa} (c_v \varrho_0 \partial_t \vartheta - \partial_y v G) + \partial_y J \frac{\partial_y \vartheta}{J},$$

and, thus, by the Hölder, Young and Gagliardo–Nirenburg inequalities and (2.20), it follows from Corollaries 2.1–2.2, and Proposition 2.9 that

$$\begin{aligned}
\|\partial_y^2 \vartheta\|_2 &\leq C(\|\sqrt{\varrho_0} \partial_t \vartheta\|_2 + \|\partial_y v\|_2 \|G\|_\infty + \|\partial_y J\|_2 \|\partial_y \vartheta\|_\infty) \\
&\leq C(1 + \|g_0\|_2 + \|h_0\|_2 + \|G\|_{H^1} + \|\partial_y \vartheta\|_2^{\frac{1}{2}} \|\partial_y^2 \vartheta\|_2^{\frac{1}{2}}) \\
&\leq \frac{1}{2} \|\partial_y^2 \vartheta\|_2 + C(1 + \|g_0\|_2 + \|h_0\|_2),
\end{aligned}$$

which gives

$$(2.25) \quad \sup_{0 \leq t \leq T} \|\partial_y^2 \vartheta\|_2^2 \leq C(1 + \|g_0\|_2^2 + \|h_0\|_2^2)$$

for a positive constant C depending only on $R, c_v, \mu, \kappa, m_1, N_1, N_2$, and T .

By calculations, one deduces

$$\begin{aligned}
\partial_y^2 \pi &= R \partial_y^2 \left(\frac{\varrho_0}{J} \vartheta \right) = R \left[\varrho_0'' \frac{\vartheta}{J} + 2 \varrho_0' \partial_y \left(\frac{\vartheta}{J} \right) + \varrho_0 \partial_y^2 \left(\frac{\vartheta}{J} \right) \right] \\
&= R \left[\varrho_0'' \frac{\vartheta}{J} + 2 \varrho_0' \left(\frac{\partial_y \vartheta}{J} - \frac{\partial_y J}{J^2} \vartheta \right) \right. \\
&\quad \left. + \varrho_0 \left(\frac{\partial_y^2 \vartheta}{J} - \frac{2}{J^2} \partial_y J \partial_y \vartheta + 2 \frac{(\partial_y J)^2}{J^3} \vartheta - \frac{\partial_y^2 J}{J^2} \vartheta \right) \right].
\end{aligned}$$

Therefore, by the Hölder and Sobolev embedding inequalities, using (2.20), (2.21), and (2.25), it follows from Corollary 2.1 and Corollary 2.2 that

$$\begin{aligned}
\|\partial_y^2 \pi\|_2 &\leq C \left[\|\varrho_0''\|_2 \|\vartheta\|_\infty + 2 \|\varrho_0'\|_\infty (\|\partial_y \vartheta\|_2 + \|\partial_y J\|_2 \|\vartheta\|_\infty) \right. \\
&\quad \left. + \|\varrho_0\|_\infty \left(\|\partial_y^2 \vartheta\|_2 + 2 \|\partial_y J\|_\infty \|\partial_y \vartheta\|_2 + 2 \|\partial_y J\|_\infty \|\partial_y J\|_2 \|\vartheta\|_\infty \right) \right. \\
(2.26) \quad &\left. + \|\varrho_0\|_\infty \|\partial_y^2 J\|_2 \|\vartheta\|_\infty \right] \leq C(1 + \|g_0\|_2 + \|h_0\|_2 + \|\partial_y^2 J\|_2)
\end{aligned}$$

for a positive constant C depending only on $R, c_v, \mu, \kappa, m_1, N_1, N_2, N_3$, and T .

Using (2.16) and (1.9), we deduce

$$\begin{aligned}
\partial_y^3 \vartheta &= \partial_y^2 \left(\frac{\partial_y \vartheta}{J} \right) J + 2 \partial_y \left(\frac{\partial_y \vartheta}{J} \right) \partial_y J + \frac{\partial_y \vartheta}{J} \partial_y^2 J \\
&= \frac{J}{\kappa} [c_v (\varrho_0 \partial_t \partial_y \vartheta + \varrho_0' \partial_t \vartheta) - \partial_y v \partial_y G - \partial_y^2 v G] \\
&\quad + \frac{2}{\kappa} \partial_y J (c_v \varrho_0 \partial_t \vartheta - \partial_y v G) + \frac{\partial_y \vartheta}{J} \partial_y^2 J
\end{aligned}$$

and

$$\begin{aligned} \partial_y^3 v &= \partial_y^2 \left(\frac{\partial_y v}{J} \right) J + 2\partial_y \left(\frac{\partial_y v}{J} \right) \partial_y J + \frac{\partial_y v}{J} \partial_y^2 J \\ &= \frac{J}{\mu} (\varrho'_0 \partial_t v + \varrho_0 \partial_t \partial_y v + \partial_y^2 \pi) + \frac{2}{\mu} \partial_y J (\varrho_0 \partial_t v + \partial_y \pi) + \frac{\partial_y v}{J} \partial_y^2 J. \end{aligned}$$

Therefore, by the Hölder and Sobolev embedding inequalities, using (2.20), (2.22), (2.24), (2.25), (2.26), Corollary 2.1, Corollary 2.2, (ii) of Proposition 2.8, and Proposition 2.9, we deduce

$$\begin{aligned} \|\partial_y^3 \vartheta\|_2 &\leq C[\|\partial_y \partial_t \vartheta\|_2 + \|\varrho'_0\|_2 \|\partial_t \vartheta\|_\infty + \|\partial_y v\|_\infty \|\partial_y G\|_2 + \|\partial_y^2 v\|_2 \|G\|_\infty \\ &\quad + \|\partial_y J\|_\infty (\|\sqrt{\varrho_0} \partial_t \vartheta\|_2 + \|\partial_y v\|_2 \|G\|_\infty) + \|\partial_y \vartheta\|_\infty \|\partial_y^2 J\|_2] \\ &\leq C[\|\partial_y \partial_t \vartheta\|_2 + \|\sqrt{\varrho_0} \partial_t \vartheta\|_2 + \|\partial_y v\|_{H^1} \|\partial_y G\|_2 + \|\partial_y^2 v\|_2 \|G\|_{H^1} \\ &\quad + \|\partial_y J\|_{H^1} (\|\sqrt{\varrho_0} \partial_t \vartheta\|_2 + \|\partial_y v\|_2 \|G\|_{H^1}) + \|\partial_y \vartheta\|_{H^1} \|\partial_y^2 J\|_2] \\ &\leq C(1 + \|g_0\|_2 + \|h_0\|_2 + \|\partial_y \partial_t \vartheta\|_2 + \|\partial_y^2 J\|_2) \\ (2.27) \quad &\leq C(1 + \|g_0\|_2 + \|h_0\|_2)(1 + \|\partial_y \partial_t \vartheta\|_2 + \|\partial_y^2 J\|_2), \end{aligned}$$

and

$$\begin{aligned} \|\partial_y^3 v\|_2 &\leq C[\|\partial_t v\|_\infty + \|\partial_y \partial_t v\|_2 + \|\partial_y^2 \pi\|_2 \\ &\quad + \|\partial_y J\|_\infty (\|\sqrt{\varrho_0} \partial_t v\|_2 + \|\partial_y \pi\|_2) + \|\partial_y v\|_\infty \|\partial_y^2 J\|_2] \\ &\leq C[\|\partial_y \partial_t v\|_2 + \|\sqrt{\varrho_0} \partial_t v\|_2 + 1 + \|g_0\|_2 + \|h_0\|_2 + \|\partial_y^2 J\|_2 \\ &\quad + \|\partial_y J\|_{H^1} (1 + \|g_0\|_2 + \|h_0\|_2) + \|\partial_y v\|_{H^1} \|\partial_y^2 J\|_2] \\ (2.28) \quad &\leq C(1 + \|g_0\|_2 + \|h_0\|_2)(1 + \|\partial_y \partial_t v\|_2 + \|\partial_y^2 J\|_2) \end{aligned}$$

for a positive constant C depending only on $R, c_v, \mu, \kappa, m_1, N_1, N_2, N_3$, and T .

Combining (2.27) with (2.28), and using (2.23), one obtains

$$\begin{aligned} \int_0^t \|(\partial_y^3 v, \partial_y^3 \vartheta)\|_2^2 d\tau &\leq C(1 + \|g_0\|_2^2 + \|h_0\|_2^2) \int_0^t (1 + \|\partial_y \partial_t v\|_2^2 + \|\partial_y^2 J\|_2^2) d\tau \\ (2.29) \quad &\leq C(1 + \|g_0\|_2^2 + \|h_0\|_2^2)^2 \left(1 + \int_0^t \|\partial_y^2 J\|_2^2 d\tau \right) \end{aligned}$$

for any $t \in [0, T]$, where C is a positive constant depending only on $R, c_v, \mu, \kappa, m_1, N_1, N_2, N_3$, and T . Using (1.8), one gets $J = 1 + \int_0^t \partial_y v d\tau$ and, thus, it follows from the Hölder inequality that

$$\|\partial_y^2 J\|_2^2(t) = \left(\left\| \int_0^t \partial_y^3 v d\tau \right\|_2 \right)^2 \leq \left(\int_0^t \|\partial_y^3 v\|_2 d\tau \right)^2 \leq t \int_0^t \|\partial_y^3 v\|_2^2 d\tau.$$

Combining this with (2.29), and applying the Gronwall inequality, one obtains

$$\int_0^T (\|\partial_y^3 v\|_2^2 + \|\partial_y^3 \vartheta\|_2^2) dt \leq C$$

and, further, that

$$\sup_{0 \leq t \leq C} \|\partial_y^2 J\|_2^2 \leq C$$

for a positive constant C depending only on $R, c_v, \mu, \kappa, m_1, N_1, N_2, N_3$, and T . \square

We summarize the a priori estimates obtained in this section as the following corollary.

COROLLARY 2.3. *Given $T \in (0, T)$ and let m_1, N_1, N_2 , and N_3 be the numbers in Propositions 2.3, 2.4, 2.6, and 2.10, respectively. Then, there are two positive constants \underline{C} and C depending only on $R, c_v, \mu, \kappa, m_1, N_1, N_2, N_3$, and T , such that*

$$\inf_{0 \leq t \leq T} \inf_{y \in (0, L)} J(y, t) \geq \underline{C},$$

and

$$\sup_{0 \leq t \leq T} \|(J, v, \vartheta)\|_{H^2}^2 + \int_0^T (\|\partial_t J\|_{H^2}^2 + \|(v, \vartheta)\|_{H^3}^2 + \|(\partial_t v, \partial_t \vartheta)\|_{H^1}^2) dt \leq C.$$

Proof. All the estimates except

$$\sup_{0 \leq t \leq T} \|v\|_2^2 + \int_0^T (\|\partial_t v\|_2^2 + \|\partial_t \vartheta\|_2^2 + \|\partial_t J\|_{H^2}^2) dt$$

are directly corollaries of Corollary 2.1, Corollary 2.2, Proposition 2.9, and Proposition 2.10. While the remaining estimates in the above follow easily from the known ones by the Poincaré inequality, using (1.8) or (ii) of Proposition 2.8. \square

3. Proof of Theorem 1.1.

Proof of Theorem 1.1. For $\varepsilon \in (0, 1)$, set

$$\varrho_{0\varepsilon} = \varrho_0 + \varepsilon, \quad \vartheta_{0\varepsilon} = \vartheta_0 + \varepsilon.$$

Let $E_{0\varepsilon}, m_{1\varepsilon}$, and $N_{i\varepsilon}$, $i = 1, 2, 3$, be the corresponding numbers as stated in section 2 for $(\varrho_{0\varepsilon}, v_0, \vartheta_{0\varepsilon})$; that is,

$$E_{0\varepsilon} := \int_0^L \left(\frac{\varrho_{0\varepsilon}}{2} v_0^2 + c_v \varrho_{0\varepsilon} \vartheta_{0\varepsilon} \right) dy, \quad m_{1\varepsilon} = \exp \left\{ \frac{2}{\mu} \sqrt{2 \|\varrho_{0\varepsilon}\|_1 E_{0\varepsilon}} \right\},$$

and $N_{i\varepsilon}$, $i = 1, 2, 3$, will be given later. We first verify that all these numbers are uniformly bounded. One can easily check that

$$\begin{aligned} \|\varrho_0\|_\infty &\leq \|\varrho_{0\varepsilon}\|_\infty \leq \|\varrho_0\|_\infty + 1, & \|\varrho_0\|_1 &\leq \|\varrho_{0\varepsilon}\|_1 \leq \|\varrho_0\|_1 + L, \\ E_0 &\leq E_{0\varepsilon} \leq E_0 + \|v_0\|_2^2 + c_v (\|\varrho_0\|_1 + \|\vartheta_0\|_1 + L) \end{aligned}$$

for $\varepsilon \in (0, 1)$. Noticing that

$$\Omega_0 := \left\{ y \in (0, L) \mid \varrho_0(y) \geq \frac{\bar{\varrho}}{2} \right\} \subseteq \Omega_{0\varepsilon} := \left\{ y \in (0, L) \mid \varrho_{0\varepsilon}(y) \geq \frac{\bar{\varrho}}{2} \right\}$$

we have

$$0 < \omega_0 := |\Omega_0| \leq \omega_{0\varepsilon} := |\Omega_{0\varepsilon}| \leq L$$

for $\varepsilon \in (0, 1)$. Therefore, we have

$$\underline{m}_1 \leq m_{1\varepsilon} \leq \bar{m}_1,$$

and

$$N_{1\varepsilon} := \frac{E_{0\varepsilon}}{c_v} + \|\varrho_{0\varepsilon}\|_\infty + \frac{1}{\|\varrho_{0\varepsilon}\|_\infty} + L + \frac{1}{L} + \frac{1}{\omega_{0\varepsilon}} + \|\varrho'_0\|_\infty \leq \bar{N}_1$$

for some positive constants $\underline{m}_1, \bar{m}_1$, and \bar{N}_1 independent of $\varepsilon \in (0, 1)$. We have

$$\begin{aligned} N_{2\varepsilon} &:= \|\sqrt{\varrho_0\varepsilon}v_0^2\|_2 + \|\sqrt{\varrho_0\varepsilon}v_0\|_2 + \|v_0'\|_2 \\ &\leq \|\sqrt{\varrho_0}v_0^2\|_2 + \sqrt{\varepsilon}\|v_0^2\|_2 + \|\sqrt{\varrho_0}v_0\|_2 + \sqrt{\varepsilon}\|v_0\|_2 + \|v_0'\|_2 \\ &\leq \|\sqrt{\varrho_0}v_0^2\|_2 + \|v_0^2\|_2 + \|\sqrt{\varrho_0}v_0\|_2 + \|v_0\|_2 + \|v_0'\|_2 \end{aligned}$$

for any $\varepsilon \in (0, 1)$. Set

$$N_{3\varepsilon} := \|g_{0\varepsilon}\|_2 + \|h_{0\varepsilon}\|_2 + \|\varrho_{0\varepsilon}''\|_2,$$

where

$$g_{0\varepsilon} = \frac{\mu v_0'' - R(\varrho_0\varepsilon\vartheta_{0\varepsilon})'}{\sqrt{\varrho_0\varepsilon}}, \quad h_{0\varepsilon} = \frac{1}{\sqrt{\varrho_0\varepsilon}}[\mu(v_0')^2 + \kappa\vartheta_{0\varepsilon}'' - Rv_0'\varrho_0\varepsilon\vartheta_{0\varepsilon}].$$

By direct calculations, and using the compatibility conditions, we have

$$\begin{aligned} \|g_{0\varepsilon}\|_2 &= \left\| \frac{1}{\sqrt{\varrho_0\varepsilon}}(\sqrt{\varrho_0}g_0 + \varepsilon(\varrho_0' + \vartheta_0')) \right\|_2 \\ &\leq \|g_0\|_2 + \sqrt{\varepsilon}(\|\varrho_0'\|_2 + \|\vartheta_0'\|_2) \leq \|g_0\|_2 + \|\varrho_0'\|_2 + \|\vartheta_0'\|_2, \end{aligned}$$

and

$$\begin{aligned} \|h_{0\varepsilon}\|_2 &= \left\| \frac{1}{\sqrt{\varrho_0\varepsilon}}[\sqrt{\varrho_0}h_0 - Rv_0'(\varepsilon\varrho_0 + \varepsilon\vartheta_0 + \varepsilon^2)] \right\|_2 \\ &\leq \|h_0\|_2 + R\|v_0'\|_2(\sqrt{\varepsilon}\|\varrho_0\|_\infty + \sqrt{\varepsilon}\|\vartheta_0\|_\infty + \varepsilon^{\frac{3}{2}}) \\ &\leq \|h_0\|_2 + R\|v_0'\|_2(\|\varrho_0\|_\infty + \|\vartheta_0\|_\infty + 1) \end{aligned}$$

for any $\varepsilon \in (0, 1)$. Therefore, we have

$$N_{3\varepsilon} \leq \bar{N}_3$$

for some positive constant \bar{N}_3 independent of $\varepsilon \in (0, 1)$.

By Proposition 2.1, for each $\varepsilon \in (0, 1)$, there is a unique global strong solution $(J_\varepsilon, v_\varepsilon, \vartheta_\varepsilon)$ to system (1.8)–(1.10), with ϱ_0 replaced by $\varrho_{0\varepsilon}$, subject to (1.11) and the initial condition

$$(J_\varepsilon, v_\varepsilon, \vartheta_\varepsilon)|_{t=0} = (1, v_0, \vartheta_{0\varepsilon}).$$

Due to the uniform boundedness of $m_{1\varepsilon}, N_{1\varepsilon}, N_{2\varepsilon}$, and $N_{3\varepsilon}$, obtained in the above, it follows from Corollary 2.3 that there are two positive constants, independent of $\varepsilon \in (0, 1)$, such that

$$(3.1) \quad \inf_{0 \leq t \leq T} \inf_{y \in (0, L)} J_\varepsilon(y, t) \geq \underline{C},$$

and

$$(3.2) \quad \sup_{0 \leq t \leq T} \|(J_\varepsilon, v_\varepsilon, \vartheta_\varepsilon)\|_{H^2}^2 + \int_0^T (\|\partial_t J_\varepsilon\|_{H^2}^2 + \|(v_\varepsilon, \vartheta_\varepsilon)\|_{H^3}^2 + \|(\partial_t v_\varepsilon, \partial_t \vartheta_\varepsilon)\|_{H^1}^2) dt \leq C$$

for any $\varepsilon \in (0, 1)$.

Thanks to (3.2), by the Banach–Alaoglu theorem, and using Cantor’s diagonal arguments, there is a subsequence, still denoted by $\{(J_\varepsilon, v_\varepsilon, \vartheta_\varepsilon)\}$, and a triple (J, v, ϑ) , such that

$$(3.3) \quad (J_\varepsilon, v_\varepsilon, \vartheta_\varepsilon) \overset{*}{\rightharpoonup} (J, v, \vartheta), \text{ in } L^\infty(0, T; H^2),$$

$$(3.4) \quad \partial_t J_\varepsilon \rightharpoonup \partial_t J, \text{ in } L^2(0, T; H^2),$$

$$(3.5) \quad (\partial_t v_\varepsilon, \partial_t \vartheta_\varepsilon) \rightharpoonup (\partial_t v, \partial_t \vartheta), \text{ in } L^2(0, T; H^1),$$

$$(3.6) \quad (v_\varepsilon, \vartheta_\varepsilon) \rightharpoonup (v, \vartheta), \text{ in } L^2(0, T; H^3),$$

and, moreover, by Aubin–Lions compactness lemma, that

$$(3.7) \quad (J_\varepsilon, v_\varepsilon, \vartheta_\varepsilon) \rightarrow (J, v, \vartheta), \text{ in } C([0, T]; H^1),$$

$$(3.8) \quad (v_\varepsilon, \vartheta_\varepsilon) \rightarrow (v, \vartheta), \text{ in } L^2(0, T; H^2),$$

where \rightharpoonup , $\overset{*}{\rightharpoonup}$, and \rightarrow , respectively, denote the weak, weak–*, and strong convergences in the corresponding spaces. Noticing that $H^1((0, L)) \hookrightarrow C([0, L])$, we then have

$$J_\varepsilon \rightarrow J \quad \text{in } C([0, L] \times [0, T])$$

and, thus, it follows from (3.1) that

$$\inf_{0 \leq t \leq T} \inf_{y \in (0, L)} J(y, t) > 0$$

for any $T \in (0, \infty)$.

Thanks to the convergences (3.3)–(3.8), one can take the limit $\varepsilon \rightarrow 0$ to show that (J, v, ϑ) is a strong solution to system (1.8)–(1.10), subject to (1.11)–(1.12), satisfying the regularities stated in Theorem 1.1.

We now prove the uniqueness. Let (J_1, v_1, ϑ_1) and (J_2, v_2, ϑ_2) be two solutions to system (1.8)–(1.10), subject to (1.11)–(1.12), satisfying the regularities stated in Theorem 1.1, with the same initial data. Denote by (J, v, ϑ) the difference of these two solutions; that is,

$$(J, v, \vartheta) = (J_1, v_1, \vartheta_1) - (J_2, v_2, \vartheta_2).$$

Then, (J, v, ϑ) satisfies the following:

$$(3.9) \quad \partial_t J = \partial_y v,$$

$$(3.10) \quad \varrho_0 \partial_t v - \mu \partial_y \left(\frac{\partial_y v}{J_1} \right) = -\mu \partial_y \left(\frac{\partial_y v_2}{J_1 J_2} J \right) + R \partial_y \left(\frac{\varrho_0 \vartheta_2}{J_1 J_2} J - \frac{\varrho_0}{J_1} \vartheta \right),$$

$$(3.11) \quad \begin{aligned} c_v \varrho_0 \partial_t \vartheta - \kappa \partial_y \left(\frac{\partial_y \vartheta}{J_1} \right) &= -\kappa \partial_y \left(\frac{\partial_y \vartheta_2}{J_1 J_2} J \right) + \left[\mu \left(\frac{\partial_y v_1}{J_1} + \frac{\partial_y v_2}{J_2} \right) - R \frac{\varrho_0 \vartheta_2}{J_2} \right] \partial_y v \\ &\quad - R \frac{\varrho_0 \partial_y v_1}{J_1} \vartheta - \frac{\partial_y v_1}{J_1 J_2} (\mu \partial_y v_2 - R \varrho_0 \vartheta_2) J. \end{aligned}$$

Multiplying (3.10) by v , integrating the resultant over $(0, L)$, and integrating by parts, it follows from the Young and Sobolev embedding inequalities, and the regularities of (J_i, v_i, ϑ_i) , $i = 1, 2$, that

$$\frac{1}{2} \frac{d}{dt} \|\sqrt{\varrho_0} v\|_2^2 + \mu \left\| \frac{\partial_y v}{\sqrt{J_1}} \right\|_2^2 = \int_0^L \left[\mu \frac{\partial_y v_2}{J_1 J_2} J + R \left(\frac{\varrho_0}{J_1} \vartheta - \frac{\varrho_0 \vartheta_2}{J_1 J_2} J \right) \right] \partial_y v dy$$

$$\begin{aligned} &\leq \frac{\mu}{2} \left\| \frac{\partial_y v}{\sqrt{J_1}} \right\|_2^2 + C \int_0^L \left(\frac{|\partial_y v_2|^2 + \varrho_0^2 \vartheta_2^2}{J_1 J_2^2} J^2 + \frac{\varrho_0^2 \vartheta^2}{J_1} \right) dy \\ &\leq \frac{\mu}{2} \left\| \frac{\partial_y v}{\sqrt{J_1}} \right\|_2^2 + C(\|(\partial_y v_2, \vartheta_2)\|_\infty^2 + 1)\|(J, \sqrt{\varrho_0} \vartheta)\|_2^2 \\ &\leq \frac{\mu}{2} \left\| \frac{\partial_y v}{\sqrt{J_1}} \right\|_2^2 + C(\|J\|_2^2 + \|\sqrt{\varrho_0} \vartheta\|_2^2) \end{aligned}$$

and, thus,

$$(3.12) \quad \frac{d}{dt} \|\sqrt{\varrho_0} v\|_2^2 + \mu \left\| \frac{\partial_y v}{\sqrt{J_1}} \right\|_2^2 \leq C(\|J\|_2^2 + \|\sqrt{\varrho_0} \vartheta\|_2^2).$$

Multiplying (3.11) by ϑ , integrating the resultant over $(0, L)$, and integrating by parts, it follows from the Hölder, Young, and Sobolev embedding inequalities, and the regularities of (J_i, v_i, ϑ_i) , $i = 1, 2$, that

$$\begin{aligned} &\frac{c_v}{2} \frac{d}{dt} \|\sqrt{\varrho_0} \vartheta\|_2^2 + \kappa \left\| \frac{\partial_y \vartheta}{\sqrt{J_1}} \right\|_2^2 \\ &= \int_0^L \left\{ \kappa \frac{\partial_y \vartheta_2}{J_1 J_2} J \partial_y \vartheta + \left[\mu \left(\frac{\partial_y v_1}{J_1} + \frac{\partial_y v_2}{J_2} \right) - R \frac{\varrho_0 \vartheta_2}{J_2} \right] \partial_y v \vartheta \right\} dy \\ &\quad - R \int_0^L \frac{\varrho_0 \partial_y v_1}{J_1} \vartheta^2 dy - \int_0^L \frac{\partial_y v_1}{J_1 J_2} (\mu \partial_y v_2 - R \varrho_0 \vartheta_2) J \vartheta dy \\ &\leq C \left\| \frac{\partial_y \vartheta}{\sqrt{J_1}} \right\|_2 \|\partial_y \vartheta_2\|_\infty \|J\|_2 + C(\|\partial_y v_1\|_2 + \|\partial_y v_2\|_2 + \|\vartheta_2\|_2) \left\| \frac{\partial_y v}{\sqrt{J_1}} \right\|_2 \|\vartheta\|_\infty \\ &\quad + C\|\partial_y v_1\|_\infty \|\sqrt{\varrho_0} \vartheta\|_2^2 + C\|\partial_y v_1\|_\infty (\|\partial_y v_2\|_2 + \|\vartheta_2\|_2) \|J\|_2 \|\vartheta\|_\infty \\ &\leq \frac{\kappa}{4} \left\| \frac{\partial_y \vartheta}{\sqrt{J_1}} \right\|_2^2 + C(\|J\|_2^2 + \|\sqrt{\varrho_0} \vartheta\|_2^2) + \left(\|J\|_2 + \left\| \frac{\partial_y v}{\sqrt{J_1}} \right\|_2 \right) \|\vartheta\|_\infty, \end{aligned}$$

from which, one obtains

$$(3.13) \quad c_v \frac{d}{dt} \|\sqrt{\varrho_0} \vartheta\|_2^2 + \frac{3\kappa}{4} \left\| \frac{\partial_y \vartheta}{\sqrt{J_1}} \right\|_2^2 \leq C\|(J, \sqrt{\varrho_0} \vartheta)\|_2^2 + \left(\|J\|_2 + \left\| \frac{\partial_y v}{\sqrt{J_1}} \right\|_2 \right) \|\vartheta\|_\infty.$$

Using the same arguments as those for the estimate $\|\vartheta\|_\infty$ in (i) of Proposition 2.4, one can show that

$$(3.14) \quad \|\vartheta\|_\infty \leq C \left(\|\sqrt{\varrho_0} \vartheta\|_2 + \left\| \frac{\partial_y \vartheta}{\sqrt{J_1}} \right\|_2 \right).$$

By the aid of this and using the Young inequality, one gets from (3.13) that

$$(3.15) \quad c_v \frac{d}{dt} \|\sqrt{\varrho_0} \vartheta\|_2^2 + \kappa \left\| \frac{\partial_y \vartheta}{\sqrt{J_1}} \right\|_2^2 \leq A \left(\|J\|_2^2 + \|\sqrt{\varrho_0} \vartheta\|_2^2 + \left\| \frac{\partial_y v}{\sqrt{J_1}} \right\|_2^2 \right),$$

where A is a positive constant.

Multiplying (3.9) by $2J$, and integrating the resultant over $(0, L)$, it follows from the Young inequality that

$$(3.16) \quad \frac{d}{dt} \|J\|_2^2 \leq A \left\| \frac{\partial_y v}{\sqrt{J_1}} \right\|_2^2 + C\|J\|_2^2.$$

Multiplying (3.12) by $\frac{2A}{\mu}$, summing the resultant with (3.15) and (3.16), one obtains

$$\begin{aligned} \frac{d}{dt} \left(\frac{2A}{\mu} \|\sqrt{\varrho_0} v\|_2^2 + c_v \|\sqrt{\varrho_0} \vartheta\|_2^2 + \|J\|_2^2 \right) + A \left\| \frac{\partial_y v}{\sqrt{J_1}} \right\|_2^2 \\ + \kappa \left\| \frac{\partial_y \vartheta}{\sqrt{J_1}} \right\|_2^2 \leq C (\|J\|_2^2 + \|\sqrt{\varrho_0} \vartheta\|_2^2), \end{aligned}$$

from which, by the Gronwall inequality, one gets

$$\sqrt{\varrho_0} v \equiv \sqrt{\varrho_0} \vartheta \equiv J \equiv \partial_y v \equiv \partial_y \vartheta \equiv 0.$$

Therefore, recalling (3.14) and its counterpart for v , we have

$$J \equiv v \equiv \vartheta \equiv 0.$$

This proves the uniqueness. \square

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